Linear Programming

Mathematical Decision-making under Constrains

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Today

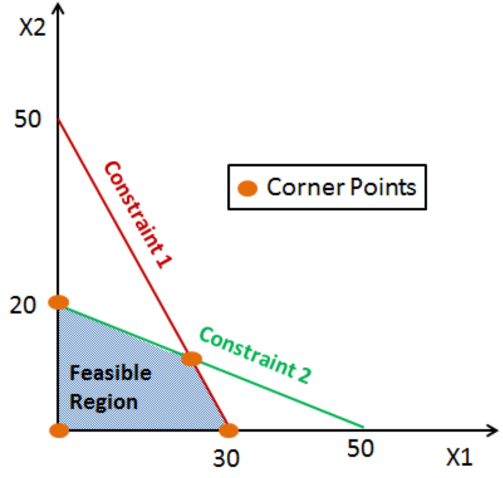
Linear Programming

- Agenda:
  - What is linear programming?
  - An overview of linear programming with some examples
  - How to formulate problems as linear programs?
  - How to optimize life everyday problems using LP?

What is Linear programming?

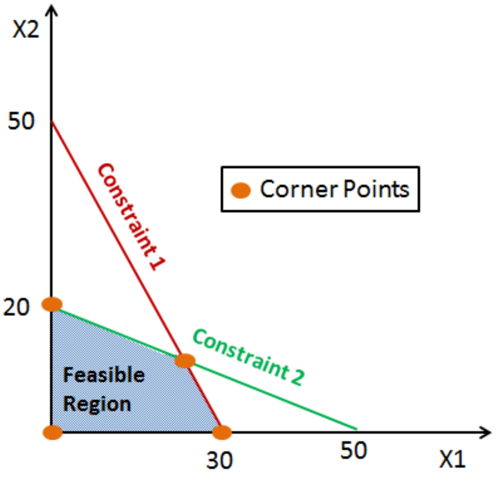
Optimization problem consisting in :

- Maximizing (or minimizing) a linear objective function of n decision variables
- Subject to a set of constraints expressed by linear equations or inequalities



What is Linear programming?

- Constraints
- Objective function
- Feasible Region
- Optimal point



Linear Programming Applications

- Routing & logistics
- Financial Planning
- Manufacturing
- Product Mix

Linear Programming Ingredients

Decision Variables

The decision variable will decide the output. It gives the ultimate solution of the problem. For any problem, the first step is to identify the decision variables.

$X_1, X_1X_2X_3, \dots, X_n$

Objective

In a problem, the objective function should be specified in a quantitative way.

$Min\ f(X_1, X_1X_2X_3, \dots, X_n)$

Or

$Max\ f(X_1, X_1X_2X_3, \dots, X_n)$

Constraints

The limitations should be expressed in the mathematical form, regarding the resource.

$f_l(X_1, X_1X_2X_3, \dots, X_n) \leq b_l$

$f_k(X_1, X_1X_2X_3, \dots, X_n) \geq b_k$

$f_m(X_1, X_1X_2X_3, \dots, X_n) = b_m$

A sample linear programming problem

- Obejective function : $z = 4x + 5y$, we want to maximize Z
- Constraints : $\begin{cases} x + y \le 20 \\ 3x + 4y \le 72 \end{cases}$

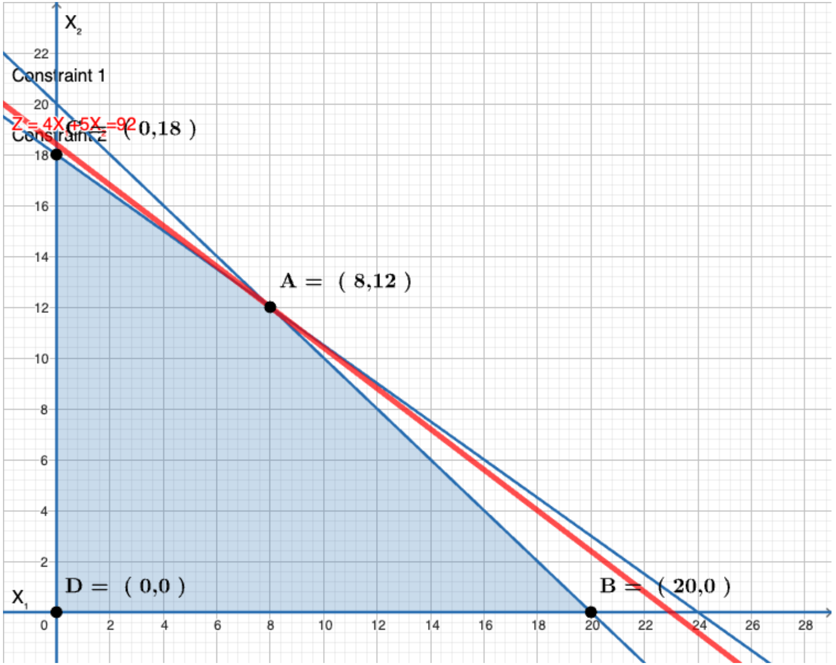
A sample linear programming problem

- Obejective function :

$$z = 4x + 5y$$

- Constraints :

$$\begin{cases} x + y \ \leq 20 \\ 3x + 4y \ \leq 72 \end{cases}$$



A sample linear programming problem

- Obejective function : $z = 4x + 5y$, we want to maximize Z
- Constraints : $\begin{cases} x + y \leq 20 \\ 3x + 4y \leq 72 \end{cases}$

Point	Coordinates (x,y)	Objective Function Value 4x+ 5y
A	(8,12)	4(8)+ 5(12) = 92
B	(20,0)	4(20)+ 5(0) = 80
C	(0,18)	4(0)+ 5(18) = 90
D	(0,0)	4(0)+ 5(0) = 0

Linear programming

Example Problem

A company recieves in sales \$20 per book and \$18 per calculator. The cost per unit to manufacture each book and calculator are \$5 and \$4 recpectively. The monthly (30 days) cost must not exceed \$27000 per month. If the manufacturing equipment used by the company takes 5 minutes to produce a book and 15b minutes to prosuce a calculator. How many books and calculator should the company make to maximize profit or sales? Determine the maximum profit or sale the company earns in a 30 day period.

Linear programming
Example Problem

	Book	Calculator
Sales	20 $	18 $
Cost	5 $	4 $
Time	5 min	15 min

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Example Problem

- Step 1 : Write the objective function
- Step 2 : Use the constraints to graph

Linear programming

Example Problem

	Book	Calculator
Sales	20 $	18 $
Cost	5 $	4 $
Time	5 min	15 min

- Objective

$$S = 20B + 18C$$

- Constraints

$$5B + 4C \leq 27000$$
$$5B + 15C \leq 43200$$

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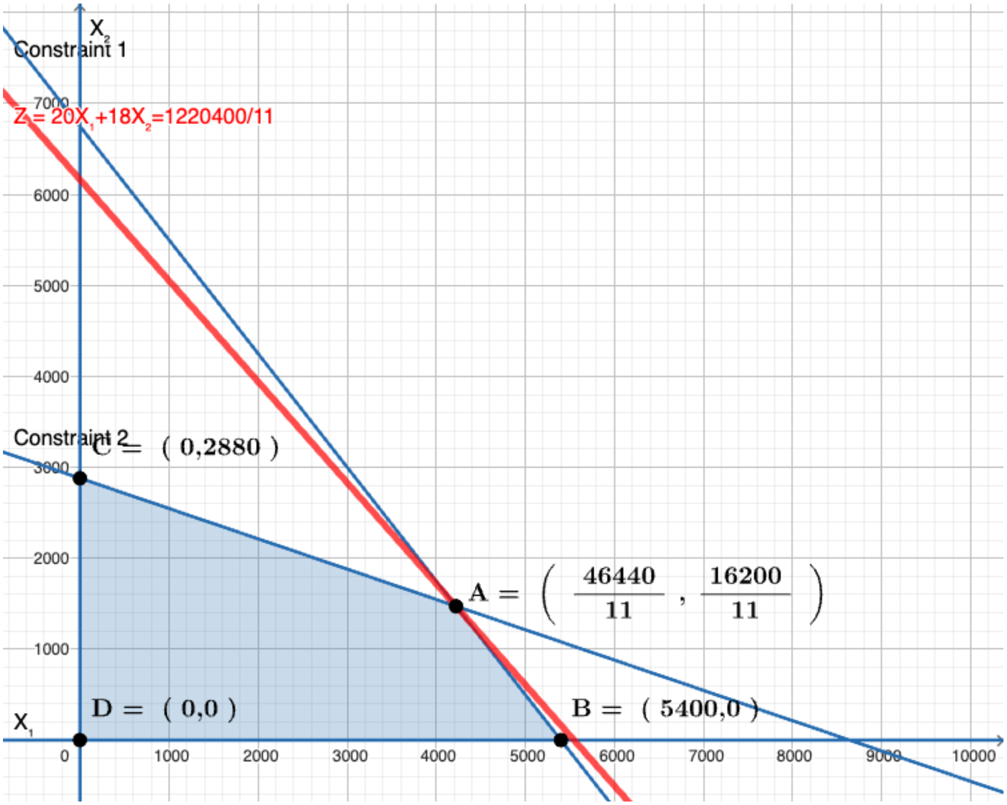
Example Problem



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Example Problem

Point	Coordinates (x,y)	Objective Function Value 20 x + 18 y
A	(46440/11,16200/11)	20(46440/11)+ 18(16200/11) = 1220400/11
B	(5400,0)	20(5400)+ 18(0) = 108000
C	(0,2880)	20(0)+ 18(2880) = 51840
D	(0,0)	20(0)+ 18(0) = 0

$$Profit = Sales - Cost$$
$$Profit = 110934 - 27000$$

$$Profit = 83934\ \$$$

Formulating problems as linear programs

Maximum Flow Problem

- Given a directed graph with a **source** vertex s and a **sink** vertex t, where each edge $u \to v$ has a **capacity** $c(u \to v) > 0$,
- A **flow** $\boldsymbol{f}$ is a set of edge labels $f(u \to v)$ such that :
  - $0 < f(u \to v) < c(u \to v)$ on every edge
  - Total flow in = total flow out, at all vertices other that s and t.
    Flow conservation

$$\boxed{\sum_{u:\, u \to v} f(u \to v) = \sum_{w:\, v \to w} f(v \to w) \quad \text{at all vertices } v \in V \setminus \{s, t\}.}$$

- And the value of the flow is :
  $value(f) = net\ flow\ out\ of\ s = net\ flow\ into\ t$

$$\boxed{\text{value}(f) = \sum_{u:\, s \to u} f(s \to u) - \sum_{u:\, u \to s} f(u \to s).}$$

Formulating problems as linear programs

Maximum Flow Problem

How to formulate a max flow problem as an LP ?

- Introduce variables to represent flow over each edge of the network
- Formulate the capacity constraints and conservation constraints
- Add an artificial feedback link from sink → source to represent the total flow :
  The objective function of the LP is the total flow (over the artificial feedback link)

Formulating problems as linear programs
Maximum Flow Problem

- The key to convert a max flow problem into a Linear Program is the use of “flow variables”.

- Flow variable: how much flow over a link:

  x_{ij} = amount of flow from i → j

Formulating problems as linear programs
Maximum Flow Problem

- A maximum flow is a flow that satisfies these constraints and maximizes the flow value, which is the total flow coming out of the source minus the total flow into the source.

- Maximize :

$$\sum_{v \in V} f_{sv} - \sum_{v \in V} f_{vs}$$

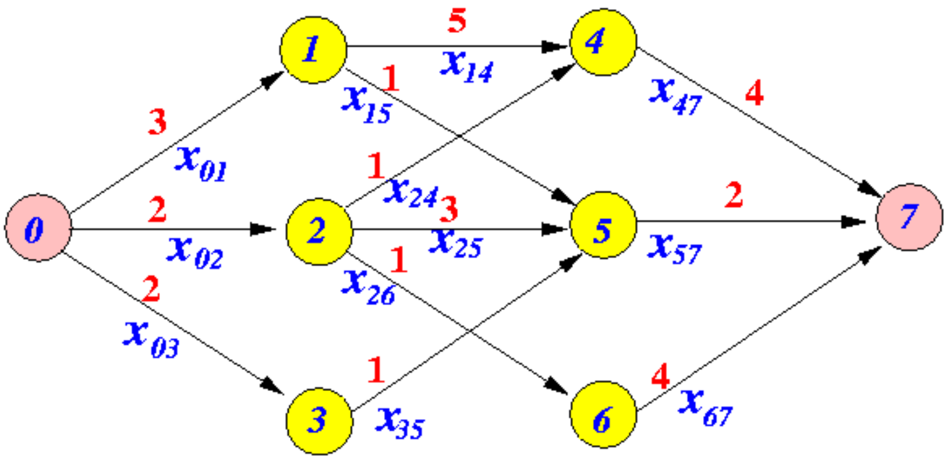
Subject to :

$$\begin{aligned} f_{uv} &\le c(u,v) && \text{for each } u, v \in V \ , \\ \sum_{v \in V} f_{vu} &= \sum_{v \in V} f_{uv} && \text{for each } u \in V - \{s,t\} \ , \\ f_{uv} &\ge 0 && \text{for each } u, v \in V \ . \end{aligned}$$

Formulating problems as linear programs

Maximum Flow Problem ; Example

- Consider the basic following network :

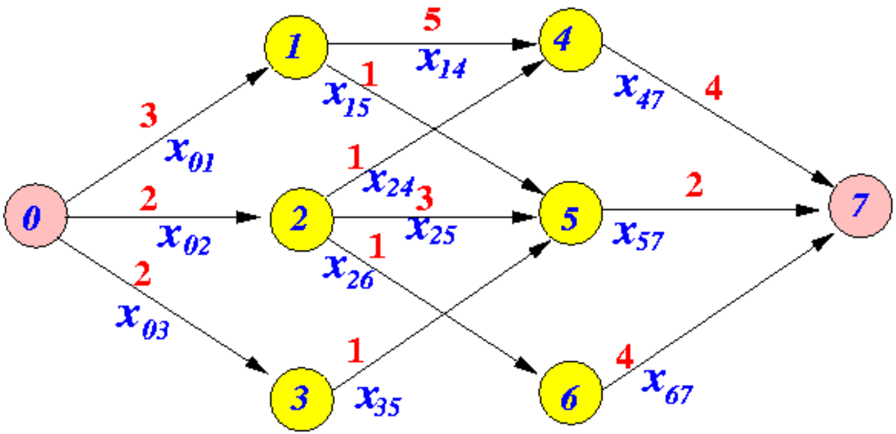


Formulating problems as linear programs

Maximum Flow Problem ; Example

- In order to formulate the max flow problem as an LP, we will need to introduce the following flow variables:

x_{01}	x_{25}
x_{02}	x_{26}
x_{03}	x_{35}
x_{14}	x_{47}
x_{15}	x_{57}
x_{24}	x_{67}



Formulating problems as linear programs

Maximum Flow Problem ; Example

- There are two types of constraints in a basic network :

    1. Capacity constraints
    2. Flow conservations constraints

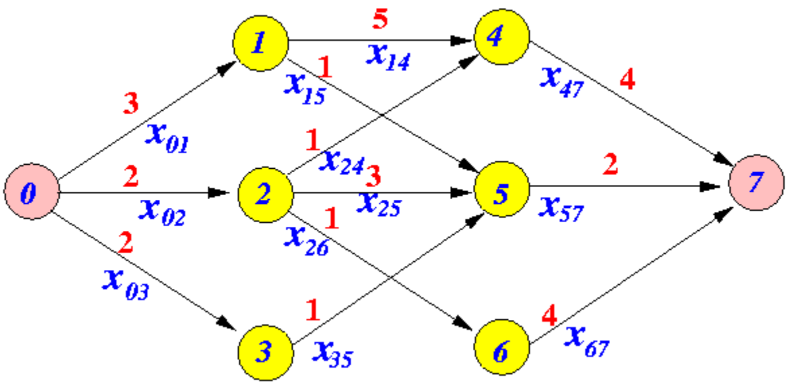
Formulating problems as linear programs

Maximum Flow Problem ; Example

1. Flow capacity constraints :
   The flow over any link cannot exceed the capacity of that link

$$f(u \rightarrow v) < c(u \rightarrow v)$$

$x_{01} \leq 3$	$x_{25} \leq 3$
$x_{02} \leq 2$	$x_{26} \leq 1$
$x_{03} \leq 2$	$x_{35} \leq 1$
$x_{14} \leq 5$	$x_{47} \leq 4$
$x_{15} \leq 1$	$x_{57} \leq 2$
$x_{24} \leq 1$	$x_{67} \leq 4$



Formulating problems as linear programs

Maximum Flow Problem ; Example

2. Flow conservations constraints :
   constraint is valid for nodes other than the source S and sink T
   Total flow flowing into a node = Total flow flowing out of a node

$$\sum_{v \in V} f_{vu} = \sum_{v \in V} f_{uv} \quad \text{for each } u \in V - \{s, t\}$$

$$f_{uv} \geq 0 \quad \text{for each } u, v \in V .$$

node 1: $x_{01} = x_{14} + x_{15}$

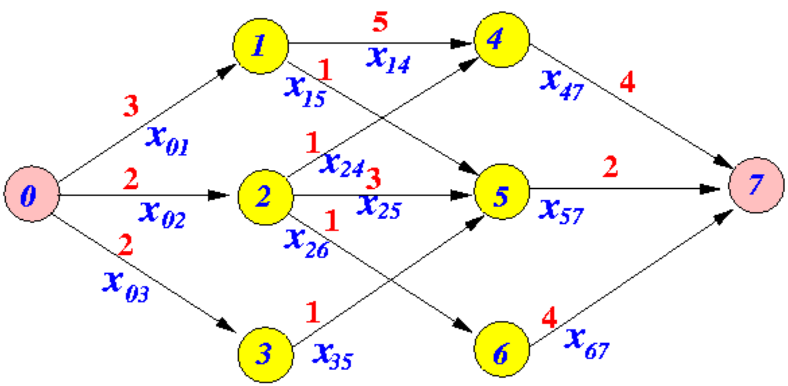
node 2: $x_{02} = x_{24} + x_{25} + x_{26}$

node 3: $x_{03} = x_{35}$

node 4: $x_{14} + x_{24} = x_{47}$

node 5: $x_{15} + x_{25} + x_{35} = x_{57}$

node 6: $x_{26} = x_{67}$



Formulating problems as linear programs

Maximum Flow Problem ; Example

- Objective function
  The objective is to maximize the flow from source node 0 to sink node 7

Objective function = sum of all flows emanating from the source S

Max : $x_{01} + x_{02} + x_{03}$