Quantum Mechanics II Project

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1 Introduction

This is my project report for Quantum Mechanics II in Spring 2019. The project goal was solving and getting Eigen Functions for Spherical Potential Well analytically.

2 Solving Schrodinger Equation

The question was

Find The Eigen Function For Given Potential For n = 1,2,3,4,5 and l = 0,1,2.

$$V(r) = \begin{cases} 0 & r < 1 \\ \infty & r > 1 \end{cases}$$

So we should consider a mass of m moving through this potential. As you can see the wave function is only non zero in the region $0 \leq r \leq a$, but we should apply boundary conditions at $r = 0$ and $r = a$. By variable separation we can write

$$\psi(r, \theta, \phi) = R_{nl}(r) Y_{lm}(\theta, \phi) \tag{1}$$

By inserting Equation (1) in Schrodinger's Equation we get

$$\frac{d^2 R_{nl}}{dr^2} + \frac{2}{r}\frac{dR_{nl}}{dr} + (k^2 - \frac{l(l+1)}{r^2})R_{nl} = 0 \tag{2}$$

where

$$k^2 = \frac{2mE}{\hbar^2} \tag{3}$$

By defining $u = kr$ we can write

$$\frac{d^2 R_{nl}}{du^2} + \frac{2}{u}\frac{dR_{nl}}{du} + (1 - \frac{l(l+1)}{u^2})R_{nl} = 0 \tag{4}$$

Solutions of this differential equation are well-known, called spherical Bessel functions. So from definition of this function we have

$$j_l(u) = u^l(-\frac{1}{u}\frac{d}{du})^l(\frac{sin\, u}{u}) \tag{5}$$

$$y_l(u) = -u^l(-\frac{1}{u}\frac{d}{du})^l(\frac{cos\, u}{u}) \tag{6}$$

because of diverging y_l at $u \to 0$ we use j_l as our solutions, By applying second boundary condition $(R_nl(a) = 0)$ at $r = a$ we get

$$ka = u_{n,l} \tag{7}$$

where $u_{n,l}$ are zeros of Bessel functions. As we see The Energy of states depends on zeros of Bessel function. It means

$$E_{n,l} = u_{n,l}^2 \frac{\hbar^2}{2ma^2} \tag{8}$$

and Eigen Functions are

$$\psi_{nlm}(r, \theta, \phi) = A_{nl}\, j_l(\frac{u_{n,l}\, r}{a})Y_{lm}(\theta, \phi) \tag{9}$$

where A_{nl} is normalization factor, as you see it depends on n and l because m and Y_{lm} are mutually normal and orthogonal.

For calculating A_{nm} we should write j_{nl} orthogonality integral in each state, by setting a = 1

$$\int_0^a A_{nl}^2 j_l(\frac{u_{nl}\, r}{a})^2 r^2 dr = 1 \tag{10}$$

So we have

$$A_{nl} = \frac{1}{\sqrt{\int_0^a j_l(\frac{u_{nl}\, r}{a})^2 r^2 dr}} \tag{11}$$

Finally our Eigen Function are

$$\psi_{nlm}(r, \theta, \phi) = \frac{1}{\sqrt{\int_0^a j_l(\frac{u_{nl}\, r}{a})^2 r^2 dr}}\, j_l(\frac{u_{n,l}\, r}{a})Y_{lm}(\theta, \phi) \tag{12}$$

with the energy of

$$E_{n,l} = u_{n,l}^2 \frac{\hbar^2}{2ma^2} \tag{13}$$