Cut Elimination in Focusing FOLL

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Abstract

This document presents the cut-elimination procedure for first-order focused linear logic. The gray boxes highlight the dependencies between rules

1 The cut-rules

$$\frac{\vdash \Theta : H, \Gamma \Downarrow F \qquad \vdash \Theta : \Delta \Downarrow H^{\perp}}{\vdash \Theta : \Gamma, \Delta \Downarrow F} [\Downarrow\text{LC}] \qquad \frac{\vdash \Theta : H, \Gamma \Uparrow S \qquad \vdash \Theta : \Delta \Downarrow H^{\perp}}{\vdash \Theta : \Gamma, \Delta \Uparrow S} [\Uparrow\text{LC}]$$

$$\boxed{\frac{\vdash \Theta : \Gamma \Uparrow H, S \qquad \vdash \Theta : \Delta \Downarrow H^{\perp}}{\vdash \Theta : \Gamma, \Delta \Uparrow S} [\Uparrow\text{C}]}$$

$$\frac{\vdash \Theta, H : \Gamma \Downarrow F \qquad \vdash \Theta : \cdot \Downarrow !H^{\perp}}{\vdash \Theta : \Gamma \Downarrow F} [\Downarrow\text{CC}] \qquad \frac{\vdash \Theta, H : \Gamma \Uparrow S \qquad \vdash \Theta : \cdot \Downarrow !H^{\perp}}{\vdash \Theta : \Gamma \Uparrow S} [\Uparrow\text{CC}]$$

- $\Uparrow(\Downarrow)$LC : the conclusion sequent is $\Uparrow(\Downarrow)$ and the cut formula goes to the linear context.
- $\Uparrow(\Downarrow)$CC : the conclusion sequent is $\Uparrow(\Downarrow)$ and the cut formula goes to the classic context.
- $\Uparrow$C : the conclusion sequent is $\Uparrow$ and the cut formula goes to the arrow context.
- H cannot be a positive atom in $\Downarrow$CC

Lemma [AbsorptionC]. If $F \in \Theta$ and $\vdash \Theta : \Gamma \Uparrow F, L$ then $\vdash \Theta : \Gamma \Uparrow L$.

Lemma [AbsorptionL]. If $A^- \in Th$ and $\vdash \Theta : A^-, \Gamma \Updownarrow L$ then $\vdash \Theta : \Gamma \Updownarrow L$.

Lemma [AbsorptionT]. If $F \in Th$ is not an atom and $\vdash \Theta : \Gamma \Uparrow F, L$ then $\vdash \Theta : \Gamma \Uparrow L$.

Lemma [Atomic Permutation]. If $\vdash \Theta, A^+ : \Gamma \Downarrow Q$ and $\vdash \Theta : \cdot \Downarrow !A^-$ then $\vdash \Theta : \Gamma \Uparrow Q$.

2 $\Downarrow$-Linear Cut Elimination

$$\dfrac{\begin{matrix}[\Pi_1]\\ \vdash H, \Gamma \Downarrow F\end{matrix} \qquad \begin{matrix}[\Pi_2]\\ \vdash \Delta \Downarrow H^{\perp}\end{matrix}}{\vdash \Gamma, \Delta \Downarrow F}\,[\Downarrow\text{LC}]$$

$$\dfrac{\dfrac{}{\vdash \Theta : X^{+} \Downarrow X^{-}} \qquad \vdash \Theta : \Delta \Downarrow X^{-}}{\vdash \Theta : \Delta \Downarrow X^{-}}\,[\Downarrow\text{LC}] \quad \rightsquigarrow \quad \vdash \Theta : \Delta \Downarrow X^{-}$$

$$\dfrac{\dfrac{\vdash \Theta : H, \Gamma_P \Downarrow P \qquad \vdash \Theta : \Gamma_Q \Downarrow Q}{\vdash \Theta : H, \Gamma \Downarrow P \otimes Q} \qquad \Pi_2}{\vdash \Theta : \Gamma, \Delta \Downarrow P \otimes Q}\,[\Downarrow\text{LC}] \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Theta : H, \Gamma_P \Downarrow P \qquad \Pi_2}{\vdash \Theta : \Gamma_P, \Delta \Downarrow P}\,[\Downarrow\text{LC}] \qquad \vdash \Theta : \Gamma_Q \Downarrow Q}{\vdash \Theta : \Gamma, \Delta \Downarrow P \otimes Q}$$

$$\dfrac{\dfrac{\vdash \Theta : \Gamma_P \Downarrow P \qquad \vdash \Theta : H, \Gamma_Q \Downarrow Q}{\vdash \Theta : H, \Gamma \Downarrow P \otimes Q} \qquad \Pi_2}{\vdash \Theta : \Gamma, \Delta \Downarrow P \otimes Q}\,[\Downarrow\text{LC}] \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Theta : H, \Gamma_Q \Downarrow Q \qquad \Pi_2}{\vdash \Theta : \Gamma_Q, \Delta \Downarrow Q}\,[\Downarrow\text{LC}] \qquad \vdash \Theta : \Gamma_P \Downarrow P}{\vdash \Theta : \Gamma, \Delta \Downarrow P \otimes Q}$$

$$\dfrac{\dfrac{\vdash \Theta : H, \Gamma \Downarrow P}{\vdash \Theta : H, \Gamma \Downarrow P \oplus Q} \qquad \Pi_2}{\vdash \Theta : \Gamma, \Delta \Downarrow P \oplus Q}\,[\Downarrow\text{LC}] \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Theta : H, \Gamma \Downarrow P \qquad \Pi_2}{\vdash \Theta : \Gamma, \Delta \Downarrow P}\,[\Downarrow\text{LC}]}{\vdash \Theta : \Gamma, \Delta \Downarrow P \oplus Q}$$

$$\dfrac{\dfrac{\vdash \Theta : H, \Gamma \Downarrow Q}{\vdash \Theta : H, \Gamma \Downarrow P \oplus Q} \qquad \Pi_2}{\vdash \Theta : \Gamma, \Delta \Downarrow P \oplus Q}\,[\Downarrow\text{LC}] \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Theta : H, \Gamma \Downarrow Q \qquad \Pi_2}{\vdash \Theta : \Gamma, \Delta \Downarrow Q}\,[\Downarrow\text{LC}]}{\vdash \Theta : \Gamma, \Delta \Downarrow P \oplus Q}$$

$$\dfrac{\dfrac{\vdash \Theta : H, \Gamma \Downarrow P[c/x]}{\vdash \Theta : H, \Gamma \Downarrow \exists x.P} \qquad \Pi_2}{\vdash \Theta : \Gamma, \Delta \Downarrow \exists x.P}\,[\Downarrow\text{LC}] \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Theta : H, \Gamma \Downarrow P[c/x] \qquad \Pi_2}{\vdash \Theta : \Gamma, \Delta \Downarrow P[c/x]}\,[\Downarrow\text{LC}]}{\vdash \Theta : \Gamma, \Delta \Downarrow \exists x.P}$$

$$\boxed{\dfrac{\dfrac{\vdash \Theta : H, \Gamma \Uparrow P}{\vdash \Theta : H, \Gamma \Downarrow P} \qquad \Pi_2}{\vdash \Theta : \Gamma, \Delta \Downarrow P}\,[\Downarrow\text{LC}] \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Theta : H, \Gamma \Uparrow P \qquad \Pi_2}{\vdash \Theta : \Gamma, \Delta \Uparrow P}\,[\Uparrow\text{LC}]}{\vdash \Theta : \Gamma, \Delta \Downarrow P}}$$

3 $\Uparrow$-Linear Cut Elimination

$$\dfrac{\begin{array}{c}[\Pi_1]\\ \vdash H, \Gamma \Uparrow S\end{array} \qquad \begin{array}{c}[\Pi_2]\\ \vdash \Delta \Downarrow H^{\perp}\end{array}}{\vdash \Gamma, \Delta \Uparrow S}\,[\Uparrow\mathrm{LC}]$$

$$\dfrac{\dfrac{\vdash H, \Gamma \Uparrow L}{\vdash H, \Gamma \Uparrow \bot, L} \qquad \Pi_2}{\vdash \Gamma, \Delta \Uparrow \bot, L}\,[\Uparrow\mathrm{LC}] \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash H, \Gamma \Uparrow L \qquad \Pi_2}{\vdash \Gamma, \Delta \Uparrow L}\,[\Uparrow\mathrm{LC}]}{\vdash \Gamma, \Delta \Uparrow \bot, L}$$

$$\dfrac{\dfrac{}{\vdash H, \Gamma \Uparrow \top, L} \qquad \Pi_2}{\vdash \Gamma, \Delta \Uparrow \top, L}\,[\Uparrow\mathrm{LC}] \qquad \rightsquigarrow \qquad \dfrac{}{\vdash \Gamma, \Delta \Uparrow \top, L}$$

$$\dfrac{\dfrac{\vdash H, \Gamma \Uparrow P, Q, L}{\vdash H, \Gamma \Uparrow P \parr Q, L} \qquad \Pi_2}{\vdash \Gamma, \Delta \Uparrow P \parr Q, L}\,[\Uparrow\mathrm{LC}] \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash H, \Gamma \Uparrow P, Q, L \qquad \Pi_2}{\vdash \Gamma, \Delta \Uparrow P, Q, L}\,[\Uparrow\mathrm{LC}]}{\vdash \Gamma, \Delta \Uparrow P \parr Q, L}$$

$$\dfrac{\dfrac{\vdash H, \Gamma \Uparrow P, L \qquad \vdash H, \Gamma \Uparrow Q, L}{\vdash H, \Gamma \Uparrow P \,\&\, Q, L} \qquad \Pi_2}{\vdash \Gamma, \Delta \Uparrow P \,\&\, Q, L}\,[\Uparrow\mathrm{LC}] \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash H, \Gamma \Uparrow P, L \qquad \Pi_2}{\vdash \Gamma, \Delta \Uparrow P, L}\,[\Uparrow\mathrm{LC}] \qquad \dfrac{\vdash H, \Gamma \Uparrow Q, L \qquad \Pi_2}{\vdash \Gamma, \Delta \Uparrow Q, L}\,[\Uparrow\mathrm{LC}]}{\vdash \Gamma, \Delta \Uparrow P \,\&\, Q, L}$$

$$\dfrac{\dfrac{\vdash H, \Gamma \Uparrow P[c/x], L}{\vdash H, \Gamma \Uparrow \forall x P, L} \qquad \Pi_2}{\vdash \Gamma, \Delta \Uparrow \forall x P, L}\,[\Uparrow\mathrm{LC}] \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash H, \Gamma \Uparrow P[c/x], L \qquad \Pi_2}{\vdash \Gamma, \Delta \Uparrow P[c/x], L}\,[\Uparrow\mathrm{LC}]}{\vdash \Gamma, \Delta \Uparrow \forall x P, L}$$

$$\dfrac{\dfrac{\vdash \Theta, P : H, \Gamma \Uparrow L}{\vdash \Theta : H, \Gamma \Uparrow ?P, L} \qquad \Pi_2}{\vdash \Theta : \Gamma, \Delta \Uparrow ?P, L}\,[\Uparrow\mathrm{LC}] \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash \Theta, P : H, \Gamma \Uparrow L \qquad \dfrac{\Pi_2}{\vdash \Theta, P : \Delta \Downarrow H^{\perp}}\,[W]}{\vdash \Theta, P : \Gamma, \Delta \Uparrow L}\,[\Uparrow\mathrm{LC}]}{\vdash \Theta : \Gamma, \Delta \Uparrow ?P, L}$$

$$\dfrac{\dfrac{\vdash P, H, \Gamma \Uparrow L}{\vdash H, \Gamma \Uparrow P, L} \qquad \Pi_2}{\vdash \Gamma, \Delta \Uparrow P, L}\,[\Uparrow\mathrm{LC}] \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\vdash H, P, \Gamma \Uparrow L \qquad \Pi_2}{\vdash P, \Gamma, \Delta \Uparrow L}\,[\Uparrow\mathrm{LC}]}{\vdash \Gamma, \Delta \Uparrow P, L}$$

$$
\dfrac{\dfrac{\vdash H, \Gamma \Downarrow P}{\vdash H, P, \Gamma \Uparrow \cdot}\,[D_1] \qquad \vdash \Delta \Downarrow H^{\perp}}{\vdash P, \Gamma, \Delta \Uparrow \cdot}\,[\Uparrow\mathrm{LC}]
\qquad \rightsquigarrow \qquad
\dfrac{\dfrac{\vdash H, \Gamma \Downarrow P \qquad \vdash \Delta \Downarrow H^{\perp}}{\vdash \Gamma, \Delta \Downarrow P}\,[\Downarrow\mathrm{LC}]}{\vdash P, \Gamma, \Delta \Uparrow \cdot}\,[D_1]
$$

$$
\dfrac{\dfrac{\vdash H, \Gamma \Downarrow Q}{\vdash H, \Gamma \Uparrow \cdot}\,[D_2] \qquad \vdash \Delta \Downarrow H^{\perp}}{\vdash \Gamma, \Delta \Uparrow \cdot}\,[\Uparrow\mathrm{LC}]
\qquad \rightsquigarrow \qquad
\dfrac{\dfrac{\vdash H, \Gamma \Downarrow Q \qquad \vdash \Delta \Downarrow H^{\perp}}{\vdash \Gamma, \Delta \Downarrow Q}\,[\Downarrow\mathrm{LC}]}{\vdash \Gamma, \Delta \Uparrow \cdot}\,[D_2]
$$

$$
\dfrac{\dfrac{\vdash H, \Gamma \Downarrow Q}{\vdash H, \Gamma \Uparrow \cdot}\,[D_3] \qquad \vdash \Delta \Downarrow H^{\perp}}{\vdash \Gamma, \Delta \Uparrow \cdot}\,[\Uparrow\mathrm{LC}]
\qquad \rightsquigarrow \qquad
\dfrac{\dfrac{\vdash H, \Gamma \Downarrow Q \qquad \vdash \Delta \Downarrow H^{\perp}}{\vdash \Gamma, \Delta \Downarrow Q}\,[\Downarrow\mathrm{LC}]}{\vdash \Gamma, \Delta \Uparrow \cdot}\,[D_3]
$$

$$
\dfrac{\dfrac{\dfrac{\vdash \Gamma \Uparrow H}{\vdash \Gamma \Downarrow H}}{\vdash H, \Gamma \Uparrow \cdot} \qquad \vdash \Delta \Downarrow H^{\perp}}{\vdash \Gamma, \Delta \Uparrow \cdot}\,[\Uparrow\mathrm{LC}]
\qquad \rightsquigarrow \qquad
\dfrac{\vdash \Gamma \Uparrow H \qquad \vdash \Delta \Downarrow H^{\perp}}{\vdash \Gamma, \Delta \Uparrow \cdot}\,[\Uparrow\mathrm{C}]
$$

$$
\dfrac{\dfrac{\vdash \Gamma \Downarrow H}{\vdash H, \Gamma \Uparrow \cdot} \qquad \dfrac{\vdash \Delta \Uparrow H^{\perp}}{\vdash \Delta \Downarrow H^{\perp}}}{\vdash \Gamma, \Delta \Uparrow \cdot}\,[\Uparrow\mathrm{LC}]
\qquad \rightsquigarrow \qquad
\dfrac{\vdash \Delta \Uparrow H^{\perp} \qquad \vdash \Gamma \Downarrow H}{\vdash \Gamma, \Delta \Uparrow \cdot}\,[\Uparrow\mathrm{C}]
$$

4 $\Uparrow$ Cut Elimination

$$\frac{\begin{array}{c}[\Pi_1] \\ \vdash \Gamma \Uparrow H, S\end{array} \qquad \begin{array}{c}[\Pi_2] \\ \vdash \Delta \Downarrow H^{\perp}\end{array}}{\vdash \Gamma, \Delta \Uparrow S}\,[\Uparrow\mathrm{C}]$$

$$\frac{\dfrac{\vdash \Gamma \Uparrow L}{\vdash \Gamma \Uparrow \bot, L} \qquad \dfrac{}{\vdash \cdot \Downarrow 1}}{\vdash \Gamma \Uparrow L}\,[\Uparrow\mathrm{C}] \quad \rightsquigarrow \quad \vdash \Gamma \Uparrow L$$

$$\frac{\dfrac{\begin{array}{c}[\Sigma_1] \\ \vdash \Gamma \Uparrow P, Q, L\end{array}}{\vdash \Gamma \Uparrow P \parr Q, L} \qquad \dfrac{\begin{array}{c}[\Sigma_2] \\ \vdash \Delta_P \Downarrow P^{\perp}\end{array} \qquad \begin{array}{c}[\Sigma_3] \\ \vdash \Delta_Q \Downarrow Q^{\perp}\end{array}}{\vdash \Delta \Downarrow P^{\perp} \otimes Q^{\perp}}}{\vdash \Gamma, \Delta \Uparrow L}\,[\Uparrow\mathrm{C}] \quad \rightsquigarrow \quad \frac{\dfrac{[\Sigma_1] \qquad [\Sigma_2]}{\vdash \Gamma, \Delta_P \Uparrow Q, L}\,[\Uparrow\mathrm{C}] \qquad [\Sigma_3]}{\vdash \Gamma, \Delta \Uparrow L}\,[\Uparrow\mathrm{C}]$$

$$\frac{\dfrac{\vdash \Gamma \Uparrow P, L \qquad \vdash \Gamma \Uparrow Q, L}{\vdash \Gamma \Uparrow P \,\&\, Q, L} \qquad \dfrac{\vdash \Delta \Downarrow P^{\perp}}{\vdash \Delta \Downarrow P^{\perp} \oplus Q^{\perp}}}{\vdash \Gamma, \Delta \Uparrow L}\,[\Uparrow\mathrm{C}] \quad \rightsquigarrow \quad \frac{\vdash \Gamma \Uparrow P, L \qquad \vdash \Delta \Downarrow P^{\perp}}{\vdash \Gamma, \Delta \Uparrow L}\,[\Uparrow\mathrm{C}]$$

$$\frac{\dfrac{\vdash \Gamma \Uparrow P, L \qquad \vdash \Gamma \Uparrow Q, L}{\vdash \Gamma \Uparrow P \,\&\, Q, L} \qquad \dfrac{\vdash \Delta \Downarrow Q^{\perp}}{\vdash \Delta \Downarrow P^{\perp} \oplus Q^{\perp}}}{\vdash \Gamma, \Delta \Uparrow L}\,[\Uparrow\mathrm{C}] \quad \rightsquigarrow \quad \frac{\vdash \Gamma \Uparrow Q, L \qquad \vdash \Delta \Downarrow Q^{\perp}}{\vdash \Gamma, \Delta \Uparrow L}\,[\Uparrow\mathrm{C}]$$

$$\frac{\dfrac{\vdash \Gamma \Uparrow P[c/x], L}{\vdash \Gamma \Uparrow \forall x.P, L} \qquad \dfrac{\vdash \Delta \Downarrow P^{\perp}[t/x]}{\vdash \Delta \Downarrow \exists x.P^{\perp}}}{\vdash \Gamma, \Delta \Uparrow L}\,[\Uparrow\mathrm{C}] \quad \rightsquigarrow \quad \frac{\vdash \Gamma \Uparrow P[c/x], L \qquad \vdash \Delta \Downarrow P^{\perp}[t/x]}{\vdash \Gamma, \Delta \Uparrow L}\,[\Uparrow\mathrm{C}]$$

$$\frac{\dfrac{\vdash \Theta, H : \Gamma \Uparrow L}{\vdash \Theta : \Gamma \Uparrow ?H, L} \qquad \vdash \Theta : \cdot \Downarrow !H^{\perp}}{\vdash \Theta : \Gamma \Uparrow L}\,[\Uparrow\mathrm{C}] \quad \rightsquigarrow \quad \frac{\vdash \Theta, H : \Gamma \Uparrow L \qquad \vdash \Theta : \cdot \Downarrow !H^{\perp}}{\vdash \Theta : \Gamma \Uparrow L}\,[\Uparrow\mathrm{CC}]$$

$$\frac{\dfrac{\vdash H, \Gamma \Uparrow L}{\vdash \Gamma \Uparrow H, L} \qquad \vdash \Delta \Downarrow H^{\perp}}{\vdash \Gamma, \Delta \Uparrow L}\,[\Uparrow\mathrm{C}] \quad \rightsquigarrow \quad \frac{\vdash H, \Gamma \Uparrow L \qquad \vdash \Delta \Downarrow H^{\perp}}{\vdash \Gamma, \Delta \Uparrow L}\,[\Uparrow\mathrm{LC}]$$

5 $\Uparrow$-Classic Cut Elimination

$$\frac{\overset{[\Pi_1]}{\vdash \Theta, H : \Gamma \Uparrow S} \qquad \overset{[\Pi_2]}{\vdash \Theta : \cdot \Downarrow !H^{\perp}}}{\vdash \Theta : \Gamma \Uparrow S}\,[\Uparrow\mathrm{CC}]$$

$$\frac{\dfrac{\vdash \Theta, H : \Gamma \Uparrow L}{\vdash \Theta, H : \Gamma \Uparrow \bot, L} \qquad \Pi_2}{\vdash \Theta : \Gamma \Uparrow \bot, L}\,[\Uparrow\mathrm{CC}] \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Theta, H : \Gamma \Uparrow L \qquad \Pi_2}{\vdash \Theta : \Gamma \Uparrow L}\,[\Uparrow\mathrm{CC}]}{\vdash \Theta : \Gamma \Uparrow \bot, L}$$

$$\frac{\dfrac{}{\vdash \Theta, H : \Gamma \Uparrow \top, L} \qquad \Pi_2}{\vdash \Theta : \Gamma \Uparrow \top, L}\,[\Uparrow\mathrm{CC}] \quad \rightsquigarrow \quad \dfrac{}{\vdash \Theta : \Gamma \Uparrow \top, L}$$

$$\frac{\dfrac{\vdash \Theta, H : \Gamma \Uparrow P, Q, L}{\vdash \Theta, H : \Gamma \Uparrow P \parr Q, L} \qquad \Pi_2}{\vdash \Theta : \Gamma \Uparrow P \parr Q, L}\,[\Uparrow\mathrm{CC}] \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Theta, H : \Gamma \Uparrow P, Q, L \qquad \Pi_2}{\vdash \Theta : \Gamma \Uparrow P, Q, L}\,[\Uparrow\mathrm{CC}]}{\vdash \Theta : \Gamma \Uparrow P \parr Q, L}$$

$$\frac{\dfrac{\overset{[\Sigma_1]}{\vdash \Theta, H : \Gamma \Uparrow P, L} \qquad \overset{[\Sigma_2]}{\vdash \Theta, H : \Gamma \Uparrow Q, L}}{\vdash \Theta, H : \Gamma \Uparrow P \,\&\, Q, L} \qquad \Pi_2}{\vdash \Theta : \Gamma \Uparrow P \,\&\, Q, L}\,[\Uparrow\mathrm{CC}] \quad \rightsquigarrow \quad \dfrac{\dfrac{\Sigma_1 \qquad \Pi_2}{\vdash \Theta : \Gamma \Uparrow P, L}\,[\Uparrow\mathrm{CC}] \qquad \dfrac{\Sigma_2 \qquad \Pi_2}{\vdash \Theta : \Gamma \Uparrow Q, L}\,[\Uparrow\mathrm{CC}]}{\vdash \Theta : \Gamma \Uparrow P \,\&\, Q, L}$$

$$\frac{\dfrac{\vdash \Theta, H, P : \Gamma \Uparrow L}{\vdash \Theta, H : \Gamma \Uparrow ?P, L} \qquad \Pi_2}{\vdash \Theta : \Gamma \Uparrow ?P, L}\,[\Uparrow\mathrm{CC}] \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Theta, P, H : \Gamma \Uparrow L \qquad \dfrac{\Pi_2}{\vdash \Theta, P : \Delta \Uparrow H^{\perp}}\,[W]}{\vdash \Theta, P : \Gamma \Uparrow L}\,[\Uparrow\mathrm{CC}]}{\vdash \Theta : \Gamma \Uparrow ?P, L}$$

$$\frac{\dfrac{\vdash \Theta, H : \Gamma \Uparrow P[c/x], L}{\vdash \Theta, H : \Gamma \Uparrow \forall x P, L} \qquad \Pi_2}{\vdash \Theta : \Gamma \Uparrow \forall x P, L}\,[\Uparrow\mathrm{CC}] \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Theta, H : \Gamma \Uparrow P[c/x], L \qquad \Pi_2}{\vdash \Theta : \Gamma \Uparrow P[c/x], L}\,[\Uparrow\mathrm{CC}]}{\vdash \Theta : \Gamma \Uparrow \forall x P, L}$$

$$\frac{\dfrac{\vdash \Theta, H : P, \Gamma \Uparrow L}{\vdash \Theta, H : \Gamma \Uparrow P, L} \qquad \Pi_2}{\vdash \Theta : \Gamma \Uparrow P, L}\,[\Uparrow\mathrm{CC}] \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Theta, H : P, \Gamma \Uparrow L \qquad \Pi_2}{\vdash \Theta : P, \Gamma \Uparrow L}\,[\Uparrow\mathrm{CC}]}{\vdash \Theta : \Gamma \Uparrow P, L}$$

H is not a positive atom.

$$\dfrac{\dfrac{\vdash \Theta, H : \Gamma \Downarrow Q}{\vdash \Theta, H : \Gamma \Uparrow \cdot}\,[D_2] \qquad \Pi_2}{\vdash \Theta : \Gamma \Uparrow \cdot}\,[\Uparrow\mathrm{CC}] \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Theta, H : \Gamma \Downarrow Q \qquad \Pi_2}{\vdash \Theta : \Gamma \Downarrow Q}\,[\Downarrow\mathrm{CC}]}{\vdash \Theta : \Gamma \Uparrow \cdot}\,[D_2]$$

$$\dfrac{\dfrac{\vdash \Theta, H : \Gamma \Downarrow Q}{\vdash \Theta, H : \Gamma \Uparrow \cdot}\,[D_3] \qquad \Pi_2}{\vdash \Theta : \Gamma \Uparrow \cdot}\,[\Uparrow\mathrm{CC}] \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Theta, H : \Gamma \Downarrow Q \qquad \Pi_2}{\vdash \Theta : \Gamma \Downarrow Q}\,[\Downarrow\mathrm{CC}]}{\vdash \Theta : \Gamma \Uparrow \cdot}\,[D_3]$$

$$\dfrac{\dfrac{\vdash \Theta, H : \Gamma \Downarrow P}{\vdash \Theta, H : P, \Gamma \Uparrow \cdot}\,[D_2] \qquad \Pi_2}{\vdash \Theta : P, \Gamma \Uparrow \cdot}\,[\Uparrow\mathrm{CC}] \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Theta, H : \Gamma \Downarrow P \qquad \Pi_2}{\vdash \Theta : \Gamma \Downarrow P}\,[\Downarrow\mathrm{CC}]}{\vdash \Theta : P, \Gamma \Uparrow \cdot}\,[D_2]$$

$$\dfrac{\dfrac{\vdash \Theta, H : \Gamma \Downarrow H}{\vdash \Theta, H : \Gamma \Uparrow \cdot} \qquad \dfrac{\vdash \Theta : \cdot \Uparrow H^{\perp}}{\vdash \Theta : \cdot \Downarrow !H^{\perp}}}{\Theta : \Gamma \Uparrow \cdot}\,[\Uparrow\mathrm{CC}] \quad \rightsquigarrow \quad \dfrac{\vdash \Theta : \cdot \Uparrow H^{\perp} \qquad \dfrac{\vdash \Theta, H : \Gamma \Downarrow H \qquad \dfrac{\vdash \Theta : \cdot \Uparrow H^{\perp}}{\vdash \Theta : \cdot \Downarrow !H^{\perp}}}{\vdash \Theta : \Gamma \Downarrow H}\,[\Downarrow\mathrm{CC}]}{\vdash \Theta : \Gamma \Uparrow \cdot}\,[\Uparrow\mathrm{C}]$$

H is a positive atom (A^+).

$$\dfrac{\dfrac{\vdash \Theta, A^+ : \Gamma \Downarrow Q}{\vdash \Theta, A^+ : \Gamma, Q \Uparrow \cdot}[D_1] \qquad \Pi_2}{\vdash \Theta : \Gamma, Q \Uparrow \cdot}[\Uparrow\text{CC}]$$

By Atomic Permutation $\vdash \Theta : \Gamma \Uparrow Q$ is derivable.
If Q is positive then $\vdash \Theta : \Gamma, Q \Uparrow \cdot$. If Q is release, then:

$$\dfrac{\dfrac{\vdash \Theta, A^+ : \Gamma \Uparrow Q}{\vdash \Theta, A^+ : \Gamma, Q \Uparrow \cdot}[D_1 R \Downarrow] \qquad \Pi_2}{\vdash \Theta : \Gamma, Q \Uparrow \cdot}[\Uparrow\text{CC}] \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Theta, A^+ : \Gamma \Uparrow Q \qquad \Pi_2}{\vdash \Theta : \Gamma \Uparrow Q}[\Uparrow\text{CC}]}{\vdash \Theta : \Gamma, Q \Uparrow \cdot}[D_1 R \Downarrow]$$

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$$\dfrac{\dfrac{\vdash \Theta, A^+ : \Gamma \Downarrow Q}{\vdash \Theta, A^+ : \Gamma \Uparrow \cdot}[D_2] \qquad \Pi_2}{\vdash \Theta : \Gamma \Uparrow \cdot}[\Uparrow\text{CC}] \quad \rightsquigarrow \quad \dfrac{\vdash \Theta : \Gamma \Uparrow Q}{\Theta : \Gamma \Uparrow \cdot}[\text{AbsorptionC}]$$

By Atomic Permutation $\vdash \Theta : \Gamma \Uparrow Q$ is derivable.

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$$\dfrac{\dfrac{\vdash \Theta, A^+ : \Gamma \Downarrow Q}{\vdash \Theta, A^+ : \Gamma \Uparrow \cdot}[D_3] \qquad \Pi_2}{\vdash \Theta : \Gamma \Uparrow \cdot}[\Uparrow\text{CC}]$$

By Atomic Permutation $\vdash \Theta : \Gamma \Uparrow Q$ is derivable.
If Q is not a negative atom, then

$$\dfrac{\vdash \Theta : \Gamma \Uparrow Q}{\Theta : \Gamma \Uparrow \cdot}[\text{AbsorptionT}]$$

If Q is a negative atom

$$\dfrac{\dfrac{\dfrac{}{\vdash \Theta, A^+ : X^+ \Downarrow X^-}}{\vdash \Theta, A^+ : X^+ \Uparrow \cdot}[D_3] \qquad \Pi_2}{\vdash \Theta : X^+ \Uparrow \cdot}[\Uparrow\text{CC}] \quad \rightsquigarrow \quad \dfrac{\dfrac{}{\vdash \Theta : X^+ \Downarrow X^-}}{\vdash \Theta : X^+ \Uparrow \cdot}[D_3]$$

$$\dfrac{\dfrac{\dfrac{}{\vdash \Theta, A^+ : \cdot \Downarrow X^-}[X^+ \in \Theta]}{\vdash \Theta, A^+ : \Gamma \Uparrow \cdot}[D_3] \qquad \Pi_2}{\vdash \Theta : \cdot \Uparrow \cdot}[\Uparrow\text{CC}] \quad \rightsquigarrow \quad \dfrac{\dfrac{}{\vdash \Theta : \cdot \Downarrow X^-}[X \in \Theta]}{\vdash \Theta : \cdot \Uparrow \cdot}[D_3]$$

$$\dfrac{\dfrac{\dfrac{}{\vdash \Theta, A^+ : \cdot \Downarrow A^-}}{\vdash \Theta, A^+ : \cdot \Uparrow \cdot}[D_3] \qquad \dfrac{\dfrac{\vdash \Theta : A^- \Uparrow \cdot}{\vdash \Theta : \cdot \Uparrow A^-}}{\vdash \Theta : \cdot \Downarrow !A^-}}{\vdash \Theta : \cdot \Uparrow \cdot}[\Uparrow\text{CC}] \quad \rightsquigarrow \quad \dfrac{\vdash \Theta : A^- \Uparrow \cdot}{\vdash \Theta : \cdot \Uparrow \cdot}[\text{AbsorptionL}]$$

6 $\Downarrow$-Classic Cut Elimination

$$\dfrac{\overset{[\Pi_1]}{\vdash \Theta, H : \Gamma \Downarrow F} \qquad \overset{[\Pi_2]}{\vdash \Theta : \cdot \Downarrow\, !H^{\perp}}}{\vdash \Theta : \Gamma \Downarrow F}\,[\Downarrow\mathrm{CC}]$$

$$\dfrac{\dfrac{}{\vdash \Theta, H : X^{+} \Downarrow X^{-}} \qquad \Pi_2}{\vdash \Theta : X^{+} \Downarrow X^{-}}\,[\Downarrow\mathrm{CC}] \quad\rightsquigarrow\quad \dfrac{}{\vdash \Theta : X^{+} \Downarrow X^{-}}$$

$$\dfrac{\dfrac{}{\vdash \Theta, H : \cdot \Downarrow X^{-}}\,[X^{+} \in \Theta] \qquad \Pi_2}{\vdash \Theta : \cdot \Downarrow X^{-}}\,[\Downarrow\mathrm{CC}] \quad\rightsquigarrow\quad \dfrac{}{\vdash \Theta : \cdot \Downarrow X^{-}}$$

$$\dfrac{\vdash \Theta, H : \cdot \Downarrow 1 \qquad \Pi_2}{\vdash \Theta : \cdot \Downarrow 1}\,[\Downarrow\mathrm{CC}] \quad\rightsquigarrow\quad \dfrac{}{\vdash \Theta : \cdot \Downarrow 1}$$

$$\dfrac{\dfrac{\overset{[\Sigma_1]}{\vdash \Theta, H : \Gamma_P \Downarrow P} \qquad \overset{[\Sigma_2]}{\vdash \Theta, H : \Gamma_Q \Downarrow Q}}{\vdash \Theta, H : \Gamma \Downarrow P \otimes Q} \qquad \Pi_2}{\vdash \Theta : \Gamma \Downarrow P \otimes Q}\,[\Downarrow\mathrm{CC}] \quad\rightsquigarrow\quad \dfrac{\dfrac{\Sigma_1 \qquad \Pi_2}{\vdash \Theta : \Gamma_P \Downarrow P}\,[\Downarrow\mathrm{CC}] \qquad \dfrac{\Sigma_2 \qquad \Pi_2}{\vdash \Theta : \Gamma_Q \Downarrow Q}\,[\Downarrow\mathrm{CC}]}{\vdash \Theta : \Gamma \Downarrow P \otimes Q}$$

$$\dfrac{\dfrac{\vdash \Theta, H : \Gamma \Downarrow P}{\vdash \Theta, H : \Gamma \Downarrow P \oplus Q} \qquad \Pi_2}{\vdash \Theta : \Gamma \Downarrow P \oplus Q}\,[\Downarrow\mathrm{CC}] \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Theta, H : \Gamma \Downarrow P \qquad \Pi_2}{\vdash \Theta : \Gamma \Downarrow P}\,[\Downarrow\mathrm{CC}]}{\vdash \Theta : \Gamma \Downarrow P \oplus Q}$$

$$\dfrac{\dfrac{\vdash \Theta, H : \Gamma \Downarrow Q}{\vdash \Theta, H : \Gamma \Downarrow P \oplus Q} \qquad \Pi_2}{\vdash \Theta : \Gamma \Downarrow P \oplus Q}\,[\Downarrow\mathrm{CC}] \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Theta, H : \Gamma \Downarrow Q \qquad \Pi_2}{\vdash \Theta : \Gamma \Downarrow Q}\,[\Downarrow\mathrm{CC}]}{\vdash \Theta : \Gamma \Downarrow P \oplus Q}$$

$$\dfrac{\dfrac{\vdash \Theta, H : \Gamma \Downarrow P[c/x]}{\vdash \Theta, H : \Gamma \Downarrow \exists x.P} \qquad \Pi_2}{\vdash \Theta : \Gamma \Downarrow \exists x.P}\,[\Downarrow\mathrm{CC}] \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Theta, H : \Gamma \Downarrow P[c/x] \qquad \Pi_2}{\vdash \Theta : \Gamma \Downarrow P[c/x]}\,[\Downarrow\mathrm{CC}]}{\vdash \Theta : \Gamma \Downarrow \exists x.P}$$

$$\dfrac{\dfrac{\vdash \Theta, H : \cdot \Uparrow P}{\vdash \Theta, H : \cdot \Downarrow\, !P} \qquad \Pi_2}{\vdash \Theta : \cdot \Downarrow\, !P}\,[\Downarrow\mathrm{CC}] \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Theta, H : \cdot \Uparrow P \qquad \Pi_2}{\vdash \Theta : \cdot \Uparrow P}\,[\Uparrow\mathrm{CC}]}{\vdash \Theta : \cdot \Downarrow\, !P}$$

$$\dfrac{\dfrac{\vdash \Theta, H : \Gamma \Uparrow P}{\vdash \Theta, H : \Gamma \Downarrow P} \qquad \Pi_2}{\vdash \Theta : \Gamma \Downarrow P}\,[\Downarrow\mathrm{CC}] \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash \Theta, H : \Gamma \Uparrow P \qquad \Pi_2}{\vdash \Theta : \Gamma \Uparrow P}\,[\Uparrow\mathrm{CC}]}{\vdash \Theta : \Gamma \Downarrow P}$$