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Cosmic Microwave Background

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Contents

1 Introduction

This is a summary from what I have learned from one of my undergraduate projects **"Cosmic Microwave Background"**. At the end of this report, we are going to write a simple script to get power spectra of Cosmic Microwave Background and talk about what we can understand from that.

2 Cosmology

Cosmology is one of the eras of physics which all parts of physics may get involved from quantum fluctuations that we have at the very early universe to separation of particles from cosmic plasma to formation of structures and .... There were two major thoughts about our universe. We know now that we and our galaxy isn't a special one in comparison to other galaxies. This principle is known for The Cosmological Principle which is the basis for Big Bang Theory which I'm going to explain it in the next part. The other popular thought about our universe is that some people thought that our universe doesn't change through time, known as Steady State Universe. But today from observations we know that this theory isn't very relatable. So let's explain what Big Bang Theory tells us about our universe from early time to now.

3 Big Bang Theory

As I say, earlier Big Bang Theory is the theory that has more consistency with observations. Big Bang is the event where all parts of cosmic plasma start to expand and getting away from each other because our world was beginning to extend! That means if we existed that time and we were in a room we started to getting away from each other not because of that we start to run away from each other also because of the tiles under our feet starts to extend. Let's explain it more with a simplified and helpful example. Imagine two ants on a balloon; Expansion of the universe is like blowing the balloon, as you see if the ants don't move they are still moving away from each other. It's important now to talk about another coordinate system that we use in our expanding universe knows as comoving coordinates. These are coordinates are along the expansion of our universe and because we used to know that the expansion is uniform we can relate two coordinates comoving coordinates and physical coordinates with this relation

$$\vec{r} = a(t)\vec{x} \tag{1}$$

where a knows as scale factor which because of our homogeneity assumption is just tion of time. Scale factor measures the expansion rate of universe. Also let's derive the Friedman Equation, one of the most important in cosmology that help us know about scale factor and how it changes in time.

Let's consider us as a observer with mass density of ρ in the universe which expand uniformly. According to cosmological principle the universe is the same from every place that we observe it and we can take anyplace as its center. So with use of Newton Theorem and considering a mass of m in the distance of r, we know this mass only feels force from is what inside and at smaller radii, by setting the total mass of this material M we can write

$$M = \frac{4\pi\rho r^3}{3} \tag{2}$$

With the expression of gravitational potential energy we have

$$V = -\frac{GMm}{r} \tag{3}$$

So the total energy can be derived by the expression of Hamiltonian as

$$U = \frac{1}{2}m\dot{r}^2 - \frac{4\pi}{3}G\rho r^2 m \tag{4}$$

With Using Equation (1) and Setting $\dot{x} = 0$

$$U = \frac{1}{2}m\dot{a}^2 x^2 - \frac{4\pi}{3}G\rho a^2 x^2 m \tag{5}$$

By rearranging Equation (5) we can write

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho + \frac{2U}{mx^2 a^2} \tag{6}$$

The fraction in the right side of relation is known as Hubble parameter (H) so we can write

$$H^2 = \frac{8\pi G}{3}\rho + \frac{2U}{mx^2 a^2} \tag{7}$$

According To Hubble's law the recession velocity is proportional to the distance this means we can write

$$\vec{v} = \dot{\vec{r}} \tag{8}$$

By writing this relation as

$$\vec{v} = \frac{\dot{\vec{r}}}{\vec{r}}\vec{r} \tag{9}$$

And using Equation (1) we have

$$\vec{v} = \frac{\dot{a}}{a}\vec{r} = H\vec{r} \tag{10}$$

this means the velocity of all components in our universe with taking distance from us increases and it may they reach to speed of light and exit from our horizon!

4 Thermal History Of Universe

Absolutely One of the most important event in our universe age is thermal history which teach us lot's of things about what's happened and how structures forms! So Let's see what's happened from first ,

Plank Scale and Inflation$(T \approx 10^{19}GeV, t \approx 10^{-43}Sec)$. Because of the high energy and temperature in this period the classical treatment of general relativity can't help us and can only understand by Quantum Theory Of Gravity.

GUT Phase Transitions$(T \approx 10^{16}GeV, t \approx 10^{-38}Sec)$.

Electroweak Phase Transition$(t \approx 10^{-12}Sec)$.

Decoupling Of Top Quark$(T \approx 100MeV,\ t \approx 10^{-12}Sec$ to $10^{-6}Sec)$.

Baryogenesis$(t \approx 10^{-11}Sec)$.

Hadron Epoch($T \approx 1MeV,\ t \approx 10^{-6}Sec$ to $1Sec$).

Neutrino Decoupling($T \approx 1MeV,\ t \approx 1Sec$).

Lepton Epoch($T \approx 100KeV$ to $1MeV,\ t \approx 10Sec$).

Big Bang Nucleosynthesis($T \approx 1KeV$ to $100KeV$, $t \approx 10^1 Sec$ to $10^3 Sec$).

Recombination and Photon Decoupling($T \approx 0.4eV$ to $t \approx 380ka$).

What we need to know about CMB depends on knowing what happens in Recombination and Decoupling, But knowing about BBN is also helpful.

4.1 Big Bang Nucleosynthesis (BBN)

Big Bang Nucleosynthesis is the process of forming the light element. Till now on this period we already have photons, neutrinos, antineutrinos, electron, positrons, protons and neutrons.
The first interaction that we have in this period is

$$n \leftrightarrow p + e^- + \overline{\nu}_{\mathrm{e}} \tag{11}$$

As we see neutrons by decomposing free beta particles. The two other particles protons and electrons interact with each other so we have another important interaction

$$p + e^- \leftrightarrow \nu_{\mathrm{e}} + n \tag{12}$$

Protons by capturing electrons transform to neutrons and because of Neutrality of total electrical charge in left side of equation the neutrino's also created. Another interaction that happened is inverse beta decay which protons and anti neutrinos interact with each other, So

$$p + \overline{\nu}_{\mathrm{e}} \leftrightarrow e^+ + n \tag{13}$$

When we want to study about BBN this three interactions; **Beta Decay**, **Electron Capture** and **Inverse Beta Decay** play a major role at that period of time and help us understand what's going on that time.

There are some variables that our so important for us, one of them is **baryon to photon ratio** that is

$$\eta_b = \frac{n_b}{n_\gamma} \tag{14}$$

By assuming that the baryons are non relativistic we can write

$$\eta_b = \frac{\varepsilon_{b0}\varepsilon_{\gamma 0}}{m_b c^2 n_{\gamma 0}} \tag{15}$$

Now the question is how much is the $\frac{\varepsilon_{\gamma 0}}{n_{\gamma 0}}$? Let's calculate it. In order to calculate the numerator of this fraction we should write distribution function, the distribution function is

$$f_\gamma = \frac{1}{exp(\frac{E}{k_B T}) - 1} = \frac{pc}{exp(\frac{pc}{k_B T}) - 1} \tag{16}$$

In order to calculate the photon energy density, we have

$$\varepsilon_\gamma = n_i \int \frac{d^3p}{(2\pi\hbar)^3} \frac{pc}{exp(\frac{pc}{k_B T}) - 1} \tag{17}$$

where n_i is the polarization factor (number of states of polarization), with $n_i = 2$ also by changing integral variable by writing $x = \frac{pc}{k_B T}$ we should have

$$\varepsilon_\gamma = \frac{c}{\pi^2 \hbar^3} (\frac{k_B T}{c})^4 \int_0^\infty dx \frac{x^3}{e^x - 1} \tag{18}$$

Finally by calculating the integral we might have

$$\varepsilon_\gamma = \frac{\pi^2}{15\,\hbar^3 c^3} (k_B T)^4 \tag{19}$$

In order to calculating photon number density we write

$$n_\gamma = \frac{1}{\pi^2 \hbar^3} \int_0^\infty dp \frac{p^2}{exp(\frac{pc}{k_B T}) - 1} \tag{20}$$

With the same change of variable we should have

$$n_\gamma = \frac{2\zeta(3)}{\pi^2 \hbar^3 c^3} (k_B T)^3 \tag{21}$$

so by inserting Equation(19) and Equation(21) in Equation(15) we can write

$$\eta_b = \frac{\pi^4 k_B T_0 \varepsilon_{b0}}{30\zeta(3) m_b c^2 \varepsilon_{\gamma 0}} = \frac{\pi^4 k_B T_0 \Omega_{b0}}{30\zeta(3) m_b c^2 \Omega_{\gamma 0}} \tag{22}$$

Now by substituting the constants we have

$$\eta_b = 5.5 \times 10^{-10} (\frac{\Omega_{b0} h^2}{0.020}) \tag{23}$$

Another event that happened in that time is **Deuterium Bottleneck** which is the event that photons destroyed the deuterium nuclei, the fact is that in that thermal bath there were billions of photons that part of them were with the energy higher than deuterium binding energy so they destroyed, We have

$$p + n \leftrightarrow D + \gamma \tag{24}$$

This event happens in $k_B T_{BBN} = 0.07MeV$. The next chemical interaction that happened in BBN are

$$p + n \to D + \gamma \tag{25}$$

$$D + D \to {}^3He + n \tag{26}$$

$$D + {}^3He \to {}^4He + p \tag{27}$$

These following interaction occurred after deuterium bottleneck, and these called Neutron Abundance. The prediction of BBN is on the abundance of ${}^4He$ because this element formed mostly in this period because of high binding energy and the short time that element has tho interact. It mean the higher elements also could formed but there are short amount of them because of short time.

4.2 Recombination and Decoupling

Recombination and Decoupling are two process that happened along each other. Recombination or more correctly "Combination" is the process that electron and protons and electrons combine, but this time because the energy level is lower instead of creating neutrons The Hydrogen atom will be produced, so we have

$$p + e^- \leftrightarrow H + \gamma \tag{28}$$

As you see by the time passing the number of free electrons might be decreased. After ε the next interaction started to happening

$$e^- + \gamma \leftrightarrow e^- + \gamma \tag{29}$$

But after t_d because the interaction rate decreasing ($\Gamma = n v_{rel} \sigma$), it reach out to the order of Hubble Parameter and photons stop interacting with free electrons and their mean free path started to become larger and we can detect them now. We called these photons, CMB Photons that came from last scattering surface.

5 Cosmic Microwave Background (CMB)

First of all, we can say Cosmic Microwave Background is our oldest and cleanest source of information that we have for early universe, that's why it's so important for us.

As we saw in previous section the photons that come from last scattering surface are CMB photons.

5.1 Observation

We can observe these photons with CMB Telescopes that we can say they are mostly like TV Antenna!
By changing the direction the antenna you measure a voltage and these voltage relate to temperature of that region, simple as that. But last two decades efforts are for increasing the precision and reducing the noises.

Why the precise measurement in this case is so important?

As like other part of physics, precise measuring help us getting closer to the fact and physics behind the story. In this case what we want to achieve is a model which can explain the whole world at large scale.
What we do in this case is that we want to fit our model (with our parameters) to the real data, so the more precise measurement, the more accurate parameter we have and at the end the better model we have.

5.2 What's behind the Physics of CMB?

The fact about cosmic events is they didn't happens suddenly at a specific time or in other word instantaneously. But we use a approximation where we assume instantaneous recombination and after that we try to improve our assumption. So by assuming instantaneous recombination we have following relation between Anisotropies at CMB (Temperature dissipation) and perturbation that we are going to discuss

$$\Theta(k.\vec{n}) = [\phi + \frac{1}{4}\delta_\nu](\tau_{rec}) + \vec{n}.\vec{v_\nu}(\tau_{rec}) + 2\int_{\tau_{rec}}^{\tau_0} \phi\prime(\tau) d\tau \tag{30}$$

I postpone describing meaning of each term to talk about these perturbations and fluctuations. ϕ, δ and v in above equation are perturbation terms which we are going to discuss about them later. Today what we get from observation is that there are inconsistencies in the structures of the universe, the most simplified example of this is that in some parts we have masses like our planet and in other part we have nothing! So these inconsistencies originated from some cosmic perturbations.

5.2.1 Perturbations

Before recombination photons and baryons (electrons and protons) were strongly coupled together via Thompson Scattering, which we know this fact from simple observations. With that being said we can we can treat them as single fluid and since this fluid and dark matter don't coupled directly to each other the stress tensor of each of them are separately conserved so we can write for each of them

$$\nabla_\mu T_i^{\mu\nu} = 0 \tag{31}$$

and by solving this relation we can get fluid equation of motion and after that we can use perturbation theory and find perturbed fluid equation of motion. The dynamics of scalar perturbations governed by continuity equation for density fluctuations and Euler equation for the divergence of the velocity field.

Here we have some important parameter that we are going to talk about them; first of all we have **density contrast**

$$\delta = \frac{\delta\rho}{\overline{\rho}} \tag{32}$$

which is the fraction of $\delta\rho$ density in a small region on $\overline{\rho}$ density average value .

As I said earlier because we focus on scalar perturbation. v peculiar velocity is velocity of particles in this fluid relative to a rest frame. This vector field according to **Helmholtz Theorem** can resolved into the sum of an curl-free and divergence-free vector field and because of our choice on using scalar perturbations we use the curl-free part of this vector field. According to Euler equation for the divergence of velocity field we have

$$\theta = \vec{\nabla}.\vec{v} \tag{33}$$

For a single uncoupled fluid with the equation of state

$$\omega = \frac{p}{\rho} \tag{34}$$

and sound speed

$$c_s^2 = \frac{\delta P}{\delta\rho} \tag{35}$$

by perturbing (first term or linear) the equations (using continuity equation) we get

$$\delta' = -(1+\omega)(\theta - 3\phi') - 3H(c_s^2 - \omega)\delta \tag{36}$$

and

$$\theta' = -H(1-3\omega)\theta - \frac{\omega'}{1+\omega}\theta + \frac{c_s^2}{1+\omega}k^2\delta + k^2\phi \tag{37}$$

where

$$H = \frac{a'}{a} \tag{38}$$

We use the metric perturbation is Newtonian Gauge which is

$$ds^2 = a^2(\tau)[-(1+2\psi)d\tau^2 + (1-2\phi)d\vec{x}^2] \tag{39}$$

where ϕ and ψ are *Bardeen Potentials* and in the absence of an anisotropic stress $\psi = \phi$ so

$$ds^2 = a^2(\tau)[-(1+2\phi)d\tau^2 + (1-2\phi)d\vec{x}^2] \tag{40}$$

where ϕ is gravitational potential.

By taking $\vec{v}$ in the form of

$$\vec{v} = -iv\hat{k} \tag{41}$$

which implies

$$\theta = kv \tag{42}$$

For cold dark matter because there is no interaction between them and cross section is equals to zero (in our model) we have

$$\omega = c_s^2 = 0 \tag{43}$$

So fluid equations of motion for cold dark matter are

$$\begin{aligned} \delta_c' &= -kv_c + 3\phi' \\ v_c' &= -Hv_c + k\phi \end{aligned} \tag{44}$$

We know photons contribute a pressure to the baryon-photon fluid which is

$$p_\nu = \frac{1}{3}\rho_\nu \tag{45}$$

then the effective equation of state and sound speed of this fluid are

$$\omega = \frac{1}{3+4R} \tag{46}$$

and

$$c_s^2 = \frac{1}{3(1+R)} \tag{47}$$

where R is

$$R = \frac{3}{4}\frac{\overline{\rho_b}}{\overline{\rho_\nu}} \tag{48}$$

by assuming tight coupling approximation which means

$$v_\nu = v_b \tag{49}$$

and

$$\delta_\nu = \frac{3}{4}\delta_b \tag{50}$$

and inserting this to fluid equations of motion for photons we get

$$\delta'_\nu = -\frac{4}{3}kv_\nu + 4\phi' \tag{51}$$

and

$$v'_\nu = \frac{-Rv_\nu + \frac{1}{4}k\delta_\nu + (1+R)k\phi}{1+R} \tag{52}$$

So these are our fluid equations of motion, now we have to be supplemented by the linearized Einstein Equation for the gravitational potential, so we have

$$k(\phi\prime + H\phi) = 4\pi Ga^2 \sum_i (\overline{\rho}_i + \overline{p_i})v_i \tag{53}$$

and from Friedman Equation from background we have

$$H^2 = (\frac{a\prime}{a})^2 = \frac{8\pi Ga^2}{3}\sum_i \overline{\rho_i} \tag{54}$$

and this relation has analytic solution which is

$$y = \frac{a}{a_{eq}} = (\alpha x)^2 + 2\alpha x \tag{55}$$

where

$$x = \frac{\tau}{\tau_r} \tag{56}$$

and

$$\alpha = \sqrt{\frac{a_{rec}}{a_{eq}}} \tag{57}$$

for numerical convenience we rescale time and momentum to

$$x = \frac{\tau}{\tau_r} \tag{58}$$

and

$$\kappa = k\tau_r \tag{59}$$

and also we define

$$\eta = \frac{a'}{a} = \frac{1}{a}\frac{da}{dx} = \frac{2\alpha(\alpha x + 1)}{(\alpha x)^2 + 2\alpha x} = \tau_r H \tag{60}$$

The fluid equations of motion then become

$$\delta_c' = -kv_c + 3\phi' \tag{61}$$
$$v_c' = -Hv_c + k\phi \tag{62}$$
$$\delta_\nu' = -\frac{4}{3}kv_\nu + 4\phi' \tag{63}$$

and

$$v_\nu' = \frac{-\frac{3}{4}y(\frac{\Omega_b}{\Omega_m})\eta v_\nu + \frac{1}{4}\kappa\delta_{nu}}{1+\frac{3}{4}y(\frac{\Omega_b}{\Omega_m})} + \kappa\phi \tag{64}$$

with

$$\phi' = -\eta\phi + \frac{3\eta^2}{2\kappa}\frac{v_\nu(\frac{3}{4}+y-y(\frac{\Omega_c}{\Omega_m})) + v_c y(\frac{\Omega_c}{\Omega_m})}{1+y} \tag{65}$$

where Ω is density parameter. Now our task is solving these differential equations for these initial conditions, by setting $\phi(x_i) = 1$ these are

$$\delta_\nu = -2\phi(x_i) \tag{66}$$
$$\delta_c = \frac{3}{4}\delta_\nu \tag{67}$$
$$v_\nu = -\frac{1}{4}\frac{\kappa}{\eta}\delta_\nu \tag{68}$$
$$v_c = v_\nu \tag{69}$$

We are going to solve these equation numerically in section 8.1.

5.2.2 Instantaneous Recombination

We know when we have Instantaneous Recombination these anisotropies relate to perturbed terms with Equation (30). Let's start this part with talking about Equation (30) and talk about those 3 terms in this relation. We have

$$\Theta(k.\vec{n}) = [\phi + \frac{1}{4}\delta_\nu](\tau_{rec}) + \vec{n}.\vec{v_\nu}(\tau_{rec}) + 2\int_{\tau_{rec}}^{\tau_0} \phi\prime(\tau)d\tau \tag{70}$$

First term known as **Ordinary Sachs-Wolfe Effect**. This term caused by two primary reason, for the first one we say when a photon passes through potential wells (masses) because of differences in potentials (masses) the effects of blue shift and red shift may be different from each other and for the second one we say because of differences and anisotropies in mass distribution we have this perturbed term.

Second term is known as **Doppler Effect**. Because of different velocities and direction of photons of last scattering surface that come to us relate to our velocity we should have different Doppler Effect for each of them which it caused these anisotropies.

And the Last term is known as **Integrated Sachs-Wolfe Effect**. One another important thing is that our potentials are related to time and this will cause these anisotropies. It means when a photon is passing through potential well the width or height of potential well will change by time, so the blue shift and red shift of photon may not neutralize each other and cause these disruptive effects.

5.3 A Simple Model !

In this part we are going to use what we learned till now to have picture of what really happens here!

The early universe was filled with Byronic matter and photons interacting in a gravitational potential. What happens here is that matter concentrations attracts each other because of gravity, but when the density increases the pressure also increases so we have repulsion here. It's like we have a spring that oscillates and through time they **compress** and **decompress**.

Our universe consists of an entire landscape of potential wells where the photon-baryon fluid (plasma) oscillates n this potential landscape.

6 What is Power Spectrum?

6.1 Fourier transforms

We know any function may be expanded to Eigen Function which these Eigen Function most times are wave shaped function like $sin(x)$ and $cos(x)$. This expansion in flat space is called **Fourier Transform**. It means can we write any function like $f(x)$ as

$$f(x) = \sum_k [a_k cos(kx) + b_k sin(kx)] = \sum_k c_k e^{ikx} \tag{71}$$

for different k we have different modes. We can find a_k, b_k and c_k very easily. Because of convenience in our calculation we use the second form so for second form by multiplying each side of relation by e^{-ikx} and integrating, so

$$c_k = \int f(x) e^{-ikx} dx \tag{72}$$

where $|c_k|$ shows the amplitude of mode k and the phase of c_k determines the position of the wave along the x axis.
A simple example it is that we can decompose a disorder wave to sum of regular waves with different amplitudes and positions. With that being said let's talk about what power spectrum means?

6.2 Power Spectra

For noise-like phenomena, what we are interested in is the amplitude of the fluctuations as a function of scale. so we define power spectrum as

$$P(k) = |c_k|^2 \tag{73}$$

6.3 Why Spherical Harmonics?

We know the required basis wave functions is found by solving Laplace equation

$$\nabla^2 \psi = 0 \tag{74}$$

Choosing the coordinates is up to us, in this case because CMB Field is defined on a sphere we are going to solve this relation in spherical coordinates, By considering a function $\psi(\theta, \phi)$ where we can separate variables we have

$$\psi(\theta, \phi) = \Theta(\theta)\Phi(\phi) \tag{75}$$

By assuming this the Laplace equation takes form of

$$\frac{1}{sin\theta}\Phi(\phi)\frac{d}{d\theta}(sin\theta\frac{d\Theta(\theta)}{d\theta}) + \frac{1}{sin^2\theta}\frac{d^2\Phi(\phi)}{d\phi^2} + l(l+1)\Theta(\theta)\Phi(\phi) = 0 \tag{76}$$

By solving this equation we find

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi} \tag{77}$$

and

$$\Theta_{lm}(cos\theta) = \sqrt{\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}}P_l^m(cos\theta) \tag{78}$$

and the product of this is our solution which is called **Spherical Harmonics**

$$\psi = Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(cos\theta)e^{im\phi} \tag{79}$$

where $l \geq 0$ and $m = -l, ..., l$

So Spherical Harmonics are wave function on the sphere completely analogous to the complex exponential (e^{ikx}), Here instead of wave number (k) these described by l and m where l determines **the wave length** of the mode (number of waves along meridian) and m determines **the shape** of the mode (number of modes along equator).

6.4 Spherical Harmonics Transform

So any function defined on the sphere as we see from Fourier Transform may be expanded into spherical Harmonics, It means we can write

$$F(\hat{n}) = \sum_{l=0}^{l_{max}}\sum_{-l_{max}}^{l_{max}} a_{lm}Y_{lm}(\hat{n}) \tag{80}$$

where expansion coefficients are

$$a_{lm} = \int_{4\pi} F(\hat{n})Y^*_{lm}(\hat{n})d\Omega \tag{81}$$

So we can expand these maps to sum of spherical wave modes with different amplitudes which a_{lm} determines for us.

6.5 The Angular Power Spectrum

The angular power spectrum measures amplitude as a function of wavelength defined as an average over m for every l so we have

$$C_l = \frac{1}{2l+1}\sum_{m=-l}^{l}|a_{lm}|^2 \tag{82}$$

7 CMB Power Spectrum

7.1 CMB Angular Power Spectrum

From section 5.2.2 we know the Instantaneous Recombination relation is

$$\Theta(k.\vec{n}) = [\phi + \frac{1}{4}\delta_\nu](\tau_{rec}) + \vec{n}.\vec{v_\nu}(\tau_{rec}) + 2\int_{\tau_{rec}}^{\tau_0} \phi\prime(\tau)d\tau \tag{83}$$

We need to relate $\Theta(\vec{n})$ to $\Theta(k)$, First note that our assumption of instantaneous recombination implies

$$\Theta(\vec{n}) = \int dr\Theta(\vec{x})\delta(r - r_*) \tag{84}$$

Where the integral is over conformal distance and r_* is the angular diameter distance between us and the last-scattering surface. Here,$\Theta(\vec{x})$ is the real space temperature field relate to our solution via

$$\Theta(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}.\vec{x}} \zeta_{\vec{k}} \Theta(k) \tag{85}$$

So

$$\Theta(\vec{n}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}.r_* n} \zeta_{\vec{k}} \Theta(k) \tag{86}$$

Where

$$e^{i\vec{k}.r_* n} = 4\pi \sum_{lm} i^l j_l(kr_*) Y^*_{lm}(\vec{k}) Y_{lm}(\vec{n}) \tag{87}$$

So, We find the CMB angular power spectrum

$$C_l^{TT} = \frac{2}{\pi} \int k^2 dk P_\zeta(k) \Delta^2_{Tl}(k) \tag{88}$$

Where

$$\Delta_{Tl}(k) = (\phi + \frac{1}{4}\delta_\nu) j_l(k[\tau_0 - \tau_{rec}]) + v_\nu j_l'(k[\tau_0 - \tau_{rec}]) + 2\int_{\tau_{rec}}^{\tau_0} d\tau j_l(k[\tau_0 - \tau_{rec}]) \frac{\phi'(\tau)}{\phi'(\tau_0)} \tag{89}$$

7.2 Silk Damping

Because of our assumptions (perfect tight coupling (mean free path = 0) and instantaneous recombination) our solution is still missing some important physics including the effects of a finite mean free path for photons and a finite duration of recombination would lead us to the damping of small fluctuations. The fact is that at large l's because of this relation

$$\theta = \frac{2\pi}{l} \tag{90}$$

The region that we looking at gets smaller and in these regions photons have enough amount of time to diffuse from the hot, over dense regions of plasma to the cold, under dense ones. This caused the fluctuations in these region become smoother and the temperatures and densities become less anisotropic , So we call it Diffusion Damping or Silk Damping (after the name of it's discoverer).

From Seljak paper we introduce a high momentum cutoff and rewriting equation (89) with this Damping term

$$C_l^{TT} \approx \frac{2}{\pi} \int k^2 dk P_\zeta(k) D(k) \Delta^2_{Tl}(k) \tag{91}$$

Where

$$D(k) = e^{(-\frac{k}{k_D})^2} \tag{92}$$

The damping scale in this approach is related to density parameters, the relation is

$$\kappa_D = \frac{1}{\sqrt{2x_s^2 + \sigma^2 x_{rec}^2}} \tag{93}$$

where

$$x_s = 0.6\,\omega_m^{1/4}\omega_b^{-1/2}a_{rec}^{3/4} \tag{94}$$

and

$$\sigma \approx 0.03 \tag{95}$$

7.3 Peaks

Inflation sets up a flat spectrum of fluctuations and shortly after inflation, the horizon is small and only small scales are processed by gravity and pressure. As the times goes by, larger and larger scales start to oscillate, then recombination happens and the CMB is **frozen**.

From section 5.3 we can say that every peaks happens when the photons and fluid compression or decompression.

Inflation, at low l's ; scales larger than $l \approx 180°$ are only weakly processed by gravity and pressure, at $l < 50$ is a direct picture of fluctuations generated by inflation.

According to General Relativity, light propagates along geodesics, In **Flat Spaces**, these are straight lines, In **Open Spaces** the geodesics diverge and in the **Closed Spaces**, the geodesics converge.

The first peak is a standard ruler for the horizon size, If

- the first peak is at $l \approx 220$, then the universe is **flat**
- the first peak is at $l > 220$, then the universe is **open**
- the first peak is at $l < 220$, then the universe is **closed**

Let's talk **higher peaks**, as we know Odd peaks relate to compression and Even peaks relate to decompression. The fact is that more baryons means heavier load of our string and this add to potential so compressions are stronger than decompressions means **Odd Peaks** are **stronger** than **Even Peaks**.

And as we see at section 7.2 at high l's we have an **exponential damping**.

8 Steps To Getting Power Spectra

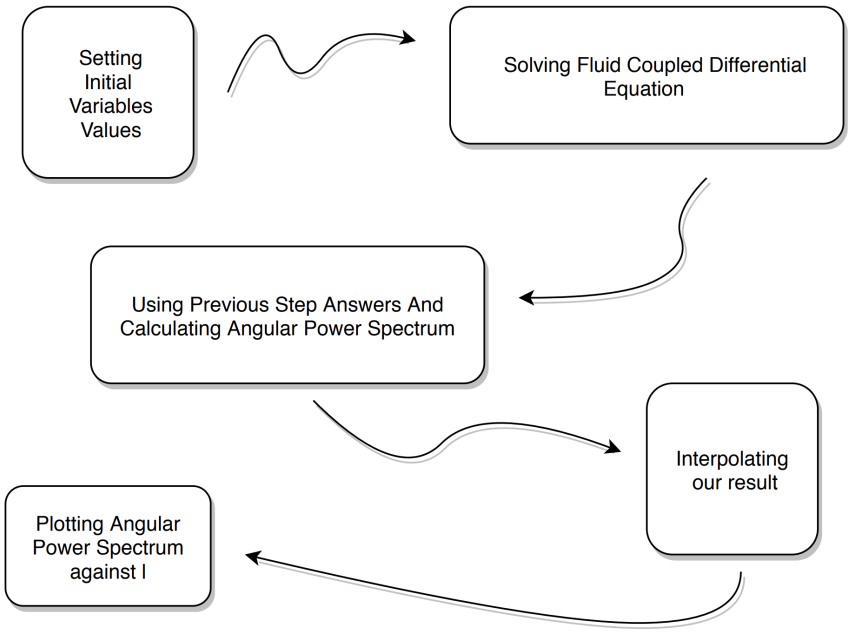


Figure 1: Steps To Getting Angular Power Spectra

In this part we are going to write a code to get the CMB angular power spectrum. All the scripts wrote in Python version 3.6.5. The complete code notebook has been uploaded to my Github, it can be find at **https://github.com/mqesmaeilian**.

As you see first of all we set our initial variables that we know their values already. After that we are going to solve Fluid Differential Equation, after solving that we are going to use our results in next part and calculating C_l^{TT} for each l. Next, we are going to use Interpolation method and trying to make our code respond faster by using less l's and finally we are going to plot angular power spectrum against l in next section.

8.1 Solving Fluid Coupled Differential Equations

For solving these coupled differential equation we are going to use Scipy Library function **odeint()**. So first of all we write a function to return initialize values of our parameters, our code is should look like this

```
def VacFlucInit(k,x):

        etha = (2*alpha) * ((alpha*x)+1) / (((alpha*x)**2)+(2*alpha*x))
        phi = 1
        d_n = -2*phi
        d_c = 3*d_n/4
        v_n = -k*d_n/(4*etha)
        v_c = v_n

        return [phi,d_c,v_c,d_n,v_n]
```

After that we write our *func* that includes our differential equations, So

```
def VacFluc(X,x,k):

        PHI = X[0]
        D_C = X[1]
        V_C = X[2]
        D_N = X[3]
        V_N = X[4]

        etha = (2*alpha) * ((alpha*x)+1) / (((alpha*x)**2)+(2*alpha*x))
        y = (2*alpha*x)+((alpha*x)**2)
        y_c = (y*OMEGA_C)/(OMEGA_M)
        y_b = (y*OMEGA_B)/(OMEGA_M)

        dPHIdx = (-etha * PHI) + ((3*(etha**2)/(2*k))*(((V_N*((4/3)+y-y_c))+(V_C*y_c))/(1+y)))
        dD_Cdx = (-k * V_C) + (3 * dPHIdx)
        dV_Cdx = (-etha * V_C) + (k * PHI)
        dD_Ndx = ((-4 * k * V_N)/3) + (4 * dPHIdx)
        dV_Ndx = ((1/(1+(3*y_b/4)))*((-3*y_b*etha*V_N/4)+(k*D_N/4)))+(k*PHI)

        return [dPHIdx, dD_Cdx, dV_Cdx, dD_Ndx, dV_Ndx]
```

Now our job is to solve these function for a range of k's, so we write another function that get k_{min} and k_{max} and at output gives us the solution as a list, it should looks like

```
def Variable_cal(k_min, k_max, step):
        x0 = 0.001
        ### this is initial x when conformal time was 0.001 of conformal time at recombination
        x = np.linspace(x0, x_rec)
        var_x_rec = []

        for i in np.arange(k_min, k_max, step):
                X0 = VacFlucInit(i,x0)
                k = i
                sol = odeint(VacFluc, X0, x, args=(k,))
                var = [k,((sol[:,0][-1])+(sol[:,3][-1]/4)),(sol[:,4][-1])]
                ### we need 'var' for calculating power spectra
                var_x_rec.append(var) ### for complete solution we can return whole 'sol'

        return var_x_rec, step, x
        ### the reason for getting 'step' and 'x' as output is we need them in next parts
```

8.2 Getting Power Spectrum

After solving differential equations we need our answers to calculating power spectrums, to find power spectrum we use equation (91) and write a function that gets l_{min}, l_{max} and our solution from previous part and give us Angular Power Spectrums as a list, the function should look like this

```
def PSpectrum(l_min, l_max, l_step, tab_var, x, step):

        L = np.arange(l_min, l_max, l_step)
        Powers = []

        for i in L:

                k = [j[0] for j in tab_var] ### calling k from our solution
                a = [j[1] for j in tab_var] ### calling (phi+1/4delta_gamma) from our solution
                b = [j[2] for j in tab_var] ### calling v_gamma from our solution
                dk = step

                func_var = list(map(lambda k,a,b:(2)*i*(i+1)*
                ((a*spherical_jn(i,(t0-tr)*k/tr,derivative=False)
                +b*spherical_jn(i,(t0-tr)*k/tr,1))**2)
                *np.exp(-(k/(tr*K_D))**2)*(k**(-1.04)), k,a,b))

                integ = np.sum([j*dk for j in func_var])
                ### I used sum of list(map(anonymous func)) for Integrating
                Powers.append(integ)

        return L, Powers
```

I use Riemann integral for calculating this numerical integral, means I divide the whole horizontal axis to very small regions and calculate $f(x)$ in that region then sum all over of them to fine integral answer.

8.3 Interpolating our results

The main advantage of using Interpolation methods is that you can with few points determined the whole behaviour of points and it's incredibly makes it faster to have the final answer.

scipy.interpolate can help us to do that! Also another important thing is that there are kinds of interpolation in this function which I used **cubic** because the result is more accurate. We can write it as a function like this

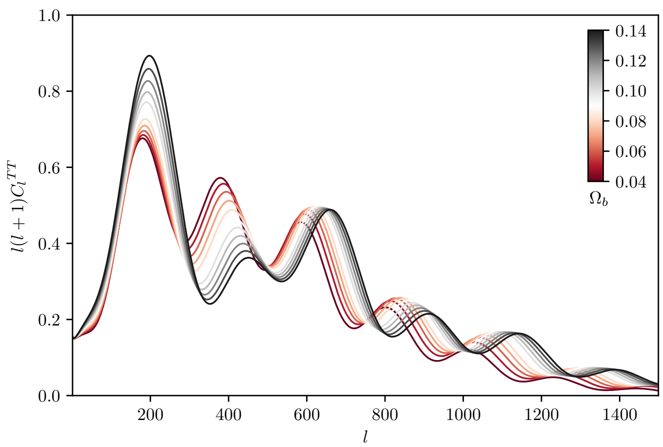
```
from scipy.interpolate import interp1d

def interp_func(l, powers):

        p = interp1d(l, powers, kind='cubic')

        return l, p
```

9 Results

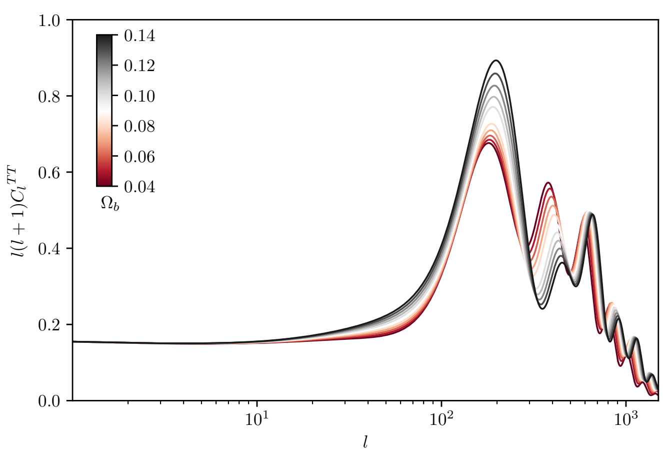
In this section we are going to plot the results by changing density parameters and see behavior of that. So each time by setting a value in a range of values for each density parameters we get the angular power spectra and finally we plot them to see how changing them effects to the final result.

9.1 Plotting Power Spectrum By changing density parameters

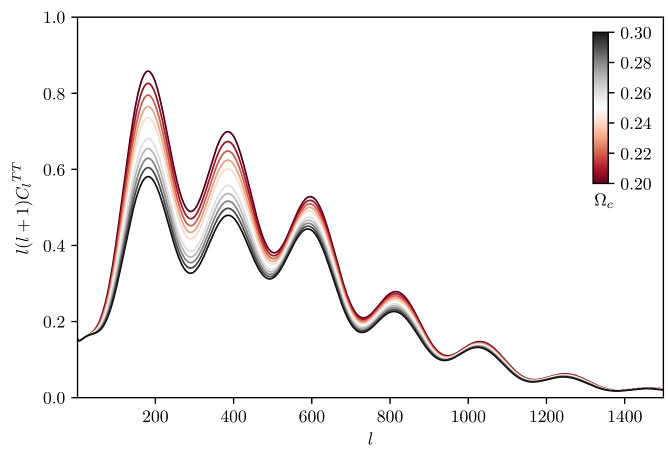
Note that the chosen range of a special density parameter in these results is just for showing the differences clearly and may not be accurate and physically true.
By changing **baryon density parameter** we have



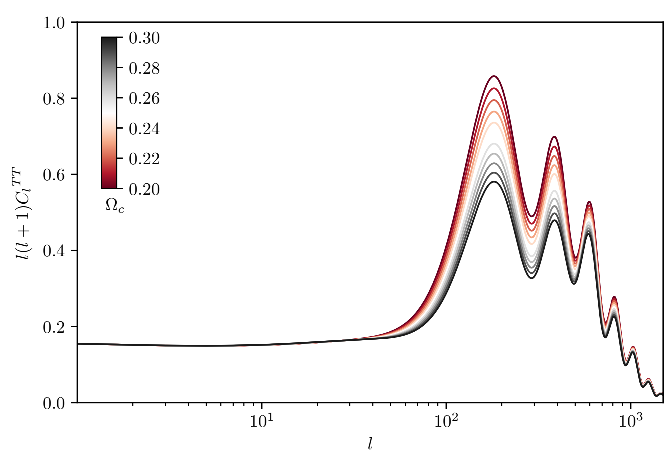
And the logarithmic plot looks like this



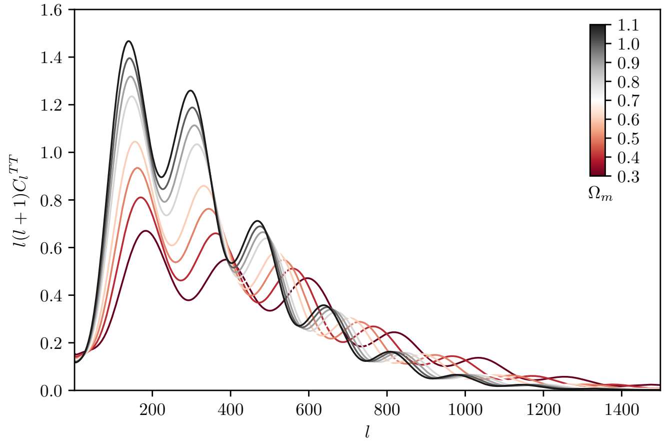
By changing **dark matter density parameter** we have



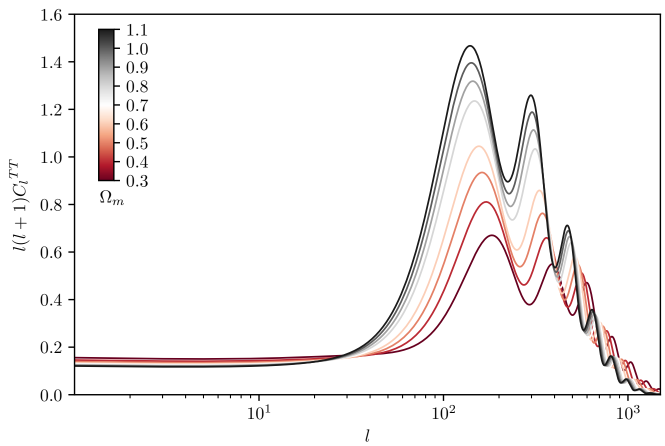
And the logarithmic plot looks like this



By changing **matter density parameter** we have



And the logarithmic plot looks like this



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