Poincaré Map, Floquet Theory, and Stability of Periodic Orbits

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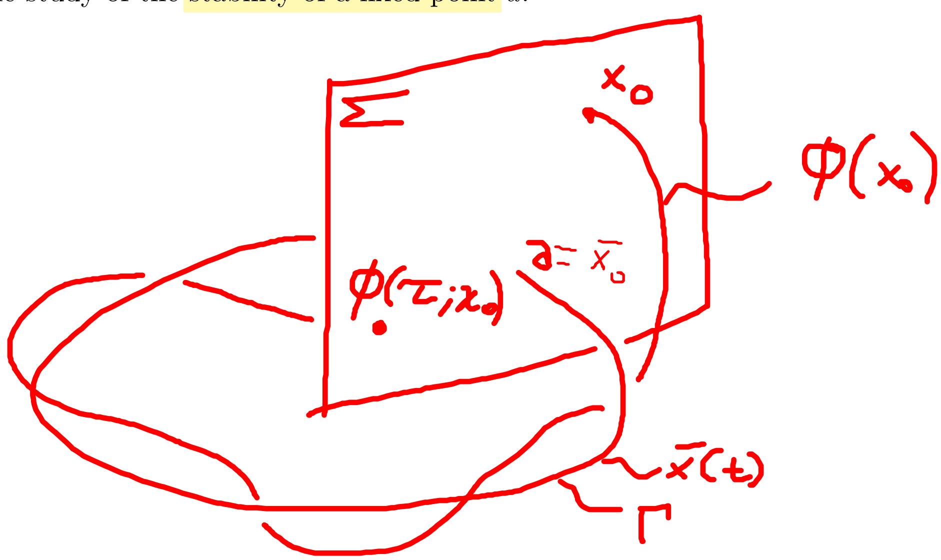
1 Poincaré Maps

Definition (Poincaré Map): Consider $\dot{x} = f(x)$ with periodic solution $\bar{x}(t)$. Construct a $(n-1)$-dimensional transversal Σ to the corresponding closed orbit Γ. Let a be the point where Γ intersects Σ. For an orbit $\phi(x_0)$ starting at $x_0 \in \Sigma$ close to a, the phase flow will return to Σ. The *first return* or *Poincaré map* $P : U \subset \Sigma \to \Sigma$ is defined by

$$P(x_0) = \phi(\tau; x_0).$$

Notice that a is a fixed point of the map P. P reduces the study of the stability of a periodic orbit $\bar{x}(t)$ to the study of the stability of a fixed point a.



Definition (Stability of Periodic Orbits): The periodic solution $\bar{x}(t)$ (the closed orbit Γ) is *stable* if for each $\epsilon > 0$, there exists a δ such that

$$||x_0 - a|| < \delta \Rightarrow ||P^n(x_0) - a|| < \epsilon.$$

Definition: The periodic solution $\bar{x}(t)$ is *asymptotically stable* if it is stable and if there exists a $\delta > 0$ such that

$$||x_0 - a|| < \delta \Rightarrow \lim_{n\to\infty} P^n(x_0) = a.$$

Example: Consider the equation of forced linear oscillations

$$\ddot{x} + \delta\dot{x} + \beta x = \gamma \cos \omega t$$

with $\delta = 2, \beta = 2, \gamma = 5, \omega = 1$.

Remarks:

- Two Poincaré maps for a periodic orbit are conjugate to each other.
- Let the linearization of the discrete map P at the fixed point a be $\frac{\partial P}{\partial x_0}(a)$. This linearized Poincaré map can be used to study the stability of the fixed point a as well as its corresponding periodic orbit.

  $P(x_0) = a + \frac{\partial P(a)}{\partial x_0}(x_0 - a)$

  $$P(x_0) - a = \frac{\partial P}{\partial x_0}(a)(x_0 - a) + O(2).$$

- Let $\lambda_1, \ldots, \lambda_{n-1}$ be the eigenvalues of this linearized map. If the moduli of all eigenvalues are small than 1, then a is stable; if the modulus of at least one eigenvalues is larger than 1, then a is unstable.
- To construct a Poincaré map analytically is not an easy task. For practical problems, numerical methods are generally needed. Poincaré sections have been widely used to study global orbit structure of low dimensional systems.

2 Floquet Theory

The stability of periodic solution can also be studied in terms of the characteristic or Floquet multiplier. We will discuss some of the major results in the Floquet theory and link them up with the Poincaré map method via

$$\frac{\partial P(a)}{\partial x_0} = \frac{\partial \phi(T;\tilde{a})}{\partial \tilde{x}_0} = \Phi(T). \tag{2.0.1}$$

$X^{-1}(0)X(T) = B$

$\dot{X}(t) = A(t)X(t)$

$\dot{x}(t) = A(t)x(t)$

Recall the Example:

$$\begin{aligned}
\dot{x} &= x - y - x(x^2+y^2),\\
\dot{y} &= x + y - y(x^2+y^2).
\end{aligned}$$

$f(\phi) =$ $\approx f(\phi) + \underbrace{\frac{\partial f_i(\phi)}{\partial x_j}}_{Df(\phi)}\underbrace{(x-\phi)}_{u}$

$x = \phi + u \; ; \; \dot{x} = \dot{\phi} + \dot{u} = f(x)$ $\Longrightarrow \dot{u} = Df(\phi)u$

Theorem 2.1 *Consider $\dot{x} = f(x)$ has a T periodic solution $\phi(t)$. Suppose that the linearized equation about $\phi(t)$*

$$\dot{u} = Df(\phi(t))u$$

$\dot{X}(t) = Df(\phi(t))X(t)$; $X(t)$ is fundamental matrix whose at $t=T$

has characteristic exponents of which one has zero real part and the rest have negative real parts. Then the periodic solution is asymptotically stable.

Theorem 2.2 *Suppose $\dot{u} = A(t)u$ has characteristic multipliers ρ_i and exponents λ_i ($\rho_i = e^{\lambda_i T}$). Then*

$X(t) = X(0)B$

eig of B are the char. mult. ($B = \Phi$ here)

$$\begin{aligned}
\rho_1\rho_2\cdots\rho_n &= e^{\int_0^T \mathrm{Tr}\,A(t)dt}\\
\lambda_1 + \lambda_2 + \cdots + \lambda_n &= \frac{1}{T}\int_0^T \mathrm{Tr}\,A(t)dt \left(\mathrm{mod}\frac{2\pi i}{T}\right)
\end{aligned}$$

?

Example: Consider

$$\ddot{x} - (1 - x^2 - \dot{x}^2)\dot{x} + x = 0.$$

Find the limit cycle and determine its stability.

$\dot{x} = y$

$\dot{y} = x + y - (x^2 + y^2)y$

$J = \begin{matrix} 0 & 1 \\ 1-2xy & 1-x^2-3y^2 \end{matrix}$

at $x,y = 0 \rightarrow J = \begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix} \rightarrow$ eigenv $\Rightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \lambda - 1 = 0$

$\lambda_{1,2} = \frac{1}{2}(1 \pm \sqrt{1+4}) \rightarrow \lambda_1 = \frac{1}{2}(1+\sqrt{5})$ $+$

$\lambda_2 = \frac{1}{2}(1-\sqrt{5})$ $-$

saddle

?

Monodromy Matrix. Stability of a periodic solution $\bar{x}$ manifests itself in the way neighboring trajectories behave. A trajectory that starts from the perturbed initial vector $\bar{x}_0 + \delta\bar{x}_0$ will after one period T be displaced by

$$\delta\bar{x}(T) = \phi(T;\bar{x}_0 + \delta\bar{x}_0) - \phi(T;\bar{x}_0).$$

$$\approx \phi(T;\bar{x}_0) + \frac{\partial \phi(T;\bar{x}_0)}{\partial x_0}\delta\bar{x}_0 + \dots$$

To first order, this displacement is given by

$$\delta\bar{x}(T) = \frac{\partial \phi(T;\bar{x}_0)}{\partial x_0}\delta\bar{x}_0,$$

Clearly, the matrix $\frac{\partial \phi(T;\bar{x}_0)}{\partial x_0}$ determines whether initial perturbations $\delta\bar{x}_0$ from the periodic orbit decay or grow. This matrix is called the *monodromy matrix.*

Fundamental Matrix Solution. Some properties of the flow ϕ help to find another representation of the monodromy matrix. Note that ϕ satisfies the autonomous equations

$$\dot{x} = f(x) \Rightarrow \overbrace{\frac{d\phi(t;x_0)}{dt}}^{\dot{x}} = f(\overbrace{\phi(t;x_0)}^{x}), \qquad \text{with} \quad \phi(0;x_0) = x_0.$$

Differentiating this identity w.r.t. x_0 yields

$$\frac{d}{dt}\frac{\partial \phi(t;x_0)}{\partial x_0} = Df(\phi)\frac{\partial \phi(t;x_0)}{\partial x_0}, \qquad \text{with} \quad \frac{\partial \overbrace{\phi(0;x_0)}^{x_0}}{\partial x_0} = I_n. = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{bmatrix}_{n \times n}$$

? it turned x to $\bar{x}$?

Hence, the monodromy matrix is the same as $\Phi(T)$ where $\Phi(t)$ is the fundamental matrix solution for the linearized equation about the periodic orbit $\bar{x}(t)$

$$\dot{\Phi} = Df(\bar{x})\Phi, \qquad \text{with} \quad \Phi(0) = I_n.$$

In summary, the monndromy matrix of a periodic solution $\bar{x}(t)$ with periodic T and initial value $\bar{x}_0$ is given by

$x^* \to \Phi$
$B \to M$

$$\boxed{\Phi(T) = \frac{\partial \phi(T;\bar{x}_0)}{\partial x_0}.} = M$$

(= B in other notations; $x^*(0)^{-1} x^*(T) = B$, can be taken I ; $x^{*\prime} = \frac{\partial f}{\partial x}\Big|_{\phi} x^*$)

Example: Find the fundamental matrix of

$$\dot{x} = Ax \qquad \dot{\Phi} = A\Phi \qquad \Phi = \frac{\partial \phi(T;x_0)}{\partial x_0}$$

where A is a matrix with constant coefficients.

$x^* = e^{At}$

Basic Results on the Monodromy Matrix. Before studying the local stability of $\bar{x}(t)$, we will first state a few basic results relating to the monodromy matrix:

- **Floquet theorem**: $\Phi(t) = P(t)e^{Rt}$ where $P(t)$ is T-periodic and R is a constant matrix.
- M has $+1$ as an eigenvalue with eigenvector $f(\bar{x}_0)$ which is tangent to the periodic orbit at $\bar{x}_0$.

The Floquet theorem can be proved as follows: Since the Jacobian $Df(\bar{x})$ is periodic, it can be easily checked that for any matrix $\Phi(t)$ that solves

$$\dot{\Phi} = Df(\bar{x})\Phi,$$

$\Phi(t+T)$ is also a solution. Hence, there is a constant nonsingular matrix C such that

$$\Phi(t+T) = \Phi(t)C.$$

By using a logarithmic operator, C can be expressed in exponential form as

$$C = e^{RT} \tag{2.0.2}$$

where R is a constant matrix. Now let $P(t) = \Phi(t)e^{-Rt}$, and one can show that $P(t)$ is periodic:

$$P(t+T) = \Phi(t+T)e^{-R(t+T)} = \Phi(t)Ce^{-RT}e^{-Rt} = \Phi(t)e^{-Rt} = P(t).$$

Thus, the Floquet theorem is proved.

As for the second result, it is obvious that if $\bar{x}(t)$ is a period T solution of autonomous equations, then the period T function $\dot{\bar{x}}(t)$ solves its linearized equations, which we can write as

$\phi = \bar{x}$ the periodic sol.

$x = \phi + y$ $\quad f(\phi)$

$\dot{x} = \dot{\phi} + \dot{y} = f(x) \rightarrow f(\phi) + \frac{\partial f_i}{\partial x_j}\Big|_{\phi} y \Rightarrow$

$$\dot{y} = Df(\bar{x})y,$$

$= \phi(t, \bar{x}_0)$

for small displacements y away from the reference periodic solution $\bar{x}$. Since every solution of this linearized problem satisfies

$$y(t) = \Phi(t)y(0),$$

Linear combinations of solutions is a solution.
But Φ includes the solutions of x, not y ??

the relation

$$1 \cdot y(0) = y(T) = \Phi(T)y(0)$$

shows that $\Phi(T)$ has 1 as an eigenvalue. The eigenvector is

$$y(0) = \dot{\bar{x}}(0) = f(\bar{x}_0).$$

?

$X \rightarrow \Phi$

$B \rightarrow M$

Example (Perko): Consider

$$
\begin{aligned}
\dot{x} &= x - y - x^3 = xy^2 \\
\dot{y} &= x + y - x^2 y - y^3 \\
\dot{z} &= \lambda z
\end{aligned}
$$

Stability of Periodic Orbits

- Since $\Phi(0) = I_n$, we have $P(0) = I_n = P(T)$. Hence,

  $$\Phi(T) = e^{RT}.$$

  It then follows that the behavior of solutions in the neighborhood of $\bar{x}(t)$ is determined by the eigenvalues of the constant matrix e^{RT}. These eigenvalues, $\rho_1, \ldots, \rho_n$ are called the *characteristic* (*Floquet*) *multipliers*. Each complex number λ_j such that $\rho_j = e^{\lambda_j T}$ is called the *characteristic exponents* of the closed orbit.

- The multiplier associated with perturbations along $\bar{x}(t)$ is always unity; let this be ρ_n. The moduli of remaining $(n-1)$ determine the stability of $\bar{x}$.

- Choosing the basis appropriately, so that the last column of e^{RT} is $(0, \ldots, 0, 1)$, the matrix $\frac{\partial P}{\partial x_0}(a)$ of the linearized Poincaré map is simply the $(n-1) \times (n-1)$ matrix obtained by deleting the n-th row and column of e^{RT}. Then the first $(n-1)$ multipliers $\rho_1, \ldots, \rho_{n-1}$ are the eigenvalues of the Poincaré map.

- To compute these eigenvalues, we need a representation of e^{RT}, and this can only be obtained by actually generating a set of n linearly independent solutions to form $\Phi(t)$. Except in special case, this is generally difficult.

- Numerical methods are needed to handle practical problems.
  - Need to use some kind of perturbation method to find an approxiamte solution of the periodic orbit. Differentiate correct to attain high accuracy.
  - Need to integrate $n + n^2$ differential equations to obtain the fundamental matrix solution and calculate the characteristic exponents of the corresponding monodromy matrix.