UNIVERSITY OF MARYLAND

ENPM667(Controls)

Formal Report

Project

Non-linear Control

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Contents

1 Dynamics(Question A)

1.1 Lagrangian

Potential Energy

The potential energy associated with m_1,M and m_2 are written obtained as follows. Setting a reference point of U= 0 at the origin we get

$$V_{m_1} = -m_1 * l_1 cos(\theta_1(t)) \tag{1}$$
$$V_{m_2} = -m_2 * l_2 cos(\theta_2(t)) \tag{2}$$
$$V_M = 0 \tag{3}$$
$$V = V_{m_1} + V_{m_2} + V_M \tag{4}$$

Kinetic Energy

The Kinetic energy associated with m_1,M and m_2 are written obtained as follows

$$T_{m_1} = \frac{1}{2}m_1(\frac{\mathrm{d}x(t)}{\mathrm{d}t})^2 + \frac{1}{2}m_1 l_1^2(\frac{\mathrm{d}\theta_1}{\mathrm{d}t})^2 - m_1 l_1 \cos\theta_1(t)\frac{\mathrm{d}\theta_1}{\mathrm{d}t}\frac{\mathrm{d}x}{\mathrm{d}t} \tag{5}$$
$$T_{m_2} = \frac{1}{2}m_2(\frac{\mathrm{d}x(t)}{\mathrm{d}t})^2 + \frac{1}{2}m_2 l_2^2(\frac{\mathrm{d}\theta_2}{\mathrm{d}t})^2 - m_2 l_2 \cos\theta_2(t)\frac{\mathrm{d}\theta_2}{\mathrm{d}t}\frac{\mathrm{d}x}{\mathrm{d}t} \tag{6}$$
$$T_M = \frac{1}{2}M(\frac{\mathrm{d}x}{\mathrm{d}t})^2 \tag{7}$$
$$T = T_{m_1} + T_{m_2} + T_M \tag{8}$$

Lagrangian

The lagrangian is calculated as $T - V$ and is obtained as follows

$$L = T - V \tag{9}$$
$$= \frac{1}{2}m_1(\frac{\mathrm{d}x(t)}{\mathrm{d}t})^2 + \frac{1}{2}m_1 l_1^2(\frac{\mathrm{d}\theta_1}{\mathrm{d}t})^2 - m_1 l_1 \cos\theta_1(t)\frac{\mathrm{d}\theta_1}{\mathrm{d}t}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{1}{2}m_2(\frac{\mathrm{d}x(t)}{\mathrm{d}t})^2 + \frac{1}{2}m_2 l_2^2(\frac{\mathrm{d}\theta_2}{\mathrm{d}t})^2 - m_2 l_2 \cos\theta_2(t)\frac{\mathrm{d}\theta_2}{\mathrm{d}t}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{1}{2}M(\frac{\mathrm{d}x}{\mathrm{d}t})^2 + m_1 * l_1 cos(\theta_1(t)) + m_2 * l_2 cos(\theta_2(t)) \tag{10}$$

1.2 Equation of Motion

$$0 = -m_1 l_1 \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} \cos\theta_1(t) + m_1 l_1^2 \frac{\mathrm{d}^2\theta_1}{\mathrm{d}t^2} + mgl_1 \sin\theta_1(t) \tag{11}$$

$$0 = -m_2 l_2 \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} \cos\theta_2(t) + m_2 l_2^2 \frac{\mathrm{d}^2\theta_2}{\mathrm{d}t^2} + mgl_2 2\sin\theta_2(t) \tag{12}$$

$$F = \frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2}[M + m_1 + m_2] - m_1 l_1 [\frac{\mathrm{d}^2\theta_1(t)}{\mathrm{d}t^2} cos(\theta_1(t)) - \sin\theta_1(t)(\frac{\mathrm{d}\theta_1}{\mathrm{d}t})^2] - m_2 l_2 [\frac{\mathrm{d}^2\theta_2}{\mathrm{d}t^2}\cos\theta_2(t) - \sin\theta_2(t)(\frac{\mathrm{d}\theta_2}{\mathrm{d}t})^2]. \tag{13}$$

NB.Refer to Appendix(Code for Dynamics) on how this was obtained

The equations obtained above can be further simplified to obtain the equation stated below

Simplification

Make $\frac{\mathrm{d}^2\theta_2(t)}{dt^2}$ and $\frac{\mathrm{d}^2\theta_1(t)}{dt^2}$ the subject of equations 11 and 12 respectively. Then substitute into equation 13 to get the equation below.

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{-\frac{g}{2}[m_1 \sin(2\theta_1(t)) + m_2 sin(2\theta_2(t))] - m_1 l_1 \sin(\theta_1(t))(\frac{\mathrm{d}\theta_1(t)}{\mathrm{d}t})^2 - m_2 l_2 \sin(\theta_2(t))(\frac{\mathrm{d}\theta_2}{\mathrm{d}t})^2 + F}{M + m_1 \sin^2(\theta_1(t)) + m_2 \sin^2(\theta_2(t))} \tag{14}$$

Resubstitue the equation above into equation 11 and 12 to get the following.

$$\frac{\mathrm{d}^2\theta_1}{\mathrm{d}t^2} = \frac{\frac{cos(\theta_1)}{l_1}(F - \frac{m_1 g \sin(2\theta_1)}{2} - \frac{m_2 g \sin(2\theta_2)}{2} - m_1 l_1 \sin\theta_1(\frac{\mathrm{d}\theta_1}{\mathrm{d}t})^2 - m_2 l_2 \sin\theta_2(\frac{\mathrm{d}\theta_2}{\mathrm{d}t})^2)}{M + m_1 \sin^2(\theta_1(t)) + m_2 \sin^2(\theta_2(t))} \tag{15}$$

$$\frac{\mathrm{d}^2\theta_2}{\mathrm{d}t^2} = \frac{\frac{cos(\theta_2)}{l_2}(F - \frac{m_1 g \sin(2\theta_1)}{2} - \frac{m_2 g \sin(2\theta_2)}{2} - m_1 l_1 \sin\theta_1(\frac{\mathrm{d}\theta_1}{\mathrm{d}t})^2 - m_2 l_2 \sin\theta_2(\frac{\mathrm{d}\theta_2}{\mathrm{d}t})^2)}{M + m_1 \sin^2(\theta_1(t)) + m_2 \sin^2(\theta_2(t))} \tag{16}$$

2 Linearization (Question B)

A non linear system,$F(X,U)$ with state space euation $\frac{\mathrm{d}X}{\mathrm{d}t} = F(X,U)$ can be linearized with the equation stated below.

$$\delta_x = A_F \frac{\mathrm{d}x(t)}{\mathrm{d}t} + B_F \frac{\mathrm{d}x(t)}{\mathrm{d}t} \tag{17}$$

where $A_F = \nabla_x F$,$B_F = \nabla_u F$, U(input) and X(State)

Given the equation above, the following state space is chosen for the Dynamic problem.

$$X = \begin{bmatrix} x & \frac{\mathrm{d}x}{\mathrm{d}t} & \theta_1 & \frac{\mathrm{d}\theta_1}{\mathrm{d}t} & \theta_2 & \frac{\mathrm{d}\theta_2}{\mathrm{d}t} \end{bmatrix}^T \tag{18}$$

Therefore, the state space equation can be written as

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ \frac{\mathrm{d}x}{\mathrm{d}t} \\ \theta_1 \\ \frac{\mathrm{d}\theta_1}{\mathrm{d}t} \\ \theta_2 \\ \frac{\mathrm{d}\theta_2}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} f_1(X,U) \\ f_2(X,U) \\ f_3(X,U) \\ f_4(X,U) \\ f_5(X,U) \\ f_6(X,U) \end{bmatrix} = F \tag{19}$$

Based on equation above, the following can be noted

$$f_1(X,U) = \frac{\mathrm{d}x}{\mathrm{d}t} \tag{20}$$

$$f_2(X,U) = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} \quad \text{Refer to Dynamics for equation} \tag{21}$$

$$f_3(X,U) = \frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{22}$$

$$f_4(X,U) = \frac{\mathrm{d}^2\theta_1}{\mathrm{d}t^2} \quad \text{Refer to Dynamics for equation} \tag{23}$$

$$f_5(X,U) = \frac{\mathrm{d}\theta_2}{\mathrm{d}t} \tag{24}$$

$$f_6(x,U) = \frac{\mathrm{d}^2\theta_2}{\mathrm{d}t^2} \quad \text{Refer to Dynamics for equation} \tag{25}$$

$$\tag{26}$$

From $A_F = \nabla_x F = \frac{\partial F}{\partial X}$ and $B_F = \nabla_u F = \frac{\partial F}{\partial U}$ we get

$$A_F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{m_1 g}{M} & 0 & -\frac{m_2 g}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{g}{l_1}(1+\frac{m_1}{M}) & 0 & \frac{-m_2 g}{l_1 M} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-m_1 g}{l_2 M} & 0 & -\frac{g}{l_1}(1+\frac{m_2}{M}) & 0 \end{bmatrix} \quad B_F = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix} \tag{27}$$

Output

Output of langrange Dynamics

```
>> proj

V(t) =

- g*l1*m1*cos(q1(t)) - g*l2*m2*cos(q2(t))


L(t) =

(M*diff(q3(t), t)^2)/2 + (m1*diff(q3(t), t)^2)/2 + (m2*diff(q3(t), t)^2)/2 + (l1^2*m1*diff(q1(t), t)^2)/2 +
(l2^2*m2*diff(q2(t), t)^2)/2 + g*l1*m1*cos(q1(t)) + g*l2*m2*cos(q2(t)) - l1*m1*cos(q1(t))*diff(q1(t), t)*diff(q3(t),
t) - l2*m2*cos(q2(t))*diff(q2(t), t)*diff(q3(t), t)


eqn1(t) =

l1^2*m1*q1_dd(t) - l1*m1*cos(q1(t))*q3_dd(t) + g*l1*m1*sin(q1(t)) == 0


eqn2(t) =

l2^2*m2*q2_dd(t) - l2*m2*cos(q2(t))*q3_dd(t) + g*l2*m2*sin(q2(t)) == 0


eqn3(t) =

l1*m1*sin(q1(t))*q1_d(t)^2 + l2*m2*sin(q2(t))*q2_d(t)^2 + M*q3_dd(t) + m1*q3_dd(t) + m2*q3_dd(t) -
l1*m1*cos(q1(t))*q1_dd(t) - l2*m2*cos(q2(t))*q2_dd(t) == F
```

Figure 1: lagrange Dynamics

3 Controllability (Question C)

The controllability matrix,C, is a 6x6 matrix and is controllable when

$$Det(C) \neq 0 \tag{28}$$

Condition for Controllability

$$Det(C) = -\frac{(g^6 l_1^2 - 2g^6 l_1 l_2 + g^6 l_2^2)}{M^6 l_1^6 l_2^6} \tag{29}$$

$$Det(C) = -\frac{g^6}{M^6 l_1^6 l_2^6}(l_1 - l_2)^2 \tag{30}$$

Therefore, Linearized system is controllable when $l_1 \neq l_2$
Refer to code for information on these where obtained

4 FeedBack and Linear Quadratic Regulator (Question D)

Linearized System with Feedback

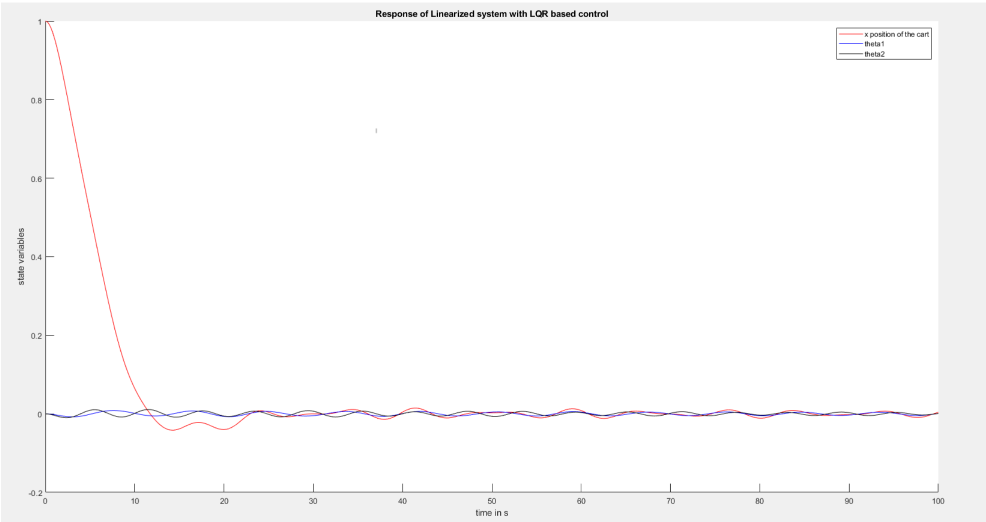

Figure 2: Linearized System with Feedback

LQR Non-Linear System

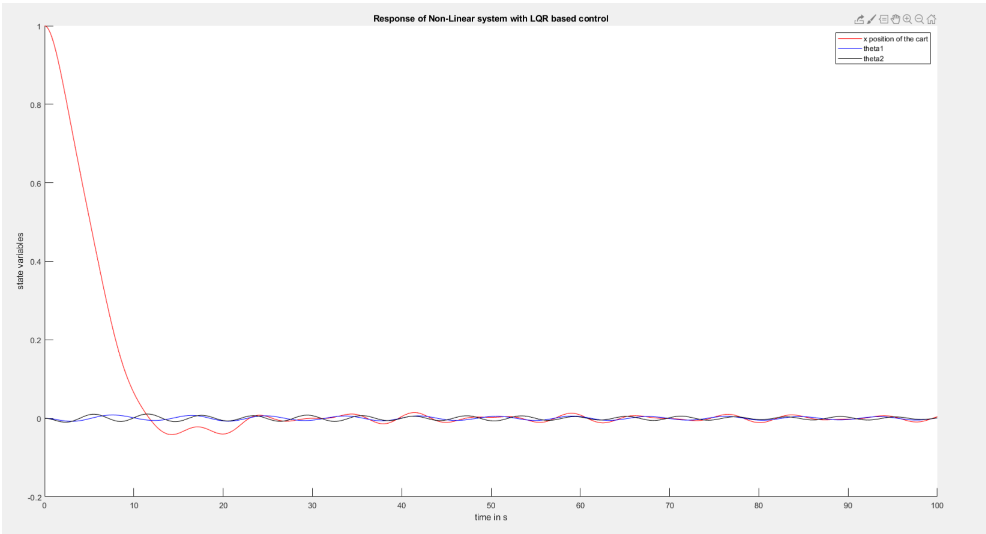

Figure 3: LQR Non linear System

Refer to code for information on how these where obtained

5 Observability (Question E)

Observability x(t)

ob1 =

1.0000	0	0	0	0	0
0	1.0000	0	0	0	0
0	0	-1.0000	0	0.6500	0
0	0	0	-1.0000	0	0.6500
0	0	-1.0000	0	1.1500	0
0	0	0	-1.0000	0	1.1500

$\theta_2(t)$ and $\theta_1(t)$

ob2 =

0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
1.0000	1.0000	0	0	-0.5500	-0.6500	0	0	0.3075
0	0	1.0000	1.0000	0	0	-0.5500	-0.6500	0
0	1.0000	0	0	-0.0500	-1.1500	0	0	0.0825
0	0	0	1.0000	0	0	-0.0500	-1.1500	0

Observability x(t) and $\theta_2(t)$

ob3 =

1.0000	0	0	0	0	0	0	0	0	0	0	0
0	0	1.0000	0	0	0	0	0	0	0	0	0
0	0	0	0	-1.0000	-0.1000	0	0	0.6500	0.1650	0	0
0	0	0	0	0	0	-1.0000	-0.1000	0	0	0.6500	0.1650
0	1.0000	0	0	-1.0000	-1.1000	0	0	1.1500	1.2150	0	0
0	0	0	1.0000	0	0	-1.0000	-1.1000	0	0	1.1500	1.2150

Observability x(t), $\theta_2(t)$ and $\theta_1(t)$

```
ob4 =

    1.0000         0         0         0         0         0         0         0         0         0         0         0         0         0         0         0         0         0
         0         0         0    1.0000         0         0         0         0         0         0         0         0         0         0         0         0         0         0
         0    1.0000         0         0         0         0   -1.0000   -0.5500   -0.1000         0         0         0    0.6500    0.3075    0.1650         0         0         0
         0         0         0         0    1.0000         0         0         0         0   -1.0000   -0.5500   -0.1000         0         0         0    0.6500    0.3075    0.1650
         0         0    1.0000         0         0         0   -1.0000   -0.0500   -1.1000         0         0         0    1.1500    0.0825    1.2150         0         0         0
         0         0         0         0         0    1.0000         0         0         0   -1.0000   -0.0500   -1.1000         0         0         0    1.1500    0.0825    1.2150
```

Observability x(t), $\theta_2(t)$ and $\theta_1(t)$

```
observability_of_C1 =

     6

Rank is 6. The system is observable for output x(t)

observability_of_C2 =

     4

Rank is 4. The system is not observable for output (t1(t),t2(t))

observability_of_C3 =

     6

The system is observable for output (x(t),t2(t))

observability_of_C4 =

     6

Rank is 6.The system is observable for output (x(t),t1(t),t2(t))
```

Refer to code for information on these where generated

6 Luenberger Observer (Question F)

State x(t) Linear Response

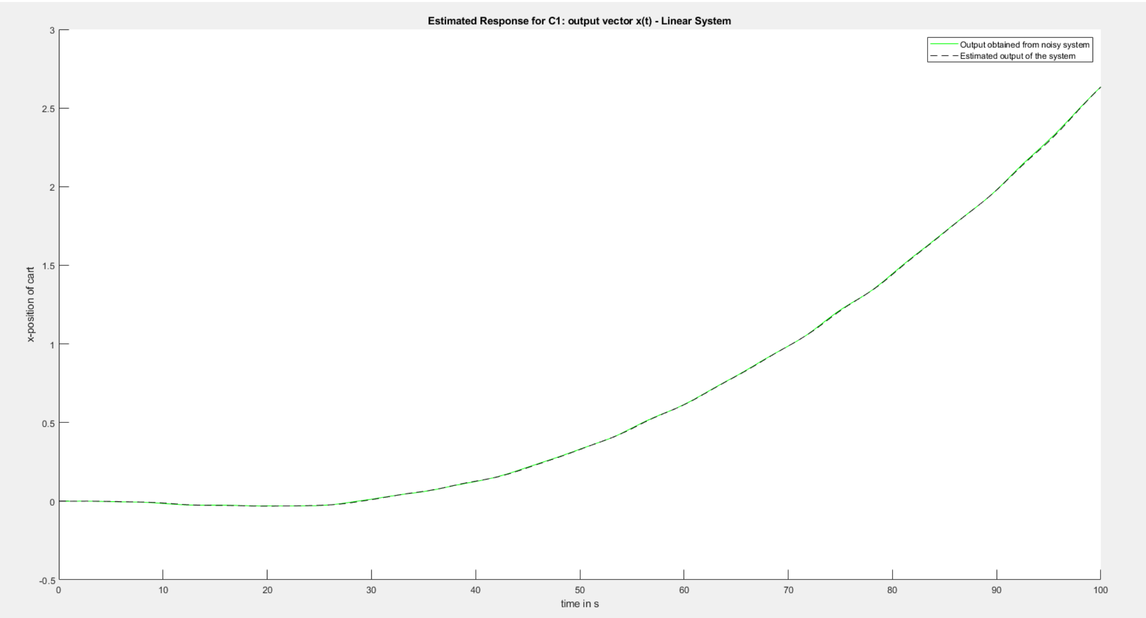

Figure 4: State x(t) Linear Response

State x(t) Non Linear Response

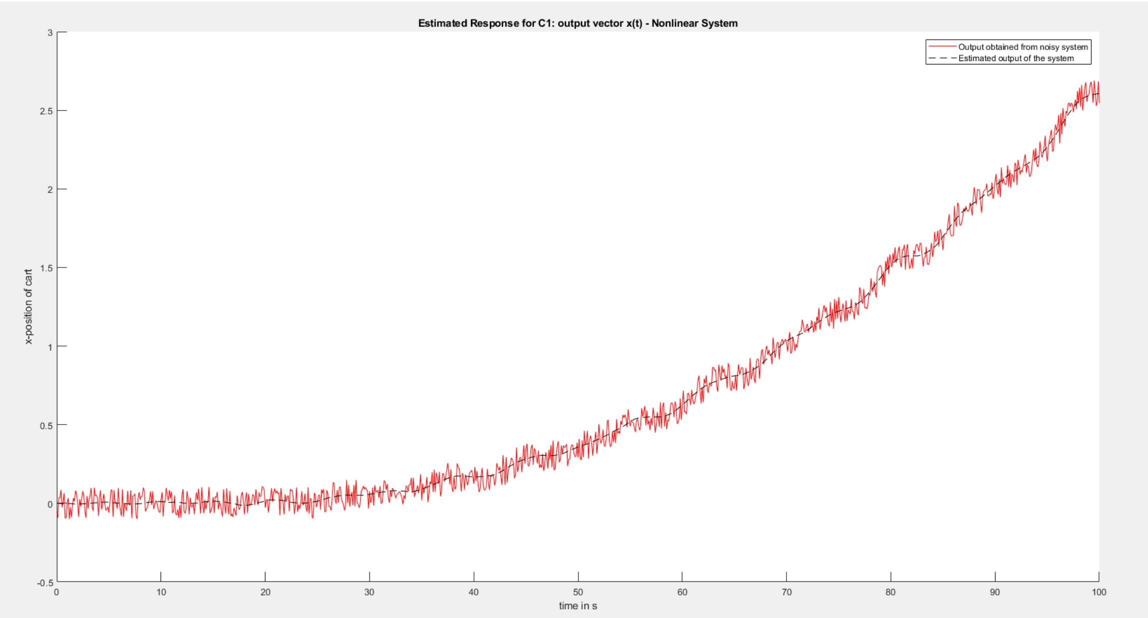

Figure 5: State x(t) Non Linear Response

State x(t) and $\theta_2(t)$ Linear Response

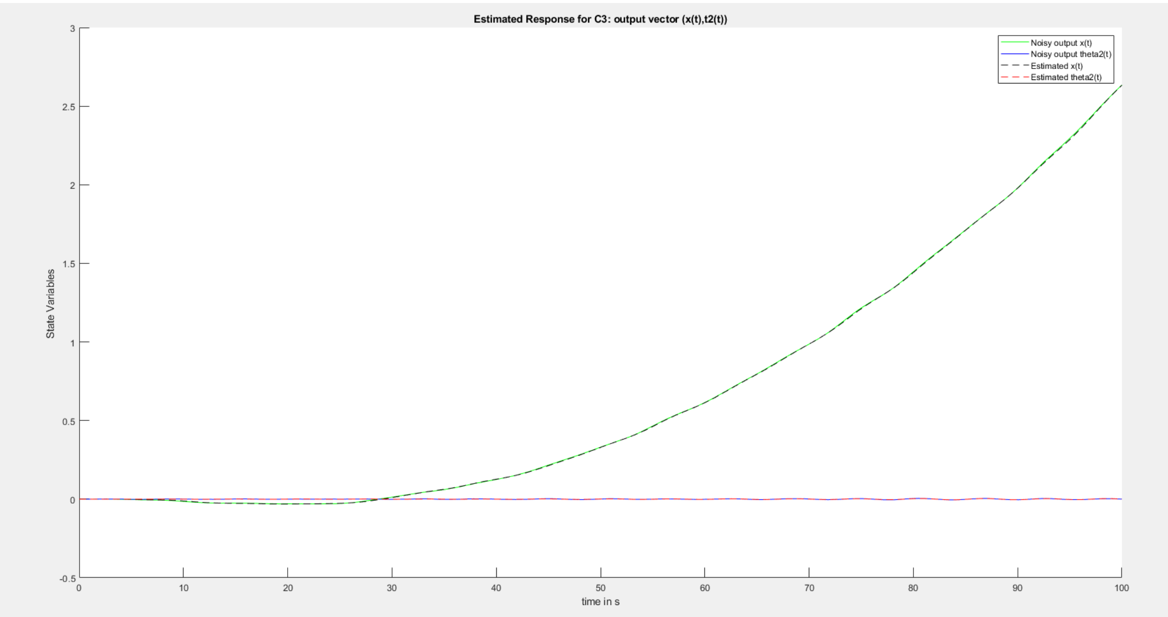

Figure 6: State x(t) and $\theta_2(t)$ Linear Response

State x(t) and $\theta_2(t)$ Non-Linear Response

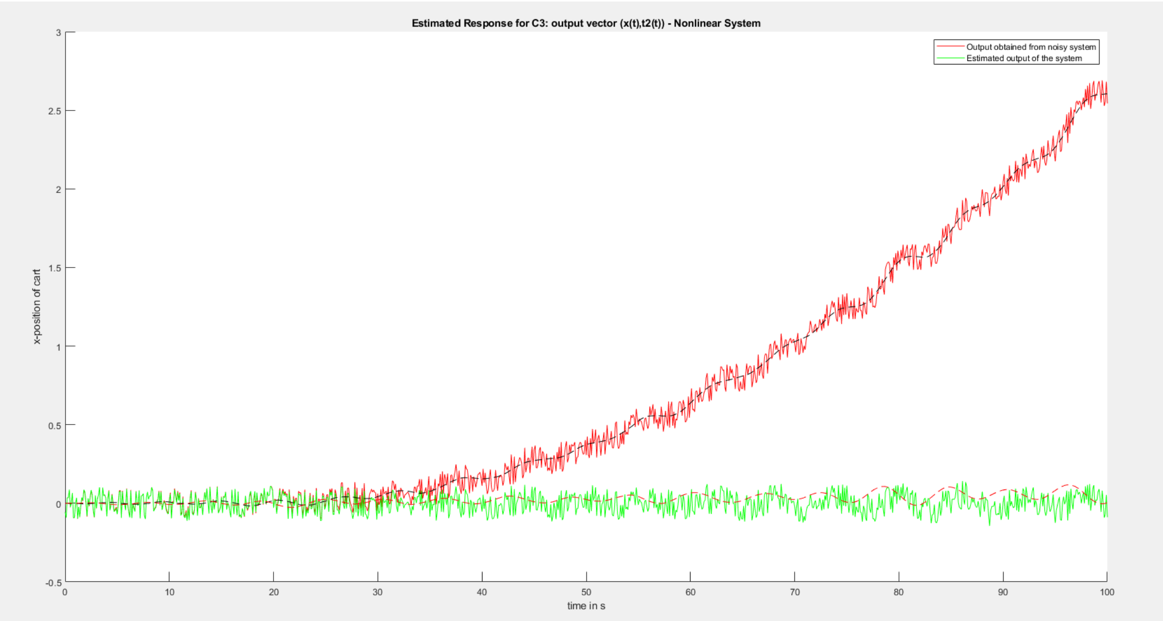

Figure 7: State x(t) and $\theta_2(t)$ Non-Linear Response

State x(t),$\theta_1(t)$ and $\theta_2(t)$ Linear Response

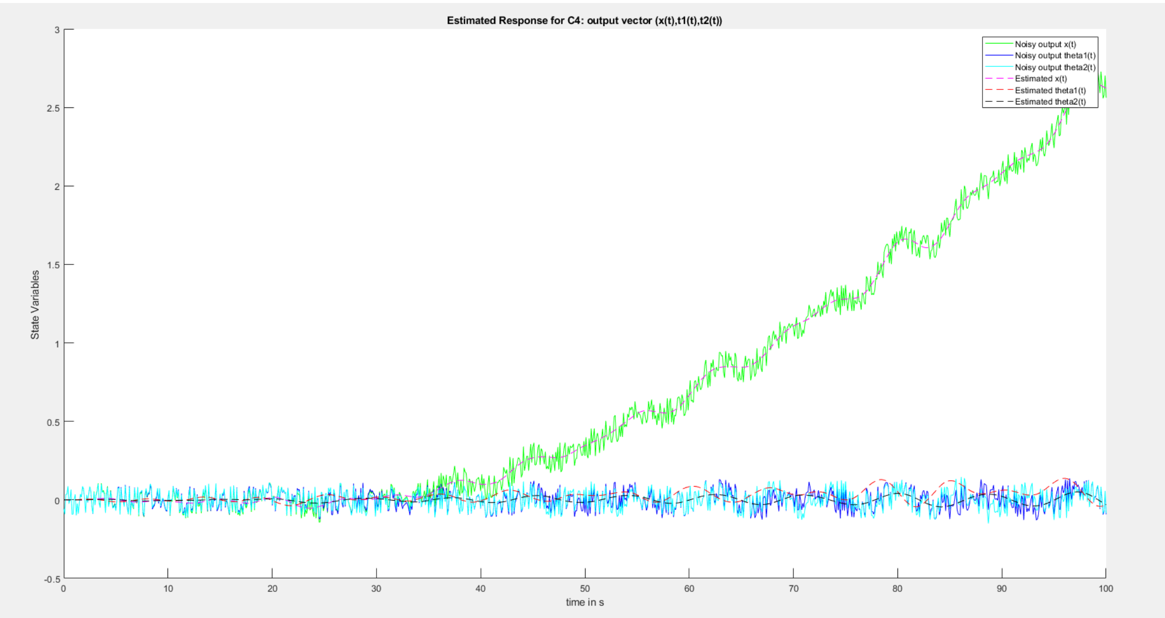

Figure 8: State x(t),$\theta_1 t()$ and $\theta_2(t)$ Linear Response

State x(t),$\theta_1(t)$ and $\theta_2(t)$ Non-Linear Response

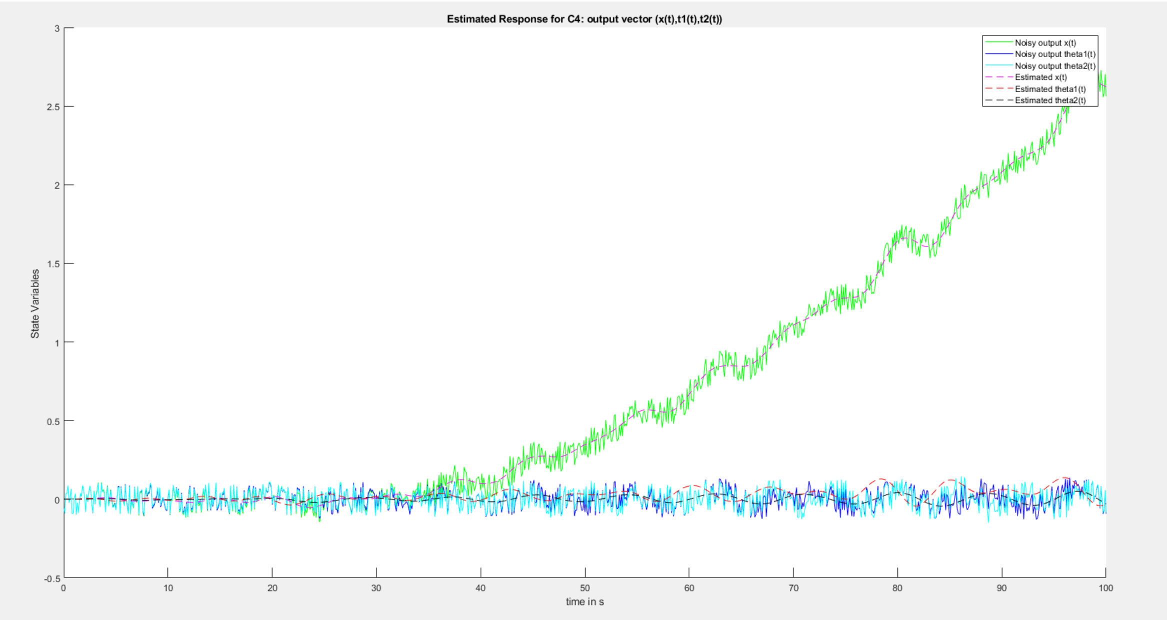

Figure 9: State x(t),$\theta_1 t()$ and $\theta_2(t)$ Non-Linear Response

Refer to code for information on how these where obtained

7 LQG Non linear

State x(t),$\theta_1(t)$ and $\theta_2(t)$ Non-Linear Response

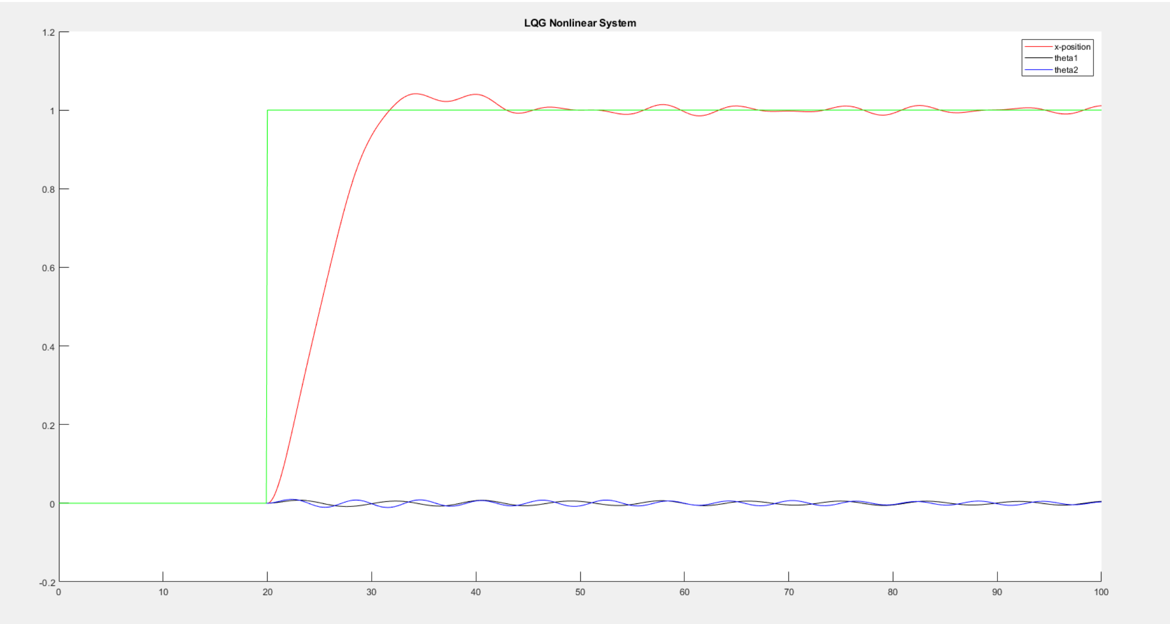

Figure 10: LQG Non linear

Refer to Code for information on how this was achieved

8 Appendix

8.1 Code for Dynamics Calculations

Output

Output of langrange Dynamics

```
>> proj

V(t) =

- g*l1*m1*cos(q1(t)) - g*l2*m2*cos(q2(t))


L(t) =

(M*diff(q3(t), t)^2)/2 + (m1*diff(q3(t), t)^2)/2 + (m2*diff(q3(t), t)^2)/2 + (l1^2*m1*diff(q1(t), t)^2)/2 +
(l2^2*m2*diff(q2(t), t)^2)/2 + g*l1*m1*cos(q1(t)) + g*l2*m2*cos(q2(t)) - l1*m1*cos(q1(t))*diff(q1(t), t)*diff(q3(t),
t) - l2*m2*cos(q2(t))*diff(q2(t), t)*diff(q3(t), t)


eqn1(t) =

l1^2*m1*q1_dd(t) - l1*m1*cos(q1(t))*q3_dd(t) + g*l1*m1*sin(q1(t)) == 0


eqn2(t) =

l2^2*m2*q2_dd(t) - l2*m2*cos(q2(t))*q3_dd(t) + g*l2*m2*sin(q2(t)) == 0


eqn3(t) =

l1*m1*sin(q1(t))*q1_d(t)^2 + l2*m2*sin(q2(t))*q2_d(t)^2 + M*q3_dd(t) + m1*q3_dd(t) + m2*q3_dd(t) -
l1*m1*cos(q1(t))*q1_dd(t) - l2*m2*cos(q2(t))*q2_dd(t) == F
```

Figure 11: lagrange Dynamics

8.2 Code for LQR

8.3 Code for LQG