Shifts Proof

September 25, 2025

We consider two main cases, namely $shifts(ms, r) = []$ and $shifts(ms, r) \neq []$.

1 Case 1: $shifts(ms, r) = []$

We want to prove

$$Strips(L(r^n))(set\, ms) = \varnothing.$$

From the induction hypothesis

$$set(shifts(ms, r)) = Strips(L(r))(set\, ms).$$

By assumption

$$set(shifts(ms, r)) = \varnothing,$$

so

$$Strips(L(r))(set\, ms) = \varnothing.$$

1.1 sub case $nullable(r)$

unfolding the shifts definition,

$$set(shifts(ms, r^n)) = set(shifts(ms, r)) \cup set(shifts(shifts(ms, r), r^{n-1})),$$

$$= Strips(L(r))(set\, ms) \cup Strips(L(r^{n-1}))(Strips(L(r))(set\, ms)),$$

$$= \varnothing \cup Strips(L(r^{n-1}))(\varnothing) = \varnothing.$$

1.2 sub case $\neg nullable(r)$

again, unfolding the shifts definition,

$$
\begin{aligned}
set(shifts(ms, r^n)) &= set(shifts(shifts(ms, r), r^{n-1})), \\
&= Strips(L(r^{n-1}))(set(shifts(ms, r))), \\
&= Strips(L(r^{n-1}))(\varnothing) = \varnothing.
\end{aligned}
$$

Therefore

$$Strips(L(r^n))(set\, ms) = \varnothing.$$

2 Case 2: $shifts(ms, r) \neq []$

We want to prove

$$set(shifts(ms, r^n)) = Strips(L(r^n))(set\, ms).$$

2.1 sub case $nullable(r)$

unfolding the shifts definition,

$$set(shifts(ms, r^n)) = a \cup b,$$

$$a := set(shifts(ms, r)), \quad b := set(shifts(shifts(ms, r), r^{n-1})).$$

by IH on $P(ms, r)$,

$$a = Strips(L(r))(set\, ms).$$

since $shifts(ms, r)$ is smaller than ms and r^{n-1} is smaller than r^n, we can use IH on $P(shifts(ms, r), r^{n-1})$,

$$b = Strips(L(r^{n-1}))(set(shifts(ms, r))) = Strips(L(r^{n-1}))(Strips(L(r))(set\, ms)).$$

therefore

$$set(shifts(ms, r^n)) = a \cup b = Strips(L(r))(set\, ms) \cup Strips(L(r^{n-1}))(Strips(L(r))(set\, ms)).$$

by Strips properties: with nullable A, and B

$$Strips(A \,@\, B)C = (Strips\, B\, (Strips\, A\, C)) \cup Strips\, A\, C \cup Strips\, B\, C$$

$$A = L(r) \;\; \texttt{and} \;\; B = L(r^{n-1})$$
$$Lr^n = L(r) \,@\, L(r^{n-1})$$

we have $Strips\, B\, (Strips\, A\, C) = Strips(L(r^{n-1}))(Strips(L(r))(set\, ms))$ and $Strips\, A\, C = Strips(L(r))(set\, ms)$

but missing the third $Strips\, B\, C = Strips(L(r^{n-1}))(set\, ms)$

2.2 sub case $\neg nullable(r)$

unfolding the shifts definition,

$$set(shifts(ms, r^n)) = b,$$

$$b := set(shifts(shifts(ms, r), r^{n-1})).$$

since $shifts(ms, r)$ is smaller than ms and r^{n-1} is smaller than r^n, we can use IH on $P(shifts(ms, r), r^{n-1})$,

$$b = Strips(L(r^{n-1}))(set(shifts(ms, r))) = Strips(L(r^{n-1}))(Strips(L(r))(set\, ms)).$$

by Strips properties: with not nullable A, and B

$$Strips(A\,@\,B)C = Strips\,B\,(Strips\,A\,C)$$

$$A = L(r) \ \texttt{and} \ B = L(r^{n-1})$$
$$Lr^n = L(r)\,@\,L(r^{n-1})$$

therefore

$$set(shifts(ms, r^n)) = Strips(L(r^n))(set\, ms)$$

3 notes

3.1 case 1

i thought i would start with two cases to consider, namely $shifts(ms, r) = []$ and $shifts(ms, r) \neq []$. similar to the urlues.

i started with first one. so first, i assumed that $shifts(ms, r) = []$ and need to prove $Strips(r^n)(setms) = \varnothing$.

since r is smaller, we can use $p(ms, r^n)$ so

$set(shifts(msr)) = Strips(Lr)(setms)$.

by assumption, lhs is $\varnothing$ so now we have $Strips(Lr)(setms) = \varnothing$.

now we need to prove $Strips(Lr^n)(setms) = \varnothing$. again two cases to consider, 1- nullable r.

unfolding shifts definition.

$$set(shiftsr^n ms) = set(shiftsrms)Uset(shifts(shiftsmsr)r^n - 1)$$

that equals to $Strip(Lr)(setms)UStrips(Lr^n - 1)(StripsLr)(setms)$.

the lhs of union $= \varnothing$

and rhs$=$ $Strips(Lr^n - 1)\varnothing(setms) = \varnothing$.

in the second case where r is not nullable:

unfolding shifts again:

$$set(shiftsr^n ms) = set(shiftsr^{n-1}(shiftsrms)) = Strip(r^{n-1})(Strip(Lrms))$$

and since $Strip(Lrms) = \varnothing$ so it becomes $Strip(r^n - 1)\varnothing$ and so $= \varnothing$.

3.2 case 2

NTIMES: for the second case, i assume $shifts(ms, r) \neq []$. i need to prove $set(shifts(ms, r^n)) = Strips(L\,r^n)(set\,ms)$. again two cases to consider, 1- nullable r.

unfolding shifts definition, $set(shifts(ms, r^n)) = a \cup b$ where $a = set(shifts(ms, r))$ $b = set(shifts(shifts(ms, r), r^{n-1}))$

for a, use IH $P(ms, r)$ to get $a = Strips(L\,r)(set\,ms)$

for b, we can use IH $P(shifts(ms, r), r^{n-1})$ since (i) $shifts(ms, r)$ is smaller than ms and (ii) r^{n-1} is smaller than r^n. hence $b = Strips(L\,r^{n-1})(set(shifts(ms, r))) = Strips(L\,r^{n-1})(Strips(L\,r)(set\,ms))$

therefore $set(shifts(ms, r^n)) = a \cup b = Strips(L\, r)(set\, ms) \cup Strips(L\, r^{n-1})(Strips(L\, r)(set\, r$

in the second case where r is not nullable:

unfolding shifts again, $set(shifts(ms, r^n)) = b$ where $b = set(shifts(shifts(ms, r), r^{n-1})) = Strips(L\, r^{n-1})(set(shifts(ms, r))) = Strips(L\, r^{n-1})(Strips(L\, r)(set\, ms))$

and since $L(r^n) = L(r) @ L(r^{n-1})$ we have $b = Strips(L\, r @ L\, r^{n-1})(set\, ms) = Strips(L\, r^n)(set\, ms)$