Proof of :r^n

We want to prove, for every $n \geq 0$ and every set of strings ms:

$$P(n, ms) \ : \ \ \textit{shifts}(ms, r^n) \ = \ \textit{Strips}(L(r^n))(ms).$$

We consider the following:

$$L(r^0) = \emptyset, \qquad L(r^1) = L(r), \qquad L(r^{n+1}) = L(r)@L(r^n).$$

1 Case $n = 0$.

By unfolding the definition:

$$\textit{shifts}(ms, r^0) = \emptyset.$$

Since $L(r^0) = \emptyset$,

$$\textit{Strips}(L(r^0))(ms) = \textit{Strips}(\emptyset)(ms) = \emptyset.$$

Thus

$$\textit{shifts}(ms, r^0) = \textit{Strips}(L(r^0))(ms),$$

Therefore $P(0, ms)$ holds.

2 Case $n = 1$.

By unfolding the definition:

$$\textit{shifts}(ms, r^1) = \textit{shifts}(ms, r).$$

By IH:

$$\textit{shifts}(ms, r) = \textit{Strips}(L(r))(ms) = \textit{Strips}(L(r^1))(ms).$$

Therefore $P(1, ms)$ holds.

3 Case $n > 1$.

we split the proof by two cases of $\textit{shifts}(ms, r)$.

3.1 $\mathit{shifts}(ms, r) = \emptyset$.

By unfolding the definition:

$$\mathit{shifts}(ms, r^n) = \emptyset.$$

Since

$$\mathit{shifts}(ms, r) = \emptyset \quad \Rightarrow \quad \mathit{Strips}(L(r))(ms) = \emptyset.$$

If $\mathit{Strips}(L(r))(ms) = \emptyset$, then

$$\mathit{Strips}(L(r^n))(ms) = \emptyset.$$

So

$$\mathit{shifts}(ms, r^n) = \emptyset = \mathit{Strips}(L(r^n))(ms),$$

Therefore $P(n, ms)$ holds in Subcase 3.1.

3.2 $\mathit{shifts}(ms, r) \neq \emptyset$.

Before unfolding further, introduce repeated stripping:

$$F_r(X) = \mathit{Strips}(L(r))(X)$$

for any set X, and define:

$$F_r^1(X) = F_r(X), \qquad F_r^{k+1}(X) = F_r(F_r^k(X)).$$

So:

$$\mathit{shifts}(ms, r) = \mathit{Strips}(L(r))(ms) = F_r(ms).$$

We now have two further cases according to $\mathit{nullable}(r)$.

3.2.1 $\neg\mathit{nullable}(r)$.

by unfolding shifts defintion:

$$\mathit{shifts}(ms, r^n) = \mathit{shifts}\big(\mathit{shifts}(ms, r),\ r^{n-1}\big).$$

By IH:

$$\mathit{shifts}\big(\mathit{shifts}(ms, r), r^{n-1}\big) = \mathit{Strips}(L(r^{n-1}))\big(\mathit{Strips}(L(r))(ms)\big).$$

Using $Strips(L(r))(ms) = F_r(ms)$, we get

$$shifts(ms, r^n) = Strips(L(r^{n-1}))\big(F_r(ms)\big).$$

By Strips properties: with not nullable A, and B

$$Strips(A\,@\,B)C = Strips\,B\,(Strips\,A\,C)$$

$$A = L(r) \;\; \texttt{and} \;\; B = L(r^{n-1})$$

$$\begin{aligned} Strips(L(r^n))(ms) = &\quad Strips(L(r)@L(r^{n-1}))(ms) \\ = &\quad Strips\big(L(r^{n-1}), Strips(L(r), ms)\big) \\ = &\quad Strips(L(r^{n-1}))(F_r(ms)). \end{aligned}$$

So

$$shifts(ms, r^n) = Strips(L(r^n))(ms),$$

Therefore $P(n, ms)$ holds in Subcase 3.2.1.

3.3 $nullable(r)$.

By unfolding shifts defintion:

$$shifts(ms, r^n) = shifts(ms, r) \;\cup\; shifts\big(shifts(ms, r),\; r^{n-1}\big),$$

By IH on $shifts(ms, r) = Strips(L(r))(ms) = F_r(ms)$, we get

$$shifts(ms, r^n) = F_r(ms) \;\cup\; shifts\big(F_r(ms),\; r^{n-1}\big).$$

By IH applied to $n-1$,

$$shifts\big(F_r(ms), r^{n-1}\big) = Strips(L(r^{n-1}))\big(F_r(ms)\big).$$

then the unfolded definition becomes:

$$shifts(ms, r^n) = F_r(ms) \cup Strips(L(r^{n-1}))\big(F_r(ms)\big).$$

We now use:

Lemma 1 (nullable r^m). *If $nullable(r)$, then for every $m \geq 1$ and every set X,*

$$Strips(L(r^m))(X) = \bigcup_{k=1}^{m} F_r^k(X).$$

by induction on m, using all nullable concatenation property of Strips. (at end of the file)

Apply Lemma 1 with $m = n - 1$ and $X = F_r(ms)$:

$$Strips(L(r^{n-1}))(F_r(ms)) = \bigcup_{k=1}^{n-1} F_r^k(F_r(ms)) = \bigcup_{k=2}^{n} F_r^k(ms).$$

So

$$shifts(ms, r^n) = F_r(ms) \cup \bigcup_{k=2}^{n} F_r^k(ms) = \bigcup_{k=1}^{n} F_r^k(ms).$$

Applying Lemma 1 again with $m = n$ and $X = ms$,

$$Strips(L(r^n))(ms) = \bigcup_{k=1}^{n} F_r^k(ms).$$

So

$$shifts(ms, r^n) = Strips(L(r^n))(ms),$$

Therefore $P(n, ms)$ holds in Subcase 3.3.

Since all cases $n = 0$, $n = 1$, and $n > 1$ (with subcases) have been included, so:

$$shifts(ms, r^n) = Strips(L(r^n))(ms).$$

4 Detailed proof of Lemma 1

Proof of Lemma 1. By induction on m.

Base case $m = 1$. Since $L(r^1) = L(r)$, we have

$$\mathit{Strips}(L(r^1))(X) = \mathit{Strips}(L(r))(X) = F_r(X) = \bigcup_{k=1}^{1} F_r^k(X).$$

So the statement holds for $m = 1$.

Induction step. Assume for some $m \geq 1$ and all sets X,

$$\mathit{Strips}(L(r^m))(X) = \bigcup_{k=1}^{m} F_r^k(X).$$

We must show, for all X, that:

$$\mathit{Strips}(L(r^{m+1}))(X) = \bigcup_{k=1}^{m+1} F_r^k(X).$$

recall:

$$L(r^{m+1}) = L(r)@L(r^m).$$

Since lemma is for $\mathit{nullable}(r)$, so $[] \in L(r)$. and for r^m we also have $[] \in L(r^m)$. So we may apply the all nullables concatenation properties of Strips with $A = L(r)$, $B = L(r^m)$, $C = X$:

$$\begin{aligned} \mathit{Strips}(L(r^{m+1}))(X) &= \mathit{Strips}(L(r)@L(r^m))(X) \\ &= \mathit{Strips}(L(r^m), \mathit{Strips}(L(r), X)) \\ &\quad \cup\ \mathit{Strips}(L(r), X) \\ &\quad \cup\ \mathit{Strips}(L(r^m), X). \end{aligned} \tag{1}$$

By IH of the lemma:

$$\mathit{Strips}(L(r^m), X) = \bigcup_{k=1}^{m} F_r^k(X).$$

We use $F_r(X)$:

$$Strips(L(r), X) = F_r(X).$$

Also, by the IH:

$$\begin{aligned} Strips(L(r^m), Strips(L(r), X)) &= Strips(L(r^m), F_r(X)) \\ &= \bigcup_{k=1}^{m} F_r^k(F_r(X)) \\ &= \bigcup_{k=1}^{m} F_r^{k+1}(X) \\ &= \bigcup_{j=2}^{m+1} F_r^j(X). \end{aligned}$$

then (1) becomes:

$$Strips(L(r^{m+1}))(X) = \left(\bigcup_{j=2}^{m+1} F_r^j(X)\right) \cup F_r(X) \cup \left(\bigcup_{k=1}^{m} F_r^k(X)\right).$$

Since $F_r(X) = F_r^1(X)$, and already contained in $\bigcup_{k=1}^{m} F_r^k(X)$.

The union of $\bigcup_{k=1}^{m} F_r^k(X)$ and $\bigcup_{j=2}^{m+1} F_r^j(X)$ contains the sets $F_r^k(X)$ for $k = 1, 2, \ldots, m+1$. Therefore

$$Strips(L(r^{m+1}))(X) = \bigcup_{k=1}^{m+1} F_r^k(X).$$

This completes the induction step and the proof of the lemma. □