

Assignment

Cut Elimination in LL with Multimodalities

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June 9, 2022

$$\begin{array}{c} \frac{\vdash \mathcal{K}_1; \Gamma, C \uparrow L_1 \quad \vdash \mathcal{K}_2; \Delta \uparrow C^\perp, L_2}{\vdash \mathcal{K}; \Gamma, \Delta \uparrow L_1, L_2} \text{ cut}_1 \quad \frac{\vdash \mathcal{K}_1; \Gamma \uparrow C, L \quad \vdash \mathcal{K}_2; \Delta \downarrow C^\perp}{\vdash \mathcal{K}; \Gamma, \Delta \uparrow L} \text{ cut}_2 \\[10pt] \frac{\vdash \mathcal{K}_1, C_i; \Gamma \uparrow L \quad \vdash \mathcal{K}_2; \cdot \downarrow !^i C^\perp}{\vdash \mathcal{K}; \Gamma \uparrow L} \text{ cut}_3 \\[10pt] \frac{\vdash \mathcal{K}_1, C_i; \Gamma \downarrow F \quad \vdash \mathcal{K}_2; \cdot \downarrow !^i C^\perp}{\vdash \mathcal{K}; \Gamma \uparrow F} \text{ cut}_4 \quad \frac{\vdash \mathcal{K}_1; \Gamma \uparrow L_1, C, L_2 \quad \vdash \mathcal{K}_2; \Delta \uparrow C^\perp, L_3}{\vdash \mathcal{K}; \Gamma, \Delta \uparrow L_1, L_2, L_3} \text{ cut}_5 \end{array}$$

Where C is the cut-formula, F is a formula, Γ and Δ are multisets of formulas, \mathcal{K}_1 and \mathcal{K}_2 are indexed contexts of formulas, and L is a list of formulas.

Moreover, \mathcal{K}_1 , \mathcal{K}_2 and \mathcal{K} have the same unbounded formulas and the linear formulas in \mathcal{K} are the multiset union of the linear formulas in \mathcal{K}_1 and \mathcal{K}_2 .

1 Exponential Rules

$i \preceq j$, $4 \notin \mathcal{U}(j)$ and j is unbounded

$$\frac{\Theta; \cdot \uparrow \cdot // i \vdash \Upsilon, \mathbf{F}_c; \cdot \uparrow L}{\Theta, \mathbf{F}_j; \cdot \uparrow \cdot // i \vdash \Upsilon; \cdot \uparrow L} [?_{Ku}]$$

$i \preceq j$, $4 \notin \mathcal{U}(j)$ and j is linear

$$\frac{\Theta; \cdot \uparrow \cdot // i \vdash \Upsilon; \cdot \uparrow L, \mathbf{F}}{\Theta, \mathbf{F}_j; \cdot \uparrow \cdot // i \vdash \Upsilon; \cdot \uparrow L} [?_{Kl}]$$

$i \preceq j$ and $4 \in \mathcal{U}(j)$

$$\frac{\Theta; \cdot \uparrow \cdot // i \vdash \Upsilon, \mathbf{F}_{j+}; \cdot \uparrow L}{\Theta, \mathbf{F}_j; \cdot \uparrow \cdot // i \vdash \Upsilon; \cdot \uparrow L} [?_{K4}]$$

Θ is unbounded and $\forall j \in \Theta, i \not\preceq j$

$$\frac{\vdash \Upsilon; \cdot \uparrow L}{\Theta; \cdot \uparrow \cdot // i \vdash \Upsilon; \cdot \uparrow L} [R_r]$$

Θ is unbounded

$$\frac{\vdash \Theta; \cdot \uparrow P}{\vdash \Theta; \cdot \downarrow !^{lc} P} [!^c]$$

$$\frac{\Theta; \cdot \uparrow \cdot // i \vdash \cdot; \cdot \uparrow F}{\Theta; \cdot \downarrow !^i F} [!^i]$$

$$\frac{\Theta; \cdot \uparrow \cdot // i \vdash \cdot; \cdot \uparrow \cdot}{\Theta; \cdot \uparrow \cdot} [D \in \mathcal{U}(i)]$$

- The subexponential c is not related with anyone
- $\mathcal{U}(c) = \{K, U, T\}$
- Let Θ be a subexp. context, $\mathcal{F}(\Theta)$ is all formulas in Θ
- Let Θ be a subexp context, Θ^u (resp. Θ^l) is the unbounded (resp. linear) context in Θ
- Note that $\Theta \equiv \Theta^u, \Theta^l$ with $\Theta^u \cap \Theta^l = \emptyset$
- $\Upsilon_{a \preceq}$ means that $\forall b : \text{subexp}, b \in \Upsilon \implies a \preceq b$
- Υ^Z means that $\forall i : \text{subexp}, i \in \Upsilon \implies Z \in \mathcal{U}(i)$
- Υ^\times means that $\forall i : \text{subexp}, i \in \Upsilon \implies Z \notin \mathcal{U}(i)$

2 Height-Preserving Lemmas

Theorem 2.1 (W). If $U \in \mathcal{U}(j)$ and $\vdash \Theta; \Delta; \Downarrow X$ then $\vdash F_j, \Theta; \Delta; \Downarrow X$

Theorem 2.2 (C). If $U \in \mathcal{U}(j)$ and $\vdash F_j, F_j, \Theta; \Delta; \Downarrow X$ then $\vdash F_j, \Theta; \Delta; \Downarrow X$

Theorem 2.3 (C_c). If $\{U, T\} \subseteq \mathcal{U}(j)$ and $\vdash F_c, F_j, \Theta; \Delta; \Downarrow X$ then $\vdash F_j, \Theta; \Delta; \Downarrow X$

Theorem 2.4 (A_c). If $\{U, T\} \subseteq \mathcal{U}(j)$ and $\vdash F_j, \Theta; F, \Delta; \Downarrow X$ then $\vdash F_j, \Theta; \Delta; \Downarrow X$

Theorem 2.5 (A_l). If $T \in \mathcal{U}(j)$ and $\vdash \Theta; F, \Delta; \Downarrow X$ then $\vdash F_j, \Theta; \Delta; \Downarrow X$

3 Elimination of cut_1

$$\frac{\vdash \mathcal{K}_1; \Gamma, C \uparrow L_1 \quad \vdash \mathcal{K}_2; \Delta \uparrow C^\perp, L_2}{\vdash \mathcal{K}; \Gamma, \Delta \uparrow L_1, L_2} \text{cut}_1$$

$$\frac{\frac{\vdash \mathcal{K}_1 : \Gamma, C \uparrow L_1}{\vdash \mathcal{K}_1 : \Gamma, C \uparrow \perp, L_1} \quad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow \perp, L_1, L_2} \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1 : \Gamma, C \uparrow L_1 \quad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow L_1, L_2}}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow \perp, L_1, L_2}$$

$$\frac{\frac{\vdash \mathcal{K}_1 : \Gamma, C \uparrow P, Q, L_1}{\vdash \mathcal{K}_1 : \Gamma, C \uparrow P \wp Q, L_1} \quad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow P \wp Q, L_1, L_2} \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1 : \Gamma, C \uparrow P, Q, L_1 \quad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow P, Q, L_1, L_2}}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow P \wp Q, L_1, L_2}$$

$$\frac{\frac{\vdash \mathcal{K}_1 : \Gamma, C \uparrow P, L_1 \quad \vdash \mathcal{K}_1 : \Gamma, C \uparrow Q, L_1}{\vdash \mathcal{K}_1 : \Gamma, C \uparrow P \& Q, L_1} \quad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow P \& Q, L_1, L_2} \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1 : \Gamma, C \uparrow P, L_1 \quad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow P, L_1, L_2} \quad \frac{\vdash \mathcal{K}_1 : \Gamma, C \uparrow Q, L_1 \quad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow Q, L_1, L_2}}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow P \& Q, L_1, L_2}$$

$$\frac{\frac{\vdash \mathcal{K}_1 : \Gamma, C \uparrow P[c/x], L_1}{\vdash \mathcal{K}_1 : \Gamma, C \uparrow \forall x P, L_1} \quad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow \forall x P, L_1, L_2} \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1 : \Gamma, C \uparrow P[c/x], L_1 \quad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow P[c/x], L_1, L_2}}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow \forall x P, L_1, L_2}$$

$$\frac{\frac{\vdash \mathcal{K}_1, P_i : \Gamma, C \uparrow L_1}{\vdash \mathcal{K}_1 : \Gamma, C \uparrow ?^i P, L_1} \quad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow ?^i P, L_1, L_2} \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1, P_i : \Gamma, C \uparrow L_1 \quad \Pi_2}{\vdash \mathcal{K}, P_i : \Gamma, \Delta \uparrow L_1, L_2}}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow ?^i P, L_1, L_2}$$

$$\frac{\frac{\vdash \mathcal{K}_1, P_i : \Gamma, C \uparrow L_1}{\vdash \mathcal{K}_1 : \Gamma, C \uparrow ?^i P, L_1} \quad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow ?^i P, L_1, L_2} \rightsquigarrow \frac{\vdash \mathcal{K}_1, P_i : \Gamma, C \uparrow L \quad \frac{\frac{\Pi_2}{\vdash \mathcal{K}_2, P_i : \Delta \uparrow C^\perp, L_2}}{\vdash \mathcal{K}, P_i : \Gamma, \Delta \uparrow L_1, L_2}}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow ?^i P, L_1, L_2} \text{W}$$

$$\frac{\frac{\vdash \mathcal{K}_1 : P, \Gamma, C \uparrow L_1}{\vdash \mathcal{K}_1 : \Gamma, C \uparrow P, L_1} \quad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow P, L_1, L_2} \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1 : P, \Gamma, C \uparrow L_1 \quad \Pi_2}{\vdash \mathcal{K} : P, \Gamma, \Delta \uparrow L_1, L_2}}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow P, L_1, L_2}$$

$$\frac{\frac{\frac{\vdash \mathcal{K}_1 : \Gamma, \mathbf{C} \Downarrow P}{\vdash \mathcal{K}_1 : \Gamma, \mathbf{C} \Downarrow P \oplus Q}}{\vdash \mathcal{K}_1 : P \oplus Q, \Gamma, \mathbf{C} \Uparrow \cdot} \Pi_2}{\vdash \mathcal{K} : P \oplus Q, \Gamma, \Delta \Uparrow L_2} \rightsquigarrow \frac{\frac{\frac{\vdash \mathcal{K}_1 : \Gamma, \mathbf{C} \Downarrow P}{\vdash \mathcal{K}_1 : P, \Gamma, \mathbf{C} \Uparrow \cdot} \Pi_2}{\vdash \mathcal{K} : P, \Gamma, \Delta \Uparrow L_2}}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow L_2, P} \mathcal{L} \oplus$$

$$\frac{\frac{\frac{\frac{\vdash \mathcal{K}_1 : \Gamma, \mathbf{C} \Uparrow P}{\vdash \mathcal{K}_1 : \Gamma, \mathbf{C} \Downarrow P}}{\vdash \mathcal{K}_1 : \Gamma, \mathbf{C} \Downarrow P \oplus Q}}{\vdash \mathcal{K}_1 : P \oplus Q, \Gamma, \mathbf{C} \Uparrow \cdot} \Pi_2}{\vdash \mathcal{K} : P \oplus Q, \Gamma, \Delta \Uparrow L_2} \rightsquigarrow \frac{\frac{\frac{\vdash \mathcal{K}_1 : \Gamma, \mathbf{C} \Uparrow P}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow P, L_2} \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow L_2, P}}{\vdash \mathcal{K} : P \oplus Q, \Gamma, \Delta \Uparrow L_2} \mathcal{L} \oplus$$

If $P \oplus Q \in \mathcal{K}_1$, we proceed similarly. For \otimes and \exists we use the same reasoning.

$$\frac{\frac{\frac{\vdash \mathcal{K}_1 : \Gamma \Downarrow \mathbf{C}}{\vdash \mathcal{K}_1 : \Gamma, \mathbf{C} \Uparrow \cdot}}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow L_2} \vdash \mathcal{K}_2 : \Delta \Uparrow \mathbf{C}^\perp, L_2}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow L_2} \rightsquigarrow \frac{\vdash \mathcal{K}_2 : \Delta \Uparrow \mathbf{C}^\perp, L_2 \quad \vdash \mathcal{K}_1 : \Gamma \Downarrow \mathbf{C}}{\vdash \mathcal{K} : \Gamma, \Delta \Uparrow \cdot} \text{cut}_2$$

4 Elimination of cut_2

$$\frac{\vdash \mathcal{K}_1; \Gamma \uparrow \mathbf{C}, \mathbf{L} \quad \vdash \mathcal{K}_2; \Delta \Downarrow \mathbf{C}^\perp}{\vdash \mathcal{K}; \Gamma, \Delta \uparrow \mathbf{L}} \text{ cut}_2$$

$$\frac{\frac{\vdash \mathcal{K} : \Gamma \uparrow L}{\vdash \mathcal{K} : \Gamma \uparrow \perp, L} \quad \frac{}{\vdash \mathcal{K}^u : \cdot \Downarrow 1}}{\vdash \mathcal{K} : \Gamma \uparrow L} \rightsquigarrow \vdash \mathcal{K} : \Gamma \uparrow L$$

$$\frac{\frac{[\Sigma_1]}{\vdash \mathcal{K}_1 : \Gamma \uparrow P, Q, L} \quad \frac{[\Sigma_2]}{\vdash \mathcal{K}'_2 : \Delta_P \Downarrow P^\perp} \quad \frac{[\Sigma_3]}{\vdash \mathcal{K}''_2 : \Delta_Q \Downarrow Q^\perp}}{\frac{\vdash \mathcal{K}_1 : \Gamma \uparrow P \wp Q, L \quad \vdash \mathcal{K}_2 : \Delta \Downarrow P^\perp \otimes Q^\perp}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow L}}$$

\rightsquigarrow

$$\frac{\frac{[\Sigma_1] \quad [\Sigma_2]}{\vdash \mathcal{K}', \Theta_2 : \Gamma, \Delta_P \uparrow Q, L} \quad [\Sigma_3]}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow L}$$

$$\frac{\frac{\vdash \mathcal{K}_1 : \Gamma \uparrow P, L \quad \vdash \mathcal{K}_1 : \Gamma \uparrow Q, L}{\vdash \mathcal{K}_1 : \Gamma \uparrow P \& Q, L} \quad \frac{\vdash \mathcal{K}_2 : \Delta \Downarrow P^\perp}{\vdash \mathcal{K}_2 : \Delta \Downarrow P^\perp \oplus Q^\perp}}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow L}$$

\rightsquigarrow

$$\frac{\vdash \mathcal{K}_1 : \Gamma \uparrow P, L \quad \vdash \mathcal{K}_2 : \Delta \Downarrow P^\perp}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow L}$$

$$\frac{\frac{\vdash \mathcal{K}_1 : \Gamma \uparrow P[c/x], L}{\vdash \mathcal{K}_1 : \Gamma \uparrow \forall x. P, L} \quad \frac{\vdash \mathcal{K}_2 : \Delta \Downarrow P^\perp[t/x]}{\vdash \mathcal{K}_2 : \Delta \Downarrow \exists x. P^\perp}}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow L}$$

\rightsquigarrow

$$\frac{\vdash \mathcal{K}_1 : \Gamma \uparrow P[c/x], L \quad \vdash \mathcal{K}_2 : \Delta \Downarrow P^\perp[t/x]}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow L}$$

$$\frac{\frac{\vdash \mathcal{K}_1, H_i : \Gamma \uparrow L}{\vdash \mathcal{K}_1 : \Gamma \uparrow ?^i H, L} \quad \vdash \mathcal{K}_2 : \cdot \Downarrow !^i H}{\vdash \mathcal{K} : \Gamma \uparrow L} \rightsquigarrow \frac{\vdash \mathcal{K}_1, H_i : \Gamma \uparrow L \quad \vdash \mathcal{K}_2 : \cdot \Downarrow !^i H}{\vdash \mathcal{K} : \Gamma \uparrow L} \text{cut}_3$$

$$\frac{\frac{\vdash \mathcal{K}_1 : H, \Gamma \uparrow L}{\vdash \mathcal{K}_1 : \Gamma \uparrow H, L} \quad \frac{\vdash \mathcal{K}_2 : \Delta \uparrow H^\perp}{\vdash \mathcal{K}_2 : \Delta \Downarrow H^\perp}}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow L} \rightsquigarrow \frac{\vdash \mathcal{K}_1 : H, \Gamma \uparrow L \quad \vdash \mathcal{K}_2 : \Delta \Downarrow H^\perp}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow L} \text{cut}_1$$

$$\frac{\frac{\vdash \mathcal{K}_1 : A^+, \Gamma \uparrow L}{\vdash \mathcal{K}_1 : \Gamma \uparrow A^+, L} \quad \vdash \mathcal{K}_2 : A^+ \Downarrow A^-}{\vdash \mathcal{K}_1 : \Gamma, A^+ \uparrow L}$$

$$\frac{\frac{\vdash \mathcal{K}_1 : A^+, \Gamma \uparrow L}{\vdash \mathcal{K}_1 : \Gamma \uparrow A^+, L} \quad \vdash \mathcal{K}_2, A_i^+ : \cdot \Downarrow A^-}{\vdash \mathcal{K}, A_i^+ : \Gamma \uparrow L} \rightsquigarrow \frac{\vdash \mathcal{K}_1 : \Gamma, A^+ \uparrow L}{\vdash \mathcal{K}, A_i^+ : \Gamma \uparrow L} A_l$$

$$\frac{\frac{\vdash \mathcal{K}_1, A_i^+ : A^+, \Gamma \uparrow L}{\vdash \mathcal{K}_1, A_i^+ : \Gamma \uparrow A^+, L} \quad \vdash \mathcal{K}_2, A_i^+ : \cdot \Downarrow A^-}{\vdash \mathcal{K}, A_i^+ : \Gamma \uparrow L} \rightsquigarrow \frac{\vdash \mathcal{K}_1, A_i^+ : \Gamma, A^+ \uparrow L}{\vdash \mathcal{K}, A_i^+ : \Gamma \uparrow L} A_c$$

5 Elimination of cut₃

$$\frac{\vdash \mathcal{K}_1, C_i; \Gamma \uparrow L \quad \vdash \mathcal{K}_2; \cdot \Downarrow !^i C^\perp}{\vdash \mathcal{K}; \Gamma \uparrow L} \text{cut}_3$$

$$\frac{\frac{\vdash \mathcal{K}_1, C_i : \Gamma \uparrow L}{\vdash \mathcal{K}_1, C_i : \Gamma \uparrow \perp, L} \quad \Pi_2}{\vdash \mathcal{K} : \Gamma \uparrow \perp, L} \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1, C_i : \Gamma \uparrow L \quad \Pi_2}{\vdash \mathcal{K} : \Gamma \uparrow L}}{\vdash \mathcal{K} : \Gamma \uparrow \perp, L}$$

$$\frac{\frac{\vdash \mathcal{K}_1, C_i : \Gamma \uparrow \top, L}{\vdash \mathcal{K} : \Gamma \uparrow \top, L} \quad \Pi_2}{\vdash \mathcal{K} : \Gamma \uparrow \top, L} \rightsquigarrow \frac{}{\vdash \mathcal{K} : \Gamma \uparrow \top, L}$$

$$\frac{\frac{\vdash \mathcal{K}_1, C_i : \Gamma \uparrow P, Q, L}{\vdash \mathcal{K}_1, C_i : \Gamma \uparrow P \wp Q, L} \quad \Pi_2}{\vdash \mathcal{K} : \Gamma \uparrow P \wp Q, L} \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1, C_i : \Gamma \uparrow P, Q, L \quad \Pi_2}{\vdash \mathcal{K} : \Gamma \uparrow P, Q, L}}{\vdash \mathcal{K} : \Gamma \uparrow P \wp Q, L}$$

$$\frac{\frac{[\Sigma_1] \quad \vdash \mathcal{K}_1, C_i : \Gamma \uparrow P, L \quad \vdash \mathcal{K}_1, C_i : \Gamma \uparrow Q, L}{\vdash \mathcal{K}_1, C_i : \Gamma \uparrow P \& Q, L} \quad \Pi_2}{\vdash \mathcal{K} : \Gamma \uparrow P \& Q, L} \rightsquigarrow \frac{\frac{\frac{\Sigma_1 \quad \Pi_2}{\vdash \mathcal{K} : \Gamma \uparrow P, L} \quad \frac{\Sigma_2 \quad \Pi_2}{\vdash \mathcal{K} : \Gamma \uparrow Q, L}}{\vdash \mathcal{K} : \Gamma \uparrow P \& Q, L}}$$

$$\frac{\frac{\vdash \mathcal{K}_1, C_i, P_a : \Gamma \uparrow L}{\vdash \mathcal{K}_1, C_i : \Gamma \uparrow ?^a P, L} \quad \Pi_2}{\vdash \mathcal{K} : \Gamma \uparrow ?^a P, L} \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1, P_a, C_i : \Gamma \uparrow L \quad \Pi_2}{\vdash \mathcal{K}, P_a : \Gamma \uparrow L}}{\vdash \mathcal{K} : \Gamma \uparrow ?^a P, L}$$

$$\frac{\frac{\vdash \mathcal{K}_1, C_i, P_a : \Gamma \uparrow L}{\vdash \mathcal{K}_1, C_i : \Gamma \uparrow ?^a P, L} \quad \Pi_2}{\vdash \mathcal{K} : \Gamma \uparrow ?^a P, L} \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1, P_a, C_i : \Gamma \uparrow L \quad \frac{\Pi_2}{\vdash \mathcal{K}_2, P_a : \cdot \Downarrow !^i H^\perp}}{\vdash \mathcal{K}, P_a : \Gamma \uparrow L} \quad \text{W}}{\vdash \mathcal{K} : \Gamma \uparrow ?^a P, L}$$

$$\frac{\frac{\vdash \mathcal{K}_1, C_i : \Gamma \uparrow P[c/x], L}{\vdash \mathcal{K}_1, C_i : \Gamma \uparrow \forall x P, L} \quad \Pi_2}{\vdash \mathcal{K} : \Gamma \uparrow \forall x P, L} \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1, C_i : \Gamma \uparrow P[c/x], L \quad \Pi_2}{\vdash \mathcal{K} : \Gamma \uparrow P[c/x], L}}{\vdash \mathcal{K} : \Gamma \uparrow \forall x P, L}$$

$$\frac{\frac{\vdash \mathcal{K}_1, C_i : P, \Gamma \uparrow L}{\vdash \mathcal{K}_1, C_i : \Gamma \uparrow P, L} \quad \Pi_2}{\vdash \mathcal{K} : \Gamma \uparrow P, L} \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1, C_i : P, \Gamma \uparrow L \quad \Pi_2}{\vdash \mathcal{K} : P, \Gamma \uparrow L}}{\vdash \mathcal{K} : \Gamma \uparrow P, L}$$

$$\begin{array}{c}
\frac{\frac{\vdash \mathcal{K}_1, C_i : \Gamma \Downarrow Q}{\vdash \mathcal{K}_1, C_i : \Gamma \Uparrow \cdot} \quad \vdash \mathcal{K}_2, C_i : \cdot \Uparrow !^i C^\perp}{\vdash \mathcal{K} : \Gamma \Uparrow \cdot} \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1, C_i : \Gamma \Downarrow Q \quad \vdash \mathcal{K}_2, C_i : \cdot \Uparrow !^i C^\perp}{\vdash \mathcal{K} : \Gamma \Uparrow Q} \text{cut}_4}{\vdash \mathcal{K} : \Gamma \Uparrow \cdot} \mathcal{L}A
\\[2em]
\frac{\frac{\vdash \mathcal{K}'_1, C_i : \Gamma \Downarrow Q}{\vdash \mathcal{K}_1, C_i : \Gamma \Uparrow \cdot} \quad \vdash \mathcal{K}_2, C_i : \cdot \Uparrow !^i C^\perp}{\vdash \mathcal{K} : \Gamma \Uparrow \cdot} \rightsquigarrow \frac{\frac{\vdash \mathcal{K}'_1, C_i : \Gamma \Downarrow Q \quad \vdash \mathcal{K}_2, C_i : \cdot \Uparrow !^i C^\perp}{\vdash \mathcal{K}' : \Gamma \Uparrow Q} \text{cut}_4}{\vdash \mathcal{K} : \Gamma \Uparrow \cdot} \mathcal{L}A^*
\\[2em]
\frac{\frac{\vdash \mathcal{K}_1, C_i : \Gamma \Downarrow P}{\vdash \mathcal{K}_1, C_i : P, \Gamma \Uparrow \cdot} \quad \vdash \mathcal{K}_2, C_i : \cdot \Uparrow !^i C^\perp}{\vdash \mathcal{K} : P, \Gamma \Uparrow \cdot} \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1, C_i : \Gamma \Downarrow P \quad \vdash \mathcal{K}_2, C_i : \cdot \Uparrow !^i C^\perp}{\vdash \mathcal{K} : \Gamma \Uparrow P} \text{cut}_4}{\vdash \mathcal{K} : P, \Gamma \Uparrow \cdot}
\end{array}$$

$$\frac{\frac{\vdash \mathcal{K}_1, C_c : \Gamma \Downarrow C}{\vdash \mathcal{K}_1, C_c : \Gamma \Uparrow \cdot} \quad \frac{\vdash \mathcal{K}_2 : \cdot \Uparrow C^\perp}{\vdash \mathcal{K}_2 : \cdot \Downarrow !^e C^\perp}}{\vdash \mathcal{K} : \Gamma \Uparrow \cdot}$$

We should analyze if C is (1) a negative ou (2) a positive formula. Note that \mathcal{K}_2 is unbounded.

$$\begin{array}{c}
1 \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1, C_c : \Gamma \Downarrow C \quad \vdash \mathcal{K}_2 : \cdot \Downarrow !^e C^\perp}{\vdash \mathcal{K}_1 : \Gamma \Uparrow C} \text{cut}_2}{\frac{\vdash \mathcal{K}_2 : \cdot \Uparrow C^\perp \quad \vdash \mathcal{K}_1 : \Gamma \Uparrow C}{\vdash \mathcal{K}_1 : \Gamma \Downarrow C} \text{cut}_2} \text{cut}_2
\\[2em]
2 \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1, C_c : \Gamma \Downarrow C \quad \vdash \mathcal{K}_2 : \cdot \Downarrow !^e C^\perp}{\vdash \mathcal{K}_1 : \Gamma \Uparrow C} \quad \frac{\vdash \mathcal{K}_2 : \cdot \Uparrow C^\perp}{\vdash \mathcal{K}_2 : \cdot \Downarrow C^\perp}}{\vdash \mathcal{K}_1 : \Gamma \Uparrow \cdot} \text{cut}_2
\end{array}$$

$$\frac{\frac{\vdash \mathcal{K}_1 : \Gamma \Downarrow C}{\vdash \mathcal{K}_1, C_i : \Gamma \Uparrow \cdot} \quad \frac{\frac{\vdash \Upsilon, \Sigma_c^u : \cdot \Uparrow C^\perp, \mathcal{F}(\Sigma^l)}{\vdash \mathcal{K}_2 : \cdot \Uparrow \cdot // i \vdash \cdot : \cdot \Uparrow C^\perp}}{\vdash \mathcal{K}_2 : \cdot \Downarrow !^i C^\perp}}{\vdash \mathcal{K} : \Gamma \Uparrow \cdot}$$

• $\mathcal{K}_2 \equiv \Upsilon^4, \Sigma^4, \Xi^U$ where $\Upsilon_{i \preceq}$ and $\Sigma_{i \preceq}$

• $T \in \mathcal{U}(i)$, therefore Υ^T and Σ^T .

$$\rightsquigarrow \frac{\frac{\vdash \Upsilon, \Sigma_c^u : \cdot \Uparrow C^\perp, \mathcal{F}(\Sigma^l)}{\vdash \Upsilon, \Sigma^u, \Xi, \Sigma_c^u : \cdot \Uparrow C^\perp, \mathcal{F}(\Sigma^l)} W \quad \frac{\vdash \mathcal{K}_1 : \Gamma \Downarrow C}{\vdash \mathcal{K}_1^l, \Upsilon^u, \Sigma^u, \Xi, \Sigma_c^u : \Gamma \Downarrow C} W}{\frac{\vdash \mathcal{K}_1^l, \Upsilon, \Sigma^u, \Xi, \Sigma_c^u : \Gamma \Uparrow \mathcal{F}(\Sigma^l)}{\vdash \mathcal{K} : \Gamma \Uparrow \cdot} A_l, C_c} \text{cut}_2$$

Note that $\mathcal{K}_1 \equiv \mathcal{K}_1^l, \Upsilon^U, \Sigma^U, \Xi$

$$\frac{\frac{\vdash \mathcal{K}_1, C_i : \Gamma \Downarrow C}{\vdash \mathcal{K}_1, C_i : \Gamma \Uparrow \cdot} \quad \frac{\frac{\vdash \Upsilon, \Sigma_c : \cdot \Uparrow C^\perp}{\vdash \mathcal{K}_2 : \cdot \Uparrow \cdot // i \vdash \cdot : \cdot \Uparrow C^\perp}}{\vdash \mathcal{K}_2 : \cdot \Downarrow !^i C^\perp}}{\vdash \mathcal{K} : \Gamma \Uparrow \cdot}$$

• $\mathcal{K}_2 \equiv \Upsilon^{U4}, \Sigma^{U4}, \Xi^U$ where $\Upsilon_{i \preceq}$ and $\Sigma_{i \preceq}$

• $T \in \mathcal{U}(i)$, therefore Υ^T and Σ^T .

We should analyze if C is (1) a negative or (2) a positive formula. Note that \mathcal{K}_2 is unbounded.

$$1 \rightsquigarrow \frac{\frac{\vdash \Upsilon, \Sigma_c : \cdot \Uparrow C^\perp}{\vdash \mathcal{K}_2, \Sigma_c : \cdot \Uparrow C^\perp} W \quad \frac{\frac{\vdash \mathcal{K}_1, C_i : \Gamma \Downarrow C}{\vdash \mathcal{K} : \Gamma \Uparrow C} \quad \vdash \mathcal{K}_2 : \cdot \Downarrow !^i C^\perp}{\vdash \mathcal{K}, \Sigma_c : \Gamma \Downarrow C} W}{\frac{\vdash \mathcal{K}, \Sigma_c : \Gamma \Uparrow \cdot}{\vdash \mathcal{K} : \Gamma \Uparrow \cdot} C_c} \text{cut}_2$$

$$2 \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1, C_i : \Gamma \Downarrow C}{\vdash \mathcal{K} : \Gamma \Uparrow C} \quad \vdash \mathcal{K}_2 : \cdot \Downarrow !^i C^\perp}{\vdash \mathcal{K}, \Sigma_c : \Gamma \Uparrow C} W \quad \frac{\frac{\vdash \Upsilon, \Sigma_c : \cdot \Uparrow C^\perp}{\vdash \mathcal{K}_2, \Sigma_c : \cdot \Downarrow C^\perp} W}{\frac{\vdash \mathcal{K}, \Sigma_c : \Gamma \Uparrow \cdot}{\vdash \mathcal{K} : \Gamma \Uparrow \cdot} C_c} \text{cut}_2$$

6 Elimination of cut_4

$$\frac{\vdash \mathcal{K}_1, C_i; \Gamma \Downarrow F \quad \vdash \mathcal{K}_2; \cdot \Downarrow !^i C^\perp}{\vdash \mathcal{K}; \Gamma \Uparrow F} \text{cut}_4$$

Note that $\vdash \Theta : \Gamma \Downarrow P$ implies $\vdash \Theta : \Gamma \Uparrow P$.

$$\frac{\vdash \mathcal{K}_1, C_i : X^+ \Downarrow X^-}{\vdash \mathcal{K} : X^+ \Uparrow X^-} \quad \vdash \mathcal{K}_2 : \cdot \Downarrow !^i C^\perp \quad \rightsquigarrow \quad \frac{\vdash \mathcal{K} : X^+ \Downarrow X^-}{\vdash \mathcal{K} : X^+ \Uparrow X^-}$$

$$\frac{\vdash \mathcal{K}_1, X_a^+, C_i : \cdot \Downarrow X^-}{\vdash \mathcal{K}, X_a^+ : \cdot \Uparrow X^-} \quad \vdash \mathcal{K}_2 : \cdot \Downarrow !^i C^\perp \quad \rightsquigarrow \quad \frac{\vdash \mathcal{K}, X_a^+ : \cdot \Downarrow X^-}{\vdash \mathcal{K}, X_a^+ : \cdot \Uparrow X^-}$$

$$\frac{\vdash \mathcal{K}_1, X_a^+, C_i : \cdot \Downarrow X^-}{\vdash \mathcal{K}, X_a^+ : \cdot \Uparrow X^-} \quad \vdash \mathcal{K}_2, X_a^+ : \cdot \Downarrow !^i C^\perp \quad \rightsquigarrow \quad \frac{\vdash \mathcal{K}, X_a^+ : \cdot \Downarrow X^-}{\vdash \mathcal{K}, X_a^+ : \cdot \Uparrow X^-}$$

$$\frac{\vdash \mathcal{K}_1, X_i^+ : \cdot \Downarrow X^-}{\vdash \mathcal{K} : \cdot \Uparrow X^-} \quad \vdash \mathcal{K}_2 : \cdot \Downarrow !^i X^- \quad \rightsquigarrow \quad \frac{\vdash \mathcal{K} : \cdot \Downarrow !^i X^-}{\vdash \mathcal{K} : \cdot \Uparrow X^-}$$

Note above that $\{U, T\} \subset \mathcal{U}(i)$: then $\vdash \mathcal{K} : \cdot \Downarrow !^i P$ implies $\vdash \mathcal{K} : \cdot \Uparrow P$, for all P . In the case below, i is linear.

$$\frac{\vdash \mathcal{K}_1, X_i^+ : \cdot \Downarrow X^- \quad \frac{\vdash \Upsilon, \Sigma_c^u : \cdot \Uparrow X^-, \mathcal{F}(\Sigma^l)}{\vdash \mathcal{K}_2 : \cdot \Downarrow !^i X^-}}{\vdash \mathcal{K} : \cdot \Uparrow X^-} \rightsquigarrow \frac{\frac{\vdash \Upsilon, \Sigma_c^u : \cdot \Uparrow X^-, \mathcal{F}(\Sigma^l)}{\vdash \mathcal{K}, \Sigma_c^u : \cdot \Uparrow X^-, \mathcal{F}(\Sigma^l)}}{\vdash \mathcal{K} : \cdot \Uparrow X^-} \text{W}_{A_l, C_c}$$

$$\frac{\vdash \mathcal{K}_1, C_i : \cdot \Downarrow 1 \quad \vdash \mathcal{K}_2 : \cdot \Downarrow !^i C^-}{\vdash \mathcal{K} : \cdot \Uparrow 1} \rightsquigarrow \frac{\vdash \mathcal{K} : \cdot \Downarrow 1}{\vdash \mathcal{K} : \cdot \Uparrow 1}$$

$$\frac{\frac{[\Sigma_1] \quad \vdash \mathcal{K}'_1, C_i : \Gamma_P \Downarrow P \quad \vdash \mathcal{K}''_1, C_i : \Gamma_Q \Downarrow Q}{\vdash \mathcal{K}_1, C_i : \Gamma \Downarrow P \otimes Q} \quad \vdash \mathcal{K}_2 : \cdot \Downarrow !^i C^\perp}{\vdash \mathcal{K} : \Gamma \Uparrow P \otimes Q} \rightsquigarrow \frac{\frac{\frac{\Sigma_1 \quad \Pi_2}{\vdash \mathcal{K}' : \Gamma_P \Uparrow P} \quad \frac{\Sigma_2 \quad \Pi_2}{\vdash \mathcal{K}'' : \Gamma_Q \Uparrow Q}}{\vdash \mathcal{K} : \Gamma, P \otimes Q \Uparrow \cdot}}{\vdash \mathcal{K} : \Gamma \Uparrow P \otimes Q} \mathcal{L}_\otimes$$

$$\frac{\frac{[\Sigma_1] \quad \vdash \mathcal{K}'_1, C_i : \Gamma_P \Downarrow P \quad \vdash \mathcal{K}''_1 : \Gamma_Q \Downarrow Q}{\vdash \mathcal{K}_1, C_i : \Gamma \Downarrow P \otimes Q} \quad \vdash \mathcal{K}_2 : \cdot \Downarrow !^i C^\perp}{\vdash \mathcal{K} : \Gamma \Uparrow P \otimes Q}$$

\rightsquigarrow

$$\frac{\frac{\Sigma_1 \quad \Pi_2}{\vdash \mathcal{K}' : \Gamma_P \uparrow P} \quad \frac{\vdash \mathcal{K}_1'' : \Gamma_Q \Downarrow Q}{\vdash \mathcal{K}_1'' : \Gamma_Q \uparrow Q}}{\frac{\vdash \mathcal{K} : \Gamma, P \otimes Q \uparrow \cdot}{\vdash \mathcal{K} : \Gamma \uparrow P \otimes Q}} \mathcal{L}_{\otimes}$$

Same reasoning for \oplus and \exists , see below.

$$\frac{\frac{\vdash \mathcal{K}_1, \mathcal{C}_i : \Gamma \Downarrow P}{\vdash \mathcal{K}_1, \mathcal{C}_i : \Gamma \Downarrow P \oplus Q} \quad \Pi_2}{\vdash \mathcal{K} : \Gamma \uparrow P \oplus Q} \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1, \mathcal{C}_i : \Gamma \Downarrow P \quad \Pi_2}{\vdash \mathcal{K} : \Gamma \uparrow P} \quad \mathcal{L}_{\oplus}}{\vdash \mathcal{K} : \Gamma \uparrow P \oplus Q}$$

$$\frac{\frac{\vdash \mathcal{K}_1, \mathcal{C}_i : \Gamma \Downarrow P[c/x]}{\vdash \mathcal{K}_1, \mathcal{C}_i : \Gamma \Downarrow \exists x.P} \quad \Pi_2}{\vdash \mathcal{K} : \Gamma \uparrow \exists x.P} \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1, \mathcal{C}_i : \Gamma \Downarrow P[c/x] \quad \Pi_2}{\vdash \mathcal{K} : \Gamma \uparrow P[c/x]} \quad \mathcal{L}_{\exists}}{\vdash \mathcal{K} : \Gamma \uparrow \exists x.P}$$

$$\frac{\frac{\vdash \mathcal{K}_1, \mathcal{C}_i : \Gamma \uparrow P}{\vdash \mathcal{K}_1, \mathcal{C}_i : \Gamma \Downarrow P} \quad \vdash \mathcal{K}_2; \cdot \Downarrow !^i \mathcal{C}^\perp}{\vdash \mathcal{K} : \Gamma \uparrow P} \rightsquigarrow \frac{\vdash \mathcal{K}_1, \mathcal{C}_i : \Gamma \uparrow P \quad \vdash \mathcal{K}_2; \cdot \Downarrow !^i \mathcal{C}^\perp}{\vdash \mathcal{K} : \Gamma \uparrow P} \text{cut}_3$$

$$\frac{\frac{\vdash \mathcal{K}_1, \mathcal{C}_i : \cdot \uparrow P}{\vdash \mathcal{K}_1, \mathcal{C}_i : \cdot \Downarrow !^e P} \quad \vdash \mathcal{K}_2; \cdot \Downarrow !^i \mathcal{C}^\perp}{\vdash \mathcal{K} : \cdot \uparrow !^e P} \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1, \mathcal{C}_i : \cdot \uparrow P \quad \vdash \mathcal{K}_2; \cdot \Downarrow !^i \mathcal{C}^\perp}{\vdash \mathcal{K} : \cdot \uparrow P} \text{cut}_3}{\frac{\vdash \mathcal{K} : \cdot \Downarrow !^e P}{\vdash \mathcal{K} : \cdot \uparrow !^e P}}$$

$a \preceq i$ and $4 \in \mathcal{U}(i)$

$$\frac{\frac{\vdash \Upsilon_+, H_{i+}, \Sigma_{lc}^u; \cdot \uparrow P, \mathcal{F}(\Sigma^l)}{\vdash \Lambda, \Theta_1, H_i; \cdot \Downarrow !^a P} \quad \frac{\vdash \Upsilon'_+; \cdot \uparrow H^\perp}{\vdash \Lambda, \Theta_2; \cdot \Downarrow !^i H^\perp}}{\vdash \Lambda, \Theta_1, \Theta_2; \cdot \uparrow !^a P} \rightsquigarrow \frac{\frac{\frac{\vdash \Upsilon_+, \Sigma_{lc}^u; \cdot \uparrow P, \mathcal{F}(\Sigma)}{\vdash (\Upsilon'_+)^u, \Upsilon_+, \Sigma_{lc}^u; \cdot \uparrow P, \mathcal{F}(\Sigma)} [W] \quad \frac{\vdash \Upsilon'_+; \cdot \Downarrow !^{i+} H^\perp}{\vdash \Upsilon'_+, (\Upsilon'_+)^u, \Sigma_{lc}^u; \cdot \Downarrow !^{i+} H^\perp} [W]}{\frac{\vdash \Upsilon_+, \Upsilon'_+, \Sigma_{lc}^u; \cdot \uparrow P, \mathcal{F}(\Sigma_1)}{\vdash \Lambda, \Theta_1, \Theta_2; \cdot \Downarrow !^a P} \text{C3}} \text{[}\uparrow\text{CC]}$$

$$\frac{\vdash \Lambda, \Theta_1, \Theta_2; \cdot \Downarrow !^a P}{\vdash \Lambda, \Theta_1, \Theta_2; !^a P \uparrow \cdot} \frac{}{\vdash \Lambda, \Theta_1, \Theta_2; \cdot \uparrow !^a P}$$

$a \preceq i$ and $4 \notin \mathcal{U}(i)$

$$\begin{array}{c}
\frac{\frac{\vdash \Upsilon_+, \Sigma_{lc}^u, H_{lc}; \cdot \uparrow P, \mathcal{F}(\Sigma)}{\vdash \Lambda, \Theta_1, H_i; \cdot \downarrow !^a P} \quad \frac{\vdash \Lambda; \cdot \uparrow H^\perp}{\vdash \Lambda; \cdot \downarrow !^i H^\perp}}{\vdash \Lambda, \Theta_1; \cdot \uparrow !^a P} \\
\\
\frac{\frac{\vdash \Upsilon_+, \Sigma_{lc}^u, H_{lc}; \cdot \uparrow P, \mathcal{F}(\Sigma)}{\vdash \Lambda, \Upsilon_+, \Sigma_{lc}^u, H_{lc}; \cdot \uparrow P, \mathcal{F}(\Sigma)} [W] \quad \frac{\vdash \Lambda; \cdot \downarrow !^{lc} H^\perp}{\vdash \Lambda, \Upsilon_+, \Sigma_{lc}^u; \cdot \downarrow !^{lc} H^\perp} [W]}{\vdash \Lambda, \Upsilon_+, \Sigma_{lc}^u; \cdot \uparrow P, \mathcal{F}(\Sigma)} [\uparrow \text{CC}] \\
\\
\frac{\vdash \Lambda, \Theta_1; \cdot \downarrow !^a P}{\vdash \Lambda, \Theta_1; !^a P \uparrow \cdot} \\
\hline
\vdash \Lambda, \Theta_1; \cdot \uparrow !^a P
\end{array}$$

$a \preceq i$ and $4 \notin \mathcal{U}(i)$

$$\begin{array}{c}
\frac{\vdash \Upsilon_+, \Sigma_{lc}^u; \cdot \uparrow P, \mathcal{F}(\Sigma_1^l), H, \mathcal{F}(\Sigma_3^l)}{\vdash \Lambda, \Theta_1, H_i; \cdot \downarrow !^a P} \quad \frac{\vdash \Upsilon'_+, (\Sigma')_{lc}^u; \cdot \uparrow H^\perp, \mathcal{K}(\Sigma_2^l)}{\vdash \Lambda, \Theta_2; \cdot \downarrow !^i H^\perp}}{\vdash \Lambda, \Theta_1, \Theta_2; \cdot \downarrow !^a P} \\
\\
\frac{\frac{\vdash \Upsilon_+, \Sigma_{lc}^u; \cdot \uparrow P, \mathcal{F}(\Sigma_1), H, \mathcal{F}(\Sigma_3)}{\vdash \Upsilon_+, \Sigma_{lc}^u, (\Upsilon')_+, (\Sigma')_{lc}^u; \cdot \uparrow P, \mathcal{F}(\Sigma_1), H, \mathcal{F}(\Sigma_3)} [W] \quad \frac{\vdash \Upsilon'_+, (\Sigma')_{lc}^u; \cdot \uparrow H^\perp, \mathcal{K}(\Sigma_2)}{\vdash \Upsilon_+, \Sigma_{lc}^u, \Upsilon'_+, (\Sigma')_{lc}^u; \cdot \uparrow H^\perp, \mathcal{K}(\Sigma_2)} [W]}{\vdash \vdash \Upsilon_+, \Sigma_{lc}^u, \Upsilon'_+, (\Sigma')_{lc}^u; \cdot \uparrow P, \mathcal{F}(\Sigma_1), \mathcal{F}(\Sigma_3), \mathcal{F}(\Sigma_2)} [\uparrow \text{C*}] \\
\\
\frac{\vdash \Lambda, \Theta_1, \Theta_2; \cdot \downarrow !^a P}{\vdash \Lambda, \Theta_1, \Theta_2; !^a P \uparrow \cdot} \\
\hline
\vdash \Lambda, \Theta_1, \Theta_2; \cdot \uparrow !^a P
\end{array}$$

7 Elimination of cut₅

$$\frac{\vdash \mathcal{K}_1; \Gamma \uparrow L_1, H, L_2 \quad \vdash \mathcal{K}_2; \Delta \uparrow H^\perp, L_3}{\vdash \mathcal{K}; \Gamma, \Delta \uparrow L_1, L_2, L_3} \text{cut}_5$$

$$\frac{\frac{\vdash \mathcal{K}_1 : \Gamma \uparrow L_1, C, L_2}{\vdash \mathcal{K}_1 : \Gamma \uparrow \perp, L_1, C, L_2} \quad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow \perp, L_1, L_2, L_3} \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1 : \Gamma \uparrow L_1, C, L_2 \quad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow L_1, L_2, L_3}}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow \perp, L_1, L_2, L_3}$$

$$\frac{\frac{\vdash \mathcal{K}_1 : \Gamma \uparrow P, Q, L_1, C, L_2}{\vdash \mathcal{K}_1 : \Gamma \uparrow P \wp Q, L_1, C, L_2} \quad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow P \wp Q, L_1, L_2, L_3} \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1 : \Gamma \uparrow P, Q, L_1, C, L_2 \quad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow P, Q, L_1, L_2, L_3}}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow P \wp Q, L_1, L_2, L_3}$$

$$\frac{\frac{\vdash \mathcal{K}_1 : \Gamma \uparrow P, L_1, C, L_2 \quad \vdash \mathcal{K}_1 : \Gamma \uparrow Q, L_1, C, L_2}{\vdash \mathcal{K}_1 : \Gamma \uparrow P \& Q, L_1, C, L_2} \quad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow P \& Q, L_1, L_2, L_3} \rightsquigarrow \frac{\frac{\frac{\vdash \mathcal{K}_1 : \Gamma \uparrow P, L_1, C, L_2 \quad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow P, L_1, L_2, L_3} \quad \frac{\vdash \mathcal{K}_1 : \Gamma \uparrow Q, L_1, C, L_2 \quad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow Q, L_1, L_2, L_3}}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow P \& Q, L_1, L_2, L_3}}$$

$$\frac{\frac{\vdash \mathcal{K}_1 : \Gamma \uparrow P[c/x], L_1, C, L_2}{\vdash \mathcal{K}_1 : \Gamma \uparrow \forall x P, L_1, C, L_2} \quad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow \forall x P, L_1, L_2, L_3} \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1 : \Gamma \uparrow P[c/x], L_1, C, L_2 \quad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow P[c/x], L_1, L_2, L_3}}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow \forall x P, L_1, L_2, L_3}$$

$$\frac{\frac{\vdash \mathcal{K}_1, P_i : \Gamma \uparrow L_1, C, L_2}{\vdash \mathcal{K}_1 : \Gamma \uparrow ?^i P, L_1, C, L_2} \quad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow ?^i P, L_1, L_2, L_3} \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1, P_i : \Gamma \uparrow L_1, C, L_2 \quad \Pi_2}{\vdash \mathcal{K}, P_i : \Gamma, \Delta \uparrow L_1, L_2, L_3}}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow ?^i P, L_1, L_2, L_3}$$

$$\frac{\frac{\vdash \mathcal{K}_1, P_i : \Gamma \uparrow L_1, C, L_2}{\vdash \mathcal{K}_1 : H, \Gamma \uparrow ?^i P, L_1, C, L_2} \quad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow ?^i P, L_1, L_2, L_3} \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1, P_i : \Gamma \uparrow L_1, C, L_2 \quad \frac{\frac{\Pi_2}{\vdash \mathcal{K}_2, P_i : \Delta \uparrow H^\perp, V}}{\vdash \mathcal{K}, P_i : \Gamma, \Delta \uparrow L_1, L_2, L_3}}}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow ?^i P, L_1, L_2, L_3} \text{W}$$

$$\frac{\frac{\vdash \mathcal{K}_1 : P, \Gamma \uparrow L_1, C, L_2}{\vdash \mathcal{K}_1 : \Gamma \uparrow P, L_1, C, L_2} \quad \Pi_2}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow P, L_1, L_2, L_3} \rightsquigarrow \frac{\frac{\vdash \mathcal{K}_1 : P, \Gamma \uparrow L_1, C, L_2 \quad \Pi_2}{\vdash \mathcal{K} : P, \Gamma, \Delta \uparrow L_1, L_2, L_3}}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow P, L_1, L_2, L_3}$$

$$\begin{array}{ccc}
\frac{\frac{\vdash \mathcal{K}_1 : \Gamma, P \uparrow L_2}{\vdash \mathcal{K}_1 : \Gamma \uparrow P, L_2} \quad \vdash \mathcal{K}_2 : \Delta \uparrow P^\perp, L_3}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow L_2, L_3} & \rightsquigarrow & \frac{\vdash \mathcal{K}_1 : P, \Gamma \downarrow L_2 \quad \vdash \mathcal{K}_2 : \Delta \uparrow P^\perp, L_3}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow L_2, L_3} \text{cut}_1 \\
\\
\frac{\vdash \mathcal{K}_1 : \Gamma \uparrow P, L_2 \quad \frac{\vdash \mathcal{K}_2 : \Delta, P^\perp \uparrow L_3}{\vdash \mathcal{K}_2 : \Delta \uparrow P^\perp, L_3}}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow L_2, L_3} & \rightsquigarrow & \frac{\vdash \mathcal{K}_1 : H^\perp, \Delta \downarrow L_3 \quad \vdash \mathcal{K}_2 : \Gamma \uparrow P, L_2}{\frac{\vdash \mathcal{K} : \Gamma, \Delta \uparrow L_3, L_2}{\vdash \mathcal{K} : \Gamma, \Delta \uparrow L_2, L_3}} \text{cut}_1
\end{array}$$