Discrete Markov chains and mixing times
Homework 1

Due 3 月 4 日（周一）at the start of class

1 Textbook problems

- Chapter 1: 1.5, 1.6, 1.10, 1.11, 1.12
- Chapter 2: 2.3, 2.5 (give two different proofs), 2.6, 2.7

Other exercises in chapters 1 and 2 may also be interesting but only the above problems are required. Others exercises are suggested for interested students for additional review and learning.

2 Additional problems

As in class, let $\mathbb{M}_n$ be the collection of $n \times n$ stochastic matrices. Let $\mathbb{M}_n^I$ be those which are irreducible. Assume $|\mathcal{X}| < \infty$.

1. Classify the elements of $\mathbb{M}_n$ for which the uniform distribution $\pi = (1/n, \ldots, 1/n)$ is stationary.
2. At the start of the proof of Proposition 1.14, we used the equivalence of $\mathbb{P}_z(\tau_z^+ < \infty) = 1$ and $\mathbb{E}_z(\tau_z^+) < \infty$ (which we proved in §1.7). Where else did the proof use the assumption $\mathbb{P}_z(\tau_z^+ < \infty) = 1$?
3. We stated in class that $x \in \mathcal{X}$ is essential iff $\mathbb{P}_x(\tau_x^+ < \infty) = 1$. Prove this.

3 Optional problems

Interested students may consider the following problem. It is not required. Let $\mathcal{P}(\mathcal{X})$ be the collection of probability measures on $\mathcal{X}$.

1. (*a*) Consider the map $S : \mathbb{M}_n^I \to \mathcal{P}(\mathcal{X})$ given by $S(P) = \pi$, where π is the (unique) stationary distribution of P. Is S injective? Is S surjective?

   (*b*) Consider the reversal map $\hat{} : \mathbb{M}_n^I \to \mathbb{M}_n^I$ given by $\hat{}(P) = \hat{P}$. It is easy to see that $\hat{}$ is a bijection. Does it possess other nice operator-theoretic properties? (For example, is it an isometry, in some sense? Does it preserve the spectrum?)