Homework 5

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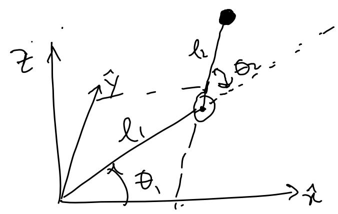
1.



$$\therefore \; r(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} (l_2 \cos\theta_2 + l_1)\cos\theta_1 \\ (l_2 \cos\theta_2 + l_1)\sin\theta_1 \\ l_2 \sin\theta_2 \end{bmatrix}$$

$$K.E = \tfrac{1}{2} M [r'(t)]^T [r'(t)]$$

$$P.E = M g\, l_2 \sin(q_2)$$

Please Refer to Code.

2

(a) $\tau = J^T(q) F$

(b) $\tau = J_v^T F$,

$$J_v \tau = J_v J_v^T F$$

$$(J_v J_v^T)^{-1} J_v \tau = F$$

Please refer to code.

3) Approach 1.

P.E

$$P.E = m_1 g\, a_{c1} \sin(q_1^*) + m_2 g (a_1 + q_2^* + a_{2c}) \sin(q_1^*).$$
$$m_3 g\big((a_1 + q_2^* + a_2)\sin(q_1^*) + a_{3c} \sin(q_3^*)\big)$$

K.E.

$$r_1(t) = \begin{bmatrix} a_{1c}\cos(q_1) \\ a_{1c}\sin q_1 \end{bmatrix} \rightsquigarrow$$ position of center of mass for joint 1

$$r_2(t) = \begin{bmatrix} (a_1 + q_2 + a_{2c})\cos(q_1) \\ (a_1 + q_2 + a_{2c})\sin(q_1) \end{bmatrix} \rightsquigarrow$$ position of center of mass for Joint 2.

$$r_3(t) = \begin{bmatrix} (a_1 + q_2 + a_2)\cos(q_1) + a_{3c}\cos(q_3) \\ (a_1 + q_2 + a_2)\sin(q_1) + a_{3c}\sin(q_3) \end{bmatrix}$$

$$\therefore \quad K.E = \frac{1}{2} m_1 [r_1'(t)]^T [r'(t)] + \frac{1}{2} m_2 [r_2'(t)]^T [r'(t)]$$
$$+ \frac{1}{2} m_3 [r'(t)]^T [r'(t)]$$

Refer to code for implementation. Code doesn't include rotational inertia of the links.

Alternative Approach2 using Christoffel Symbols.

1. $$r_1(t) = \begin{bmatrix} a_{1c}\cos(q_1) \\ a_{1c}\sin(q_1) \end{bmatrix} \quad \therefore \quad r'(t) = \begin{bmatrix} -\dot{q}_1 a_{1c}\sin(q_1) \\ \dot{q}_1 a_{1c}\cos(q_1) \end{bmatrix}$$

$$J_{vc_1} = \begin{bmatrix} -a_{1c}\sin(q_1) & 0 & 0 \\ a_{1c}\cos q_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

$$r_2(t) = \begin{bmatrix} (a_1 + q_2 + a_{2c})\cos(q_1) \\ (a_1 + q_2 + a_{2c})\sin(q_1) \end{bmatrix} \rightsquigarrow$$ position of center of mass for Joint 2.

$$r_2'(t) = \begin{bmatrix} -\dot{q}_1\sin(q_1)(a_1 + q_2 + a_{2c}) + \cos(q_1)(\dot{q}_2) \\ \dot{q}_1\cos(q_1)(a_1 + q_2 + a_{2c}) + \dot{q}_2\sin(q_1) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} & 0 & \end{bmatrix}$$

$$J_{v_{c_2}} = \begin{bmatrix} -\sin(q_1)(a_1 + q_2 + a_{2c}) & \cos(q_1) & 0 \\ \cos(q_1)(a_1 + q_2 + a_{2c}) & \sin q_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

$$r_3(t) = \begin{bmatrix} (q_1 + q_2 + a_2)\cos(q_1) + a_{3c}\cos(q_3) \\ (a_1 + q_2 + a_2)\sin(q_1) + a_{3c}\sin(q_3) \end{bmatrix}$$

$$r'_3(t) = \begin{bmatrix} -\dot{q}_1\sin(q_1)(a_1 + q_2 + a_2) + \dot{q}_2\cos(q_1) - \dot{q}_3 a_{3c}\sin(q_3) \\ \dot{q}_1\cos(q_1)(a_1 + q_2 + a_2) + \dot{q}_2\sin(q_1) + \dot{q}_3 a_{3c}\cos(q_3) \\ 0 \end{bmatrix}$$

$$J_{v_{c_3}} = \begin{bmatrix} -\sin(q_1)(a_1 + q_2 + a_2) & \cos(q_1) & -a_{3c}\sin(q_3) \\ \cos(q_1)(a_1 + q_2 + a_2) & \sin(q_1) & a_{3c}\cos(q_3) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

Translational K.E $= \frac{1}{2}\dot{q}^T \left[m_1 J_{v_{c_1}}^T J_{v_{c_1}} + m_2 J_{v_{c_2}}^T J_{v_{c_2}} + m_2 J_{v_{c_3}}^T J_{v_{c_3}} \right] \dot{q}_1$

Rotational K.E

$\omega_1 = \dot{q}_1 k \qquad \omega_2 = 0 \qquad \omega_3 = (\dot{q}_1 + \dot{q}_3)k$

$$J_{\omega_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} \quad J_{\omega_2} = \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} \quad J_{\omega_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

let

${}^1I$ -> Rotational intertia of link 1 wrt to inertia frame.

${}^2I$ -> Rotational inertia of link 2 wrt to inertia frame

${}^{1}I \rightarrow$ Rotational inertia of link 1 ...

${}^{2}I \rightarrow$ Rotational inertia of link 2 wrt to inertia frame

${}^{3}I \rightarrow$ Rotational inertia of link 3 wrt to inertia frame.

$$
{}^{i}I = \begin{bmatrix} {}^{i}I_{xx} & {}^{i}I_{xy} & {}^{i}I_{xz} \\ {}^{i}I_{yx} & {}^{i}I_{yy} & {}^{i}I_{yz} \\ {}^{i}I_{zx} & {}^{i}I_{zy} & {}^{i}I_{zz} \end{bmatrix} \quad \text{for } i = 1, 2, 3
$$

Rotational K.E

$$
K = \frac{1}{2}\dot{q}_1^T \left[J_{\omega_1}^T \, {}^{1}I \, J_{\omega_1} + J_{\omega_2}^T \, {}^{2}I \, J_{\omega_2} + J_{\omega_3}^T \, {}^{3}I \, J_{\omega_3} \right] \dot{q}_1
$$

$$
J_{\omega_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad J_{\omega_2} = \begin{bmatrix} 0 \end{bmatrix} \qquad J_{\omega_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}
$$

D

$$
D(q) = m_1 J_{v_{c_1}}^T J_{v_{c_1}} + m_2 J_{v_{c_2}}^T J_{v_{c_2}} + m_3 J_{v_{c_3}}^T J_{v_{c_3}}
$$

$$
+ J_{\omega_1}^T \, {}^{1}I \, J_{\omega_1} + J_{\omega_2}^T \, {}^{2}I \, J_{\omega_2} + J_{\omega_3}^T \, {}^{3}I \, J_{\omega_3}.
$$

$$
P = m_1 g \, a_{c_1} \sin(q_1^*) + m_2 g \left(a_1 + q_2^* + a_{2c}\right) \sin(q_1^*)
$$

$$
m_3 g \left((a_1 + q_2^* + a_2) \sin(q_1^*) + a_{3c} \sin(q_3^*) \right)
$$

Euler - lagrange Equation can be written as

$$
\sum_{j} D_{kj}(q)\ddot{q}_j + \sum_{i,j} c_{ijk}(q)\dot{q}_i \dot{q}_j + \phi_k(q) = \tau_k
$$

where $K = 1 \ldots n$ & $\phi_k = \frac{\partial P}{\partial q_{k}}$

and $C_{ijk} \quad \frac{1}{2}\left(\frac{\partial D_{kj}}{\partial q_i} + \frac{\partial D_{ki}}{\partial q_j} - \frac{\partial D_{ij}}{\partial q_k}\right)$

The christofel symbols C_{ijk}, the Inertia matrix D and Φ_k (represented as dP in the code) have been calucualted. Please refer to the code.