Strategies for Algorithm Design

Greedy Algorithms

This week

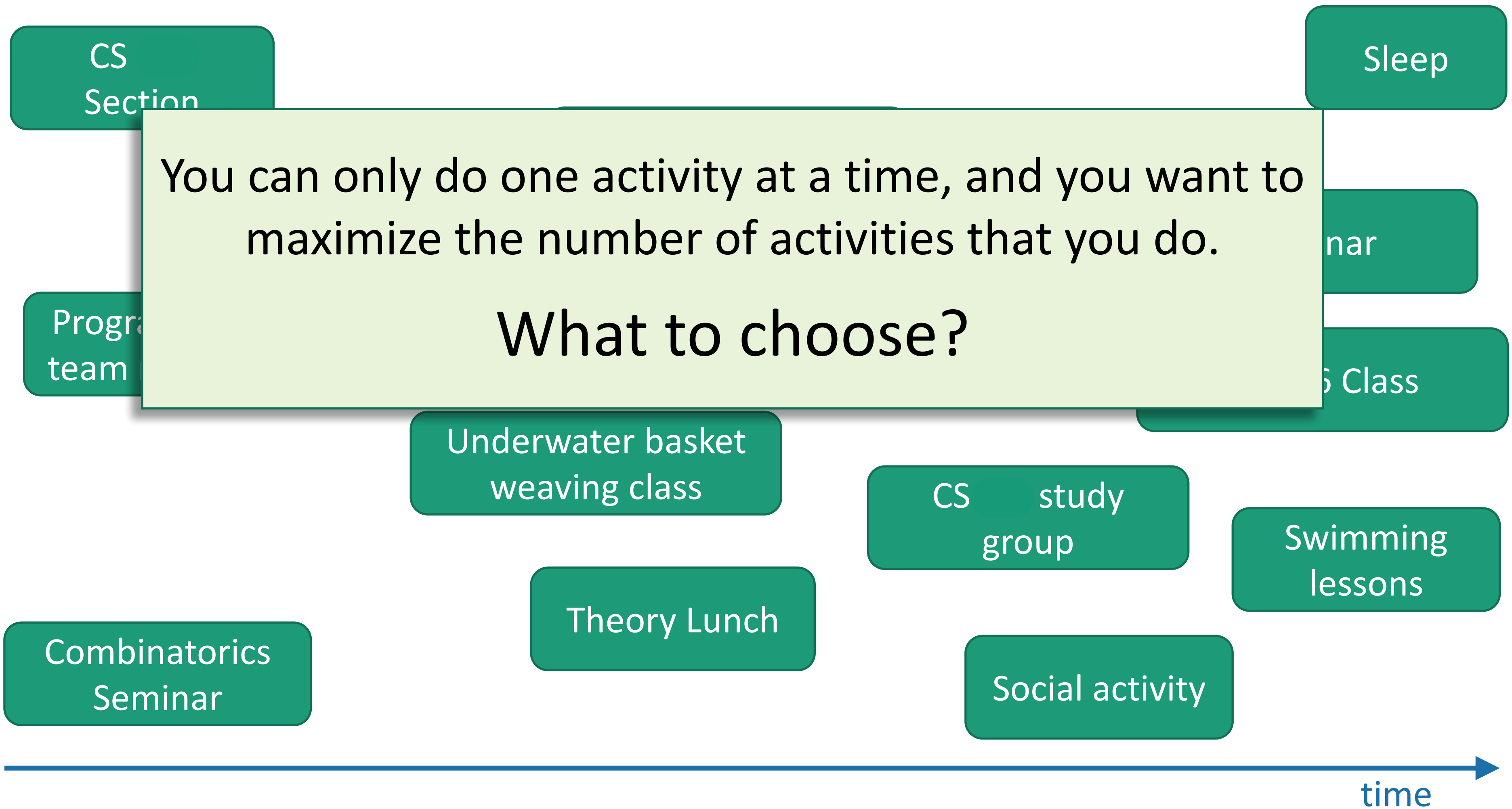
- Greedy algorithms!
- Builds on our ideas from dynamic programming

Example

Activity selection

You can only do one activity at a time, and you want to maximize the number of activities that you do.

What to choose?

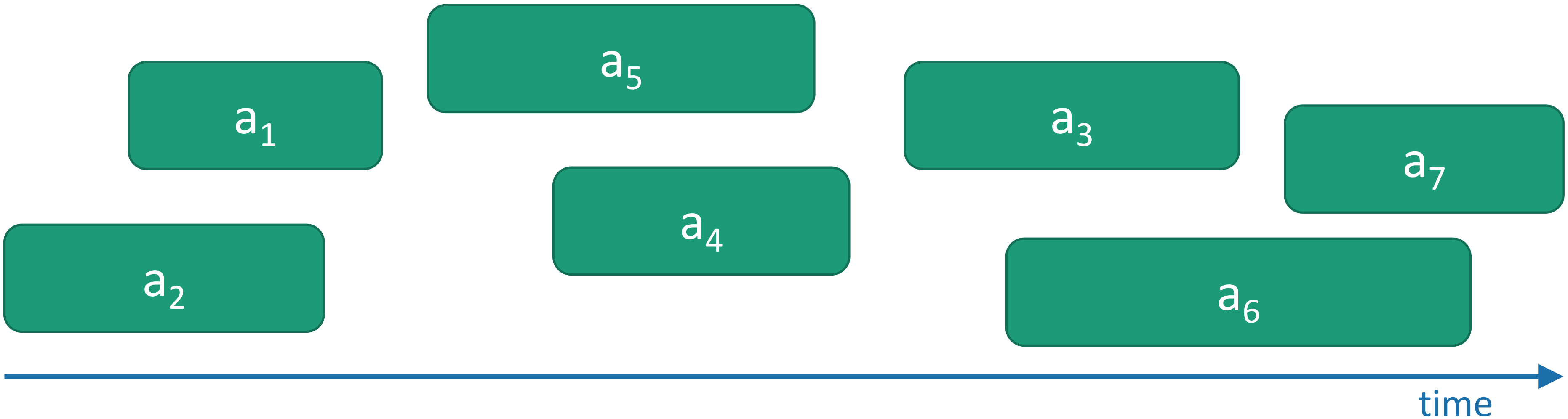


Activity selection

- Input:
  - Activities $a_1, a_2, \ldots, a_n$
  - Start times $s_1, s_2, \ldots, s_n$
  - Finish times $f_1, f_2, \ldots, f_n$

- Output:
  - How many activities can you do today?

Greedy Algorithm



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- Include it in your activity list.
- Repeat.

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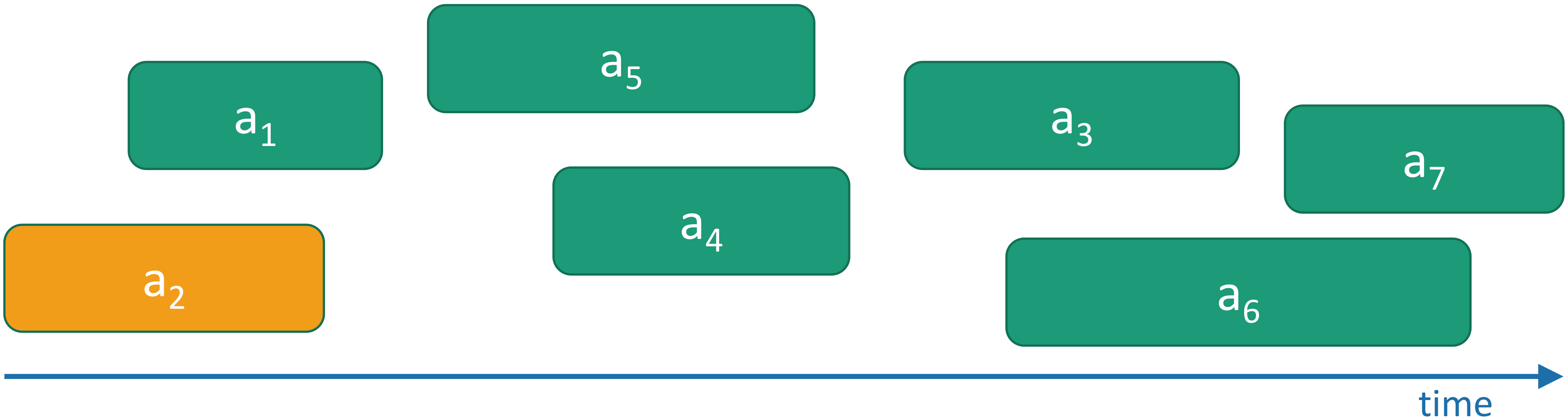

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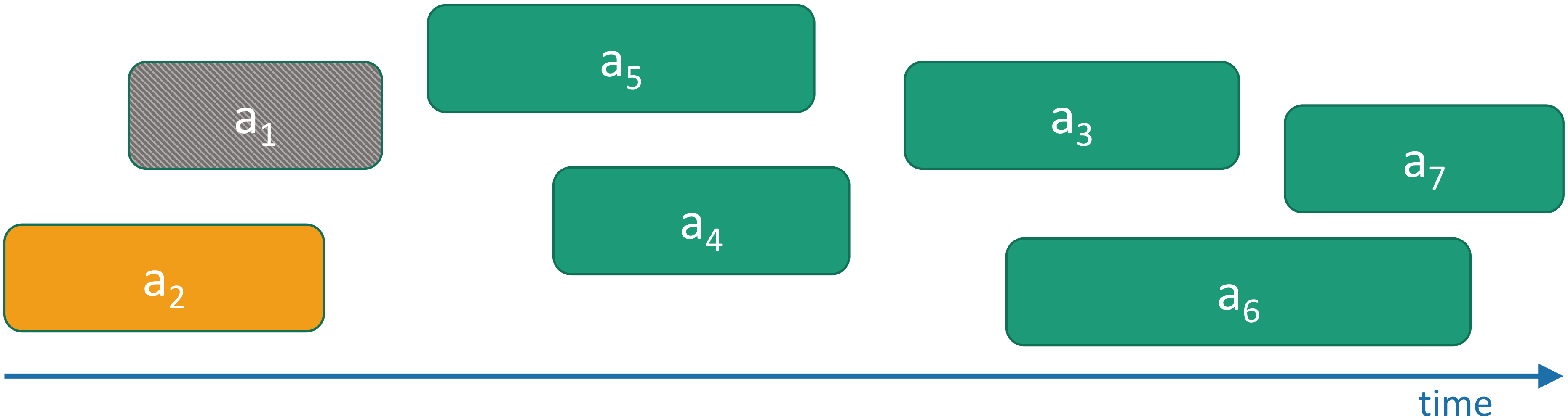

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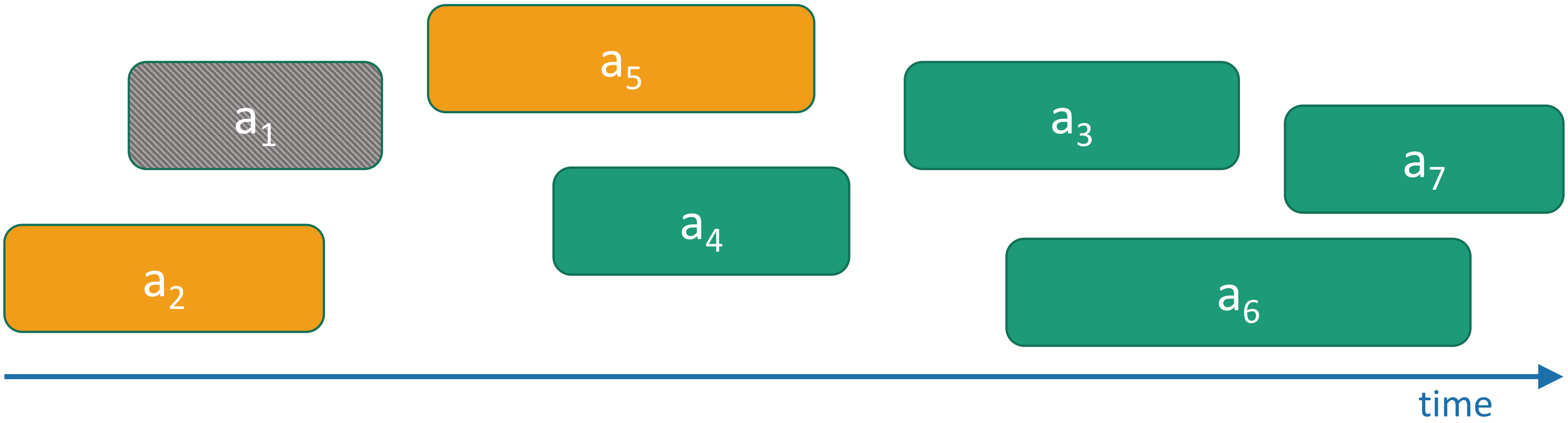

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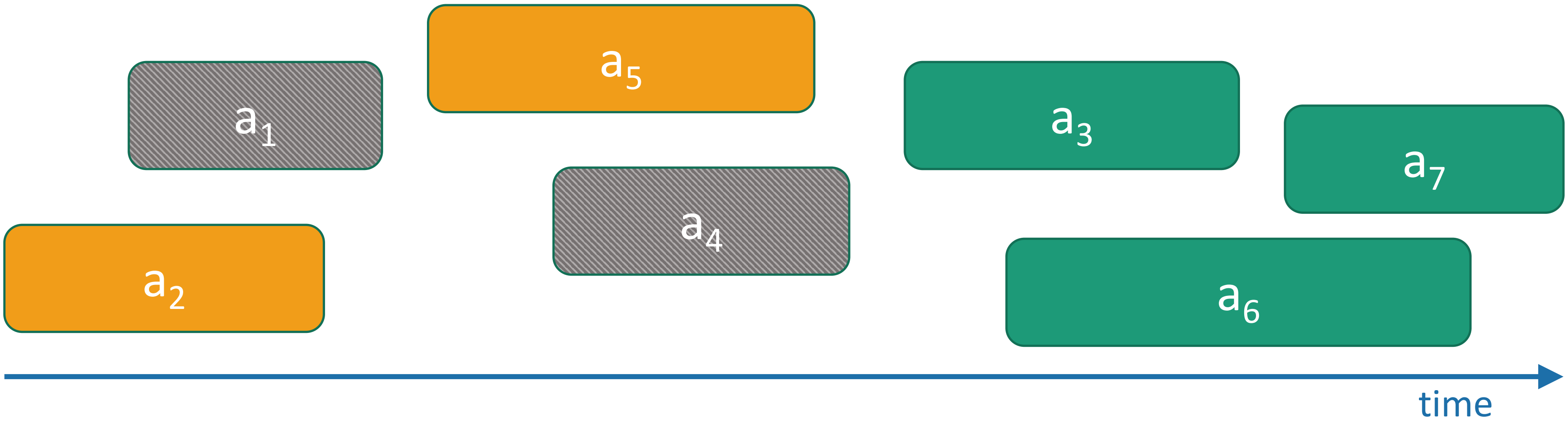

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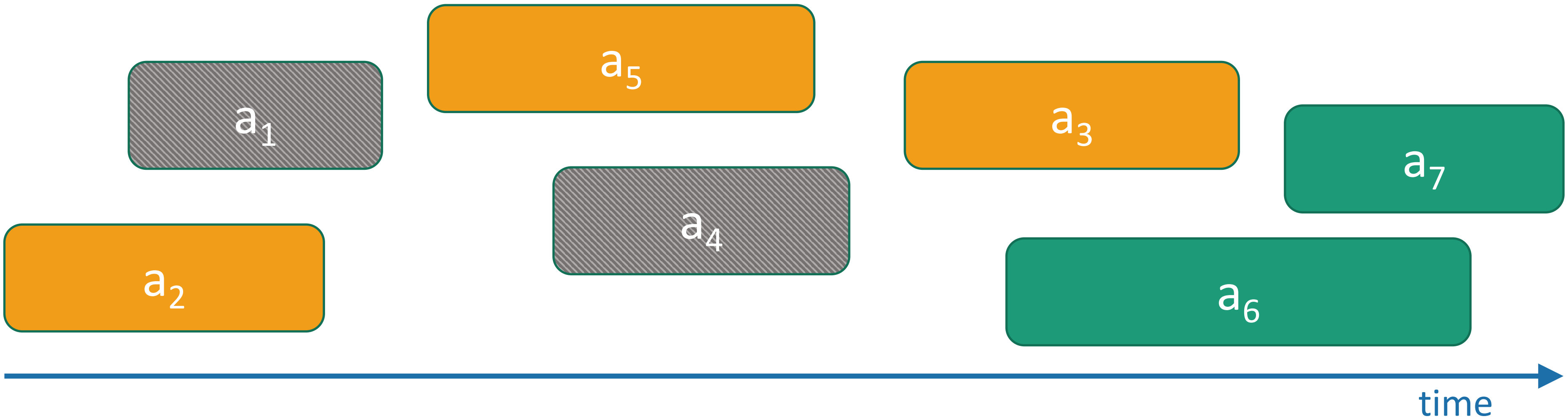

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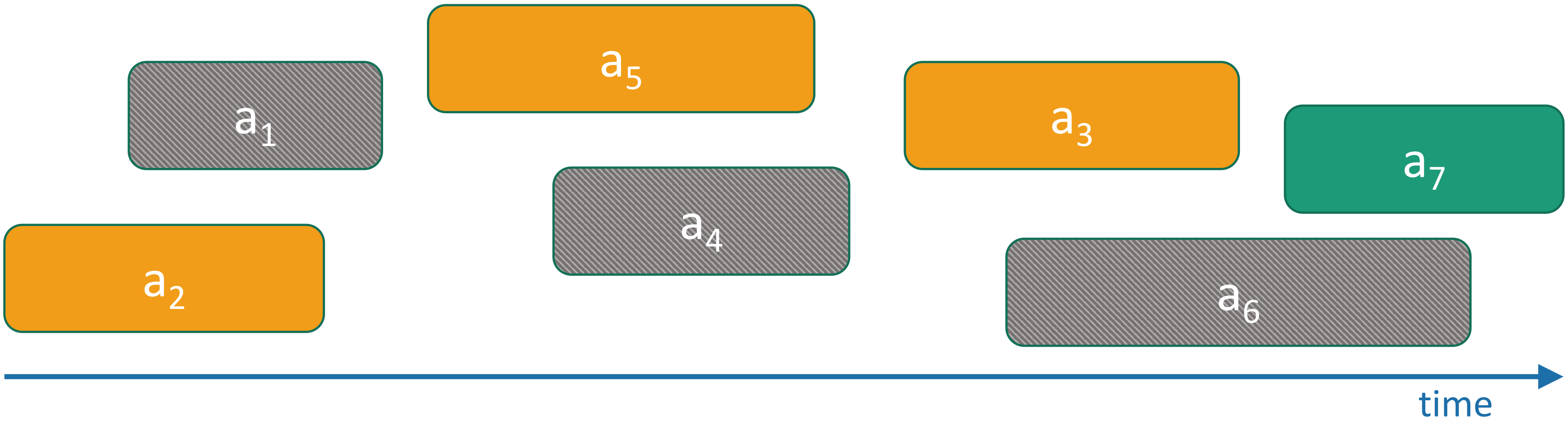

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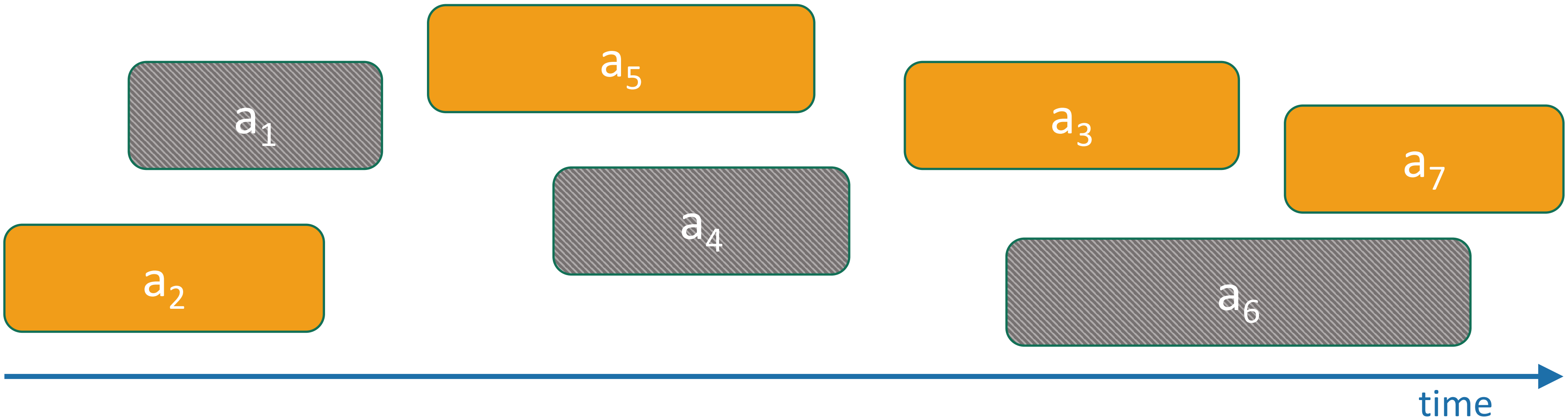

- Pick the activity you can add
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That seems like a reasonable thing to do…

- Running time:
  - O(n) if the activities are already sorted by finish time.
  - Otherwise O(nlog(n)) if you have to sort them first.
- Does it work?
  - We'll see soon.

This is an example of a greedy algorithm

- At each step in the algorithm, make a choice.
  - Hey, I can increase my activity set by one,
  - And leave lots of room for future choices,
  - Let’s do that and hope for the best!!!

- **Hope** that at the end of the day, this results in a globally optimal solution.

Three questions

1. Does this greedy algorithm for activity selection work?

2. In general, when are greedy algorithms a good idea?

3. The “greedy” approach is often the first you’d think of…

Answers

1. Does this greedy algorithm for activity selection work?
   - Yes.
2. In general, when are greedy algorithms a good idea?
   - When they exhibit especially nice optimal substructure.

3. The “greedy” approach is often the first you’d think of…

DP view of activity selection

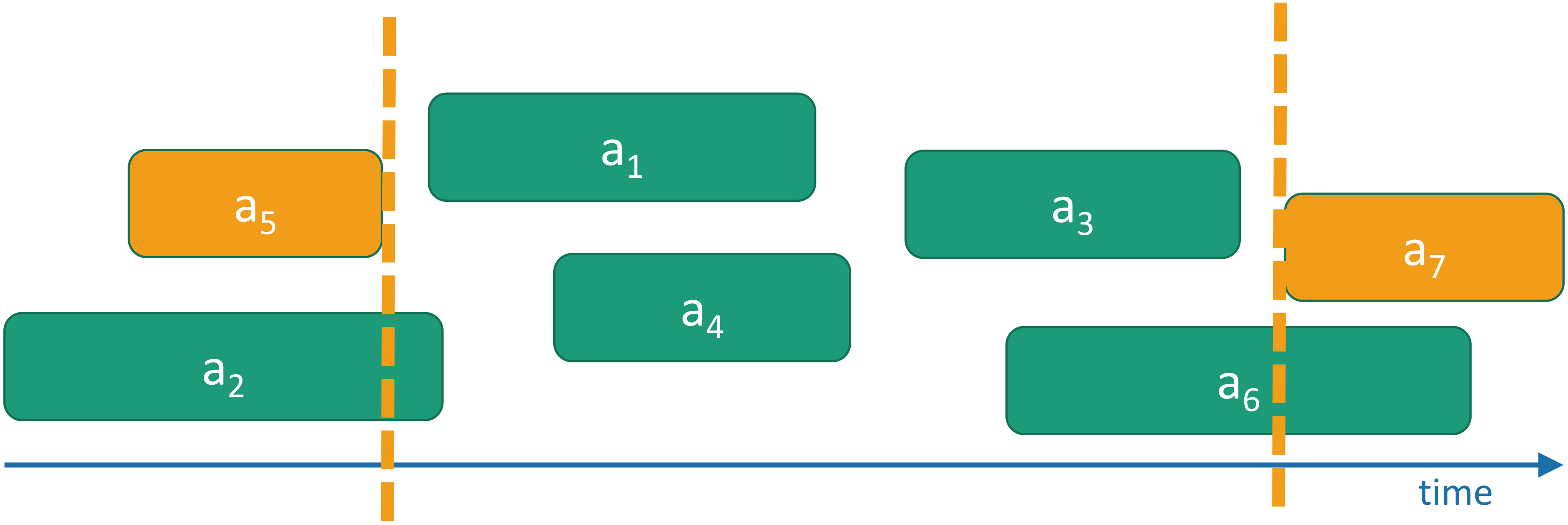
Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up like a reasonable person.

Optimal substructure

- Subproblems:

number of activities you can squeeze in after Activity i finishes and before Activity j starts



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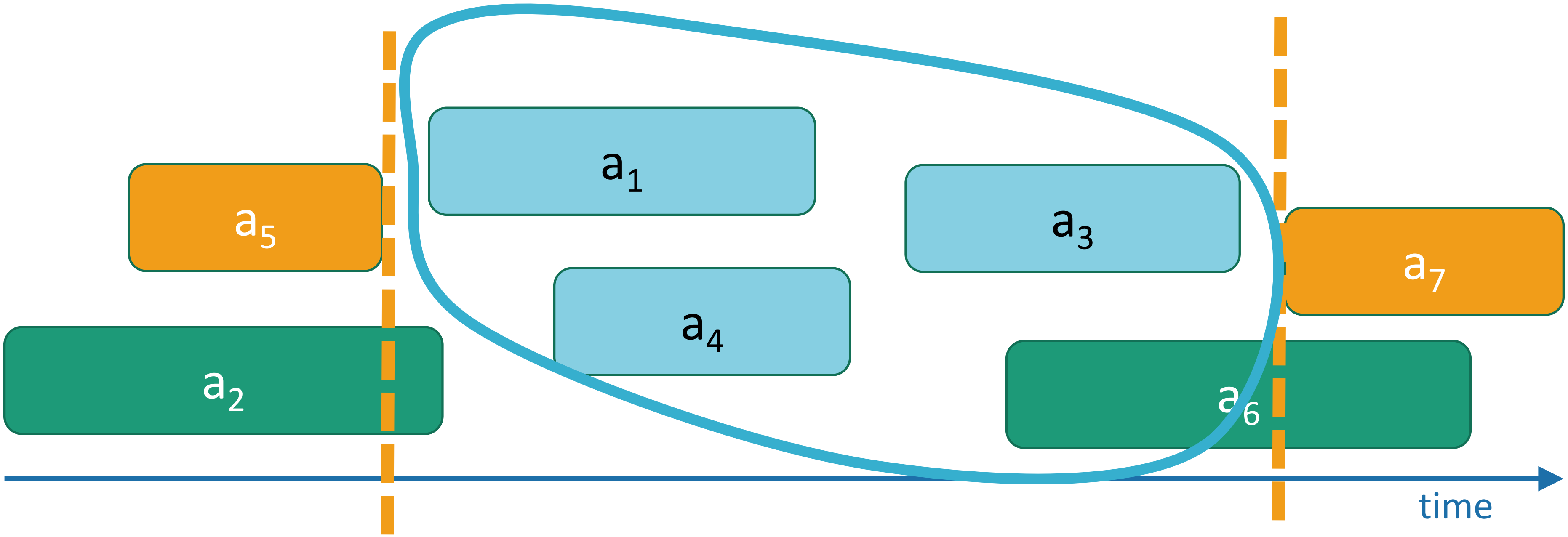


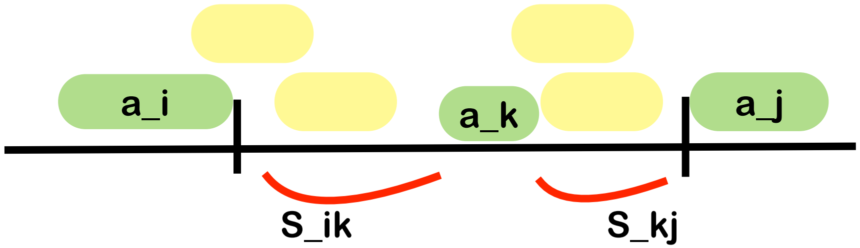
Now let's define an optimal solution, i.e., a maximal set of mutualy compatible activities between i,j.

Details for optimal substructure property

Then the candidate activities to consider are those that start after ai and end before aj:

Now let's define Aij to be an optimal solution, i.e., a maximal set of mutualy compatible activities in Sij. What is the structure of this solution?

At some point we will need to make a choice to include some activity ak, two sets of compatible candidates after ak is taken out:



Sik : activities that start after ai finishes, and finish before ak starts
Skj : activities that start after ak finishes, and finish aj starts

Note that Sij may be a proper superset of Sik ∪ {ak} ∪ Skj, as activities incompatible with ak are excluded.)

Using the same notation as above, define the optimal solutions to these subproblems to be:

Aik = Aij∩ Sik: the optimal solution to Sik
Akj = Aij ∩ Skj: the optimal solution to Skj

So the structure of an optimal solution Aij is:

Aij = Aik ∪ {ak} ∪ Akj

Recipe for applying Dynamic Programming

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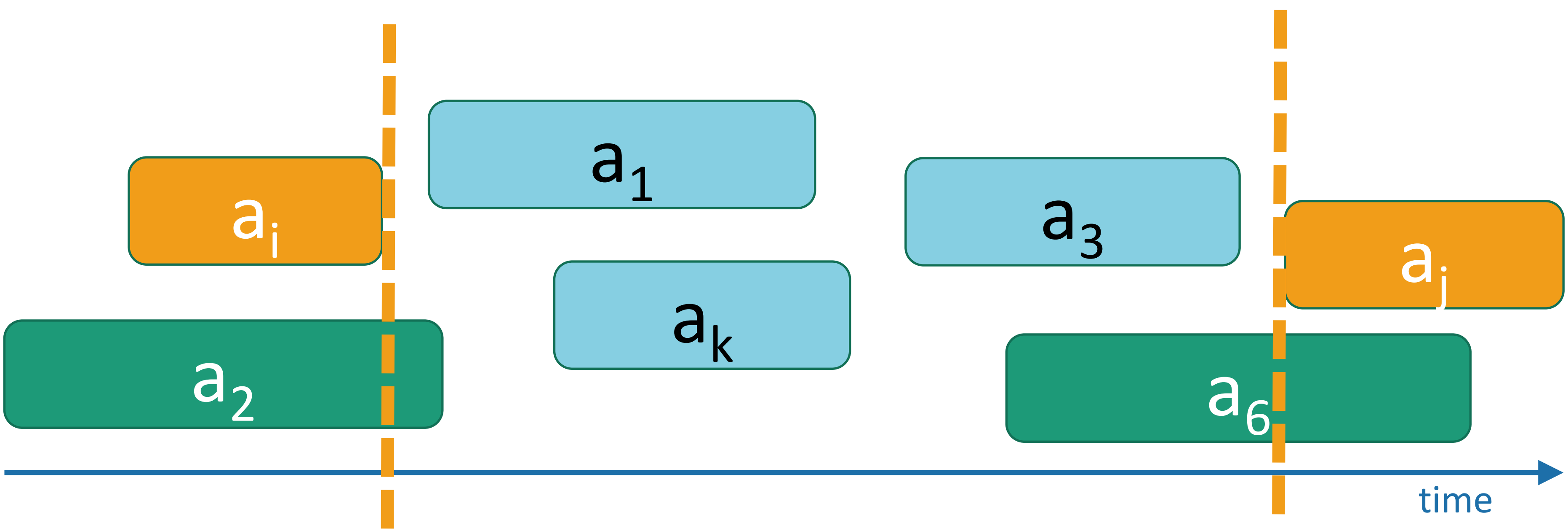
This satisfies a nice recursive relationship

A[i,j] = number of activities you can squeeze in after Activity i finishes and before Activity j starts

- $A[i,j] = \max_k \{ A[i,k] + 1 + A[k,j] \}$
  - The maximum is over all k so that Activity k fits in between Activities i and j.



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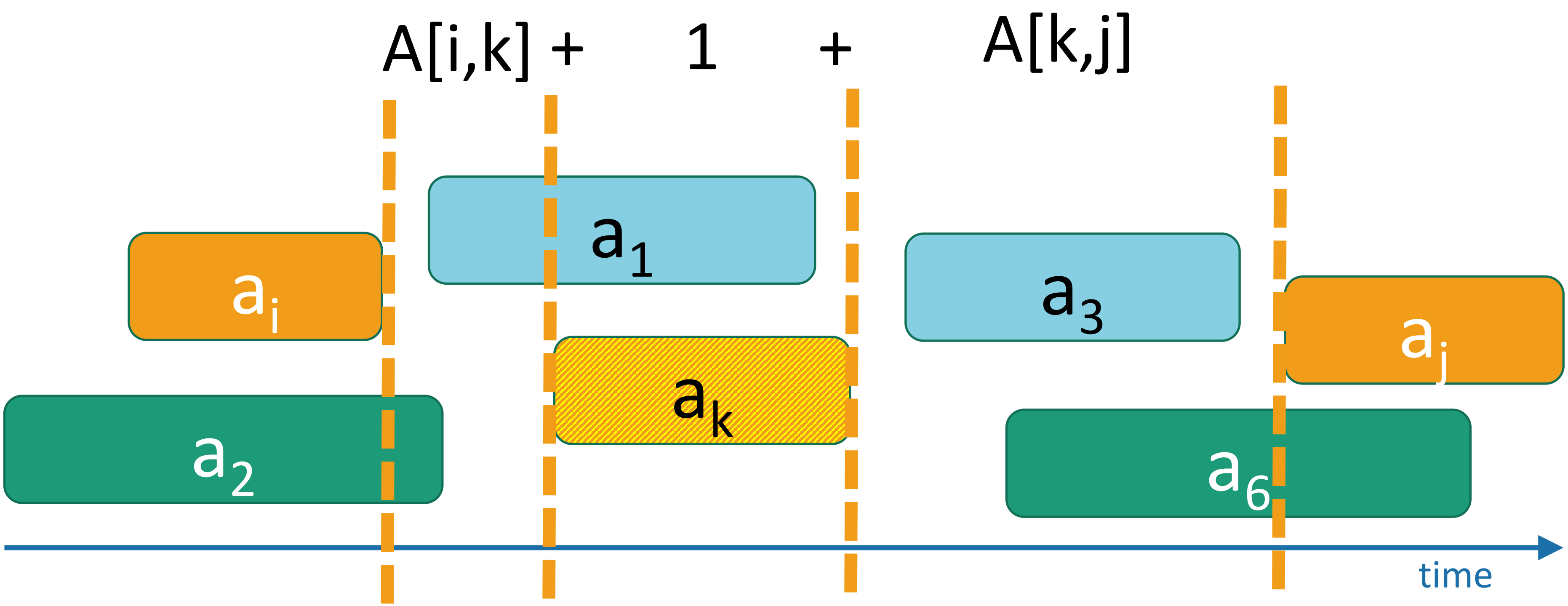
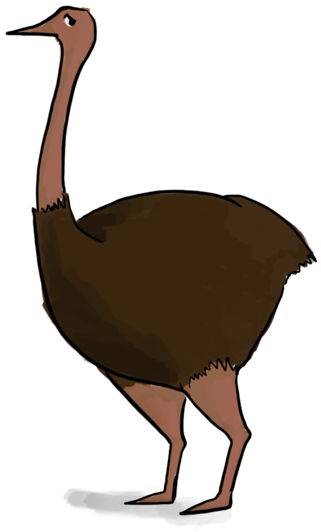


We could turn this into a DP algorithm

- .Would take time something like $O(n^3)$
  - Fill out an n-by-n table.
  - For each entry search over maybe n possiblities for k.
- But this would be wasteful!
  - we just saw an algorithm that takes time $O(n\log(n))$, if it's correct…

Try it!
It builds character!



The thing that's wasteful

$A[i,j]$ = number of activities you can squeeze in after Activity i finishes and before Activity j starts

- Actually, we should know in advance what subproblem to look at.

- Lemma:
  - Suppose that k is the activity you can squeeze in after i **with the smallest finishing time.**
  - Then there is an **optimal solution to A[i..n+1]** that **extends the optimal solution to A[k..n+1].**

Let's add an additional activity a_{n+1} that starts "tomorrow".

This is abusing notation…technically A[i..n+1] is a value, not a problem.

A[i,j] = number of activities you can squeeze in after Activity i finishes and before Activity j starts

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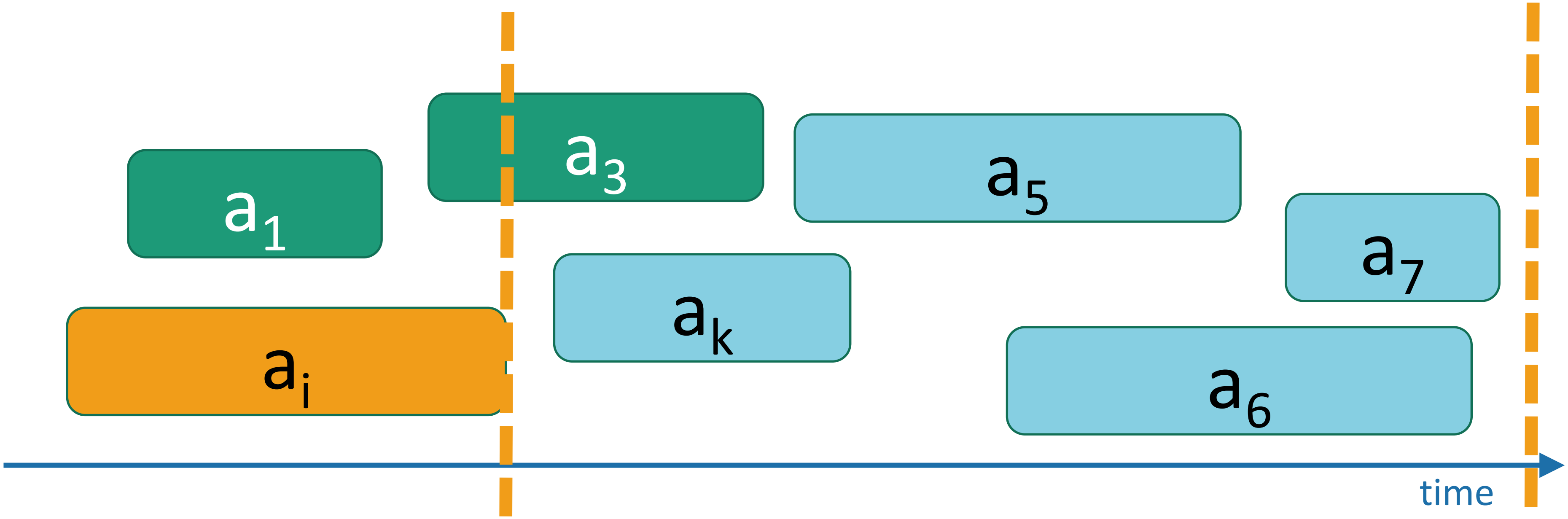

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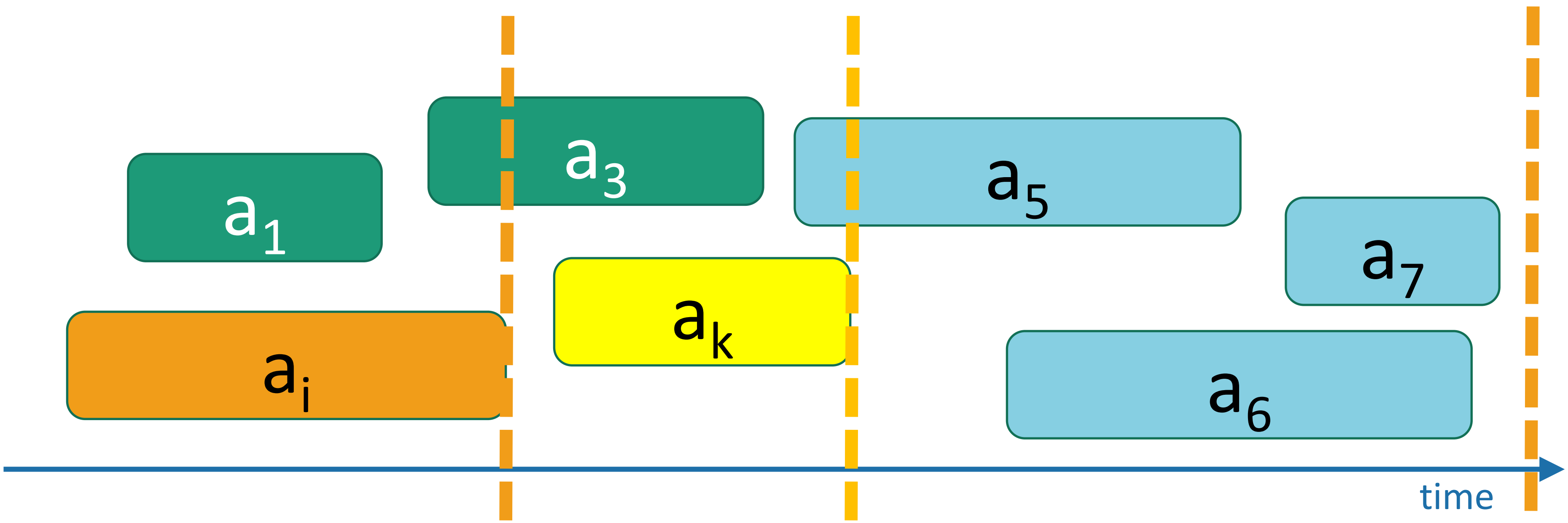

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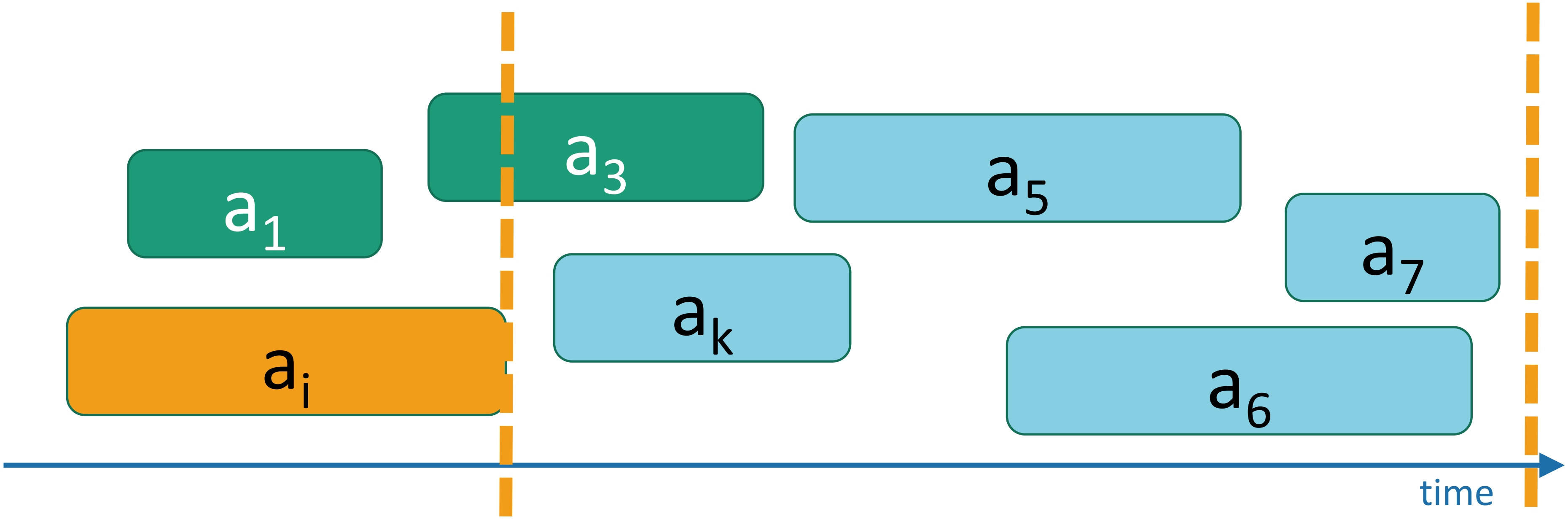

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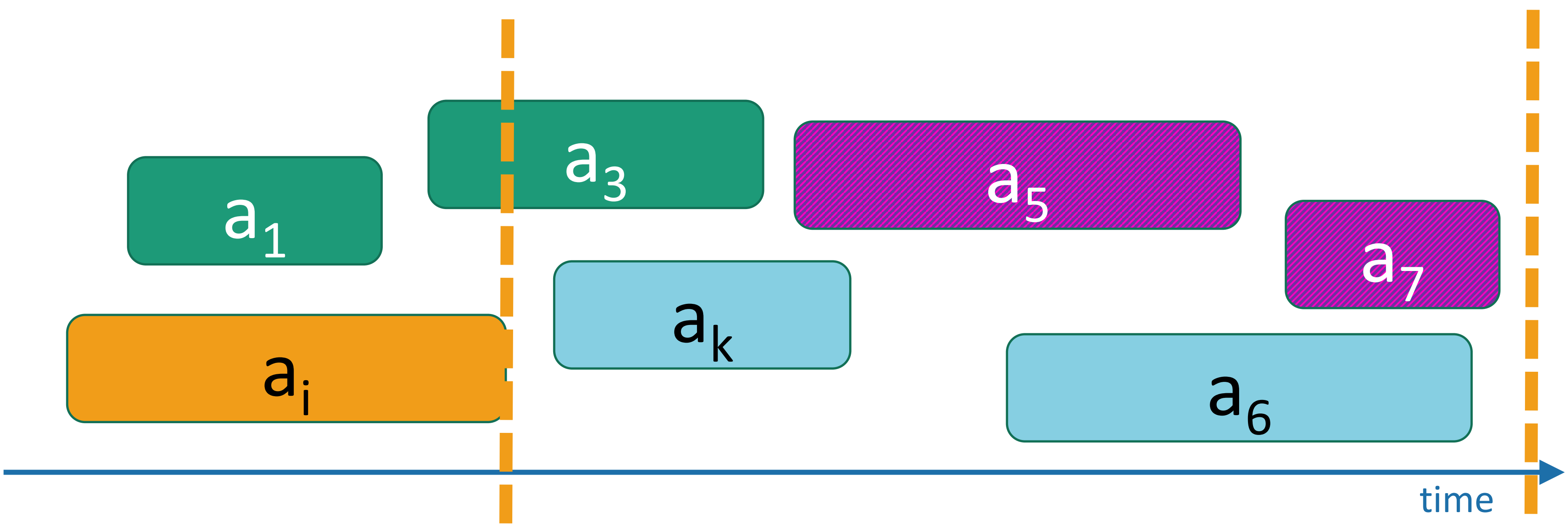

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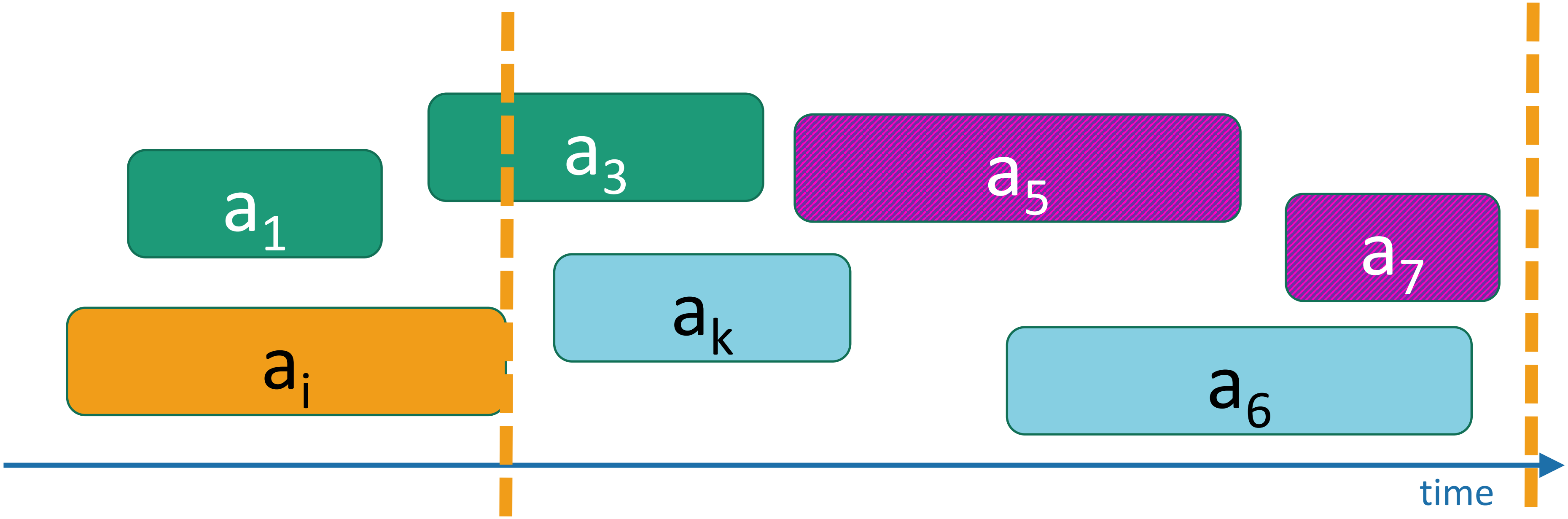

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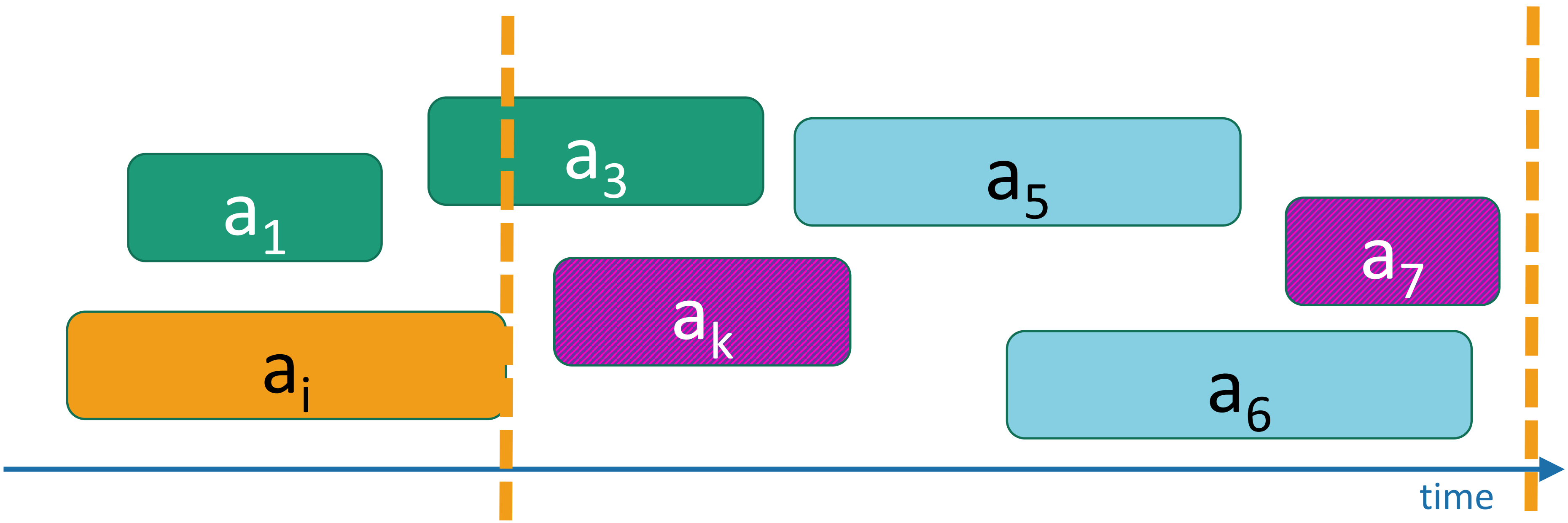

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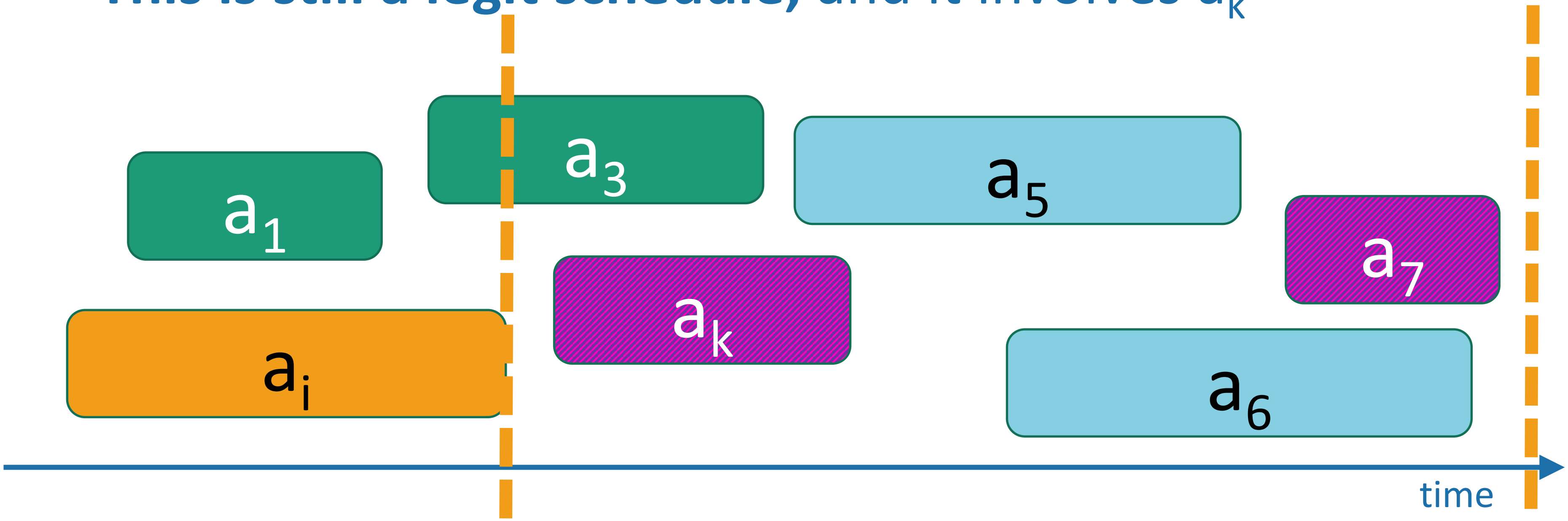
- **This is still a legit schedule,** and it involves a_k



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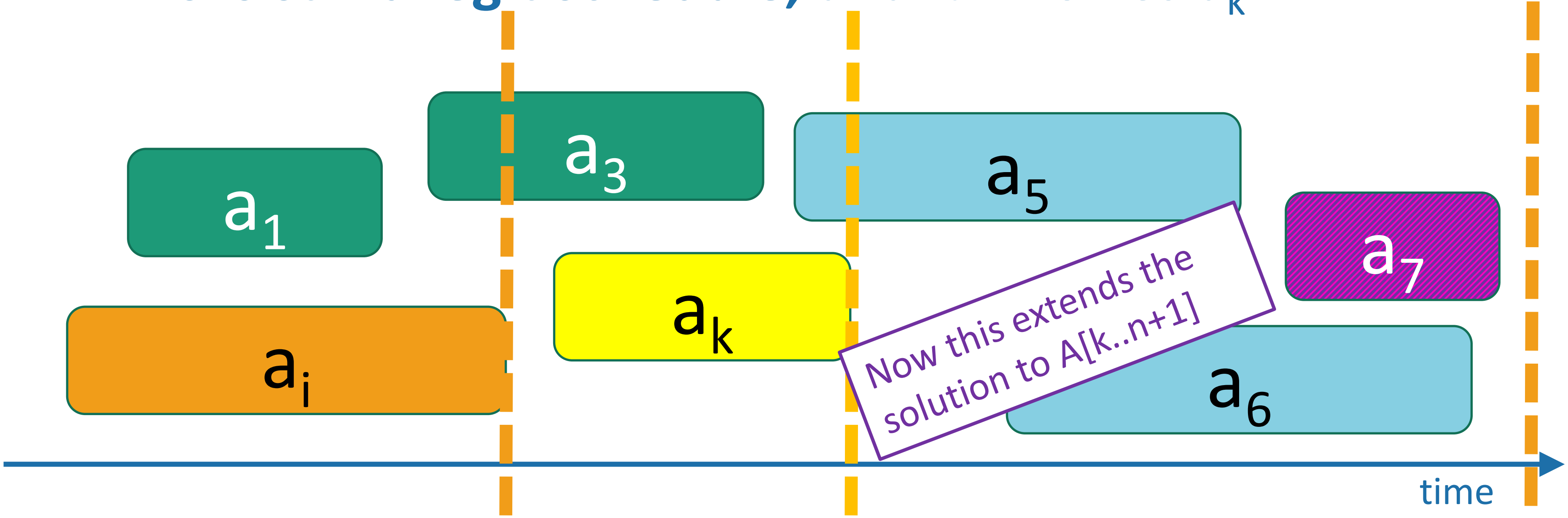

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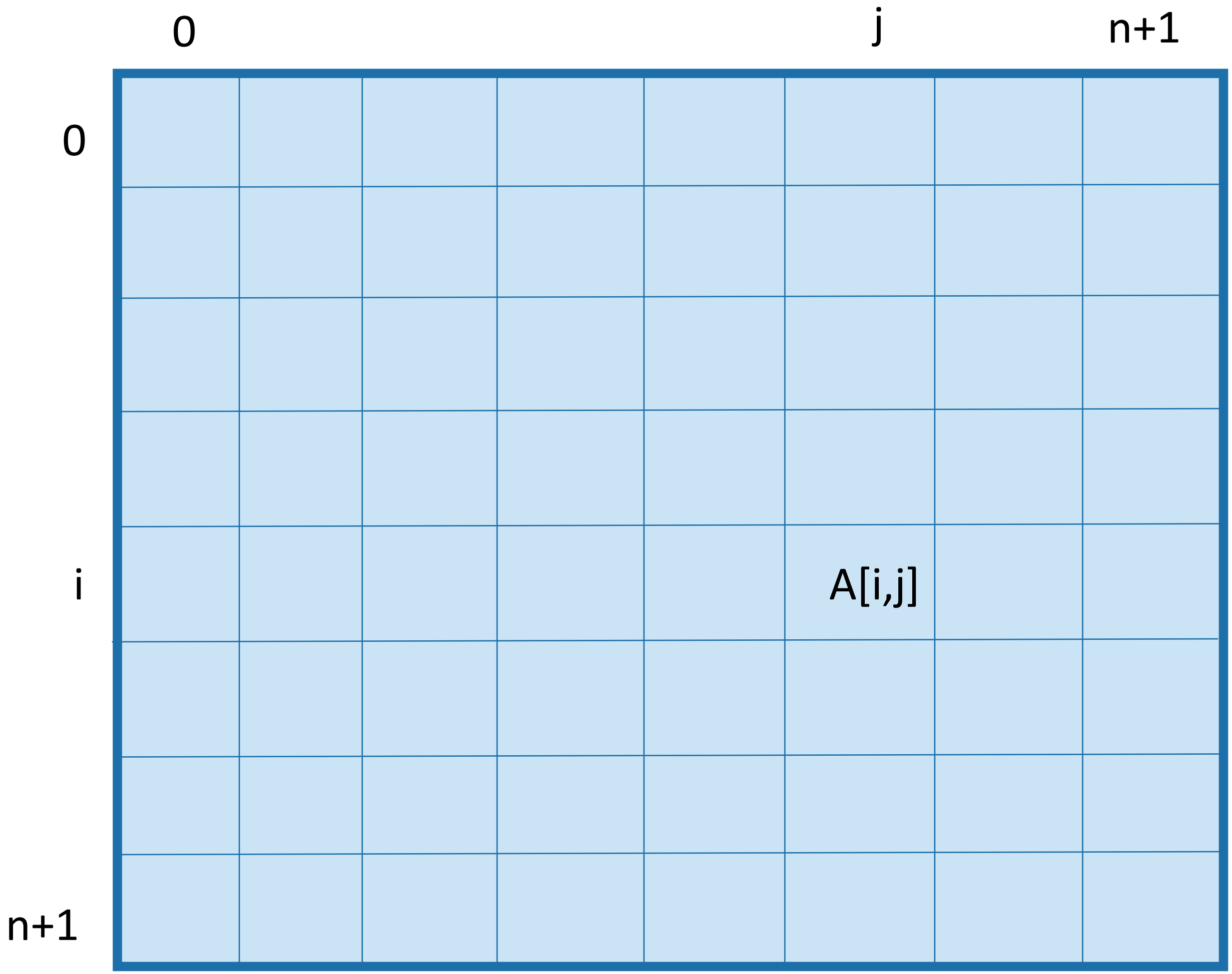

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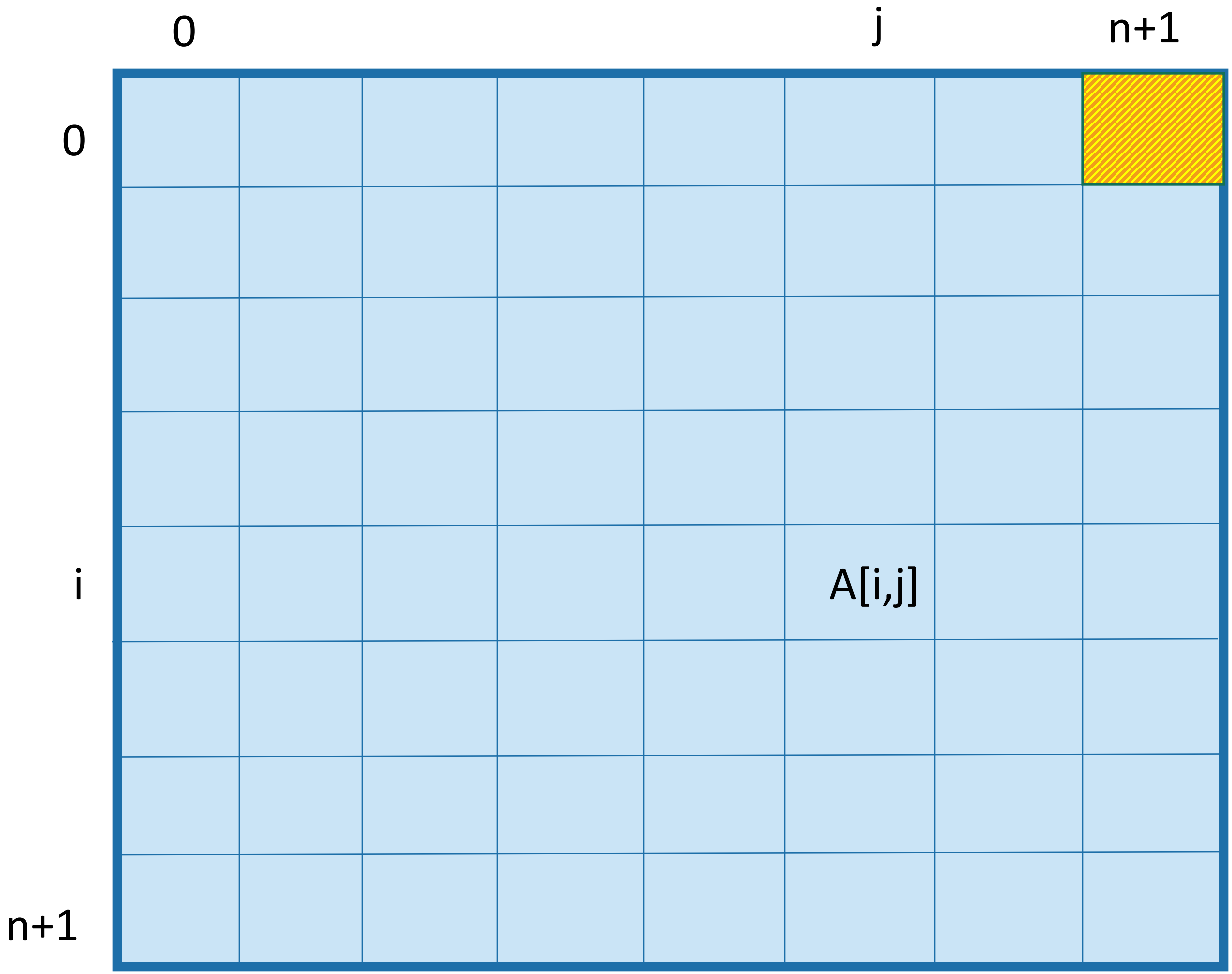

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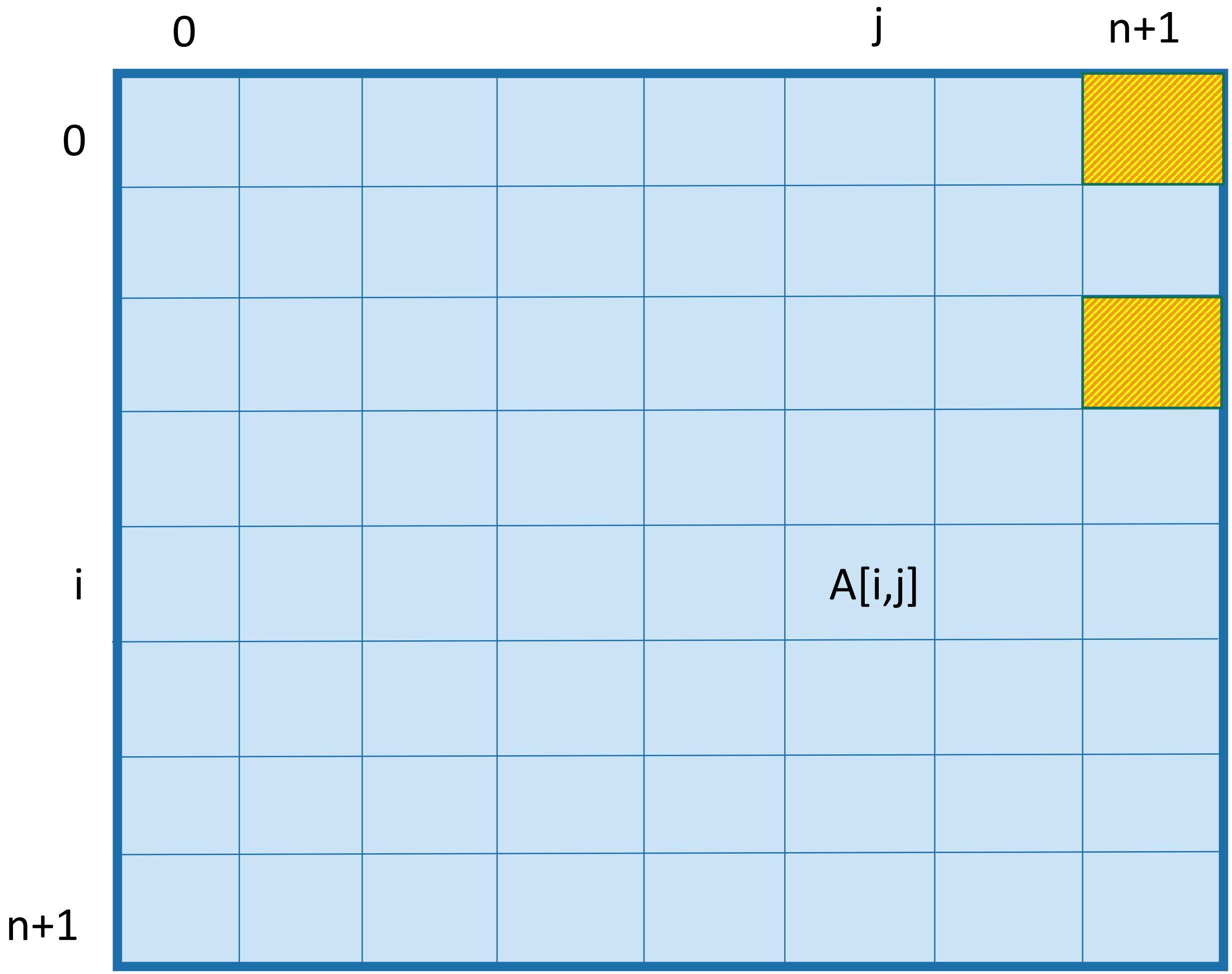

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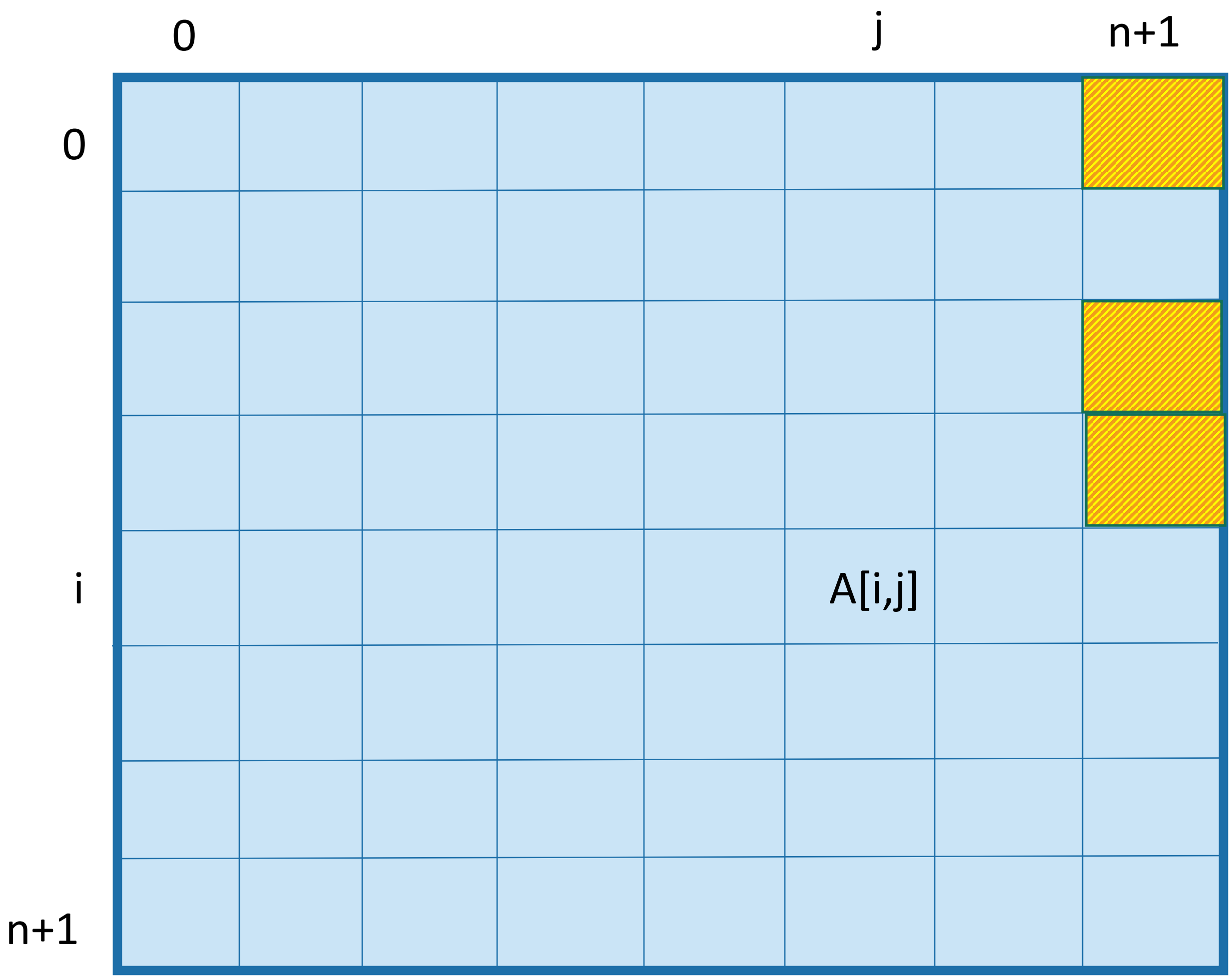

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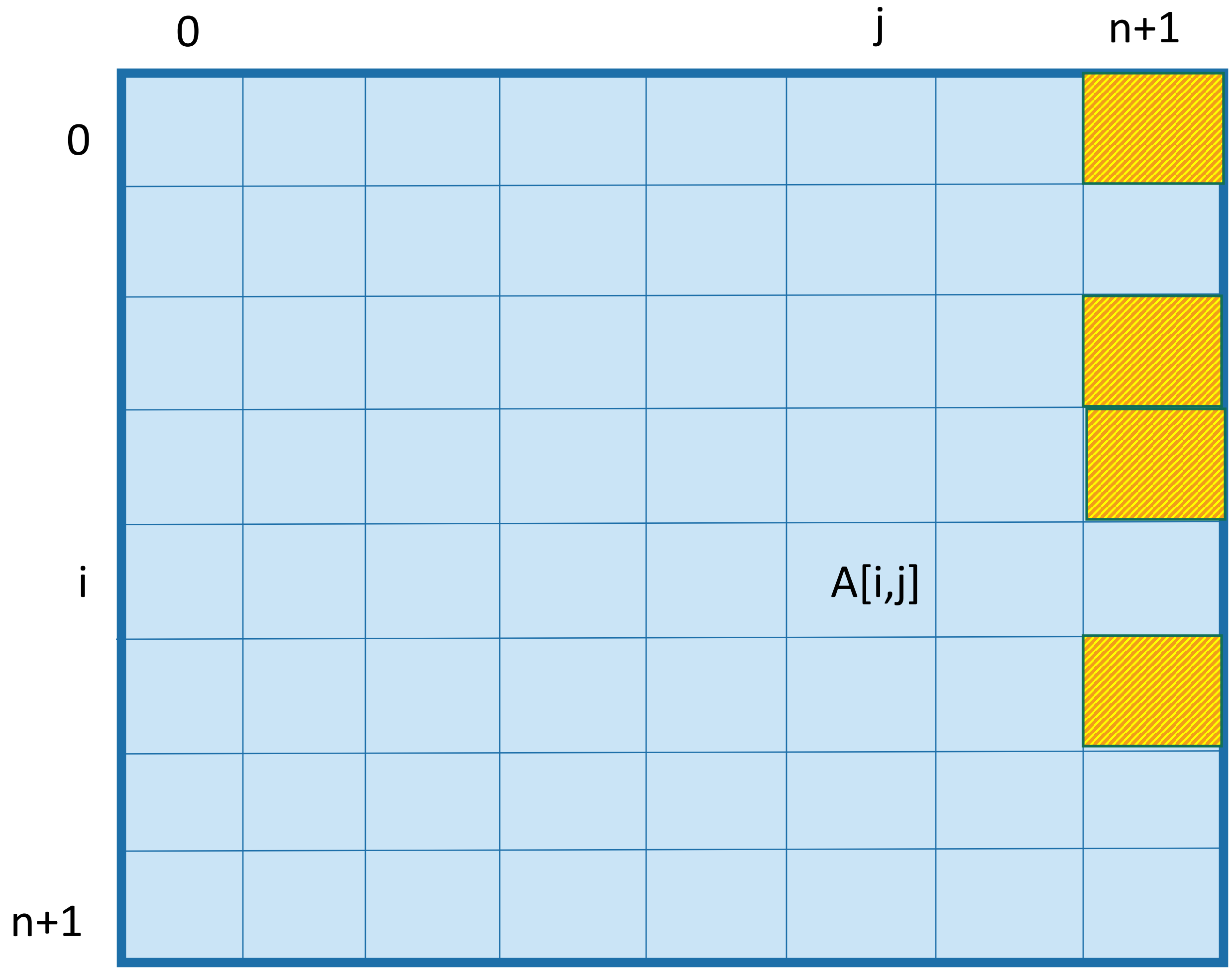

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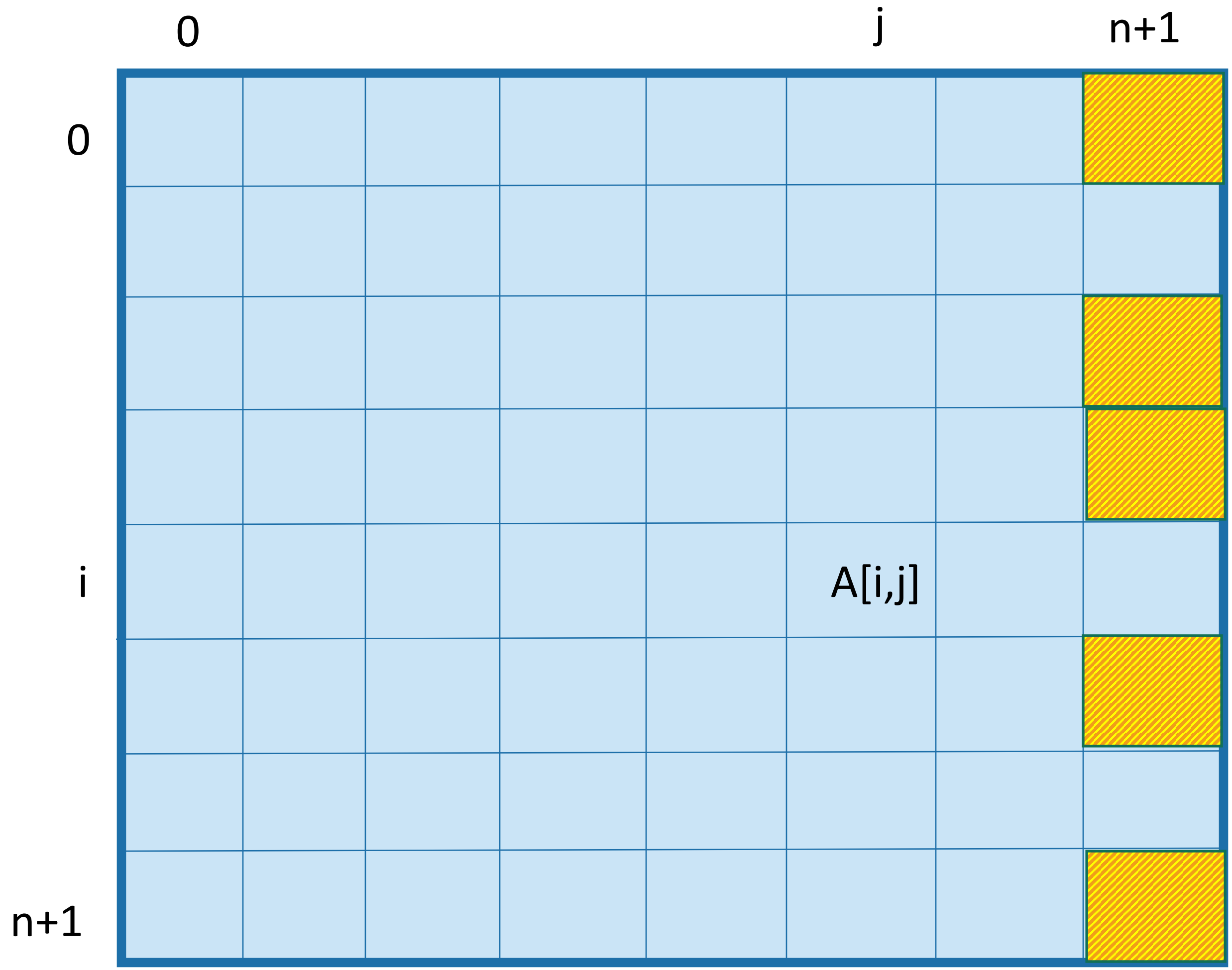

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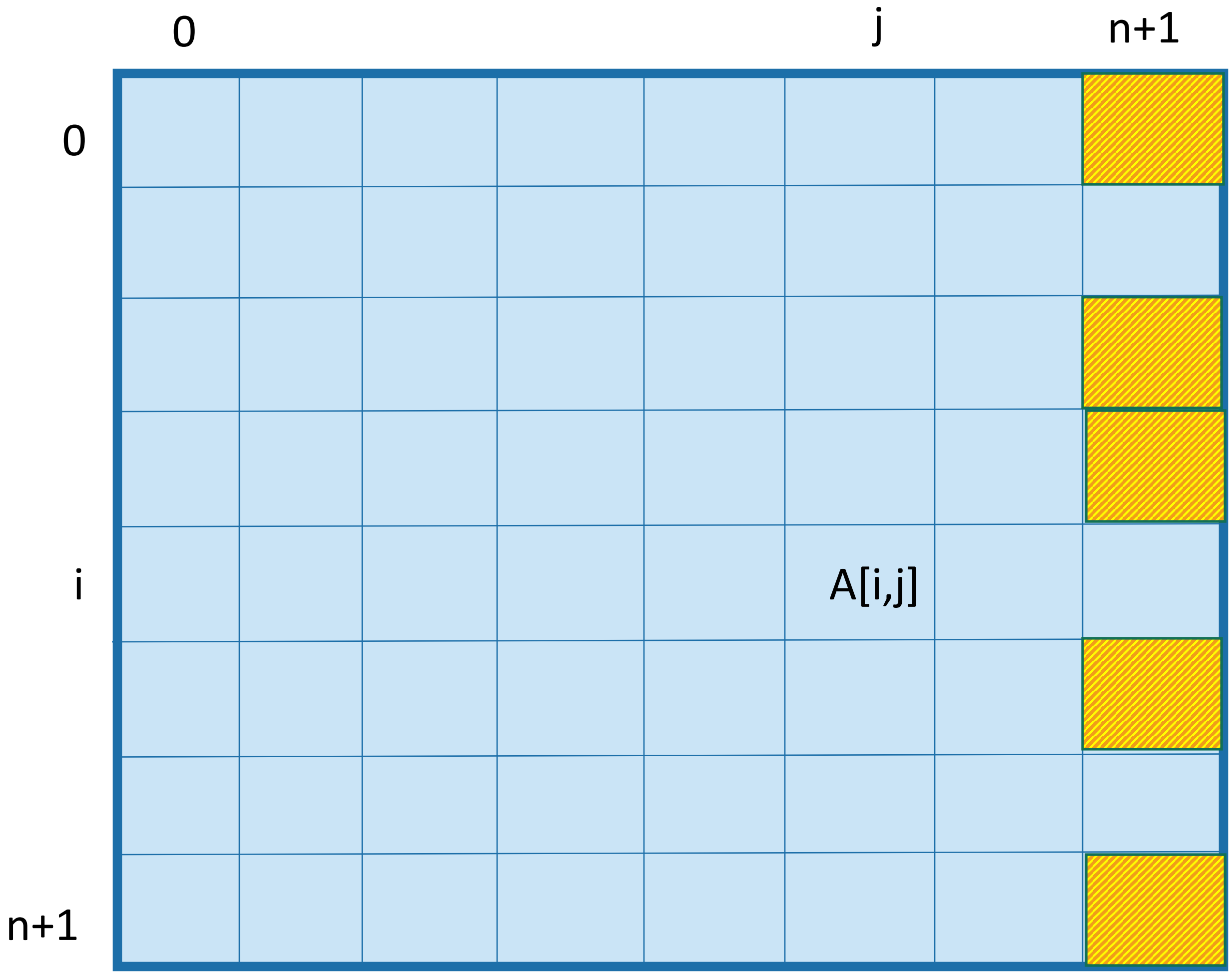

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etc.

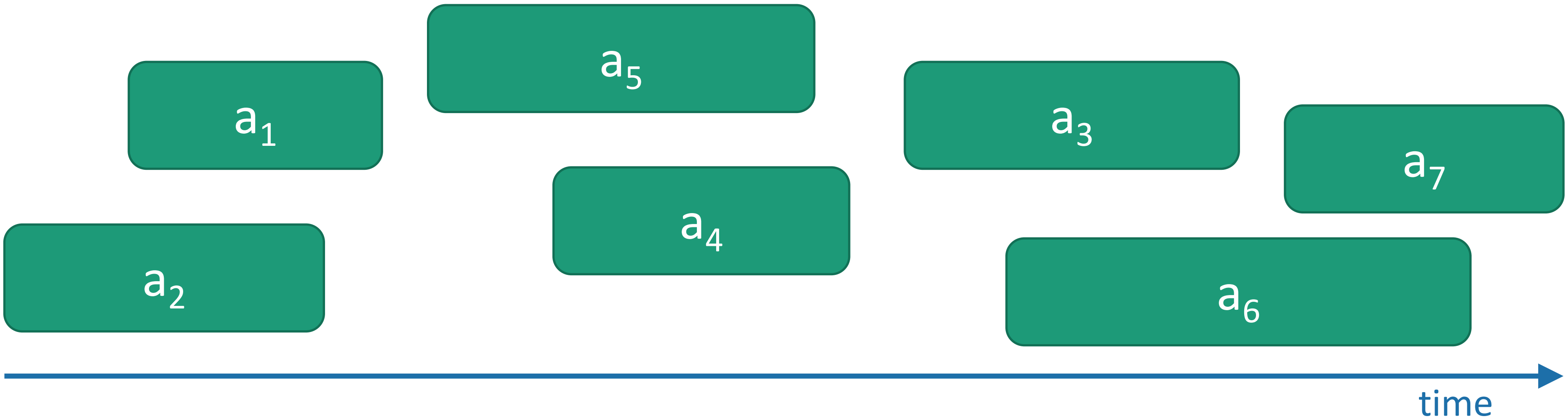
There's no reason we have to look at the whole table!

Instead, let's use this insight to make a greedy algorithm

- Suppose the activities are sorted by finishing time
  - if not, sort them.
- mySchedule = []
- **for** k = 1,…,n:
  - **if** I can fit in Activity k after the last thing in mySchedule:
    - mySchedule.append(Activity k)
- return mySchedule

This is the same thing we saw before

Greedy Algorithm



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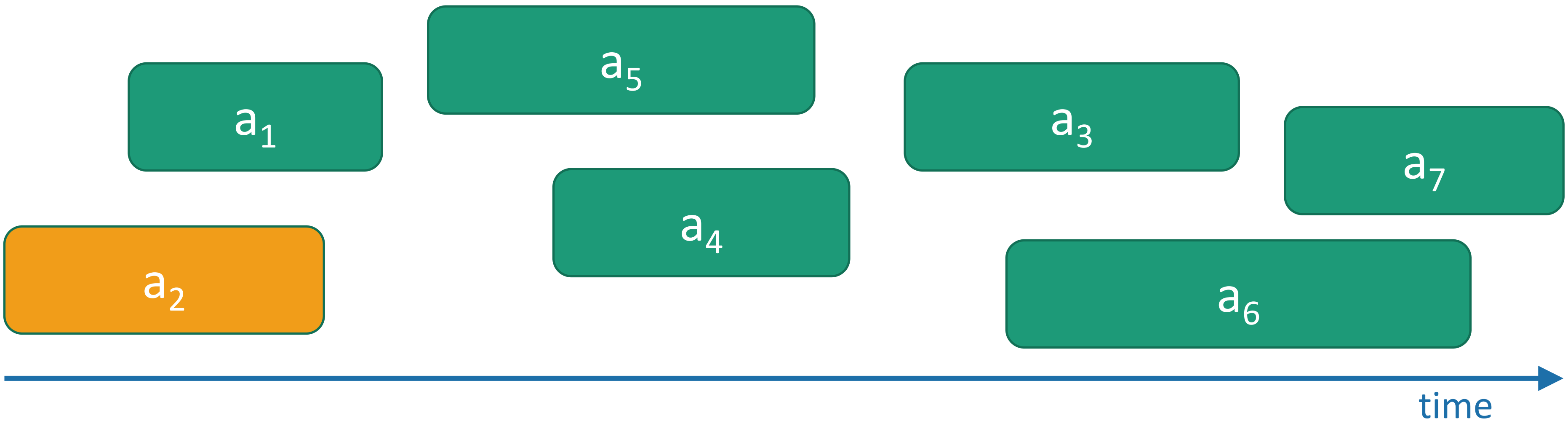

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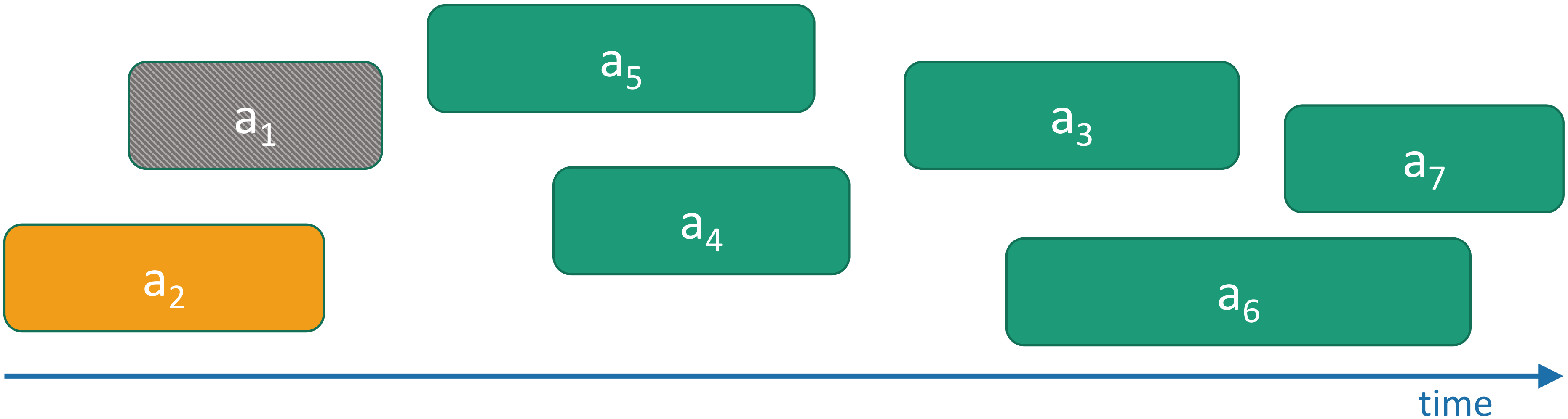

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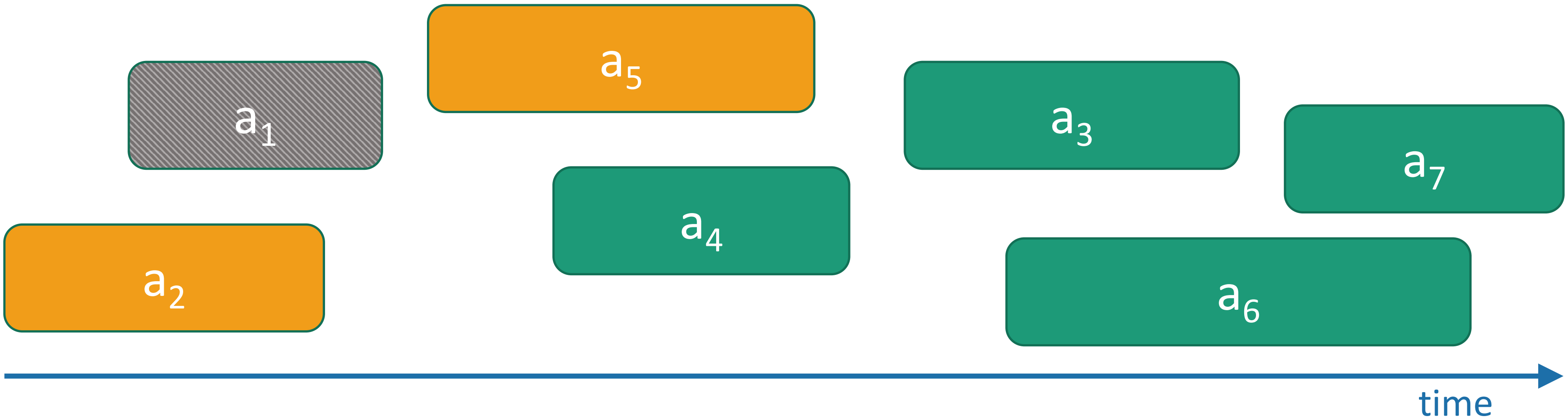

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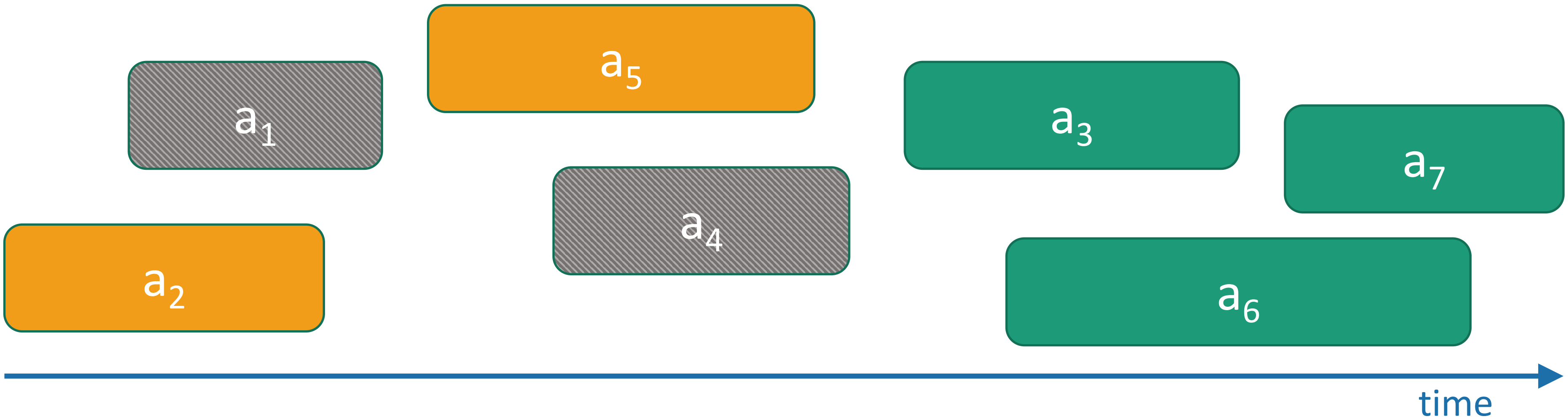

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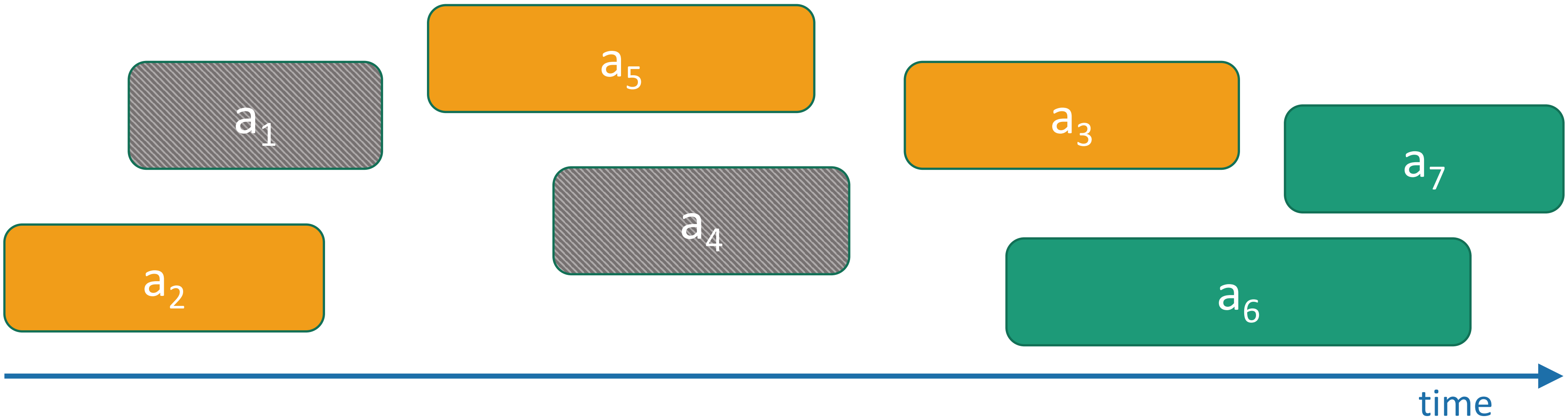

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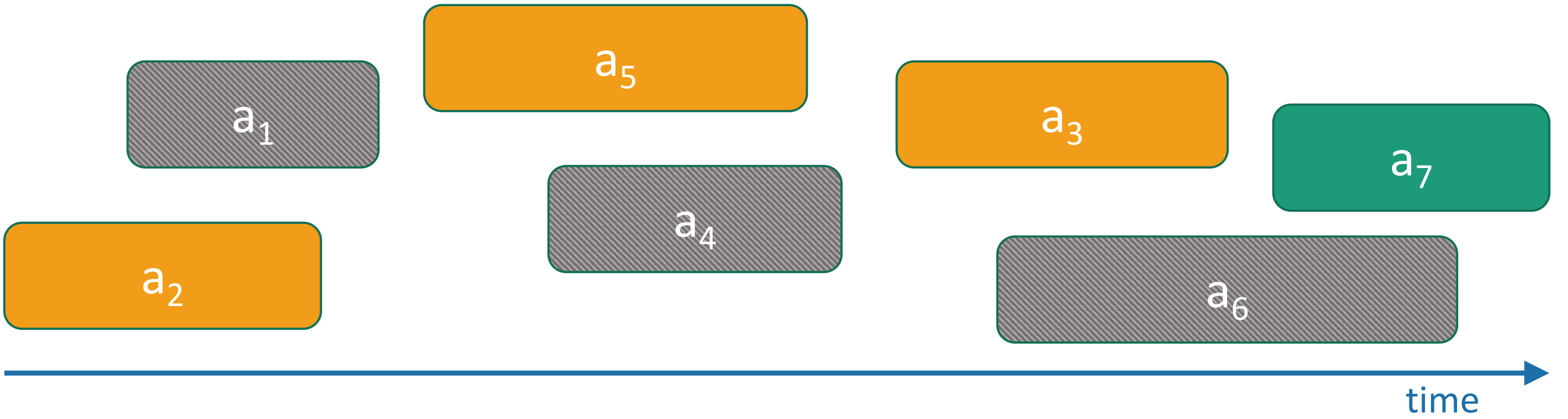

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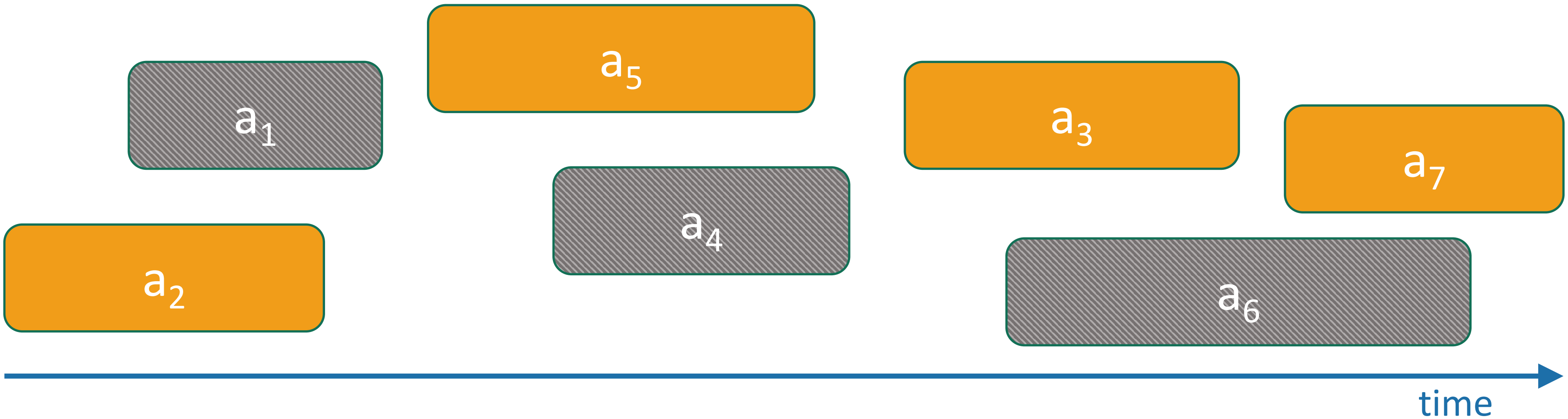

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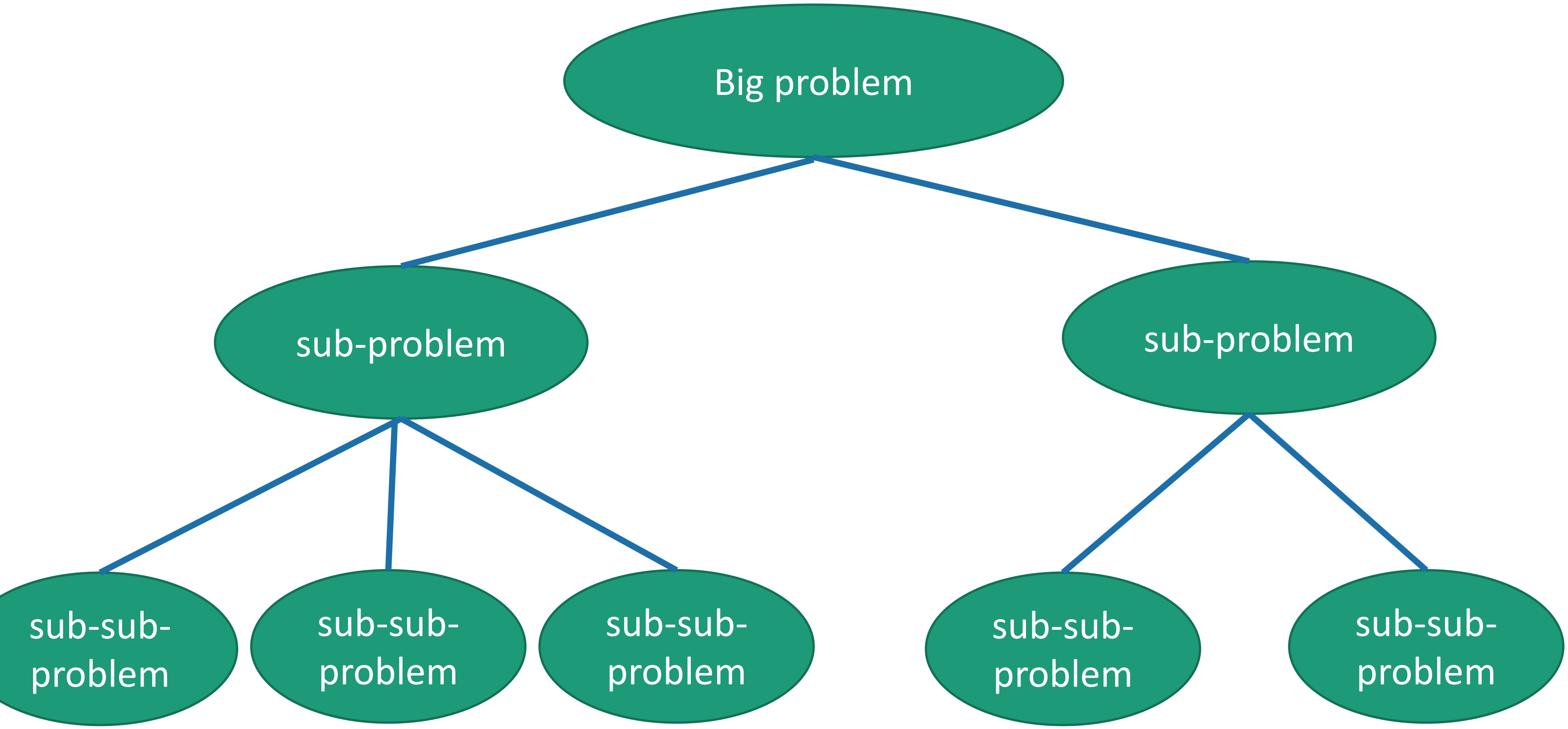
Why does this work?

- At each step, **we make a choice**
  - Include activity k
- We can show that this choice **will never rule out an optimal solution.**
  - Formally: There is an optimal solution to A[i..n+1] that contains A[k..n+1].
- So when we reach the **end of the argument**:
  - we haven't ruled out an optimal solution
  - and we only have one solution left
  - **so it must be optimal**.

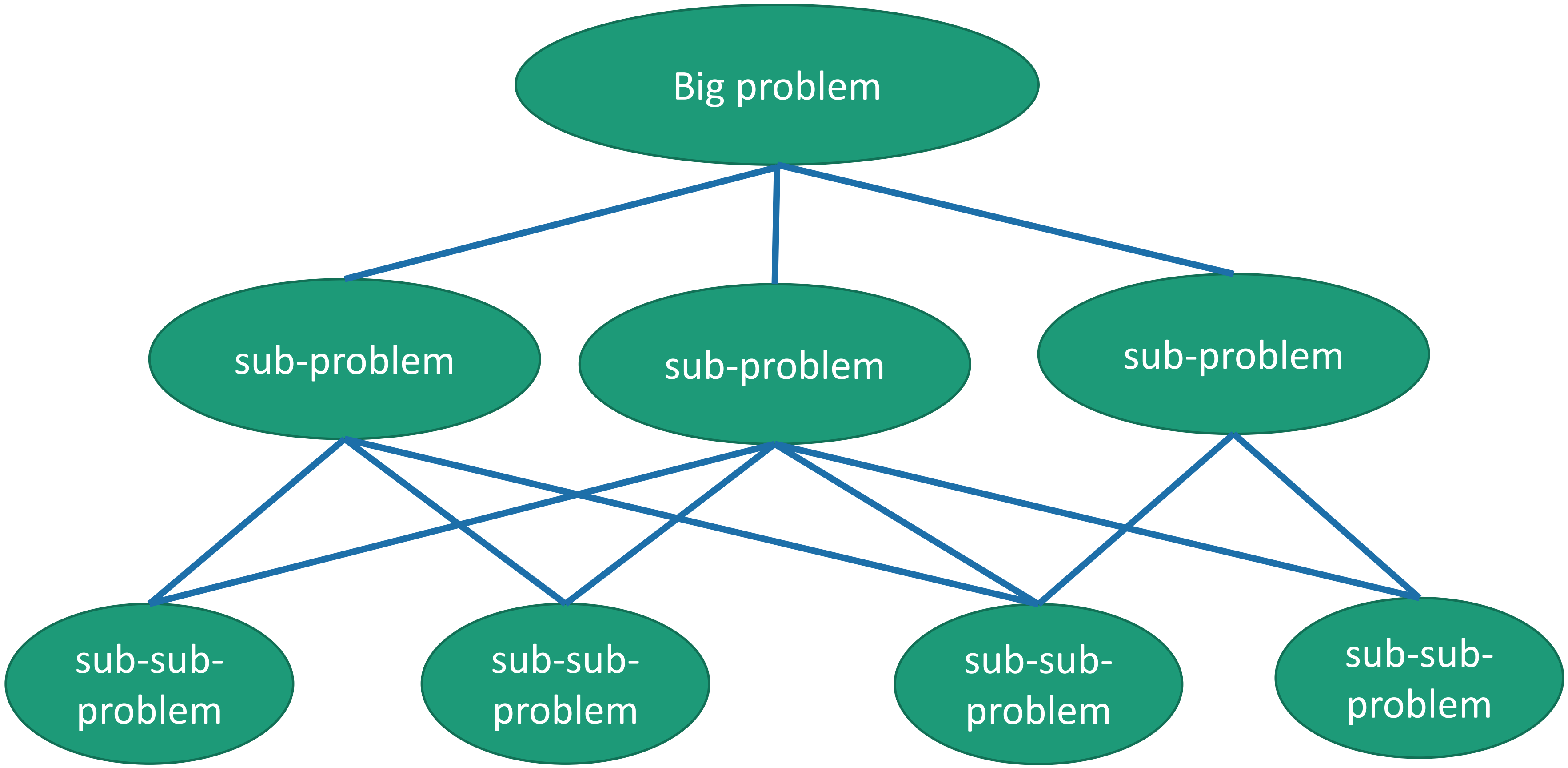
Sub-problem graph view

- Divide-and-conquer:



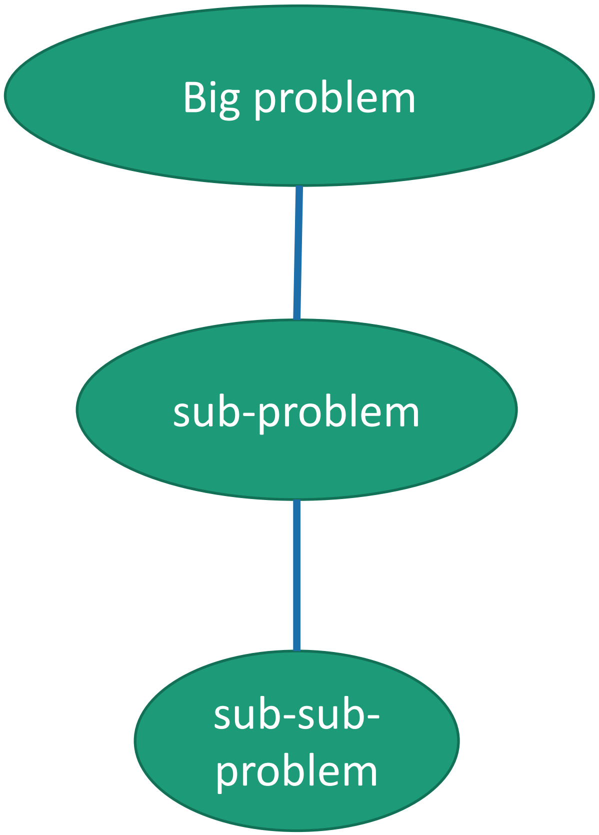
Sub-problem graph view

- Dynamic Programming:



Sub-problem graph view

- Greedy algorithms:



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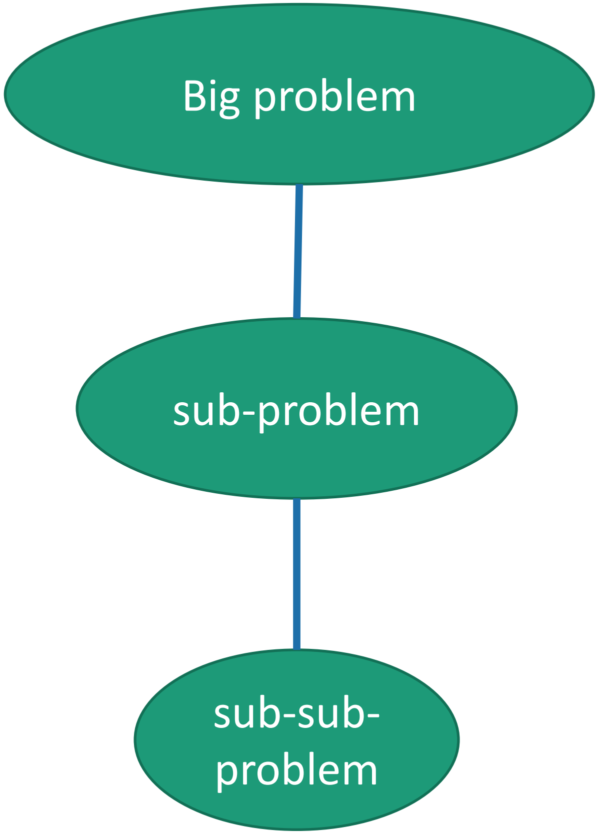


- Not only is there **optimal sub-structure:**
  - optimal solutions to a problem are made up from optimal solutions of sub-problems
- but each problem **depends on only one sub-problem**.

What have we learned?

- If we come up with a DP solution, and it turns out that we really only care about one sub-problem, then maybe we can use a greedy algorithm.
- One example was activity selection.
- In order to come up with a greedy algorithm, we:
  - Made a **series of choices**
  - Proved that our choices **will never rule out an optimal solution.**
  - Conclude that **our solution at the end is optimal.**

One more example

Huffman coding

- everyday english sentence
- 01100101 01110110 01100101 01110010 01111001 01100100 01100001 01111001 00100000 01100101 01101110 01100111 01101100 01101001 01110011 01101000 00100000 01110011 01100101 01101110 01110100 01100101 01101110 01100011 01100101
- qwertyui_opasdfg+hjklzxcv
- 01110001 01110111 01100101 01110010 01110100 01111001 01110101 01101001 01011111 01101111 01110000 01100001 01110011 01100100 01100110 01100111 00101011 01101000 01101010 01101011 01101100 01111010 01111000 01100011 01110110

One more example

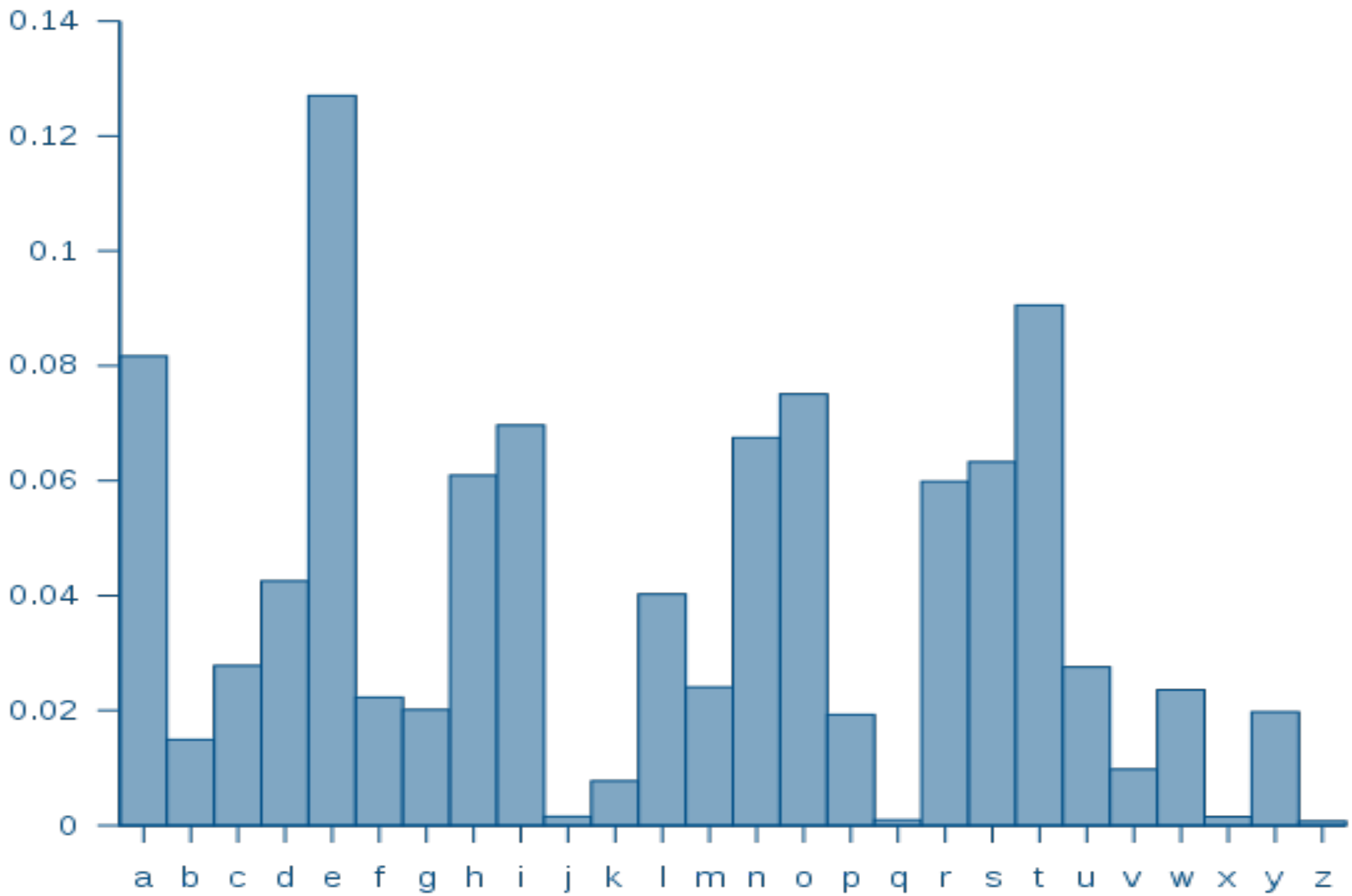
Huffman coding

ASCII is pretty wasteful. If **e** shows up so often, we should have a more parsimonious way of representing it!

- **e**v**e**ryday **e**nglish s**e**nt**e**nc**e**
- **01100101** 01110110 **01100101** 01110010 01111001 01100100 01100001 01111001 00100000 **01100101** 01101110 01100111 01101100 01101001 01110011 01101000 00100000 01110011 **01100101** 01101110 01110100 **01100101** 01101110 01100011 **01100101**

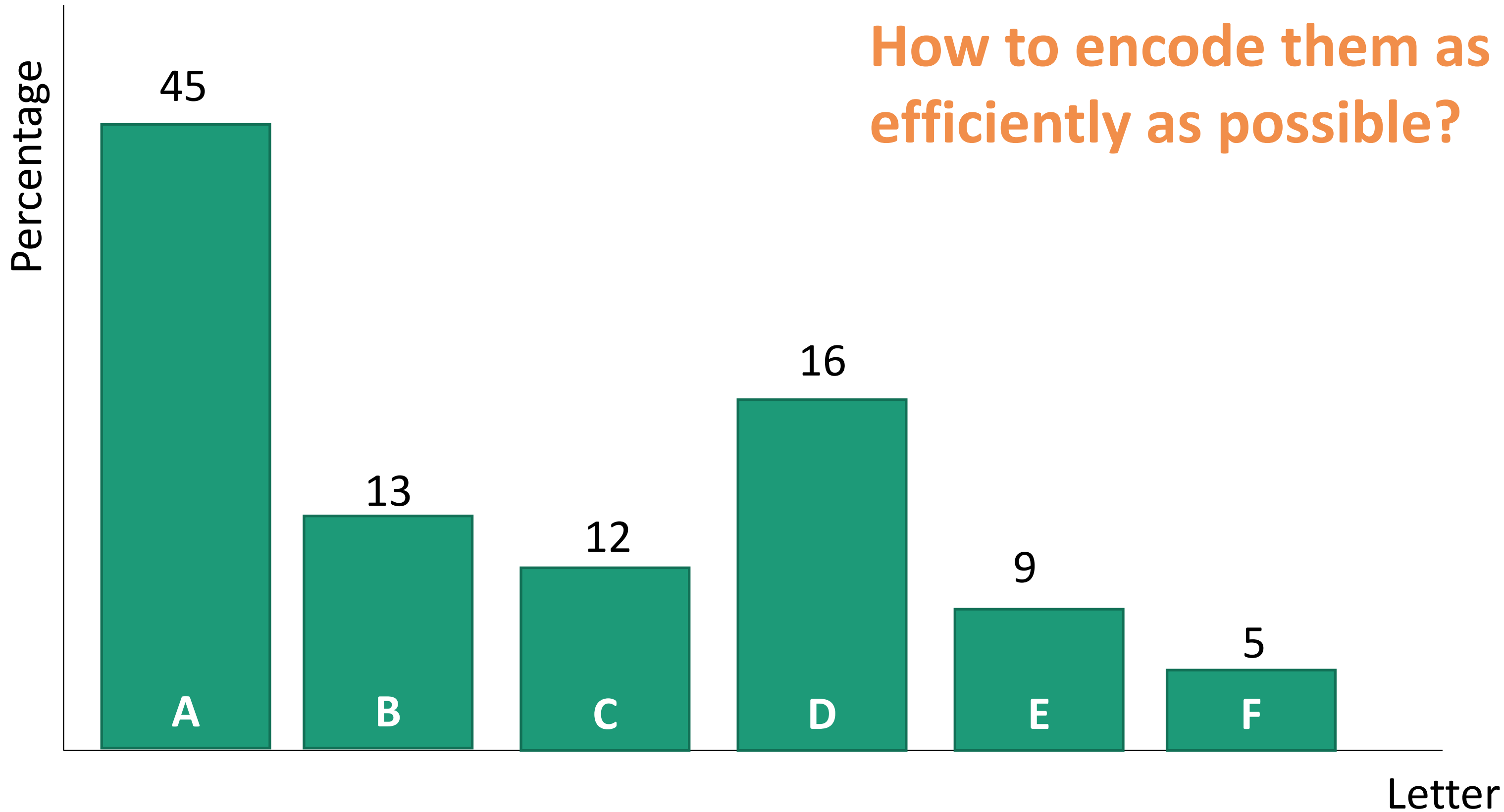
- qwertyui_opasdfg+hjklzxcv
- 01110001 01110111 01100101 01110010 01110100 01111001 01110101 01101001 01011111 01101111 01110000 01100001 01110011 01100100 01100110 01100111 00101011 01101000 01101010 01101011 01101100 01111010 01111000 01100011 01110110

Suppose we have some distribution on characters



Suppose we have some distribution on characters

For simplicity, let’s go with this made-up example

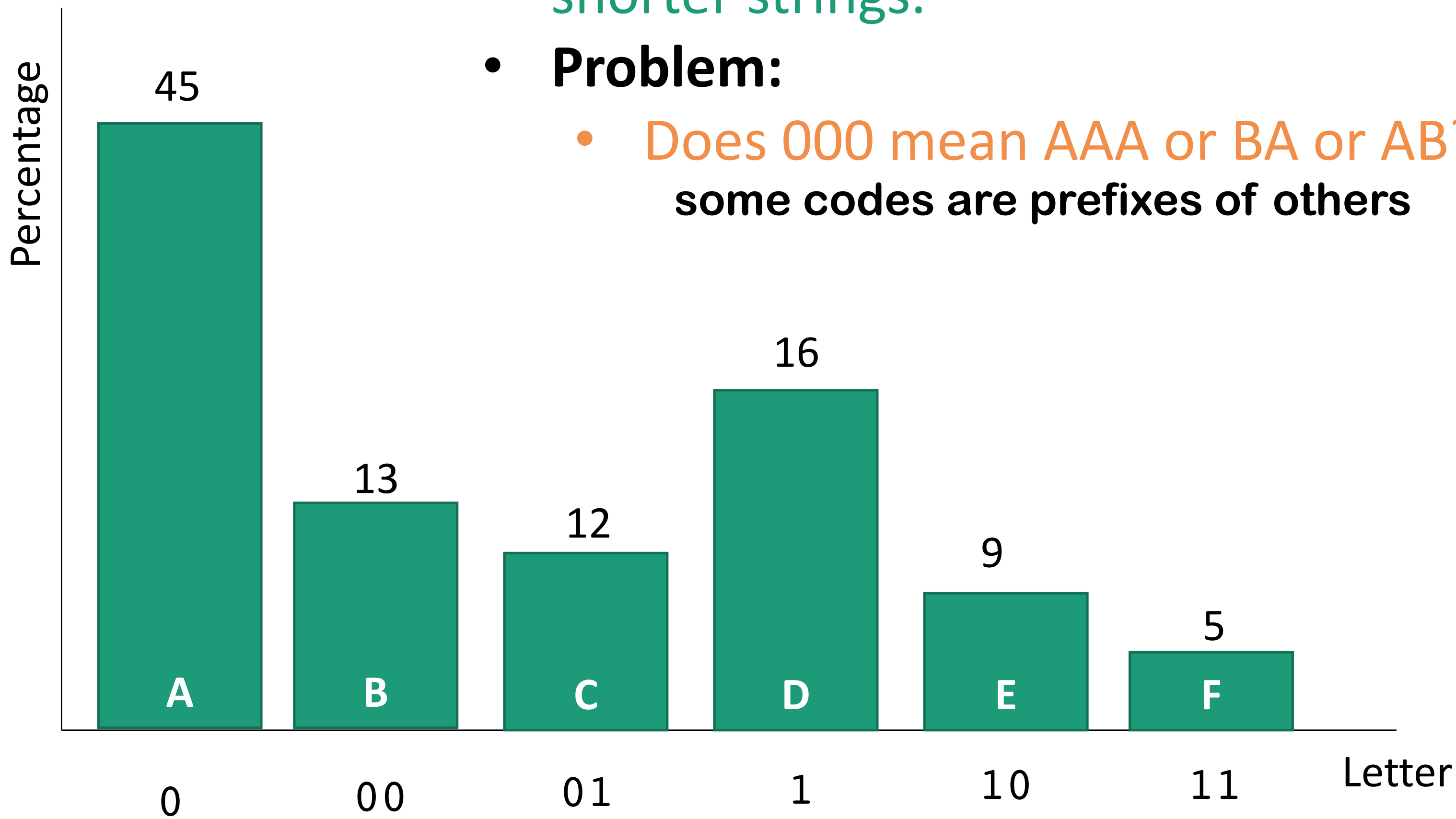


How to encode them as efficiently as possible?

Try 1

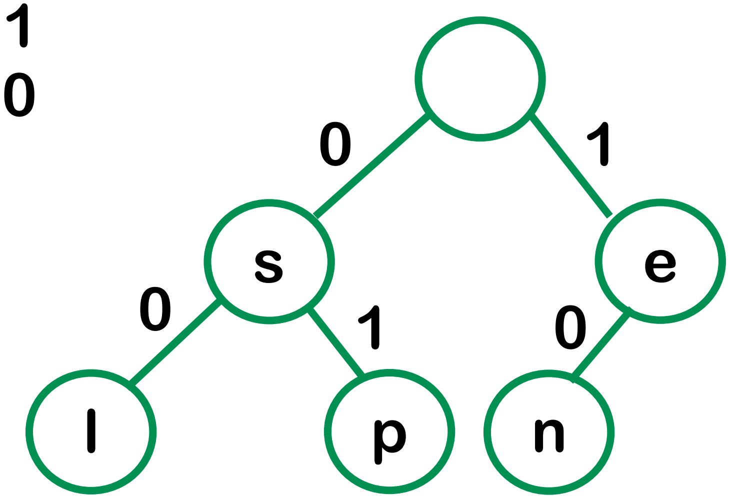
- Every letter is assigned a **binary string** of one or two bits.
- The more frequent letters get the shorter strings.
- **Problem:**
  - Does 000 mean AAA or BA or AB?

some codes are prefixes of others



example text : Sleeplessness

characters	frequency	code
s	5	0
e	4	1
l	2	00
p	1	01
n	1	10



0 is the prefix of 00,01
1 is the prefix of 10

character node that has children
it’s code is the prefix of another code

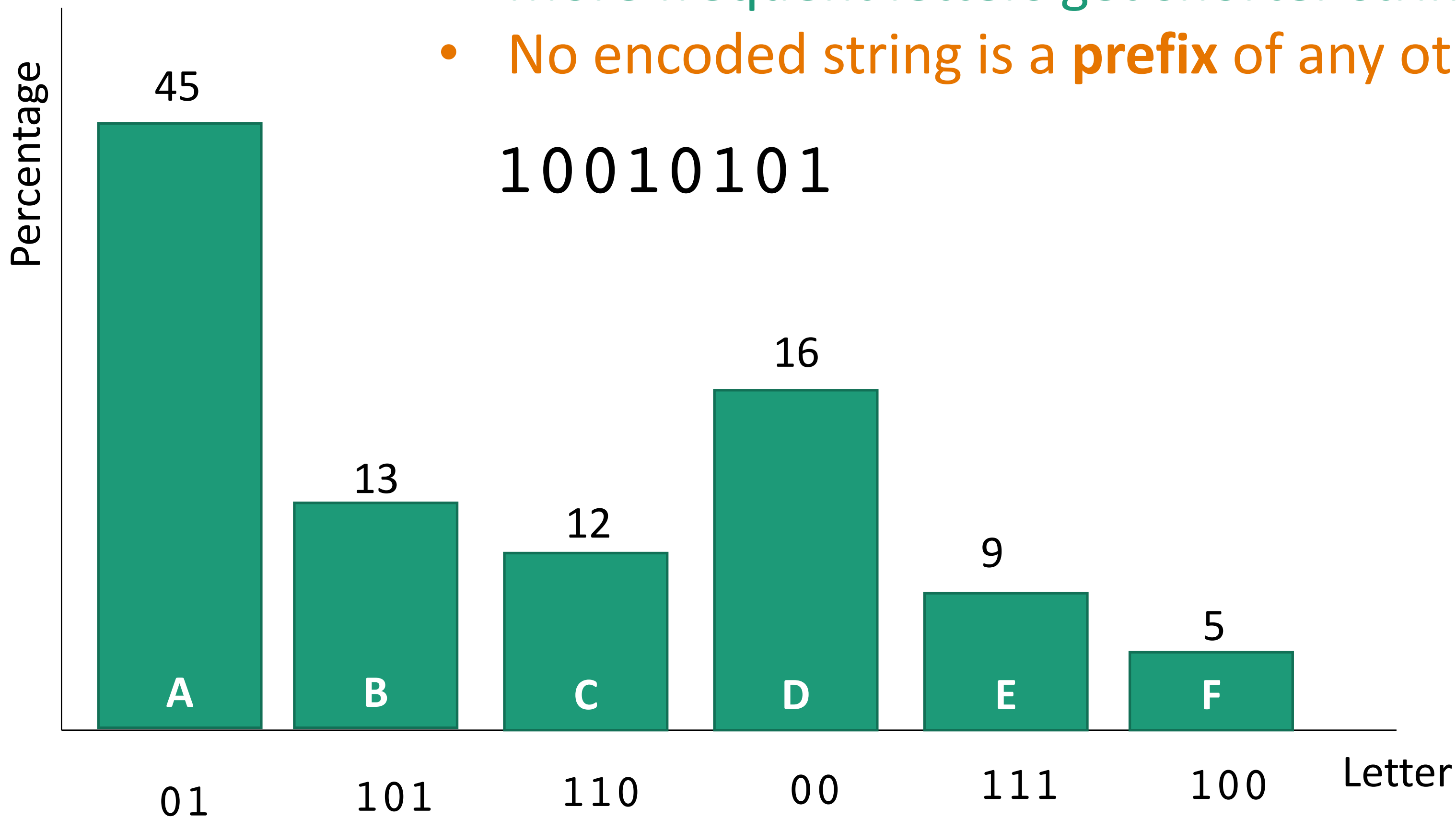
! all characters must be leaves in order not to be prefixes of another

Confusingly, "prefix-free codes" are also sometimes called "prefix codes" (including in CLRS).

Try 2: prefix-free coding

- Every letter is assigned a **binary string.**
- More frequent letters get shorter strings.
- No encoded string is a **prefix** of any other.

10010101



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Try 2: prefix-free coding

- Every letter is assigned a **binary string.**
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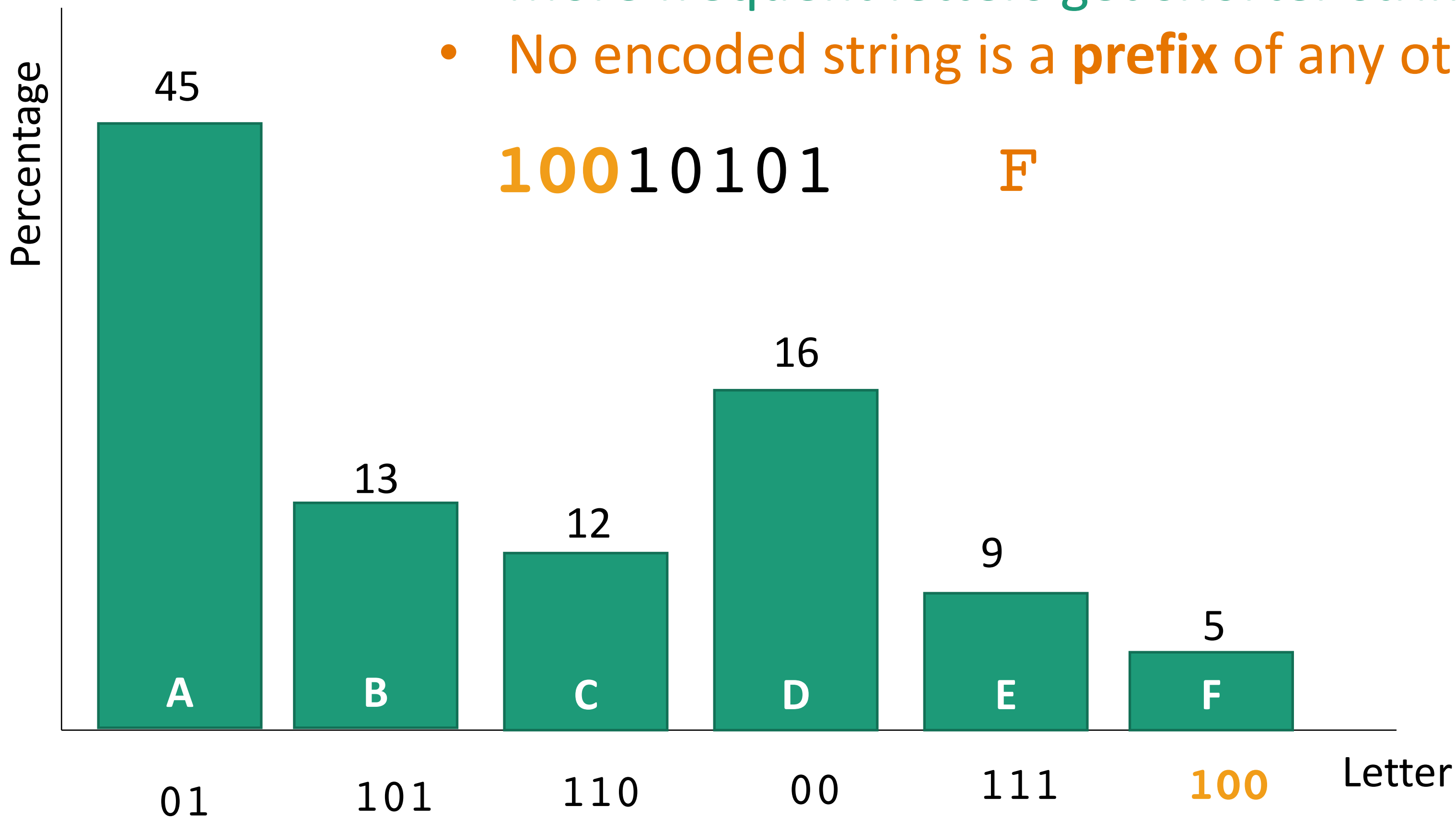
- No encoded string is a **prefix** of any other.

10010101 **F**



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Try 2: prefix-free coding

- Every letter is assigned a **binary string.**
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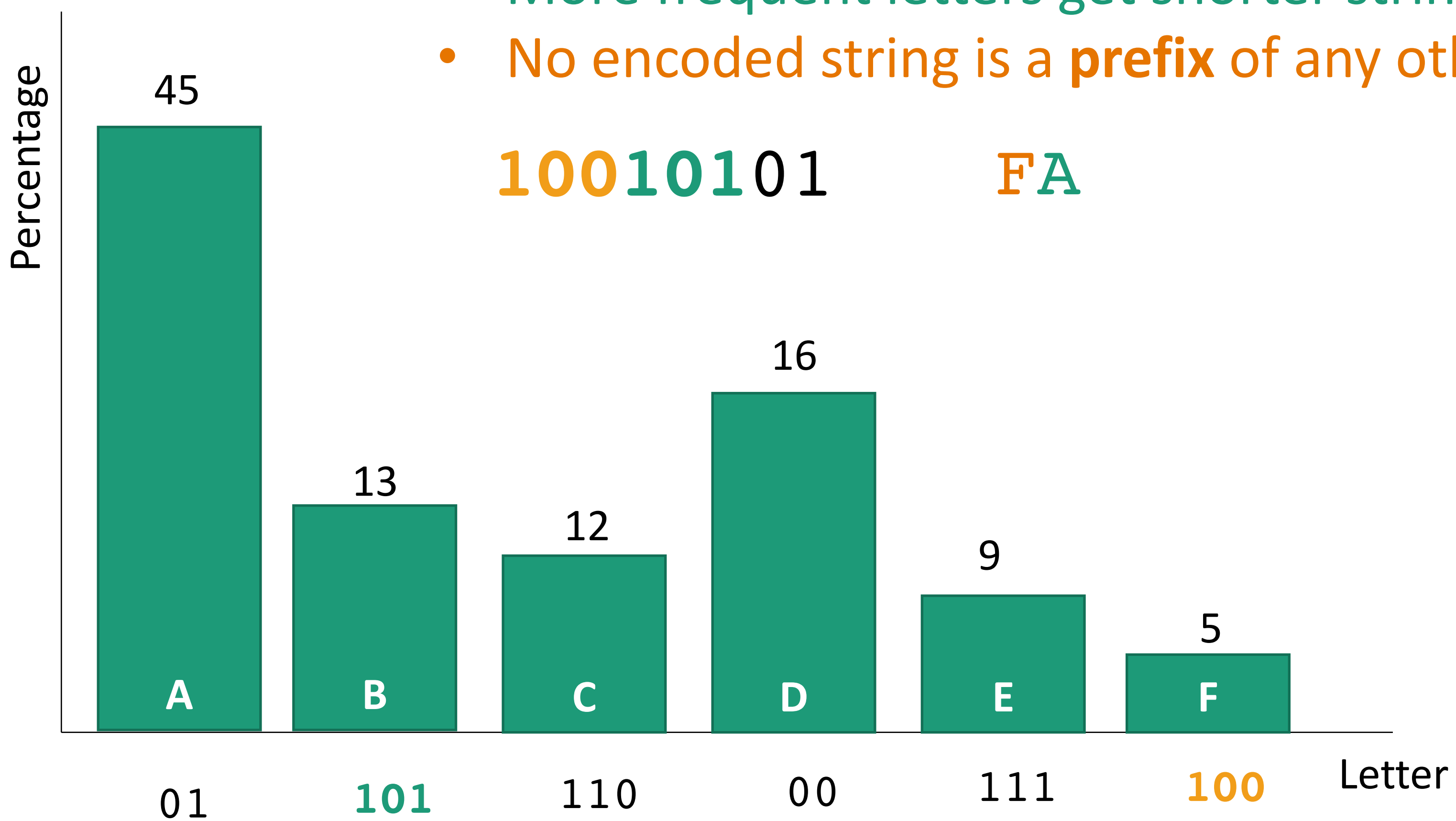
- No encoded string is a **prefix** of any other.

10010101 FA



Confusingly, “prefix-free codes” are also sometimes called “prefix codes” (including in CLRS).

Try 2: prefix-free coding

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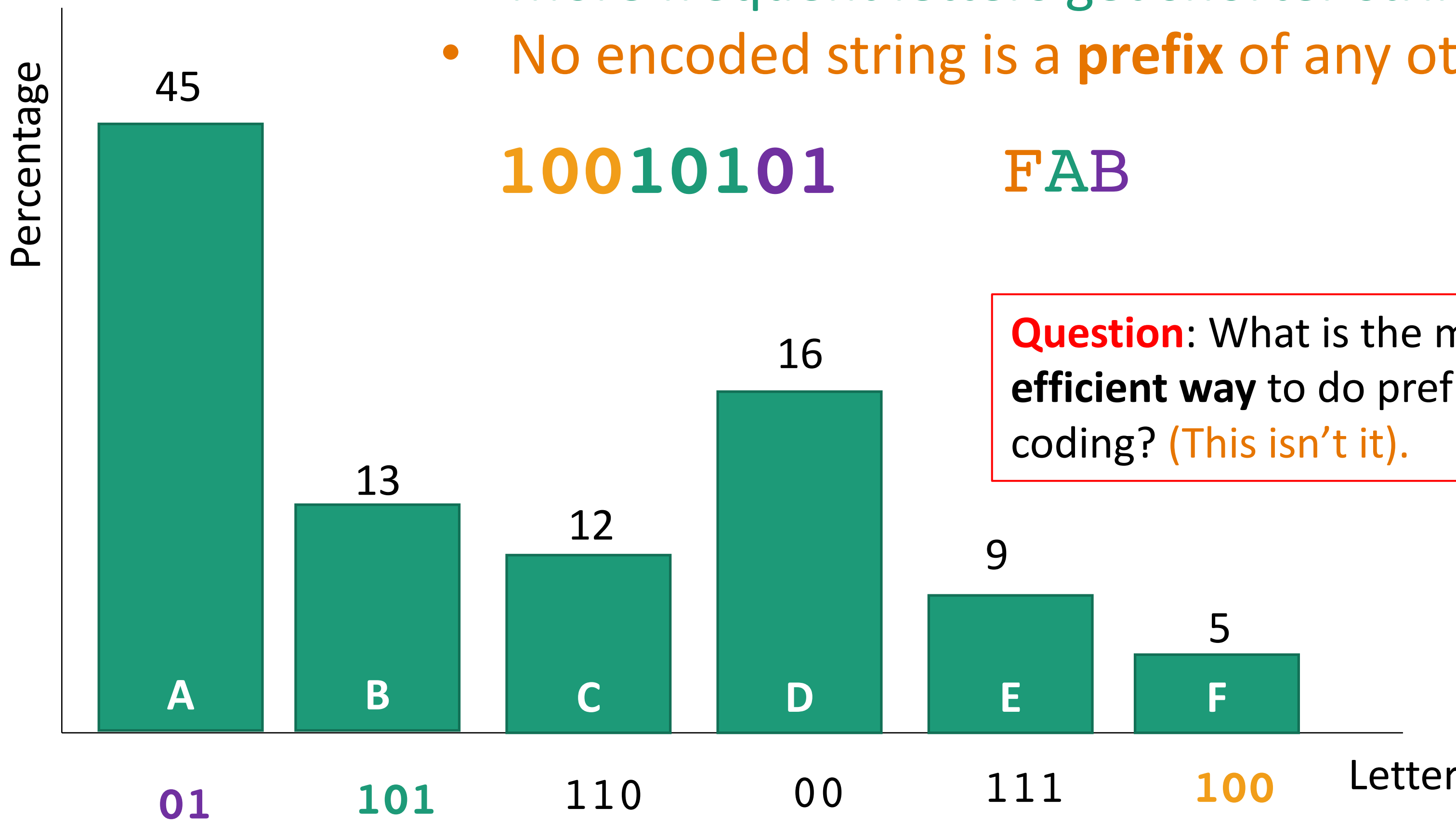
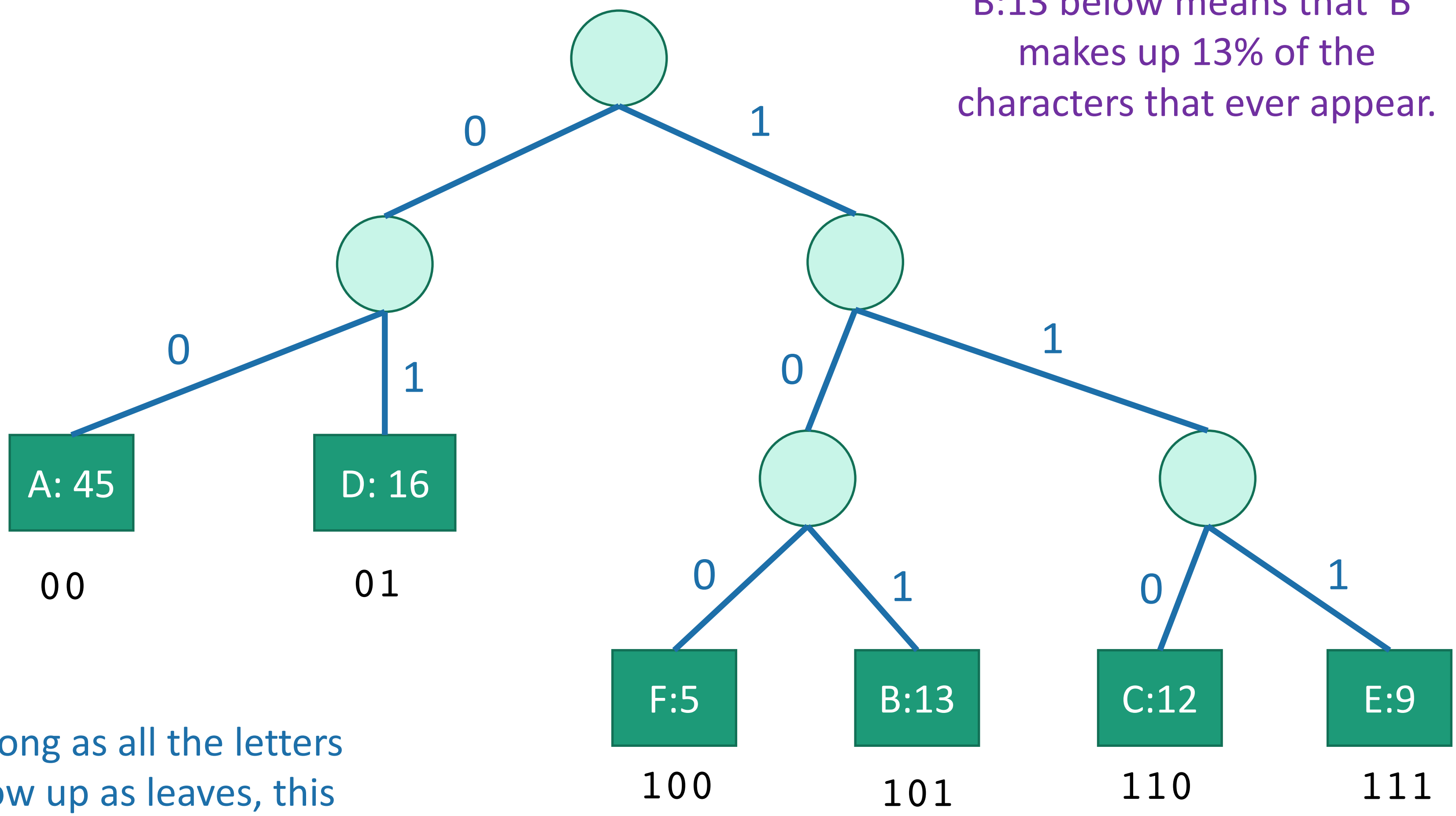
10010101 FAB



Question: What is the most **efficient way** to do prefix-free coding? (This isn’t it).

A prefix-free code is a tree

B:13 below means that 'B' makes up 13% of the characters that ever appear.



As long as all the letters show up as leaves, this code is **prefix-free**.

Some trees are better than others

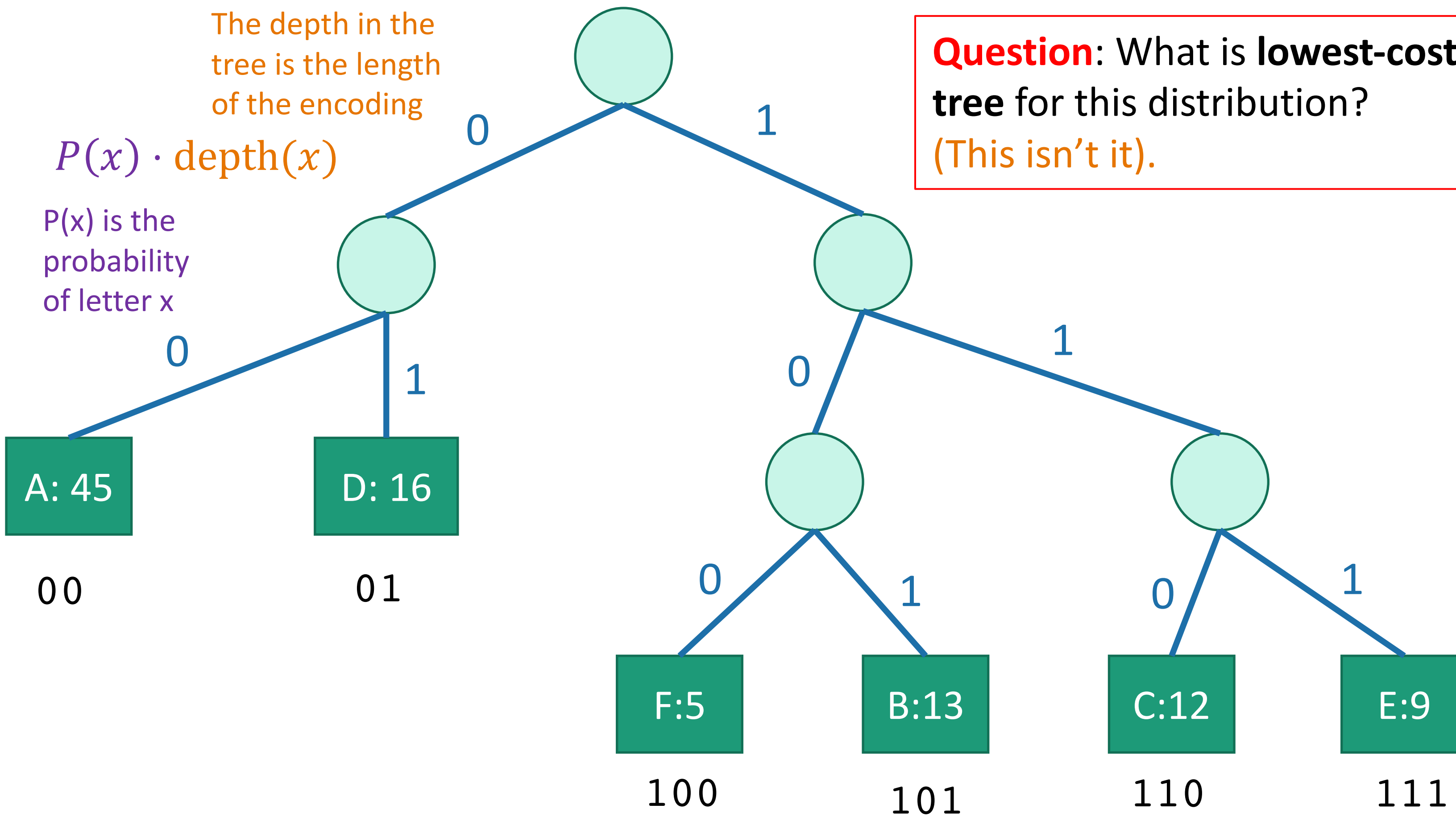
- Imagine choosing a letter at random from the language.
  - Not uniform, but according to our histogram!
- The **cost of a tree** is the expected length of the encoding of that letter.

Cost = $\sum_{leaves\ x} P(x) \cdot \mathrm{depth}(x)$

The depth in the tree is the length of the encoding

P(x) is the probability of letter x

Question: What is **lowest-cost tree** for this distribution?
(This isn't it).

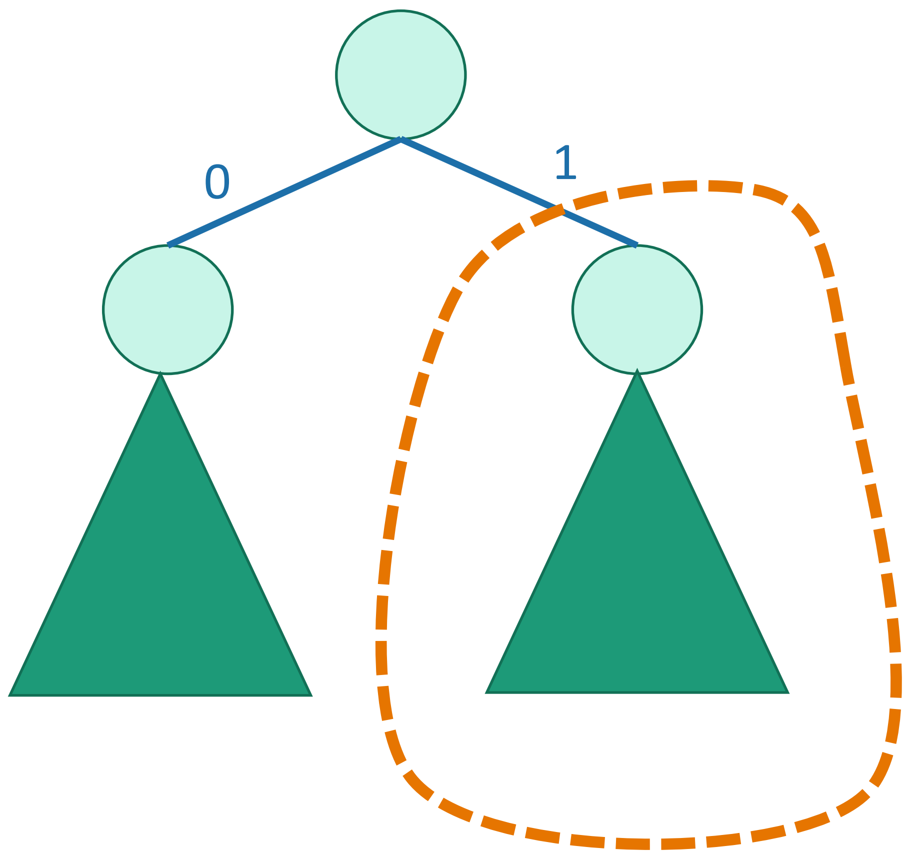


Expected cost of encoding a letter with this tree:

$$\mathbf{2(0.45 + 0.16) + 3(\ 0.05 + 0.13 + 0.12 + 0.09) = 2.39}$$

Optimal sub-structure

- Suppose this is an optimal tree:

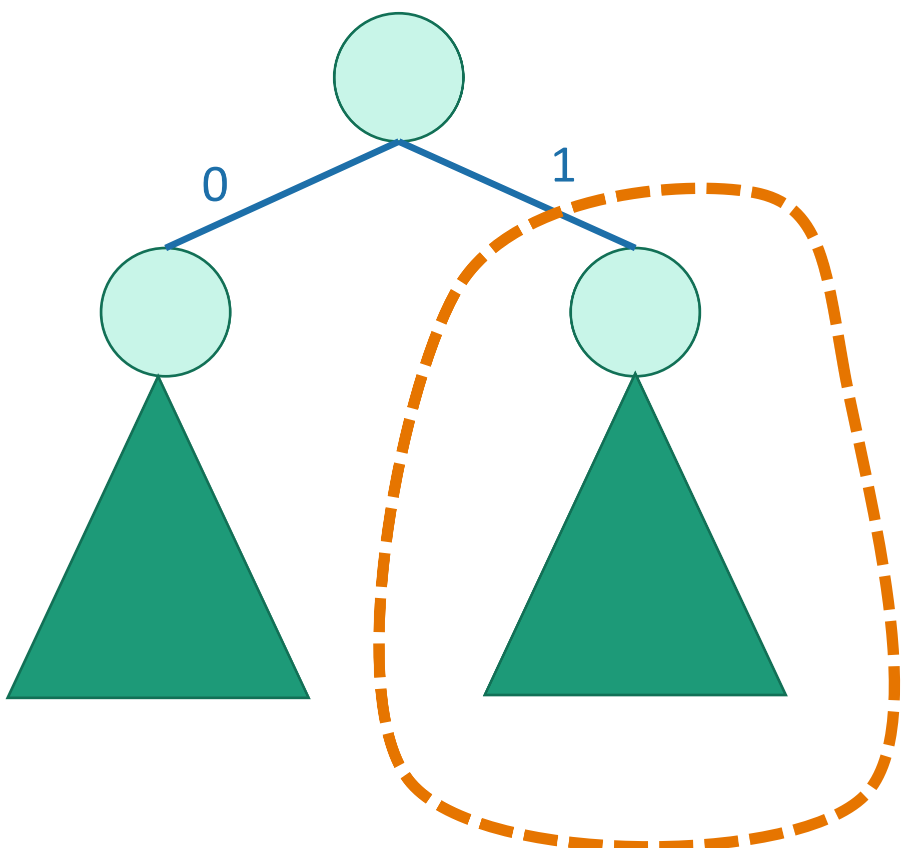


Then this is an optimal tree on fewer letters.

Otherwise, we could change this sub-tree and end up with a better overall tree.

In order to design a greedy algorithm

- Think about what letters belong in this sub-problem...



What's a **safe choice** to make for these lower sub-trees?

Infrequent elements!

We want them as low down as possible.

Solution

greedily build subtrees, starting with the infrequent letters

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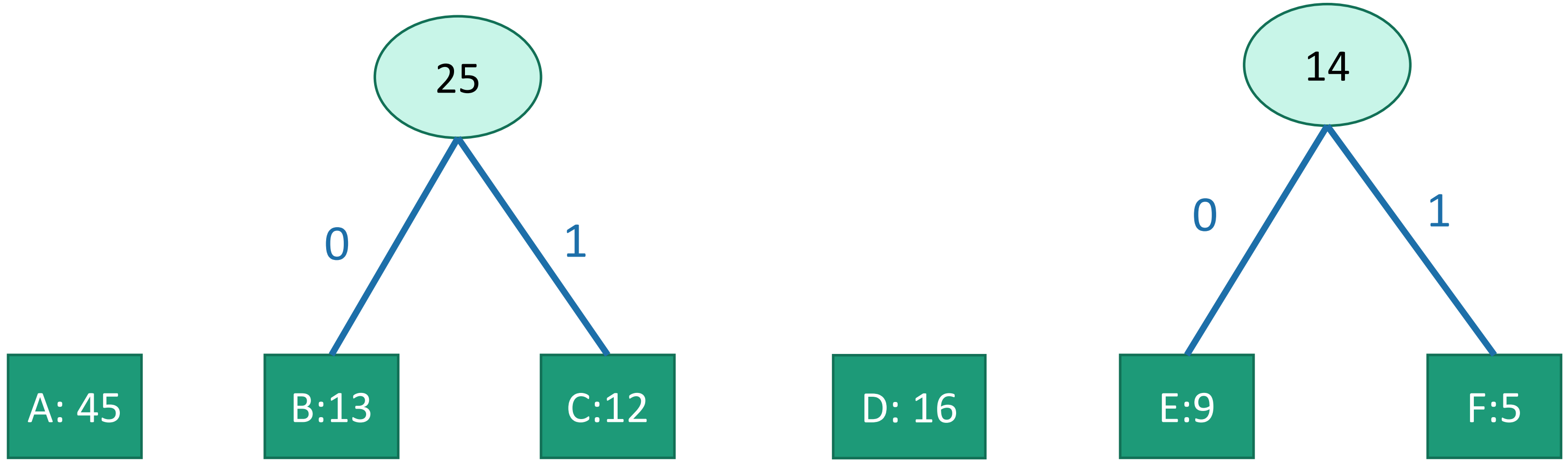

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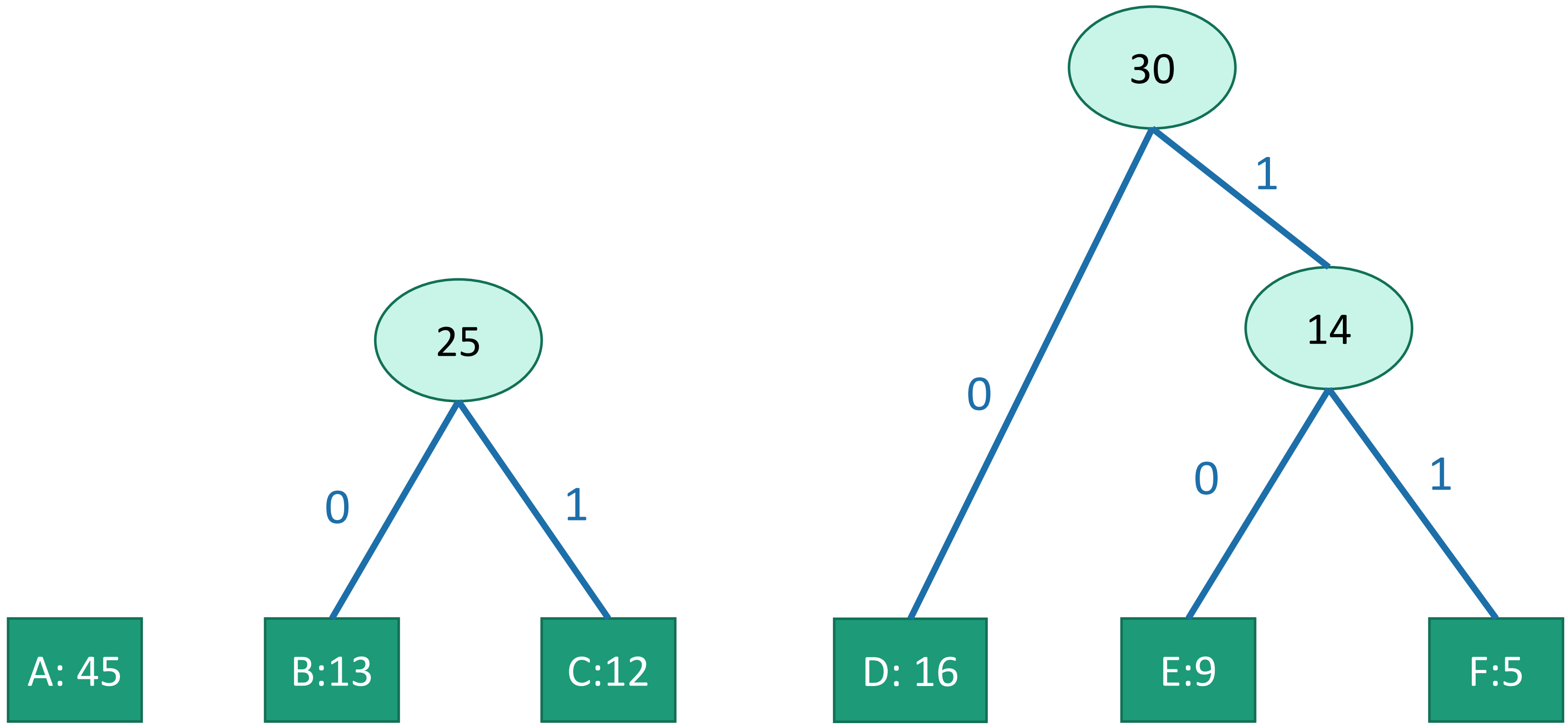
greedily build subtrees, starting with the infrequent letters



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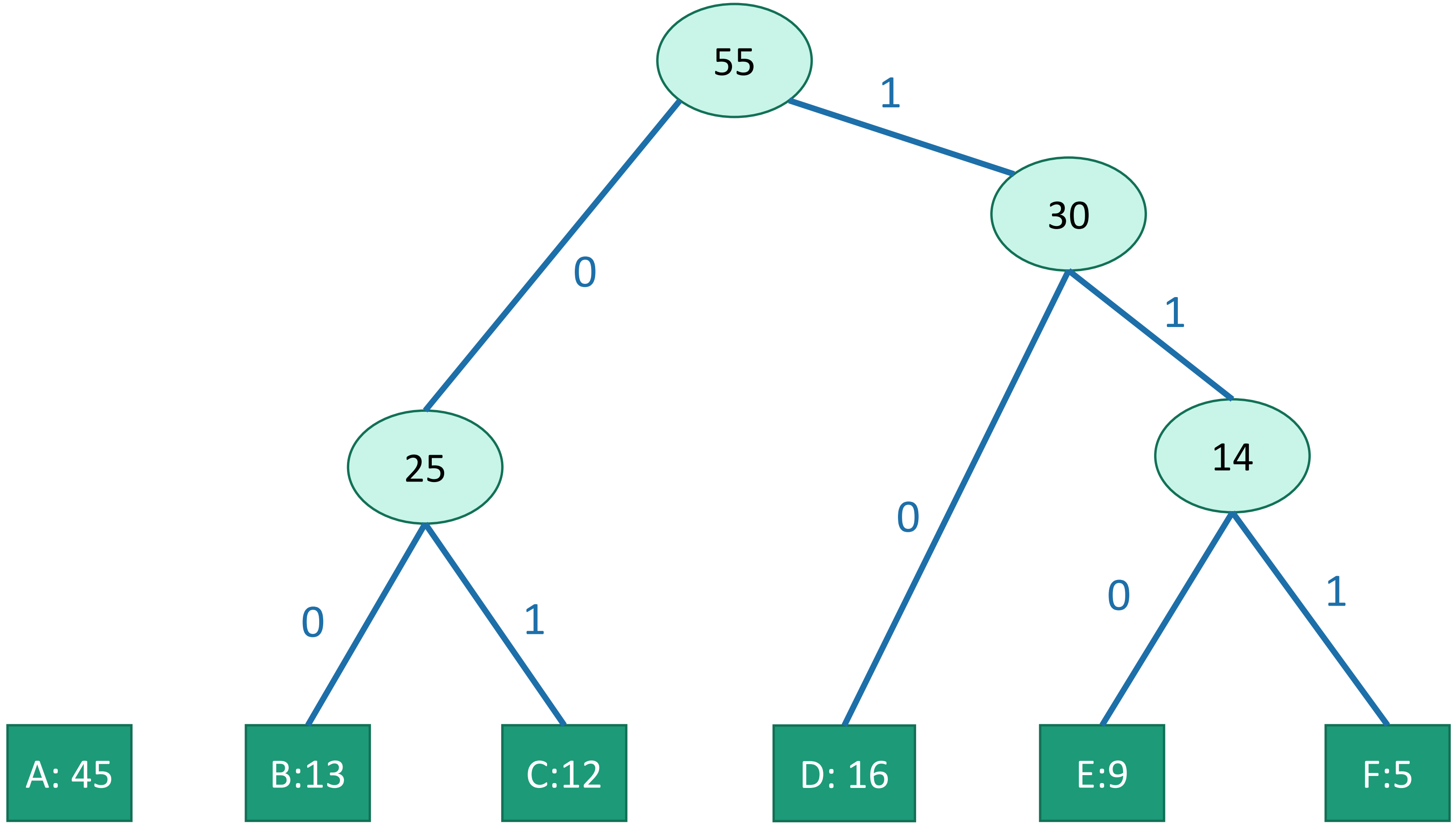

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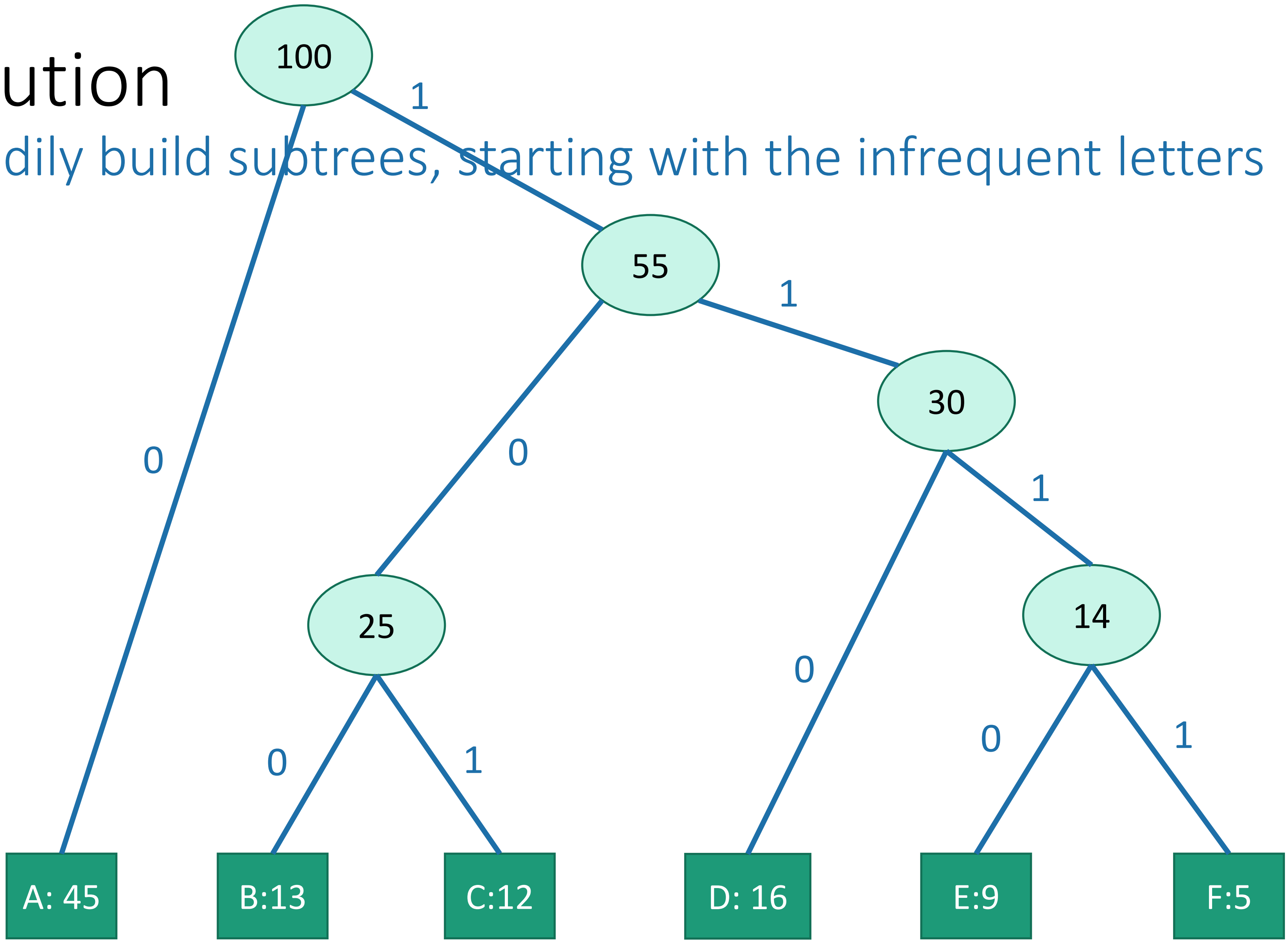

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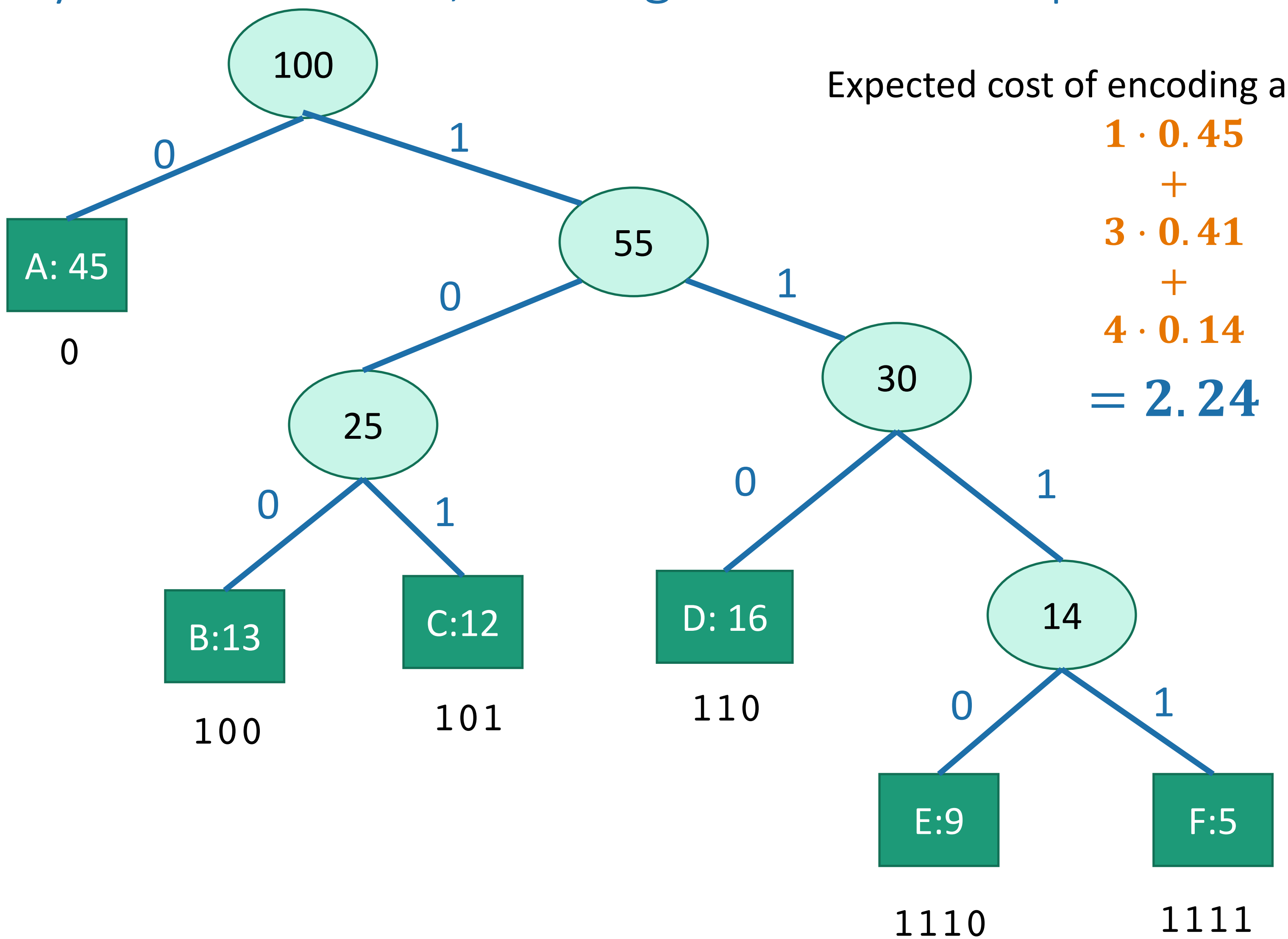
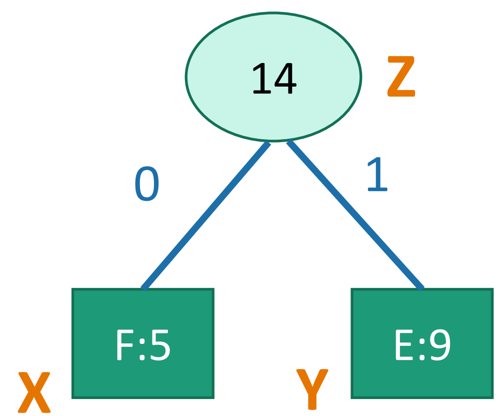


Expected cost of encoding a letter:

$$\mathbf{1 \cdot 0.45} + \mathbf{3 \cdot 0.41} + \mathbf{4 \cdot 0.14} = \mathbf{2.24}$$

What exactly was the algorithm?

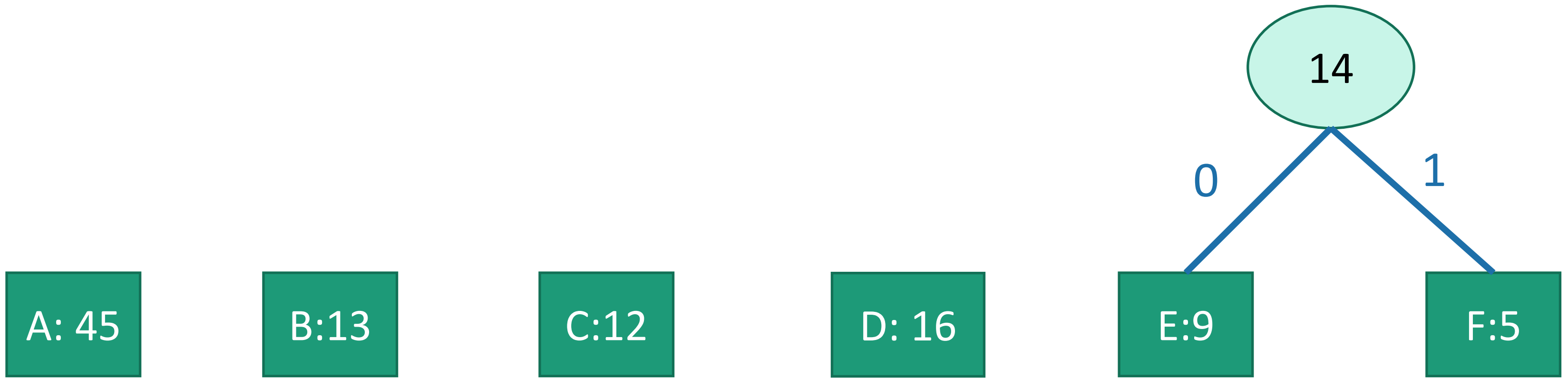
- Create a node like D: 16 for each letter/frequency
  - The key is the frequency (16 in this case)
- Let **CURRENT** be the list of all these nodes.
- **while** len(**CURRENT**) > 1:
  - **X** and **Y** ← the nodes in **CURRENT** with the smallest keys.
  - Create a new node **Z** with **Z.key = X.key + Y.key**
  - Set **Z.left = X, Z.right = Y**
  - Add **Z** to **CURRENT** and remove **X** and **Y**
- return **CURRENT**[0]



Proof strategy

just like before

- Show that at each step, the choices we are making **won't rule out** an optimal solution.
- Lemma:
  - Suppose that x and y are the two least-frequent letters. Then there is an optimal tree where x and y are siblings.

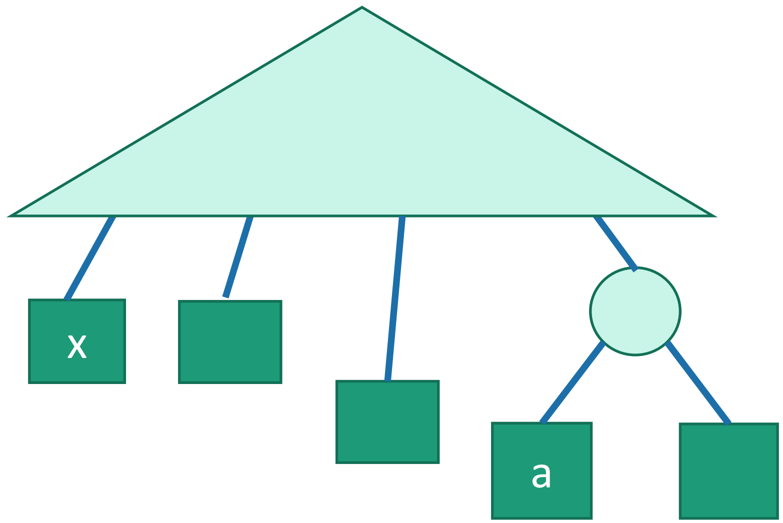


Lemma

proof idea

If x and y are the two least-frequent letters, there is an optimal subtree where x and y are siblings.

- Say that an optimal tree looks like this:



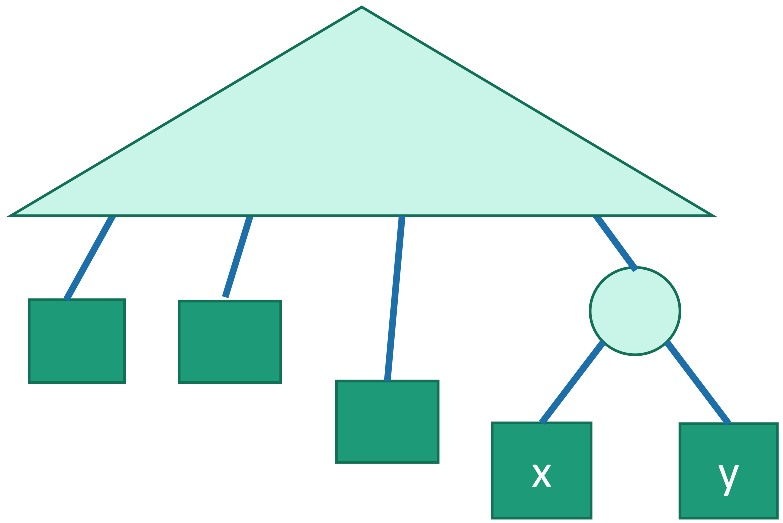
Lowest-level sibling nodes: at least one of them is neither x nor y

- What happens to the cost if we swap x for a?
  - the cost can't increase; a was more frequent than x, and we just made its encoding shorter.
- Repeat this logic until we get an optimal tree with x and y as siblings.

Lemma
proof idea

If x and y are the two least-frequent letters, there is an optimal subtree where x and y are siblings.

- Say that an optimal tree looks like this:



Lowest-level sibling nodes: at least one of them is neither x nor y

- What happens to the cost if we swap x for a?
  - the cost can't increase; a was more frequent than x, and we just made its encoding shorter.

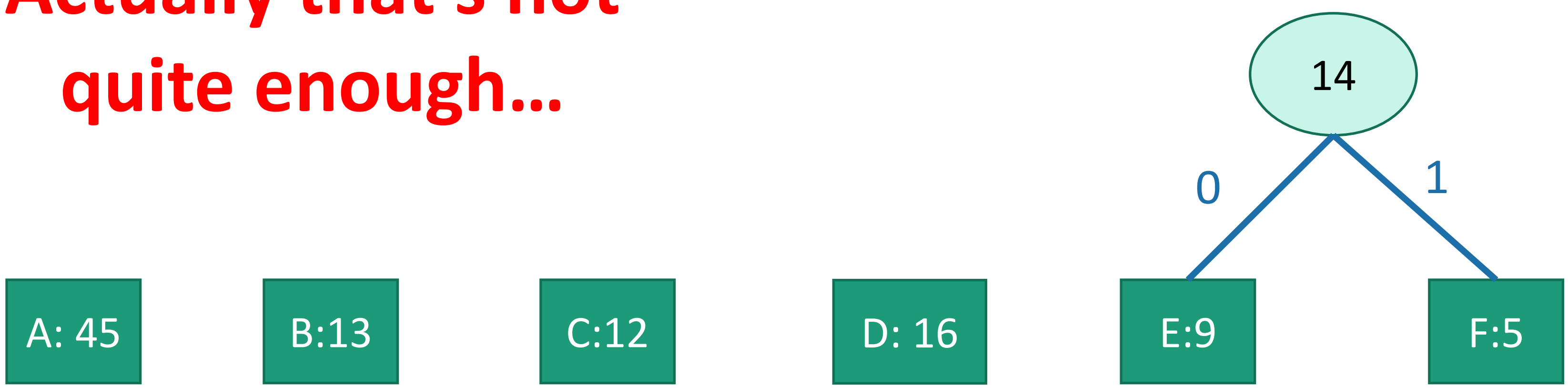
let's name it T and T' :
cost (T) - cost (T') = (p_a - p_x) Delta_1 + (p_b - p_y) Delta_2 >=0

Proof strategy

just like last time

- Show that at each step, the choices we are making **won't rule out** an optimal solution.
- Lemma:
  - Suppose that x and y are the two least-frequent letters. Then there is an optimal tree where x and y are siblings.

Actually that's not quite enough…



Proof strategy

just like last time

Our argument before just showed that we made the right choice at the first step, when everything was a leaf. What about once we start grouping stuff?

- Show that at each step, the choices we are making **won't rule out** an optimal solution.
- Lemma:
  - Suppose that x and y are the two least-frequent letters. Then there is an optimal tree where x and y are siblings.

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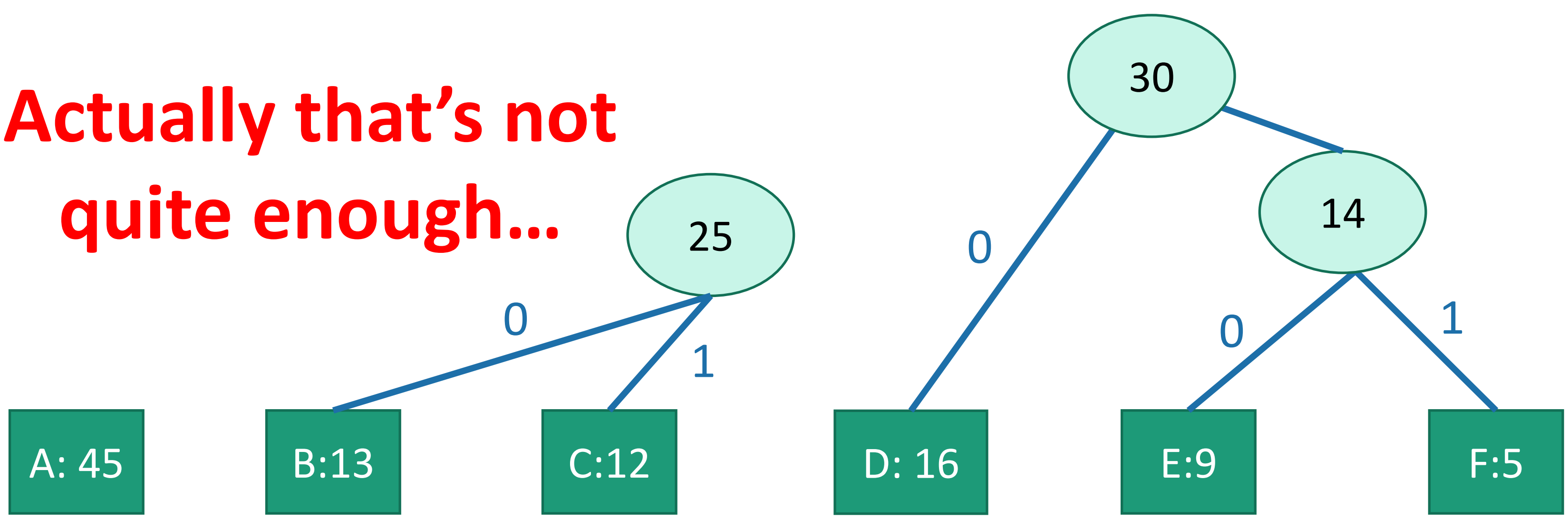
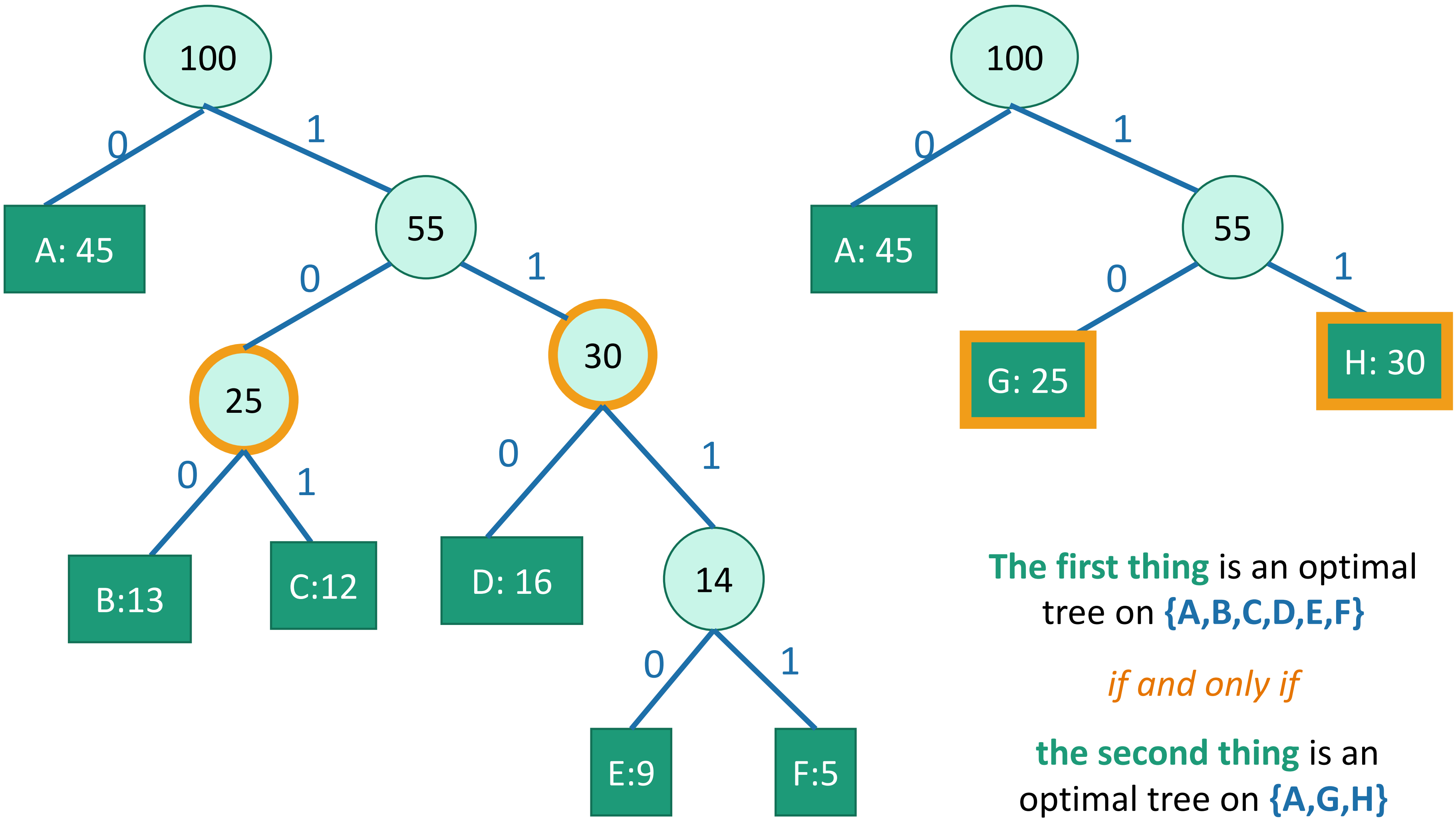


Lemma 2

this distinction doesn't really matter



The first thing is an optimal tree on **{A,B,C,D,E,F}**

if and only if

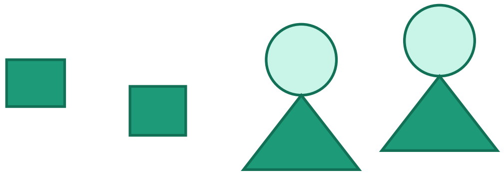
the second thing is an optimal tree on **{A,G,H}**

Together

- Lemma 1:
  - Suppose that x and y are the two least-frequent letters. Then there is an optimal tree where x and y are siblings.
- Lemma 2:
  - We may as well imagine that CURRENT contains only leaves.
- These imply:
  - At each step, our choice doesn't rule out an optimal tree.

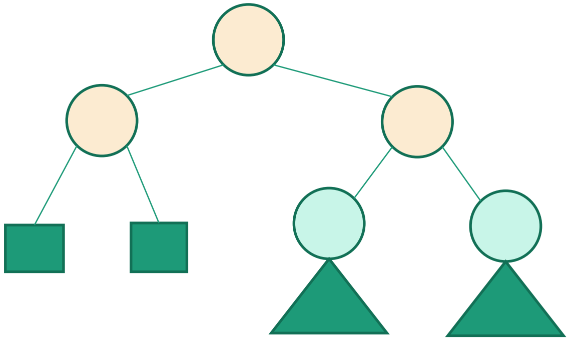
Formally, we'd use induction

After the t'th step, we've got a bunch of current sub-trees:

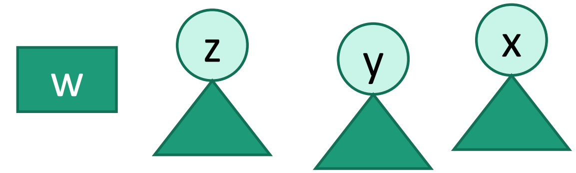


- Inductive hypothesis:
  - after the t'th step,
    - there is an optimal tree containing the current subtrees as "leaves"
- Base case:
  - after the 0'th step,
    - there is an optimal tree containing all the characters.
- Inductive step:
  - **TO DO**
- Conclusion:
  - after the last step,
    - there is an optimal tree containing this whole tree as a subtree.
  - **aka,**
    - after the last step the tree we've constructed is optimal.

Inductive hyp. asserts that our subtrees can be assembled into an optimal tree:

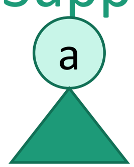


We've got a bunch of current sub-trees:



say that x and y are the two smallest.

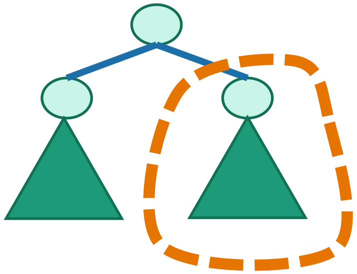
Inductive step

- Suppose that the inductive hypothesis holds for t-1
  - After t-1 steps, there is an optimal tree containing all the current sub-trees as "leaves."
- Want to show:
  - After t steps, there is an optimal tree containing all the current sub-trees as leaves.
- <u>Two ingredients:</u>
  - **Lemma 1**: If x and y are the two least-frequent letters, there is an optimal subtree where x and y are siblings.
  - **Lemma 2**: Suppose that there is an optimal tree containing 

 as a subtree. Then we may as well replace it with a new letter with frequency 

What have we learned?

- ASCII isn’t an optimal way to encode English, since the distribution on letters isn’t uniform.
- Huffman Coding is an optimal way!
- To come up with an optimal scheme for any language efficiently, we can use a greedy algorithm.



- To come up with a greedy algorithm:
  - Identify **optimal substructure**
  - Find a way to make “safe” choices that **won’t rule out an optimal solution.**
    - Create subtrees out of the smallest two current subtrees.

Recap I

- Greedy algorithms!

  examples:

  - Activity Selection

  - Huffman Coding

Recap II

- Greedy algorithms!
- Often easy to write down
  - But may be hard to come up with and hard to justify
- The natural greedy algorithm may not always be correct

- A problem is a good candidate for a greedy algorithm if:
  - it has optimal substructure
  - that optimal substructure is **REALLY NICE**
    - solutions depend on just one other sub-problem.