THEORY OF COMPLEX NETWORKS

Lecture notes for 6CCMCS02/7CCMCS02

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September 2010

Course: MSc in Complex Systems Modelling

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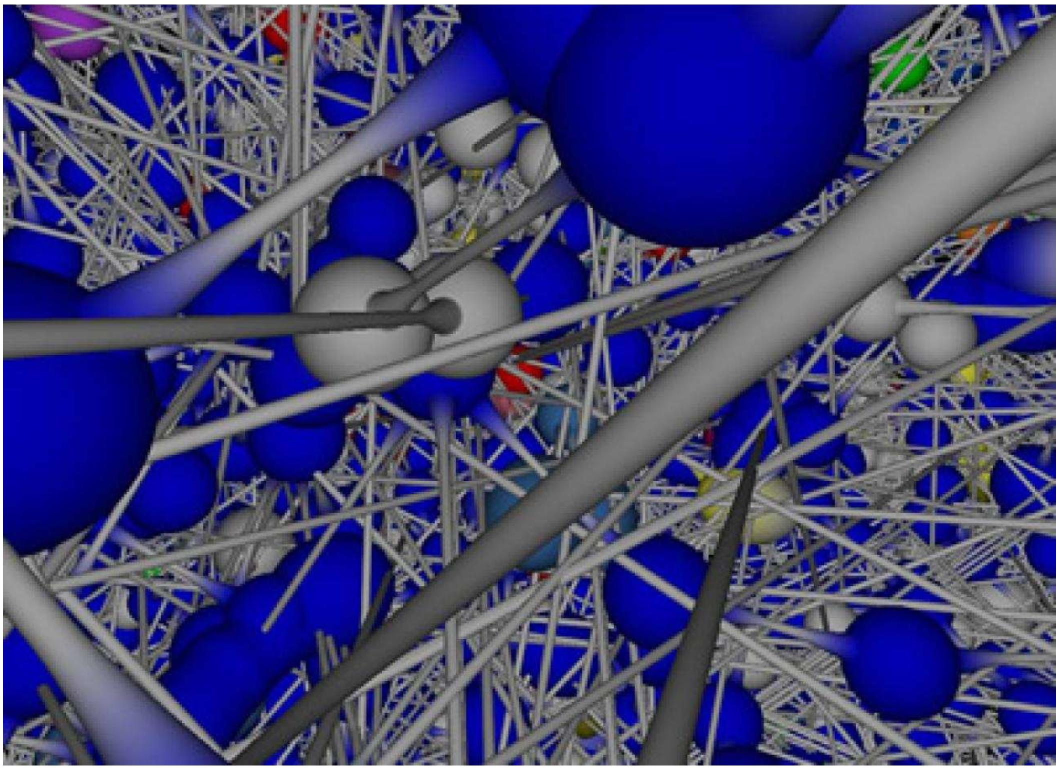

Figure 1: Modelling the internet (www.visualcomplexity.com)

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Chapter 1

Concepts and definitions

In this chapter we introduce the concept of a network and how to characterize this mathematically by means of a single matrix. Basic structures such as the number of connections of a node, called the *node degree*, are constructed from this matrix. Probability structures in terms of the node degrees are then defined by simple counting processes for the matrix entries.

1.1 The connectivity matrix

Consider a set of N points possibly connected by lines. The points are called *nodes* or *vertices* (or *sites*). The lines are called *links*, *edges* or sometimes *interactions*. The collection of points and lines is called a *network*. Here we shall make no distinction between the notion of a *network* and a *graph*.

The N nodes are labeled by the index i (or j) with $i, j = 1..N$. In order to define whether or not a link exist between the nodes i and j, we define a so-called *connectivity matrix* $\boldsymbol{c} = \{c_{ij}\}$ for the network which is an $N \times N$ matrix, also-called *adjacency matrix*. We set $c_{ij} = 1$ if the link between the nodes i and j exists, and $c_{ij} = 0$ if the link between the nodes i and j does not exist. Per definition the connectivity matrix defines the *local* characteristics (or also called *local measures*) of a network. For a so-called *non-directed* network we have $c_{ij} = c_{ji} \; \forall\, i, j$ (syymetric connectivity matrix). For a so-called *directed* network we have that $\exists (i, j)$ such that $c_{ij} \neq c_{ji}$. Our convention here is that in a *directed* network c_{ij} labels the link from node j to node i. So-called *self-interactions* exist when $c_{ii} \neq 0$ for some i. Our focus in the lectures shall be mainly on non-directed graphs with no self-interactions.

For a *directed* network consisting of N nodes we have a *maximum* number of $N(N-1)$ links; i.e. start with one node, make $N-1$ links to all other nodes, then do that for all nodes, counting N times $N-1$ links in total. Clearly, for a *non-directed* network we have then a *maximum* number of $\frac{1}{2}N(N-1)$ links in total. A network that contains this maximum number of links is called a *fully connected network*.

When *only* the number of nodes N is specified, the total possible number of *different* directed networks or *network configurations* (i.e. for which at least one entry c_{ij} differs) is $2^{N(N-1)}$. This number is made up from the 2 possibities (link or no-link) for each of the node combinations, of whom there are $N(N-1)$ in total. To give an idea of how large this number can be we compute e.g. for a network of $N = 1000$ nodes approximately $10^{150,364}$ different networks. To put such number in perspective, there are estimated to be 'only' 10^{82} atoms in the entire universe. Such vast numbers of possible network configurations forces us to drastically refine the characterization of a network structure and bring in probability structures. Such probability structures are mainly formulated in terms of a so-called *node degree* which we shall define next.

1.2 Further local measures of network structure

1.2.1 Node-degree

As mentioned before, introducing probability distributions on a network with connectivity matrix $\boldsymbol{c} = \{c_{ij}\}$ needs further refinement of the mathematical structure. Given the connectivity matrix, one defines a further measure called the *node-degree.* This local measure for a particular node simply counts the number of links attached to the node, or in other words the number of attached other nodes (called *neighboring* nodes). In this simple definition the node-degree quantifies the nodes' ability if you like to connect to other members/nodes of the network. As such it quantifies the nodes' *connectivity.*

To compute this number we may now use the connectivity matrix as follows. For a *non-directed* network we denote this *degree* for node i by k_i and compute,

$$k_i = \sum_j c_{ij} \equiv \sum_{j=1}^{N} c_{ij} \,. \tag{1.1}$$

For a *directed* network we denote the *in-degree* (i.e. the number of in-going links) for node i by k_i^{in} and the *out-degree* (i.e. the number of out-going links) for node i by k_i^{out} and compute,

$$k_i^{in} = \sum_j c_{ij} \,, \tag{1.2}$$

$$k_i^{out} = \sum_j c_{ji} \,. \tag{1.3}$$

Instead of k_i and k_i^{in}, k_i^{out} we could also write $k_i(\boldsymbol{c})$ and $k_i^{in}(\boldsymbol{c}), k_i^{out}(\boldsymbol{c})$ to emphasize that we are *given a particular network configuration* $\boldsymbol{c}$. We shall see later the importance of this. For the moment we shall not explicitely write this extra label ($\boldsymbol{c}$).

One may now also compute the *average node-degrees*, which characterizes the network simply by *one number* for a non-directed network and simply by *two numbers* for a directed network,

$$\overline{k} = \frac{1}{N}\sum_i k_i = \frac{1}{N}\sum_{ij} c_{ij}\,, \tag{1.4}$$

$$\overline{k^{in}} = \frac{1}{N}\sum_i k_i^{in} = \frac{1}{N}\sum_{ij} c_{ij}\,, \tag{1.5}$$

$$\overline{k^{out}} = \frac{1}{N}\sum_i k_i^{out} = \frac{1}{N}\sum_{ij} c_{ji}\,. \tag{1.6}$$

These number are then no more *local measures* but *global measures* of the network. Again, instead of $\overline{k}$ and $\overline{k^{in}}, \overline{k^{out}}$ we could also write $\overline{k}(\boldsymbol{c})$ and $\overline{k^{in}}(\boldsymbol{c}), \overline{k^{out}}(\boldsymbol{c})$ to emphasize that we are *given a particular network configuration* $\boldsymbol{c}$.

We have now refined the mathematical structure of the network by adding the new structure *node degree* using a simple counting process. As such this new structure is indeed an obvious extension of the network structure introduced in section 1.1. As we saw from the above definitions this new local structure is *fully included* in the specification of the connectivity matrix $\boldsymbol{c}$. This means that the collection of all k_i or k_i^{in}, k_i^{out} *follow* from the given connectivity matrix (or network configuration) $\boldsymbol{c}$. But this does *not* mean that we may reverse this, i.e. that the connectivity matrix $\boldsymbol{c}$ is fully determined by specifying the collection of all k_i or k_i^{in}, k_i^{out}. We shall see in chapter 5 that specifying the collection of all k_i for a non-directed network specifies a *whole family* or distribution of possible network configurations denoted as $\{\boldsymbol{c}\}$, *each member* in the family defining the *same* set of node-degrees $\{k_i\}$. Later on we shall denote the set or collection of node-degrees $\{k_i\}$ by a vector of degrees $\boldsymbol{k} = (k_1, k_2, .., k_i, .., k_N)$.

To end this section we mention that in a *real network* – which is specified by its connectivity matrix $\boldsymbol{c}$! – each node is attched to at least one other node ($k_i \geq 1$, $\forall i$). Later on we shall introduce formal simplified *mathematical networks* (using random structures) modelling i.e. approximating the real networks. For such random networks we may formally also allow for a *zero* degree.

PROBLEM 1.1 *Show that for a directed network* $\overline{k^{in}} = \overline{k^{out}}$.

1.2.2 Clustering coefficient

In the concept of node-degree we have a simple counting structure that quantifies each nodes' connectivity, as we saw in the previous section.

Next, we would like to define a suitable local measure that characterizes a further feature, namely the *clustering* 'around' a particular node. With this we mean the extend to which neighboring nodes (i.e. nodes that are connected to the node in question) are *connected themselves*. We like to define this feature as the *fraction* of such *existing* connections relative to the number of *all possible* connections between the neighboring nodes. We speak then of a clustering-*coefficient*.

To count the number of *existing* connections between neighbors of node i we first label these neighbors by k and ℓ (with $k \neq \ell$), then require them to be connected to node i, i.e. *both* c_{ik} and $c_{i\ell}$ have to be equal to 1, and additionally we require $c_{k\ell} = 1$ to assure the connection between node k and ℓ. We count the total of such connections by summing over all possible k and ℓ, i.e. $\sum_{k=1}^{N}$ and $\sum_{\ell=1}^{N}$ or in other notation $\sum_{k\neq\ell}$. This counting process defines a numerator $\sum_{k\neq\ell} c_{ik}c_{i\ell}c_{k\ell}$. Clearly, the denominator counting *all possible* connections (i.e. not only the *existing*) has $c_{k\ell} = 1$ *for all* k, ℓ. So, the clustering-coefficient for node i is,

$$C_i = \frac{\sum_{k\neq\ell} c_{ik}\, c_{i\ell}\, c_{k\ell}}{\sum_{k\neq\ell} c_{ik}\, c_{i\ell} \cdot 1}\,. \tag{1.7}$$

The denominator, counting the number of all possible links between neighbors of node i, is of course also equal to $k_i(k_i - 1)$ since there are k_i neighboring nodes. Thus instead of (2.8) we may also write,

$$C_i = \frac{\sum_{k\neq\ell} c_{ik}\, c_{i\ell}\, c_{k\ell}}{k_i(k_i - 1)}\,. \tag{1.8}$$

<u>*PROBLEM*</u> <u>1.2</u> *Verify carefully for yourself the following.*

(a) The numerator of (2.8) counts the number of directed links between neighbors of node i that actually exist.

(b) The denominator of (2.8) counts the number of all possible *directed links between neighbors of node i. This number equals $k_i(k_i - 1)$.*

(c) The quotient C_i of the numerator and denominator represents then the fraction of 'connected links' between neighbors of node i. Or in other words, the quotient C_i represenst the fraction of neighbors of node i that are themselves mutually connected.

Note that for the case of a non-directed network (symmetric connectivity matrix) we can write the numerator of (2.8) in a matrix notation as follows,

$$\sum_{k\neq\ell} c_{ik}\, c_{i\ell}\, c_{k\ell} = \sum_{k\neq\ell} c_{ik}\, c_{k\ell}\, c_{\ell i} = (\boldsymbol{c}^3)_{ii}\,. \tag{1.9}$$

Such notations will become handy as we will see later in this chapter.

PROBLEM 1.3 *Show that for a non-directed network* $k_i = (\boldsymbol{c}^2)_{ii}$.

To end this section, we note that the *average clustering-coefficient* is defined as,

$$\overline{C} = \frac{1}{N}\sum_i C_i\,. \tag{1.10}$$

1.3 Probabilities and distributions

1.3.1 Degree distribution

Consider a non-directed network of N nodes with connectivity matrix $\boldsymbol{c} = \{c_{ij}\}$ giving rise to node-degrees $\{k_i\}$. We want to compute the probability distribution for the degrees of this network which we shall denote by $P(k)$ (in later chapters we also write $p(k)$ meaning the same. Again, we could denote such distribution by $P(k|\boldsymbol{c})$ since it is computed *given* the network (or network configuration) $\boldsymbol{c}$. We shall use this notation later in chapter 5 and onwards. For the moment we only work with one particular given network $\boldsymbol{c}$ and we will not mention it in the notation.

Clearly, the probability of a randomly selected node to have degree k is computed by the simple counting process,

$$P(k) = \frac{\text{number of nodes having exactly a degree } k}{\text{total number of nodes}}\,. \tag{1.11}$$

This can be written in mathematical form as,

$$P(k) = \frac{1}{N}\sum_{i=1}^{N}\delta_{k,k_i}\,, \tag{1.12}$$

where δ_{k,k_i} represents the *Kronecker-delta* yielding 1 when $k = k_i$ and 0 otherwise. Thus the sum $\sum_i$ 'counts' a contribution 1 when there is a 'hit' $k = k_i$ and a zero otherwise. Again, formally we should write,

$$P(k|\boldsymbol{c}) = \frac{1}{N}\sum_{i=1}^{N} \delta_{k,k_i(\boldsymbol{c})}\,, \tag{1.13}$$

but we shall do that later in chapter 5 and onwards.

<u>*PROBLEM* 1.4</u> *Show that* $\sum_{k\geq 1} P(k) = 1$, *using* $\sum_k \delta_{k,k_i} = 1$.

The degree distribution $P(k)$ is a *global* characterization or measure of the network $\boldsymbol{c}$ since it includes details (namely the degrees) of *all* nodes.

For a directed network with node degrees $\{k_i^{in}\}$ and $\{k_i^{out}\}$ we have for the *joint* distribution of in- and out-degrees,

$$P(k_{in}, k_{out}) = \frac{1}{N}\sum_i \delta_{k_{in},k_i^{in}}\ \delta_{k_{out},k_i^{out}}\,. \tag{1.14}$$

<u>*PROBLEM* 1.5</u> *Convince yourself that the sum* $\sum_i \delta_{k_{in},k_i^{in}}\ \delta_{k_{out},k_i^{out}}$ *represents the number of nodes that have in-degree* k_{in} and *at the same time out-degree* k_{out}.

1.3.2 Degree correlations

We consider again a non-directed network of N nodes with connectivity matrix $\boldsymbol{c} = \{c_{ij}\}$ giving rise to node-degrees $\{k_i\}$. From now on we shall only deal with *non-directed networks* and not always mention this specifically.

Having a prescription (1.12) to compute the probability distribution $P(k)$ of degrees k in the network, a further refinement is to look also at *correlations* between degrees. To describe correlations we introduce two basic functions. Both functions are used in the literature and it is good to be familiar with each of them. In the next section we shall prove a relation between the two. We first give the mathematical definitions and in a following problem discuss the interpretations. Consider two values for the degrees k and k' and define,

$$W(k,k') = \frac{\sum_{i\neq j} c_{ij}\ \delta_{k,k_i}\ \delta_{k',k_j}}{\sum_{i\neq j} c_{ij}}\,, \qquad (1.15)$$

$$\tilde{W}(k,k') = \frac{\sum_{i\neq j} c_{ij}\ \delta_{k,k_i}\ \delta_{k',k_j}}{\sum_{i\neq j} \delta_{k,k_i}\ \delta_{k',k_j}}\,. \qquad (1.16)$$

Note that instead of $\sum_{i\neq j} c_{ij}$ we may equally write $N\overline{k}$.
Again here, instead of $W(k,k')$ and $\tilde{W}(k,k')$ we could write $W(k,k'|\boldsymbol{c})$ and $\tilde{W}(k,k'|\boldsymbol{c})$ to emphasize that we are *given a particular network configuration* $\boldsymbol{c}$. We shall use this full notation in chapter 5 and onwards.
Clearly, from (1.15) and (1.16) we have that the defined functions are symmetric in the degrees, i.e. $W(k,k') = W(k',k)$ and $\tilde{W}(k,k') = \tilde{W}(k',k)$.

PROBLEM 1.6 *Verify carefully for yourself the following.*

(a) The numerator of both (1.15) and (1.16) count the number of links in the network connecting two nodes, where one node has degree k and the other node degree k'.

(b) The denominator of (1.15) counts the number of existing links connecting two vertices.

(c) Then $W(k,k')$ represents the probability that when picking a link at

random, we find degree k on one side of the link and degree k' on the other side of the link.

(d) The denominator of (1.16) counts the number of 'two-node combinations', where one node has degree k and the other node degree k', (irrespective of whether they are connected by a link).

(e) Then $\tilde{W}(k,k')$ represents the probability that two randomly drawn nodes are connected, where one node has degree k and the other node degree k'.

<u>*PROBLEM*</u> <u>1.7</u> *Compute the marginal distributions, i.e. show that,*

$$W(k) \equiv \sum_{k'\geq 1} W(k,k') = P(k)\, k/\overline{k}\,, \tag{1.17}$$

$$W(k') \equiv \sum_{k\geq 1} W(k,k') = P(k')\, k'/\overline{k}. \tag{1.18}$$

Here you need to use that $\sum_k \delta_{k,k_i} = 1$, $\sum_{ij} c_{ij} = \sum_i k_i$ *and* $P(k) = \frac{1}{N}\sum_i \delta_{k,k_i}$.

It is important to see that $P(k)$ automatically 'comes out' when considering $W(k,k')$ summed over k'. This means that at 'the refinement level' of degree-correlations we have automatically included the degree-distribution. More on this later in chapter 5 and onwards.

Note also from (1.17) that $W(k) \neq P(k)$. This is because $W(k)$ is computed from $W(k,k')$ for which the 'chosen link' is to be regarded as *deterministic*, while in the computation of $P(k)$ all links are *probabilistic*.

<u>*PROBLEM*</u> <u>1.8</u> *Show that,*

$$\sum_{k,k'\geq 1} W(k,k') = 1\,, \tag{1.19}$$

$$\sum_{k,k'\geq 1} \tilde{W}(k,k')P(k)P(k') = \overline{k}/N \ (in\ leading\ order\ of\ N) \tag{1.20}$$

In the literature one usually denotes,

$$\text{Prob}[\text{conn.}|k,k'] \equiv \tilde{W}(k,k')\,, \tag{1.21}$$

or alternatively (which we shall do later on) denoting this as Prob[conn.$|k,k',\boldsymbol{c}$], when also mentioning the given network configuration $\boldsymbol{c}$.
We may now also define Prob[conn.], i.e. without specifying the degrees k and k', which is then the probability that two randomly drawn nodes are connected by a link, irrespective of their degrees.

<u>*PROBLEM*</u> <u>1.9</u> *Show that,*

$$\text{Prob}[\text{conn.}] = \frac{nr.\ of\ existing\ links}{nr.\ of\ possible\ links} = \frac{\sum_{i\neq j} c_{ij}}{N(N-1)} \tag{1.22}$$

Next, we want to define fractions rather then probabilities such as $W(k,k')$ and $\tilde{W}(k,k')$. Using the marginal introduced in (1.17) and the notation (1.21) and (1.22) we define,

$$\Pi(k,k') = \frac{W(k,k')}{W(k)\,W(k')}\,, \tag{1.23}$$

$$\tilde{\Pi}(k,k') = \frac{\text{Prob}[\text{conn.}|k,k']}{\text{Prob}[\text{conn.}]}\,, \tag{1.24}$$

or alternatively (which we shall do later on) denoting these as $\Pi(k,k'|\boldsymbol{c})$ and $\tilde{\Pi}(k,k'|\boldsymbol{c})$, when specifying *the given network configuration* $\boldsymbol{c}$.

Note that for networks with *uncorrelated* degrees we necessarily have $W(k,k') = W(k)\,W(k')$ so that from (1.23) follows,

$$\text{Uncorrelated degrees} \quad \Leftrightarrow \quad \Pi(k,k') = 1\,. \tag{1.25}$$

1.3.3 Relation between $\Pi(k,k')$ and $\tilde{\Pi}(k,k')$

Here we prove the following relation between the previously introduced functions $\Pi(k,k')$ and $\tilde{\Pi}(k,k')$,

$$\tilde{\Pi}(k,k') = \Pi(k,k')\,kk'/\overline{k}^2\,, \tag{1.26}$$

for which we need to assume that N is very large which is usually the case. For a *simple network* with *uncorrelated degrees* we see immediately that,

$$\tilde{\Pi}(k,k') = kk'/\overline{k}^2 \quad \text{(for abscent degree-correlations)}\,, \tag{1.27}$$

where we have used (1.25).

Proof of (1.26):

Using the various definitions given above we have,

$$\frac{\tilde{\Pi}(k,k')}{kk'/\overline{k}^2} = \frac{\sum_{i\neq j} c_{ij}\ \delta_{k,k_i}\ \delta_{k',k_j}}{\big(kk'/\overline{k}(N-1)\big)\sum_{i\neq j}\delta_{k,k_i}\ \delta_{k',k_j}}\,. \tag{1.28}$$

The numerator of (1.28) already matches the numerator of $\Pi(k,k')$, since we have from (1.23) and other definitions given before that,

$$\Pi(k,k') = \frac{\sum_{i\neq j} c_{ij}\ \delta_{k,k_i}\ \delta_{k',k_j}}{\big(kk'/\overline{k}^2\big)P(k)P(k')\underbrace{\sum_{i\neq j} c_{ij}}_{=N\overline{k}}}\,. \tag{1.29}$$

All we need to show now is that the denominators of (1.28) and (1.29) match, i.e. that,

$$\frac{kk'}{\overline{k}(N-1)}\sum_{i\neq j}\delta_{k,k_i}\ \delta_{k',k_j}=\frac{kk'}{\overline{k}}\,N\,P(k)P(k')\,. \tag{1.30}$$

To see this we write,

$$\begin{aligned}
\frac{kk'}{\overline{k}(N-1)}\sum_{i\neq j}\delta_{k,k_i}\ \delta_{k',k_j} &= \frac{kk'}{\overline{k}(N-1)}\Big\{\sum_{ij}\delta_{k,k_i}\ \delta_{k',k_j}-\sum_{i}\delta_{k,k_i}\ \delta_{k',k_i}\Big\}\\
&= \frac{kk'}{\overline{k}(N-1)}\Big\{\sum_{ij}\delta_{k,k_i}\ \delta_{k',k_j}-\delta_{k,k'}\sum_{i}\delta_{k',k_i}\Big\}\\
&= \frac{kk'}{\overline{k}(N-1)}\Big\{N\,P(k)P(k')-N\delta_{k,k'}P(k)\Big\}\\
&= \frac{kk'}{\overline{k}}\Big(\frac{N^2}{N-1}\Big)\Big\{P(k)P(k')-\frac{1}{N}\delta_{k,k'}P(k)\Big\}\\
&= \frac{kk'}{\overline{k}}\,N\,P(k)P(k')\quad(\text{for } N\to\infty)
\end{aligned} \tag{1.31}$$

□

1.4 Network assortativity

We define the following *global* measure for the presence of degree-correlations within a given network, namely the overall *correlation-coefficient* for the random variables k and k',

$$a=\frac{\mathrm{Cov}(k,k')}{\sigma_k^2}=\frac{\langle kk'\rangle_W-\langle k\rangle_W\langle k'\rangle_W}{\langle k^2\rangle_{W_{marg.}}-\langle k\rangle^2_{W_{marg.}}}\,, \tag{1.32}$$

where $\langle(..)\rangle_W\equiv\sum_{kk'}W(kk')(..)$ and $\langle..\rangle_{W_{marg.}}\equiv\sum_k W(k)(..)$. This correlation-coefficient is otherwise known as the *network-assortativity.*

PROBLEM 1.10 *Use the definitions and the results previously ob-*

tained to show that the network-assortativity satisfies,

$$a = \frac{\frac{\overline{k}}{N}\sum_{i\neq j} c_{ij} k_i\, k_j \;-\; (\overline{k^2})^2}{\overline{k}\,\overline{k^3} - (\overline{k^2})^2}\,. \tag{1.33}$$

To interprete this assortativity we distinguish between $a > 0$ and $a < 0$ as follows,

Assortative networks (a>0): The degrees on either side of a randomly chosen link are (on average) *positively* correlated, i.e. high-degree nodes prefer to connect to high-degree nodes and also low-degree nodes prefer to connect to low-degree nodes.

Dissortative networks (a<0): The degrees on either side of a randomly chosen link are (on average) *negatively* correlated, i.e. high-degree nodes prefer to connect to low-degree nodes.

To end this section we mention here the general fact from probability theory that in the abscence of degree correlations we have $W(k, k') = W(k)\, W(k') \Rightarrow a = 0$, but this cannot be reversed in general.

1.5 Higher order degrees, paths and loops

We consider non-directed networks with connectivity matrix $\boldsymbol{c} = \{c_{ij}\}$ having no self-interactions.

At the degree-level of network structure we have thus far only looked at the node-degree k_i, i.e. the number of neighboring nodes of vertex i also called the number of *nearest-neighbors*. Extending this one may also consider for instance the notion of 'second neighboring nodes' usually called *second-nearest-neighbors*. The set of second-nearest-neighbors is the collection of all nodes each of which is attached to at least one of the nearest-neighbors of the original node i. And of course this can be extended to the notion of *third-nearest-neighbors* and in general th*-nearest-neighbors*. We call the *number* of n^{th}-nearest-neighbors of node i the n^{th} *order degree* which we denote by $k_i^{(n)}$. We shall come back to these concepts in more detail in chapter 7 when introducing the so-called theory of generating functions.

The number of n^{th}-nearest-neighbors of node i can be expressed mathematically as,

$$\begin{aligned} k_i^{(1)} &= \sum_{r_1} c_{ir_1} \equiv k_i \\ k_i^{(2)} &= \sum_{r_1 r_2} c_{ir_1}\, c_{r_1 r_2} \\ k_i^{(3)} &= \sum_{r_1 r_2 r_3} c_{ir_1}\, c_{r_1 r_2}\, c_{r_2 r_3} \\ &.. \\ k_i^{(n)} &= \sum_{r_1 r_2 r_3 .. r_n} c_{ir_1}\, c_{r_1 r_2}\, c_{r_2 r_3} \ldots c_{r_{n-1} r_n}\,. \end{aligned} \tag{1.34}$$

<u>*PROBLEM*</u> <u>1.11</u> *Show that in matrix language we have* $k_i^{(n)} = \sum_j (\boldsymbol{c}^n)_{ij}$

Next, we look at loops. Modify the right hand side of (1.34) by deleting the sum over the last index r_n and instead setting $r_n = i$. We denote the result by $m_i^{(n)}$,

$$m_i^{(n)} = \sum_{r_1 r_2 r_3 .. r_{n-1}} c_{ir_1}\, c_{r_1 r_2}\, c_{r_2 r_3} \ldots c_{r_{n-1} i} = (\boldsymbol{c}^n)_{ii}\,. \tag{1.35}$$

This quantity counts the number of closed paths or *loops* of length n going out from node i. With the *length of a path* we mean here the number of links that are counted over this path.

PROBLEM 1.12 *Draw part of a network (around node i) for which $m_i^{(3)} = 2$, i.e. the node should have 2 loops of length 3.*

Now we modify (1.35) by changing the second index i into some arbitrary index j,

$$(\boldsymbol{c}^n)_{ij} = \sum_{r_1 r_2 r_3 .. r_{n-1}} c_{ir_1}\, c_{r_1 r_2}\, c_{r_2 r_3} \dots c_{r_{n-1} j} \, . \tag{1.36}$$

PROBLEM 1.13 *Convince yourself that (1.36) defines the number of existing paths of length n between node i and node j.*

1.6 Distance, average pathlength and diameter

In the previous section we have seen that $(\boldsymbol{c}^\ell)_{ij}$ defines the number of paths between node i and node j that have length ℓ. Clearly, if $(\boldsymbol{c}^\ell)_{ij} \geq 1$ then a path between the nodes i and j *exists*. Let us now look at the following construction,

$$\theta\big[(\boldsymbol{c}^\ell)_{ij} - \frac{1}{2}\big], \tag{1.37}$$

where $\theta[x]$ denotes the step function, i.e. $\theta[x < 0] = 0$ and $\theta[x \geq 0] = 1$. What does this function stand for? Well,

- if $(\boldsymbol{c}^\ell)_{ij} \geq 1$, a path between i and j exists, and $\theta\big[(\boldsymbol{c}^\ell)_{ij} - \frac{1}{2}\big] = 1$,

• if $(\boldsymbol{c}^\ell)_{ij} = 0$, *no* path between i and j exists, and $\theta\big[(\boldsymbol{c}^\ell)_{ij} - \frac{1}{2}\big] = 0$.

Note that for the number $\frac{1}{2}$ in (1.37) we could have equally well chosen any number $\in (0, 1]$.

Now we multiply (1.37) by ℓ resulting in,

• a path between i and j exists $\Rightarrow \ell \cdot \theta\big[(\boldsymbol{c}^\ell)_{ij} - \frac{1}{2}\big] = \ell$,

• *no* path between i and j exists $\Rightarrow \ell \cdot \theta\big[(\boldsymbol{c}^\ell)_{ij} - \frac{1}{2}\big] = 0$,

We now define the *distance* d_{ij} (also called *shortest distance*) as the length of the *shortest path* between the nodes i and j. If such a path exists then clearly we have,

$$d_{ij} = \min_{\ell \le N} \; \ell \cdot \theta\big[(\boldsymbol{c}^\ell)_{ij} - \frac{1}{2}\big]\,. \tag{1.38}$$

The *average pathlength* $\overline{\ell}$ is then defined as,

$$\overline{\ell} = \frac{\text{sum over all possible distances}}{\text{total nr. of possible links}} = \frac{\sum_{i \neq j} d_{ij}}{N(N-1)} \tag{1.39}$$

We see now why we added the construction $\theta\big[(\boldsymbol{c}^\ell)_{ij} - \frac{1}{2}\big]$ in (1.38). It ensures that we have a well-defined value for the sum $\sum_{i \neq j} d_{ij}$.

For a fully connected network where $c_{ij} = 1\,\forall (i,j)$, we have of course $\overline{\ell} = 1$.

We also define the so-called *diameter* d of a network as the maximum distance that exists in the network,

$$d = \max_{(i,j)} \; d_{ij}\,. \tag{1.40}$$

<u>*PROBLEM*</u> <u>1.14</u> *The average pathlength $\overline{\ell}$ is an important global characteristic of a network. Convince yourself that you have understood all details that lead to (1.39).*

Chapter 2

Example networks

In this chapter we briefly look at some basic but important *random* network structures. Furthermore, to practice working with the connectivity matrix $\boldsymbol{c} = \{c_{ij}\}$ introduced in chapter 1 we shall compute the clustering-coefficient of a simple non-random network.

2.1 The 'classical random network' (Erdös-Rényi model)

We introduce here the important concept of *random networks*, for which the entries of the connectivity matrix $\boldsymbol{c} = \{c_{ij}\}$ are *random variables* and consequently also the network degrees $\{k_i\}$.

We start with a type of network called the *classical random network* or better known as the *Erdös-Rényi network* or the *Erdös-Rényi model* introduced in 1959 [1]. This is a non-directed network of N nodes (no self-inteactions) with a uniform probability p of any link to exist, i.e. for all nodes i and j we have that $c_{ij} = 1$ with probability p and consequently that

$c_{ij} = 0$ with probability $1 - p$. Although indeed a simple network, it has many applications.

Degree-distribution of the Erdös-Rényi network:

<u>*PROBLEM 2.1*</u> *Show that the degree distribution of the Erdös-Rényi (ER) network is a binomial distribution,*

$$P_{ER}(k) = p^k \, (1-p)^{N-1-k} \binom{N-1}{k}, \tag{2.1}$$

with average degree $\overline{k} = (N-1)p \approx Np$*, so that* $p = \frac{\overline{k}}{N}$.

For large N such the binomial distribution is approximated by a *Poisson-distribution*,

$$P_{ER}(k) = \frac{\overline{k}^k}{k!} \, \mathrm{e}^{\overline{k}} \quad (N \to \infty, \; p \to 0, \text{ such that } \overline{k} = Np = \text{finite}) \tag{2.2}$$

<u>*PROBLEM 2.2*</u> *For an Erdös-Rényi network the entries of the connectivity matrix* $\{c_{ij}\}$ *are random numbers by definition. Convince yourself of the folowing facts.*
(1) The probability disribution for the entry c_{ij}*, which we denote here as* $p_{ER}(c_{ij})$*, has the mathematical form,*

$$p_{ER}(c_{ij}) = \frac{\overline{k}}{N} \delta_{c_{ij},1} \; + \; \left(1 \; - \; \frac{\overline{k}}{N}\right) \delta_{c_{ij},0} \, . \tag{2.3}$$

(2) The probability distribution of the entire network configuration $\mathbf{c} = \{c_{ij}\}$ *is then expressed as,*

$$p_{ER}(\mathbf{c}) = \prod_{i<j} \left\{ \frac{\overline{k}}{N} \delta_{c_{ij},1} \; + \; \left(1 \; - \; \frac{\overline{k}}{N}\right) \delta_{c_{ij},0} \right\} . \tag{2.4}$$

(Although simple in form and easy to understand, this is an important expression that you should be familiar with. It will appear on numerous occasions in the lectures).

Clustering-coefficient of the Erdös-Rényi network:

The clustering coefficient for node i is of course a random variable since the entries of the connectivity matrix $\{c_{ij}\}$ are random variables. So wish to compute the *average* clustering coefficient.

PROBLEM 2.3 *Show that the average clustering coefficient of the Erdös-Rényi network equals* p*, i.e. the uniform probability of a link to exist.*

Degree-correlations of the Erdös-Rényi network:

Since each link in the network exists with equal probability p it is clear that there are no degree-correlations in such network, i.e. the Erdös-Rényi network is an uncorrelated network.

PROBLEM 2.4 *Although it is clear that the Erdös-Rényi network is an uncorrelated network, analyse numerator and denominator in the definition (1.15) of* $W(k,k')$ *and conclude that* $W(k,k') = W(k)\,W(k')$ *resulting in* $\Pi(k'k') = 1$.

PROBLEM 2.5 *Analyse also numerator and denominator in the definition (1.16) of* $\tilde{W}(k,k')$ *and conclude that* $\tilde{W}(k,k') = k\,k'/\overline{k}^2$.

Average pathlength of the Erdös-Rényi network:

Here we shall discuss the important fact that the average pathlength of the Erdös-Rényi network scales as $\log N$. This is a striking property of many types of networks, which is also known as the so-called *small-world-property* of networks. To show that $\overline{\ell} \sim \log N$ we start with selecting one node at random. On average, the number of nodes at distance 1 of this node is $\overline{k}$. The average number of nodes *up till* distance 2 of the node is approximately[1] $\overline{k} + \overline{k}^2$, since for each nearest neighbor we have $\overline{k}$ second nearest neighbors. Then the average number of nodes *up till* distance ℓ is approximately equal to $\overline{k} + \overline{k}^2 + .. + \overline{k}^{\ell}$. Taking for ℓ the average pathlength of the network $\overline{\ell}$ and setting this sum equal to N gives us implicitely an approximate for $\overline{\ell}$.

<u>*PROBLEM*</u> <u>2.6</u> *Use this understanding and the expression for the geometric series to approximate* $\overline{k}^{\overline{\ell}} \approx N$ *and show that* $\overline{\ell} \sim \log N / \log \overline{k}$.

We end this section with the following concluding remarks. The *average pathlegth* of the Erdös-Rényi network scales as $\log N$. This *small-world-property* is shared by many *real-world* networks and as such the Erdös-Rényi model is a useful approximation. On the other hand, the average clustering-coefficient for a real-world network is usually much larger than that of the Erdös-Rényi network, which prompt us to consider also other models with more realistic values for the average clustering-coefficent. Networks that dispays *both* the small-world-property and a large value for the average clustering-coefficient are usually called *small-world-networks*.

[1] Here we neglect the possible occurrence of loops.

2.2 Small-world networks (Watts-Strogatz model)

Consider a set of N nodes and fix their positions on a closed ring. Such network is called a one-dimensional ring lattice, an example of which is shown in figure 2.1. Here each node is simply linked to 2 neighboring nodes ($k_i = 2$, $\forall i$). For this network the average pathlength is $\mathcal{O}(N)$. Now consider

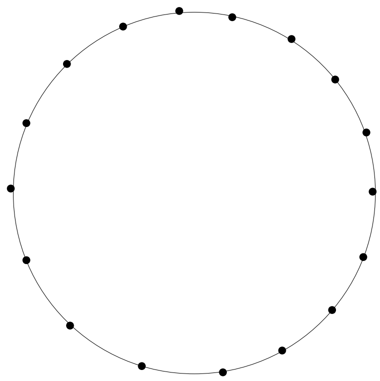

Figure 2.1: A one-dimensional ring lattice $N = 17$ with $k_i = 2$.

the clustering-coefficient for node i. The nearest neighbors on each side of this node are not linked so that $C_i = 0$. Clearly, this simple network is *not* a small-world-network.

Next, we modify this network by linking each node, not to 2 but to K nearest nodes on the ring (K is even). For example see figure 2.2, where each node is linked to 4 neighboring nodes, 2 on each side ($k_i = 4$, $\forall i$). The clustering-coefficient is now $C_i = \frac{3}{4(4-1)/2} = \frac{1}{2}$, since there are 3 existing links and 4 connected neighbors of node i who could be mutually connected by a maximum of $4(4-1)/2 = 6$ links. The value of the clustering-coefficient of this model network is large and this is good, since real-world networks generally do have such large clustering-coefficients. Although a little more work to compute, the average pathlength is still $\mathcal{O}(N)$, so also this model

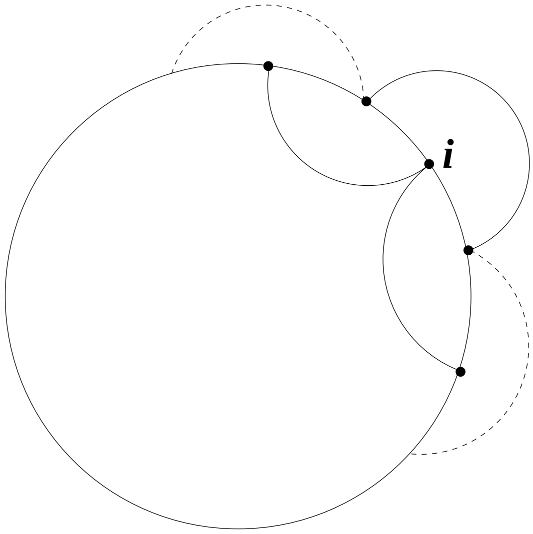

Figure 2.2: Part of a one-dimensional ring lattice with $k_i = 4$.

network is *not* a small-world-network.

Thusfar we have not considered any probabilistic structure for the model networks described above. We shall see that by adding appropriate randomness – according to the so-called *Watts-Strogatz model* – to these ring networks, we can indeed engineer *both* a large value of the average clustering-coefficient *and* a small average pathlength, thus resulting in a model small-world-network.

The Watts-Strogatz model:

(1) Consider a one-dimensional ring lattice with N nodes, and link each node to its first K neighbors (e.g. like in figure 2.2).
(2) Next, we randomize this structure according the following simple recipe. Randomly *rewire each link* with probability p, excluding the possibility of

self-connections and duplicates.

Clearly, when the value of p increases we expect the average pathlength to decrease, since on average there will be increasingly more links between 'far apart' vertices. Let us consider a large value of K and look at the following limit cases:

p=0: The average pathlength $\overline{\ell} = \mathcal{O}(N)$ as we saw above. The average clustering-coefficient $\overline{C} = 3/4$, (we shall prove this section 2.5).

p=1: The average pathlength $\overline{\ell} = \mathcal{O}(\log N)$, (same proof as for the Erdös-Rényi network, see problem 2.6). The average clustering-coefficient $\overline{C} \approx K/N$.

PROBLEM 2.7 *Show that for the Watts-Strogatz model network in the limit case of unit rewiring probability ($p = 1$) the average clustering-coefficient approximately equals K/N.*

We see here that both limiting cases do *not* produce the desired small-world-network, but that small values of p produce the wanted large clustering-coefficient and unfortunately a large average pathlength, and that large values of p produce the wanted small average pathlength and unfortunately a small average clustering-coefficient. The big question is if there is a range within $0 < p < 1$ such that both wanted features emerge, namely a small average pathlength and a large average clustering-coefficient. The answer to this question is yes! Simulations of Watts-Strogatz model networks leads to a graph of the kind as shown in figure 2.3. Indeed, for this particular graph we find for instance in the region $0.005 \lesssim p \lesssim 0.05$ suitably *small* ranges for the value of the average pathlength and suitably *large* values for the average clustering-coefficient.

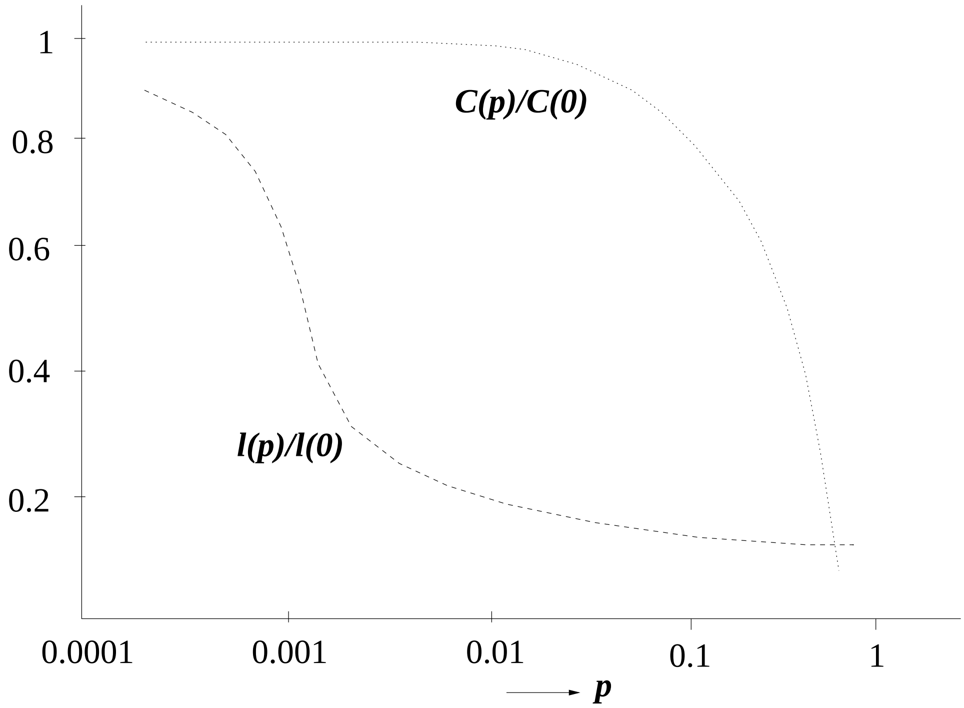

Figure 2.3: Average pathlengths $\ell(p)$ and average clustering-coefficients $C(p)$ plotted for various values of the rewiring probability p (log-scale). For more details see [2].

2.3 Scale-free networks (Barabási-Albert model)

Here we only make some introductory remarks for this type of network. Later in chapter 8 we shall treat such networks in more detail. For now we just mention the most important charcteristic of scale-free networks, the *power-law* degree-distribution,

$$P(k) \sim k^{-\gamma}\,. \tag{2.5}$$

Typical values of the parameter γ are 2–4. In chapter 8 we show how to contruct networks with this property.

This is an important type of network; many real-world networks indeed show this kind of degree-distribution. The recipe to construct random networks with this property is to let networks *evolve* (grow) according to the principle of *preferential attachment*. The idea is that a newly added vertex would prefer to connect to nodes with a high degree than to node with a low degree, a simple principle that is often seen in real-world networks. Mathematically we prescribe that a newly introduced node attaches to say an existing node i with a probability that increases with the degree k_i. For the case of *linear* increase, i.e. the probability of connecting to node i is proportional to k_i, one speaks of the *Barabási-Albert model*.

It should be noted that the power-law (2.5) is often only valid for a limited range of the degrees k. For instance, at large k one may see an exponential decay becoming dominant over the power-law decay. More on this in chapter 8.

2.4 Caylee trees and regular lattices

Here we only present some very brief remarks on these type of networks. Firstly, the *Caylee-tree* is a network of N nodes with no loops. Each node is attached to $z \in \mathbb{N}$ neighboring nodes, except the nodes at the 'edge or end' of the network which are attached to only one node. An example of a Caylee-tree is drawn in figure 2.4. In the case that $N \to \infty$ (no 'dead ends')

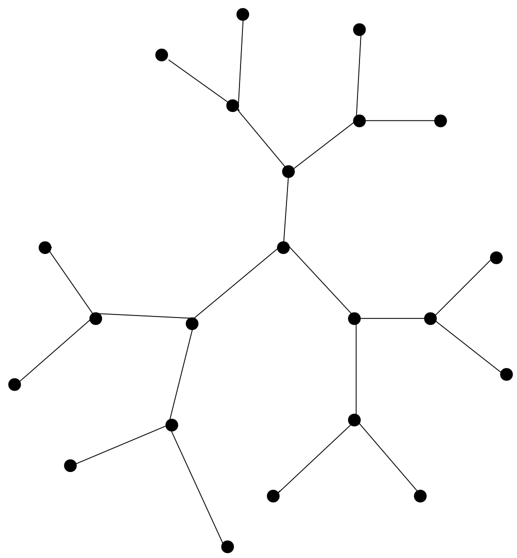

Figure 2.4: Caylee tree with $z = 3$ and $N = 22$.

one speaks of a so-called *Bethe-lattice*. As a random network we define that each link is present with uniform probability p.

<u>*PROBLEM* 2.8</u> *Show that for* $N \to \infty$, $P(k) = Bin(p, z)$ *and* $C_i = 0$.

We also briefly mention the so-called *regular lattice* in d dimensions (see figure 2.5).

<u>*PROBLEM* 2.9</u> *Show that the average pathlength* $\overline{\ell}$ *of a regular lattice in* d *dimensions scales as* $N^{1/d}$.

2.5 Example clustering coefficient of a ring lattice

Consider a one-dimensional ring-lattice with N nodes where each node is linked to K nearby nodes ($\frac{1}{2}K$ on each side of the node, i.e. $k_i = K$), such

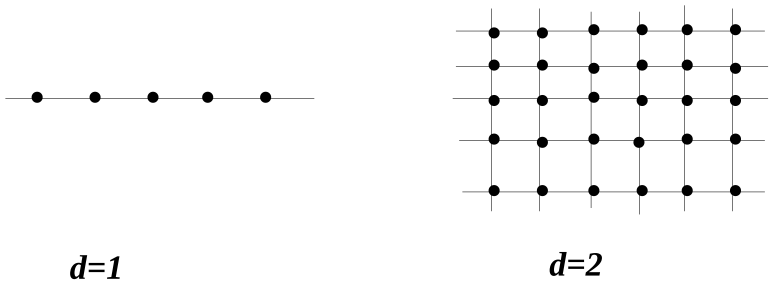

Figure 2.5: Regular lattices in 1 and 2 dimensions.

as for example drawn in figure 2.6. For the case that $K = 4$ we compute

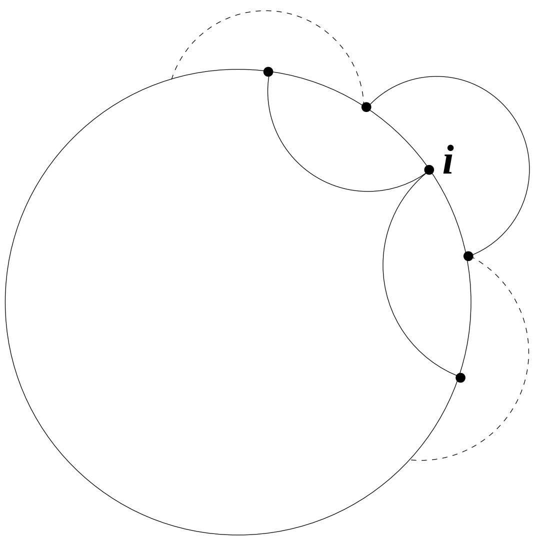

Figure 2.6: Part of a one-dimensional ring lattice with $K = 4$.

the clustering coefficient for each node i as

$$C_i = \frac{3}{4(4-1)/2} = \frac{1}{2}\,, \tag{2.6}$$

since there are 3 existing links and 4 connected neighbors of node i who could be mutually connected by a maximum of $4(4-1)/2 = 6$ links.

In this section we compute as a matter of exercise the clustering-coefficient for the general case of node degree K (with $K \leq N$) with result,

$$C = C_i = \frac{3(K-2)}{4(K-1)}\,. \tag{2.7}$$

The starting point is the defining relation (1.8) for the clustering-coefficient C_i of the network nodes,

$$C_i = \frac{\sum_{k\ell} c_{ik}\, c_{i\ell}\, c_{k\ell}}{k_i(k_i-1)}\,. \tag{2.8}$$

The first task is to write down an expression for the connectivities $\{c_{ij}\}$. Let us first give the expression and then justify it,

$$c_{ij} = \sum_{\sigma=\pm1} \sum_{\ell=1}^{\frac{1}{2}K} \delta_{\,i,\,j+\sigma\,\ell}\;. \tag{2.9}$$

Consider the situation drawn in figure 2.7, where the node j is 'far away' form the node i, meaning further away from i than the node labeled as $i + 1/2K$. Then clearly, no matter what value ℓ and σ take in its range of the sum in (2.9) we always have $i \neq j + \sigma\ell$ so that $c_{ij=0}$. Of course this is what we want since there is *no* link between the nodes i and j.

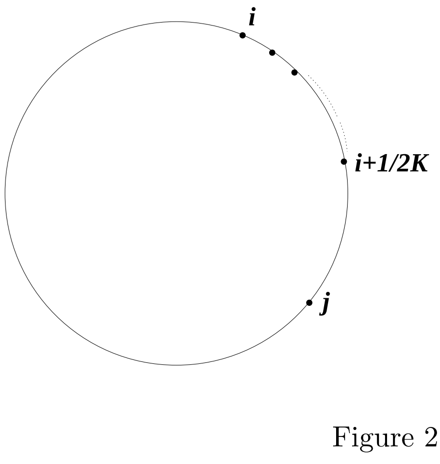

Figure 2.7: $j > i + \frac{1}{2}K$.

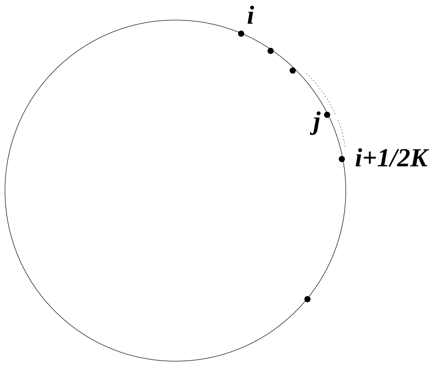

Figure 2.8: $j < i + \frac{1}{2}K$.

Next, consider a different situation drawn in figure 2.8, where the node j is closer by i than the node labeled as $i + 1/2K$. Now there is *one* particular value of ℓ and σ such that $i = j + \sigma\ell$, resulting of course in $c_{ij} = 1$. But this is just what we wanted since there is indeed a link between the nodes i and such j.

When changing the positions of the node j like for the examples as depicted here, we easily see that we always have $c_{ij} = 1$ when there is an existing link between the nodes i and j and $c_{ij} = 0$ otherwise, which justifies (2.9).

Next is to substitute (2.9) in the expression for the clustering-coefficient (2.8). This gives,

$$C_i = \frac{1}{K(K-1)} \sum_{jk} \sum_{\sigma_1\sigma_2\sigma_3=\pm 1} \sum_{\ell_1\ell_2\ell_3=1}^{1/2K} \delta_{i,\,j+\sigma_1\ell_1}\ \delta_{i,\,k+\sigma_2\ell_2}\ \delta_{j,\,k+\sigma_3\ell_3}\ , \tag{2.10}$$

where we have used that $k_i = K$.

PROBLEM 2.10 *Convince yourself of the following,*

$$\sum_{jk} \delta_{i,j+\sigma_1\ell_1}\,\delta_{i,k+\sigma_2\ell_2}\,\delta_{j,k+\sigma_3\ell_3} = \delta_{i-\sigma_1\ell_1,i-\sigma_2\ell_2+\sigma_3\ell_3} = \delta_{\sigma_3\ell_3,\sigma_2\ell_2-\sigma_1\ell_1}\,. \tag{2.11}$$

Note that when using this C_i in (2.10) has become independent of the index i, which is no surprise.

We may exchange the sums in (2.10),

$$\sum_{jk}\ \sum_{\sigma_1\sigma_2\sigma_3=\pm1}\ \sum_{\ell_1\ell_2\ell_3=1}^{1/2K} .. = \sum_{\sigma_1\sigma_2\sigma_3=\pm1}\ \sum_{\ell_1\ell_2\ell_3=1}^{1/2K}\ \sum_{jk} ..$$

Doing this and substituting (2.11) into (2.10) gives,

$$C = C_i = \frac{1}{K(K-1)} \sum_{\sigma_1\sigma_2\sigma_3=\pm1}\ \sum_{\ell_1\ell_2\ell_3=1}^{1/2K} \delta_{\sigma_3\ell_3,\sigma_2\ell_2-\sigma_1\ell_1}\,. \tag{2.12}$$

Next, we perform the sum $\sum_{\sigma_3}$ in (2.12). We can write this sum out explicitely as,

$$\begin{aligned}\sum_{\sigma_3=\pm1} \delta_{\sigma_3\ell_3,\sigma_2\ell_2-\sigma_1\ell_1} &= \delta_{\ell_3,\sigma_2\ell_2-\sigma_1\ell_1} + \delta_{-\ell_3,\sigma_2\ell_2-\sigma_1\ell_1} \\ &= \delta_{\ell_3,\sigma_2\ell_2-\sigma_1\ell_1} + \delta_{\ell_3,-\sigma_2\ell_2+\sigma_1\ell_1} \end{aligned} \tag{2.13}$$

But since σ_1 and σ_2 in (2.13) both take on the values ±1, clearly both terms in (2.13) contribute the same amount to the sum in (2.12). So we perform the sum over σ_3 and (2.12) becomes then,

$$C = \frac{2}{K(K-1)} \sum_{\sigma_1\sigma_2=\pm1}\ \sum_{\ell_1\ell_2\ell_3=1}^{1/2K} \delta_{\ell_3,\sigma_2\ell_2-\sigma_1\ell_1}\,. \tag{2.14}$$

Now we explicitely write out the sums over σ_1 and σ_2 in (2.14) giving,

$$C = \frac{2}{K(K-1)} \sum_{\ell_1\ell_2\ell_3=1}^{1/2K} \Big\{ \delta_{\ell_3,\ell_1+\ell_2} + \delta_{\ell_3,\ell_1-\ell_2} + \delta_{\ell_3,\ell_2-\ell_1} + \delta_{\ell_3,-\ell_1-\ell_2} \Big\}\,. \tag{2.15}$$

PROBLEM 2.11 *Show that*

$$\delta_{\ell_3,\,-\ell_1-\ell_2} = 0,\ \forall(\ell_1,\ell_2,\ell_3)\,. \tag{2.16}$$

PROBLEM 2.12 *Show that,*

$$\delta_{\ell_3,\,\ell_1-\ell_2} + \delta_{\ell_3,\,\ell_2-\ell_1} = 2\,\delta_{\ell_3,\,\ell_2-\ell_1}\,. \tag{2.17}$$

Next, we focus on the sums $\sum_{\ell_2\ell_3}$ over the *first* term in (2.15). We separate this sum as follows (carefully check this for yourself!),

$$\begin{aligned}
\sum_{\ell_2\ell_3=1}^{1/2K} \delta_{\ell_3,\,\ell_1+\ell_2} &= \sum_{\ell_2=1}^{1/2K-\ell_1} \underbrace{\sum_{\ell_3=1}^{1/2K} \delta_{\ell_3,\,\ell_1+\ell_2}}_{=1} + \sum_{\ell_2=1/2K-\ell_1+1}^{1/2K} \underbrace{\sum_{\ell_3=1}^{1/2K} \delta_{\ell_3,\,\ell_1+\ell_2}}_{=0} \\
&= \sum_{\ell_2=1}^{1/2K-\ell_1} 1\,.
\end{aligned} \tag{2.18}$$

Next, we focus on the sums $\sum_{\ell_2\ell_3}$ over the *second* and *third* term in (2.15) (which give equal contributions as we saw in problem 2.12). We separate this sum as follows (carefully check this for yourself!),

$$\begin{aligned}
2\sum_{\ell_2\ell_3=1}^{1/2K} \delta_{\ell_3,\,\ell_1-\ell_2} &= 2\sum_{\ell_2=1}^{\ell_1} \underbrace{\sum_{\ell_3=1}^{1/2K} \delta_{\ell_3,\,\ell_1-\ell_2}}_{=0} + 2\sum_{\ell_2=\ell_1+1}^{1/2K} \underbrace{\sum_{\ell_3=1}^{1/2K} \delta_{\ell_3,\,\ell_1-\ell_2}}_{=1} \\
&= 2\sum_{\ell_2=\ell_1+1}^{1/2K} 1\,.
\end{aligned} \tag{2.19}$$

Then using (2.18) and (2.19) we get for (2.15),

$$\begin{aligned} C &= \frac{2}{K(K-1)} \sum_{\ell_1=1}^{1/2K} \Big\{ \sum_{l_2=1}^{1/2K-\ell_1} 1 \ + \ 2 \sum_{\ell_2=\ell_1+1}^{1/2K} 1 \Big\} \\ &= \frac{2}{K(K-1)} \sum_{\ell_1=1}^{1/2K-1} \Big\{ \frac{1}{2}K - \ell_1 \ + \ 2\big(\frac{1}{2}K - \ell_1\big) \Big\} \\ &= \frac{6}{K(K-1)} \sum_{\ell_1=1}^{1/2K-1} \Big(\frac{1}{2}K - \ell_1 \Big) . \end{aligned} \tag{2.20}$$

In the second line we have used the fact that for $\ell_1 = \frac{1}{2}K$ the sum vanishes.

<u>*PROBLEM*</u> <u>2.13</u> *Show that*

$$\sum_{\ell_1=1}^{1/2K-1} \Big(\frac{1}{2}K - \ell_1 \Big) = \sum_{\ell_1=1}^{1/2K-1} \ell_1 \tag{2.21}$$

Hint: Expand this sum.

With the result (2.21) we get for (2.20),

$$\begin{aligned} C &= \frac{6}{K(K-1)} \sum_{\ell_1=1}^{1/2K-1} \ell_1 \\ &= \frac{3}{K(K-1)} \Big[\ \frac{1}{2}\big(\frac{1}{2}K - 1\big)^2 \ + \ \frac{1}{2}\big(\frac{1}{2} - 1\big) \Big] \\ &= \frac{3(K-2)}{4(K-1)} , \end{aligned} \tag{2.22}$$

which is (2.7) that we wanted to prove.

Chapter 3

Eigenvalue spectra

3.1 Eigenvalues of the connectivity matrix

For eigenvector $\boldsymbol{x}$ and eigenvalue λ the eigenvalue equation for the matrix $\boldsymbol{c}$ is,

$$\boldsymbol{c}\boldsymbol{x} = \lambda \boldsymbol{x} \,. \tag{3.1}$$

In terms of matrix entries and eigenvector components this reads,

$$\sum_j c_{ij} x_j = \lambda \, x_i \,. \tag{3.2}$$

We consider here *non*-directed networks with no self-interactions so that $\boldsymbol{c}$ is a symmetric matrix ($c_{ij} = c_{ji}$) and $c_{ii} = 0, \forall\, i$. From the theory of Linear Algebra we know that a symmetric matrix has *real eigenvalues* λ and orthogonal eigenvectors.

We prove here two basic properties of the eigenvalues. Firstly,

$$\lambda_{\max} \geq \overline{k} \,, \tag{3.3}$$

i.e. the largest eigenvalue is larger or equal then the average degree.

PROBLEM 3.1 *Show that,*

$$\lambda_{max} \geq \frac{\boldsymbol{x} \cdot \boldsymbol{c}\boldsymbol{x}}{\boldsymbol{x}^2}\Big|_{\boldsymbol{x}=(1,1,..,1)} = \frac{1}{N}\sum_{ij} c_{ij} \equiv \overline{k}\,, \tag{3.4}$$

You may use the identity [1]

$$\lambda_{max} = max_{\boldsymbol{x}} \left[\frac{\boldsymbol{x} \cdot \boldsymbol{c}\boldsymbol{x}}{\boldsymbol{x}^2}\right], \tag{3.5}$$

where '$max_{\boldsymbol{x}}$' means that we select the particular vector $\boldsymbol{x} \in \mathbb{R}^N$ in the argument [..] *for which this argument is maximal.*

The next basic property we consider is the relation between the largest eigenvalue and the largest degree,

$$\lambda_{\max} \leq k_{\max}\,. \tag{3.6}$$

Prove: Let $\boldsymbol{x}$ be the eigenvector of the largest eigenvalue $\lambda_{\max}$. We know from the eigenvalue equation that an eigen*vector* is determined up to a scalar $\in \mathbb{R}$. We can always choose this scalar such that $\exists i$ for which $x_i > 0$, otherwise all $\{x_i\}$ would vanish. Now consider the largest component of $\boldsymbol{x}$ and call it x_m, which is of course also > 0. Property (3.6) follows then from

[1] This is proved as follows. For *any* vector $\boldsymbol{x} \in \mathbb{R}^N$ we may decompose $\boldsymbol{x} = \sum_\mu x_\mu \boldsymbol{e}_\mu$, where $\{\boldsymbol{e}_\mu\}$ is a basis of eigenvectors. This implies that $\frac{\boldsymbol{x}\cdot\boldsymbol{c}\boldsymbol{x}}{\boldsymbol{x}^2} = \frac{\sum_\mu x_\mu^2 \lambda_\mu}{\sum_\mu x_\mu^2} \leq \frac{\sum_\mu x_\mu^2 \lambda_{\max}}{\sum_\mu x_\mu^2} = \frac{\lambda_{\max}\sum_\mu x_\mu^2}{\sum_\mu x_\mu^2} = \lambda_{\max}$ for *all* vectors $\boldsymbol{x}$. For *one particular* vector $\boldsymbol{x}$ we know that $\frac{\boldsymbol{x}\cdot\boldsymbol{c}\boldsymbol{x}}{\boldsymbol{x}^2}$ equals the maximum eigenvalue $\lambda_{\max}$, which is of course clear since for that particular (eigen)vector $\boldsymbol{x}$ we simply have $\frac{\boldsymbol{x}\cdot\boldsymbol{c}\boldsymbol{x}}{\boldsymbol{x}^2} = \frac{\boldsymbol{x}\cdot\lambda_{\max}\boldsymbol{x}}{\boldsymbol{x}^2} = \lambda_{\max}$. Then we may indeed conclude that $\lambda_{\max} = \max_{\boldsymbol{x}}\,(\frac{\boldsymbol{x}\cdot\boldsymbol{c}\boldsymbol{x}}{\boldsymbol{x}^2})$.

analyzing the eigenvalue equation for this eigenvector component x_m,

$$
\begin{aligned}
\lambda_{\max}\, x_m &= \sum_\ell c_{m\ell}\, x_l \le \sum_\ell c_{m\ell}\, x_m = x_m \underbrace{\sum_\ell c_{m\ell}}_{\equiv k_m} \\
\Rightarrow \qquad & (\lambda_{\max} - k_m) \overbrace{x_m}^{>0} \le 0 \\
\Rightarrow \qquad & \lambda_{\max} - k_m \le 0 \\
\Rightarrow \qquad & \lambda_{\max} \le k_m \le k_{\max}
\end{aligned}
\tag{3.7}
$$

$\square$

3.2 The Laplacian

The so-called *Laplacian* matrix $\boldsymbol{L} = \{L_{ij}\}$ is defined from the connectivity matrix and the degrees,

$$L_{ij} = k_i\, \delta_{ij} - c_{ij}\,, \tag{3.8}$$

where δ_{ij} is the Kronecker-delta, i.e. to construct the Laplacian we replace the zeros on the diagonal of the connectivity matrix (non-directed network) by the degrees $\{k_i\}$, and replace the off-diagonals $\{c_{i\neq j}\} \to \{-c_{i\neq j}\}$. For $\boldsymbol{x} \in \mathbb{R}^N$ we have then,

$$
\begin{aligned}
\sum_j L_{ij}\, x_j &= \sum_j k_i \delta_{ij}\, x_j - \sum_j c_{ij} x_j \\
&= k_i\, x_i - \sum_j c_{ij} x_j \\
&= \overbrace{\sum_j C_{ij}}^{\equiv k_i} x_i - \sum_j c_{ij}\, x_j \\
&= \sum_j c_{ij} (x_i - x_j)\,,
\end{aligned}
\tag{3.9}
$$

We shall use (3.9) in the next section. We also have the identity,

$$\sum_j L_{ij}\,x_j = k_i\Big(\,x_i - \frac{\sum_j c_{ij}\,x_j}{\sum_j c_{ij}}\Big)\,. \tag{3.10}$$

3.3 Eigenvalues of the Laplacian

Just as $\boldsymbol{c}$, the Laplacian matrix $\boldsymbol{L}$ is symmetric and thus also has real eigenvalues with orthogonal eigenvectors. Let $\boldsymbol{x}$ be an eigenvector of $\boldsymbol{L}$ with eigenvalue λ. Then the eigenvalue equation $\sum_j L_{ij}\,x_j = \lambda\,x_i$ implies,

$$\sum_{ij} x_i L_{ij} x_j = \sum_i x_i\,\lambda\,x_i = \lambda \underbrace{\sum_i x_i^2}_{>0}\,. \tag{3.11}$$

From this we may conclude the following. If we can prove that for an eigenvector $\boldsymbol{x}$ (with eigenvalue λ) $\sum_{ij} x_i L_{ij} x_j \geq 0$, then $\lambda \geq 0$.
Proof of $\sum_{ij} x_i L_{ij} x_j \geq 0$:

<u>*PROBLEM* 3.2</u> *Show that,*

$$\sum_{ij} x_j\,c_{ij}(x_i - x_j) = -\sum_{ij} x_i\,c_{ij}(x_i - x_j)\,. \tag{3.12}$$

Then using (3.9) and (3.12), we have,

$$\begin{aligned}
\sum_{ij} x_i\, L_{ij}\, x_j &= \sum_{ij} x_i\, c_{ij}\, (x_i - x_j) \\
&= \sum_{ij} (x_i - x_j)\, c_{ij} (x_i - x_j) + \sum_{ij} x_j\, c_{ij} (x_i - x_j) \\
&= \sum_{ij} (x_i - x_j)\, c_{ij} (x_i - x_j) - \sum_{ij} x_i\, c_{ij} (x_i - x_j) \\
&= \sum_{ij} c_{ij}\, (x_i - x_j)^2 - \sum_{ij} x_i\, c_{ij} (x_i - x_j) \\
\Rightarrow \qquad & \sum_{ij} x_i\, L_{ij}\, x_j = \frac{1}{2} \sum_{ij} \underbrace{c_{ij}}_{\in\{0,1\}} (x_i - x_j)^2 \;\geq 0 \qquad (3.13)
\end{aligned}$$

□

And this proves indeed that,

$\boxed{\lambda \geq 0, \textit{ the eigenvalues of } \boldsymbol{L} \textit{ are non-negative.}}$

3.4 Kernel of the Laplacian

Consider the eigenvalue $\lambda = 0$ of the Laplacian $\boldsymbol{L}$. The Kernel of the Laplacian is the eigenvector space belonging to eigenvalue zero, i.e.,

$$\text{Ker}\ \boldsymbol{L} = \{\, \boldsymbol{x} \mid \boldsymbol{L}\,\boldsymbol{x} = \boldsymbol{0} \} \,. \qquad (3.14)$$

Of course we have,

$$\begin{aligned}
\boldsymbol{L}\,\boldsymbol{x} = \boldsymbol{0} \qquad &\Rightarrow \qquad \boldsymbol{x} \cdot \boldsymbol{L}\,\boldsymbol{x} = \boldsymbol{0} \\
&\Rightarrow \quad \sum_{ij} x_i\, L_{ij}\, x_j = 0 \qquad (3.15)
\end{aligned}$$

Using the result of (3.13) $\sum_{ij} x_i\, L_{ij}\, x_j = \frac{1}{2} \sum_{ij} c_{ij} (x_i - x_j)^2$, we get,

$$\sum_{ij} c_{ij} (x_i - x_j)^2 = 0 \,. \qquad (3.16)$$

From (3.16) we can draw important conclusions. First, we define the notion of a connected component. The collection of nodes $\{i, j\}$ for which there exist a *path* (i.e. a set of consecutive links) between the nodes i and j is called a *connected component.* Since *all* nodes in the network are linked to at least one other node (otherwise the node would not be part of this network) clearly, the entire network can be seen as build up from a set of connected components. Per definition each connected component is *not* connected to any other component, so that we can say that the entire network consists of a set of *disjoint connected components.* Although somewhat confusingly, one also speaks of a *disconnected component* instead of a *connected component* in the sense of the definition given above.

PROBLEM 3.3 *Let* $\boldsymbol{x} \in Ker\,\boldsymbol{L}$. *Show from (3.16) that,*

$$x_i \;=\; x_j\,, \tag{3.17}$$

if node i and j are connected by a path. (Hint: it helps to construct and draw a simple example of a connected component that contains these two nodes.)

Assume we have M connected components in the network. We label the connected components by the index μ with $\mu = 1..M$. Say, the connected component μ has C_μ nodes. Next, we choose to label the *nodes* starting in the first connected component, then continue the labeling in the second component etc. Clearly, from (3.17) we then have that each vector $\boldsymbol{x}$ in the Kernel of $\boldsymbol{L}$ has the following structure,

$$\boldsymbol{x} = (\,\underbrace{z_1, z_1, z_1, .., z_1}_{C_1\times}, \underbrace{z_2, z_2, z_2, .., z_2}_{C_2\times}, ..., \underbrace{z_M, z_M, z_M, .., z_M}_{C_M\times}\,)\,. \tag{3.18}$$

<u>PROBLEM 3.4</u> *Convince yourself that the collection of vectors* $\{\boldsymbol{x}\}$ *in (3.18) define an eigenvector space of dimension* M. *Hint: Choose a convenient basis and 'count' the basis vectors,*

$$\boldsymbol{b}_\mu = (\,0,0,..,0\ \mathbf{,}\ 0,0,..,0\ \mathbf{,}\ \cdots\cdots\ \mathbf{,}\ \underbrace{1,1,..,1}_{C_\mu \times}\ \mathbf{,}\ \cdots\cdots\ \mathbf{,}\ 0,0,..,0\,)\,. \tag{3.19}$$

Our conclusion is thus,

The number of 'disconnected components' in a network equals Dim Ker L

<u>PROBLEM 3.5</u> *Show that the Laplacian has the following block-structure,*

$$\boldsymbol{L} = \begin{pmatrix} (C_1 \times C_1) & 0 & 0 & 0 & 0 \\ 0 & (C_2 \times C_2) & 0 & 0 & 0 \\ 0 & 0 & (C_3 \times C_3) & 0 & 0 \\ 0 & 0 & 0 & .. & .. \\ 0 & 0 & 0 & .. & (C_M \times C_M) \end{pmatrix}, \tag{3.20}$$

where $(C_\mu \times C_\mu)$ *denotes a* $C_\mu \times C_\mu$*–matrix.*

We end this section with considering some further consequences of (3.17). Let $\boldsymbol{x^0}$ be an eigenvector of $\boldsymbol{L}$ for eigenvalues $\lambda = 0$ and let $\boldsymbol{x}$ be an eigenvector of $\boldsymbol{L}$ for eigenvalues $\lambda > 0$. Since eigenvectors of *different* eigenvalues are always orthogonal to each other ($\boldsymbol{L}$ is symmetric) we have,

$$\boldsymbol{x} \cdot \boldsymbol{x^0} = 0\,. \tag{3.21}$$

If we take for $\boldsymbol{x^0}$ the μ^{th} basis vector (3.19), then $x_i = 1$ for $i \in \mu$ and $x_i = 0$ otherwise[2]. With this (3.21) becomes,

$$\sum_{i \in \mu} x_i = 0, \quad (\mu = 1..M) \tag{3.22}$$

On the other hand, from the eigenvalue equation $\sum_j c_{ij}\, x_j = \lambda\, x_i$ $(\lambda > 0)$, we have,

$$\sum_{j \in \mu} c_{ij}\, x_j = \lambda\, x_i \,. \tag{3.23}$$

From (3.22) and (3.23) follows then,

$$0 = \lambda \sum_{i \in \mu} x_i = \sum_{i,j \in \mu} c_{ij}\, x_j = \sum_{j \in \mu} k_j\, x_j \quad \Rightarrow \quad \sum_{j \in \mu} k_j\, x_j = 0\,, \tag{3.24}$$

which is a useful identity.

[2]with $i \in \mu$ we mean that the node i is part of the connected component labeled by μ.

Chapter 4

Eigenvalue spectra of Gaussian random matrices

4.1 The Dirac-delta

We list here some idendities for delta-functions. They will be used extensively in the lectures. First the *Kronecker-delta*, already used in the previous chapters (see figure 4.1),

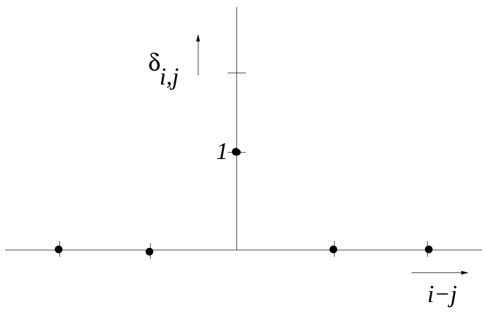

Figure 4.1: Graph of the Kronecker-delta: $\delta_{ii} = 1,\ \delta_{i\neq j} = 0$.

$$\delta_{i,j} = 1,\ \text{if}\ i = j \quad \text{and} \quad \delta_{i,j} = 0,\ \text{if}\ i \neq j\ , \tag{4.1}$$

with properties,

$$\sum_{i=1}^{N} \delta_{i,j} = 1 \ (\forall j \in \mathbb{Z}) \tag{4.2}$$

$$\sum_{i=1}^{N} f_i \ \delta_{i,j} = f_j \,, \tag{4.3}$$

for some function f of the index j.

Generalized to continuous variables x we have the *Dirac-delta* (see figure 4.2),

$$\delta(x - x^0) = \infty, \ \text{if } x = x^0 \quad \text{and} \quad \delta(x - x^0) = 0, \ \text{if } x \neq x^0 \,, \tag{4.4}$$

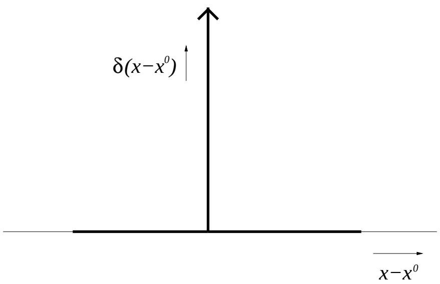

Figure 4.2: Graph of the Dirac-delta: $\delta(0) = \infty, \ \delta(\neq 0) = 0$.

with required properties,

$$\int_{-\infty}^{+\infty} \delta(x - x^0)\, dx = 1 \ (\forall x^0 \in \mathbb{R}) \tag{4.5}$$

$$\int_{-\infty}^{+\infty} f(x) \ \delta(x - x^0)\, dx = f(x^0) \,, \tag{4.6}$$

for some function f of the variable x. The identity (4.6) should rather be seen as *defining* $\delta(x - x^0)$. As such the Dirac-delta function defines a *distribution* of values for the variable x. From this definition follows (setting

$x^0 = 0$) that,

$$\delta(x) = \delta(-x)\,. \tag{4.7}$$

There are various so-called *representations* of the Dirac-delta $\delta(x)$. We mention here the *Fourier-representation*,

$$\delta(x) = \mathcal{F}\{1\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{i\,\xi\,x}\,d\xi\,, \tag{4.8}$$

where ξ is the *Fourier-conjugate* variable of x. Clearly, this satisfies (4.4). Also the property (4.6) is easily checked[1],

$$\begin{aligned}\int_{-\infty}^{+\infty} f(x)\ \delta(x)\,dx &= \frac{1}{2\pi}\Big\{\int_{-\infty}^{+\infty} f(x)\,\mathrm{e}^{i\,\xi\,x}\,dx\Big\}\,d\xi \\ &\equiv \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\xi)\,d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\xi)\,\mathrm{e}^{-i\,\xi\,0}\,d\xi\ \equiv\ f(0)\end{aligned} \tag{4.9}$$

□

We now extend the property (4.5) to the for us important *multi-dimensional* case. Although this is almost trivial, it is good to explicitely list these multi-dimensional extensions,

$$\begin{aligned} & \int dx\,\delta(x-x^0) = 1 \\ \Rightarrow\quad & \int dx\,\delta(x-x^0)\cdot\int dy\,\delta(y-y^0) = 1 \\ \Rightarrow\quad & \int\Big(\prod_{\alpha=1}^{n} dx_\alpha\Big)\ \prod_{\alpha=1}^{n}\delta(x_\alpha - x^0_\alpha) = 1\,,\end{aligned} \tag{4.10}$$

where n is some integer. We may instead of the n vector-components $\{x_\alpha\}$ equally well take n^2 matrix-components $\{q_{\alpha\beta}\}$ $(\alpha,\beta = 1..n)$ and multiply

[1] We check this here for the case $x^0 = 0$ but this is easily extended to $x^0 \neq 0$.

$n^2 \times 1 = 1$. Then we have the following *representation of the unit* 1,

$$1 = \int \underbrace{\Big(\prod_{\alpha\beta} dq_{\alpha\beta}\Big)} \prod_{\alpha\beta} \delta(q_{\alpha\beta} - q^0_{\alpha\beta}) \tag{4.11}$$

$$\Big(\prod_{\alpha\beta} dq_{\alpha\beta}\Big) \equiv \{dq_{\alpha\beta}\} \tag{4.12}$$

which is true for *any* fixed set of values $\{q^0_{\alpha\beta}\}$ for the matrix entries. Loosely speaking, while integrating (i.e. summing) over *all* possible matrix entries $\{q_{\alpha\beta}\}$ we will always 'hit' the chosen fixed set $\{q^0_{\alpha\beta}\}$ and for that particular 'hit' the Dirac-delta results in 1, and in 0 for all other possible values of the matrix entries $\{q_{\alpha\beta}\}$.

Furthermore, we may write instead of $\prod_{\alpha\beta} \delta(q_{\alpha\beta} - q^0_{\alpha\beta})$ in (4.11) a *product of Fourier-representations*, using (4.8) for each of the n^2 Fourier-representations. Denoting the Fourier-conjugates of $\{q_{\alpha\beta}\}$ by $\{\hat{q}_{\alpha\beta}\}$ we have then,

$$\prod_{\alpha\beta} \delta(q_{\alpha\beta} - q^0_{\alpha\beta}) = \int \underbrace{\Big(\prod_{\alpha\beta} \frac{d\hat{q}_{\alpha\beta}}{2\pi}\Big)} \prod_{\alpha\beta} \mathrm{e}^{i\,\hat{q}_{\alpha\beta}(q_{\alpha\beta} - q^0_{\alpha\beta})}$$

$$\Big(\prod_{\alpha\beta} \frac{d\hat{q}_{\alpha\beta}}{2\pi}\Big) = \frac{1}{(2\pi)^{n^2}} \prod_{\alpha\beta} d\hat{q}_{\alpha\beta} \equiv \{d\hat{q}_{\alpha\beta}\} \tag{4.13}$$

$$= \int \{d\hat{q}_{\alpha\beta}\}\ \mathrm{e}^{i\sum_{\alpha\beta}\ \hat{q}_{\alpha\beta}(q_{\alpha\beta} - q^0_{\alpha\beta})} \tag{4.14}$$

Finally substituting (4.14) into (4.11) gives us the following representation of the unit as a multi-dimensional integral *over exponentials*,

$$1 = \int \{dq_{\alpha\beta}\}\{d\hat{q}_{\alpha\beta}\}\ \mathrm{e}^{i\sum_{\alpha\beta}\ \hat{q}_{\alpha\beta}(q_{\alpha\beta} - q^0_{\alpha\beta})}\,, \tag{4.15}$$

where we may substitute for $q^0_{\alpha\beta}$ *any* expression that 'depends on' the labels α and β.

Note that the Fourier-representation of the one-dimensional case (4.8) can be modified by introducing a further constant N, for which we take an integer,

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\xi \;\; \mathrm{e}^{i\xi\, x} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d(N\xi) \;\; \mathrm{e}^{i\,N\xi\, x} \tag{4.16}$$

We can do this also for the multi-dimensional which modifies (4.15) into,

$$1 \;=\; \int \{dq_{\alpha\beta}\}\{d\hat{q}_{\alpha\beta}\} \;\; \mathrm{e}^{i\,N\sum_{\alpha\beta}\;\hat{q}_{\alpha\beta}(q_{\alpha\beta}-q^0_{\alpha\beta})}\,, \tag{4.17}$$

where we now re-define the integration measure (4.13) as,

$$\{d\hat{q}_{\alpha\beta}\} \equiv \Big(\prod_{\alpha\beta}\frac{d\hat{q}_{\alpha\beta}}{2\pi/N}\Big) = \frac{1}{(2\pi/N)^{n^2}}\prod_{\alpha\beta} d\hat{q}_{\alpha\beta}\,, \tag{4.18}$$

and the other measure remains as it was in (4.12),

$$\{dq_{\alpha\beta}\} \equiv \Big(\prod_{\alpha\beta} dq_{\alpha\beta}\Big)\,. \tag{4.19}$$

The expression (4.17) as *multi-dimensional representation of the unit 1*, including this extra constant N, is a useful identity that we shall use later on.

To end this section we give without proof another representation of the Dirac-delta,

$$\delta(z) = \frac{1}{\pi}\lim_{\epsilon\downarrow 0}\;\mathrm{Re}\left\{\frac{\int dx\;x^2\,\mathrm{e}^{-\frac{1}{2}(\epsilon+iz)x^2}}{\int dx\;\mathrm{e}^{-\frac{1}{2}(\epsilon+iz)x^2}}\right\}. \tag{4.20}$$

This representation will be used in the next section.

4.2 The spectral density representation

We consider a general *symmetric* matrix and denote this as $\boldsymbol{C}$. Later in this chapter we shall take for the entries $\{C_{ij}\}$ independent random Gaussians with zero mean. Since the connectivity matrix $\boldsymbol{c}$ introduced before has entries $c_{ij} \in \{0, 1\}$ we do not *directly* look at the connectivity matrix in this chapter. Yet the results are closely related to models for connectivity matrices. We shall not explain here how this works, which is a next step in this theory. We refer to e.g. [3], although this is *not* part of the examinable material.

Let $\{\mu_i\}$ be the eigenvalues of the general symmetric $N \times N$ matrix $\boldsymbol{C}$. Our aim in this section is to find a workable[2] expression for the distribution of eigenvalues of this matrix. If N is a finite natural number then the finite set of eigenvalues $\{\mu_i\}$ $(i = 1..N)$ would have the following distribution,

$$\rho(\mu) = \frac{1}{N} \sum_{i=1}^{N} \delta_{\mu,\mu_i} \,. \tag{4.21}$$

This simply defines a counting process, where the index i 'scans' over all values $1..N$ and contribute 1 to the sum on a 'hit' $\mu = \mu_i$ and 0 otherwise. This defines of course a *discrete* distribution since only those values of μ equal to one (or more) eigenvalues μ_i give a non-zero value for $\rho(\mu)$. Clearly, summing over all these values of μ results in,

$$\sum_{\mu} \rho(\mu) = 1 \,, \tag{4.22}$$

as it should.

If $N \to \infty$, which will be the case of interest in this chapter, the number of eigenvalues will also be ∞. The sum in equation (4.22) then changes into an integration over all values of μ,

$$\int_{-\infty}^{+\infty} d\mu \; \rho(\mu) = 1 \,. \tag{4.23}$$

But then the Kronecker-delta in (4.21) has to be replaced by a Dirac-delta in order to assure that (4.23) is valid. In this case we have a *density distribution*

[2] It will become clear in the course of this chapter what we mean with 'workable'.

of eigenvalues or also called *spectral density*,

$$\rho(\mu) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta(\mu - \mu_i)\,. \tag{4.24}$$

<u>*PROBLEM*</u> <u>4.1</u> *Convince yourself of the validity of (4.23) and (4.24).*

Although formally correct, (4.24) says very little about the matrix $\boldsymbol{C}$, except that it formally mentions the infinitely many eigenvalues $\{\mu_i\}$. We shall now derive two workable representations of (4.24). The first representation of the eigenvalue distribution is,

$$\rho(\mu) \;=\; \lim_{N\to\infty} \frac{2}{N\pi} \; \lim_{\epsilon\downarrow 0} \; \mathrm{Im}\, \frac{d}{d\mu} \,\ln Z(\mu)\,, \tag{4.25}$$

$$Z(\mu) = \int d\vec{\Phi} \;\; \mathrm{e}^{-\frac{1}{2} i\, \vec{\Phi}\cdot[\boldsymbol{C}-(\mu+i\epsilon)\mathbb{1}]\,\vec{\Phi}}\,. \tag{4.26}$$

The integration variable $\vec{\Phi}$ in (4.26) stands for a N-dimensional vector $\vec{\Phi} = (\Phi_1, \Phi_2, ..\Phi_N)$ so that,

$$\int d\vec{\Phi} \equiv \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdot\cdot \int_{-\infty}^{+\infty} d\Phi_1 d\Phi_2 \cdot\cdot d\Phi_N\,. \tag{4.27}$$

The second representation is,

$$\rho(\mu) = \lim_{N\to\infty} \frac{1}{N\pi} \; \lim_{\epsilon\downarrow 0} \; \mathrm{Im}\,\mathrm{Tr}\,\left[\boldsymbol{C} - \mathbb{1}\big(\mu + i\epsilon\big)\right]^{-1}, \tag{4.28}$$

where $\mathbb{1}$ denotes the $N \times N$ unit matrix.

Proof of (4.25):

Consider the $\vec{\Phi}$-integral (4.26) first. We first define,

$$\boldsymbol{M} = \boldsymbol{C} - \big(\mu + i\epsilon\big)\mathbb{1}\,, \tag{4.29}$$

and let $\{\boldsymbol{e}_j\}$ be set of orthonormal eigenvectors of this (symmetric) matrix spanning the N-dimensional space, i.e. $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$. When integrating over the vector $\vec{\Phi}$ in (4.26), we integrate over all possible vectors in the N-dimensional space, i.e. they span the entire space. Instead, we could also integrate over all *linear combinations of eigenvectors* $\boldsymbol{e}_j$ since these also span the N-dimensional space. We write these linear combinations as follows,

$$\vec{\Phi} = \sum_{j=1}^{N} x_j \, \boldsymbol{e}_j \, . \tag{4.30}$$

where the $\{x_j\}$ are of course scalars. Clearly, when the vector $\vec{\Phi}$ is varying over the space so do these components $\{x_j\}$, since the eigenvectors $\{\boldsymbol{e}_j\}$ are fixed. Then, instead of integrating over the vector $\vec{\Phi}$ via $\int d\vec{\Phi}$, we may also integrate over $\{x_j\}$ via $\int \prod_{i=1}^{N} dx_j$. But, changing integration variables in a multi-dimensional integral introduces a *Jacobian* determinant. Since the change of variables is linear according to (4.30), the Jacobian will be a *constant matrix*. So we have,

$$\int d\vec{\Phi} = \text{Constant} \cdot \int \prod_{i=1}^{N} dx_j \, . \tag{4.31}$$

But this constant does not effect (4.25) since $\frac{d}{d\mu}(\text{Const} \times ..) = \frac{d}{d\mu}(..)$. Thus we may ignore the constant, so that instead of (4.26) we may write,

$$Z(\mu) = \int d\vec{\Phi} \;\; \mathrm{e}^{-\frac{1}{2} i \, \vec{\Phi} \cdot \boldsymbol{M} \, \vec{\Phi}} = \int \Big(\prod_{i=1}^{N} dx_j \Big) \; \mathrm{e}^{\frac{1}{2} i \, \sum_{j=1}^{N} \lambda_j \, x_j^2} \, , \tag{4.32}$$

where $\{\lambda_j\}$ are the eigenvalues of the matrix M.

<u>*PROBLEM* 4.2</u> *Verify (4.32), using* $\boldsymbol{M} x_j \boldsymbol{e}_j = \lambda_j x_j \boldsymbol{e}_j$. *Also check that* $\lambda_j = \mu_j - \mu - i\,\epsilon$, *where* $\{\mu_j\}$ *are the eigenvalues of the matrix* $\boldsymbol{C}$.

Next we simplify (4.32) as follows,

$$\begin{aligned} Z(\mu) &= \int \Big(\prod_{i=1}^{N} dx_j\Big)\ \mathrm{e}^{\frac{1}{2} i \sum_{j=1}^{N} \lambda_j\, x_j^2} \\ &= \prod_{j=1}^{N} \left(\int dx_j\ \ \mathrm{e}^{\frac{1}{2} i\, \lambda_j\, x_j^2} \right) \\ &= \prod_{j=1}^{N} \left(\int dx\ \ \mathrm{e}^{\frac{1}{2} i\, \lambda_j\, x^2} \right) \qquad (4.33) \\ &= \prod_{j=1}^{N} \sqrt{\frac{\pi}{\frac{1}{2} i\, \lambda_j}} \qquad (4.34) \end{aligned}$$

The last step is the result of a *complex Gaussian integration*. More on this type of integrations later. For the moment we shall only use (4.33) and use (4.34) in the proof of (4.28), see below.

Then substituting (4.33) into (4.25) concludes the proof,

$$\begin{aligned} \rho(\mu) &= \lim_{N\to\infty} \frac{2}{N\pi}\ \lim_{\epsilon\downarrow 0}\ \mathrm{Im}\, \frac{d}{d\mu}\ \ln Z(\mu) \\ &= \lim_{N\to\infty} \frac{2}{N\pi}\ \lim_{\epsilon\downarrow 0}\ \mathrm{Im}\, \frac{d}{d\mu}\ \ln \left\{ \prod_{j=1}^{N} \left(\int dx_j\ \ \mathrm{e}^{\frac{1}{2} i\, \lambda_j\, x_j^2} \right) \right\} \\ &= \lim_{N\to\infty} \frac{2}{N\pi}\ \lim_{\epsilon\downarrow 0}\ \mathrm{Im}\, \frac{d}{d\mu} \sum_{j=1}^{N} \ln \left\{ \int dx_j\ \ \mathrm{e}^{\frac{1}{2} i\, \lambda_j\, x_j^2} \right\} \\ &= \lim_{N\to\infty} \frac{2}{N\pi} \sum_{j=1}^{N} \lim_{\epsilon\downarrow 0}\ \mathrm{Im}\ \frac{1}{2} \left\{ i\, \frac{\int dx\ x^2\, \mathrm{e}^{-\frac{1}{2}\left(\epsilon + i(\mu_j - \mu)\right) x^2}}{\int dx\ \mathrm{e}^{-\frac{1}{2}\left(\epsilon + i(\mu_j - \mu)\right) x^2}} \right\} \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \underbrace{\frac{1}{\pi} \lim_{\epsilon\downarrow 0}\ \mathrm{Re} \left\{ \frac{\int dx\ x^2\, \mathrm{e}^{-\frac{1}{2}\left(\epsilon + i(\mu_j - \mu)\right) x^2}}{\int dx\ \mathrm{e}^{-\frac{1}{2}\left(\epsilon + i(\mu_j - \mu)\right) x^2}} \right\}}_{=\ \delta(\mu_j - \mu)\ \ \text{see (4.20)}} \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \delta(\mu_j - \mu)\,. \qquad (4.35) \end{aligned}$$

which matches (4.24) □.

Proof of (4.28):

We start with (4.25) which is already proven above. Instead of using (4.33) we use now (4.34) and substitute this into (4.25). But let us proceed step by step. We have from (4.34) that,

$$\begin{aligned} Z(\mu) &= \prod_j \sqrt{\frac{\pi}{\frac{1}{2} i\,\lambda_j}} \\ &\equiv \frac{(2\pi)^{N/2}}{\sqrt{\prod_j i\lambda_j}}\,, \end{aligned} \tag{4.36}$$

with $\lambda_j = \mu_j - \mu - i\epsilon$ the eigenvalues of $\boldsymbol{M} = \boldsymbol{C} - \big(\mu + i\epsilon\big)\mathbb{1}$ as we have defined in the previous proof. Note that (4.36) can be written as

$$\begin{aligned} Z(\mu) = \frac{(2\pi)^{N/2}}{\mathrm{Det}\,(i\boldsymbol{M})} &= \frac{(2\pi)^{N/2}}{\mathrm{Det}\, i\big(\boldsymbol{C} - (\mu + i\epsilon)\mathbb{1}\big)} \\ &= \frac{(2\pi)^{N/2}}{\mathrm{Det}\,\big(\epsilon\mathbb{1} + i(\boldsymbol{C} - \mu\mathbb{1})\big)}\,, \end{aligned} \tag{4.37}$$

since a product of eigenvalues of a matrix can be represented as the determinant of the matrix.

PROBLEM 4.3 *Verify that (4.37) is a representation of (4.36) (change of basis etc.).*

Sunstituting (4.37) into the argument of the logarithm in (4.25) gives,

$$\begin{aligned} \ln\left\{\frac{(2\pi)^{N/2}}{\mathrm{Det}\,\big(\epsilon\mathbb{1} + i(\boldsymbol{C} - \mu\mathbb{1})\big)}\right\} &= \frac{N}{2}\ln 2\pi - \frac{1}{2}\ln \mathrm{Det}\,\big(\epsilon\mathbb{1} + i(\boldsymbol{C} - \mu\mathbb{1})\big) \\ &= \frac{N}{2}\ln 2\pi - \frac{1}{2}\mathrm{Tr}\ln\,\big(\epsilon\mathbb{1} + i(\boldsymbol{C} - \mu\mathbb{1})\big)\,, \end{aligned} \tag{4.38}$$

where we have used the identity $\ln \mathrm{Det}\, A = \mathrm{Tr}\, \ln A$ for some matrix A, a prove of which was given in the lectures[3].

Then substituting (4.38) in (4.25) gives,

$$\begin{aligned}
\rho(\mu) &= \lim_{N\to\infty} \frac{2}{N\pi} \lim_{\epsilon\downarrow 0} \operatorname{Im} \frac{d}{d\mu} \ln Z(\mu) \\
&= \lim_{N\to\infty} \frac{2}{N\pi} \lim_{\epsilon\downarrow 0} \operatorname{Im} \frac{d}{d\mu} \left\{ \frac{N}{2} \ln 2\pi - \frac{1}{2} \operatorname{Tr} \ln \left(\epsilon \mathbb{1} + i(\boldsymbol{C} - \mu \mathbb{1}) \right) \right\} \\
&= \lim_{N\to\infty} \frac{1}{N\pi} \lim_{\epsilon\downarrow 0} \operatorname{Im} \left\{ i \operatorname{Tr} \left[\epsilon \mathbb{1} + i(\boldsymbol{C} - \mu \mathbb{1}) \right]^{-1} \right\} \\
&= \lim_{N\to\infty} \frac{1}{N\pi} \lim_{\epsilon\downarrow 0} \operatorname{Im} \operatorname{Tr} \left[\boldsymbol{C} - \mathbb{1}(\mu + i\epsilon) \right]^{-1}, \qquad (4.39)
\end{aligned}$$

which proves (4.28) □.

4.3 Spectral density for Gaussian random matrices, introduction to the replica-approach

We shall now take for the matrix entries $\{C_{ij}\}$ *independent zero-mean Gaussians* and use the representation (4.25)+(4.26) to compute the *average* spectral density $\overline{\rho(\mu)}$ for the matrix $\boldsymbol{C}$. Under certain further assumptions this will result in the so-called *Wigner's semi-circular law.* The rest of this chapter will be devoted to proving this result.

We start with writing down the distribution of the random matrix entry C_{ij}, the zero-mean Gaussian,

$$p(C_{ij}) \sim \mathrm{e}^{-\frac{1}{2}\left(\frac{C_{ij}}{\sigma/\sqrt{N}}\right)^2}. \qquad (4.40)$$

The standard deviation is taken to be $\sigma/\sqrt{N}$. For this choice[4] we will obtain

[3]See chapter 4 - page 7 of the handwritten notes.

[4]Alternatively, one could of course start of with a general standard deviation σ', work this trough and then at the end observe that the choice $\sigma' = \sigma/\sqrt{N}$ makes $\rho(\mu)$ independent of N.

the desired end-result that $\overline{\rho(\mu)}$ does not depend on N (of which the limit ∞ is taken).

Next we transform (4.40) into a *normal* Gaussian z_{ij} by defining,

$$\begin{aligned} C_{ij} &= \frac{\sigma}{\sqrt{N}} z_{ij} \,, & (4.41) \\ \Rightarrow\ p(z_{ij}) &= \frac{1}{\sqrt{2\pi}}\ \mathrm{e}^{-\frac{1}{2} z_{ij}^2} \,. & (4.42) \end{aligned}$$

Since the random matrix entries $\{C_{ij}\}$ are assumed independent, we have for the distribution of the entire set,

$$\prod_{i<j} p(z_{ij}) = \Big(\prod_{i<j} \frac{1}{\sqrt{2\pi}}\Big)\ \mathrm{e}^{-\frac{1}{2}\sum_{i<j} z_{ij}^2} \,. \qquad (4.43)$$

Here we need $i < j$ to avoid double counting[5].

In (4.25) we have that $\ln Z(\mu)$ depends on the random matrix entries $\{C_{ij}\}$. To compute the average spectral density $\overline{\rho(\mu)}$ we would need to compute,

$$\begin{aligned} \overline{\rho(\mu)} &= \overline{\lim_{N\to\infty} \frac{2}{N\pi}\ \lim_{\epsilon\downarrow 0}\ \mathrm{Im}\, \frac{d}{d\mu}\ \ln Z(\mu)} \\ &= \lim_{N\to\infty} \frac{2}{N\pi}\ \lim_{\epsilon\downarrow 0}\ \mathrm{Im}\, \frac{d}{d\mu}\, \overline{\ln Z(\mu)} \,, & (4.44) \end{aligned}$$

where the 'bar' stands for the average over the entire distribution (4.43). As we know such average is computed as,

$$\begin{aligned} \overline{(..)} &= \int \prod_{i<j} p(z_{ij})\ \ (..) \,, \\ &= \int \Big(\prod_{i<j} \frac{dz_{ij}}{\sqrt{2\pi}}\Big)\ \mathrm{e}^{-\frac{1}{2}\sum_{i<j} z_{ij}^2}\ \ (..) \,. & (4.45) \end{aligned}$$

Without going into details, an average such as $\overline{\ln Z(\mu)}$ is impossible to compute. This problem is solved by considering the formal identity,

$$\overline{\ln Z(\mu)} = \lim_{n\to 0} \frac{1}{n}\ \ln \overline{Z^n} \,. \qquad (4.46)$$

[5] We mention here that $\prod_{i<j} \equiv \prod_i \prod_{j(i<j)}$ and $\sum_{i<j} \equiv \sum_i \sum_{j(i<j)}$.

It turns out that $\overline{Z^n}$ *is computable*, but only if we assume $n \in \mathbb{N}$. We shall do this here, regarding n as *integer-valued* right through and then at the end divert from this and take the *real-valued* limit $n \to 0$. This is an often used technique (approximation) known as the *replica-method* or also called the *replica-trick*. In this method the integer n counts for the number of (multiplicative) *replicas* or repetitions of the function Z in (4.46). We shall label these replicas as Z^α ($\alpha = 1..n$) which are random variables when the matrix entries $\{C_{ij}\}$ are taken random, the case we have here[6].

<u>*PROBLEM*</u> <u>4.3</u> *Proof (4.46) using first-order Tailor expansions for the logarithmic- and exponential functions.*

Using (4.46) we get for (4.44),

$$\begin{aligned}\overline{\rho(\mu)} &= \lim_{N\to\infty} \frac{2}{N\pi} \lim_{\epsilon\downarrow 0} \operatorname{Im} \frac{d}{d\mu} \lim_{n\to 0} \frac{1}{n} \ln \overline{Z^n} \\ &= \frac{2}{\pi} \lim_{\epsilon\downarrow 0} \operatorname{Im} \frac{d}{d\mu} \lim_{n\to 0} \lim_{N\to\infty} \frac{1}{nN} \ln \overline{Z^n} \qquad (4.47)\end{aligned}$$

Note that in (4.47) we have *exchanged* the limites for n and N. This is by no means trivial but we shall not go into the details here. It is part of the *replica-method* and common practice. We shall later see that in order to compute $\overline{Z^n}$ for *integer* values n, we indeed need the limit $N \to \infty$ taken before the limit $n \to 0$.

[6] A key assumption will be that these replicas $\{Z^\alpha\}$ are taken statistically *dependent*, which is implemented through the so-called *replica-structure* of certain correlation functions. We shall come back to this later.

Substuting (4.26) for Z in (4.47) gives for the average spectral density,

$$\overline{\rho(\mu)} = \frac{2}{\pi} \lim_{\epsilon \downarrow 0} \mathrm{Im} \frac{d}{d\mu} \lim_{n \to 0} \lim_{N \to \infty} \frac{1}{nN} \ln \overline{\left[\int d\vec{\Phi} \quad \mathrm{e}^{-\frac{1}{2} i \, \vec{\Phi} \cdot [\boldsymbol{C} - (\mu + i\epsilon) \mathbb{1}] \, \vec{\Phi}} \right]^n} . \tag{4.48}$$

We concentrate now on computing the part in (4.48) that contains the average,

$$\overline{\left[\int d\vec{\Phi} \quad \mathrm{e}^{-\frac{1}{2} i \, \vec{\Phi} \cdot [\boldsymbol{C} - (\mu + i\epsilon) \mathbb{1}] \, \vec{\Phi}} \right]^n} \tag{4.49}$$

We start with explicitely labeling the replicas ($\alpha = 1..n$),

$$\begin{aligned}
&\overline{\left[\int d\vec{\Phi} \quad \mathrm{e}^{-\frac{1}{2} i \, \vec{\Phi} \cdot [\boldsymbol{C} - (\mu + i\epsilon) \mathbb{1}] \, \vec{\Phi}} \right]^n} \\
= &\overline{\int \Big(\prod_{\alpha=1}^{n} d\vec{\Phi}^{\alpha} \Big) \; \prod_{\alpha=1}^{n} \mathrm{e}^{-\frac{1}{2} i \, \vec{\Phi}^{\alpha} \cdot [\boldsymbol{C} - (\mu + i\epsilon) \mathbb{1}] \, \vec{\Phi}^{\alpha}}} \\
= &\overline{\int \Big(\prod_{\alpha=1}^{n} d\vec{\Phi}^{\alpha} \Big) \quad \mathrm{e}^{-\frac{1}{2} i \sum_{\alpha=1}^{n} \, \vec{\Phi}^{\alpha} \cdot [\boldsymbol{C} - (\mu + i\epsilon) \mathbb{1}] \, \vec{\Phi}^{\alpha}}} \\
= &\int \Big(\prod_{\alpha=1}^{n} d\vec{\Phi}^{\alpha} \Big) \quad \underbrace{\mathrm{e}^{-\frac{1}{2} (\epsilon - i\mu) \sum_{\alpha=1}^{n} \, \vec{\Phi}^{\alpha} \cdot \vec{\Phi}^{\alpha}}}_{\text{does not depend on } \boldsymbol{C}} \quad \cdot \; \overline{\mathrm{e}^{-\frac{1}{2} i \sum_{\alpha=1}^{n} \, \vec{\Phi}^{\alpha} \cdot \boldsymbol{C} \, \vec{\Phi}^{\alpha}}}
\end{aligned} \tag{4.50}$$

Next we write,

$$\vec{\Phi}^{\alpha} \cdot \vec{\Phi}^{\alpha} \equiv \sum_{i=1}^{N} \Phi_i^{\alpha} \Phi_i^{\alpha} \tag{4.51}$$

$$\begin{aligned}
\vec{\Phi}^{\alpha} \cdot \boldsymbol{C} \vec{\Phi}^{\alpha} &\equiv \sum_{i,j} \Phi_i^{\alpha} C_{ij} \Phi_j^{\alpha} \\
&= 2 \sum_{i<j} \Phi_i^{\alpha} C_{ij} \Phi_j^{\alpha}
\end{aligned} \tag{4.52}$$

<u>*PROBLEM 4.4*</u> *Convince yourself that you have understood all expressions listed so far. They are essential for what is to come next.*

Now we are going to work on the last term in expression (4.50), the term with the 'bar'. To evaluate this average we use (4.45) with the z_{ij} defined in (4.41). The result is, also using (4.52),

$$\overline{\mathrm{e}^{-\frac{1}{2} i \sum_{\alpha=1}^{n} \vec{\Phi}^{\alpha} \cdot \boldsymbol{C} \vec{\Phi}^{\alpha}}} = \int \Big(\prod_{i<j} \frac{dz_{ij}}{\sqrt{2\pi}} \Big) \; \mathrm{e}^{-\frac{1}{2} \sum_{i<j} z_{ij}^2 - i \sum_{i<j} \Phi_i^{\alpha} \overbrace{\frac{\sigma}{\sqrt{N}} z_{ij}}^{=C_{ij}} \Phi_j^{\alpha}} \tag{4.53}$$

<u>*PROBLEM*</u> <u>4.5</u> *Using the standard Gaussian integral.*

$$\int_{-\infty}^{+\infty} dz \; e^{-a\,z^2} = \sqrt{\frac{\pi}{a}}\,, \tag{4.54}$$

show that after 'completing the squares' we get the Gaussian integral,

$$\int_{-\infty}^{+\infty} \frac{dz}{2\pi} \; e^{-\frac{1}{2}\,z^2 \pm b\,z} = e^{b^2/2}\,. \tag{4.55}$$

Generalize this to a product of Gaussian integrals, i.e. verify that,

$$\begin{aligned} e^{\frac{1}{2}\sum_{i<j} b_{ij}^2} = \prod_{i<j} e^{\frac{1}{2} b_{ij}^2} &= \prod_{i<j} \int_{-\infty}^{+\infty} \frac{dz}{\sqrt{2\pi}} \; e^{-\frac{1}{2}\,z^2 \pm b_{ij}\,z} \\ &= \int_{-\infty}^{+\infty} \Big(\prod_{i<j} \frac{dz_{ij}}{\sqrt{2\pi}} \Big) \; e^{-\frac{1}{2}\sum_{i<j} z_{ij}^2 \,\pm\, \sum_{i<j} b_{ij}\, z_{ij}} \end{aligned} \tag{4.56}$$

Comparing (4.53) and (4.56) we identify,

$$\begin{aligned} b_{ij} &\equiv i\,\frac{\sigma}{\sqrt{N}} \sum_{\alpha} \Phi_i^{\alpha} \Phi_j^{\alpha}\,, \\ \Rightarrow \frac{1}{2}\, b_{ij}^2 &\equiv \frac{-\sigma^2}{2N} \Big(\sum_{\alpha} \Phi_i^{\alpha} \Phi_j^{\alpha} \Big)^2\,, \end{aligned} \tag{4.57}$$

so that (4.53) becomes,

$$\begin{aligned}
\overline{\mathrm{e}^{-\frac{1}{2}i\sum_{\alpha=1}^{n}\vec{\Phi}^{\alpha}\cdot\boldsymbol{C}\vec{\Phi}^{\alpha}}} &= \mathrm{e}^{\frac{-\sigma^2}{2N}\sum_{i<j}\left(\sum_{\alpha}\Phi_i^{\alpha}\Phi_j^{\alpha}\right)^2} \\
&= \mathrm{e}^{\frac{-\sigma^2}{4N}\sum_{ij}\left(\sum_{\alpha}\Phi_i^{\alpha}\Phi_j^{\alpha}\right)\left(\sum_{\beta}\Phi_i^{\beta}\Phi_j^{\beta}\right)} \\
&= \mathrm{e}^{\frac{-\sigma^2}{4N}\sum_{\alpha\beta}\left(\sum_{i}\Phi_i^{\alpha}\Phi_i^{\beta}\right)\left(\sum_{j}\Phi_j^{\alpha}\Phi_j^{\beta}\right)} \\
&= \mathrm{e}^{\frac{-\sigma^2}{4N}\sum_{\alpha\beta}\left(\sum_{i}\Phi_i^{\alpha}\Phi_i^{\beta}\right)^2} \\
&= \mathrm{e}^{-N\frac{\sigma^2}{4}\sum_{\alpha\beta}\left(\frac{1}{N}\sum_{i}\Phi_i^{\alpha}\Phi_i^{\beta}\right)^2} \qquad (4.58)
\end{aligned}$$

Now we substitute this result (4.58) back into (4.50), and (4.50) into (4.48) giving the result of this section,

$$\begin{aligned}
\overline{\rho(\mu)} &= \frac{2}{\pi}\lim_{\epsilon\downarrow 0}\,\mathrm{Im}\,\frac{d}{d\mu}\lim_{n\to 0}\lim_{N\to\infty}\frac{1}{nN}\ln\left\{\overline{Z(\mu)^n}\right\} \\
&= \frac{2}{\pi}\lim_{\epsilon\downarrow 0}\,\mathrm{Im}\,\frac{d}{d\mu}\lim_{n\to 0}\lim_{N\to\infty}\frac{1}{nN}\ln\left\{\int\left(\prod_{\alpha=1}^{n}d\vec{\Phi}^{\alpha}\right)\mathrm{e}^{-\frac{1}{2}(\epsilon-i\mu)\sum_{\alpha=1}^{n}\vec{\Phi}^{\alpha}\cdot\vec{\Phi}^{\alpha}}\right. \\
&\qquad\qquad\left.\times\,\overline{\mathrm{e}^{-\frac{1}{2}i\sum_{\alpha=1}^{n}\vec{\Phi}^{\alpha}\cdot\boldsymbol{C}\vec{\Phi}^{\alpha}}}\right\} \\
&= \frac{2}{\pi}\lim_{\epsilon\downarrow 0}\,\mathrm{Im}\,\frac{d}{d\mu}\lim_{n\to 0}\lim_{N\to\infty}\frac{1}{nN}\ln\left\{\int\left(\prod_{\alpha=1}^{n}d\vec{\Phi}^{\alpha}\right)\mathrm{e}^{-\frac{1}{2}(\epsilon-i\mu)\sum_{\alpha=1}^{n}\sum_i\Phi_i^{\alpha}\Phi_i^{\alpha}}\right. \\
&\qquad\qquad\left.\times\,\mathrm{e}^{-N\frac{\sigma^2}{4}\sum_{\alpha\beta}\left(\frac{1}{N}\sum_{i}\Phi_i^{\alpha}\Phi_i^{\beta}\right)^2}\right\}
\end{aligned} \qquad (4.59)$$

In the next two sections we will see how to deal with the multi-dimensional integral.

<u>PROBLEM 4.6</u> *Thoroughly convince yourself that you have understood all details in (4.59) before continuing. Also note that the square in the second exponential of the integrals makes that such multi-dimensional integral cannot be computed and that linearization is necessary (subject of the next section).*

4.4 Saddle-point equations and the delta-function method

Here we shall use (4.17) to linearize the square in the last exponential of (4.59). Let us rewrite (4.59) as follows,

$$\overline{\rho(\mu)} = \frac{2}{\pi} \lim_{\epsilon\downarrow 0} \mathrm{Im} \frac{d}{d\mu} \lim_{n\to 0} \lim_{N\to\infty} \frac{1}{nN} \ln\Big(\overline{Z(\mu)^n}\Big) \tag{4.60}$$

$$\overline{Z(\mu)^n} = \int \Big(\prod_{i=1}^{N}\prod_{\alpha=1}^{n} d\Phi_i^\alpha\Big)\ \mathrm{e}^{-\frac{1}{2}(\epsilon-i\mu)\sum_{i=1}^N\sum_{\alpha=1}^n(\Phi_i^\alpha)^2\ -\ \frac{1}{4}N\sigma^2\sum_{\alpha,\beta=1}^n\left(\frac{1}{N}\sum_{i=1}^N\Phi_i^\alpha\Phi_i^\beta\right)^2}. \tag{4.61}$$

We are now going to use the so-called *delta-function method* (or 'delta-function trick'), i.e. utilizing (4.17), to show that $\overline{Z(\mu)^n}$ in (4.61) can be cast into the form

$$\overline{Z(\mu)^n} = \int \{dq_{\alpha\beta}\}\{d\hat{q}_{\alpha\beta}\}\ \mathrm{e}^{N\Psi(\{q_{\alpha\beta},\hat{q}_{\alpha\beta}\},\mu)}\,, \tag{4.62}$$

where $\{dq_{\alpha\beta}\}$ and $\{d\hat{q}_{\alpha\beta}\}$ are defined in (4.18) and (4.19), and

$$\Psi(\{q_{\alpha\beta},\hat{q}_{\alpha\beta}\},\mu) = i\sum_{\alpha,\beta=1}^{n}\hat{q}_{\alpha\beta}\,q_{\alpha\beta} - \frac{1}{2}(\epsilon-i\mu)\sum_{\alpha=1}^{n}q_{\alpha\alpha} - \frac{1}{4}\sigma^2\sum_{\alpha,\beta=1}^{n}q_{\alpha\beta}^2 + \ln Z_s \tag{4.63}$$

$$Z_s = \int\Big(\prod_{\alpha=1}^{n}d\Phi^\alpha\Big)\ \mathrm{e}^{-i\sum_{\alpha,\beta=1}^n\hat{q}_{\alpha\beta}\,\Phi^\alpha\Phi^\beta}\,. \tag{4.64}$$

Note that in this result, which we shall derive below in full detail, there is no more mention of the site-index i, i.e. there appears no more $\sum_{i=1}^N$. In fact there in 'only one site left' for which the integral has to be computed. As such the result (4.62) can be seen as an *effective single-site* formulation. This is a big improvement on (4.61) where, through the square in the last exponential, we still had $(\sum_i)^2.. = \sum_i\sum_j\cdots$

In (4.61) we had $N \times n$ complicated integrals to deal with. In (4.62) we have only n integrals. The cost is the introduction of the new set of variables $\{q_{\alpha\beta}\}$ and $\{\hat{q}_{\alpha\beta}\}$. These are $n \times n$ matrices for which we do not know any structure. More on that later. Let us first prove (4.62). We start with (4.61) and use the representation of the unit discussed in section 4.1,

$$
\begin{aligned}
\overline{Z(\mu)^n} &= \int \Big(\prod_{i=1}^{N}\prod_{\alpha=1}^{n} d\Phi_i^\alpha\Big)\ \mathrm{e}^{-\frac{1}{2}(\epsilon - i\mu)\sum_{i=1}^{N}\sum_{\alpha=1}^{n}(\Phi_i^\alpha)^2\ -\ \frac{1}{4}N\sigma^2\sum_{\alpha,\beta=1}^{n}\left(\frac{1}{N}\sum_{i=1}^{N}\Phi_i^\alpha\Phi_i^\beta\right)^2} \\
&= \int \Big(\prod_{i=1}^{N}\prod_{\alpha=1}^{n} d\Phi_i^\alpha\Big)\ \mathrm{e}^{-\frac{1}{2}(\epsilon - i\mu)\sum_{i=1}^{N}\sum_{\alpha=1}^{n}(\Phi_i^\alpha)^2\ -\ \frac{1}{4}N\sigma^2\sum_{\alpha,\beta=1}^{n}\left(\frac{1}{N}\sum_{i=1}^{N}\Phi_i^\alpha\Phi_i^\beta\right)^2} \\
&\quad \times 1 \\
&= \int \Big(\prod_{i=1}^{N}\prod_{\alpha=1}^{n} d\Phi_i^\alpha\Big)\ \mathrm{e}^{-\frac{1}{2}(\epsilon - i\mu)\sum_{i=1}^{N}\sum_{\alpha=1}^{n}(\Phi_i^\alpha)^2\ -\ \frac{1}{4}N\sigma^2\sum_{\alpha,\beta=1}^{n}\left(\frac{1}{N}\sum_{i=1}^{N}\Phi_i^\alpha\Phi_i^\beta\right)^2} \\
&\quad \times \int \{dq_{\alpha\beta}\}\ \prod_{\alpha\beta}\delta\Big(q_{\alpha\beta} - \frac{1}{N}\sum_i \Phi_i^\alpha\Phi_i^\beta\Big) \\
&= \int \{dq_{\alpha\beta}\}\prod_{i,\alpha} d\Phi_i^\alpha\ \mathrm{e}^{-\frac{1}{2}(\epsilon - i\mu)\sum_{i=1}^{N}\sum_{\alpha=1}^{n}(\Phi_i^\alpha)^2\ -\ \frac{1}{4}N\sigma^2\sum_{\alpha,\beta=1}^{n}\left(\frac{1}{N}\sum_{i=1}^{N}\Phi_i^\alpha\Phi_i^\beta\right)^2} \\
&\quad \times \prod_{\alpha\beta}\delta\Big(q_{\alpha\beta} - \frac{1}{N}\sum_i \Phi_i^\alpha\Phi_i^\beta\Big) \\
&= \int \{dq_{\alpha\beta}\}\prod_{i,\alpha} d\Phi_i^\alpha\ \mathrm{e}^{-\frac{1}{2}(\epsilon - i\mu)N\sum_\alpha q_{\alpha\alpha}\ -\ \frac{1}{4}N\sigma^2\sum_{\alpha,\beta} q_{\alpha\beta}^2} \\
&\quad \times \prod_{\alpha\beta}\delta\Big(q_{\alpha\beta} - \frac{1}{N}\sum_i \Phi_i^\alpha\Phi_i^\beta\Big) \quad \text{(simply by the working of the deltas!)} \\
&= \int \{dq_{\alpha\beta}\}\prod_{i,\alpha} d\Phi_i^\alpha\ \mathrm{e}^{-\frac{1}{2}(\epsilon - i\mu)N\sum_\alpha q_{\alpha\alpha}\ -\ \frac{1}{4}N\sigma^2\sum_{\alpha\beta} q_{\alpha\beta}^2} \\
&\quad \times \int \{d\hat{q}_{\alpha\beta}\}\ \mathrm{e}^{i\,N\sum_{\alpha\beta}\hat{q}_{\alpha\beta}\left(q_{\alpha\beta} - \frac{1}{N}\sum_i \Phi_i^\alpha\Phi_i^\beta\right)} \\
&= \int \{dq_{\alpha\beta}\}\{d\hat{q}_{\alpha\beta}\}\ \mathrm{e}^{-\frac{1}{2}(\epsilon - i\mu)N\sum_\alpha q_{\alpha\alpha}\ -\ \frac{1}{4}N\sigma^2\sum_{\alpha\beta} q_{\alpha\beta}^2 + iN\sum_{\alpha\beta}\hat{q}_{\alpha\beta}q_{\alpha\beta}} \\
&\quad \times \int \Big(\prod_{i,\alpha} d\Phi_i^\alpha\Big)\ \mathrm{e}^{-i\sum_{\alpha\beta}\hat{q}_{\alpha\beta}\sum_i \Phi_i^\alpha\Phi_i^\beta}\,.
\end{aligned} \tag{4.65}
$$

The integral in the second line of (4.65) can be simplified as follows,

$$\begin{aligned}
\int\Big(\prod_{i,\alpha} d\Phi_i^\alpha\Big)\ \mathrm{e}^{-i\sum_{\alpha\beta}\hat{q}_{\alpha\beta}\sum_i \Phi_i^\alpha\Phi_i^\beta} &= \prod_i\left(\int\Big(\prod_\alpha d\Phi_i^\alpha\Big)\ \mathrm{e}^{-i\sum_{\alpha\beta}\hat{q}_{\alpha\beta}\Phi_i^\alpha\Phi_i^\beta}\right)\\
&= \prod_i\left(\int\Big(\prod_\alpha d\Phi^\alpha\Big)\ \mathrm{e}^{-i\sum_{\alpha\beta}\hat{q}_{\alpha\beta}\Phi^\alpha\Phi^\beta}\right)\\
&= \mathrm{e}^{N\ln\left[\int\left(\prod_\alpha d\Phi^\alpha\right)\ \mathrm{e}^{-i\sum_{\alpha\beta}\hat{q}_{\alpha\beta}\Phi^\alpha\Phi^\beta}\right]}\,.
\end{aligned} \tag{4.66}$$

Putting (4.66) into (4.65) gives then,

$$\begin{aligned}
\overline{Z(\mu)^n} &= \int\{dq_{\alpha\beta}\}\{d\hat{q}_{\alpha\beta}\}\\
&\quad\times\mathrm{e}^{N\left\{-\frac{1}{2}(\epsilon-i\mu)\sum_\alpha q_{\alpha\alpha}\ -\ \frac{1}{4}\sigma^2\sum_{\alpha\beta}q_{\alpha\beta}^2+i\sum_{\alpha\beta}\hat{q}_{\alpha\beta}q_{\alpha\beta}+\ln Z_s\right\}}\\
&\equiv \int\{dq_{\alpha\beta}\}\{d\hat{q}_{\alpha\beta}\}\ \mathrm{e}^{N\Psi(\{q_{\alpha\beta},\hat{q}_{\alpha\beta}\},\mu)}\,.\\
Z_s &= \int\big(\prod_\alpha d\Phi^\alpha\big)\ \mathrm{e}^{-i\sum_{\alpha\beta}\hat{q}_{\alpha\beta}\Phi^\alpha\Phi^\beta}\,,
\end{aligned}$$

which is what we wanted to prove.

<u>*PROBLEM*</u> <u>4.7</u> *(The Saddle-Point Method) Prove for the one-dimensional case that,*

$$\lim_{N\to\infty}\int_{-\infty}^{+\infty} dx\ e^{Nf(x)} \sim e^{Nf(x_0)}\,, \tag{4.67}$$

where $f'(x_0)=0$ *and* $f''(x_0)<0$. *Hint: Expand* $f(x)$ *around* x_0 *to* 3^{rd} *order.*

In (4.62) we have integrals of the type (4.69) so that we may write (for $N\to\infty$),

$$\overline{Z(\mu)^n} = \int\{dq_{\alpha\beta}\}\{d\hat{q}_{\alpha\beta}\}\ \mathrm{e}^{N\Psi(\{q_{\alpha\beta},\hat{q}_{\alpha\beta}\},\mu)} = \mathrm{e}^{N\Psi(\{q_{\alpha\beta}^{SP},\hat{q}_{\alpha\beta}^{SP}\},\mu)}\,, \tag{4.68}$$

where $\{q_{\alpha\beta}^{SP}\}$ and $\{\hat{q}_{\alpha\beta}^{SP}\}$ denote the values of $\{q_{\alpha\beta}\}$ and $\{\hat{q}_{\alpha\beta}\}$ evaluated at the saddle point of $\Psi(\{q_{\alpha\beta}, \hat{q}_{\alpha\beta}\}, \mu)$. These are solutions of the following *Saddle-Point equations*,

$$\begin{aligned}\frac{\partial\Psi(\{q_{\alpha\beta}, \hat{q}_{\alpha\beta}\}, \mu)}{\partial q_{\alpha\beta}} &= 0 \qquad (4.69)\\ \frac{\partial\Psi(\{q_{\alpha\beta}, \hat{q}_{\alpha\beta}\}, \mu)}{\partial \hat{q}_{\alpha\beta}} &= 0 \qquad (4.70)\end{aligned}$$

With (4.68) this we get for (4.60),

$$\begin{aligned}\overline{\rho(\mu)} &= \frac{2}{\pi} \lim_{\epsilon\downarrow 0} \operatorname{Im} \frac{d}{d\mu} \lim_{n\to 0} \lim_{N\to\infty} \frac{1}{nN} \ln\Big(\overline{Z(\mu)^n}\Big)\\ &= \frac{2}{\pi} \lim_{\epsilon\downarrow 0} \operatorname{Im} \frac{d}{d\mu} \lim_{n\to 0} \lim_{N\to\infty} \frac{1}{nN} \ln\Big(\mathrm{e}^{N\Psi(\{q_{\alpha\beta}^{SP}, \hat{q}_{\alpha\beta}^{SP}\}, \mu)}\Big)\\ &= \frac{2}{\pi} \lim_{\epsilon\downarrow 0} \operatorname{Im} \frac{d}{d\mu} \lim_{n\to 0} \frac{1}{n} \Psi(\{q_{\alpha\beta}^{SP}, \hat{q}_{\alpha\beta}^{SP}\}, \mu) \qquad (4.71)\end{aligned}$$

<u>*PROBLEM* 4.8</u> *Convince yourself that you understand the following (use* $\frac{\partial q_{\alpha\beta}}{\partial q_{\delta\gamma}} = \delta_{\alpha\delta}\delta_{\beta\gamma}$*),*

$$\begin{aligned}\frac{\partial}{\partial q_{\delta\gamma}} \sum_{\alpha\beta} \hat{q}_{\alpha\beta}\, q_{\alpha\beta} &= \hat{q}_{\delta\gamma} \qquad (4.72)\\ \frac{\partial}{\partial q_{\delta\gamma}} \sum_{\alpha} q_{\alpha\alpha} &= \delta_{\delta\gamma} \qquad (4.73)\\ \frac{\partial}{\partial q_{\delta\gamma}} \sum_{\alpha\beta} q_{\alpha\beta}^2 &= 2\, q_{\delta\gamma} \qquad (4.74)\end{aligned}$$

<u>*PROBLEM* 4.9</u> *Prove that the Saddle-Point values of* $\{q_{\alpha\beta}\}$ *and*

$\{\hat{q}_{\alpha\beta}\}$ *are (the so-called* self-consistent equations*),*

$$i\hat{q}^{SP}_{\alpha\beta} = \frac{1}{2}(\epsilon - i\mu)\delta_{\alpha\beta} + \frac{1}{2}\sigma^2\, q^{SP}_{\alpha\beta} \tag{4.75}$$

$$q^{SP}_{\alpha\beta} = \frac{\int \big(\prod_\alpha d\Phi^\alpha\big)\ \Phi^\alpha\Phi^\beta\ \ e^{-i\sum_{\alpha\beta}\hat{q}^{SP}_{\alpha\beta}\Phi^\alpha\Phi^\beta}}{\int \big(\prod_\alpha d\Phi^\alpha\big)\ \ e^{-i\sum_{\alpha\beta}\hat{q}^{SP}_{\alpha\beta}\Phi^\alpha\Phi^\beta}} \tag{4.76}$$

PROBLEM 4.10 *Prove that (4.71) implies,*

$$\overline{\rho(\mu)} = \lim_{n\to 0}\frac{1}{n\pi}\ Re\ \sum_\alpha q^{SP}_{\alpha\alpha}\,. \tag{4.77}$$

4.5 Saddle-point equations and Hubbard-Stratonovich transformations

In this section we will deal with the square in the second exponential of (4.59) in an alternative way, using so-called Hubbard-Stratonovich transformations.

First we prove the identity (multivariate Gaussian integral),

$$\mathrm{e}^{-\frac{1}{2}\sum_{\alpha,\beta=1}^n b^2_{\alpha\beta}} = \int\{dq_{\alpha\beta}\}\ \mathrm{e}^{-\frac{N\sigma^2}{4}\sum_{\alpha,\beta=1}^n q^2_{\alpha\beta}\ -\ i\,\sigma\sqrt{\frac{N}{2}}\sum_{\alpha,\beta=1}^n b_{\alpha\beta}\,q_{\alpha\beta}}\ , \tag{4.78}$$

where,

$$\begin{aligned}\{dq_{\alpha\beta}\} &= \prod_{\alpha\beta}\frac{dq_{\alpha\beta}}{\sigma^{-1}\sqrt{\pi/N}} \\ &= \frac{1}{(\sigma^{-1}\sqrt{\pi/N})^{n^2}}\Big(\prod_{\alpha\beta}dq_{\alpha\beta}\Big)\,.\end{aligned} \tag{4.79}$$

This follows from the Gaussian identities,

$$\begin{aligned}
\mathrm{e}^{\frac{1}{2}\,b^2} &= \int_{-\infty}^{+\infty} \frac{dz}{\sqrt{2\pi}}\ \mathrm{e}^{-\frac{1}{2}z^2 \pm b\,z} \\
\Rightarrow \mathrm{e}^{\frac{1}{2}\,(i\,b)^2} &= \int_{-\infty}^{+\infty} \frac{dz}{\sqrt{2\pi}}\ \mathrm{e}^{-\frac{1}{2}z^2 \pm i\,b\,z} \\
\Rightarrow \mathrm{e}^{-\frac{1}{2}\,b^2} &= \int_{-\infty}^{+\infty} \frac{dz}{\sqrt{2\pi}}\ \mathrm{e}^{-\frac{1}{2}z^2 \pm i\,b\,z} \\
& \quad \text{(now take multivariate case with } \alpha,\beta = 1..n) \Rightarrow \\
\mathrm{e}^{-\frac{1}{2}\sum_{\alpha\beta} b_{\alpha\beta}^2} &= \int \Big(\prod_{\alpha\beta} \frac{dz_{\alpha\beta}}{\sqrt{2\pi}} \Big)\ \mathrm{e}^{-\sum_{\alpha\beta} \frac{1}{2} z_{\alpha\beta}^2 \,\pm\, i \sum_{\alpha\beta} b_{\alpha\beta}\, z_{\alpha\beta}} \\
& \quad \Big(\text{now define } z_{\alpha\beta} = \sigma\sqrt{\frac{N}{2}}\ q_{\alpha\beta}\Big) \Rightarrow \\
\mathrm{e}^{-\frac{1}{2}\sum_{\alpha\beta} b_{\alpha\beta}^2} &= \int \Big(\prod_{\alpha\beta} \frac{dq_{\alpha\beta}}{\sigma^{-1}\sqrt{\pi/N}} \Big)\ \mathrm{e}^{-\frac{N\sigma^2}{4}\sum_{\alpha\beta} q_{\alpha\beta}^2 \,\pm\, i\sigma\sqrt{\frac{N}{2}}\sum_{\alpha\beta} b_{\alpha\beta}\, q_{\alpha\beta}} \\
\Rightarrow \mathrm{e}^{-\frac{1}{2}\sum_{\alpha\beta} b_{\alpha\beta}^2} &= \int \{dq_{\alpha\beta}\}\ \mathrm{e}^{-\frac{N\sigma^2}{4}\sum_{\alpha\beta} q_{\alpha\beta}^2 \,\pm\, i\sigma\sqrt{\frac{N}{2}}\sum_{\alpha\beta} b_{\alpha\beta}\, q_{\alpha\beta}}\,, \qquad (4.80)
\end{aligned}$$

which we wanted to prove.

Now we use (4.78) to show that $\overline{Z(\mu)^n}$ in (4.59) can be put into the form

$$\overline{Z(\mu)^n} = \int \{dq_{\alpha\beta}\}\ \mathrm{e}^{N\Psi(\{q_{\alpha\beta}\},\,\mu)}\,, \qquad (4.81)$$

where

$$\Psi(\{q_{\alpha\beta}\},\mu) = -\frac{1}{4}\sigma^2 \sum_{\alpha,\beta=1}^{n} q_{\alpha\beta}^2 \,+\, \ln Z_s(\mu) \qquad (4.82)$$

$$Z_s(\mu) = \int \Big(\prod_{\alpha=1}^{n} d\Phi^\alpha \Big)\ \mathrm{e}^{-\frac{1}{2}(\epsilon - i\mu)\sum_{\alpha=1}^{n}(\Phi^\alpha)^2 \,-\, \frac{1}{2}i\sigma^2\sum_{\alpha,\beta=1}^{n} q_{\alpha\beta}\,\Phi^\alpha\Phi^\beta}\,. \qquad (4.83)$$

This goes as follows,

$$
\begin{aligned}
\mathrm{e}^{-\frac{1}{4}N\sigma^2\sum_{\alpha\beta}\left(\frac{1}{N}\sum_{i=1}^{N}\Phi_i^\alpha\Phi_i^\beta\right)^2} &= \mathrm{e}^{-\frac{1}{2}\sum_{\alpha\beta}\big(\overbrace{\frac{\sigma}{\sqrt{2N}}\sum_{i=1}^{N}\Phi_i^\alpha\Phi_i^\beta}^{\equiv b^2_{\alpha\beta}}\big)^2} \\
&= \int\{dq_{\alpha\beta}\}\ \mathrm{e}^{-\frac{N\sigma^2}{4}\sum_{\alpha\beta}q^2_{\alpha\beta}\,-\,i\,\sigma\sqrt{\frac{N}{2}}\sum_{\alpha\beta}b_{\alpha\beta}\,q_{\alpha\beta}} \\
&= \int\{dq_{\alpha\beta}\}\ \mathrm{e}^{-\frac{N\sigma^2}{4}\sum_{\alpha\beta}q^2_{\alpha\beta}\,-\,\frac{i\sigma^2}{2}\sum_{\alpha\beta}\sum_i q_{\alpha\beta}\Phi_i^\alpha\Phi_i^\beta}
\end{aligned}
$$

We have now a *linearizatized* form. Using this for (4.59) gives,

$$
\begin{aligned}
\overline{Z(\mu)^n} &= \int\Big(\prod_{i\alpha}d\Phi_i^\alpha\Big)\ \mathrm{e}^{-\frac{1}{2}(\epsilon-i\mu)\sum_{i\alpha}(\Phi_i^\alpha)^2}\int\{dq_{\alpha\beta}\}\ \mathrm{e}^{-\frac{N\sigma^2}{4}\sum_{\alpha\beta}q^2_{\alpha\beta}-\frac{i\sigma^2}{2}\sum_{\alpha\beta}\sum_i q_{\alpha\beta}\Phi_i^\alpha\Phi_i^\beta} \\
&= \int\{dq_{\alpha\beta}\}\ \mathrm{e}^{-\frac{N\sigma^2}{4}\sum_{\alpha\beta}q^2_{\alpha\beta}}\int\Big(\prod_{i\alpha}d\Phi_i^\alpha\Big)\mathrm{e}^{-\frac{1}{2}(\epsilon-i\mu)\sum_{i\alpha}(\Phi_i^\alpha)^2-\frac{i\sigma^2}{2}\sum_{\alpha\beta}\sum_i q_{\alpha\beta}\Phi_i^\alpha\Phi_i^\beta} \\
&= \int\{dq_{\alpha\beta}\}\ \mathrm{e}^{-\frac{N\sigma^2}{4}\sum_{\alpha\beta}q^2_{\alpha\beta}}\prod_i\left(\int\Big(\prod_\alpha d\Phi^\alpha\Big)\mathrm{e}^{-\frac{1}{2}(\epsilon-i\mu)\sum_\alpha(\Phi^\alpha)^2-\frac{i\sigma^2}{2}\sum_{\alpha\beta}q_{\alpha\beta}\Phi^\alpha\Phi^\beta}\right) \\
&= \int\{dq_{\alpha\beta}\}\ \mathrm{e}^{N\left[-\frac{\sigma^2}{4}\sum_{\alpha\beta}q^2_{\alpha\beta}+\ln\int\left(\prod_\alpha d\Phi^\alpha\right)\mathrm{e}^{-\frac{1}{2}(\epsilon-i\mu)\sum_\alpha(\Phi^\alpha)^2-\frac{i\sigma^2}{2}\sum_{\alpha\beta}q_{\alpha\beta}\Phi^\alpha\Phi^\beta}\right]} \\
&= \int\{dq_{\alpha\beta}\}\ \mathrm{e}^{N\left[-\frac{\sigma^2}{4}\sum_{\alpha\beta}q^2_{\alpha\beta}+\ln Z_s(\mu)\right]} \\
&\equiv \int\{dq_{\alpha\beta}\}\ \mathrm{e}^{N\Psi(\{q_{\alpha\beta},\mu)}\,. \\
&\quad \text{with}\quad Z_s(\mu)=\int\big(\prod_\alpha d\Phi^\alpha\big)\mathrm{e}^{-\frac{1}{2}(\epsilon-i\mu)\sum_\alpha(\Phi^\alpha)^2-\frac{i\sigma^2}{2}\sum_{\alpha\beta}q_{\alpha\beta}\Phi^\alpha\Phi^\beta}\,,
\end{aligned}
$$

as we wanted to show.

Next we look at $\{q^{SP}_{\alpha\beta}\}$, denoting the values of $\{q_{\alpha\beta}\}$ evaluated at the relevant saddle point of $\Psi(\{q_{\alpha\beta},\mu)$. The general *self-consistent* expression (i.e. contaning $q^{SP}_{\alpha\beta}$ implicitly) is determined as follows.

Firtsty, the *saddle-point equations* are,

$$\frac{\partial\Psi(\{q_{\alpha\beta}\},\,\mu)}{\partial q_{\alpha\beta}}=0\,.$$

Applying this we get from simple differentiation of (4.82) the self-consitency relation that $q_{\alpha\beta}^{SP}$ has to satisfy,

$$iq_{\alpha\beta}^{SP} \quad = \quad \frac{\int \left(\prod_\alpha d\Phi^\alpha\right) \;\; \Phi^\alpha \Phi^\beta \;\; \mathrm{e}^{-\frac{1}{2}(\epsilon - i\mu)\sum_\alpha (\Phi^\alpha)^2 - \frac{1}{2}\sigma^2 i \sum_{\alpha\beta} q_{\alpha\beta}^{SP} \Phi^\alpha \Phi^\beta}}{\int \left(\prod_\alpha d\Phi^\alpha\right) \;\; \mathrm{e}^{-\frac{1}{2}(\epsilon - i\mu)\sum_\alpha (\Phi^\alpha)^2 - \frac{1}{2}\sigma^2 i \sum_{\alpha\beta} q_{\alpha\beta}^{SP} \Phi^\alpha \Phi^\beta}} \tag{4.84}$$

Finally, we get from (4.59) for $\overline{\rho(\mu)}$,

$$\begin{aligned} \overline{\rho(\mu)} \quad &= \quad \frac{2}{\pi} \lim_{\epsilon \downarrow 0} \; \mathrm{Im} \, \frac{d}{d\mu} \lim_{n \to 0} \lim_{N \to \infty} \frac{1}{nN} \ln\Big(\overline{Z(\mu)^n}\Big) \\ &= \quad \frac{2}{\pi} \lim_{\epsilon \downarrow 0} \; \mathrm{Im} \, \frac{d}{d\mu} \lim_{n \to 0} \lim_{N \to \infty} \frac{1}{nN} \ln\Big(\mathrm{e}^{N\Psi(\{q_{\alpha\beta}^{SP}\}, \, \mu)}\Big) \\ &= \quad \frac{2}{\pi} \lim_{\epsilon \downarrow 0} \; \mathrm{Im} \, \frac{d}{d\mu} \lim_{n \to 0} \frac{1}{n} \, \Psi(\{q_{\alpha\beta}^{SP}\}, \, \mu) \\ &= \quad \frac{2}{\pi} \lim_{\epsilon \downarrow 0} \; \mathrm{Im} \, \frac{d}{d\mu} \lim_{n \to 0} \frac{1}{n} \left(-\frac{1}{4}\sigma^2 \sum_{\alpha,\beta=1}^{n} q_{\alpha\beta}^2 \; + \; \ln Z_s(\mu) \right) \end{aligned} \tag{4.85}$$

With the expression for $Z_s(\mu)$ given in (4.83) and,

$$\begin{aligned} \frac{d}{d\mu}\Psi(\{q_{\alpha\beta}^{SP}\}, \, \mu) \quad &= \quad \frac{\partial}{\partial \mu}\Psi(\{q_{\alpha\beta}^{SP}\}, \, \mu) + \sum_{\alpha\beta} \frac{\partial \Psi}{\partial q_{\alpha\beta}} \frac{\partial q_{\alpha\beta}}{\partial \mu} \\ &= \quad \frac{\partial}{\partial \mu}\Psi(\{q_{\alpha\beta}^{SP}\}, \, \mu) \, , \end{aligned} \tag{4.86}$$

one can check that we get for $\overline{\rho(\mu)}$,

$$\overline{\rho(\mu)} = \lim_{n \to 0} \frac{1}{n\pi} \; \mathrm{Re} \; \sum_\alpha i \, q_{\alpha\alpha}^{SP} \, . \tag{4.87}$$

It is easily checked that $i\, q_{\alpha\alpha}^{SP}$ defined by (4.84) is exactly equal to $q_{\alpha\alpha}^{SP}$ of (4.76), with substitution of $\hat{q}_{\alpha\beta}$ given in (4.76). This must be the case of course. The method of this section should give the same final answer for $\overline{\rho(\mu)}$ as the method of the previous section. So indeed (4.87) matches (4.77), as it should.

In the next section we shall take the so-called *replica-symmetry assumption* (approximation), when choosing a simple structure for $\hat{q}_{\alpha\beta}$. That is we shall assume,

$$q_{\alpha\beta} = Q\delta_{\alpha\beta} + q(1 - \delta_{\alpha\beta}) \tag{4.88}$$

(when using the results of the *previous section*) ,

$$i\, q_{\alpha\beta} = Q\delta_{\alpha\beta} + q(1 - \delta_{\alpha\beta}) \tag{4.89}$$

(when using the results of *this section*),

where Q and q are real numbers that can be found from solving the appropriate saddle-point equations.

4.6 Replica symmetry

4.7 Final result: Proof of Wigner's semi-circular law

4.8 Notes on Gaussian integration

Chapter 5

Tailored random graph ensembles - 1 (handwritten notes available)

5.1 Classification of networks according to their complexity; constraints and ensembles

5.2 Maximum entropy ensembles

5.3 Example, the Erdös-Renyi ensemble

5.4 Constraint degree correlations

Chapter 6

Tailored random graph ensembles - 2 (handwritten notes available)

6.1 Ensemble for constrained degree correlations

6.2 Entropy and complexity

6.3 Entropy for Erdös-Renyi networks

6.4 Entropy for networks with constrained degree distribution

6.5 Computing $\frac{1}{N}\sum_{\mathbf{k}} p(\mathbf{k}) \log \langle \prod_{\mathbf{i}} \delta_{\mathbf{k_i},\mathbf{k_i(c)}} \rangle_{\bar{\mathbf{k}}}$

Chapter 7

Generating functions - random networks with arbitrary degree distributions (handwritten notes available)

7.1 Generating function for the vertex degree distribution

7.2 Excess degree and its distribution q_k

7.3 Powers of generating functions

7.4 Higher order nearest neighbors

7.5 Examples (Poisson, exponential, power-law)

7.6 Giant components

7.7 Average pathlength

Chapter 8

Theory of evolving networks, preferential attachment (handwritten notes available)

8.1 Power-law degree distributions, the Albert-Barabasi model

8.2 Continuum approach

8.3 Master equation approach

8.4 Rate equation approach

8.5 Beyond the Barabasi-Albert model

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