Specification of context constraints for focused linear logic

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When searching for a proof in linear logic, it is common to have conditions over the content of the context(s) so that a rule can be applied. In fact, this is what allows a greater control over the resources and therefore increases this logic's expressiveness.

When generating macro rules, the content of the context is not known. Therefore, it is necessary to generate some constraints over the resulting context of the macro rule that will guarantee that all "micro" rules (the single inference rules that were applied to generate that macro) are valid.

The aim of this document is to list all these constraints for each rule of linear logic's sequent calculus. The final constraints of the macro rule will be a combination of the constraints of each individual inference rule used.

For every formula below, we assume that Θ_i are unbounded contexts, Γ_i are bounded contexts, Δ_i are sets of formulas which we are aware that exists in the context i (this is necessary since some rules put formulas on the context). The contexts are separated by a colon (:). The number of bounded and unbounded contexts will depend on each specification.

Inference rules for formulas focused on the right of the sequent

Synchronous phase

$$\frac{\Theta_i : \Gamma_{1i} \gg A \quad \Theta_i : \Gamma_{2i} \gg B}{\Theta_i : \Gamma_i, \Delta_i \gg A \otimes B} \otimes_R$$

Constraints:

- $\Gamma_i \cup \Delta_i = \Gamma_{1i} \cup \Gamma_{2i}$: $\mathrm{eqctx}(\{\Gamma_i, \Delta_i\}, \{\Gamma_{1i}, \Gamma_{2i}\})$

$$\frac{\Theta_i : \Gamma_i \gg A_i}{\Theta_i : \Gamma_i \gg A_1 \oplus B_2} \oplus_R$$

Constraints:

- NONE. Contexts remain unchanged.

$$\frac{\Theta_i : \Gamma_i \Longrightarrow A}{\Theta_i : \Gamma_i \gg !^l A} \; !^l_R$$

Constraints:

- $\Gamma_x = \emptyset$ if $l \not\preceq x$: emp(Γ_x)
- If there exists any Δ_x, the rule should fail.

$$\frac{}{\Theta_i : \Gamma_i \gg 1} \; 1_R$$

Constraints:

- $\Gamma_i = \emptyset$: emp(Γ_i)

$$\frac{\Theta_i : \Gamma_i \gg A[x \backslash t]}{\Theta_i : \Gamma_i \gg \exists x.A} \; \exists_R$$

Constraints:

- NONE. Contexts remain unchanged.

$$\frac{}{\Theta_i : \Gamma_i \gg P} \; init(P \text{ is a positive atom.})$$

Constraints:

- $\Gamma_x = \{P\} \wedge \forall y \neq x \Gamma_y = \emptyset$: $elin(P, \Gamma_x) \wedge \forall y \neq x.emp(\Gamma_y)$
- $P \in \Theta_x \wedge \Gamma_i = \emptyset$: $mctx(P, \Theta_x) \wedge \forall i.emp(\Gamma_i)$

It is important to note that this case generates several different macro rules. One for each linear subexponential (where we consider that P is the only formula of one of them and the rest is empty) and one for the case that P is on an unbounded context and in this case, all the linear contexts are empty.

Asynchronous phase

$$\frac{\Theta_i : \Gamma_i \Longrightarrow A \quad \Theta_i : \Gamma_i \Longrightarrow B}{\Theta_i : \Gamma_i \Longrightarrow A \& B} \; \&_R$$

Constraints:

- NONE. Contexts remain unchanged.

$$\frac{\Theta_i : \Gamma_i' \Longrightarrow B}{\Theta_i : \Gamma_i \Longrightarrow A \multimap^l B} \multimap_R$$

Constraints:

- $\Gamma_i' = \Gamma_i \cup \{\Delta_l \cup A\}$: This does not characterize a constraint in itself. Just states that the formula A is added to context l. Actually, this constraint will have to be included in the case A is an atom.

$$\frac{}{\Theta_i : \Gamma_i \Longrightarrow \top} \top_R$$

Constraints:

- NONE. Contexts remain unchanged.

$$\frac{\Theta_i : \Gamma_i \Longrightarrow A[x\backslash t]}{\Theta_i : \Gamma_i \Longrightarrow \forall x.A} \forall_R$$

Constraints:

- NONE. Contexts remain unchanged.

Inference rules for formulas focused on the left of the sequent

Synchronous phase

$$\frac{\Theta_i : \Gamma_i; A_i \ll \gamma}{\Theta_i : \Gamma_i; A_1 \& A_2 \ll \gamma} \&_L$$

Constraints:

- NONE. Contexts remain unchanged.

$$\frac{\Theta_i : \Gamma_{1i}; B \ll \gamma \quad \Theta_i : \Gamma_{2i} \gg A}{\Theta_i : \Gamma_i; A \multimap B \ll \gamma} \multimap_L$$

Constraints:

- $\Gamma_i = \Gamma_{1i} \cup \Gamma_{2i}$

$$\frac{\Theta_i : \Gamma_i; A[x\backslash t] \ll \gamma}{\Theta_i : \Gamma_i; \forall x.A \ll \gamma} \forall_L$$

Constraints:

- NONE. Contexts remain unchanged.

$$\frac{}{\Theta_i : \Gamma_i; N \ll \gamma} \; init (N \text{ is a negative atom.})$$

Constraints:

- $\Gamma_i = \emptyset : \mathrm{emp}(\Gamma_i)$
- $\gamma = N$

Asynchronous phase