Finding r(θ) for some of well know potentials

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1 Finding r(θ)

First of all for simplification we use a form of derivation that is very useful. With the definition of central force we can write

$$F(r) = m\ddot{r} - \frac{L^2}{mr^3} \tag{1}$$

Then we have

$$\ddot{r} = \frac{1}{m}F(r) + \frac{L^2}{mr^3} \tag{2}$$

By changing variable $u = \frac{1}{r}$ we can write

$$\dot{r} = \frac{d}{dt}(\frac{1}{u}) = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -(r^2)\dot{\theta}\frac{du}{d\theta} = -(r^2\dot{\theta})\frac{du}{d\theta} \tag{3}$$

and because $mr^2\dot{\theta} = L$ we have

$$\dot{r} = -\frac{L}{m}\frac{du}{d\theta} \tag{4}$$

So

$$\ddot{r} = -\frac{L}{m}\frac{d}{dt}(\frac{du}{d\theta}) = -\frac{L}{m}\frac{d\theta}{dt}\frac{d}{d\theta}(\frac{du}{d\theta}) = -\frac{L}{m}\ddot{\theta}\frac{d^2u}{d\theta^2} = -\frac{L}{m}(\frac{L}{mr^2})\frac{d^2u}{d\theta^2} = -\frac{L^2}{m^2}u^2\frac{d^2u}{d\theta^2} \tag{5}$$

By inserting this to Equation (2) we get

$$-(\frac{L^2}{m^2})u^2\frac{d^2u}{d\theta^2} = \frac{1}{m}F(r) + \frac{L^2}{m^2r^3} \tag{6}$$

So

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2}\frac{1}{u^2}F(\frac{1}{u}) \tag{7}$$

Wea are going to use this Equation to find paths in next section.

1.1 $V(r) = \frac{-k}{r}$

By writing $F(r) = -\frac{dV}{dr} = -\frac{k}{r^2} = -ku^2$ we have

$$\frac{d^2u}{d\theta} + u = -\frac{m}{L^2}\frac{1}{u^2}(-ku^2) = \frac{mk}{L^2} \tag{8}$$

By changing variable $w = u - \frac{mk}{L^2}$ we can write

$$\frac{d^2w}{d\theta^2} + w = 0 \tag{9}$$

This equation has well known answer as

$$w = C\cos(\theta - \theta_0) \tag{10}$$

So

$$w = u - \frac{mk}{L^2} \longrightarrow C\cos(\theta - \theta_0) = u - \frac{mk}{L^2} \longrightarrow u = \frac{mk}{L^2} + C\cos(\theta - \theta_0) \tag{11}$$

Then

$$r(\theta) = \frac{1}{\frac{mk}{L^2} + C\cos(\theta - \theta_0)} \tag{12}$$

By setting $k = GMm$

$$u = \frac{GMm^2}{L^2} + C\cos(\theta - \theta_0) \longrightarrow r(\theta) = \frac{1}{\frac{GMm^2}{L^2} + C\cos(\theta - \theta_0)} \tag{13}$$

And finally we can write

$$r(\theta) = \frac{\frac{L^2}{GMm^2}}{1 + e\cos(\theta - \theta_0)} \tag{14}$$

where

$$e = \frac{CL^2}{km} \tag{15}$$

1.2 $V(r) = \frac{-k}{2r^2}$

By writing $F(r) = -\frac{dV}{dr} = -\frac{k}{r^3}$ we have

$$\frac{d^2u}{d\theta} + u = -\frac{m}{L^2}\frac{1}{u^2}(-ku^3) \longrightarrow \frac{d^2u}{d\theta^2} + (1 - \frac{mk}{L^2})u = 0 \tag{16}$$

Then we have

$$r(\theta) = \frac{1}{C\cos(\sqrt{1 - \frac{mk}{L^2}}(\theta - \theta_0))} \tag{17}$$

and by setting $k = GMm$

$$r(\theta) = \frac{1}{C\cos(\sqrt{1 - \frac{GMm^2}{L^2}}(\theta - \theta_0))} \tag{18}$$

1.3 $V(r) = \frac{-k}{r} - \frac{\alpha}{r^2}$ (Bound State)

The effective potential is

$$V_{eff}(r) = V(r) + \frac{L^2}{2mr^2} \longrightarrow V_{eff}(r) = -\frac{k}{r} - \frac{\alpha}{r^2} + \frac{L^2}{2mr^2} = -\frac{k}{r} + \frac{L^2 - 2m\alpha}{2mr^2} \tag{19}$$

we can use this equation to find the path

$$\theta(r) = \theta_0 + \frac{L}{\sqrt{2m}} \int \frac{dr}{r^2\sqrt{E - V_{eff}(r)}} \tag{20}$$

So by inserting equation (19) to equation (20)

$$\theta = \theta_0 + \frac{L}{\sqrt{2m}} \int \frac{du}{\sqrt{-(\frac{\beta^2 L^2}{2m})u^2 + ku + E}} = \theta_0 - \frac{1}{\beta}\cos^{-1}\left(\frac{-(\frac{\beta^2 L^2}{m})u + k}{\sqrt{k^2 + (\frac{2\beta^2 L^2 E}{m})}}\right) \tag{21}$$

Where $\beta^2 = 1 - \frac{2m\alpha}{L^2}$, So

$$r(\theta) = \frac{r_0}{1 - e\cos\beta(\theta - \theta_0)} \tag{22}$$

Where

$$r_0 = \beta^2 \frac{L^2}{mk}, \qquad e = \sqrt{1 + \frac{2\beta^2 L^2 E}{mk^2}} \tag{23}$$

2 Plotting orbits for $V(r) = \frac{-k}{r} - \frac{\alpha}{r^2}$ at bound state

We are going to see that variables e and β determine the shape of our orbit or if the orbit is closed or open.

If $e = 0$ the orbit is circular.
If $e \neq 0$ if β is rational then the orbit is closed otherwise if β is irrational then the orbit is open.

The orbits are in the next page .