19/08/2019

UNIT 3, 4, 5 - COMPLEX ANALYSIS

Complex Numbers Set: $\mathbb{C} = \{ z = x+iy \mid x, y \in \mathbb{R}, \sqrt{-1} = i, i^2 = -1 \}$.

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$.

Equality of Complex Number:
① $z_1 = z_2 \Leftrightarrow x_1 = x_2$ and $y_1 = y_2$

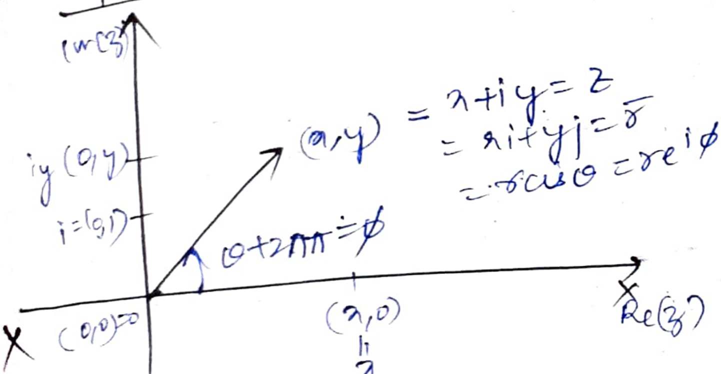
Addition of Complex Numbers. ② $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

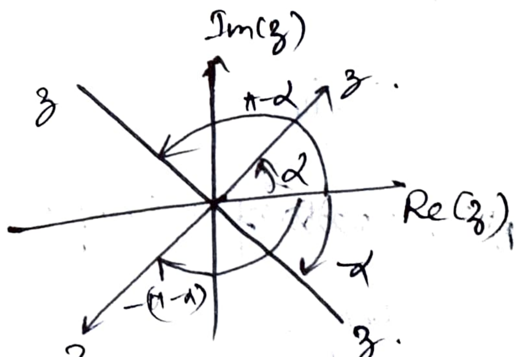
Multiplication of Complex number ③ $z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

Note $x = Re(z)$ $y = Im(z)$

Notation:
① $(0,1) = i$ ② $i^2 = -1$ ③ $(x,y) = x+iy = z$ ④ $|z| = |mag(z)| = \sqrt{x^2+y^2}$
⑤ $\phi = amp(z) = arg(z)$

Complex plane / xy plane / Argand Diagram (2D plane)





$x+iy$ – complex/imaginary
iy – purely imaginary.

Principal Argument of a Complex Number:

$$Arg(z) = \begin{cases} \text{undefined} & x=0, y=0 \\ 0 & x>0, y=0 \\ \pi/2 & x=0, y>0 \\ \pi & x<0, y=0 \\ -\pi/2 & x=0, y<0 \\ \alpha = \tan^{-1}|y/x| & x>0, y>0 \\ \pi-\alpha & x<0, y>0 \\ -(\pi-\alpha) & x<0, y<0 \\ -\alpha & x>0, y<0 \end{cases}$$

$arg(z) = Arg(z) + 2n\pi$, $n \in \mathbb{Z}$

$\phi = \theta + 2n\pi$
where
$-\pi < \theta \leq \pi$

Find argument and principal argument of following complex numbers

Complex Number (z)	Principal Argument $\theta = Arg(z) \in (-\pi, \pi]$	Argument $\phi = \theta + 2n\pi, n \in Z$
0	Undefined	Undefined
2	0	$0 + 2n\pi$ $+2n\pi$
i	$\pi/2$	$\pi/2 + 2n\pi$
-3	π	$\pi + 2n\pi = (2n+1)\pi$
$-i/2$	$-\pi/2$	$-\pi/2 + 2n\pi$
$1+i\sqrt{3}$	$\pi/3 = \alpha$	$\pi/3 + 2n\pi$
$-1+i\sqrt{3}$	$\pi - \alpha \left(Tan^{-1}\left\|\frac{\sqrt{3}}{-1}\right\|\right)$ $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$	$\frac{2\pi}{3} + 2n\pi$
$-1-i\sqrt{3}$	$-(\pi - \alpha) = -(+\pi - \pi/3)$ $= -\frac{2\pi}{3}$	$-\frac{2\pi}{3} + 2n\pi$
$1-i\sqrt{3}$	$-\alpha = -\pi/3$	$-\pi/3 + 2n\pi$

nth roots of a complex number :

Let $z = r = e^{i\phi} = re^{i(\theta + 2k\pi)}$, $k \in Z$

$\sqrt[n]{z} = z^{1/n} = [re^{i(\theta+2k\pi)}]^{1/n}$, $k \in Z$

$= r^{1/n} e^{i\left(\frac{\theta + 2k\pi}{n}\right)}$, $k = 0, 1, \ldots n-1$

Thus, nth root of a non-zero complex number have n non-zero roots.

Note: nth root of a complex no geometrically represents vertices of a n sided regular polygon which is inscribed in a circle centred at origin and radius $(r)^{1/n}$ where $r = |z|$

① $\sqrt[3]{\sqrt{3} - i} = (\sqrt{3} - i)^{1/3} = 2^{1/3} i\left(\frac{-\pi/6 + 2k\pi}{3}\right)$ $k = 0, 1, 2 \ldots$

② $\sqrt[3]{1} = 1^{1/3} e^{i\left(\frac{0 + 2k\pi}{3}\right)} = e^{i\frac{2k\pi}{3}}$ $k = 0, 1, 2$

$= \left(e^{i\frac{2\pi}{3}}\right)^k = 1, e^{i\frac{2\pi}{3}}, \left(e^{i\frac{2\pi}{3}}\right)^2$

$= 1, \omega, \omega^2$

Logarithm

for $z \neq 0$, $\ln(z) = \ln|z| + i\arg(z) = \ln r + i\phi$

Principal Logarithm

for $z \neq 0$, $\operatorname{Ln} z = \ln|z| + i\operatorname{Arg}(z) = \ln r + i\theta$ where $-\pi < \theta < \pi$

$\ln(-1) = \ln|-1| + i(\pi + 2n\pi)$

$\ln(-1) = (2n+1)\pi i$, $n \in \mathbb{Z}$

$\operatorname{Ln}(-1) = \pi i$

General power

for $z \neq 0$, for any c $z^c = e^{c\ln z}$.

Principal value of $z^c = e^{c\operatorname{Ln} z}$

Principal value $i^i = e^{i\operatorname{Ln}(i)} = e^{i[\ln|i| + i\operatorname{Arg}(i)]} = e^{i[i(\pi/2)]} = e^{-\pi/2}$

General value of $i^i = e^{i[\ln|i| + i\arg(i)]}$

$= e^{i[0 + i(\frac{\pi}{2} + 2n\pi)]} = e^{-(\frac{\pi}{2} + 2n\pi)}$

UNIT 3

ANALYTICITY OF COMPLEX FUNCTIONS:

- Limit
- Continuity
- Differentiability
- Analyticity of Complex Functions and its properties
- Cauchy-Riemann equations in Cartesian and Polar coordinates
- Harmonic functions
- Milne-Thompson method.

2nd Chapter (Continuation)

23rd August 2019

FIRST ORDER NON-LINEAR PDE (Standard type):

Type-1: $f(p,q)=0$

Let $p=a \Rightarrow$ substitute $p=a$ in $f=0$

$\Rightarrow f(a,q)=0 \Rightarrow q=\phi(a)$

$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy \Rightarrow dz=p\,dx+q\,dy=a\,dx+\phi(a)\,dy$

IOBS

$\Rightarrow z=ax+\phi(a)y+c=ax+by+c$

where $f(a,b)=0$

Type-2: $f(z,p,q)=0$

Assume $q=ap \Rightarrow f(z,p,ap)=0$

$\Rightarrow p=\phi(z,a) \Rightarrow q=a.\phi(z,a)$

$\frac{dz}{\phi(z)}=\frac{p\,dx+q\,dy}{\phi(z)}=\frac{\phi(z,a)dx+a.\phi(z,a)dy}{\phi(z)}$

$\Rightarrow \frac{dz}{\phi(z)}=dx+a\,dy$

IOBS

$\int\frac{1}{\phi(z)}dz=x+ay+b.$

Type-3: $f(x,p)=g(y,q)$

Let $f(x,p)=g(y,q)=a$

$f(x,p)=a \Rightarrow p=f_1(x,a)=f_1(x)$ ($\because$ a is arbitrary constant)

$g(y,q)=a \Rightarrow q=g_1(y,a)=g_1(y)$

$dz=p\,dx+q\,dy=f_1(x)dx+g_1(y)dy$

$z=\int f_1\,dx+\int g_1(y)dy+b$

Type 4: Clairaut Equation

$z=px+qy+f(p,q)$

Put $p=a, q=b$. Then, $z=ax+by+f(a,b).$

Solve the following PDE

01) $p^3-q^3=0$ (Type 1)

put $p=a \Rightarrow q^3=a^3 \Rightarrow q=a$

$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy = pdx+qdy = adx+\phi(a)dy$

$dz = adx+ady = a(dx+dy)$ IOBS $z=a(x+y)+c$

02) $p^2+q^2=npq$ (T-1)

Let $p=a \Rightarrow a^2+q^2=naq \Rightarrow q^2-naq=-a^2$ $q(q-na)=-a^2$

$\Rightarrow q^2-naq+a^2=0$ Let $q=\phi(a)=b$ $= \frac{na \pm \sqrt{n^2a^2-4a^2}}{2}$

$dz = pdx+qdy = adx+bdy$

IOBS

$z=ax+by+c$ where $a^2+b^2=nab$

03) $zpq=p+q$ (T-2)

put $q=ap \Rightarrow zp(ap)=p+ap \Rightarrow p^2az=p(a+1)$ $p=\frac{a+1}{a \cdot z}$

$q=\frac{1+a}{z}$ $dz=pdx+qdy = \left(\frac{a+1}{az}\right)dx+\left(\frac{1+a}{z}\right)dy$

$zdz = \frac{(a+1)}{a}dx+(a+1)dy$

IOBS

$\frac{z^2}{2} = \left(\frac{a+1}{a}\right)x+(a+1)y+c$

04) $p^2z^2+q^2=p^2q$ (T-2)

Put $q=ap \Rightarrow p^2z^2+a^2p^2=p^2(ap) \Rightarrow p^2z^2+a^2p^2=p^3a$

$\Rightarrow p=\frac{z^2+a^2}{a}$; $q=z^2+a^2 \Rightarrow dz=pdx+qdy=\frac{z^2+a^2}{a}dx+(z^2+a^2)dy$

$\Rightarrow \frac{dz}{z^2+a^2} = \frac{1}{a}dx+dy$ IOBS $\Rightarrow \frac{1}{a}\tan^{-1}\left(\frac{z}{a}\right) = \frac{x}{a}+y+c$

5) $yp+xq+pq=0$ (T-3)

Dividing with pq

$\frac{y}{q}+\frac{x}{p}+1=0 \Rightarrow \frac{x}{p}+1 = -\frac{y}{q} = a$ (say)

Let $\frac{x}{p}+1=a \Rightarrow x+p=ap \Rightarrow p=\frac{x}{a-1}$

$-\frac{y}{q} = a \Rightarrow q = -\frac{y}{a}$

$dz = p\,dx + q\,dy = \frac{x}{a-1}dx + -\frac{y}{a}dy$

IOBS

$z = \frac{x^2}{2(a-1)} - \frac{y^2}{2(a)} + c$ //

06) $p^2+q^2 = x^2+y^2$ (T-3)

$-x^2+p^2 = -q^2+y^2 = a$

$\Rightarrow p = \sqrt{x^2+a}$; $q = \sqrt{y^2-a}$

$dz = p\,dx + q\,dy = \sqrt{x^2+a^2}\,dx + \sqrt{y^2-a}\,dy$

IOBS

$z = \frac{x}{a}\sqrt{x^2+a^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{y}{a}\sqrt{y^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right) + c$

07) $p^2+q^2 = x+y$ (T-3)

$p^2-a = y-q^2 = a \Rightarrow p = \sqrt{x+a}$, $q = \sqrt{y-a}$

$dz = p\,dx + q\,dy = \sqrt{x+a}\,dx + \sqrt{y-a}\,dy$

$z = \frac{(x+a)^{3/2}}{3/2} + \frac{(y-a)^{3/2}}{3/2} + c$

08) $z = px+qy+\ln pq$ (T-4)

Put $p=a$, $q=b$.

$\Rightarrow z = ax+by+\ln ab$.

09) $(p-q)(z-px-qy) = 1$ (T-4)

$z-px-qy = \frac{1}{p-q}$

$p=a$, $q=b$

$z-ax-by = \frac{1}{a-b} \Rightarrow z = ax+by+\frac{1}{a-b}$ //

10) $pqz = p^2(qx+p^2)+q^2(py+q^2)$

$z = \frac{p^2qx+p^4+q^2py+q^4}{pq}$

$z = px + qy + \frac{p^3}{q} + \frac{q^3}{p}$

$z = ax + by + \frac{a^3}{b} + \frac{b^3}{a}$.

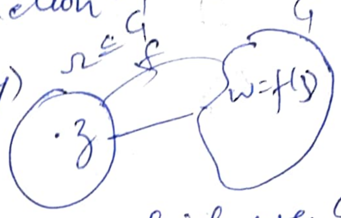
26th August 2019

Chapter 3

Topological Concepts in Complex Function theory.

Complex Function:- Let Ω be a region in $\mathbb{C}$, then any function $f: \Omega \to \mathbb{C}$ is said to be a complex function.

Notation: $w = f(z) = u + iv$ where $u = u(x, y)$



Note:- The graph of a complex function $w = f(z)$ is a four dimensional figure which we cannot imagine.

$w = f(z) = u + iv$

$u = u(x, y) = Re(f(z))$

$v = v(x, y) = Im(f(z))$

Ex: $f(z) = z^2 = (x + iy)^2 = x^2 + (iy)^2 + 2xyi$ = Real $(f(z))$ + Imag(z).

$f(z) = (x^2 - y^2) + i(2xy)$

Limit: $\lim_{z \to z_0} f(z) = L$

i.e, $\boxed{z \to z_0} \Rightarrow \boxed{f(z) \to L}$.

Continuity: $\lim_{z \to z_0} f(z) = L = f(z_0) \Rightarrow f(z)$ is continuous at z_0.

Note: $f(z_0)$ undefined $\Rightarrow f(z)$ is discontinuous at z_0.

Derivative:

$w = f(z)$

$$\left.\frac{dw}{dz}\right|_{z=z_0} = \left.\frac{df}{dz}\right|_{z=z_0} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = L = f'(z_0).$$

Note: Differentiability implies continuity.

discontinuity implies non differentiability.

Continuity is a necessary condition

$f(z)$ differentiable at z_0 implies $f(z)$ continuous at z_0.

Note:-

(i) $f(z)$ discontinuous at $z_0 \Rightarrow f(z)$ not differentiable at z_0.

(ii) $f(z_0)$ undefined $\Rightarrow f(z)$ not differentiable at z_0.

(iii) $f(z)$ continuous at $z_0 \not\Rightarrow f(z)$ differentiable at z_0.

Alternate definition of Derivative:

$z - z_0 = \Delta z$

$\boxed{z \to z_0}$ $\quad \Delta z \to 0$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

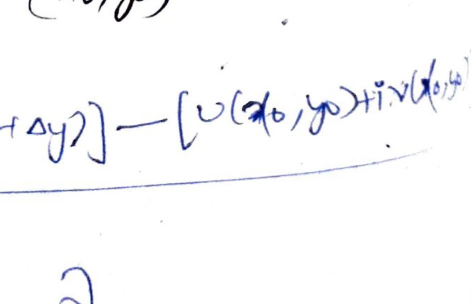
29th August 2019

Theorem: If $f(z) = u + iv$ is differentiable at $z_0 = x_0 + iy_0$, then u, v satisfy Cauchy Riemann Equations ie

$u_x = v_y$, $u_y = -v_x$ at $z_0 = x_0 + iy_0$.

Proof:

Suppose $f(z) = u + iv$ be differentiable at a point $z_0 = x_0 + iy_0$



$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \left[\frac{u(x_0+\Delta x, y_0+\Delta y) + iv(x_0+\Delta x, y_0+\Delta y)] - [u(x_0,y_0) + iv(x_0,y_0)}{\Delta x + i\Delta y} \right]$$

$$f'(z_0) = \left[\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0)}{\Delta x + i\Delta y} \right]$$

$$+ i \left[\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{v(x_0+\Delta x, y_0+\Delta y) - v(x_0, y_0)}{\Delta x + i\Delta y} \right]$$

Case 1 $f'(z_0)$ along path -1

$$f'(z_0) = \left[\lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \frac{u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0)}{\Delta x + i\Delta y}\right] + i\left[\lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \frac{v(x_0+\Delta x, y_0+\Delta y) - v(x_0, y_0)}{\Delta x + i\Delta y}\right]$$

$$= \left[\lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0, y_0)}{\Delta x}\right] + i\left[\lim_{\Delta x \to 0} \frac{v(x_0+\Delta x, y_0) - v(x_0, y_0)}{\Delta x}\right]$$

$$f'(z) = u_x(x_0, y_0) + i\, v_x(x_0, y_0)$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{at } z_0 = x_0 + iy_0 \quad \text{—— (1)}$$

Case 2 $f(z_0)$ along path -1

$$f'(z_0) = \left[\lim_{\Delta y \to 0} \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0)}{\Delta x + i\Delta y}\right] + i\left[\lim_{\Delta y \to 0} \lim_{\Delta x \to 0} \frac{v(x_0+\Delta x, y_0+\Delta y) - v(x_0, y_0)}{\Delta x + i\Delta y}\right]$$

$$= \left[\lim_{\Delta y \to 0} \frac{u(x_0, y_0+\Delta y) - u(x_0, y_0)}{i\Delta y}\right] + i\left[\lim_{\Delta y \to 0} \frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{i\Delta y}\right]$$

$$f'(z_0) = \frac{1}{i} u_y(x_0, y_0) + v_y(x_0, y_0)$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad \text{at } z_0 = x_0 + iy_0 \longrightarrow (2)$$

From (1), (2)

$$f'(z_0) = u_x + iv_x = v_y - iu_y \quad \text{at } z_0 = x_0 + iy_0$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \quad \text{at } z_0 = x_0 + iy_0$$

Cauchy-Riemann Equation

(CR Equations)

Note: If u and v ~~doesnt~~ do not satisfy the CR equations, at point z_0, then $f(z) = u + iv$ is not differentiable at z_0. [Converse need not be true]

29th August 2019

ANALYTICITY

If $f(z)$ is differentiable at z_0, then $f(z)$ is continuous at z_0.

Proof:

Suppose $f(z)$ is differentiable at z_0.

i.e., $\lim_{z \to z_0} f(z) = f(z_0)$

$\iff \lim_{z \to z_0} [f(z) - f(z_0)] = 0$

$$LHS = \lim_{z \to z_0} [f(z) - f(z_0)] = \lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right)(z - z_0)$$

$$= \left(\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \right) \times \lim_{z \to z_0} (z - z_0)$$

$$= f'(z_0) \times 0 = 0.$$

Analyticity

A complex function $f(z)$ is said to be analytic at a point z_0 if there exist a neighbourhood of z_0 such that $f(z)$ is differentiable at all points in the neighbourhood of z_0.

Entire function

If $f(z)$ is analytic at all points in the complex plane then $f(z)$ is said to be an entire function.

$f(z)$ is entire $\Rightarrow$ $f(z)$ is analytic at z_0 $\Rightarrow$ $f(z) = u + iv$ is differentiable at z_0.

$\Downarrow$ u, v satisfy C.R equations ($u_x = v_y$, $u_y = -v_x$) at $z = x_0 + iy_0$

$\Downarrow$ $f(z)$ is continuous at z_0 $\Rightarrow f(z_0)$ is defined

Equivalently,

** $f(z_0)$ is not defined $\Rightarrow$ $f(z)$ is not continuous at z_0

$\Downarrow$

$f(z)$ is not entire. $\Leftarrow$ $f(z)$ is not analytic at z_0. $\Leftarrow$ $f(z)$ is not differentiable at z_0.

$\Uparrow$

U, V not satisfying C-R equations at z_0.
($U_x \neq V_y$ or $V_y \neq -V_x$)

Result:-

If $f(z) = U + iV$ is analytic at a point $z_0 = x_0 + iy_0$ then U and V satisfies Cauchy Riemann equations at z_0 i.e,

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \text{ and } \frac{\partial V}{\partial y} = \frac{-\partial U}{\partial x}$$ at a point z_0.

Proof: Suppose $f(z)$ is analytic at z_0.
$\Rightarrow$ $f(z)$ is differentiable at z_0.

Note:- U and V satisfying CR conditions at a point z_0 need not imply that $f(z) = U + iV$ is differentiable at z_0.

U and V satisfying CR equations at z_0 need not imply that $f(z) = U + iV$ is analytic at z_0

Example: $$f(z) = \begin{cases} \frac{2xy(x+iy)}{x^2+y^2} & z \neq 0 \\ 0 & z = 0. \end{cases}$$

$$\Rightarrow U(x,y) = \begin{cases} \frac{2x^2y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\text{and } V(x,y) = \begin{cases} \frac{2xy^2}{x^2+y^2} & (x,y) = (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

$$\left.\frac{\partial u}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{u(0+h,0)-u(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h} = \lim_{h\to 0} \frac{\left(\frac{2h^2(0)}{h^2+0}\right)-0}{h}$$

$$= \lim_{h\to 0} \frac{0}{h} = 0 \qquad \therefore \underline{u_x(0,0)=0}$$

$$\left.\frac{\partial v}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{v(0+h,0)-v(0,0)}{h} = \lim_{h\to 0} \frac{v(h,0)-v(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{2h(0^2)}{h^2+0^2}\right)-0}{k} = \lim_{h\to 0} \frac{0}{h} = 0 \quad \therefore \underline{v_x(0,0)=0.}$$

$$\left.\frac{\partial u}{\partial y}\right|_{(0,0)} = \lim_{k\to 0} \frac{u(0,0+k)-u(0,0)}{k} = \lim_{k\to 0} \frac{u(0,k)-u(0,0)}{h}$$

$$= \lim_{k\to 0} \frac{\left(\frac{2(0)^2k}{0^2+k^2}\right)-0}{h} = \lim_{k\to 0} \frac{0}{k} = 0. \quad \therefore \underline{u_y = 0.}$$

$$\left.\frac{\partial v}{\partial y}\right|_{(0,0)} = \lim_{k\to 0} \frac{v(0,0+k)-v(0,0)}{k} = \lim_{k\to 0} \frac{v(0,k)-v(0,0)}{h}$$

$$= \lim_{k\to 0} \frac{\left(\frac{2(0)(k^2)}{0^2+k^2}\right)-0}{k} = \lim_{k\to 0} \frac{0}{k} = 0 \quad \therefore \underline{v_y = 0}$$

At $z=0$

$u_x = v_y$ and $u_y = -v_x$

$\Rightarrow u, v$ satisfies CR equations at $z=0$

Consider

$f'(0)$

$$= \lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{(x,y)\to(0,0)} \left[\frac{\frac{2xy(x+iy)}{x^2+y^2}-0}{x+iy}\right]$$

$$= \lim_{(x,y)\to(0,0)} \left[\frac{2xy(x+iy)}{(x+iy)(x^2+y^2)}\right]$$

Let $y = mx$

$x\to 0 \Rightarrow y\to 0.$

$$= \lim_{x\to 0}\left[\frac{2x\cdot(mx)}{x^2+mx^2}\right] = \lim_{x\to 0}\frac{x^2(2m)}{x^2(1+m^2)} = \lim_{x\to 0}\frac{2m}{1+m^2} = \frac{2m}{1+m^2}$$

which is not unique.

therefore, $f'(0)$ doesnt exist.

$\Rightarrow$ ~~$f'(0)$~~ $f(z)$ is not differentiable at $z=0$.

Thus, CR equations are necessary but not sufficient conditions for the function to be differentate.

04 | September | 2019

H.W 01) $f(z) = \sqrt{|xy|}$ show that u and v satisfy CR equations at origin but $f(z)$ is not analytic

02) $f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$

H.W 03) $f(z) = \begin{cases} \dfrac{(x^3-y^3)+i(x^3+y^3)}{x^2+y^2} & , z \neq 0 \\ 0 & , z=0. \end{cases}$

Sol

$$f(z) = u+iv = \begin{cases} \dfrac{(x^2-y^2)-i2xy}{x+iy} & (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

$\Rightarrow$ ~~$u(x,y)$~~

$$f(z) = u+iv = \begin{cases} \dfrac{[(x^2-y^2)-i2xy](x-iy)}{x^2+y^2} & z \neq 0 \\ 0 & , z=0 \end{cases}$$

$$u(x,y) = \begin{cases} \dfrac{x(x^2-y^2)-2xy^2}{x^2+y^2} & , x,y \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

$$v(x,y) = \begin{cases} \dfrac{y(y^2-x^2)-2x^2y}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

$$\left.\frac{\partial u}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{u(0+h,0)-u(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{h(h^2-0)-2(0)}{h^2} - 0}{h} = 1 \qquad \left.\frac{\partial u}{\partial x}\right|_{(0,0)} = 1$$

$$\left.\frac{\partial v}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{v(0+h,0)-v(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{v(h,0)-v(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0. \qquad \left.\frac{\partial v}{\partial x}\right|_{(0,0)} = 0$$

$$\left.\frac{\partial u}{\partial y}\right|_{(0,0)} = \lim_{h\to 0} \frac{u(0,0+k)-u(0,0)}{k}$$

$$= \lim_{h\to 0} \frac{u(0,k)-u(0,0)}{k}$$

$$= \lim_{h\to 0} \frac{0-0}{k} = 0 \qquad \left.\frac{\partial u}{\partial y}\right|_{(0,0)} = 0$$

$$\left.\frac{\partial v}{\partial y}\right|_{(0,0)} = \lim_{h\to 0} \frac{v(0,0+k)-v(0,0)}{k}$$

$$= \lim_{h\to 0} \frac{v(0,k)-v(0,0)}{k}$$

$$= \frac{\frac{k^3}{k^2}-0}{k} = 1 \qquad \left.\frac{\partial v}{\partial y}\right|_{(0,0)} = 1$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$u_x = v_y$ & $u_y = -v_x$

u, v satisfies CR equations at $z = 0$.

Consider $f'(0) = \lim_{z \to 0} \dfrac{f(z) - f(0)}{z - 0}$

$$= \lim_{z \to 0} \left[\frac{\frac{(\bar{z})^2 - 0}{z}}{z} \right] = \lim_{z \to 0} \frac{(\bar{z})^2}{z^2}$$

$$= \lim_{(x,y) \to (0,0)} \frac{(x^2 - y^2) - i2xy}{(x^2 - y^2) + i2xy}$$

on $y = mx$.

$x \to 0 \quad y \to 0$.

$$f'(0) = \lim_{x \to 0} \frac{[x^2 - (mx)^2] - i[2x(mx)]}{[x^2 - (mx)^2] + i[2xmx]} = \lim_{x \to 0} \frac{(1 - m^2) - i2m}{1 - m^2 + i2m}.$$

$= \dfrac{(1 - m^2) - i2m}{1 - m^2 + i2m}$ which is not unique !

$\therefore f(z)$ is not diff. at $z = 0$

$\Rightarrow f(z)$ is not analytic at $z = 0$

Q) Test the analyticity of the following functions.

(i) $\bar{z}$ Let $f(z) = u + iv = \bar{z} = x - iy$

$\Rightarrow u = x, \; v = -y$

$u_x = 1, \; u_y = 0$

$v_x = 0, \; v_y = -1$

$u_x \neq v_y \quad \forall (x, y)$

$\Rightarrow f(z) = u + iv$ not analytic $\forall z$

($\because u, v$, not satisfying C-R equation)

$\therefore f(z)$ is nowhere analytic.

Sufficient Conditions for $f(z) = u + iv$ to be differentiable at a point z_0.

(01). u and v satisfies CR equations at z_0.

(02). All the first order Partial derivatives are continuous at z_0.

$\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$

Sufficient conditions for $f(z) = u+iv$ to analytic in an open region are

01) u, v satisfy CR equations at all points in the region

02) All the 1st order Partial DE u_x, u_y, v_x, v_y are continuous at all points in the region R.

02) Re(z)

$f(z) = u+iv = Re(z) = x.$

$u = x, \quad v = 0$

$u_x = 1 \quad u_y = 0, \quad v_x = 0, \quad v_y = 0.$

$u_x \neq v_y \quad \forall z.$

$\Rightarrow$ u, v not satisfying CR equations $\forall z$.

f(z) is not analytic ~~$\forall$~~ z.

$\therefore$ Re(z) is nowhere analytic

03) Im(z)

$f(z) = u+iv \qquad Im(z) = y$

$u = 0 \quad v = y$

$u_x = 0, \quad u_y = 0 \quad v_x = 0 \quad v_y = 1$

$u_x \neq v_y \quad \forall z$

$\Rightarrow$ u, v not satisfying CR equations $\forall z$

f(z) is not analytic $\forall z$

$\therefore$ Im(z) is nowhere analytic.

04) $|z|^2 = z\bar{z}$

$f(z) = (x^2+y^2) + \cancel{i(2xy)} = u+iv = |z|^2$

$u = (x^2+y^2) \quad v = 0$

$u_x = 2x, \quad u_y = 2y \quad v_x = 0 \quad v_y = 0$

$u_x = v_y \Rightarrow 2x = 0 \Rightarrow x = 0$

$u_y = -v_x \Rightarrow 2y = 0 \Rightarrow y = 0$

$\Rightarrow u, v$ satisfies C-R equations only at $z=0 \rightarrow (*)$

$\Rightarrow u, v$ not satisfying C-R equations for all $z \neq 0$.

$\Rightarrow f(z) = u+iv$ not diff $\forall z \neq 0 \rightarrow (1)$

$\Rightarrow f(z) = u+iv$ not analytic $\forall z \neq 0 \rightarrow (2)$

Note that u_x, u_y, v_x, v_y are continuous at $z=0 \rightarrow (**)$

$\therefore$ from $(*)$, $(**)$

$f(z) = u+iv$ is differentiable at $z=0 \longrightarrow (3)$

$\therefore$ from (1); (3)

$f(z) = u+iv$ is differentiable only at $z=0$.

Thus there does not exist any neighbourhood of $z=0$ such that $f(z)$ is diff in the nbd of $z=0$

$\Rightarrow f(z)$ is not analytic at $z=0$

from (2), (4) $f(z)$ is nowhere analytic.

Hence $f(z) = |z|^2 = z\bar{z}$ is differentiable at $z=0$ but not analytic at $z=0$.

05/September/2019

Elementary Complex functions:

01) Polynomial:

$P_n(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n$ where $a_0, a_1, \dots, a_n$ are complex constants with $a_0 \neq 0$ is an n^{th} degree polynomial in the complex variable z.

$P_n'(z) = n a_0 z^{n-1} + (n-1) a_1 z^{n-2} + \dots + a_{n-1}$ $\forall z$.

As $P_n(z)$ is differentiable $\forall z$, it is an entire function.

02) Exponential function:

$$e^z = e^{x+iy} = e^x[\cos y + i \sin y].$$

$$u = e^x \cos y, \quad v = e^x \sin y$$

$$f(z) = u + iv.$$

$$f'(z) = u_x + i v_x = e^x \cos y + i e^x \sin y = e^x(\cos y + i \sin y)$$

$$f'(z) = e^z \ \forall z.$$

Note :-

Let $f(z) = u + iv$ be differentiable

$$f'(z) = u_x + iv_x$$
$$= v_y - iu_y$$
$$= u_x - iu_y$$
$$= v_y + iv_x.$$

Let $f(z) = e^z$, $f(z+T) = f(z)\ \forall z$

$e^{z+T} = e^z$ to be periodic

e^z is periodic with principal period $i2\pi$

$Re(e^z) = u$

Bounded

$|f(z)| \leq k\ \forall z.$

Unbounded

$|f(z)| \to \infty$ as $z \to z_0$

$$|e^z| = |e^x(\cos y + i\sin y)|$$
$$= |e^x|\,|\cos y + i\sin y|$$
$$= e^x.$$

05) Trigonometric functions :

(i) $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$ (ii) $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

$$\frac{d}{dz}\sin z = \frac{1}{2i}\left[ie^{iz} + ie^{-iz}\right] = \frac{e^{iz} + e^{-iz}}{2} = \cos z\ \forall z.$$

$$\frac{d}{dz}\cos z = -\sin z\ \forall z.$$

$\therefore$ $\sin z$ and $\cos z$ are entire functions

$\sin z$ and $\cos z$ are (unbounded) periodic functions with period $T = 2\pi$.

(iii) $\tan z = \dfrac{\sin z}{\cos z}$.

(iv) $\operatorname{cosec} z = \dfrac{1}{\sin z}$.

(v) $\sec z = \dfrac{1}{\cos z}$

(vi) $\cot z = \dfrac{1}{\tan z}$

→ All the formulae of real trigonometric functions are also valid for complex trigonometric functions.

Ex: $\sin^2 z + \cos^2 z = 1$

$\sin 2z = 2\sin z \cos z$.

All the formulae of real hyperbolic functions are also valid for complex hyperbolic functions

(i) $-\sin h^2 z + \cosh^2 z = 1$ etc.

(4) Hyperbolic functions:

(i) $\sinh z = \dfrac{e^z - e^{-z}}{2}$

(ii) $\cosh z = \dfrac{e^z + e^{-z}}{2}$

$\dfrac{d}{dz}\sinh z = \cosh z \ \forall z$

$\dfrac{d}{dz}\cosh z = \sinh z \ \forall z$.

Therefore, $\sinh z$ & $\cosh z$ are entire functions.

$\boxed{T = 2\pi i}$.

∴ $\sinh z$ & $\cosh z$ are unbounded functions

Relation between Trigonometric and hyperbolic functions.

1) $\sin(iz) = i\sinh z$

2) $\cos(iz) = \cosh z$

3) $\sinh(iz) = i\sin z$

4) $\cosh(iz) = \cos z$

$\sin z$

$= \sin(x+iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i\cos x \sinh y$.

$\operatorname{Re}(\sin z) = \sin x \cosh y$ $\quad$ $\operatorname{Im}(\sin z) = \cos x \sinh y$

$$|\sin z| = \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y}$$

$$= \sqrt{(\sin^2 x)\left[\frac{e^{y}+e^{-y}}{2}\right]^2 + \cos^2 x\left[\frac{e^{y}-e^{-y}}{2}\right]^2}$$

$\to \infty$ as $z \to \infty$.

$\therefore$ $\sin z$ is unbounded !!!

Trigonometric functions $\sin z$ & $\cos z$ are unbounded.

$$\cos z = \cos(x+iy)$$

$$= \cos x \cos iy - \sin x \sin iy$$

$$= \cos x \cosh y - i \sin x \sinh y$$

$Re(\cos z) = \cos x \cosh y$ $\qquad$ $Im(\cos z) = -\sin x \sinh y$.

$$|\cos z| = \sqrt{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}$$

$$= \sqrt{\cos^2 x\left[\frac{e^{y}+e^{-y}}{2}\right]^2 + \sin^2 x\left[\frac{e^{y}-e^{-y}}{2}\right]^2}$$

$\to \infty$ as $z \to \infty$

$\therefore$ $\cos z$ is unbounded !

$\rightarrow$ W

Let $f(z) = \sqrt{|xy|} = u(x,y) + iv(x,y)$

then $u(x,y) = \sqrt{|xy|}$, $v(x,y) = 0$

At the origin we have

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

i.e, C.R equations are satisfied at the origin.

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{z\to 0} \frac{f(z) - f(0)}{z - z_0} = \lim_{z\to 0} \frac{\sqrt{|xy|} - 0}{x + iy}$$

$$= \lim_{x\to 0} \frac{\sqrt{|mx^2|} - 0}{x(1+im)} \qquad \text{Along } y = mx$$

$$= \frac{\sqrt{|m|}}{1+im} \text{ which is not unique.}$$

$\therefore f'(0)$ does not exist.

Hence, $f(z)$ is not analytic at the origin

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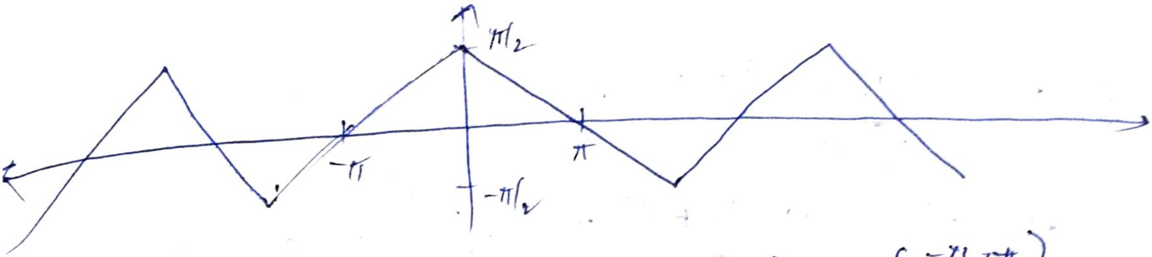
$f(x)$

- **PERIODIC**
  - $T = 2\pi$
    - $(-\pi, \pi)$
      - Even
        - $\frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi x}{L}\right)$
        - $L = \pi$.
        - $a_0 = \frac{2}{L}\int_0^L f(x)\,dx$.
        - $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$.
      - Odd
        - $\sum b_n \sin\left(\frac{n\pi x}{L}\right)$
        - $L = \pi$
        - $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$.
      - Neither Even nor odd
        - A
        - $\alpha = -\pi$
        - $L = \pi$
    - $(0, 2\pi)$
      - A
  - $T = 2L$
    - $(-L, L)$
      - A $a_0 + \sum a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$
      - $\boxed{L = \pi}$ $\boxed{\alpha = 0}$
      - $a_0 = \frac{1}{\cancel{\pi}}\int_\alpha^{\alpha+2L} f(x)\,dx$
      - $a_n = \frac{1}{L}\int_\alpha^{\alpha+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$.
      - $b_n = \frac{1}{L}\int_\alpha^{\alpha+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$.
        - Even B
        - Odd C
        - Neither even nor odd. A
    - $(\alpha, \alpha + 2L)$
      - A
- **NON-PERIODIC**

1) a) Find the Fourier Series of $f(x) = \begin{cases} x + \frac{\pi}{2} & ; -\pi < x < 0 \\ \frac{\pi}{2} - x & ; 0 < x < \pi \end{cases}$

Hence show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.



Since $f(x)$ is even periodic function on $(-\pi, \pi)$
$f(x)$ has Fourier Cosine Series

Fourier Series for $f(x) = \frac{a_0}{2} + \Sigma a_n \cos nx$.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2} - x\right)dx = \frac{2}{\pi}\left[\frac{\pi x}{2} - \frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}\left[\left(\frac{\pi^2}{2} - \frac{\pi^2}{2}\right) - (0-0)\right] = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)(\cos nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2} - x\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2}\cos nx - x\cos nx\right)dx$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{2}\frac{\sin nx}{n}\right)_0^{\pi} - \int_0^{\pi} x\cos nx\,dx\right.$$

$$= \frac{2}{\pi}\left[\cancel{\frac{\pi}{2n}\left(\sin n\pi - 0\right)}\right] -$$

$$= \frac{2}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\pi(0) + \frac{(-1)^n}{n^2}\right) - \left(0 + \frac{1}{n^2}\right)\right]$$

$$= \frac{2\left[(-1)^n + 1\right]}{\pi n^2} = \frac{2}{\pi n^2}\left[1 - (-1)^n\right]$$

$$f(x) = \frac{a_0}{2} + \sum a_n \cos nx.$$

$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{[1-(-1)^n]}{n^2} \cos nx.$$

$$f(0) = \pi/2 = \frac{2}{\pi} \sum \frac{1-(-1)^n}{n^2} \cos nx.$$

$$\Rightarrow \frac{\pi^2}{4} = \sum_{\cancel{n=odd}} \frac{1-(-1)^n}{n^2} \cos nx.$$ $n \Rightarrow even \to \Sigma = 0$

$n = e$

$$\frac{\pi^2}{4} = \sum_{n=odd} \frac{2}{n^2}$$

$$\sum_{n=odd} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

Q1) ~~Q(b)~~ $f(x) = x \cos x \to (0, 2\pi)$

~~$f(x) = x \cos x$~~

~~odd~~ ~~even~~

~~odd~~

~~f(x) is odd~~

$f(x) = x \cos x$.

Fourier series for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx.$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\,dx.$$

$$= \frac{1}{\pi} \int_0^{\pi} x \cos x\,dx$$

$$= \frac{1}{\pi} (x \sin x + \cos x)_0^{\pi} = \frac{1}{\pi} [[\pi(0) + (-1)] - [0 + 1]]$$

$$\boxed{a_0 = -\frac{2}{\pi}}$$

$$a_n = \frac{1}{\pi} \int_0^{\pi} f(x) \cos nx\,dx.$$

$$= \frac{1}{\pi} \int_0^{\pi} x \cos x \cos nx\,dx.$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} x\,[2\cos nx \cos x]\,dx$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} x\,[\cos(n+1)x + \cos(n-1)x]\,dx.$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x\cos(n+1)x\,dx + \int_0^{\pi} x\cos(n-1)x\,dx\right]$$

$$= \frac{1}{2\pi}\left[\left[x\frac{\sin(n+1)x}{n+1} + \frac{\cos(n+1)x}{(n+1)^2}\right]_0^{2\pi} + \left[x\frac{\sin(n-1)x}{n-1} + \frac{\cos(n-1)x}{(n-1)^2}\right]_0^{2\pi}\right] \quad (n \neq 1)$$

$$= \frac{1}{2\pi}\left[\left[\pi(0) + \frac{(-1)^{n+1}}{(n+1)^2}\right] - \left(0 - \frac{1}{(n+1)^2}\right)\right] + \left[\left(\pi(0) + \frac{(-1)^{n-1}}{(n-1)^2}\right) - \left(0 + \frac{1}{(n-1)^2}\right)\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{\cancel{(-1)^{n+1} + 1}}{(n+1)^2}\right)^{0} + \left(\frac{\cancel{(-1)^{n-1} - 1}}{(n-1)^2}\right)^{0}\right] = 0.$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos^2 x\,dx.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\,(1+\cos 2x)\,dx.$$

$$a_1 = \frac{1}{2\pi}\left[\int_0^{2\pi} x + \int_0^{2\pi} x\cos 2x\right]$$

(b) $f(x) = x^2$ in $(-1, 1)$

$f(-x) = x^2 = f(x)$

$f(x)$ is even periodic function on $(-1,1)$ with period

$T = 2l = 2 \Rightarrow l = 1$

F.S becomes F.C.S

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{1}\right)$$

$$a_0 = \frac{2}{1}\int_0^1 x^2\,dx = 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3}$$

$$a_n = \frac{2}{1}\int_0^1 x^2 \cos n\pi x\,dx$$

$$= 2\left[x^2\frac{\sin n\pi x}{n\pi} - 2x\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + 2\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left[0 + 2\frac{(-1)^n}{n^2\pi^2} + 0\right] - (0)$$

$$= \frac{4(-1)^n}{n^2\pi^2}$$

$\Rightarrow$ F.C.S is $f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2} \cos n\pi x.$

Let $x = 0 \Rightarrow f(x) = 0$

$$\Rightarrow 0 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2} \cdot (1)$$

$$\frac{-1}{3} = \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2}$$

$$\frac{\pi^2}{12} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\frac{\pi^2}{12} = -\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right]$$

$$\Rightarrow \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

Hence, Proved!

03)

a) Solve $z(z^2+xy)(px-qy)=x^4$

$[xz(z^2+xy)]p+[-yz(z^2+xy)]q=x^4$

$$\frac{dx}{xz(z^2+xy)}=\frac{dy}{-yz(z^2+xy)}=\frac{dz}{x^4}$$

Consider,
$$\frac{dx}{xz(z^2+xy)}=\frac{dy}{-yz(z^2+xy)}$$

$$\frac{dx}{x}=\frac{dy}{-y}$$

$\ln x=-\ln y+\ln c_1$

$\ln x+\ln y=\ln c_1$

$xy=c_1$

$\Rightarrow u=c_1$

Consider,
$$\frac{dx}{xz(z^2+xy)}=\frac{dz}{x^4}$$

$$\frac{dx}{z(z^2+c_1)}=\frac{dz}{x^3}.$$

$x^3dx=(z^3+c_1z)dz$

$$\frac{x^4}{4}-\frac{z^4}{4}-\frac{c_1z^2}{2}=c_1$$

$$\frac{x^4}{4}-\frac{z^2}{24}-\frac{z^2xy}{2}=c_1$$

$\Rightarrow v=c_2$

G.S is $F(u,v)=0$

$F\left(xy,\ \frac{x^4}{4}-\frac{z^4}{4}-xy\frac{z^2}{2}\right)=0$

b) Solve $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$

Sol. $\dfrac{dx}{x(y^2+z)} = \dfrac{dy}{-y(x^2+z)} = \dfrac{dz}{z(x^2-y^2)}$

Choose $P_1 = x,\ Q_1 = y,\ R_1 = -1$ as Lagrange's multipliers

then

$P_1 dx + Q_1 dy + R_1 dz$.

$= x^2y^2 + x^2z - x^2y^2 - zy^2 - x^2z + y^2z = 0.$

Integrate $P_1 dx + Q_1 dy + R_1 dz = 0$

$x dx + y dy - dz = 0$

$d\left(\frac{x^2}{2} + \frac{y^2}{2} - z\right) = 0$

$\frac{x^2}{2} + \frac{y^2}{2} - z = c_1$

$\Rightarrow u = c_1$

Choose $P_2 = \frac{1}{x},\ Q_2 = \frac{1}{y},\ R_1 = \frac{1}{z}$ as Lagrange's multipliers. then

$P_2 dx + Q_2 dy + R_2 dz = y^2 + z - x^2 - z + x^2 - y^2$

$= 0.$

Integrate $P_2 dx + Q_2 dy + R_2 dz = 0$

$\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz = 0$

$\ln x + \ln y + \ln z = 0$

$\ln xyz = \ln c_2$

$xyz = c_2$

$\Rightarrow v = c_2$

$\therefore$ G.S is $F(u, v) = 0$

$\Rightarrow F\left[\frac{x^2}{2} + \frac{y^2}{2} - z,\ xyz\right) = 0.$

23rd September 2019

Properties of Analytic functions.

01) If $f(z)$ and $g(z)$ are analytic, then $f(z)+g(z)$, $f(z)-g(z)$, $f(z)\times g(z)$, $\frac{f(z)}{g(z)}$ $(g(z)\neq 0)$ are all analytic

Also $f(g(z))$ is analytic.

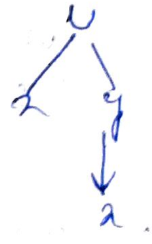
02) If $f(z)=u+iv$, is analytic then the family of curves $u=c_1$ and $v=c_2$ are ~~of~~ orthogonal to each other

Proof: Suppose $f(z)=u+iv$ is analytic

$\Rightarrow u, v$ satisfy C-R equations

i.e., $u_x=v_y$, $u_y=-v_x$.

Consider the families of curves $u(x,y)=c_1$ and $v(x,y)=c_2$



$$u(x,y)=c_1$$

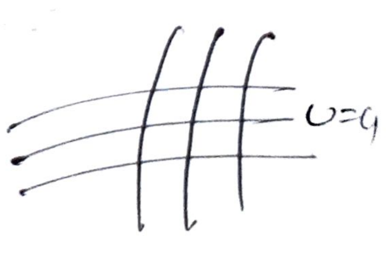
$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-u_x}{u_y}=m_1$$

$$v(x,y)=c_2$$

$$\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{v_x}{v_y}=m_2$$



$$m_1\cdot m_2=\left(\frac{-u_x}{u_y}\right)\left(\frac{-v_x}{v_y}\right)$$

$$=\frac{u_x}{u_y}\times\frac{-u_y}{u_x}=-1$$

$\therefore m_1\times m_2=-1$

$\Rightarrow u(x,y)=c_1$, $v(x,y)=c_2$ are orthogonal to each other

(3) If $f(z) = u+iv$ is analytic then u and v are harmonic functions.

Harmonic functions
↳ Solutions of Laplace functn.

Conditions
① . $u_{xx} + u_{yy} = 0$
②

Proof: Suppose $f(z) = u+iv$ is analytic
→ u, v satisfy C-R equation
i.e., $u_x = v_y$, $u_y = -v_x$.

we have
$$u_x = v_y \qquad u_y = -v_x$$
$$u_{xx} = v_{yx} \qquad \Rightarrow u_{yy} = -v_{xy}$$

$\therefore u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$ → iff u & v are first order perm continuous.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$\Rightarrow u(x,y)$ ~~and v(x,y) are~~ is harmonic function

similarly $v(x,y)$
$$v_x = -u_y \qquad v_y = u_x$$
$$v_{xx} = -u_{yx} \qquad v_{yy} = u_{xy}.$$
$$v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0$$
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$
$\Rightarrow v(x,y)$ is harmonic function.

$\therefore$ u and v are harmonic functions //

Cauchy Riemann equations in Polar Co-ordinates:-

Let $f(z) = u+iv$ be analytic.

$f(z) = u(x,y) + iv(x,y)$

Let $x = r\cos\theta$, $y = r\sin\theta$

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$\therefore f(re^{i\theta}) = u(r,\theta) + iv(r,\theta) \longrightarrow (*)$$

differentiate (*) wrt r

$f' \cdot e^{i\theta} = u_r + i v_r \longrightarrow (1)$

diff (*) wrt θ

$f' \cdot r e^{i\theta}(i) = u_\theta + i v_\theta$

$f' \cdot e^{i\theta} = \frac{1}{ir}(u_\theta + i v_\theta)$

$f' e^{i\theta} = \frac{1}{i} \times \frac{1}{r} u_\theta + \frac{1}{r} v_\theta \longrightarrow (2).$

$f' e^{i\theta} = u_r + i v_r$ [by (1)]

$f' e^{i\theta} = \frac{-i}{r} u_\theta + \frac{1}{r} v_\theta$ [by (2)]

$\therefore u_r + i v_r = \frac{1}{r} v_\theta - i \frac{1}{r} u_\theta$

$u_x = v_y$

$u_y = -v_x$

r, θ

$\Rightarrow \boxed{u_r = \frac{1}{r} v_\theta} \quad \boxed{v_r = \frac{-1}{r} u_\theta}$

HOW

(10) If $f(z) = u + iv$ is analytic then show that u and v are harmonic functions in polar coordinates

(9)

If $f(z)$ is analytic. $f(z) = u(r, \theta) + i v(r, \theta)$

$\therefore u, v$ satisfy CR equations.

$\nabla^2 = ?$

25/09/2019

$u_r = \frac{1}{r} v_\theta, \; v_r = \frac{1}{r} u_\theta$

$u_r = \frac{1}{r} v_\theta$

$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$

$\frac{\partial^2 u}{\partial \theta \partial r} = \frac{1}{r} \frac{\partial^2 v}{\partial \theta^2}$

$u_\theta = -r v_r$

$\frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}$

$\frac{\partial^2 u}{\partial r \partial \theta} = -r \frac{\partial^2 v}{\partial r^2}$

$$\frac{\partial^2 u}{\partial\theta\partial r} = \frac{\partial^2 u}{\partial r\partial\theta}$$

$$\Rightarrow \frac{1}{r}\frac{\partial^2 v}{\partial\theta^2} = -\frac{\partial v}{\partial r} - r\frac{\partial^2 v}{\partial r^2}$$

$$\Rightarrow r\frac{\partial^2 v}{\partial r^2} + \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial^2 v}{\partial\theta^2} = 0$$

$$\boxed{\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}}$$

$$\Rightarrow \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} = 0$$

$$\Rightarrow \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right] v = 0$$

$\nabla^2 v = 0 \Rightarrow v$ is harmonic function

Similarly, $\nabla^2 u = 0$

$\therefore$ u is also harmonic.

Result: If $f(z)$ is analytic in a open region D such that $f'(z) = 0$ $\forall z \in D$, then $f(z)$ is constant.

Proof: Let $f(z) = u + iv$ be analytic in an open region D. u & v satisfy CR equation for all z in D.

Assume, $f'(z) = 0 \quad \forall z \in D$

$\cancel{f(z) = c}$ $\Rightarrow u_x + i v_x = 0$

$\Rightarrow u_x = 0, \; v_x = 0$

$u_x = 0 \Rightarrow u$ is independent of x

$v_x = 0 \Rightarrow -u_y = 0 \Rightarrow u_y = 0 \Rightarrow u$ is independent of y

u is independent of x and y

$\Rightarrow u$ is constant

$u_x = 0 \Rightarrow v_y = 0 \Rightarrow v$ is independent of y

$v_x = 0 \Rightarrow v$ is independent of x.

$\therefore$ v is independent of x and y

$\Rightarrow v = \text{Constant} = c_2 \quad \forall z \in D$

$\therefore f(z) = u + iv = c_1 + ic_2 = \text{Constant} \quad \forall z \in D$

Q7) Show that an analytic function in an open region D with constant modulus is constant:

Let $f(z) = u + iv$ be analytic in an open region D.

$\Rightarrow u, v$ satisfy CR equations for all z in D.

$$u_x = v_y, \quad u_y = -v_x$$

Assume,

$$|f(z)| = c \text{ (Constant)}$$

$$\Rightarrow |u + iv| = c$$

$$\Rightarrow \sqrt{u^2 + v^2} = c$$

$$\Rightarrow u^2 + v^2 = c^2 \rightarrow (*)$$

diff (*) w.r.t x

$$2uu_x + 2vv_x = 0$$

$$uu_x + vv_x = 0$$

$$\Rightarrow uu_x - vu_y = 0$$

$$u_y = \frac{u}{v} u_x.$$

diff (*) w.r.t y

$$2uu_y + 2vv_y = 0$$

$$uu_y + vv_y = 0$$

$$\rightarrow uu_y + vu_x = 0$$

$$u\left[\frac{u}{v} u_x\right] + vu_x = 0$$

$$(u^2 + v^2) u_x = 0$$

$$u_x = 0$$

Similarly we can show

$$v_x = 0.$$

$f(z) = u + iv$, $f'(z) = u_x + iv_x \Rightarrow f'(z) = 0 + i(0)$

$f'(z) = 0 \;\forall z \in D \Rightarrow f(z) = \text{Constant} \;\forall z \in D$

(1) Show that if $f(z)$ and $\overline{f(z)}$ are analytic then $f(z)$ is constant.

(2) If $f(z)$ is analytic, then show that $\left(\frac{\partial}{\partial x}|f|\right)^2+\left(\frac{\partial}{\partial y}|f|\right)^2 = |f'(z)|^2$

$f(z) = u+iv$

$f'(z) = u_x + iv_x$.

$|f'(z)|^2 = u_x^2+v_x^2$

Proof: Let $f(z) = u+iv$

$|f(z)| = \sqrt{u^2+v^2}$

$|f(z)|^2 = u^2+v^2$

$$\frac{\partial}{\partial x}|f| = \frac{1}{2\sqrt{u^2+v^2}}[2uu_x+2vv_x] = \frac{uu_x+vv_x}{\sqrt{u^2+v^2}}$$

$$\frac{\partial}{\partial y}|f| = \frac{1}{2\sqrt{u^2+v^2}}[2uu_y+2vv_y] = \frac{uu_y+vv_y}{\sqrt{u^2+v^2}}$$

LHS:

$$\left(\frac{\partial}{\partial x}|f|\right)^2+\left(\frac{\partial}{\partial y}|f|\right)^2 = \frac{(uu_x+vv_x)^2}{u^2+v^2}+\frac{(uu_y+vv_y)^2}{u^2+v^2}$$

$$= \frac{u^2u_x^2+v^2v_x^2+2uvu_xv_x+u^2u_y^2+v^2v_y^2+2uvv_yu_y}{u^2+v^2}$$

$u_x = v_y$

$v_x = -u_y$.

$$= \frac{u^2[u_x^2+u_y^2]+v^2[v_x^2+v_y^2]}{u^2+v^2}$$

$$= \frac{u^2[v_x^2+u_y^2]+v^2[v_x^2+u_y^2]}{u^2+v^2}$$

$$= \frac{(u^2+v^2)(v_x^2+u_y^2)}{u^2+v^2} = v_x^2+u_y^2 = u_x^2+v_x^2$$

$= |f'(z)|^2$

$|f(z)|^2 = u_x^2+v_x^2$ //

Hence, Proved!

(3) Find and analytic function $f(z)$ whose imaginary part is $e^{-x}(y\sin y+x\cos y)$ by using Milne-Thompson method.

(01) $f(z)$ is analytic $f(z) = u + iv$ is analytic

$\overline{f(z)} = u - iv$ is analytic

u & v satisfy CR equations.

$$u_x = v_y, \quad u_y = -v_x.$$

$$\overline{f(z)} = u - iv = u + iV$$

$$u_x = V_y \qquad u_y = -V_x$$

Verify that the given function is harmonic and find its conjugate harmonic function. Express $u+iv$ as an analytic function.

① $u = x^2 - y^2 - y$.

Sol $u_x = 2x \Rightarrow u_{xx} = 2$

$u_y = -2y-1 \Rightarrow u_{yy} = -2$

$u_{xx} + u_{yy} = 2-2 = 0$

$\therefore u$ is harmonic.

$\Rightarrow$ There exists conjugate harmonic of u such that $f(z) = u+iv$ is analytic.

$\Rightarrow u, v$ satisfies C-R equations

i.e., $u_x = v_y$, $u_y = -v_x$

We have, $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$

$\Rightarrow \frac{\partial v}{\partial y} = 2x$

$v = 2xy + g(x)$

$\frac{\partial v}{\partial x} = 2y + g'(x)$

But $\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$

$2y + g'(x) = -(-2y-1)$

$2y + g'(x) = 2y + 1$

$g'(x) = 1 \Rightarrow g(x) = x + c$

$\therefore v = 2xy + x + c$

Put $c = 0$

$\therefore v = 2xy + c$ is a conjugate harmonic of u

Such that $f(z) = u+iv$ is analytic

$f(z) = (x^2 - y^2 - y) + i(2xy + x)$

$f(z) = [x^2 - y^2 + i2xy] + i(x+iy)$

$f(z) = z^2 + iz$.

(2) Find an analytic function $f(z)$ whose real part is x^2-y^2-y. By using Milne - Thompson method.

Sol Given that $f(z)$ is analytic such that $Re[f(z)]$

$= U = x^2-y^2-y$.

Let $Im[f(z)] = V$

Then $f(z) = U+iV$

$\Rightarrow f'(z) = U_x + iV_x$.

$f'(z) = U_x - iU_y$.

$f'(z) = [2x] - i[-2y-1]$

$f'(z) = 2x + i(2y+1)$

Put $x = z,\ y=0$

$f'(z) = 2z + i$

$f(z) = z^2 + iz + C$

Put $C=0$

$f(z) = z^2 + iz$

$f(z) = (x+iy)^2 + i(x+iy)$

$f(z) = (x^2-y^2-y) + i(2xy+x)$

$f(z) = U+iV$

$\therefore Im[f(z)] = V = 2xy + x$.

P.Q

(1) If $V = x^2-y^2-y$, then find its conjugate harmonic u such that $f(z) = U+iV$ is analytic.

(2) Find an analytic function $f(z)$ whose imaginary part is x^2-y^2-y.

(01) $v = x^2 - y^2 - y$ $\quad u = ?$ $\quad f(z) = u + iv$ is analytic

H.W

$v = x^2 - y^2 - y$

$v_x = 2x \Rightarrow v_{xx} = 2$

$v_y = -2y - 1 \Rightarrow v_{yy} = -2$

$v_{xx} + v_{yy} = 2 - 2 = 0$

$\therefore v$ is harmonic

$\Rightarrow$ There exists a conjugate harmonic of v such that $f(z) = u + iv$ is analytic

$\Rightarrow u$ & v satisfy C-R equations.

i.e., $u_x = v_y$, $u_y = -v_x$

We have $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$

$$\frac{\partial u}{\partial x} = -2y - 1$$

$$u = -2xy - x + g(y)$$

$$\frac{\partial u}{\partial y} = -2x + g'(y)$$

$$-2x = -2x + g'(y)$$

$$g'(y) = 0$$

$$g(x) = c$$

$\therefore u = -2xy - x + c$

Put $c = 0$

$\therefore u = -2xy - x$ is a conjugate harmonic of v.

Such that $f(z) = u + iv$ is analytic

$f(z) = (-2xy - x) + i(x^2 - y^2 - y)$ //

(Q) ~~$v = x^2 - y^2 - y$~~

Imag $[f(z)] = v = x^2 - y^2 - y$

Let Real ~~part~~ $[f(z)] = u$

$v_x = 2x$

$v_y = -2y - 1$

$u_x = v_y$

$v_y = -u_x$.

Then $f(z) = u + iv$

$\Rightarrow f'(z) = u_x + i v_x$

$f'(z) = \cancel{u_x} + i v_x$

$= (-2y - 1) + i(2x)$

$f'(z) = (-2y - 1) + i(2x)$

Put $x = z, y = 0$

$f'(z) = -1 + i2z$

$f'(z) = +2zi - 1$

$f(z) = +z^2 i - z + C$

put $C = 0$

$u_x = -2x - 1$

$u_y = 2y$

$v_x = -2y$

$v_y = -1 - 2x$.

$f(z) = +z^2 i - z$

$f(z) = +(x+iy)^2 i - (x+iy)$

$= -x - iy + [x^2 - y^2 + 2ixy]$

$= -x - iy + [x^2 - y^2 + 2ixy]$

$= -x - iy + (x^2 - y^2 + 2ixy) i$

$= \cancel{(-x^2 + y^2 - x)} + i(-y - 2xy)$

$f(z) = u + iv \quad (x^2 - y^2 - x)$.

$\therefore$ Real $[f(z)] = -x^2 + y^2 - x$.

$= -x - iy + x^2 i - y^2 i - 2xy$

$f(z) = (-x - 2xy) + i(x^2 - y^2 - y)$ is the analytic function

(i) Given that $v = e^{-x}(y\sin y + x\cos y)$. Show that v is harmonic and find a conjugate harmonic of v.

$v_x = -e^{-x}[y\sin y + x\cos y] + e^{-x}[\cos y]$

$v_{xx} = e^{-x}[y\sin y + x\cos y] - e^{-x}[\cos y) - e^{-x}\cos y$

$v_y = e^{-x}[\sin y + y\cos y - x\sin y]$

$v_{yy} = e^{-x}[\cos y + \cos y - y\sin y - x\cos y]$

$v_{xx} + v_{yy} = 0$

$\Rightarrow$ v is harmonic function.

There exists (∃) conjugate harmonic u, of v such that $f(z) = u + iv$ is analytic.

$\Rightarrow$ u, v satisfy C-R equations.

i.e., $u_x = v_y$, $u_y = -v_x$

We have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\rightarrow \frac{\partial u}{\partial x} = e^{-x}\sin y + e^{-x}y\cos y - xe^{-x}\sin y$.

$u = -e^{-x}\cancel{\sin y} - e^{-x}y\cos y + e^{-x}(x+1)\sin y + g(y)$

$\frac{\partial u}{\partial y} = -e^{-x}\cos y + e^{-x}y\sin y - e^{-x}\cos y + e^{-x}(x+1)\cos y + g'(y)$

We have

$-\frac{\partial v}{\partial x} = -[e^{-x}\cos y - e^{-x}(y\sin y + x\cos y)]$

W.K.T $u_y = -v_x$

$-e^{-x}\cos y + e^{-x}y\sin y - e^{-x}\cos y + e^{-x}x\cos y + e^{-x}\cos y + g'(y)$

$= -e^{-x}\cos y + e^{-x}y\sin y + e^{-x}x\cos y$

$\Rightarrow g'(y) = 0$

$\Rightarrow g(y) = 0$.

$\therefore v = e^{-x} y\cos y + e^{-x} x \sin y$.

$v = e^{-x}[x\sin y - y\cos y]$ (by taking $c=0$)

H.W

(Q) Find an analytic function $f(z)$ whose imaginary part is $e^{-x}(y\sin y + x\cos y)$ by using Milne-thompson method.

$V = e^{-x}(y\sin y + x\cos y)$

$\text{Imag}[f(z)] = v = e^{-x}(y\sin y + x\cos y)$

$f(z) = u + iv$ $\qquad\qquad$ $u_x = v_y$

$f'(z) = u_x + iv_x$ $\qquad\qquad$ $u_y = -v_x$.

$V_x = e^{-x}(\cos y) - (y\sin y + x\cos y) e^{-x}$.

$V_y = e^{-x}(y\cos y + \sin y - x\sin y)$

$f'(z) = u_x + iv_x = v_y + iv_x$.

$= e^{-x}(y\cos y + \sin y - x\sin y) + i e^{-x}[\cos y - x\cos y - y\sin y]$

put $x = z, y = 0$

$f'(z) = e^{-z}[0 + 0 - 0] + i e^{-z}[(1) - z(1).]$

$f'(z) = [\cancel{e^{-z} - z}]\, i \quad e^{-z} i\, [1 - z]$

$f(z) = \cancel{e^{-z} i - \frac{z^2}{2} i + c}$ $\quad$ ~~$e^{-z} i\,(-1) - (1-z) e^{-z} i$~~

put $c = 0$ $\quad f'(z) = i e^{-z} - i z e^{-z} = i e^{-z}(1-z)$

$\cancel{f(z) = -e^{-(x+iy)} i - \frac{(x+iy)^2}{2}}$ ~~$f(z) = -i e^{-z} = i$~~

$f(z) = i \int e^{-z}(1-z)$

$= i\, (1-z) \frac{e^{-z}}{-1} - \int (-1)(-e^{-z})$

$= i[(1-z) - e^{-z} + e^{-z}] + c = +iz[e^{-z}] + c$

$= +i(x+iy)\, e^{-(x+iy)} + c$ put $c=0$.

$= +i(x+iy)\, e^{-x}[\cos y - i\sin y]$

$= +i\, e^{-x}[x\cos y - ix\sin y + iy\cos y + y\sin y]$

$= +i\, e^{-x}[x\cos y + y\sin y + i(y\cos y - x\sin y)]$

$= +ie^{-x}[x\cos y + y\sin y] + \cancel{i}e^{-x}[y\cos y - x\sin y]$

$= -e^{-x}[y\cos y - x\sin y] + i[e^{-x}(x\cos y + y\sin y)]$

$v = -e^{-x}[y\cos y - x\sin y]$

$= e^{-x}[x\sin y - y\cos y]$

30th ~~October~~ September 2019

(Q1) If $u = e^{-x}(x\sin y - y\cos y)$, find v such that $f(z) = u+iv$ is analytic by Milne Thompson method.

$u = e^{-x}(x\sin y - y\cos y)$

Let v be the conjugate harmonic of u.

show that $f(z) = u+iv$ is analytic

$\Rightarrow u, v$ satisfy C-R equations.

i.e, $u_x = v_y$, $u_y = -v_x$.

$f(z) = u+iv$

$f'(z) = u_x + iv_x$.

$f'(z) = u_x - iu_y$.

$f'(z) = [-e^{-x}(x\sin y - y\cos y) + e^{-x}(\sin y)] -$
$i[e^{-x}(x\cos y - \cos y + y\sin y)]$

Apply Milne Thompson method.

put $x = z$, $y = 0$

$f'(z) = -e^{-z}[0+0] - i[e^{-z}(z-1)]$

$f'(z) = -ie^{-z}(z-1)$

$f(z) = i\int (z-1)e^{-z}\,dz$

$= i[(z-1)[-e^{-z}] - 1[e^{-z}]] + c$

$= ie^{-z}(z-1+1) = ize^{-z} + c$

$= i(x+iy)e^{-(x+iy)}$ put $c=0$

$= i e^{-x}(x+iy)[\cos y - i\sin y]$.

$= +i e^{-x}[x\cos y - xi\sin y + iy\cos y + y\sin y]$

$= +i e^{-x}[x\cos y + y\sin y + (y\cos y - x\sin y)i]$

$= +i e^{-x}[x\cos y + y\sin y]$

$= e^{-x}(y\cos y - x\sin y) + i e^{-x}[x\cos y + y\sin y]$

$f(z) = e^{-x}(x\sin y - y\cos y) + i e^{-x}[x\cos y + y\sin y]$.

$u = e^{-x}(x\sin y - y\cos y)$

$v = e^{-x}[x\cos y + y\sin y]$

(b) Determine the constant k such that $u = e^{kx}\cos 5y$ is harmonic and hence find its conjugate harmonic.

$u = e^{kx}\cos 5y$

$u_x = ke^{kx}\cos 5y$

$u_{xx} = k^2 e^{kx}\cos 5y$

$u_y = -5e^{kx}\sin 5y$

$u_{yy} = -25e^{kx}\cos 5y$.

$u \Rightarrow$ harmonic

$\Rightarrow u_{xx} + u_{yy} = 0$

$k^2 e^{kx}\cos 5y - 25e^{kx}\cos 5y = 0$.

$e^{kx}\cos 5y[k^2 - 25] = 0$

$k^2 - 25 = 0$

$\Rightarrow k = \pm 5$.

Let $k = 5$

$\therefore u = e^{5x}\cos 5y$.

Let v be the conjugate harmonic of u.

Show that $f(z) = u + iv$ is analytic

$\Rightarrow u, v$ satisfy C.R equations

i.e, $u_x = v_y$, $u_y = -v_x$

$f(z) = u + iv$

$f'(z) = u_x + iv_x$.

$f'(z) = u_x - iu_y$

$f'(z) = 5e^{5x}\cos 5y + 5e^{5x}\sin 5y$

Apply Milne Thompson method

Put $x = z,\ y = 0$

$f'(z) = 5e^{5z}$

$f(z) = \frac{5e^{5z}}{5} + c$

$f(z) = e^{5z} + c$

put $c = 0$.

$f(z) = e^{5z}$

$f(z) = e^{5(x+iy)}$

$u + iv = e^{5x} \cdot e^{5iy}$

$= e^{5x} \cdot [\cos 5y + i \sin 5y]$

$u + iv = e^{5x}\cos 5y + i e^{5x}\sin 5y$

$\therefore\ u = e^{5x}\cos 5y$

$v = e^{5x}\sin 5y$.

(03) $u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

Let v be conjugate harmonic of u. such that $f(z) = u + iv$ is analytic

$\Rightarrow u, v$ satisfy C-R equations

ie $u_x = v_y,\ u_y = -v_x$

$f(z) = u + iv$

$f'(z) = u_x + i v_x$

$f'(z) = u_x - i u_y$.

$f'(z) = u_x - i u_y$.

$$f'(z) = \left[\frac{(\cosh 2y - \cos 2x)2\cos 2x - \sin 2x\,[0 + 2\sin 2x]}{(\cosh 2y - \cos 2x)^2}\right]$$

$$-i\left[\frac{(\cosh 2y - \cos 2x)(0) - \sin 2x\,(2\sinh 2y - 0)}{(\cosh 2y - \cos 2x)^2}\right].$$

~~Put~~ Apply Milne Thompson method put $x = z, y = 0$

$$\therefore f'(z) = \left[\frac{(1-\cos 2z)(2\cos 2z) - 2\sin^2 2z}{(1-\cos 2z)^2}\right] - i\left[0\right].$$

$$f'(z) = \left[\frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1-\cos 2z)^2}\right]$$

$$= \frac{2\cos 2z - 2}{(1-\cos 2z)^2}$$

$$= \frac{-2(1-\cos 2z)}{(1-\cos 2z)^2} = \frac{-2}{(1-\cos 2z)}$$

$$f'(z) = \frac{-2}{1-\cos 2z} = \frac{2}{\cos 2z - 1} = \frac{2}{\cos 2(\cancel{x+iy}) - 1} = \frac{2}{(1 - 2\sin^2 z - 1)}$$

$$= \frac{-1}{\sin^2 z} = -\operatorname{cosec}^2 z.$$

$$f'(z) = -\operatorname{cosec}^2 z.$$

$$f(z) = \cot z + c$$

Put $c = 0$

$$\therefore f(z) = \cot z = \cot(x+iy) = \frac{\cos(x+iy)}{\sin(x+iy)}$$

$$= \frac{\cos z}{\sin z}.$$

$$= \frac{2\cos(x+iy)\sin(x-iy)}{2\sin(x+iy)\sin(x-iy)}$$

$$= \frac{\sin(2x) - \sin(i2y)}{\cos(i2y) - \cos(2x)}$$

$$= \frac{\sin 2x - i \sinh 2y}{\cosh 2y - \cos 2x}$$

$$= \frac{\sin 2x}{\cosh 2y - \cos 2x} + i \frac{\sinh 2y}{\cos 2x - \cosh 2y}$$

$f(z) = u + iv$

$\therefore v = \frac{\sinh 2y}{\cos 2x - \cosh 2y}$.

H.W Notes! Separate real & imaginary parts of $\tan z$, $\sec z$ and $\operatorname{cosec} z$.

(04) If $u - v = \frac{\cos x + \sin y - e^{-y}}{2(\cos x - \cosh y)}$, $f(\pi/2) = 0$, determine the analytic function $f(z) = u - iv$.

$f(z) = u + iv$.

such that $u - v = \frac{\cos x + \sin y - e^{-y}}{2(\cos x - \cosh y)}$

$f(z) = u + iv$

$if(z) = iu - v$

$f(z) + if(z) = (u - v) + i(u + v)$

$(1+i)(f(z)) = (u - v) + i(u + v) = U + iV = F(z)$

where $U = u - v$

$V = u + v$:

Since $f(z)$ is analytic

$F(z) = (1+i)f(z) = U + iV$ also analytic.

$\Rightarrow U, V$ satisfy CR equations.

$F(z) = u + iv$.

$\Rightarrow F'(z) = u_x + i v_x$.

$F'(z) = u_x - i u_y$. [by CR equations.

$$F'(z) = \frac{1}{2}\left[\left[\frac{(\cos x - \cosh y)(-\sin x) - (\cos x + \sin y - e^{-y})(-\sin x)}{(\cos x - \cosh y)^2}\right]\right.$$

$$\left. - i\left[\frac{(\cos x - \cosh y)(\cos y + e^{-y}) - (\cos x + \sin y - e^{-y})(-\sinh y)}{(\cos x - \cosh y)^2}\right]\right]$$

Applying Milne thompson method.

put $x = z,\ y = 0$.

$$F'(z) = \frac{1}{2}\left[\left[\frac{(\cos z - 1)(-\sin z) - (\cos z + 0 - 1)(-\sin z)}{(\cos z - 1)^2}\right]\right.$$

$$\left. - i\left[\frac{(\cos z - 1)(1 + 1) - (\cos z + 0 - 1)(0)}{(\cos z - 1)^2}\right]\right]$$

$$F'(z) = \frac{1}{2}\left[\cancel{(-\sin z)[\cos z - 1} \quad 0 \quad - \frac{i2(\cos z - 1)}{(\cos z - 1)^2}\right]$$

$$= -\frac{1}{2} \times i \times 2 \frac{\cos z - 1}{(\cos z - 1)^2} = \frac{-i}{\cos z - 1}$$

$$\frac{d}{dz}[(1+i) f(z)] = \frac{+i}{1 - \cos z}.$$

$$(1+i) f'(z) = \frac{i}{1 - \cos z} = \frac{i}{1 - (1 - 2\sin^2 \frac{z}{2})} = \frac{i}{2\sin^2 \frac{z}{2}}.$$

$$(1+i) f'(z) = \frac{i}{2} \operatorname{cosec}^2 \frac{z}{2}$$

Integrate on both sides,

$$\Rightarrow (1+i) f(z) = i\left(-\cot \frac{z}{2}\right) + c$$

$$f(z) = \frac{-i}{1+i} \cot \frac{z}{2} + \frac{c}{1+i}$$

$$= \frac{-i}{1+i} \cot \frac{z}{2} + d \qquad \text{where } d = \frac{c}{1+i}$$

$$f\left(\frac{\pi}{2}\right) = 0.$$

$$\Rightarrow \quad 0 = \frac{-i}{1+i} \cot\left(\frac{\pi}{4}\right) + d$$

$$d = \frac{i}{1+i} (1)$$

$$d = \frac{i(1-i)}{1-i^2} = \frac{i+1}{2}$$

$$f(z) = \frac{-i}{1+i} \cot \frac{z}{2} + \frac{i+1}{2}$$

$$= -\left(\frac{i+1}{2}\right) \cot \frac{z}{2} + \left(\frac{i+1}{2}\right)$$

$$f(z) = \left(\frac{i+1}{2}\right)\left[1 - \cot \frac{z}{2}\right]$$

03/Oct/2019

MBC Mon-3rd, Mon-3, Wed-6

If $f(z)$ is analytic, show that $\left[\frac{\partial}{\partial x} |f|\right]^2 + \left[\frac{\partial}{\partial y} |f|\right]^2 = |f'(z)|^2$.

Let $f(z) = u + iv$ be analytic $\quad \left[\frac{\partial}{\partial x} |f|\right]^2 + \left[\frac{\partial}{\partial y} |f|\right]^2 = [|f'(z)|]^2$

$\Rightarrow u$ & v satisfy CR equations

i.e., $u_x = v_y$, $u_y = -v_x$.

Consider $f(z) = u + iv$

$$|f(z)| = \sqrt{u^2+v^2}$$

$$|f|^2 = u^2 + v^2$$

$$\frac{\partial}{\partial x} |f|^2 = 2uu_x + 2vv_x = 2(uu_x + vv_x)$$

$$\frac{\partial}{\partial y} |f|^2 = 2uu_y + 2vv_y = 2(uu_y + vv_y)$$

$\frac{\partial}{\partial x}|f|^2 + \frac{\partial}{\partial y}|f|^2 = 2(uu_x + vv_y + vv_x + vv_y)$ $\qquad u_x = v_y$

$= 2(u_x(u+v) + v_y(u-v))$ $\qquad u_y = -v_x$

$f(z) = \sqrt{u^2+v^2}$

$f'(z) = \frac{1}{2\sqrt{u^2+v^2}}$ $\quad 2uu_x + 2vv_x = \frac{uu_x + vv_x}{\sqrt{u^2+v^2}}$

$u_x(u+v) +$

Q2. Show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(|f(z)|)^n = n^2 |f(z)|^{n-2} |f'(z)|^2$

Let $f(z) = u + iv$ be analytic

$\Rightarrow u, v$ satisfy CR equations

i.e., $u_x = v_y$, $u_y = -v_x$.

$f(z) = u + iv \Rightarrow |f(z)| = \sqrt{u^2+v^2} \Rightarrow |f(z)|^n = (u^2+v^2)^{n/2}$

$\frac{\partial}{\partial x} \cancel{|f(z)| = \frac{1}{2\sqrt{u^2+v^2}}}$ $\cancel{2uu_x} + 2v$

$\frac{\partial}{\partial x}|f(z)|^n = \frac{n}{2}(u^2+v^2)^{\frac{n}{2}-1}[2uu_x + 2vv_x]$

$= n(u^2+v^2)^{n/2-1}[uu_x + vv_x]$

$\frac{\partial^2}{\partial x^2}|f(z)|^n = n\left(\frac{n}{2}-1\right)(u^2+v^2)^{\frac{n}{2}-2}[2uu_x + 2vv_x][uu_x + vv_x]$

$+ n(u^2+v^2)^{\frac{n}{2}-1}[(u_x)^2 + uu_{xx} + (v_x)^2 + vv_{xx}]$.

$= n(n-2)(u^2+v^2)^{\frac{n}{2}-2}[uu_x + vv_x]^2$

$\frac{\partial^2}{\partial x^2}|f(z)|^n = n(n-2)(u^2+v^2)^{\frac{n}{2}-2}[uu_x + vv_x]^2$

$+ n(u^2+v^2)^{\frac{n}{2}-1}[(u_x)^2 + (v_x)^2 + uu_{xx} + vv_{xx}]$ — (1)

vv_y

$\frac{\partial^2}{\partial y^2}|f(z)|^n = n(n-2)(u^2+v^2)^{\frac{n}{2}-2}[uu_y + vv_y]^2$

$+ n(u^2+v^2)^{\frac{n}{2}-1}[(u_y)^2 + (v_y)^2 + uu_{yy} + vv_{yy}]$ — (2)

(1) + (2)

$$\Rightarrow n(n-2)(U^2+V^2)^{\frac{n}{2}-2}\left[(UU_x+VV_x)^2+(UU_y+VU_y)^2\right]$$

$$+n(U^2+V^2)^{\frac{n}{2}-1}\left[(U_x)^2+(V_x)^2+UU_{xx}+VV_{xx}+(U_y)^2+(V_y)^2+UU_{yy}+VV_{yy}\right]$$

$$\Rightarrow n(n-2)(U^2+V^2)^{\frac{n}{2}-2}\left[U^2U_x^2+V^2V_x^2+\cancel{2UVU_xV_x}+U^2U_y^2+V^2V_y^2+\cancel{2UVU_yV_y}\right]$$

$$+n(U^2+V^2)^{\frac{n}{2}-1}\left[U_x^2+V_x^2+UU_{xx}+VV_{xx}+U_y^2+V_y^2+UU_{yy}+VV_{yy}\right]$$

$$U_x=V_y,\quad U_y=-V_x.$$

$$\Rightarrow n(n-2)(U^2+V^2)^{\frac{n}{2}-2}\left[U_x^2(U^2+V^2)+U_y^2(U^2+V^2)\right]$$

$$+n(U^2+V^2)^{\frac{n}{2}-1}\left[2U_x^2+2U_y^2+UU_{xx}+UU_{yy}+VV_{xx}+VV_{yy}\right]$$

$$\Rightarrow n(n-2)(U^2+V^2)^{\frac{n}{2}-2}\left[(U_x^2+U_y)^2(U^2+V^2)\right]$$

$$+n(U^2+V^2)^{\frac{n}{2}-1}\left[2U_x^2+2U_y^2+U(\cancel{U_{xx}+U_{yy}})^{0}+V(\cancel{V_{xx}+V_{yy}})^{0}\right]$$

$$\Rightarrow n(n-2)(U^2+V^2)^{\frac{n}{2}-2}(U^2+V^2)*(U_x^2+U_y^2)$$

$$+n(U^2+V^2)^{\frac{n}{2}-1}\times 2\times(U_x^2+U_y)^2$$

$$\Rightarrow n(n-2)(U^2+V^2)^{\frac{n}{2}-1}(U_x^2+U_y)^2+2n(U^2+V^2)^{\frac{n}{2}-1}(U_x^2+U_y)^2$$

$$\Rightarrow n(U^2+V^2)^{\frac{n}{2}-1}(U_x^2+U_y^2)(n-2+2)$$

$$\Rightarrow n^2(U^2+V^2)^{\frac{n}{2}-1}(U_x^2+U_y)^2 \quad \text{—— (3)}$$

RHS:

$$n^2|f(z)|^{n-2}|f'(z)|^2$$

$$|f(z)|=\sqrt{U^2+V^2}\qquad |f(z)|^{n-2}=(U^2+V^2)^{\frac{n-2}{2}}$$

$|f'(z)|$

$f'(z) = U_x + iV_x.$

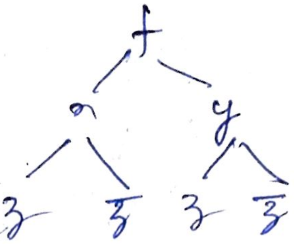
$|f'(z)| = \sqrt{U_x^2 + V_x^2} \Rightarrow |f'(z)|^2 = U_x^2 + V_x^2.$

$\Rightarrow n^2 (U^2+V^2)^{\frac{n-2}{2}} (U_x^2 + V_x^2)^2$

$= n^2 (U^2+V^2)^{\frac{n}{2}-1} (U_x^2 + V_y^2)^2$ (4) $\qquad \boxed{U_x = V_y}.$

(3) = (4) $\Rightarrow$ LHS = RHS $\Rightarrow$ Hence, Verified!

If $f(z)$ is analytic, show that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$



Let $f(z) = U(x,y) + iV(x,y)$ such $f(z)$ is analytic,

Let $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$

$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial z}\right) + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y},$

$= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)f.$

$\therefore \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$

$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y}$

$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f$

$\therefore \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

Consider $4\frac{\partial^2 f}{\partial z \partial \bar{z}} = 4\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \bar{z}}\right)$

$= 4\left[\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\right]$

1st order continuous

$= \frac{\partial^2 f}{\partial x^2} + i\frac{\partial^2 f}{\partial x \partial y} - i\frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 f}{\partial y^2} \qquad$ ∀

$4\frac{\partial}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \qquad = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ ~~= RHS~~ //

Q) Show that an analytic function of constant real part is constant.
H.W

Q) Show that an analytic function of constant imaginary part is constant.
H.W

$\overline{f(z)} = f(\bar{z}) \Rightarrow ?$

(1) $Re[z] = k$

$u = k$

$u_x = 0$

$u_y = 0$

$u_x = v_y$

$u_y = -v_x$

$f(z) = u + iv$

$f'(z) = u + iv$

$f'(z) = u_x + iv_x$.

$f'(z) = iv_x$

$f'(z) = 0$.

$f(z) = c$ constant //.

Imaginary $[z] = k$

$v = k$

$v_x = 0$

$v_y = 0$

$u_x = v_y$

$u_y = -v_x$.

$f(z) = u + iv$

$f'(z) = u_x + iv_x$

$f'(z) = u_x + 0$

$f'(z) = 0$

$f(z) = c$ constant.

Q) If $f(z)$ is analytic, show that

$$\left[\frac{\partial}{\partial x}|f|\right]^2 + \left[\frac{\partial}{\partial y}|f|\right]^2 = |f'|^2$$

$|f(z)| = |u(x,y) + iv(x,y)| = \sqrt{u^2+v^2}$

partially differentiating wrt x and y

$$\frac{\partial}{\partial x}|f| = \frac{1}{2\sqrt{u^2+v^2}}\left[2uu_x + 2vv_x\right] = \frac{uu_x + vv_x}{|f|}$$

Similarly,

$$\frac{\partial}{\partial y}|f| = \frac{uu_y + vv_y}{|f|}$$

Squaring and adding, we get

$$\left[\frac{\partial}{\partial x}|f|\right]^2 + \left[\frac{\partial}{\partial y}|f|\right]^2 = \frac{(uu_x + vv_x)^2 + (uu_y + vv_y)^2}{|f|^2}$$

$$= \frac{[u^2u_x^2 + v^2v_x^2 + 2uu_xvv_x + u^2u_y^2 + v^2v_y^2 + 2uu_yvv_y]}{|f|^2}$$

Since f is analytic, C-R conditions are satisfied.

So $u_x = v_y$, $u_y = -v_x$.

Then $2uv\,u_xv_x = -2uv\,u_yv_y$

$$= \frac{(u^2+v^2)(u_x^2+v_x^2)}{|f|^2} = u_x^2 + v_x^2 = |f'|^2$$

Since $f' = u_x + iv_x$ and $|f'| = \sqrt{u_x^2 + v_x^2}$