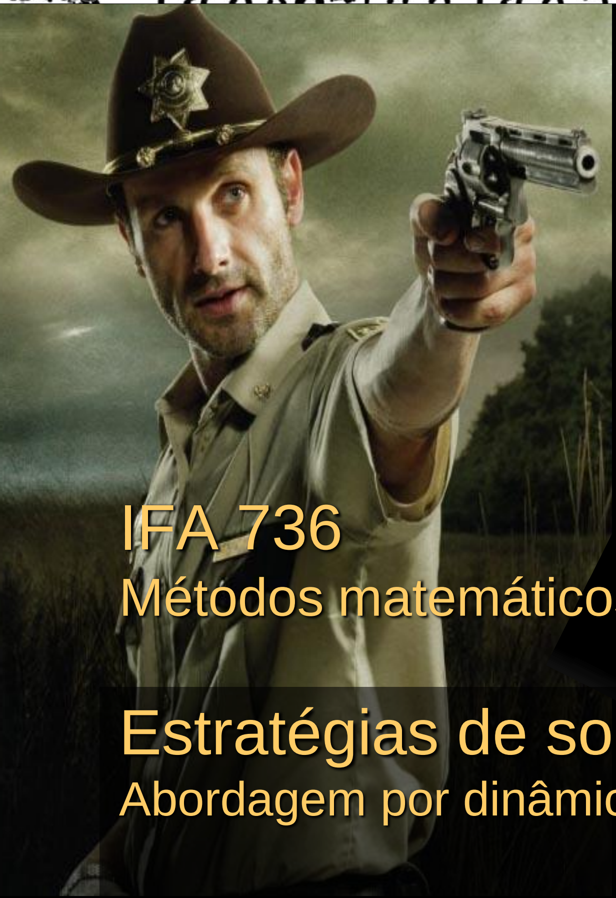

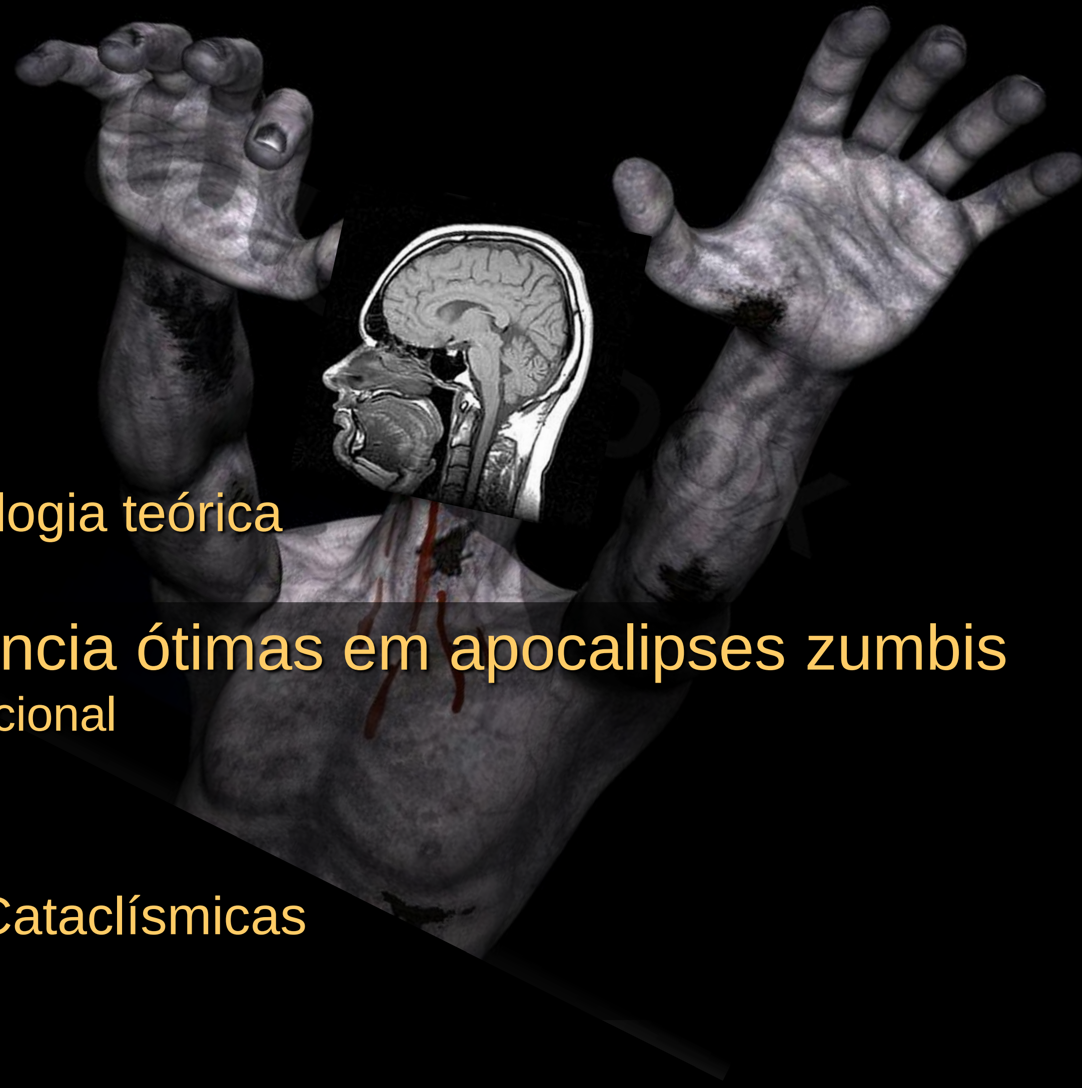

IFA 736
Métodos matemáticos em biologia teórica

Estratégias de sobrevivência ótimas em apocalipses zumbis
Abordagem por dinâmica populacional

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PREPAREDNESS

Because those goddamn zombies aren't going to kill themselves

HUMANITY

IT'S WHAT'S FOR DINNER

ZOMBIE COSTUME

You're doing it right!

UNITY

IF YOU CAN'T EAT THEM, JOIN THEM.

TECHNOLOGY

TELEVISION KILLS MORE THAN JUST BRAIN CELLS

ZOMBIES

A little warning goes a long way.

- Zumbis são mortos-vivos/infectados/amaldiçoados que voltam a vida/são reanimados devido a virus/magia/radiação/roteiristas preguiçosos

- Zumbis não tem mente própria, se movem espasmodicamente, e não sentem fome ou frio; só param se o cérebro for destruído
- Zumbis ‘caçam’ seres humanos. A mordida de um zumbi torna o humano infectado, e este se torna um zumbi irreversivelmente após um tempo de incubação que depende da conveniência do roteiro
- Os militares são capazes de derrotar invasões alienígenas e robôs gigantes armados de lasers, mas por alguma razão corpos cambaleantes, sem inteligência ou armas, estão além das suas capacidades
- Por outro lado, enfermeiras, guardas de shopping e donas de casa se tornam matadores de zumbis extremamente proficientes se sobreviverem por mais de 15 minutos. Nomes, linhas de diálogo flashbacks aumentam ainda mais esta proficiência
- **Como modelar matematicamente uma infestação de zumbis? E qual a aplicabilidade de tal modelo?**

Population dynamics – The Malthusian catastrophe

- Population growth rate is constant
- Population increase in a small time incremente is proportional to existing population. E.g.

- 2 rabbits beget 4 rabbits that beget 8 ... 16 ... 32 ... 64 ... 128 ...

$\rightarrow$ Overpopulation

$$N_m^{Rabbit} = N_0^{Rabbit} 2^m \quad = N_0^{Rabbit} e^{\log 2\, m} \qquad \gamma_{Rabbit} = \log 2$$

- 64 Pandas beget 32 pandas that beget 16 ... 8 ... 4 ... 2 ...

$\rightarrow$ Extintion

$$N_m^{Panda} = N_0^{Panda} 2^{-m} \quad = N_0^{anda} e^{-\log 2\, m} \qquad \gamma_{panda} = -\log 2$$

- For very small increments in time,

$$\frac{dN}{dt} = \gamma N \quad \Rightarrow \; N(t) = N_0 e^{\gamma t}$$

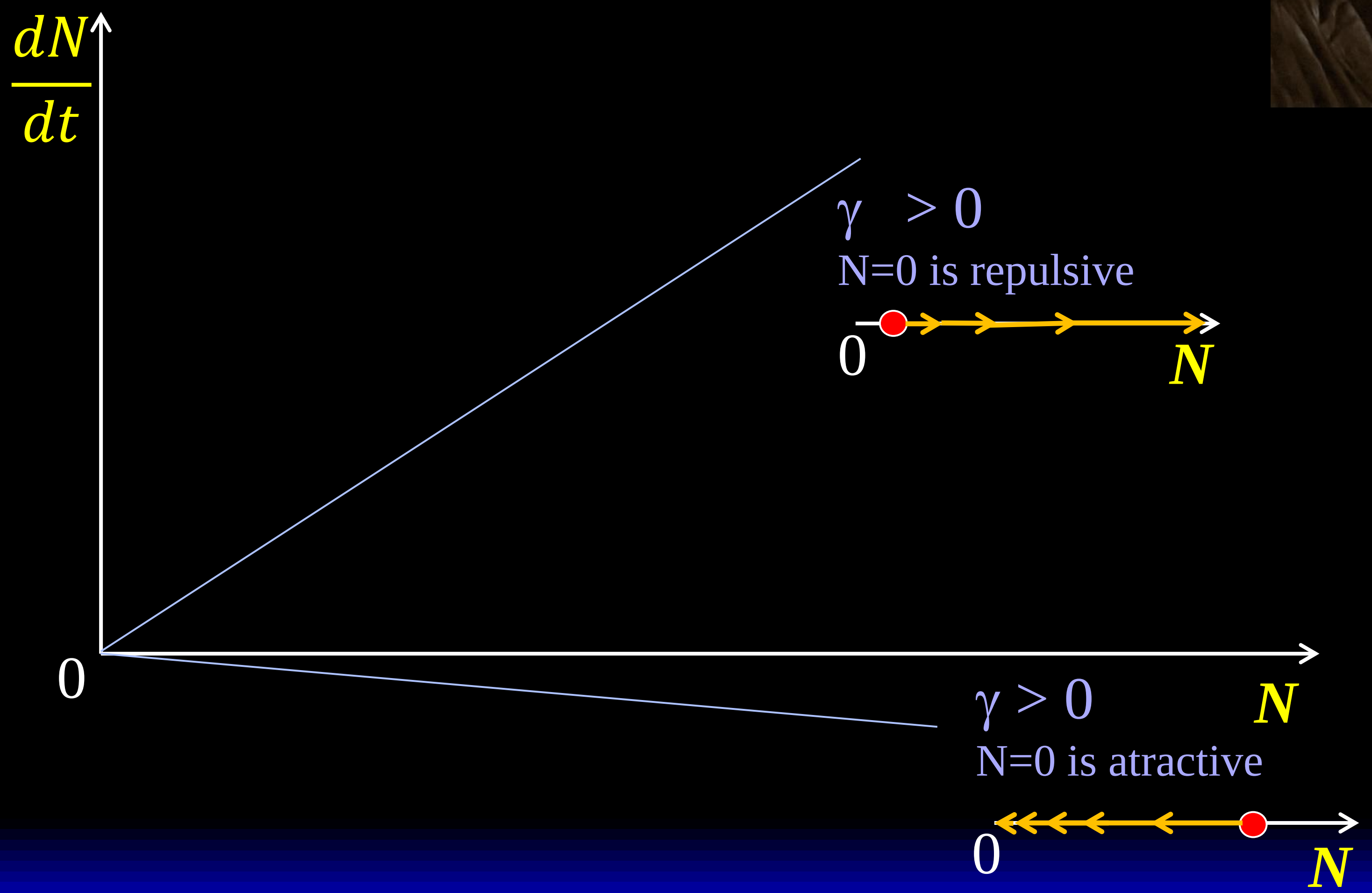
- Population is stable if $\gamma = 0$ (all N are fixed points) or $N_0=0$ (a fixed point)

Population dynamics – The Malthusian catastrophe

$$\frac{dN}{dt} = \gamma N \quad \Rightarrow N(t) = N_0 e^{\gamma t}$$

Growth rate

- *N=0* is a fixed point

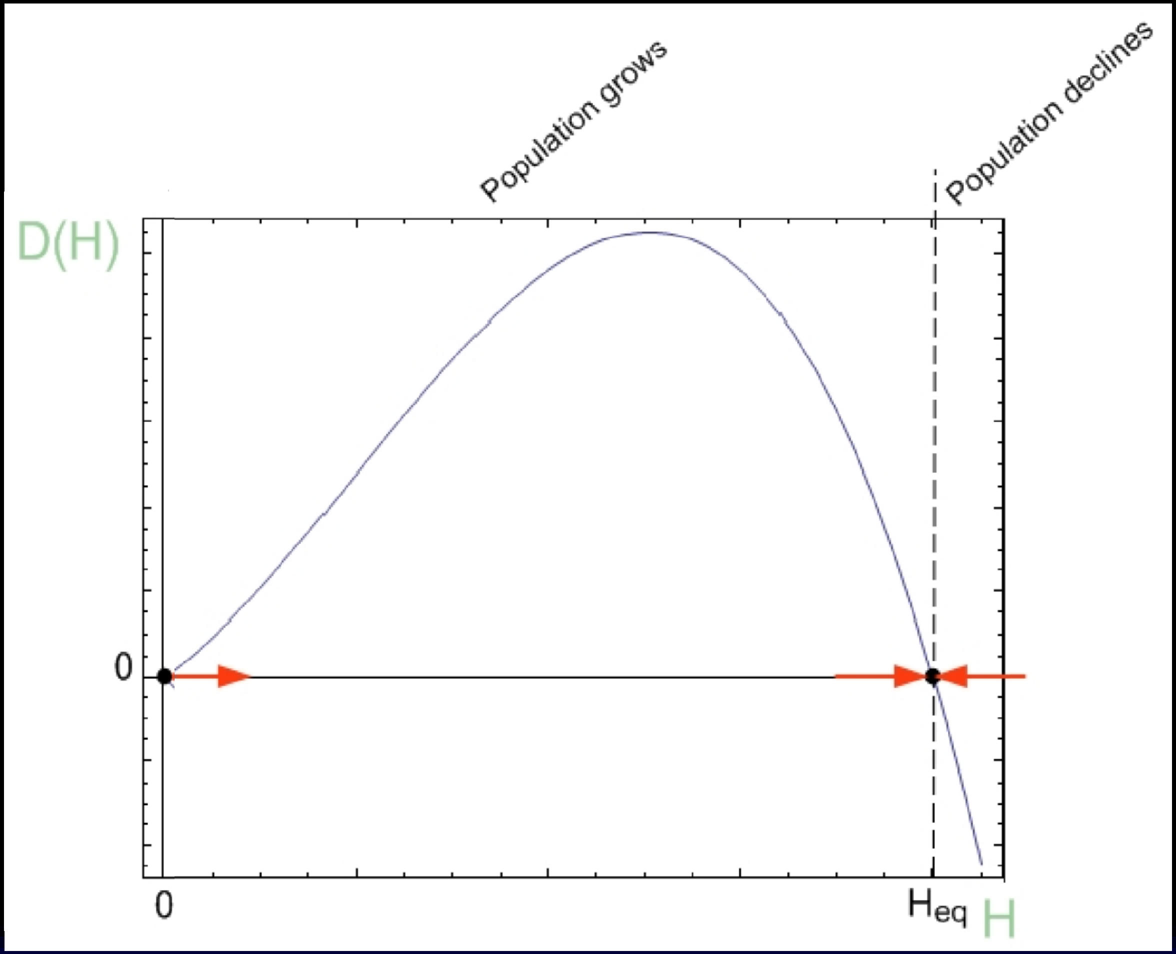


Evitando a catástrofe maltusiana – o modelo logístico

$$\frac{dH}{dt} = \gamma H \left(1 - \frac{H}{H_{eq}}\right)$$

Growth rate

Carrying capacity



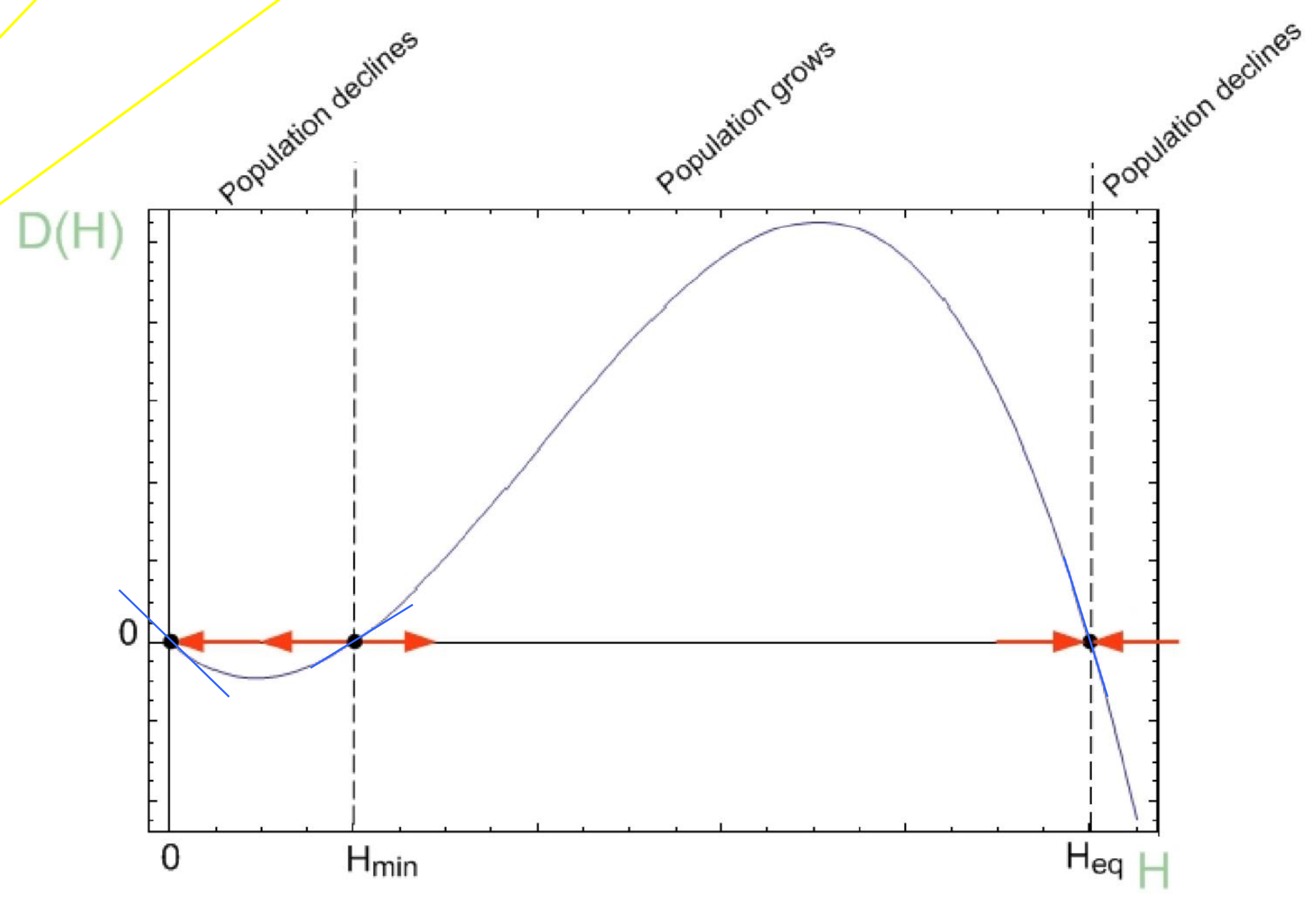
The logistical growth, modified

$$\frac{dH}{dt} = \gamma H(H - H_{min})\left(1 - \frac{H}{H_{eq}}\right)$$

Growth rate

Min. viable pop.

Carrying capacity



- Linearize around fixed points

$$\lambda_0 = -kH_{min} < 0$$

$$\lambda_{H_{min}} = kH_{min}\left(1 - \frac{H_{min}}{H_{eq}}\right) > 0$$

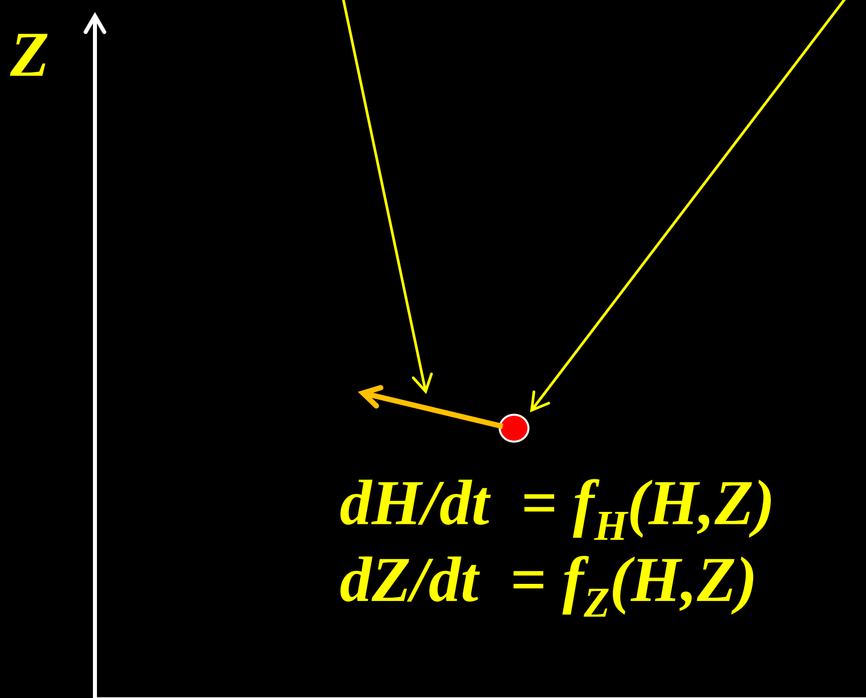
$$\lambda_{H_{eq}} = -k\left(H_{eq} - H_{min}\right) < 0$$

- Atractors have basins of attraction

Interacting populations

- H(t) is the population of humans
- Z(t) is the population of zombies

- A population dynamics models relates where you are (H, Z) to where you are going (H', Z'):



$$dH/dt = f_H(H,Z)$$
$$dZ/dt = f_Z(H,Z)$$

H

- State of population is a vector
$$\vec{P} = \{H, Z\}$$
so that
$$\frac{d\vec{P}}{dt} = \vec{f}(\vec{P})$$

- Linearization means turning $\vec{f}$ into a matrix near fixed points
$$\frac{dP_i}{dt} = \sum_j \frac{\partial f_i}{\partial P_j} P_j$$
$$\frac{d\vec{P}}{dt} = M_{fp}.\vec{P}$$

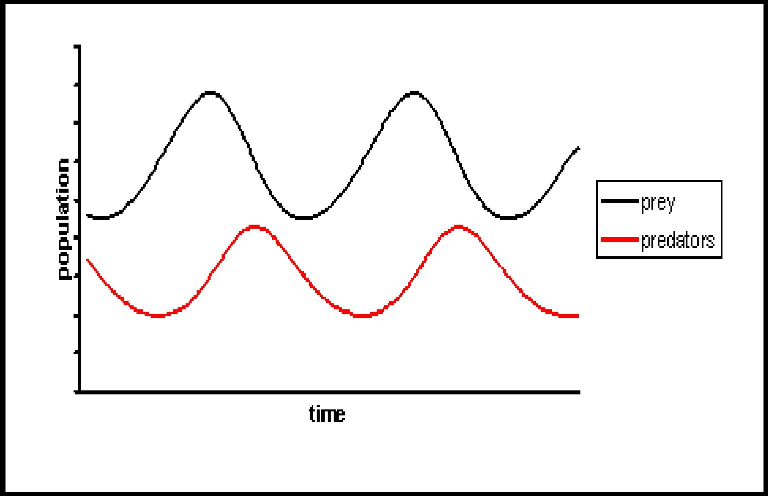
- Dynamics driven by eigenvector/value structure of M_{fp}

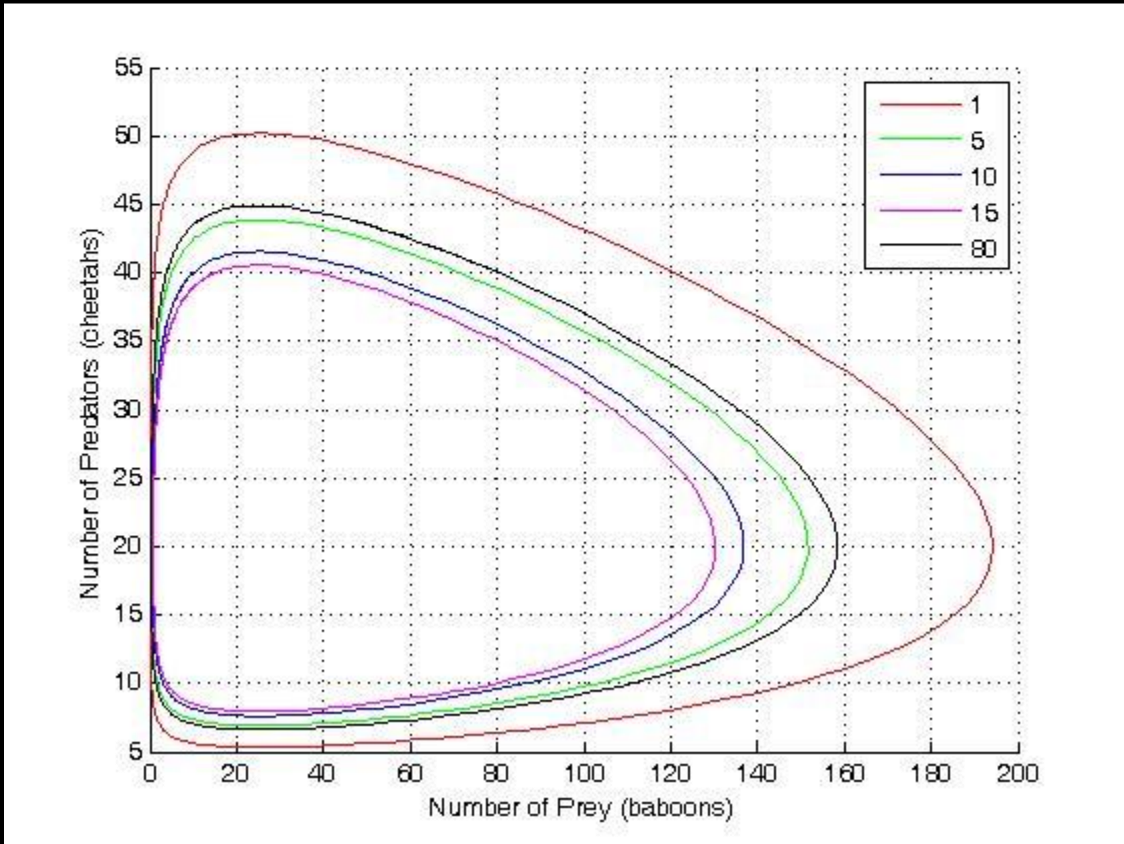
Qual a dinâmica apropriada: um exemplo
O modelo Lotke-Volterra para predador-presa

- C(t) é a população de coelhos no tempo t
- R(t) é a população de raposas no tempo t

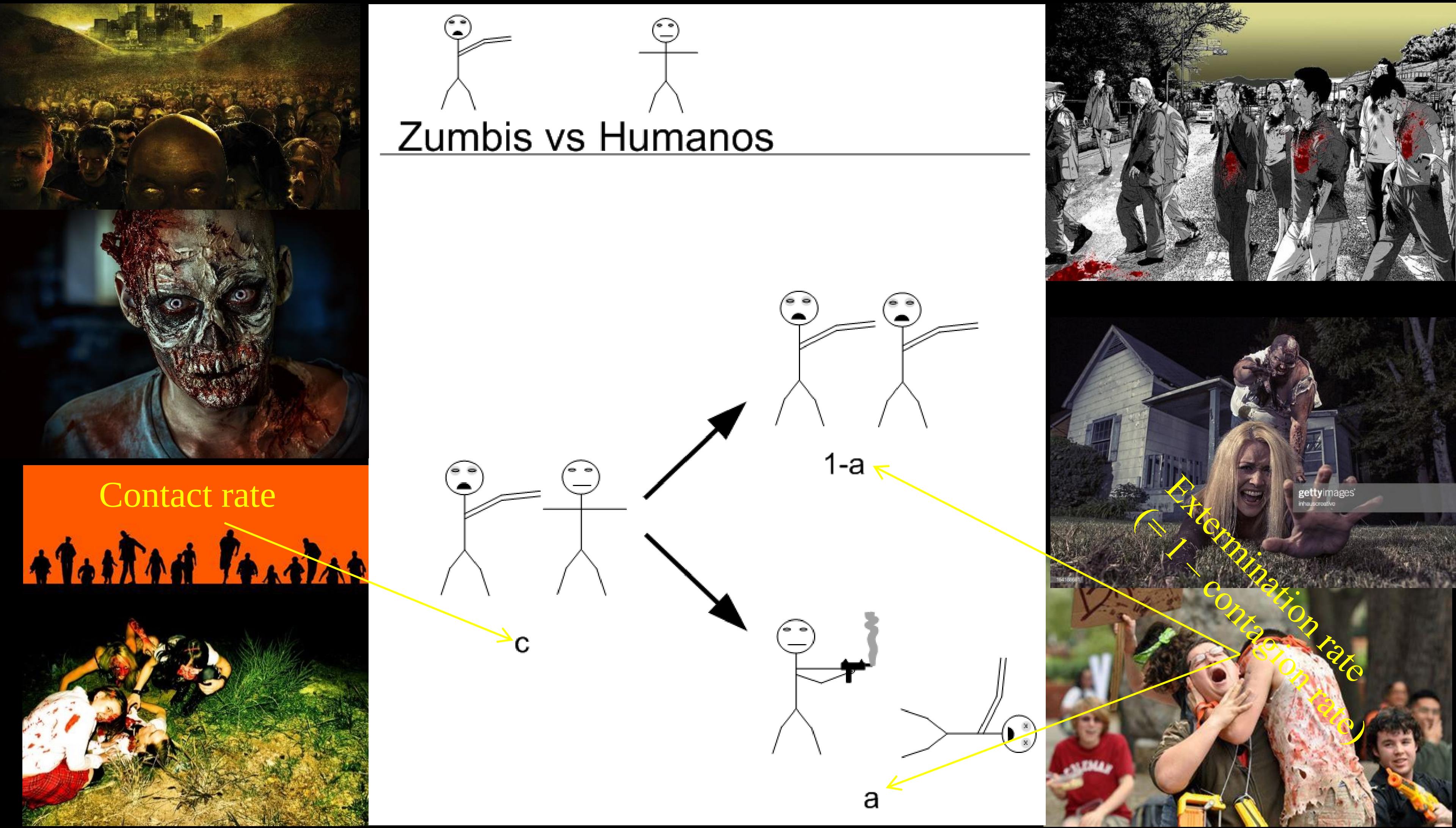
- O único atrator no caso é um ciclo-limite

$$dC/dt = C(a - b\,R)$$
$$dR/dt = R(d\,C - c)$$





Zombies vs Humans



Zombies vs humans, acute phase

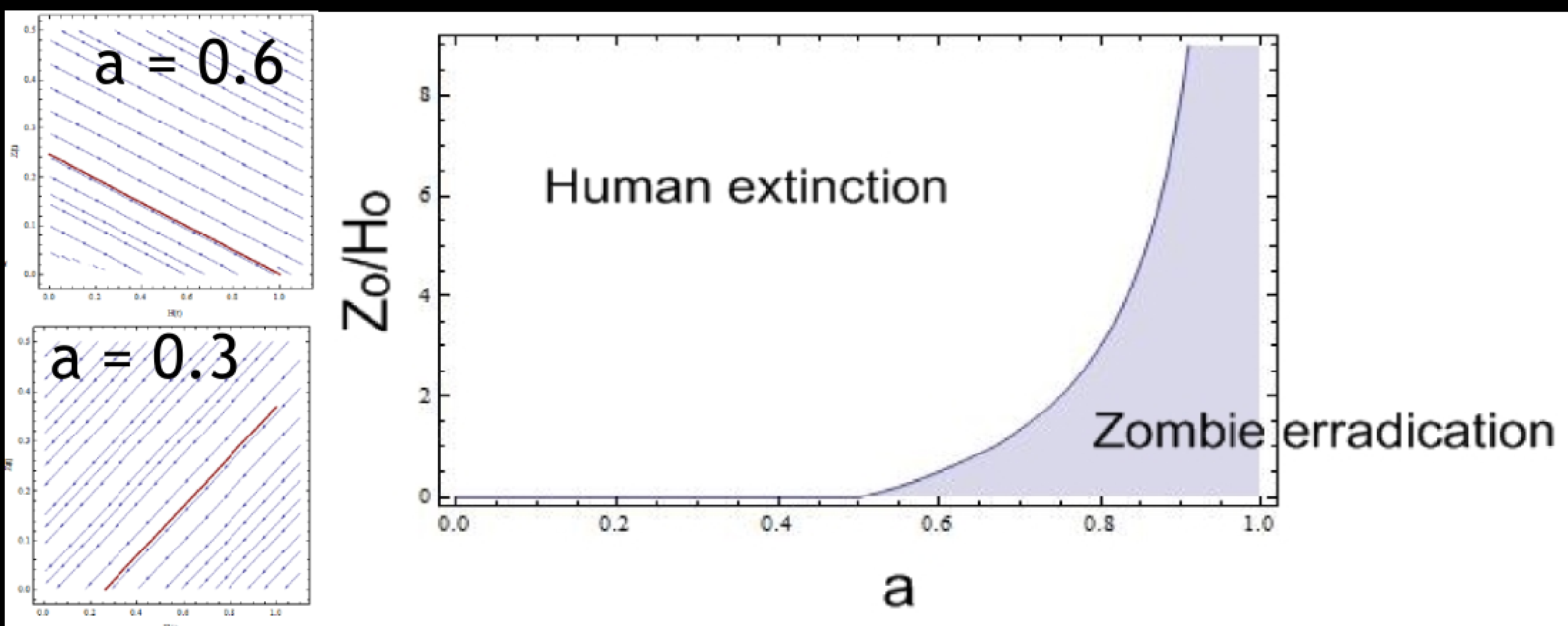
$$\frac{dH}{dt} = -\underline{c(1-a)HZ}$$

$$\frac{dZ}{dt} = \underline{c(1-a)HZ} - \underline{caHZ}$$

Humans turned into zombies

Zombies destroyed by humans

- No self-interaction (demographic) terms
- Analytical solution:
  - For any initial condition (H_o, Z_o,), if $a < 1/2$, the human population suffers extinction. Humans survive if $Zo/Ho < (2a-1)/(1-a)$



Zombies vs humans, long term

$$\frac{dH}{dt} = -c(1-a)HZ + \underline{D(H)}$$

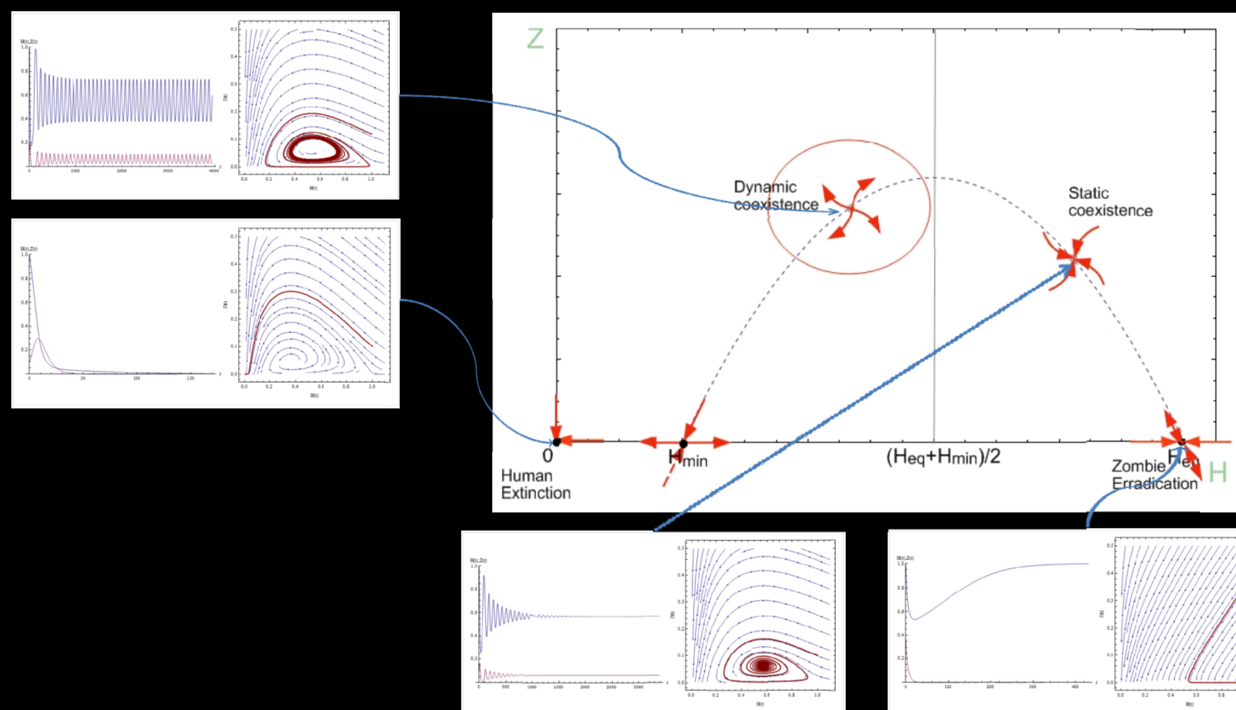
$$\frac{dZ}{dt} = c(1-a)HZ \; - caHZ - \underline{rZ}$$

Human demography

Zombie attrition

If no zombies (Z=0)

$$\frac{dH}{dt} = r(H - H_{min})\left(1 - \frac{H}{H_{eq}}\right)$$

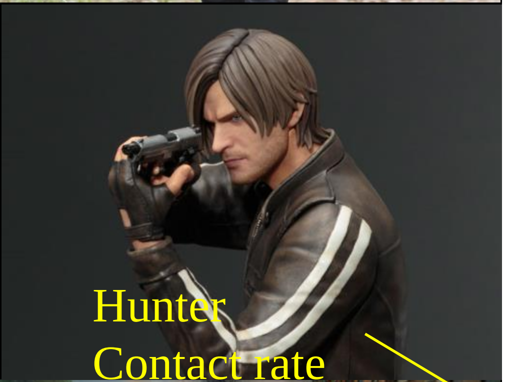


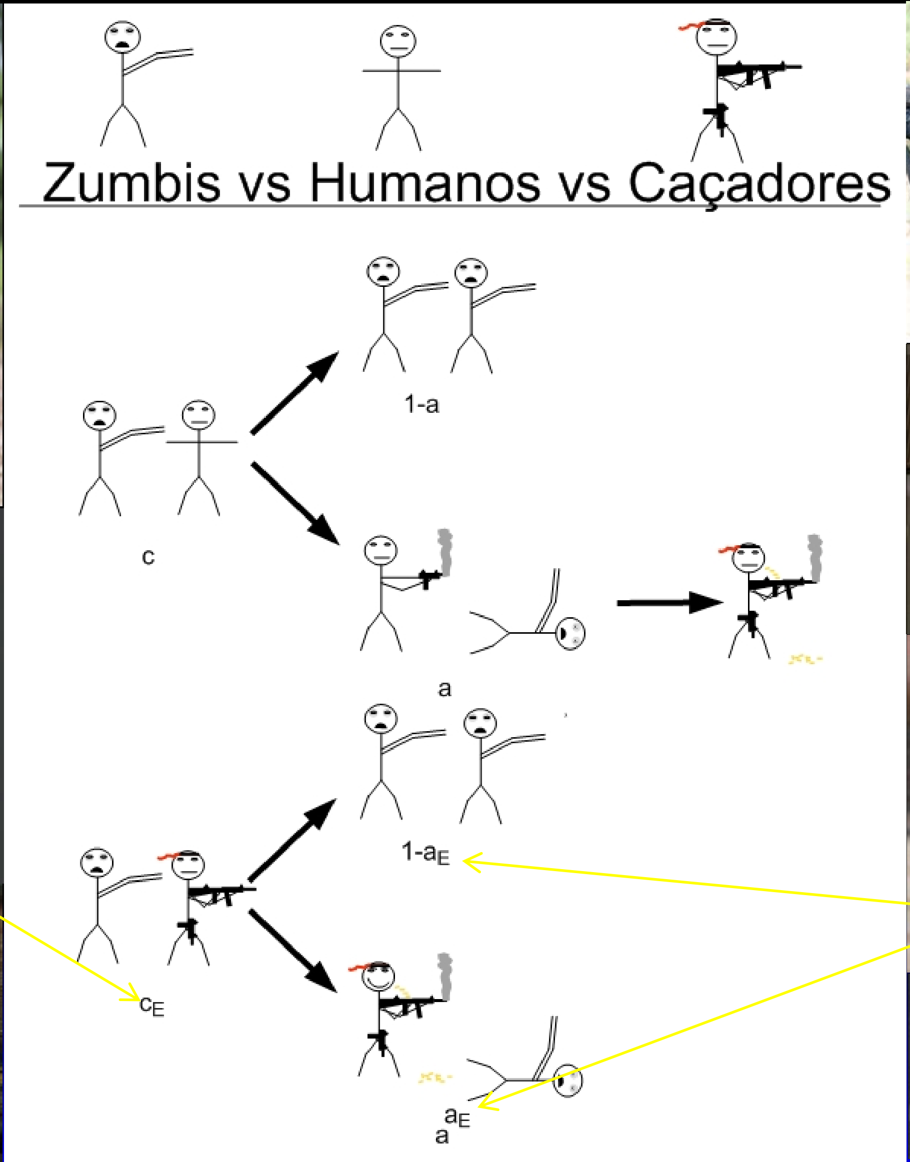
- We assume the infestation starts small ($Z_o << H_o$), amongst a human population is in equilibrium ($H(0) = H_{eq}$)
- Upon encountering a zombie, a human can either
  - a) destroy the zombie with probability a
  - b) be turned into a zombie with probability 1-a
- There are 4 fixed points:
  - a) $Z_E = 0$, $H_E = 0$: Human extinction, always stable
  - b) $Z_U = 0$, $H_U = H_{min}$: Always unstable
  - c) $Z_C = 0$, $H_C = H_{eq}$: Zombie erradication
  - d) $Z_M = D(H_M)/H_M\, c(1-a)$, $H_M = r/c(1-2a)$: Coexistence

-If $a > 1/2 - r/c\, H_{eq}$ the erradication fixed point is stable and zombies are erradicated for benign initial condition

- If $2r/c\,(H_{eq} + H_{min}) < a - 1/2 < r/c\, H_{eq}$ the coexistence point is stable and the erradication point is unstable. Humans and zombies coexist
- If $r/c\, H_{min} < a - 1/2 < 2r/c\,(H_{eq} + H_{min})$ the coexistence point is unstable, but an stable limit cicle forms around it; the erradication point is unstable. Zombies and human coexist in dynamical equilibrium
- If $a < 1/2 - r/c\, H_{min}$ only the extinction point is stable. Humans go extinct.

Zombie vs Humans vs Hunters



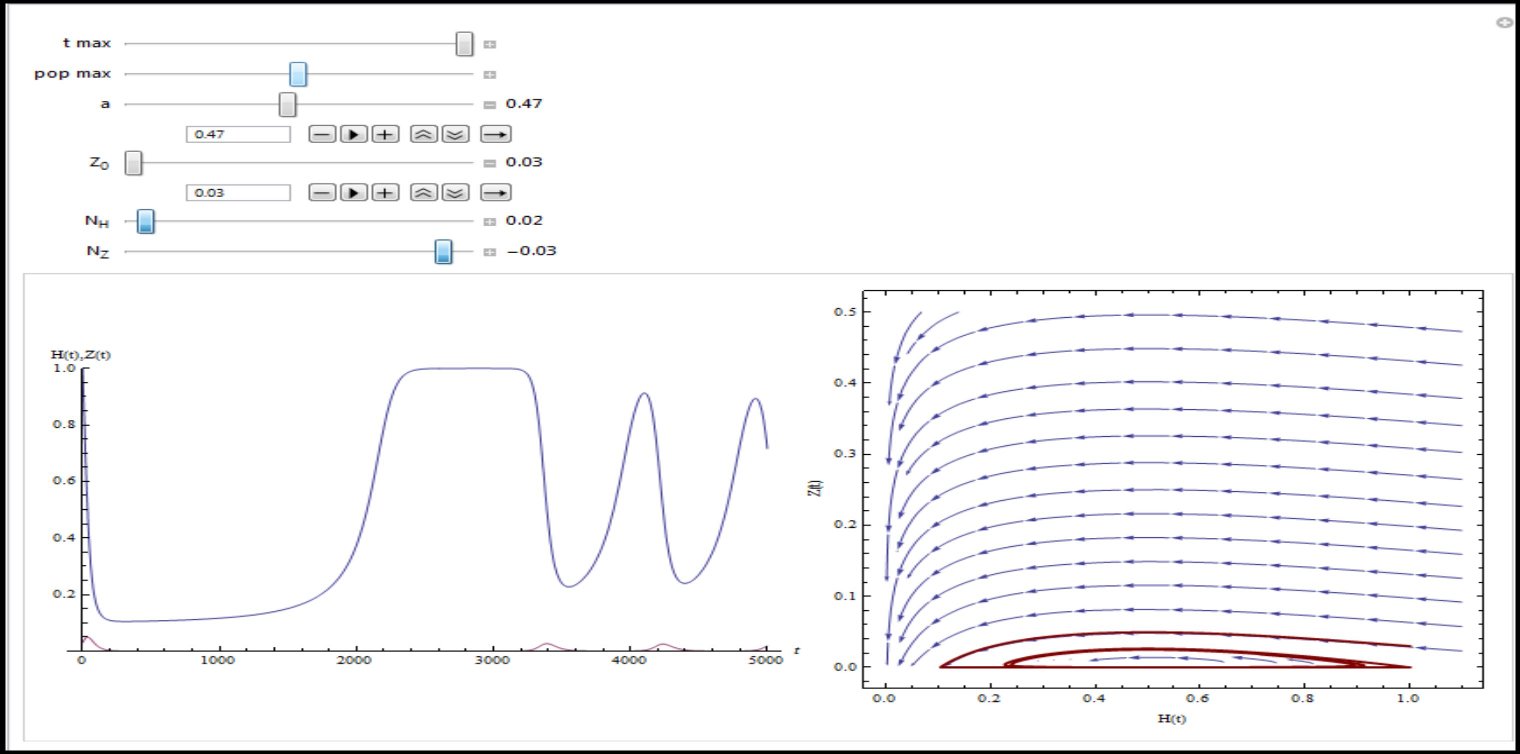


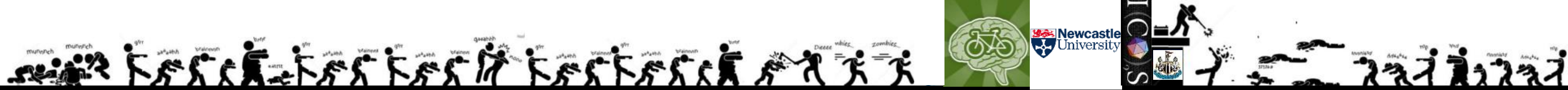
A dinâmica com os zumbis 2

- H(t) é a população de humanos no tempo t
- Z(t) é a população de zumbis no tempo t

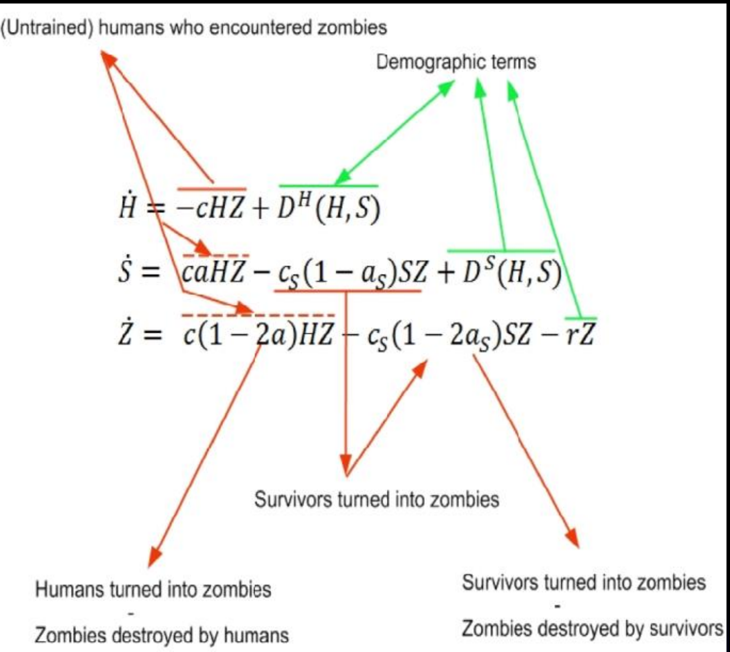
- Um modelo de dinâmica populacional relaciona onde você está (H, Z) com onde você está indo (H’, Z’):

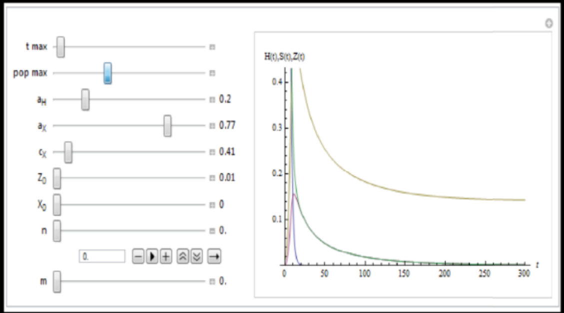
$$dH/dt = -H(H_{min}-H)(H-H_{max}) - c(1-a)\,HZ$$
$$dZ/dt = c(1-a)\,HZ - ca\,HZ - hZ$$

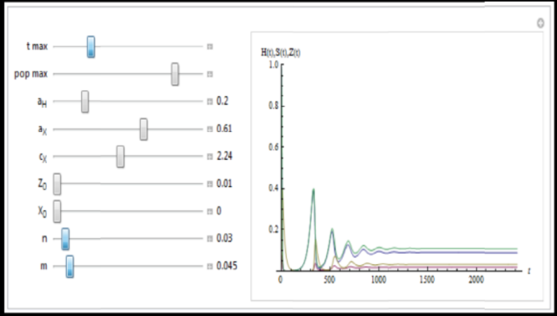


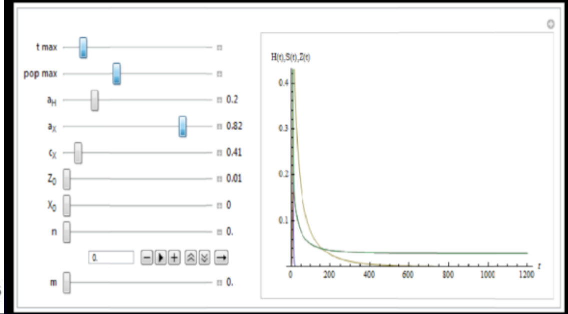


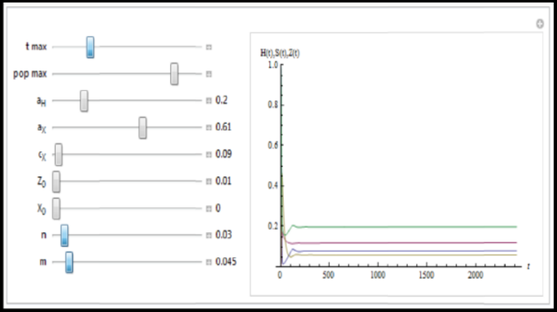
Zombie vs Humans vs Hunters











- We now assume a human who successfully fought a zombie will become more proficient in the future
- These 'survivors' have a higher probability of killing other zombies, $a_S > a$
- Survivors are also more proficient at both avoiding and hunting zombies; i.e., they can adjust the value of c_S
- We show that the criteria for zombie erradication is unchanged, i.e., if $a > 1/2 - r/c\ H_{eq}$ the erradication fixed point is stable and <u>zombies are erradicated</u> for benign initial condition
- There are no new fixed points although we went from 2 to 3 populations

- We show that, if $a + a_S + r/c\ H_{eq} > 1$ populations reach a state of coexistence
- In this case, the stable coexisting total human population increases with c_S, i.e., hunting zumbies by maximizing c_S pays off for survivors
- Conversely, if $a + a_S + r/c\ H_{eq} < 1$ human extinction occurs if k and r are small, and/or if cs is small
- Thus, in the latter case, it pays for survivors to avoid zombies as much as possible by minimizing c_S

Optimal survival strategies

- The various possible outcomes (extinction, coexistence and eradication). Humanity should employ limited resources efficiently to reach the best outcome. <u>Our model can be used to calculate the benefits of changing the value of each parameter, to be balanced against cost.</u>

- When survival probability doesn't increase with the number of encounters humans should avoid zombies rather than hunt them down

- There non-afflicted human population growth rate has no influence on the asymptotic human population level. <u>Learn how to fight, run or both rather than make babies</u>

- Zombie attrition rates are fundamental in defining outcomes. <u>Humans should seek refuge in environments that promote zombie attrition</u> (e.g. extreme cold)

- The addition of more proficient humans can improve human population levels during coexistence, but won't by itself achieve eradication. Only an overall increase of zombie killing skill levels can. <u>Broad competence is more useful than localized brilliance</u>

- Nonetheless, there is a specific criterium that determines whether survivors should fight or hide from zombies. <u>Identifying these condition is critical during an outbreak</u>

[1] This present work was in part inspired by <u>*When Zombies Attack!: Mathematical Modelling of an Outbreak of Zombie Infection*</u>, *Philip Munz, Ioan Hudea, Joe Imad and Robert J. Smith?. In "Infectious Disease Modelling Research Progress," eds. J.M. Tchuenche and C. Chiyaka, Nova Science Publishers, Inc. pp. 133-150, 2009*, but the implementation of these ideas therein is somewhat unrealistic