Computational Physics Project 6

Jack Messerli-Wallace

March 24, 2023

1 Solving the Laplace Equation

1.1 Introduction

In this work, we use variations on the relaxation method to obtain solutions to the Laplace Equation for the potential between two plates, as discussed in class. Three methods are explored; the Jacobi method, the Gauss-Siedel method, and the over-relaxation method. Changes made to the base program `point-3D-fast.f` and computational results will be discussed.

1.2 Jacobi Method

The Jacobi method is the default relaxation technique and is the same method utilized in the `point-3D-fast.f` program. Changes made to the base program to solve our parallel plate problem include changes to the dimension of the calculations and initial conditions. The third dimension was removed to make the problem two-dimensional. This meant we had to change the relaxation calculation line from

$$V_{\text{new}}(i,j,k) = \frac{1}{6}[V(i+1,j,k) + V(i-1,j,k) + V(i,j+1,k) + V(i,j-1,k) + V(i,j,k+1) + V(i,j,k-1)]$$

in the base program to

$$V_{\text{new}}(i,j) = \frac{1}{4}[V(i+1,j) + V(i-1,j) + V(i,j+1) + V(i,j-1)]$$

in our Jacobi solution program. Initial conditions were changed such that, instead of a point charge, our parallel plates were represented according to instructions laid in the assignment (the total computational box is ten times larger than the width of the plates and the distance between the plates is half their width). Note that the potential at the location of the parallel plates is updated between iterations because they are kept at a constant potential difference. The grid size was 400 by 400 points for this and all subsequent methods.

The program ran for a maximum of 100,000 iterations, or until the total accumulated absolute error between two subsequent iterations was below 10^{-5} volts.

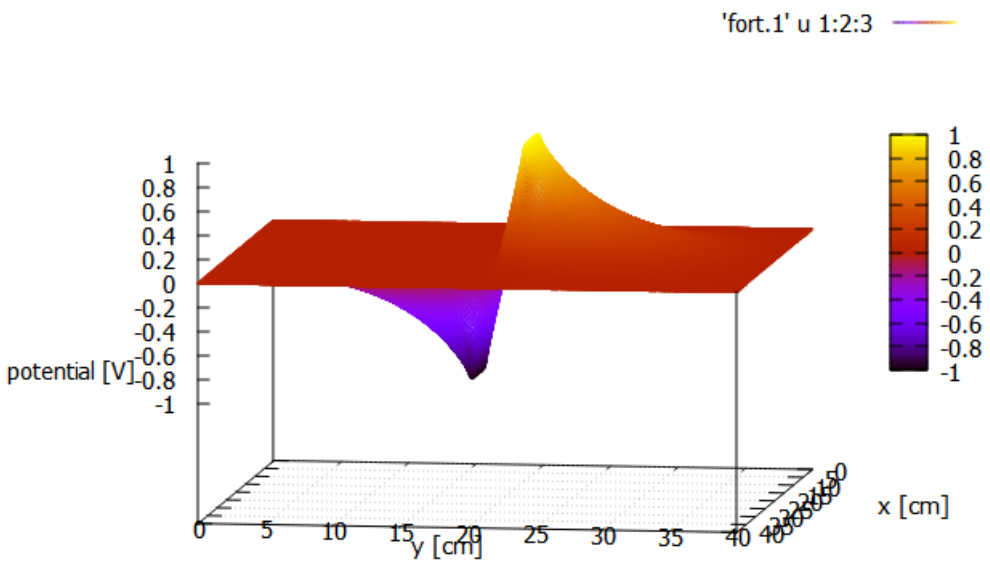

Figure 1: Plot of potential calculated using the Jacobi method

Figure 1 shows the plot of the potential as a function of the two spacial dimensions calculated over. This program ran through 32389 iterations before reaching the desired error and took 53.377 seconds to run.

1.3 Gauss-Siedel Method

The Gauss-Siedel method uses the current iteration's new values in conjunction with previous values to double the computational efficiency of the relaxation method. This allowed us to rewrite the Jacobi program with small changes, including changing the calculation equation from

$$V_{\text{new}}(i,j) = \frac{1}{4}[V(i+1,j) + V(i-1,j) + V(i,j+1) + V(i,j-1)]$$

in the Jacobi program to

$$V_{\text{new}}(i,j) = \frac{1}{4}[V(i+1,j) + V_{\text{new}}(i-1,j) + V(i,j+1) + V_{\text{new}}(i,j-1)]$$

in the Gauss-Siedel program. This program ran through 24314 iterations before reaching the desired error (10^{-5} volts) and took 30.755 seconds to run.

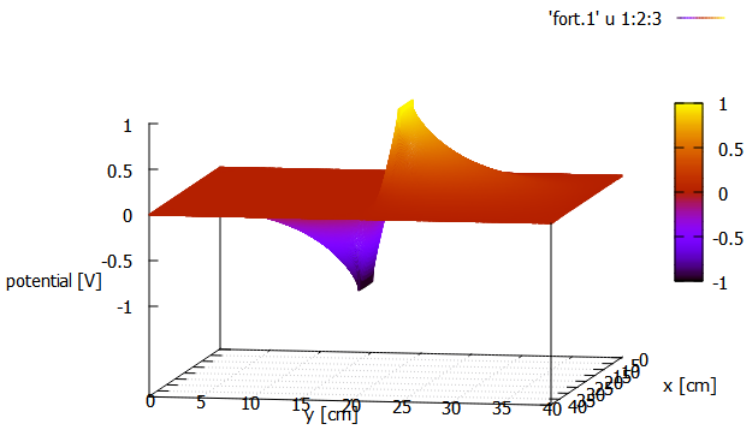

Figure 2: Plot of potential calculated using the Gauss-Siedal method

1.4 Over-Relaxation Method

The over-relaxation method changes the updating technique of the Gauss-Siedal method to greatly improve efficiency. Instead of taking

$$V_*(i,j) = \frac{1}{4}[V(i+1,j) + V_{\text{new}}(i-1,j) + V(i,j+1) + V_{\text{new}}(i,j-1)]$$

as V_{new}, this V_* is used as follows.

$$V_{\text{new}}(i,j) = \alpha(V_*(i,j) - V_{\text{old}}(i,j)) + V_{\text{old}}$$

where $\alpha = \frac{2}{1+\pi/L}$. Note that L is the length of the box, which is 40 in our case (the grid size is 400 by 400 and the step size is 0.1). For our box, $\alpha \simeq 1.7285$. This program ran through 7204 iterations before reaching the desired error and took 13.451 seconds to run.

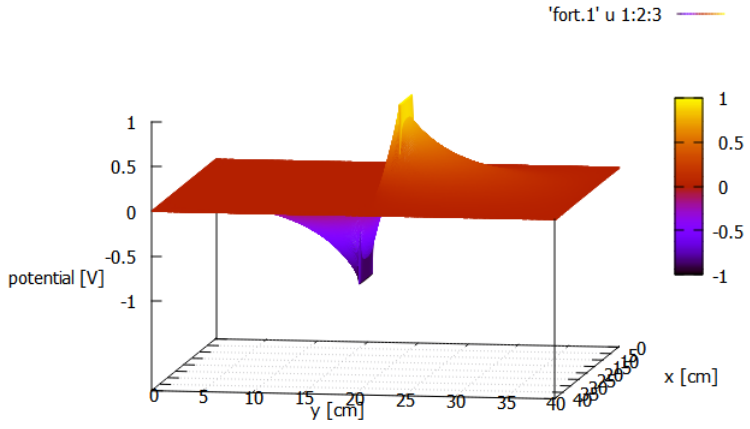

Figure 3: Plot of potential calculated using the over-relaxation method

1.5 Conclusion

Solutions to our parallel plate problem have been computed using the Jacobi method, the Gauss-Seidel method, and the over-relaxation method. Model convergence results are as follows.

- Jacobi method: 32389 iterations, 53.377 seconds
- Gauss-Seidel method: 24314 iterations, 30.755 seconds
- Over-relaxation method: 7204 iterations, 13.451 seconds

The Gauss-Seidel method took about 75% as many iterations and just over half the time to converge compared to the Jacobi method. This is within our expectations. The over-relaxation took far less iterations and time to converge, as expected.