



ENPM667(CONTROLS)

FORMAL REPORT

PROJECT

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# Non-linear Control

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# 1 Dynamics(Question A)

## 1.1 Lagrangian

### Potential Energy

The potential energy associated with  $m_1, M$  and  $m_2$  are written obtained as follows. Setting a reference point of  $U = 0$  at the origin we get

$$V_{m_1} = -m_1 * l_1 \cos(\theta_1(t)) \quad (1)$$

$$V_{m_2} = -m_2 * l_2 \cos(\theta_2(t)) \quad (2)$$

$$V_M = 0 \quad (3)$$

$$V = V_{m_1} + V_{m_2} + V_M \quad (4)$$

### Kinetic Energy

The Kinetic energy associated with  $m_1, M$  and  $m_2$  are written obtained as follows

$$T_{m_1} = \frac{1}{2}m_1\left(\frac{dx(t)}{dt}\right)^2 + \frac{1}{2}m_1l_1^2\left(\frac{d\theta_1}{dt}\right)^2 - m_1l_1 \cos \theta_1(t) \frac{d\theta_1}{dt} \frac{dx}{dt} \quad (5)$$

$$T_{m_2} = \frac{1}{2}m_2\left(\frac{dx(t)}{dt}\right)^2 + \frac{1}{2}m_2l_2^2\left(\frac{d\theta_2}{dt}\right)^2 - m_2l_2 \cos \theta_2(t) \frac{d\theta_2}{dt} \frac{dx}{dt} \quad (6)$$

$$T_M = \frac{1}{2}M\left(\frac{dx}{dt}\right)^2 \quad (7)$$

$$T = T_{m_1} + T_{m_2} + T_M \quad (8)$$

### Lagrangian

The lagrangian is calculated as  $T - V$  and is obtained as follows

$$L = T - V \quad (9)$$

$$\begin{aligned} &= \frac{1}{2}m_1\left(\frac{dx(t)}{dt}\right)^2 + \frac{1}{2}m_1l_1^2\left(\frac{d\theta_1}{dt}\right)^2 - m_1l_1 \cos \theta_1(t) \frac{d\theta_1}{dt} \frac{dx}{dt} + \\ &\quad \frac{1}{2}m_2\left(\frac{dx(t)}{dt}\right)^2 + \frac{1}{2}m_2l_2^2\left(\frac{d\theta_2}{dt}\right)^2 - m_2l_2 \cos \theta_2(t) \frac{d\theta_2}{dt} \frac{dx}{dt} + \\ &\quad \frac{1}{2}M\left(\frac{dx}{dt}\right)^2 + m_1 * l_1 \cos(\theta_1(t)) + m_2 * l_2 \cos(\theta_2(t)) \end{aligned} \quad (10)$$

## 1.2 Equation of Motion

$$0 = -m_1 l_1 \frac{d^2 x}{dt^2} \cos \theta_1(t) + m_1 l_1^2 \frac{d^2 \theta_1}{dt^2} + m g l_1 \sin \theta_1(t) \quad (11)$$

$$0 = -m_2 l_2 \frac{d^2 x}{dt^2} \cos \theta_2(t) + m_2 l_2^2 \frac{d^2 \theta_2}{dt^2} + m g l_2 \sin \theta_2(t) \quad (12)$$

$$F = \frac{d^2 x(t)}{dt^2} [M + m_1 + m_2] - m_1 l_1 \left[ \frac{d^2 \theta_1(t)}{dt^2} \cos(\theta_1(t)) - \sin \theta_1(t) \left( \frac{d\theta_1}{dt} \right)^2 \right] \\ - m_2 l_2 \left[ \frac{d^2 \theta_2}{dt^2} \cos \theta_2(t) - \sin \theta_2(t) \left( \frac{d\theta_2}{dt} \right)^2 \right]. \quad (13)$$

**NB.Refer to Appendix(Code for Dynamics) on how this was obtained**

The equations obtained above can be further simplified to obtain the equation stated below

### Simplification

Make  $\frac{d^2 \theta_2(t)}{dt^2}$  and  $\frac{d^2 \theta_1(t)}{dt^2}$  the subject of equations 11 and 12 respectively. Then substitute into equation 13 to get the equation below.

$$\frac{d^2 x}{dt^2} = \frac{-\frac{g}{2} [m_1 \sin(2\theta_1(t)) + m_2 \sin(2\theta_2(t))] - m_1 l_1 \sin(\theta_1(t)) \left( \frac{d\theta_1(t)}{dt} \right)^2 - m_2 l_2 \sin(\theta_2(t)) \left( \frac{d\theta_2}{dt} \right)^2 + F}{M + m_1 \sin^2(\theta_1(t)) + m_2 \sin^2(\theta_2(t))} \quad (14)$$

Resubstitue the equation above into equation 11 and 12 to get the following.

$$\frac{d^2 \theta_1}{dt^2} = \frac{\frac{\cos(\theta_1)}{l_1} \left( F - \frac{m_1 g \sin(2\theta_1)}{2} - \frac{m_2 g \sin(2\theta_2)}{2} - m_1 l_1 \sin \theta_1 \left( \frac{d\theta_1}{dt} \right)^2 - m_2 l_2 \sin \theta_2 \left( \frac{d\theta_2}{dt} \right)^2 \right)}{M + m_1 \sin^2(\theta_1(t)) + m_2 \sin^2(\theta_2(t))} \quad (15)$$

$$\frac{d^2 \theta_2}{dt^2} = \frac{\frac{\cos(\theta_2)}{l_2} \left( F - \frac{m_1 g \sin(2\theta_1)}{2} - \frac{m_2 g \sin(2\theta_2)}{2} - m_1 l_1 \sin \theta_1 \left( \frac{d\theta_1}{dt} \right)^2 - m_2 l_2 \sin \theta_2 \left( \frac{d\theta_2}{dt} \right)^2 \right)}{M + m_1 \sin^2(\theta_1(t)) + m_2 \sin^2(\theta_2(t))} \quad (16)$$

## 2 Linearization (Question B)

A non linear system,  $F(X, U)$  with state space equation  $\frac{dX}{dt} = F(X, U)$  can be linearized with the equation stated below.

$$\delta_x = A_F \frac{dx(t)}{dt} + B_F \frac{du(t)}{dt} \quad (17)$$

where  $A_F = \nabla_x F, B_F = \nabla_u F$ ,  $U$ (input) and  $X$ (State)

Given the equation above, the following state space is chosen for the Dynamic problem.

$$X = \begin{bmatrix} x & \frac{dx}{dt} & \theta_1 & \frac{d\theta_1}{dt} & \theta_2 & \frac{d\theta_2}{dt} \end{bmatrix}^T \quad (18)$$

Therefore, the state space equation can be written as

$$\frac{dX}{dt} = \frac{d}{dt} \begin{bmatrix} x \\ \frac{dx}{dt} \\ \theta_1 \\ \frac{d\theta_1}{dt} \\ \theta_2 \\ \frac{d\theta_2}{dt} \end{bmatrix} = \begin{bmatrix} f_1(X, U) \\ f_2(X, U) \\ f_3(X, U) \\ f_4(X, U) \\ f_5(X, U) \\ f_6(X, U) \end{bmatrix} = F \quad (19)$$

Based on equation above, the following can be noted

$$f_1(X, U) = \frac{dx}{dt} \quad (20)$$

$$f_2(X, U) = \frac{d^2x}{dt^2} \quad \text{Refer to Dynamics for equation} \quad (21)$$

$$f_3(X, U) = \frac{d\theta}{dt} \quad (22)$$

$$f_4(X, U) = \frac{d^2\theta_1}{dt^2} \quad \text{Refer to Dynamics for equation} \quad (23)$$

$$f_5(X, U) = \frac{d\theta_2}{dt} \quad (24)$$

$$f_6(x, U) = \frac{d^2\theta_2}{dt^2} \quad \text{Refer to Dynamics for equation} \quad (25)$$

$$(26)$$

From  $A_F = \nabla_x F = \frac{\partial F}{\partial X}$  and  $B_F = \nabla_u F = \frac{\partial F}{\partial U}$  we get

$$A_F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{m_1 g}{M} & 0 & -\frac{m_2 g}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{g}{l_1} \left(1 + \frac{m_1}{M}\right) & 0 & \frac{-m_2 g}{l_1 M} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-m_1 g}{l_2 M} & 0 & -\frac{g}{l_1} \left(1 + \frac{m_2}{M}\right) & 0 \end{bmatrix} \quad B_F = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{M l_1} \\ 0 \\ \frac{1}{M l_2} \end{bmatrix} \quad (27)$$

**Output**

## Output of langrange Dynamics

```

>> proj
V(t) =
- g*l1*m1*cos(q1(t)) - g*l2*m2*cos(q2(t))

L(t) =
(M*diff(q3(t), t)^2)/2 + (m1*diff(q3(t), t)^2)/2 + (m2*diff(q3(t), t)^2)/2 + (l1^2*m1*diff(q1(t), t)^2)/2 +
(l2^2*m2*diff(q2(t), t)^2)/2 + g*l1*m1*cos(q1(t)) + g*l2*m2*cos(q2(t)) - l1*m1*cos(q1(t))*diff(q1(t), t)*diff(q3(t),
t) - l2*m2*cos(q2(t))*diff(q2(t), t)*diff(q3(t), t)

eqn1(t) =
l1^2*m1*q1_dd(t) - l1*m1*cos(q1(t))*q3_dd(t) + g*l1*m1*sin(q1(t)) == 0

eqn2(t) =
l2^2*m2*q2_dd(t) - l2*m2*cos(q2(t))*q3_dd(t) + g*l2*m2*sin(q2(t)) == 0

eqn3(t) =
l1*m1*sin(q1(t))*q1_d(t)^2 + l2*m2*sin(q2(t))*q2_d(t)^2 + M*q3_dd(t) + m1*q3_dd(t) + m2*q3_dd(t) -
l1*m1*cos(q1(t))*q1_dd(t) - l2*m2*cos(q2(t))*q2_dd(t) == F

```

Figure 1: lagrange Dynamics

### 3 Controllability (Question C)

The controllability matrix,  $C$ , is a  $6 \times 6$  matrix and is controllable when

$$\text{Det}(C) \neq 0 \quad (28)$$

#### Condition for Controllability

$$\text{Det}(C) = -\frac{(g^6 l_1^2 - 2g^6 l_1 l_2 + g^6 l_2^2)}{M^6 l_1^6 l_2^6} \quad (29)$$

$$\text{Det}(C) = -\frac{g^6}{M l_1^6 l_2^6} (l_1 - l_2)^2 \quad (30)$$

Therefore, Linearized system is controllable when  $l_1 \neq l_2$

**Refer to code for information on these where obtained**

## 4 Feedback and Linear Quadratic Regulator (Question D)

### Linearized System with Feedback

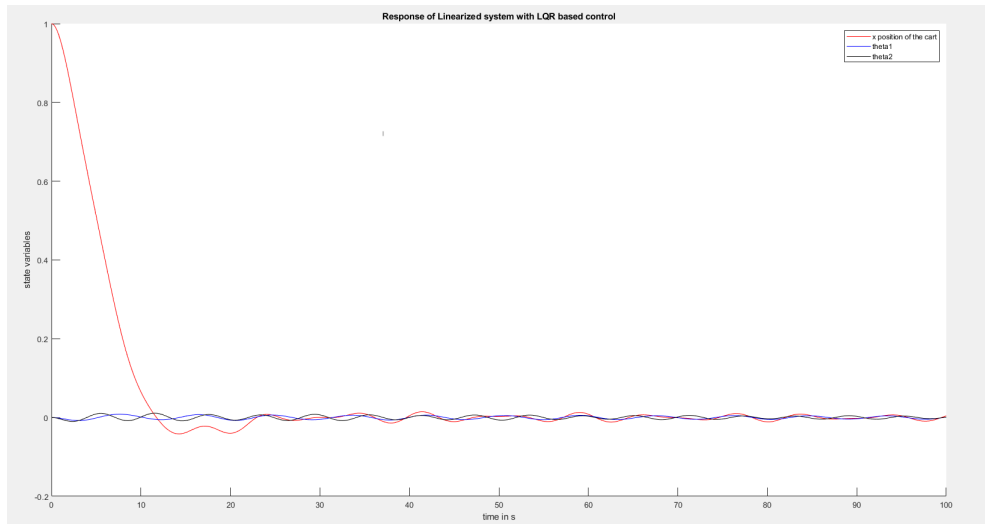


Figure 2: Linearized System with Feedback

### LQR Non-Linear System

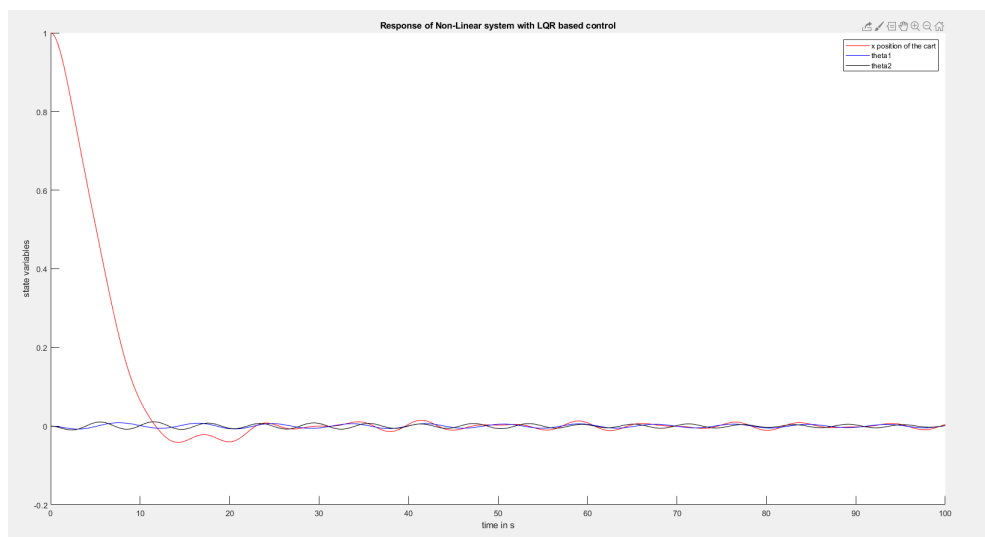


Figure 3: LQR Non linear System



Refer to code for information on how these were obtained

## 5 Observability (Question E)

## 6 Luenberger Observer (Question F)

State  $x(t)$  Linear Response

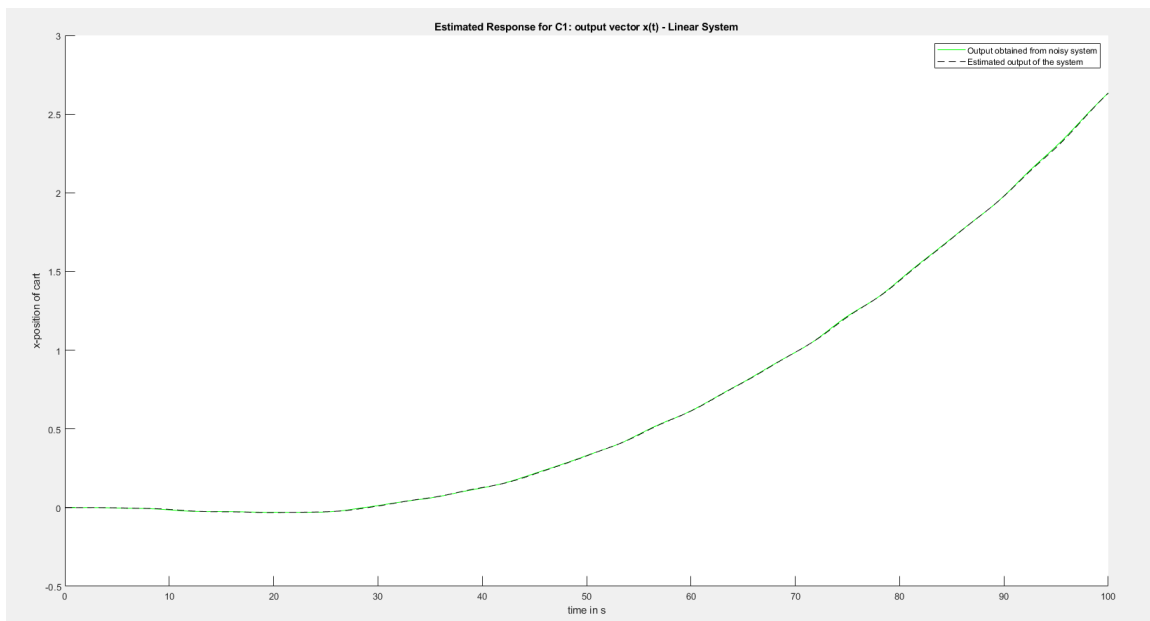
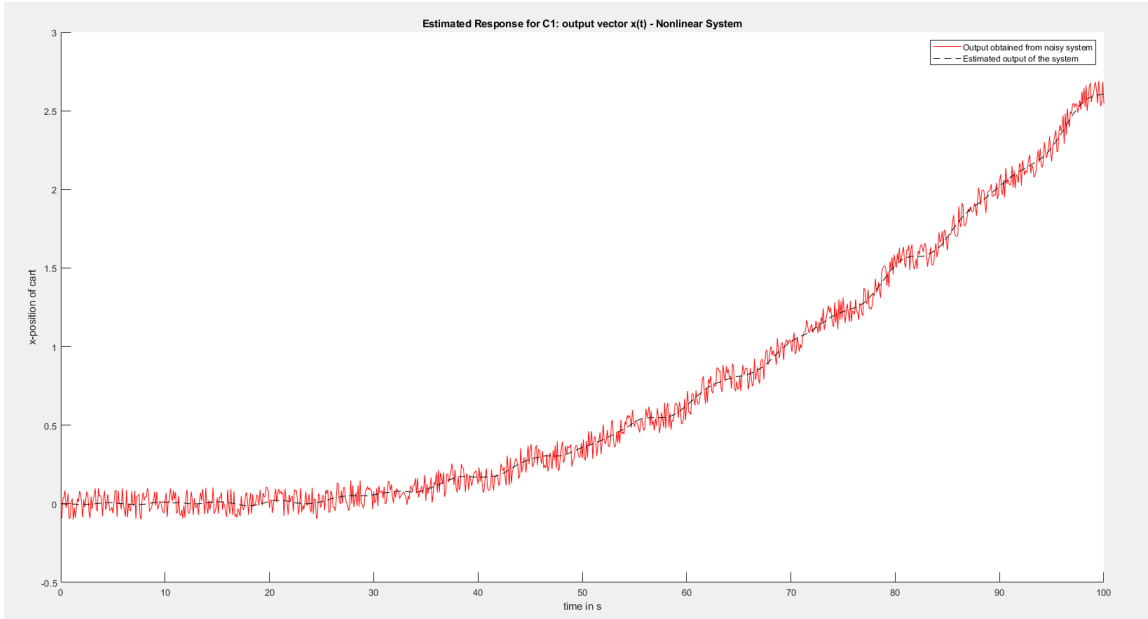
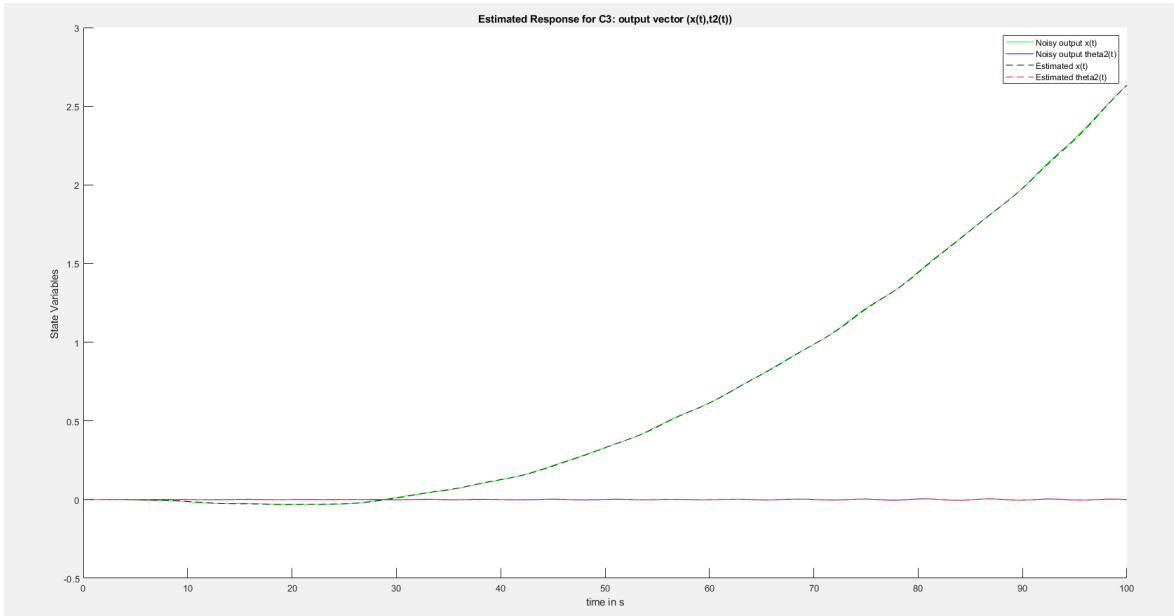
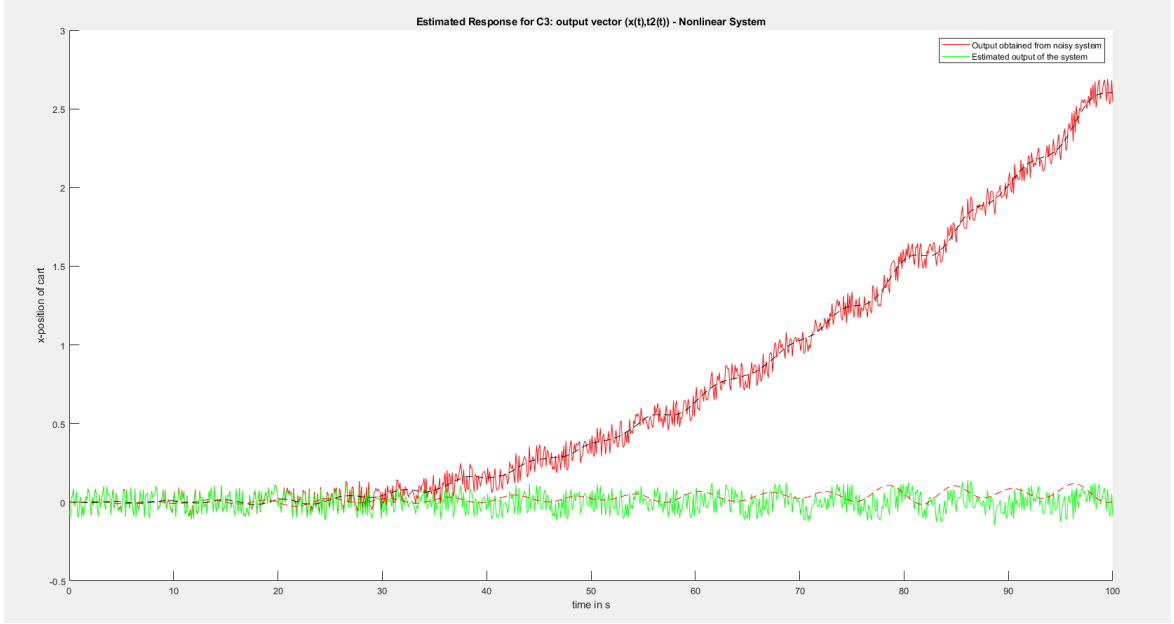
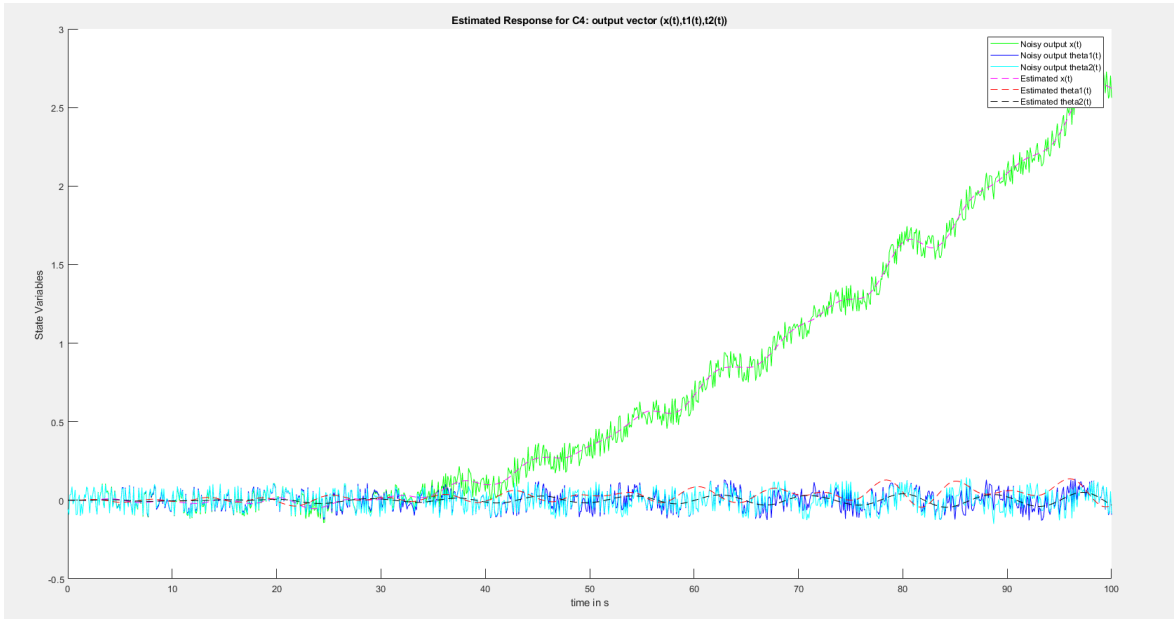
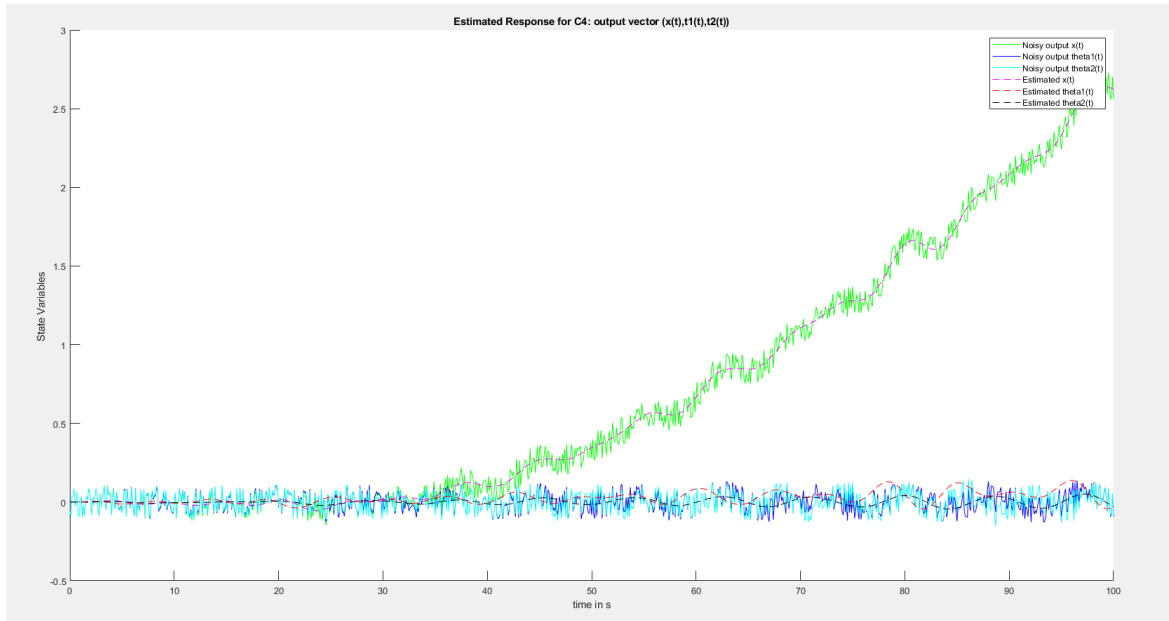


Figure 4: State  $x(t)$  Linear Response

State  $x(t)$  Non Linear ResponseFigure 5: State  $x(t)$  Non Linear ResponseState  $x(t)$  and  $\theta_2(t)$  Linear ResponseFigure 6: State  $x(t)$  and  $\theta_2(t)$  Linear Response

**State  $x(t)$  and  $\theta_2(t)$  Non-Linear Response**Figure 7: State  $x(t)$  and  $\theta_2(t)$  Non-Linear Response**State  $x(t), \theta_1(t)$  and  $\theta_2(t)$  Linear Response**Figure 8: State  $x(t), \theta_1(t)$  and  $\theta_2(t)$  Linear Response

**State  $x(t), \theta_1(t)$  and  $\theta_2(t)$  Non-Linear Response**Figure 9: State  $x(t), \theta_1(t)$  and  $\theta_2(t)$  Non-Linear Response

Refer to code for information on how these were obtained

**7 LQG (Question G)****8 Appendix****8.1 Code for Dynamics Calculations**

Output

## Output of langrange Dynamics

```
>> proj
V(t) =
- g*l1*m1*cos(q1(t)) - g*l2*m2*cos(q2(t))

L(t) =
(M*diff(q3(t), t)^2)/2 + (m1*diff(q3(t), t)^2)/2 + (m2*diff(q3(t), t)^2)/2 + (l1^2*m1*diff(q1(t), t)^2)/2 +
(l2^2*m2*diff(q2(t), t)^2)/2 + g*l1*m1*cos(q1(t)) + g*l2*m2*cos(q2(t)) - l1*m1*cos(q1(t))*diff(q1(t), t)*diff(q3(t),
t) - l2*m2*cos(q2(t))*diff(q2(t), t)*diff(q3(t), t)

eqn1(t) =
l1^2*m1*q1_dd(t) - l1*m1*cos(q1(t))*q3_dd(t) + g*l1*m1*sin(q1(t)) == 0

eqn2(t) =
l2^2*m2*q2_dd(t) - l2*m2*cos(q2(t))*q3_dd(t) + g*l2*m2*sin(q2(t)) == 0

eqn3(t) =
l1*m1*sin(q1(t))*q1_d(t)^2 + l2*m2*sin(q2(t))*q2_d(t)^2 + M*q3_dd(t) + m1*q3_dd(t) + m2*q3_dd(t) -
l1*m1*cos(q1(t))*q1_dd(t) - l2*m2*cos(q2(t))*q2_dd(t) == F
```

Figure 10: lagrange Dynamics

## 8.2 Code for LQR

## 8.3 Code for LQG