PrSAT: Some Examples

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First, load in the `PrSAT` package

See my **`PrSAT` website** for instructions on downloading and installing **`PrSAT`** (assuming you have *Mathematica* installed).

```
<< PrSAT`
```

Example #1

The first example of a probability model that we saw was the following:

$$\textbf{MODEL1} = \textbf{PrSAT}\left[\left\{\textbf{Pr}[X \wedge Y] == \frac{1}{6}, \textbf{Pr}[X \wedge \neg Y] == \frac{1}{4}, \textbf{Pr}[\neg X \wedge Y] == \frac{1}{8}, \textbf{Pr}[\neg X \wedge \neg Y] == \frac{11}{24}\right\}\right]$$

$$\left\{\{X \to \{a_2, a_4\}, Y \to \{a_3, a_4\}, \Omega \to \{a_1, a_2, a_3, a_4\}\}, \left\{a_1 \to \frac{11}{24}, a_2 \to \frac{1}{4}, a_3 \to \frac{1}{8}, a_4 \to \frac{1}{6}\right\}\right\}$$

`PrSAT` will show us an STT representation of **`MODEL1`**:

```
TruthTable[MODEL1]
```

X	Y	var	Pr
T	T	a_4	$\frac{1}{6}$
T	F	a_2	$\frac{1}{4}$
F	T	a_3	$\frac{1}{8}$
F	F	a_1	$\frac{11}{24}$

We can use **`PrSAT`** to calculate probability, using **`MODEL1`**:

$$\textbf{EvaluateProbability}[\{\textbf{Pr}[X \vee Y], \textbf{Pr}[X], \textbf{Pr}[Y]\}, \textbf{MODEL1}]$$

$$\left\{\frac{13}{24}, \frac{5}{12}, \frac{7}{24}\right\}$$

We can also check arbitrary claims to see if they are *true on* **`MODEL1`**:

$$\textbf{EvaluateProbability}[\textbf{Pr}[X \mid Y] > \textbf{Pr}[X], \textbf{MODEL1}]$$

```
True
```

Example #2

The second example we saw was an algebraic proof of the following theorem:

$$\mathbf{Pr}(X \vee Y) = \mathbf{Pr}(X) + \mathbf{Pr}(Y) - \mathbf{Pr}(X \wedge Y)$$

PrSAT easily verifies this theorem (note: it does not present a readable proof).

```
PrSAT[{Pr[X ∨ Y] ≠ Pr[X] + Pr[Y] - Pr[X ∧ Y]}]
```

PrSAT::srchfail : Search phase failed; attempting FindInstance

{}

This output means there are no probability models on which **Pr[X ∨ Y] ≠ Pr[X] + Pr[Y] - Pr[X ∧ Y]**. That "proves" that the above statement is a theorem of probability calculus.

Example #3

The second example we saw was an algebraic proof of the following theorem:

$$\mathbf{Pr}(X) = \mathbf{Pr}(X \wedge Y) + \mathbf{Pr}(X \wedge \neg Y)$$

PrSAT easily verifies this theorem (note: it does not present a readable proof).

```
PrSAT[{Pr[X] ≠ Pr[X ∧ Y] + Pr[X ∧ ¬ Y]}]
```

PrSAT::srchfail : Search phase failed; attempting FindInstance

{}

This output means there are no probability models on which **Pr[X] ≠ Pr[X ∧ Y] + Pr[X ∧ ¬ Y]**. That "proves" that the above statement is a theorem of probability calculus.

Example #4

The next example involves the following theorem:

$$\mathbf{Pr}(X \to Y) \geq \mathbf{Pr}(Y \mid X)$$

PrSAT easily verifies this theorem (note: it does not present a readable proof). First, we need to define the conditional operator.

```
X_ → Y_ := ¬ X ∨ Y;
PrSAT[{Pr[X → Y] < Pr[Y | X]}]
```

PrSAT::srchfail : Search phase failed; attempting FindInstance

{}

This output means there are no probability models on which **Pr[X → Y] ≥ Pr[Y | X]**. That "proves" that the above statement is a theorem of probability calculus.

Example #5

The next example involves the fact that the following is NOT a theorem:

$$\mathbf{Pr}(X \mid Y \vee Z) = \mathbf{Pr}(X \mid Y \wedge Z)$$

PrSAT easily finds a counter-model to this claim.

```
PrSAT[{Pr[X | Y ∨ Z] ≠ Pr[X | Y ∧ Z]}]
```

$$\left\{\{X \to \{a_2, a_5, a_6, a_8\}, Y \to \{a_3, a_5, a_7, a_8\}, Z \to \{a_4, a_6, a_7, a_8\}, \Omega \to \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}\}, \left\{a_1 \to 0, a_2 \to 0, a_3 \to \frac{1}{28}, a_4 \to \frac{13}{28}, a_5 \to 0, a_6 \to 0, a_7 \to 0, a_8 \to \frac{1}{2}\right\}\right\}$$

The model **PrSAT** finds by defualt is *non*-regular. We can force it to find a *regular* counter-model, as follows:

```
MODEL2 = PrSAT[{Pr[X | Y ∨ Z] ≠ Pr[X | Y ∧ Z]}, Probabilities → Regular]
```

$$\left\{\{X \to \{a_2, a_5, a_6, a_8\}, Y \to \{a_3, a_5, a_7, a_8\}, Z \to \{a_4, a_6, a_7, a_8\}, \Omega \to \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}\}, \left\{a_1 \to \frac{2959}{5\,227\,434}, a_2 \to \frac{1}{999}, a_3 \to \frac{1}{999}, a_4 \to \frac{1}{999}, a_5 \to \frac{8}{27}, a_6 \to \frac{19}{94}, a_7 \to \frac{83}{167}, a_8 \to \frac{1}{999}\right\}\right\}$$

Here is an STT representation of **MODEL2**:

```
TruthTable[MODEL2]
```

X	Y	Z	var	Pr
T	T	T	a_8	$\frac{1}{999}$
T	T	F	a_5	$\frac{8}{27}$
T	F	T	a_6	$\frac{19}{94}$
T	F	F	a_2	$\frac{1}{999}$
F	T	T	a_7	$\frac{83}{167}$
F	T	F	a_3	$\frac{1}{999}$
F	F	T	a_4	$\frac{1}{999}$
F	F	F	a_1	$\frac{2959}{5\,227\,434}$

We can calculate the values of **Pr[X | Y ∧ Z]**, **Pr[X | Y ∨ Z]** on this model as follows:

```
EvaluateProbability[{Pr[X | Y ∧ Z], Pr[X | Y ∨ Z]}, MODEL2]
```

$$\left\{\frac{167}{83\,084}, \frac{7\,832\,133}{15\,657\,727}\right\}$$

We can look at decimal representations of these exact real numbers, as follows:

```
% // N
```

{0.00201001, 0.500209}

We gave a different model in the lecture notes. We can enter that model in by hand, and then verify it has the desired properties, as follows:

MODEL3 = **PrSAT**$\left[\left\{\Pr[X\wedge Y\wedge Z]==\frac{1}{6},\ \Pr[X\wedge Y\wedge\neg Z]==\frac{1}{6},\ \Pr[X\wedge\neg Y\wedge Z]==\frac{1}{4},\ \Pr[X\wedge\neg Y\wedge\neg Z]==\frac{1}{16},\ \Pr[\neg X\wedge Y\wedge Z]==\frac{1}{6},\ \Pr[\neg X\wedge Y\wedge\neg Z]==\frac{1}{12},\ \Pr[\neg X\wedge\neg Y\wedge Z]==\frac{1}{24},\ \Pr[\neg X\wedge\neg Y\wedge\neg Z]==\frac{1}{16}\right\}\right]$

$\{\{X\to\{a_2, a_5, a_6, a_8\},\ Y\to\{a_3, a_5, a_7, a_8\},\ Z\to\{a_4, a_6, a_7, a_8\},\ \Omega\to\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}\},$
$\{a_1\to\frac{1}{16},\ a_2\to\frac{1}{16},\ a_3\to\frac{1}{12},\ a_4\to\frac{1}{24},\ a_5\to\frac{1}{6},\ a_6\to\frac{1}{4},\ a_7\to\frac{1}{6},\ a_8\to\frac{1}{6}\}\}$

TruthTable[**MODEL3**]

X	Y	Z	var	Pr
T	T	T	a_8	$\frac{1}{6}$
T	T	F	a_5	$\frac{1}{6}$
T	F	T	a_6	$\frac{1}{4}$
T	F	F	a_2	$\frac{1}{16}$
F	T	T	a_7	$\frac{1}{6}$
F	T	F	a_3	$\frac{1}{12}$
F	F	T	a_4	$\frac{1}{24}$
F	F	F	a_1	$\frac{1}{16}$

EvaluateProbability[{**Pr**[**X** | **Y** ∧ **Z**], **Pr**[**X** | **Y** ∨ **Z**]}, **MODEL3**]

$\left\{\frac{1}{2}, \frac{2}{3}\right\}$

We can also see the algebraic form of an expression, as follows:

AlgebraicForm[**Pr**[**X** | **Y** ∧ **Z**] == **Pr**[**X** | **Y** ∨ **Z**], {**X**, **Y**, **Z**}]

$$\frac{a_8}{a_7 + a_8} == \frac{a_5 + a_6 + a_8}{a_3 + a_4 + a_5 + a_6 + a_7 + a_8}$$

Note that **PrSAT** uses different conventions (*i.e.*, a different ordering in the truth-table) for the a_i than I use in the lecture notes.