Particle Physics Project - Pionic Atoms

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1 Overview

A pionic atom is system contain a quark and an anti-quark i.e. is a meson. We have three pions in definition π^0, π^+ and π^-. Pions are the lightests mesons. They are unstable with the life time and rest mass of

	π^0	π^+	π^-
Life Time	$8.4 \times 10^{-17} s$	$2.6033 \times 10^{-8} s$	$2.6033 \times 10^{-8} s$
Rest Mass	$134.9766 \mathrm{MeV}/c^2$	$139.57018 \mathrm{MeV}/c^2$	$139.57018 \mathrm{MeV}/c^2$

Pions, which are mesons with zero spin, are composed of first-generation quarks. In the quark model, an up quark and an anti-down quark make up a π^+, whereas a down quark and an anti-up quark make up the π^-, and these are the antiparticles of one another. The neutral pion π^0 is a combination of an up quark with an anti-up quark or a down quark with an anti-down quark. The two combinations have identical quantum numbers, and hence they are only found in superpositions. The lowest-energy superposition of these is the π^0, which is its own antiparticle. Together, the pions form a triplet of isospin. Each pion has isospin ($I = 1$) and third-component isospin equal to its charge ($I_z = +1, 0, 1$). Also the well-known decays of pions are listed below

π^0	π^+	π^-
$\pi^0 \to 2\gamma$	$\pi^+ \to \mu^+ + v_\mu$	$\pi^- \to \mu^- + \bar{v}_\mu$
$\pi^0 \to \gamma + e^- + e^+$	$\pi^+ \to e^+ + v_e$	$\pi^- \to e^- + \bar{v}_e$
$\pi^0 \to e^- + e^+ + e^- + e^+$	$\pi^+ \to \pi^0 + e^+ + v_e$	$\pi^- \to \pi^0 + e^- + \bar{v}_e$

In the next part we are going to obtain the wavefunctions and energy levels in these systems.

2 Energy Levels and Wave functions

The General form of Lagrangian Density is

$$\mathcal{L} = \frac{\hbar^2}{2m_0}\left[\left(\partial_\mu - \frac{iq}{\hbar c}A_\mu\right)\psi^*\left(\partial^\mu + \frac{iq}{\hbar c}A^\mu\right)\psi - \frac{m^2c^2}{\hbar^2}\psi^*\psi - \frac{U_s^2}{c^2\hbar^2}\psi^*\psi\right] - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \quad (1)$$

Now for a simple pionic atom we choose

$$A^\mu = \left(\frac{Z|e|}{r}, 0\right) \quad (2)$$

So by this definition we solve the Euler-Lagrange equation which for ψ is

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \psi\right)} - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \quad (3)$$

and for ψ^* is

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \psi^*\right)} - \frac{\partial \mathcal{L}}{\partial \psi^*} = 0 \quad (4)$$

By substituting relation (1) in Euler-Lagrange equation for ψ^*

$$\partial_\mu \left[\partial^\mu + \frac{iq}{\hbar c}A^\mu\right]\psi + \frac{iq}{\hbar c}A_\mu \left[\partial^\mu + \frac{iq}{\hbar c}A^\mu\right]\psi + \frac{m^2c^2}{\hbar^2}\psi + \frac{U_s^2}{\hbar^2 c^2}\psi = 0 \tag{5}$$

which we can write it as

$$\left[D_\mu D^\mu + \frac{m^2c^2}{\hbar^2} + \frac{U_{(S)}^2}{\hbar^2 c^2}\right]\psi = 0 \tag{6}$$

and similarly for ψ we should have

$$\left[\left[\partial_\mu - \frac{iq}{\hbar c}A_\mu\right]\left[\partial^\mu - \frac{iq}{\hbar c}A^\mu\right] + \frac{m^2c^2}{\hbar^2} + \frac{U_s^2}{\hbar^2 c^2}\right]\psi^* = 0 \tag{7}$$

which is

$$\left[D'_\mu D'^\mu + \frac{m^2c^2}{\hbar^2} + \frac{U_{(S)}^2}{\hbar^2 c^2}\right]\psi^* = 0 \tag{8}$$

These are knowns as Klein-Gorden Equations. Because in this case our scaler and vector potential don't depend to time we can seperate our solution. So we can write

$$\left[(\frac{1}{c}\partial_t + \frac{iq}{\hbar c}A^0)(\frac{1}{c}\partial_t + \frac{iq}{\hbar c}A^0) - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right]\psi(\vec{x},t) = 0 \tag{9}$$

which means we can write

$$\psi(\vec{x},t) = \psi(\vec{x})e^{-\frac{i\epsilon t}{\hbar}} \tag{10}$$

Now we have

$$\left[(\frac{1}{c}\partial_t + \frac{iq}{\hbar c}A^0)(\frac{1}{c}\partial_t + \frac{iq}{\hbar c}A^0) - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right]\psi(\vec{x})e^{-\frac{i\epsilon t}{\hbar}} = 0 \tag{11}$$

where in spherical coordinates we have

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{1}{r^2 \sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\right) + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2 \tag{12}$$

For the complete form of equation we can write

$$\begin{aligned} E'\psi = &-\frac{\hbar^2}{\mu_0+\mu}\nabla^2\psi + \frac{2\mu}{\mu_0+\mu}U\psi - \frac{U^2}{(\mu_0+\mu)\,c^2}\psi \\ &+ \frac{1}{(\mu_0+\mu)\left[(\mu_0+\mu)\,c^2 - U\right]}\frac{dU}{dr}\left(\frac{2}{r}\mathbf{S}\cdot\mathbf{L}\psi - \hbar^2\frac{\partial\psi}{\partial r}\right) \end{aligned} \tag{13}$$

which for simplifying this equation we can write this as

$$\begin{aligned} E'\psi = &-\frac{\hbar^2}{m_0+m}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\psi \\ &- \frac{2m}{m_0+m}\frac{Ze_s^2}{r}\psi - \frac{1}{(m_0+m)\,c^2}\frac{Z^2e_s^4}{r^2}\psi \end{aligned} \tag{14}$$

Now from this equation, becuase our potential has spherical symmetry we can seperate the radial part and spherical part of it i.e.

$$\psi(r,\theta,\phi) = R(r)Y(\theta,\phi) \tag{15}$$

So

$$\begin{aligned} &\frac{(m_0+m)\,E'r^2}{\hbar^2} + \frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + 2m\frac{Ze_s^2}{\hbar^2}r + \frac{Z^2e_s^4}{\hbar^2c^2} \\ &\quad = -\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right] = \lambda \end{aligned} \tag{16}$$

and

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{(m_0+m)\,E'}{\hbar^2}R+\left[\frac{2m}{\hbar^2}\frac{Ze_s^2}{r}+\left(\frac{Z^2e_s^4}{\hbar^2c^2}-\lambda\right)\frac{1}{r^2}\right]R=0 \tag{17}$$

Also for spatial part we have

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}+\lambda Y=0 \tag{18}$$

where λ denote $l(l+1)$ for $l=0,1,2,...$ and certainly we find that the solutions spherical part are spherical harmonics. Now for radial part, let's talk about a condition where we are in the bound states $E'<0$, Let

$$\alpha'=\left[\frac{4\,(m_0+m)\,|E'|}{\hbar^2}\right]^{1/2}, \quad \beta=\frac{2mZe_s^2}{\alpha'\hbar^2} \tag{19}$$

Also by letting $R(r)=\frac{u(r)}{r}$,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=\frac{1}{r}\frac{d^2}{dr^2}(rR) \tag{20}$$

and by using $\rho=\alpha' r$, then the equation will be

$$\frac{d^2u}{d\rho^2}+\left[\frac{\beta}{\rho}-\frac{1}{4}-\frac{l(l+1)-Z^2\alpha^2}{\rho^2}\right]u=0 \tag{21}$$

Now by studing the asymptotic behaviour of this equation, when $\rho\to\infty$ we can write

$$\frac{d^2u}{d\rho^2}-\frac{1}{4}u=0, \quad u(\rho)=e^{\pm\rho/2} \tag{22}$$

So the general solution will be $u(\rho)=e^{-\frac{\rho}{2}}f(\rho)$, (we choose negative sign for convergency)

$$\frac{d^2f}{d\rho^2}-\frac{df}{d\rho}+\left[\frac{\beta}{\rho}-\frac{l(l+1)-Z^2\alpha^2}{\rho^2}\right]f=0 \tag{23}$$

The solution for this equation can be shown as a series form i.e.

$$f(\rho)=\sum_{\nu=0}^{\infty}b_\nu\rho^{s+\nu}, \quad b_0\neq 0 \tag{24}$$

Now by substitiuting (24) in (23) as the coefficient of $\rho^{s+\nu-1}$ is equal to zero which we get,

$$b_{\nu+1}=\frac{s+\nu-\beta}{(s+\nu)(s+\nu+1)-l(l+1)+Z^2\alpha^2}b_\nu \tag{25}$$

If the series are infinite series, then when $\nu\to\infty$ we have $b_{\nu+1}/b_\nu\to 1/\nu$. Therefore, when $\rho\to\infty$, the behaviour of the series is the same as that of e^ρ, then we have

$$R=\frac{\alpha'}{\rho}u(\rho)=\frac{\alpha'}{\rho}e^{-\rho/2}f(\rho) \tag{26}$$

Where $f()\rho$ goes to infinity when $\rho\to\infty$, which is in conflict with the finite conditions of wave functions. There for, the series should only have finite terms. Let $b_{n_r}\rho^{s+n_r}$ be the highest-order term, then $b_{n_r+1}=0$, by substituting $\nu=n_r$ into (25) we have $\beta=n_r+s$. On the other hand, the series starts from $\nu=0$, and doesn't have the term $\nu=-1$, therefore $b_{-1}=0$. Substituting this in (25) and considering b_0 is not zero we have $s(s-1)=l(l+1)-Z^2\alpha^2$. Denoting $n=n_r+l+1$ then the following set of equations can be solved for s and β,

$$\begin{cases} s(s-1)=l(l+1)-Z^2\alpha^2 \\ \beta=n_r+s \\ n=n_r+l+1 \end{cases} \tag{27}$$

where we have $s = 1/2 \pm \sqrt{(l+1/2)^2 - Z^2\alpha^2}$ which we take the positive sign. So

$$\beta = n_r + s = n - l - 1/2 + \sqrt{(l+1/2)^2 - Z^2\alpha^2} = n - \sigma_l \tag{28}$$

where

$$\sigma_l = l + 1/2 - \sqrt{(l+1/2)^2 - Z^2\alpha^2} \tag{29}$$

and according to (19) we have

$$\beta = \frac{2mZe_s^2}{\alpha'\hbar^2} = \frac{Ze_s^2}{\hbar}\left[\frac{m^2}{(m_0+m)\,|E'|}\right]^{1/2} \quad \text{or} \quad (n-\sigma_l)^2 = \frac{Z^2e_s^4}{\hbar^2}\frac{m^2}{(m_0+m)\,|E'|} \tag{30}$$

Considering $m = m_0 - \frac{|E'|}{c^2}$, then $|E'|$ satisfies the following second-order algebric equation

$$\frac{Z^2\alpha^2 + (n-\sigma_l)^2}{c^2}|E'|^2 - 2m_0\left[Z^2\alpha^2 + (n-\sigma_l)^2\right]|E'| + Z^2\alpha^2 m_0^2 c^2 = 0 \tag{31}$$

As we see, the expression of $|E'|$ obtained by solving the equation is related to both $n = 1, 2, ...$ and $l = 0, 1, ..., n-1$ thus $E = m_0c^2 - |E'|$ can be denoted by E_n then we have

$$\begin{aligned} E_n &= m_0c^2 - |E'| \\ &= m_0c^2 - m_0c^2\left(1 \pm \frac{n-\sigma_l}{\sqrt{Z^2\alpha^2 + (n-\sigma_l)^2}}\right) = \mp\frac{(n-\sigma_l)\,m_0c^2}{\sqrt{Z^2\alpha^2 + (n-\sigma_l)^2}} \end{aligned} \tag{32}$$

by taking positive sign

$$E_n = \frac{m_0c^2}{\sqrt{1 + Z^2\alpha^2/(n-\sigma_l)^2}} \tag{33}$$

If we ignore the effects of nuclear motion, then its relativistic energy level is

$$\begin{aligned} E_n &= m_0c^2\left[1 + \frac{Z^2\alpha^2}{(n-(l+1/2) + \sqrt{(l+1/2)^2 - Z^2\alpha^2})^2}\right]^{-1/2} \\ &= m_0c^2\left[1 - \frac{Z^2\alpha^2}{2n^2} - \frac{Z^4\alpha^4}{2n^4}\left(\frac{n}{l+1/2} - \frac{3}{4}\right) + \cdots\right] \end{aligned} \tag{34}$$

As $E_n = mc^2$, the system mass corresponding to the positive solution is

$$m = \frac{m_0}{\sqrt{1 + Z^2\alpha^2/(n-\sigma_l)^2}} \tag{35}$$

Solving the radial equation, we found that if we substitute the solutions $s = l + 1 - \sigma_l$ and $\beta = n - \sigma_l$ into equation (25) we have,

$$b_{\nu+1} = \frac{\nu + l + 1 - n}{(\nu + 1 - \sigma_l)(2l + 2 + \nu - \sigma_l) + Z^2\alpha^2}b_\nu \tag{36}$$

and

$$\begin{aligned} b_\nu &= \frac{(l-n+\nu)(l-n+\nu-1)\cdots(l-n+2)(l-n+1)}{\prod_{k=1}^{\nu}\left[(k-\sigma_l)(2l+1+k-\sigma_l) + (Z\alpha)^2\right]}b_0 \\ &= \frac{(-1)^\nu(n-l-1)(n-l-2)\cdots(n-l-\nu)}{\prod_{k=1}^{\nu}\left[(k-\sigma_l)(2l+1+k-\sigma_l)\right]\prod_{k=1}^{\nu}\left(1 + \frac{Z^2\alpha^2}{(k-\sigma_l)(2l+k-\sigma_l)}\right)}b_0 \\ &= \frac{(-1)^\nu(n-l-1)!}{(n-l-1-\nu)!\prod_{k=1}^{\nu}\left[(k-\sigma_l)(2l+1+k-\sigma_l)\right]\eta(l,\nu)}b_0 \end{aligned} \tag{37}$$

where b_0 is a constant

$$b_0 = -\frac{[(n+l)!]^2}{(n-l-1)!\,\Gamma(1-\sigma_l)\,\Gamma(2l+2-\sigma_l)}\frac{N_{nl}}{\alpha'} \tag{38}$$

Thus we have

$$b_\nu = \frac{(-1)^{\nu+1}[(n+l)!]^2}{(n-l-1-\nu)!\Gamma\left(\nu+1-\sigma_l\right)\Gamma\left(2l+2+\nu-\sigma_l\right)\eta(l,\nu)}\frac{N_{nl}}{\alpha'} \tag{39}$$

By substituting into (24) we get

$$R_{nl}(r) = N_{nl}e^{-\rho/2}\sum_{\nu=0}^{n-l-1}\frac{(-1)^{\nu+1}[(n+l)!]^2\rho^{l-\sigma_l+\nu}}{(n-l-1-\nu)!\Gamma\left(\nu+1-\sigma_l\right)\Gamma\left(2l+2+\nu-\sigma_l\right)\eta(l,\nu)} \tag{40}$$

So the total solution is

$$\Psi(\mathbf{r},t) = \psi_{nlm}(r,\theta,\varphi)e^{-iE_nt/\hbar} = R_{nl}(r)Y_{lm}(\theta,\varphi)e^{-iE_nt/\hbar} \tag{41}$$

By normalizing we might get

$$\begin{gathered} R_{nl}(r) = N_{nl}e^{-\frac{Z}{(n-\sigma_l)a_0}r}r\left(\frac{2Z}{\left(n-\sigma_l\right)a_0}r\right)^{l-\sigma_l}L_{n+l}^{2l+1-\sigma_l}\left(\frac{2Z}{\left(n-\sigma_l\right)a_0}r\right) \\ \sigma_l = l+\frac{1}{2}-\left[\left(l+\frac{1}{2}\right)^2-Z^2\alpha^2\right]^{1/2} = \sum_{k=1}^{\infty}\frac{2^{k-1}(2k-3)!!}{k!(2l+1)^{2k-1}}(Z\alpha)^{2k} \end{gathered} \tag{42}$$

and now we got the orthonormality relation as

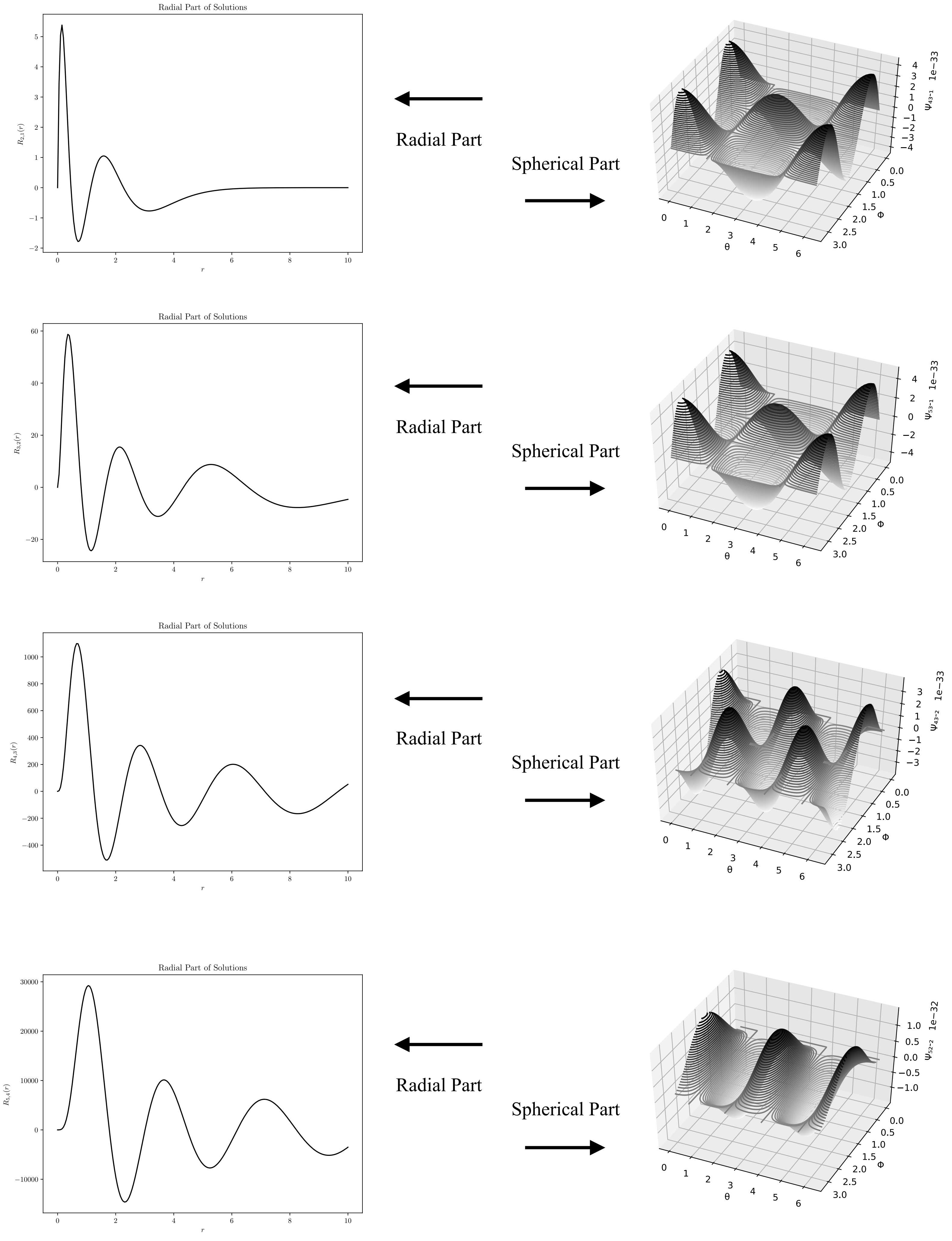
$$\int_0^\infty R_{nl}^2(r)r^2dr = 1 \quad \text{and} \quad \int_0^\pi\int_0^{2\pi}Y_{lm}^*(\theta,\varphi)Y_{lm}(\theta,\varphi)\sin\theta d\theta d\varphi = 1 \tag{43}$$

3 Results

I solved these equations completely and the code is available at my github page. For instance the below table shows some of the energy levels

	π^0	π^+	π^-
(1, 0)	$1.799952045397237 \times 10^{-13}$	$2.399936060529649 \times 10^{-13}$	$2.399936060529649 \times 10^{-13}$
(2, 0)	$1.7999880116287428 \times 10^{-13}$	$2.39998401550499 \times 10^{-13}$	$2.39998401550499 \times 10^{-13}$
(2, 1)	$1.799988012054546 \times 10^{-13}$	$2.399984016072728 \times 10^{-13}$	$2.399984016072728 \times 10^{-13}$
(3, 0)	$1.79999467190005 \times 10^{-13}$	$2.3999928958667334 \times 10^{-13}$	$2.3999928958667334 \times 10^{-13}$
(3, 1)	$1.799994672026214 \times 10^{-13}$	$2.399992896034952 \times 10^{-13}$	$2.399992896034952 \times 10^{-13}$
(3, 2)	$1.7999946720514467 \times 10^{-13}$	$2.3999928960685954 \times 10^{-13}$	$2.3999928960685954 \times 10^{-13}$
(4, 0)	$1.7999970029645693 \times 10^{-13}$	$2.3999960039527587 \times 10^{-13}$	$2.3999960039527587 \times 10^{-13}$
(4, 1)	$1.7999970030177948 \times 10^{-13}$	$2.399996004023726 \times 10^{-13}$	$2.399996004023726 \times 10^{-13}$
(4, 2)	$1.7999970030284397 \times 10^{-13}$	$2.3999960040379193 \times 10^{-13}$	$2.3999960040379193 \times 10^{-13}$
(4, 3)	$1.799997003033002 \times 10^{-13}$	$2.3999960040440025 \times 10^{-13}$	$2.3999960040440025 \times 10^{-13}$

and some of eigen-functions are in the next page.


Radial Part of Solutions
$R_{2,1}(r)$
r
Radial Part
Spherical Part
$\Psi_{43^{-1}}$ 1e−33
θ
Φ
Radial Part of Solutions
$R_{3,2}(r)$
r
Radial Part
Spherical Part
$\Psi_{53^{-1}}$ 1e−33
θ
Φ
Radial Part of Solutions
$R_{4,3}(r)$
r
Radial Part
Spherical Part
$\Psi_{43^{-2}}$ 1e−33
θ
Φ
Radial Part of Solutions
$R_{5,4}(r)$
r
Radial Part
Spherical Part
$\Psi_{52^{-2}}$ 1e−32
θ
Φ