Derivation of equations (4.15) and (4.16) from equation (4.14).

Taken from section 4.1.2 of Werner's book.

of summing over 575 757 states, there is now an equivalent sum over $6^{35} \simeq 1.72 \times 10^{27}$ terms. However, eqn (4.13) has been put together from an unrestricted sum and a constraint (a Kronecker δ-function), and the equation can again be simplified using the providential integral representation given in Subsection 4.1.1 (see eqn (4.3)):

$$Z_{\text{btm}}(\beta) = \int_{-\pi}^{\pi} \frac{d\lambda}{2\pi} e^{-iN\lambda} \times \underbrace{\left(\sum_{n_0} e^{n_0(-\beta E_0 + i\lambda)} \right)}_{f_0(\beta,\lambda)} \cdots \underbrace{\left(\sum_{n_{34}} e^{n_{34}(-\beta E_{34} + i\lambda)} \right)}_{f_{34}(\beta,\lambda)}. \quad (4.14)$$

In this difficult problem, the summations have again become independent, and can be performed. As written in eqn (4.13), the sums should go from 0 to N for all states. However, for the excited states ($E > 0$), the absolute value of $\Upsilon = e^{-\beta E + i\lambda}$ is smaller than 1:

$$|\Upsilon| = \left| e^{-\beta E} e^{i\lambda} \right| = \underbrace{\left| e^{-\beta E} \right|}_{\substack{<1 \text{ if} \\ E>0}} \underbrace{\left| e^{i\lambda} \right|}_{1},$$

which allows us to take the sum to infinity:

$$\sum_{n=0}^{N} \Upsilon^n = \frac{1 - \Upsilon^{N+1}}{1 - \Upsilon} \quad \xrightarrow[|\Upsilon|<1]{N \to \infty} \quad \frac{1}{1 - \Upsilon}.$$

The Kronecker δ-function picks up the correct terms even from an infinite sum, and we thus take the sums in eqn (4.14) for the excited states to infinity, but treat the ground state differently, using a finite sum. We also take into account the fact that the f_k depend only on the energy E_k, and not explicitly on the state number:

$$f_E(\beta, \lambda) = \frac{1 - \exp[i(N+1)\lambda]}{1 - \exp(i\lambda)}, \quad E = 0, \text{ (ground state)}, \quad (4.15)$$

$$f_E(\beta, \lambda) = \frac{1}{1 - \exp(-\beta E + i\lambda)}, \quad E > 0, \text{ (excited state)}. \quad (4.16)$$

(The special treatment of the ground state is "naive", that is, appropriate for a first try. In Subsection 4.1.3, we shall move the integration contour for λ in the complex plane, and work with infinite sums for all energies.)

The partition function is finally written as

$$Z_N(\beta) = \int_{-\pi}^{\pi} \frac{d\lambda}{2\pi} \underbrace{e^{-iN\lambda} \prod_{E=0}^{E_{\max}} [f_E(\beta, \lambda)]^{\mathcal{N}(E)}}_{Z_N(\beta,\lambda)}. \quad (4.17)$$

This equation is generally useful for an N-particle problem. For the five-boson bounded trap model, we can input $\mathcal{N}(E) = \frac{1}{2}(E+1)(E+2)$