Cut Elimination in LL with Multimodalities

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Abstract

This document contains all the needed proof transformation to prove cut-admissibility for the focused system for linear logic with multimodalities.

1 The system and the cut rules

The rules of the system are in Figure 1. The cut-elimination procedure requires the cut-rules presented below. In the following section, it is shown how the are (mutually) eliminated.

$$\frac{\vdash \mathcal{K}_1; \Gamma, \mathsf{C} \Uparrow \mathsf{L}_1 \quad \vdash \mathcal{K}_2; \Delta \Uparrow \mathsf{C}^\perp, \mathsf{L}_2}{\vdash \mathcal{K}; \Gamma, \Delta \Uparrow \mathsf{L}_1, \mathsf{L}_2}\ cut_1 \qquad \frac{\vdash \mathcal{K}_1; \Gamma \Uparrow \mathsf{C}, \mathsf{L} \quad \vdash \mathcal{K}_2; \Delta \Downarrow \mathsf{C}^\perp}{\vdash \mathcal{K}; \Gamma, \Delta \Uparrow \mathsf{L}}\ cut_2$$

$$\frac{\vdash \mathcal{K}_1, \mathsf{C_i}; \Gamma \Uparrow \mathsf{L} \quad \vdash \mathcal{K}_2; \cdot \Downarrow\ !^i\mathsf{C}^\perp}{\vdash \mathcal{K}; \Gamma \Uparrow \mathsf{L}}\ \mathsf{cut}_3$$

$$\frac{\vdash \mathcal{K}_1, \mathsf{C_i}; \Gamma \Downarrow \mathsf{F} \quad \vdash \mathcal{K}_2; \cdot \Downarrow\ !^i\mathsf{C}^\perp}{\vdash \mathcal{K}; \Gamma \Uparrow \mathsf{F}}\ cut_4 \qquad \frac{\vdash \mathcal{K}_1; \Gamma \Uparrow \mathsf{L}_1, \mathsf{C}, \mathsf{L}_2 \quad \vdash \mathcal{K}_2; \Delta \Uparrow \mathsf{C}^\perp, \mathsf{L}_3}{\vdash \mathcal{K}; \Gamma, \Delta \Uparrow \mathsf{L}_1, \mathsf{L}_2, \mathsf{L}_3}\ \mathsf{cut}_5$$

Where C is the cut-formula, F is a formula, Γ and Δ are multisets of formulas, $\mathcal{K}_1$ and $\mathcal{K}_2$ are indexed contexts of formulas, and L is a list of formulas.

Moreover, $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}$ have the same unbounded formulas and the linear formulas in $\mathcal{K}$ are the multiset union of the linear formulas in $\mathcal{K}_1$ and $\mathcal{K}_2$.

Negative rules:

$$\frac{}{\vdash \Theta; \Gamma \Uparrow \top, L}\ \top \qquad \frac{\vdash \Theta; \Gamma \Uparrow L}{\vdash \Theta; \Gamma \Uparrow \bot, L}\ \bot \qquad \frac{\vdash \Theta; \Gamma \Uparrow F, G, L}{\vdash \Theta; \Gamma \Uparrow F \parr G, L}\ \parr \qquad \frac{\vdash \Theta; \Gamma, S \Uparrow L}{\vdash \Theta; \Gamma \Uparrow S, L}\ \mathsf{store}$$

$$\frac{\vdash \Theta; \Gamma \Uparrow F, L \quad \vdash \Theta; \Gamma \Uparrow G, L}{\vdash \Theta; \Gamma \Uparrow F \mathbin{\&} G, L}\ \& \qquad \frac{\vdash \Theta; \Gamma \Uparrow F[y/x], L}{\vdash \Theta; \Gamma \Uparrow \forall x.F, L}\ \forall \qquad \frac{\vdash \Theta, i : F; \Gamma \Uparrow L}{\vdash \Theta; \Gamma \Uparrow ?^{i}F, L}\ \mathsf{store_s}$$

Positive rules:

$$\frac{\vdash \Theta^{u}, \Theta^{l}_{1}; \Gamma_1 \Downarrow F \quad \vdash \Theta^{u}, \Theta^{l}_{2}; \Gamma_2 \Downarrow G}{\vdash \Theta^{u}, \Theta^{l}_{1}, \Theta^{l}_{2}; \Gamma_1, \Gamma_2 \Downarrow F \otimes G}\ \otimes \qquad \frac{\vdash \Theta; \Gamma \Downarrow F_1}{\vdash \Theta; \Gamma \Downarrow F_1 \oplus F_2}\ \oplus_1 \qquad \frac{\vdash \Theta; \Gamma \Downarrow F_2}{\vdash \Theta; \Gamma \Downarrow F_1 \oplus F_2}\ \oplus_2$$

$$\frac{\vdash \Theta; \Gamma \Downarrow F[t/x]}{\vdash \Theta; \Gamma \Downarrow \exists x.F}\ \exists \qquad \frac{}{\vdash \Theta; \cdot \Downarrow 1}\ 1$$

Structural:

$$\frac{}{\vdash \Theta^{u}; A \Downarrow A^{\perp}}\ \mathsf{I_l} \qquad \frac{}{\vdash \Theta^{u}, i : A; \cdot \Downarrow A^{\perp}}\ \mathsf{I_s}$$

$$\frac{\vdash \Theta; \Gamma \Downarrow P}{\vdash \Theta; \Gamma, P \Uparrow \cdot}\ \mathsf{D_l} \qquad \frac{\vdash \Theta, i : P_a; \Gamma \Downarrow P_a}{\vdash \Theta, i : P_a; \Gamma \Uparrow \cdot}\ \mathsf{D^{u}_{s}} \qquad \frac{\vdash \Theta; \Gamma \Downarrow P_a}{\vdash \Theta, i : P_a; \Gamma \Uparrow \cdot}\ \mathsf{D^{l}_{s}} \qquad \frac{\vdash \Theta; \Gamma \Uparrow N}{\vdash \Theta; \Gamma \Downarrow N}\ \mathsf{R_n}$$

Modal:

$$\frac{\vdash \Theta; \cdot \Uparrow \cdot \mathbin{/\!/}^{i} \vdash \cdot; \cdot \Uparrow F}{\vdash \Theta; \cdot \Downarrow !^{i}F}\ !^{i} \qquad \frac{\vdash \Upsilon; \cdot \Uparrow L}{\vdash \Theta^{u}; \cdot \Uparrow \cdot \mathbin{/\!/}^{i} \vdash \Upsilon; \cdot \Uparrow L}\ \mathsf{R_r} \qquad \frac{\vdash \Theta; \cdot \Uparrow \cdot \mathbin{/\!/}^{i} \vdash \cdot; \cdot \Uparrow \cdot}{\vdash \Theta; \cdot \Uparrow \cdot}\ \mathsf{D_d} \qquad \frac{\vdash \Theta^{u}; \cdot \Uparrow F}{\vdash \Theta^{u}; \cdot \Downarrow !^{c}F}\ !^{c}$$

$$\frac{\vdash \Theta; \Gamma \Uparrow \cdot \mathbin{/\!/}^{i} \vdash \Upsilon, j+ : F; \cdot \Uparrow L}{\vdash \Theta, j : F; \Gamma \Uparrow \cdot \mathbin{/\!/}^{i} \vdash \Upsilon; \cdot \Uparrow L}\ ?^{i}{}_{4} \qquad \frac{\vdash \Theta; \cdot \Uparrow \cdot \mathbin{/\!/}^{i} \vdash \Upsilon; \cdot \Uparrow L, F}{\vdash \Theta, j : F; \cdot \Uparrow \cdot \mathbin{/\!/}^{i} \vdash \Upsilon; \cdot \Uparrow L}\ ?^{i}{}_{kl} \qquad \frac{\vdash \Theta; \cdot \Uparrow \cdot \mathbin{/\!/}^{i} \vdash \Upsilon, c : F; \cdot \Uparrow L}{\vdash \Theta, j : F; \cdot \Uparrow \cdot \mathbin{/\!/}^{i} \vdash \Upsilon; \cdot \Uparrow L}\ ?^{i}{}_{ku}$$

Figure 1: End-active focused system $\mathsf{LNS_{FSLL}}$. Θ^{u} (resp. Θ^{l}) contains only unbounded (resp. linear) subexponentials. In $\mathsf{I_s}$ and $\mathsf{I_l}$, A is atomic. In $\forall$, y is fresh. In store, S is a literal or a positive formula. In $\mathsf{R_n}$, N is a negative formula. In $\mathsf{D_l}$, P is positive, and in $\mathsf{D^{u}_{s}}, \mathsf{D^{l}_{s}}$, P_a is not atomic. In $\mathsf{D^{u}_{s}}, \mathsf{D^{l}_{s}}$ and $\mathsf{I_s}$, $\mathsf{T} \in \mathcal{U}(i)$. In all question-marked rules $i \preceq j$. Moreover, $i \neq \mathsf{c}$ in $!^{i}$; $\mathsf{D} \in \mathcal{U}(i)$ in $\mathsf{D_d}$; $4 \in \mathcal{U}(j)$ in $?^{i}_{4}$; $\{4, \mathsf{C}, \mathsf{W}\} \cap \mathcal{U}(j) = \emptyset$ in $?^{i}_{kl}$; $4 \notin \mathcal{U}(i)$ and $\mathsf{U} \subseteq \mathcal{U}(i)$ in $?^{i}_{ku}$ and in $\mathsf{D^{u}_{s}}$.

2 Exponential Rules

$i \preceq j$, $4 \notin \mathcal{U}(j)$ and j is unbounded

$$\frac{\Theta; \cdot \Uparrow \cdot /\!/ i \vdash \Upsilon, F_c; \cdot \Uparrow L}{\Theta, F_j; \cdot \Uparrow \cdot /\!/ i \vdash \Upsilon; \cdot \Uparrow L}\ [?_{Ku}]$$

$i \preceq j$, $4 \notin \mathcal{U}(j)$ and j is linear

$$\frac{\Theta; \cdot \Uparrow \cdot /\!/ i \vdash \Upsilon; \cdot \Uparrow L, F}{\Theta, F_j; \cdot \Uparrow \cdot /\!/ i \vdash \Upsilon; \cdot \Uparrow L}\ [?_{Kl}]$$

$i \preceq j$ and $4 \in \mathcal{U}(j)$

$$\frac{\Theta; \cdot \Uparrow \cdot /\!/ i \vdash \Upsilon, F_{j+}; \cdot \Uparrow L}{\Theta, F_j; \cdot \Uparrow \cdot /\!/ i \vdash \Upsilon; \cdot \Uparrow L}\ [?_{K4}]$$

Θ is unbounded and $\forall j \in \Theta, i \not\preceq j$

$$\frac{\vdash \Upsilon; \cdot \Uparrow L}{\Theta; \cdot \Uparrow \cdot /\!/ i \vdash \Upsilon; \cdot \Uparrow L}\ [R_r]$$

Θ is unbounded

$$\frac{\vdash \Theta; \cdot \Uparrow P}{\vdash \Theta; \cdot \Downarrow !^{lc} P}\ [!^c]$$

$$\frac{\Theta; \cdot \Uparrow \cdot /\!/ i \vdash \cdot; \cdot \Uparrow F}{\Theta; \cdot \Downarrow !^i F}\ [!^i]$$

$$\frac{\Theta; \cdot \Uparrow \cdot /\!/ i \vdash \cdot; \cdot \Uparrow \cdot}{\Theta; \cdot \Uparrow \cdot}\ [D \in \mathcal{U}(i)]$$

- The subexponential c is not related with anyone
- $\mathcal{U}(c) = \{K, U, T\}$
- Let Θ be a subexp. context, $\mathcal{F}(\Theta)$ is all formulas in Θ
- Let Θ be a subexp context, Θ^u (resp. Θ^l) is the unbounded (resp. linear) context in Θ
- Note that $\Theta \equiv \Theta^u, \Theta^l$ with $\Theta^u \cap \Theta^l = \emptyset$
- $\Upsilon_{a\preceq}$ means that $\forall b : subexp, b \in \Upsilon \implies a \preceq b$
- Υ^Z means that $\forall i : subexp, i \in \Upsilon \implies Z \in \mathcal{U}(i)$
- $\Upsilon^{\cancel{Z}}$ means that $\forall i : subexp, i \in \Upsilon \implies Z \notin \mathcal{U}(i)$

3 Height-Preserving Lemmas

Theorem 3.1 (W). *If* $U \in \mathcal{U}(j)$ *and* $\vdash \Theta; \Delta; \Updownarrow X$ *then* $\vdash F_j, \Theta; \Delta; \Updownarrow X$

Theorem 3.2 (C). *If* $U \in \mathcal{U}(j)$ *and* $\vdash F_j, F_j, \Theta; \Delta; \Updownarrow X$ *then* $\vdash F_j, \Theta; \Delta; \Updownarrow X$

Theorem 3.3 (C_c). *If* $\{U, T\} \subseteq \mathcal{U}(j)$ *and* $\vdash F_c, F_j, \Theta; \Delta; \Updownarrow X$ *then* $\vdash F_j, \Theta; \Delta; \Updownarrow X$

Theorem 3.4 (A_c). *If* $\{U, T\} \subseteq \mathcal{U}(j)$ *and* $\vdash F_j, \Theta; F, \Delta; \Updownarrow X$ *then* $\vdash F_j, \Theta; \Delta; \Updownarrow X$

Theorem 3.5 (A_l). *If* $T \in \mathcal{U}(j)$ *and* $\vdash \Theta; F, \Delta; \Updownarrow X$ *then* $\vdash F_j, \Theta; \Delta; \Updownarrow X$

4 Elimination of cut_1

$$\frac{\vdash \mathcal{K}_1;\Gamma,\mathsf{C}\Uparrow \mathsf{L}_1 \quad \vdash \mathcal{K}_2;\Delta\Uparrow \mathsf{C}^{\perp},\mathsf{L}_2}{\vdash \mathcal{K};\Gamma,\Delta\Uparrow \mathsf{L}_1,\mathsf{L}_2}\ cut_1$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1:\Gamma,\mathsf{C}\Uparrow L_1}{\vdash \mathcal{K}_1:\Gamma,\mathsf{C}\Uparrow \bot,L_1} \quad \Pi_2}{\vdash \mathcal{K}:\Gamma,\Delta\Uparrow \bot,L_1,L_2} \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \mathcal{K}_1:\Gamma,\mathsf{C}\Uparrow L_1 \quad \Pi_2}{\vdash \mathcal{K}:\Gamma,\Delta\Uparrow L_1,L_2}}{\vdash \mathcal{K}:\Gamma,\Delta\Uparrow \bot,L_1,L_2}$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1:\Gamma,\mathsf{C}\Uparrow P,Q,L_1}{\vdash \mathcal{K}_1:\Gamma,\mathsf{C}\Uparrow P\parr Q,L_1} \quad \Pi_2}{\vdash \mathcal{K}:\Gamma,\Delta\Uparrow P\parr Q,L_1,L_2} \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \mathcal{K}_1:\Gamma,\mathsf{C}\Uparrow P,Q,L_1 \quad \Pi_2}{\vdash \mathcal{K}:\Gamma,\Delta\Uparrow P,Q,L_1,L_2}}{\vdash \mathcal{K}:\Gamma,\Delta\Uparrow P\parr Q,L_1,L_2}$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1:\Gamma,\mathsf{C}\Uparrow P,L_1 \quad \vdash \mathcal{K}_1:\Gamma,\mathsf{C}\Uparrow Q,L_1}{\vdash \mathcal{K}_1:\Gamma,\mathsf{C}\Uparrow P\mathbin{\&}Q,L_1} \quad \Pi_2}{\vdash \mathcal{K}:\Gamma,\Delta\Uparrow P\mathbin{\&}Q,L_1,L_2}$$

$$\rightsquigarrow\quad \dfrac{\dfrac{\vdash \mathcal{K}_1:\Gamma,\mathsf{C}\Uparrow P,L_1 \quad \Pi_2}{\vdash \mathcal{K}:\Gamma,\Delta\Uparrow P,L_1,L_2} \quad \dfrac{\vdash \mathcal{K}_1:\Gamma,,\mathsf{C}\Uparrow Q,L_1 \quad \Pi_2}{\vdash \mathcal{K}:\Gamma,\Delta\Uparrow Q,L_1,L_2}}{\vdash \mathcal{K}:\Gamma,\Delta\Uparrow P\mathbin{\&}Q,L_1,L_2}$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1:\Gamma,\mathsf{C}\Uparrow P[c/x],L_1}{\vdash \mathcal{K}_1:\Gamma,\mathsf{C}\Uparrow \forall xP,L_1} \quad \Pi_2}{\vdash \mathcal{K}:\Gamma,\Delta\Uparrow \forall xP,L_1,L_2} \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \mathcal{K}_1:\Gamma,\mathsf{C}\Uparrow P[c/x],L_1 \quad \Pi_2}{\vdash \mathcal{K}:\Gamma,\Delta\Uparrow P[c/x],L_1,L_2}}{\vdash \mathcal{K}:\Gamma,\Delta\Uparrow \forall xP,L_1,L_2}$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1,P_i:\Gamma,\mathsf{C}\Uparrow L_1}{\vdash \mathcal{K}_1:\Gamma,\mathsf{C}\Uparrow ?^iP,L_1} \quad \Pi_2}{\vdash \mathcal{K}:\Gamma,\Delta\Uparrow ?^iP,L_1,L_2} \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \mathcal{K}_1,P_i:\Gamma,\mathsf{C}\Uparrow L_1 \quad \Pi_2}{\vdash \mathcal{K},P_i:\Gamma,\Delta\Uparrow L_1,L2}}{\vdash \mathcal{K}:\Gamma,\Delta\Uparrow ?^iP,L_1,L_2}$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1,P_i:\Gamma,\mathsf{C}\Uparrow L_1}{\vdash \mathcal{K}_1:\Gamma,\mathsf{C}\Uparrow ?^iP,L_1} \quad \Pi_2}{\vdash \mathcal{K}:\Gamma,\Delta\Uparrow ?^iP,L} \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \mathcal{K}_1,P_i:\Gamma,\mathsf{C}\Uparrow L \quad \dfrac{\Pi_2}{\vdash \mathcal{K}_2,P_i:\Delta\Uparrow \mathsf{C}^{\perp},L_2}\ \mathsf{W}}{\vdash \mathcal{K},P_i:\Gamma,\Delta\Uparrow L_1,L_2}}{\vdash \mathcal{K}:\Gamma,\Delta\Uparrow ?^iP,L_1,L_2}$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1:P,\Gamma,\mathsf{C}\Uparrow L_1}{\vdash \mathcal{K}_1:\Gamma,\mathsf{C}\Uparrow P,L_1} \quad \Pi_2}{\vdash \mathcal{K}:\Gamma,\Delta\Uparrow P,L_1,L_2} \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \mathcal{K}_1:P,\Gamma,\mathsf{C}\Uparrow L_1 \quad \Pi_2}{\vdash \mathcal{K}:P,\Gamma,\Delta\Uparrow L_1,L_2}}{\vdash \mathcal{K}:\Gamma,\Delta\Uparrow P,L_1,L_2}$$

$$\dfrac{\dfrac{\dfrac{\vdash \mathcal{K}_1 : \Gamma, \mathsf{C} \Downarrow P}{\vdash \mathcal{K}_1 : \Gamma, \mathsf{C} \Downarrow P \oplus Q}}{\vdash \mathcal{K}_1 : P \oplus Q, \Gamma, \mathsf{C} \Uparrow \cdot} \qquad \Pi_2}{\vdash \mathcal{K} : P \oplus Q, \Gamma, \Delta \Uparrow L_2} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\vdash \mathcal{K}_1 : \Gamma, \mathsf{C} \Downarrow P}{\vdash \mathcal{K}_1 : P, \Gamma, \mathsf{C} \Uparrow \cdot} \qquad \Pi_2}{\vdash \mathcal{K} : P, \Gamma, \Delta \Uparrow L_2}}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow L_2, P}}{\vdash \mathcal{K} : P \oplus Q, \Gamma, \Delta \Uparrow L_2} \mathcal{L}\oplus$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \mathcal{K}_1 : \Gamma, \mathsf{C} \Uparrow P}{\vdash \mathcal{K}_1 : \Gamma, \mathsf{C} \Downarrow P}}{\vdash \mathcal{K}_1 : \Gamma, \mathsf{C} \Downarrow P \oplus Q}}{\vdash \mathcal{K}_1 : P \oplus Q, \Gamma, \mathsf{C} \Uparrow \cdot} \qquad \Pi_2}{\vdash \mathcal{K} : P \oplus Q, \Gamma, \Delta \Uparrow L_2} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\vdash \mathcal{K}_1 : \Gamma, \mathsf{C} \Uparrow P \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow P, L_2}}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow L_2, P}}{\vdash \mathcal{K} : P \oplus Q, \Gamma, \Delta \Uparrow L_2} \mathcal{L}\oplus$$

If $P \oplus Q \in \mathcal{K}_1$, we proceed similarly. For $\otimes$ and $\exists$ we use the same reasoning.

$$\dfrac{\dfrac{\vdash \mathcal{K}_1 : \Gamma \Downarrow \mathsf{C}}{\vdash \mathcal{K}_1 : \Gamma, \mathsf{C} \Uparrow \cdot} \qquad \vdash \mathcal{K}_2 : \Delta \Uparrow \mathsf{C}^{\perp}, L_2}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow L_2} \quad \rightsquigarrow \quad \dfrac{\vdash \mathcal{K}_2 : \Delta \Uparrow \mathsf{C}^{\perp}, L_2 \qquad \vdash \mathcal{K}_1 : \Gamma \Downarrow \mathsf{C}}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow \cdot} \mathsf{cut}_2$$

5 Elimination of cut_2

$$\frac{\vdash \mathcal{K}_1; \Gamma \Uparrow \mathsf{C}, \mathsf{L} \qquad \vdash \mathcal{K}_2; \Delta \Downarrow \mathsf{C}^{\perp}}{\vdash \mathcal{K}; \Gamma, \Delta \Uparrow \mathsf{L}}\ cut_2$$

$$\dfrac{\dfrac{\vdash \mathcal{K} : \Gamma \Uparrow L}{\vdash \mathcal{K} : \Gamma \Uparrow \bot, L} \qquad \dfrac{}{\vdash \mathcal{K}^u : \cdot \Downarrow 1}}{\vdash \mathcal{K} : \Gamma \Uparrow L} \qquad \rightsquigarrow \qquad \vdash \mathcal{K} : \Gamma \Uparrow L$$

$$\dfrac{\dfrac{\begin{matrix}[\Sigma_1]\\ \vdash \mathcal{K}_1 : \Gamma \Uparrow P, Q, L\end{matrix}}{\vdash \mathcal{K}_1 : \Gamma \Uparrow P \mathbin{⅋} Q, L} \qquad \dfrac{\begin{matrix}[\Sigma_2]\\ \vdash \mathcal{K}_2' : \Delta_P \Downarrow P^{\perp}\end{matrix} \qquad \begin{matrix}[\Sigma_3]\\ \vdash \mathcal{K}_2'' : \Delta_Q \Downarrow Q^{\perp}\end{matrix}}{\vdash \mathcal{K}_2 : \Delta \Downarrow P^{\perp} \otimes Q^{\perp}}}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow L}$$

$$\rightsquigarrow \qquad \dfrac{\dfrac{[\Sigma_1] \qquad [\Sigma_2]}{\vdash \mathcal{K}', \Theta_2 : \Gamma, \Delta_P \Uparrow Q, L} \qquad [\Sigma_3]}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow L}$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1 : \Gamma \Uparrow P, L \qquad \vdash \mathcal{K}_1 : \Gamma \Uparrow Q, L}{\vdash \mathcal{K}_1 : \Gamma \Uparrow P \mathbin{\&} Q, L} \qquad \dfrac{\vdash \mathcal{K}_2 : \Delta \Downarrow P^{\perp}}{\vdash \mathcal{K}_2 : \Delta \Downarrow P^{\perp} \oplus Q^{\perp}}}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow L}$$

$$\rightsquigarrow \qquad \dfrac{\vdash \mathcal{K}_1 : \Gamma \Uparrow P, L \qquad \vdash \mathcal{K}_2 : \Delta \Downarrow P^{\perp}}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow L}$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1 : \Gamma \Uparrow P[c/x], L}{\vdash \mathcal{K}_1 : \Gamma \Uparrow \forall x.P, L} \qquad \dfrac{\vdash \mathcal{K}_2 : \Delta \Downarrow P^{\perp}[t/x]}{\vdash \mathcal{K}_2 : \Delta \Downarrow \exists x.P^{\perp}}}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow L}$$

$$\rightsquigarrow \qquad \dfrac{\vdash \mathcal{K}_1 : \Gamma \Uparrow P[c/x], L \qquad \vdash \mathcal{K}_2 : \Delta \Downarrow P^{\perp}[t/x]}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow L}$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1, H_i : \Gamma \Uparrow L}{\vdash \mathcal{K}_1 : \Gamma \Uparrow ?^i H, L} \qquad \vdash \mathcal{K}_2 : \cdot \Downarrow !^i H}{\vdash \mathcal{K} : \Gamma \Uparrow L} \quad \rightsquigarrow \quad \dfrac{\vdash \mathcal{K}_1, H_i : \Gamma \Uparrow L \qquad \vdash \mathcal{K}_2 : \cdot \Downarrow !^i H}{\vdash \mathcal{K} : \Gamma \Uparrow L}\,\mathsf{cut}_3$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1 : H, \Gamma \Uparrow L}{\vdash \mathcal{K}_1 : \Gamma \Uparrow H, L} \qquad \dfrac{\vdash \mathcal{K}_2 : \Delta \Uparrow H^\perp}{\vdash \mathcal{K}_2 : \Delta \Downarrow H^\perp}}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow L} \quad \rightsquigarrow \quad \dfrac{\vdash \mathcal{K}_1 : H, \Gamma \Uparrow L \qquad \vdash \mathcal{K}_2 : \Delta \Downarrow H^\perp}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow L}\,\mathsf{cut}_1$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1 : A^+, \Gamma \Uparrow L}{\vdash \mathcal{K}_1 : \Gamma \Uparrow A^+, L} \qquad \dfrac{}{\vdash \mathcal{K}_2 : A^+ \Downarrow A^-}}{\vdash \mathcal{K}_1 : \Gamma, A^+ \Uparrow L}$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1 : A^+, \Gamma \Uparrow L}{\vdash \mathcal{K}_1 : \Gamma \Uparrow A^+, L} \qquad \dfrac{}{\vdash \mathcal{K}_2, A_i^+ : \cdot \Downarrow A^-}}{\vdash \mathcal{K}, A_i^+ : \Gamma \Uparrow L} \quad \rightsquigarrow \quad \dfrac{\vdash \mathcal{K}_1 : \Gamma, A^+ \Uparrow L}{\vdash \mathcal{K}, A_i^+ : \Gamma \Uparrow L}\,\mathsf{A}_l$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1, A_i^+ : A^+, \Gamma \Uparrow L}{\vdash \mathcal{K}_1, A_i^+ : \Gamma \Uparrow A^+, L} \qquad \dfrac{}{\vdash \mathcal{K}_2, A_i^+ : \cdot \Downarrow A^-}}{\vdash \mathcal{K}, A_i^+ : \Gamma \Uparrow L} \quad \rightsquigarrow \quad \dfrac{\vdash \mathcal{K}_1, A_i^+ : \Gamma, A^+ \Uparrow L}{\vdash \mathcal{K}, A_i^+ : \Gamma \Uparrow L}\,\mathsf{A}_c$$

6 Elimination of cut_3

$$\dfrac{\vdash \mathcal{K}_1, \mathsf{C_i}; \Gamma \Uparrow \mathsf{L} \qquad \vdash \mathcal{K}_2; \cdot \Downarrow\, !^i \mathsf{C}^\perp}{\vdash \mathcal{K}; \Gamma \Uparrow \mathsf{L}}\ \mathsf{cut}_3$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Uparrow L}{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Uparrow \bot, L} \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma \Uparrow \bot, L} \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Uparrow L \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma \Uparrow L}}{\vdash \mathcal{K} : \Gamma \Uparrow \bot, L}$$

$$\dfrac{\dfrac{}{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Uparrow \top, L} \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma \Uparrow \top, L} \quad \rightsquigarrow \quad \dfrac{}{\vdash \mathcal{K} : \Gamma \Uparrow \top, L}$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Uparrow P, Q, L}{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Uparrow P \mathbin{⅋} Q, L} \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma \Uparrow P \mathbin{⅋} Q, L} \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Uparrow P, Q, L \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma \Uparrow P, Q, L}}{\vdash \mathcal{K} : \Gamma \Uparrow P \mathbin{⅋} Q, L}$$

$$\dfrac{\dfrac{\begin{matrix}[\Sigma_1] \\ \vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Uparrow P, L\end{matrix} \qquad \begin{matrix}[\Sigma_2] \\ \vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Uparrow Q, L\end{matrix}}{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Uparrow P \,\&\, Q, L} \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma \Uparrow P \,\&\, Q, L}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\Sigma_1 \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma \Uparrow P, L} \qquad \dfrac{\Sigma_2 \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma \Uparrow Q, L}}{\vdash \mathcal{K} : \Gamma \Uparrow P \,\&\, Q, L}$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1, \mathsf{C}_i, P_a : \Gamma \Uparrow L}{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Uparrow ?^a P, L} \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma \Uparrow ?^a P, L} \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \mathcal{K}_1, P_a, \mathsf{C}_i : \Gamma \Uparrow L \qquad \Pi_2}{\vdash \mathcal{K}, P_a : \Gamma \Uparrow L}}{\vdash \mathcal{K} : \Gamma \Uparrow ?^a P, L}$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1, \mathsf{C}_i, P_a : \Gamma \Uparrow L}{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Uparrow ?^a P, L} \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma \Uparrow ?^a P, L} \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \mathcal{K}_1, P_a, \mathsf{C}_i : \Gamma \Uparrow L \qquad \dfrac{\Pi_2}{\vdash \mathcal{K}_2, P_a : \cdot \Downarrow !^i H^\perp}\ \mathsf{W}}{\vdash \mathcal{K}, P_a : \Gamma \Uparrow L}}{\vdash \mathcal{K} : \Gamma \Uparrow ?^a P, L}$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Uparrow P[c/x], L}{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Uparrow \forall x P, L} \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma \Uparrow \forall x P, L} \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Uparrow P[c/x], L \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma \Uparrow P[c/x], L}}{\vdash \mathcal{K} : \Gamma \Uparrow \forall x P, L}$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1, \mathsf{C}_i : P, \Gamma \Uparrow L}{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Uparrow P, L} \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma \Uparrow P, L} \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \mathcal{K}_1, \mathsf{C}_i : P, \Gamma \Uparrow L \qquad \Pi_2}{\vdash \mathcal{K} : P, \Gamma \Uparrow L}}{\vdash \mathcal{K} : \Gamma \Uparrow P, L}$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Downarrow Q}{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Uparrow \cdot} \qquad \vdash \mathcal{K}_2, \mathsf{C}_i : \cdot \Uparrow !^i \mathsf{C}^\perp}{\vdash \mathcal{K} : \Gamma \Uparrow \cdot} \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Downarrow Q \qquad \vdash \mathcal{K}_2, \mathsf{C}_i : \cdot \Uparrow !^i \mathsf{C}^\perp}{\vdash \mathcal{K} : \Gamma \Uparrow Q}\,\mathsf{cut}_4}{\vdash \mathcal{K} : \Gamma \Uparrow \cdot}\,\mathcal{L}A$$

$$\dfrac{\dfrac{\vdash \mathcal{K}'_1, \mathsf{C}_i : \Gamma \Downarrow Q}{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Uparrow \cdot} \qquad \vdash \mathcal{K}_2, \mathsf{C}_i : \cdot \Uparrow !^i \mathsf{C}^\perp}{\vdash \mathcal{K} : \Gamma \Uparrow \cdot} \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \mathcal{K}'_1, \mathsf{C}_i : \Gamma \Downarrow Q \qquad \vdash \mathcal{K}_2, \mathsf{C}_i : \cdot \Uparrow !^i \mathsf{C}^\perp}{\vdash \mathcal{K}' : \Gamma \Uparrow Q}\,\mathsf{cut}_4}{\vdash \mathcal{K} : \Gamma \Uparrow \cdot}\,\mathcal{L}A*$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Downarrow P}{\vdash \mathcal{K}_1, \mathsf{C}_i : P, \Gamma \Uparrow \cdot} \qquad \vdash \mathcal{K}_2, \mathsf{C}_i : \cdot \Uparrow !^i \mathsf{C}^\perp}{\vdash \mathcal{K} : P, \Gamma \Uparrow \cdot} \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Downarrow P \qquad \vdash \mathcal{K}_2, \mathsf{C}_i : \cdot \Uparrow !^i \mathsf{C}^\perp}{\vdash \mathcal{K} : \Gamma \Uparrow P}\,\mathsf{cut}_4}{\vdash \mathcal{K} : P, \Gamma \Uparrow \cdot}$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1, \mathsf{C}_c : \Gamma \Downarrow \mathsf{C}}{\vdash \mathcal{K}_1, \mathsf{C}_c : \Gamma \Uparrow \cdot} \qquad \dfrac{\vdash \mathcal{K}_2 : \cdot \Uparrow \mathsf{C}^\perp}{\vdash \mathcal{K}_2 : \cdot \Downarrow !^c \mathsf{C}^\perp}}{\vdash \mathcal{K} : \Gamma \Uparrow \cdot}$$

We should analyze if C is (1) a negative ou (2) a positive formula. Note that $\mathcal{K}_2$ is unbounded.

$$1 \rightsquigarrow \quad \dfrac{\vdash \mathcal{K}_2 : \cdot \Uparrow \mathsf{C}^\perp \qquad \dfrac{\dfrac{\vdash \mathcal{K}_1, \mathsf{C}_c : \Gamma \Downarrow \mathsf{C} \qquad \vdash \mathcal{K}_2 : \cdot \Downarrow !^c \mathsf{C}^\perp}{\vdash \mathcal{K}_1 : \Gamma \Uparrow \mathsf{C}}\,\mathsf{cut}_2}{\vdash \mathcal{K}_1 : \Gamma \Downarrow \mathsf{C}}}{\vdash \mathcal{K}_1 : \Gamma \Uparrow \cdot}\,\mathsf{cut}_2$$

$$2 \rightsquigarrow \quad \dfrac{\dfrac{\vdash \mathcal{K}_1, \mathsf{C}_c : \Gamma \Downarrow \mathsf{C} \qquad \vdash \mathcal{K}_2 : \cdot \Downarrow !^c \mathsf{C}^\perp}{\vdash \mathcal{K}_1 : \Gamma \Uparrow \mathsf{C}} \qquad \dfrac{\vdash \mathcal{K}_2 : \cdot \Uparrow \mathsf{C}^\perp}{\vdash \mathcal{K}_2 : \cdot \Downarrow \mathsf{C}^\perp}}{\vdash \mathcal{K}_1 : \Gamma \Uparrow \cdot}\,\mathsf{cut}_2$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1 : \Gamma \Downarrow \mathsf{C}}{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Uparrow \cdot} \qquad \dfrac{\dfrac{\vdash \Upsilon, \Sigma_c^u : \cdot \Uparrow \mathsf{C}^\perp, \mathcal{F}(\Sigma^l)}{\vdash \mathcal{K}_2 : \cdot \Uparrow \cdot /\!/ i \vdash \cdot : \cdot \Uparrow \mathsf{C}^\perp}}{\vdash \mathcal{K}_2 : \cdot \Downarrow\, !^i \mathsf{C}^\perp}}{\vdash \mathcal{K} : \Gamma \Uparrow \cdot}$$

- $\mathcal{K}_2 \equiv \Upsilon^4, \Sigma^{\bcancel{4}}, \Xi^U$ where $\Upsilon_{i\preceq}$ and $\Sigma_{i\preceq}$
- $T \in \mathcal{U}(i)$, therefore Υ^T and Σ^T.

$$\rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\vdash \Upsilon, \Sigma_c^u : \cdot \Uparrow \mathsf{C}^\perp, \mathcal{F}(\Sigma^l)}{\vdash \Upsilon, \Sigma^u, \Xi, \Sigma_c^u : \cdot \Uparrow \mathsf{C}^\perp, \mathcal{F}(\Sigma^l)}\ \mathsf{W} \qquad \dfrac{\vdash \mathcal{K}_1 : \Gamma \Downarrow \mathsf{C}}{\vdash \mathcal{K}_1^l, \Upsilon^u, \Sigma^u, \Xi, \Sigma_c^u : \Gamma \Downarrow \mathsf{C}}\ \mathsf{W}}{\vdash \mathcal{K}_1^l, \Upsilon, \Sigma^u, \Xi, \Sigma_c^u : \Gamma \Uparrow \mathcal{F}(\Sigma^l)}\ \mathsf{cut}_2}{\vdash \mathcal{K} : \Gamma \Uparrow \cdot}\ \mathsf{A}_l,\ \mathsf{C}_c$$

Note that $\mathcal{K}_1 \equiv \mathcal{K}_1^l, \Upsilon^U, \Sigma^U, \Xi$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Downarrow \mathsf{C}}{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Uparrow \cdot} \qquad \dfrac{\dfrac{\vdash \Upsilon, \Sigma_c : \cdot \Uparrow \mathsf{C}^\perp}{\vdash \mathcal{K}_2 : \cdot \Uparrow \cdot /\!/ i \vdash \cdot : \cdot \Uparrow \mathsf{C}^\perp}}{\vdash \mathcal{K}_2 : \cdot \Downarrow\, !^i \mathsf{C}^\perp}}{\vdash \mathcal{K} : \Gamma \Uparrow \cdot}$$

- $\mathcal{K}_2 \equiv \Upsilon^{U4}, \Sigma^{U\bcancel{4}}, \Xi^U$ where $\Upsilon_{i\preceq}$ and $\Sigma_{i\preceq}$
- $T \in \mathcal{U}(i)$, therefore Υ^T and Σ^T.

We should analyze if C is (1) a negative ou (2) a positive formula. Note that $\mathcal{K}_2$ is unbounded.

$$1 \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\vdash \Upsilon, \Sigma_c : \cdot \Uparrow \mathsf{C}^\perp}{\vdash \mathcal{K}_2, \Sigma_c : \cdot \Uparrow \mathsf{C}^\perp}\ \mathsf{W} \qquad \dfrac{\dfrac{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Downarrow \mathsf{C} \qquad \vdash \mathcal{K}_2 : \cdot \Downarrow\, !^i \mathsf{C}^\perp}{\vdash \mathcal{K} : \Gamma \Uparrow \mathsf{C}}}{\vdash \mathcal{K}, \Sigma_c : \Gamma \Downarrow \mathsf{C}}\ \mathsf{W}}{\vdash \mathcal{K}, \Sigma_c : \Gamma \Uparrow \cdot}\ \mathsf{cut}_2}{\vdash \mathcal{K} : \Gamma \Uparrow \cdot}\ \mathsf{C}_c$$

$$2 \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Downarrow \mathsf{C} \qquad \vdash \mathcal{K}_2 : \cdot \Downarrow\, !^i \mathsf{C}^\perp}{\vdash \mathcal{K} : \Gamma \Uparrow \mathsf{C}}}{\vdash \mathcal{K}, \Sigma_c : \Gamma \Uparrow \mathsf{C}}\ \mathsf{W} \qquad \dfrac{\vdash \Upsilon, \Sigma_c : \cdot \Uparrow \mathsf{C}^\perp}{\vdash \mathcal{K}_2, \Sigma_c : \cdot \Downarrow \mathsf{C}^\perp}\ \mathsf{W}}{\vdash \mathcal{K}, \Sigma_c : \Gamma \Uparrow \cdot}\ \mathsf{cut}_2}{\vdash \mathcal{K} : \Gamma \Uparrow \cdot}\ \mathsf{C}_c$$

7 Elimination of cut_4

$$\frac{\vdash \mathcal{K}_1, \mathsf{C_i}; \Gamma \Downarrow \mathsf{F} \quad \vdash \mathcal{K}_2; \cdot \Downarrow\, !^i \mathsf{C}^\perp}{\vdash \mathcal{K}; \Gamma \Uparrow \mathsf{F}}\ cut_4$$

Note that $\vdash \Theta : \Gamma \Downarrow P$ implies $\vdash \Theta : \Gamma \Uparrow P$.

$$\frac{\dfrac{}{\vdash \mathcal{K}_1, \mathsf{C}_i : X^+ \Downarrow X^-} \quad \vdash \mathcal{K}_2 : \cdot \Downarrow\, !^i \mathsf{C}^\perp}{\vdash \mathcal{K} : X^+ \Uparrow X^-} \quad \rightsquigarrow \quad \frac{\dfrac{}{\vdash \mathcal{K} : X^+ \Downarrow X^-}}{\vdash \mathcal{K} : X^+ \Uparrow X^-}$$

$$\frac{\dfrac{}{\vdash \mathcal{K}_1, X_a^+, \mathsf{C}_i : \cdot \Downarrow X^-} \quad \vdash \mathcal{K}_2 : \cdot \Downarrow\, !^i \mathsf{C}^\perp}{\vdash \mathcal{K}, X_a^+ : \cdot \Uparrow X^-} \quad \rightsquigarrow \quad \frac{\dfrac{}{\vdash \mathcal{K}, X_a^+ : \cdot \Downarrow X^-}}{\vdash \mathcal{K}, X_a^+ : \cdot \Uparrow X^-}$$

$$\frac{\dfrac{}{\vdash \mathcal{K}_1, X_a^+, \mathsf{C}_i : \cdot \Downarrow X^-} \quad \vdash \mathcal{K}_2, X_a^+ : \cdot \Downarrow\, !^i \mathsf{C}^\perp}{\vdash \mathcal{K}, X_a^+ : \cdot \Uparrow X^-} \quad \rightsquigarrow \quad \frac{\dfrac{}{\vdash \mathcal{K}, X_a^+ : \cdot \Downarrow X^-}}{\vdash \mathcal{K}, X_a^+ : \cdot \Uparrow X^-}$$

$$\frac{\dfrac{}{\vdash \mathcal{K}_1, X_i^+ : \cdot \Downarrow X^-} \quad \vdash \mathcal{K}_2 : \cdot \Downarrow\, !^i X^-}{\vdash \mathcal{K} : \cdot \Uparrow X^-} \quad \rightsquigarrow \quad \frac{\vdash \mathcal{K} : \cdot \Downarrow\, !^i X^-}{\vdash \mathcal{K} : \cdot \Uparrow X^-}$$

Note above that $\{U, T\} \subset \mathcal{U}(i)$: then $\vdash \mathcal{K} : \cdot \Downarrow\, !^i P$ implies $\vdash \mathcal{K} : \cdot \Uparrow P$, for all P. In the case below, i is linear.

$$\frac{\dfrac{}{\vdash \mathcal{K}_1, X_i^+ : \cdot \Downarrow X^-} \quad \dfrac{\vdash \Upsilon, \Sigma_c^u : \cdot \Uparrow X^-, \mathcal{F}(\Sigma^l)}{\vdash \mathcal{K}_2 : \cdot \Downarrow\, !^i X^-}}{\vdash \mathcal{K} : \cdot \Uparrow X^-} \quad \rightsquigarrow \quad \frac{\dfrac{\vdash \Upsilon, \Sigma_c^u : \cdot \Uparrow X^-, \mathcal{F}(\Sigma^l)}{\vdash \mathcal{K}, \Sigma_c^u : \cdot \Uparrow X^-, \mathcal{F}(\Sigma^l)}\ \mathsf{W}}{\vdash \mathcal{K} : \cdot \Uparrow X^-}\ \mathsf{A}_l,\ \mathsf{C}_c$$

$$\frac{\vdash \mathcal{K}_1, \mathsf{C}_i : \cdot \Downarrow 1 \quad \vdash \mathcal{K}_2 : \cdot \Downarrow\, !^i \mathsf{C}^-}{\vdash \mathcal{K} : \cdot \Uparrow 1} \quad \rightsquigarrow \quad \frac{\dfrac{}{\vdash \mathcal{K} : \cdot \Downarrow 1}}{\vdash \mathcal{K} : \cdot \Uparrow 1}$$

$$\frac{\dfrac{\begin{matrix}[\Sigma_1] \\ \vdash \mathcal{K}_1', \mathsf{C}_i : \Gamma_P \Downarrow P\end{matrix} \quad \begin{matrix}[\Sigma_2] \\ \vdash \mathcal{K}_1'', \mathsf{C}_i : \Gamma_Q \Downarrow Q\end{matrix}}{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Downarrow P \otimes Q} \quad \vdash \mathcal{K}_2 : \cdot \Downarrow\, !^i \mathsf{C}^\perp}{\vdash \mathcal{K} : \Gamma \Uparrow P \otimes Q}$$

$$\rightsquigarrow \quad \frac{\dfrac{\dfrac{\Sigma_1 \quad \Pi_2}{\vdash \mathcal{K}' : \Gamma_P \Uparrow P} \quad \dfrac{\Sigma_2 \quad \Pi_2}{\vdash \mathcal{K}'' : \Gamma_Q \Uparrow Q}}{\vdash \mathcal{K} : \Gamma, P \otimes Q \Uparrow \cdot}\ \mathcal{L}\otimes}{\vdash \mathcal{K} : \Gamma \Uparrow P \otimes Q}$$

$$\frac{\dfrac{\begin{matrix}[\Sigma_1] \\ \vdash \mathcal{K}_1',, \mathsf{C}_i : \Gamma_P \Downarrow P\end{matrix} \quad \begin{matrix}[\Sigma_2] \\ \vdash \mathcal{K}_1'' : \Gamma_Q \Downarrow Q\end{matrix}}{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Downarrow P \otimes Q} \quad \vdash \mathcal{K}_2 : \cdot \Downarrow\, !^i \mathsf{C}^\perp}{\vdash \mathcal{K} : \Gamma \Uparrow P \otimes Q}$$

$$\rightsquigarrow \qquad \cfrac{\cfrac{\Sigma_1 \qquad \Pi_2}{\vdash \mathcal{K}' : \Gamma_P \Uparrow P} \qquad \cfrac{\vdash \mathcal{K}_1'' : \Gamma_Q \Downarrow Q}{\vdash \mathcal{K}_1'' : \Gamma_Q \Uparrow Q}}{\cfrac{\vdash \mathcal{K} : \Gamma, P \otimes Q \Uparrow \cdot}{\vdash \mathcal{K} : \Gamma \Uparrow P \otimes Q}}\ \mathcal{L}\otimes$$

Same reasoning for $\oplus$ and $\exists$, see below.

$$\cfrac{\cfrac{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Downarrow P}{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Downarrow P \oplus Q} \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma \Uparrow P \oplus Q} \qquad \rightsquigarrow \qquad \cfrac{\cfrac{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Downarrow P \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma \Uparrow P}\ \mathcal{L}\oplus}{\cfrac{\vdash \mathcal{K} : \Gamma, P \oplus Q \Uparrow \cdot}{\vdash \mathcal{K} : \Gamma \Uparrow P \oplus Q}}$$

$$\cfrac{\cfrac{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Downarrow P[c/x]}{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Downarrow \exists x.P} \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma \Uparrow \exists x.P} \qquad \rightsquigarrow \qquad \cfrac{\cfrac{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Downarrow P[c/x] \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma \Uparrow P[c/x]}\ \mathcal{L}\exists}{\cfrac{\vdash \mathcal{K} : \Gamma, \exists x.P \Uparrow \cdot}{\vdash \mathcal{K} : \Gamma \Uparrow \exists x.P}}$$

$$\cfrac{\cfrac{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Uparrow P}{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Downarrow P} \qquad \vdash \mathcal{K}_2; \cdot \Downarrow\ !^i\mathsf{C}^\perp}{\vdash \mathcal{K} : \Gamma \Uparrow P} \qquad \rightsquigarrow \qquad \cfrac{\vdash \mathcal{K}_1, \mathsf{C}_i : \Gamma \Uparrow P \qquad \vdash \mathcal{K}_2; \cdot \Downarrow\ !^i\mathsf{C}^\perp}{\vdash \mathcal{K} : \Gamma \Uparrow P}\ \mathsf{cut}_3$$

$$\cfrac{\cfrac{\vdash \mathcal{K}_1, \mathsf{C}_i : \cdot \Uparrow P}{\vdash \mathcal{K}_1, \mathsf{C}_i : \cdot \Downarrow !^c P} \qquad \vdash \mathcal{K}_2; \cdot \Downarrow\ !^i\mathsf{C}^\perp}{\vdash \mathcal{K} : \cdot \Uparrow !^c P} \qquad \rightsquigarrow \qquad \cfrac{\vdash \mathcal{K}_1, \mathsf{C}_i : \cdot \Uparrow P \qquad \vdash \mathcal{K}_2; \cdot \Downarrow\ !^i\mathsf{C}^\perp}{\cfrac{\vdash \mathcal{K} : \cdot \Uparrow P}{\cfrac{\vdash \mathcal{K} : \cdot \Downarrow !^c P}{\vdash \mathcal{K} : \cdot \Uparrow !^c P}}}\ \mathsf{cut}_3$$

$a \preceq i$ and $4 \in \mathcal{U}(i)$

$$\cfrac{\cfrac{\vdash \Upsilon_+, H_{i+}, \Sigma^u_{lc}; \cdot \Uparrow P, \mathcal{F}(\Sigma^l)}{\vdash \Lambda, \Theta_1, H_i; \cdot \Downarrow !^a P} \qquad \cfrac{\vdash \Upsilon'_+; \cdot \Uparrow H^\perp}{\vdash \Lambda, \Theta_2; \cdot \Downarrow !^i H^\perp}}{\vdash \Lambda, \Theta_1, \Theta_2; \cdot \Uparrow !^a P}$$

$$\rightsquigarrow \qquad \cfrac{\cfrac{\vdash \Upsilon_+, \Sigma^u_{lc}; \cdot \Uparrow P, \mathcal{F}(\Sigma)}{\vdash (\Upsilon'_+)^u, \Upsilon_+, \Sigma^u_{lc}; \cdot \Uparrow P, \mathcal{F}(\Sigma)}\ [W] \qquad \cfrac{\vdash \Upsilon'_+; \cdot \Downarrow !^{i+} H^\perp}{\vdash \Upsilon'_+, (\Upsilon_+)^u, \Sigma^u_{lc}; \cdot \Downarrow !^{i+} H^\perp}\ [W]}{\cfrac{\vdash \Upsilon_+, \Upsilon'_+, \Sigma^u_{lc}; \cdot \Uparrow P, \mathcal{F}(\Sigma_1)}{\cfrac{\vdash \Lambda, \Theta_1, \Theta_2; \cdot \Downarrow !^a P}{\cfrac{\vdash \Lambda, \Theta_1, \Theta_2; !^a P \Uparrow \cdot}{\vdash \Lambda, \Theta_1, \Theta_2; \cdot \Uparrow !^a P}}}\ \mathrm{C3}}\ [\Uparrow\mathrm{CC}]$$

$a \preceq i$ and $4 \notin \mathcal{U}(i)$

$$\dfrac{\dfrac{\vdash \Upsilon_+, \Sigma^u_{lc}, H_{lc}; \cdot \Uparrow P, \mathcal{F}(\Sigma)}{\vdash \Lambda, \Theta_1, H_i; \cdot \Downarrow !^a P} \qquad \dfrac{\vdash \Lambda; \cdot \Uparrow H^\perp}{\vdash \Lambda; \cdot \Downarrow !^i H^\perp}}{\vdash \Lambda, \Theta_1; \cdot \Uparrow !^a P}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash \Upsilon_+, \Sigma^u_{lc}, H_{lc}; \cdot \Uparrow P, \mathcal{F}(\Sigma)}{\vdash \Lambda, \Upsilon_+, \Sigma^u_{lc}, H_{lc}; \cdot \Uparrow P, \mathcal{F}(\Sigma)}\,[W] \qquad \dfrac{\vdash \Lambda; \cdot \Downarrow !^{lc} H^\perp}{\vdash \Lambda, \Upsilon^u_+, \Sigma^u_{lc}; \cdot \Downarrow !^{lc} H^\perp}\,[W]}{\vdash \Lambda, \Upsilon_+, \Sigma^u_{lc}; \cdot \Uparrow P, \mathcal{F}(\Sigma)}\,[\Uparrow\mathrm{CC}]}{\vdash \Lambda, \Theta_1; \cdot \Downarrow !^a P}}{\vdash \Lambda, \Theta_1; !^a P \Uparrow \cdot}}{\vdash \Lambda, \Theta_1; \cdot \Uparrow !^a P}$$

$a \preceq i$ and $4 \notin \mathcal{U}(i)$

$$\dfrac{\dfrac{\vdash \Upsilon_+, \Sigma^u_{lc}; \cdot \Uparrow P, \mathcal{F}(\Sigma^l_1), H, \mathcal{F}(\Sigma^l_3)}{\vdash \Lambda, \Theta_1, H_i; \cdot \Downarrow !^a P} \qquad \dfrac{\vdash \Upsilon'_+, (\Sigma')^u_{lc}; \cdot \Uparrow H^\perp, \mathcal{K}(\Sigma^l_2)}{\vdash \Lambda, \Theta_2; \cdot \Downarrow !^i H^\perp}}{\vdash \Lambda, \Theta_1, \Theta_2; \cdot \Downarrow !^a P}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash \Upsilon_+, \Sigma^u_{lc}; \cdot \Uparrow P, \mathcal{F}(\Sigma_1), H, \mathcal{F}(\Sigma_3)}{\vdash \Upsilon_+, \Sigma^u_{lc}, (\Upsilon')^u_+, (\Sigma')^u_{lc}; \cdot \Uparrow P, \mathcal{F}(\Sigma_1), H, \mathcal{F}(\Sigma_3)}\,[W] \qquad \dfrac{\vdash \Upsilon'_+, (\Sigma')^u_{lc}; \cdot \Uparrow H^\perp, \mathcal{K}(\Sigma_2)}{\vdash \Upsilon^u_+, \Sigma^u_{lc}, \Upsilon'_+, (\Sigma')^u_{lc}; \cdot \Uparrow H^\perp, \mathcal{K}(\Sigma_2)}\,[W]}{\vdash \ \vdash \Upsilon_+, \Sigma^u_{lc}, \Upsilon'_+, (\Sigma')^u_{lc}; \cdot \Uparrow P, \mathcal{F}(\Sigma_1), \mathcal{F}(\Sigma_3), \mathcal{F}(\Sigma_2)}\,[\Uparrow\mathrm{C}^*]}{\vdash \Lambda, \Theta_1, \Theta_2; \cdot \Downarrow !^a P}}{\vdash \Lambda, \Theta_1, \Theta_2; !^a P \Uparrow \cdot}}{\vdash \Lambda, \Theta_1, \Theta_2; \cdot \Uparrow !^a P}$$

8 Elimination of cut_5

$$\dfrac{\vdash \mathcal{K}_1; \Gamma \Uparrow \mathsf{L}_1, \mathsf{H}, \mathsf{L}_2 \qquad \vdash \mathcal{K}_2; \Delta \Uparrow \mathsf{H}^{\perp}, \mathsf{L}_3}{\vdash \mathcal{K}; \Gamma, \Delta \Uparrow \mathsf{L}_1, \mathsf{L}_2, \mathsf{L}_3}\ \mathsf{cut}_5$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1 : \Gamma \Uparrow L_1, \mathsf{C}, L_2}{\vdash \mathcal{K}_1 : \Gamma \Uparrow \bot, L_1, \mathsf{C}, L_2} \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow \bot, L_1, L_2, L_3} \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \mathcal{K}_1 : \Gamma \Uparrow L_1, \mathsf{C}, L_2 \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow L_1, L_2, L_3}}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow \bot, L_1, L_2, L_3}$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1 : \Gamma \Uparrow P, Q, L_1, \mathsf{C}, L_2}{\vdash \mathcal{K}_1 : \Gamma \Uparrow P \parr Q, L_1, \mathsf{C}, L_2} \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow P \parr Q, L_1, L_2, L_3} \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \mathcal{K}_1 : \Gamma \Uparrow P, Q, L_1, \mathsf{C}, L_2 \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow P, Q, L_1, L_2, L_3}}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow P \parr Q, L_1, L_2, L_3}$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1 : \Gamma \Uparrow P, L_1, \mathsf{C}, L_2 \qquad \vdash \mathcal{K}_1 : \Gamma \Uparrow Q, L_1, \mathsf{C}, L_2}{\vdash \mathcal{K}_1 : \Gamma \Uparrow P \,\&\, Q, L_1, \mathsf{C}, L_2} \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow P \,\&\, Q, L_1, L_2, L_3}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\vdash \mathcal{K}_1 : \Gamma \Uparrow P, L_1, \mathsf{C}, L_2 \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow P, L_1, L_2, L_3} \qquad \dfrac{\vdash \mathcal{K}_1 : \Gamma \Uparrow Q, L_1, \mathsf{C}, L_2 \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow Q, L_1, L_2, L_3}}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow P \,\&\, Q, L_1, L_2, L_3}$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1 : \Gamma \Uparrow P[c/x], L_1, \mathsf{C}, L_2}{\vdash \mathcal{K}_1 : \Gamma \Uparrow \forall x P, L_1, \mathsf{C}, L_2} \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow \forall x P, L_1, L_2, L_3} \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \mathcal{K}_1 : \Gamma \Uparrow P[c/x], L_1, \mathsf{C}, L_2 \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow P[c/x], L_1, L_2, L_3}}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow \forall x P, L_1, L_2, L_3}$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1, P_i : \Gamma \Uparrow L_1, \mathsf{C}, L_2}{\vdash \mathcal{K}_1 : \Gamma \Uparrow ?^i P, L_1, \mathsf{C}, L_2} \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow ?^i P, L_1, L_2, L_3} \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \mathcal{K}_1, P_i : \Gamma \Uparrow L_1, \mathsf{C}, L_2 \qquad \Pi_2}{\vdash \mathcal{K}, P_i : \Gamma, \Delta \Uparrow L_1, L_2, L_3}}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow ?^i P, L_1, L_2, L_3}$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1, P_i : \Gamma \Uparrow L_1, \mathsf{C}, L_2}{\vdash \mathcal{K}_1 : H, \Gamma \Uparrow ?^i P, L_1, \mathsf{C}, L_2} \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow ?^i P, L_1, L_2, L_3} \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \mathcal{K}_1, P_i : \Gamma \Uparrow L_1, \mathsf{C}, L_2 \qquad \dfrac{\Pi_2}{\vdash \mathcal{K}_2, P_i : \Delta \Uparrow H^{\perp}, V}\ \mathsf{W}}{\vdash \mathcal{K}, P_i : \Gamma, \Delta \Uparrow L_1, L_2, L_3}}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow ?^i P, L_1, L_2, L_3}$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1 : P, \Gamma \Uparrow L_1, \mathsf{C}, L_2}{\vdash \mathcal{K}_1 : \Gamma \Uparrow P, L_1, \mathsf{C}, L_2} \qquad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow P, L_1, L_2, L_3} \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \mathcal{K}_1 : P, \Gamma \Uparrow L_1, \mathsf{C}, L_2 \qquad \Pi_2}{\vdash \mathcal{K} : P, \Gamma, \Delta \Uparrow L_1, L_2, L_3}}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow P, L_1, L_2, L_3}$$

$$\dfrac{\dfrac{\vdash \mathcal{K}_1 : \Gamma, P \Uparrow L_2}{\vdash \mathcal{K}_1 : \Gamma \Uparrow P, L_2} \qquad \vdash \mathcal{K}_2 : \Delta \Uparrow P^{\perp}, L_3}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow L_2, L_3} \quad \rightsquigarrow \quad \dfrac{\vdash \mathcal{K}_1 : P, \Gamma \Downarrow L_2 \qquad \vdash \mathcal{K}_2 : \Delta \Uparrow P^{\perp}, L_3}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow L_2, L_3}\,\mathsf{cut}_1$$

$$\dfrac{\vdash \mathcal{K}_1 : \Gamma \Uparrow P, L_2 \qquad \dfrac{\vdash \mathcal{K}_2 : \Delta, P^{\perp} \Uparrow L_3}{\vdash \mathcal{K}_2 : \Delta \Uparrow P^{\perp}, L_3}}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow L_2, L_3} \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \mathcal{K}_1 : P^{\perp}, \Delta \Downarrow L_3 \qquad \vdash \mathcal{K}_2 : \Gamma \Uparrow P, L_2}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow L_3, L_2}\,\mathsf{cut}_1}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow L_2, L_3}$$