PHIL 424: Practice Mid-Term *Solutions*

October 18, 2014

The actual mid-term eaxm will have the same structure as this practice mid-term. You will have the full class period on Tuesday (10/21) to complete the actual exam. You may use a calculator.

1 Proving A Probability Theorem Algebraically

Prove the following theorem, by (a) translating it into algebra (using the stochastic truth-table below), and then (b) showing that the resulting algebraic inequality must be true (assuming, as always, that a, b, c, d are each on $[0, 1]$ and that they sum to one).

> **Theorem 1.** $\mathrm{cr}(X \supset Y) \geq \mathrm{cr}(Y \mid X)$.

Please use the following stochastic truth-table to prove **Theorem 1**.

X	Y	$\mathrm{cr}(\cdot)$
T	**T**	a
T	**F**	b
F	**T**	c
F	**F**	d

Solution. Suppose **Theorem 1** is *false*. That is, suppose $\mathrm{cr}(X \supset Y) < \mathrm{cr}(Y \mid X)$. We can then reason to an absurdity/contradiction, as follows.

$$\begin{aligned}
\mathrm{cr}(X \supset Y) = 1 - b &< \frac{a}{a+b} = \mathrm{cr}(Y \mid X) \\
(a+b)(1-b) &< a \\
a - ab + b - b^2 &< a \\
ab + b - b^2 &< 0 \\
b(a + (1-b)) &< 0
\end{aligned}$$

Because b, a, and $1 - b$ are all *probabilities*, they must all be *non-negative*. Hence, $b(a + (1 - b))$ must be *non-negative* (*i.e.*, we must have $b(a + (1 - b)) \geq 0$). So, our assumption that **Theorem 1** is *false* has led to a contradiction. Therefore, **Theorem 1** must be true.

2 Finding a Probability Distribution

Find a probability distribution (*i.e.*, an assignment of numbers to a, b, c, d, which are each on $[0, 1]$ and which sum to one — as in the above stochastic truth-table) which satisfies the following three constraints. Explain how you found the solution, and why it is correct.

1. $\mathrm{cr}(X \supset Y) = \mathrm{cr}(Y \mid X)$.
2. $\mathrm{cr}(X) = 1/2$.
3. $\mathrm{cr}(Y) = 5/8$.

Solution. There is a *unique* solution to these three equations. This solution can be found by reasoning as follows. First, from (1), we know that:

$$\mathrm{cr}(X \supset Y) = 1 - b = \frac{a}{a+b} = \mathrm{cr}(Y \mid X)$$

Expanding and simplifying (1) yields:

$$\begin{aligned}(a+b)(1-b) &= a\\ a - ab + b - b^2 &= a\\ b(a + (1-b)) &= 0\end{aligned}$$

From (2), we know that $a + b = 1/2$. Hence, b must be between zero and $1/2$. So, if $b > 0$, then $1 - b > 0$ and $b(a + (1 - b)) > 0$. As a result, it follows from (1) that b must be equal to zero. Therefore, $a = 1/2$. Finally, from (3), we know that $a + c = 5/8$. Therefore, $c = 1/8$. This yields the following *unique* probability distribution (since $d = 1 - (a + b + c)$).

X	Y	$\mathrm{cr}(\cdot)$
T	**T**	$1/2$
T	**F**	0
F	**T**	$1/8$
F	**F**	$3/8$

3 Verifying Properties of a Probability Distribtion

Here is a (stochastic truth-table representation of a) probability distribution over the algebra generated by the three atomic sentences H, E, K.

H	E	K	$\mathrm{cr}(\cdot)$
T	**T**	**T**	$a \coloneqq 49/256$
T	**T**	**F**	$b \coloneqq 1/16$
T	**F**	**T**	$c \coloneqq 31/256$
T	**F**	**F**	$d \coloneqq 1/8$
F	**T**	**T**	$e \coloneqq 31/256$
F	**T**	**F**	$f \coloneqq 1/8$
F	**F**	**T**	$g \coloneqq 17/256$
F	**F**	**F**	$h \coloneqq 3/16$

Use this table to verify the following three claims about this distribution.[1]

1. $\mathrm{cr}(H \mid E) > \mathrm{cr}(H)$.
2. $\mathrm{cr}(H \mid E \,\&\, K) < \mathrm{cr}(H \mid K)$.
3. $\mathrm{cr}(H \mid E \,\&\, {\sim}K) < \mathrm{cr}(H \mid {\sim}K)$.

Solution. This is a straightforward *plug-and-chug* problem. So, I will just do the first one.

$$\begin{aligned}\mathrm{cr}(H \mid E) = \frac{a+b}{a+b+e+f} &= \frac{49/256 + 1/16}{49/256 + 1/16 + 31/256 + 1/8} \\ &= \frac{65}{128} \\ &> \frac{64}{128} = 49/256 + 1/16 + 31/256 + 1/8 = a+b+c+d = \mathrm{cr}(H)\end{aligned}$$

4 Proving Another Probability Theorem Algebraically

Prove the following theorem, by (a) translating it into algebra (using the stochastic truth-table below), and then (b) showing that the resulting algebraic statement must be true (assuming, as always, that a, b, c, d, e, f, g, h are each on $[0, 1]$ and that they sum to one).

Theorem 2. If $\mathrm{cr}(X \mid Y \,\&\, Z) = 1$, then $\mathrm{cr}(X \mid Y) \geq \mathrm{cr}(Z \mid Y)$.

Please use the following stochastic truth-table to prove **Theorem 2**.

X	Y	Z	$\mathrm{cr}(\cdot)$
T	**T**	**T**	a
T	**T**	**F**	b
T	**F**	**T**	c
T	**F**	**F**	d
F	**T**	**T**	e
F	**T**	**F**	f
F	**F**	**T**	g
F	**F**	**F**	h

Solution. This one looks complicated, but it's actually very simple. First, we translate **Theorem 2** into algebra.

Theorem 2 (algebraically). If $\frac{a}{a+e} = 1$, then $\frac{a+b}{a+b+e+f} \geq \frac{a+e}{a+b+e+f}$.

[1] This is a case in which (1) E is positiviely relevant to H, *unconditionally*; but, (2) E is negatively relevant to H, *conditional upon* K and (3) E is negatively relevant to H, *conditional upon* ${\sim}K$. What is this kind of case called? Hint: it's got "paradox" in the name. **Solution**: This is an instance of *Simpson's Paradox*.

Proof. If $\dfrac{a}{a+e} = 1$, then $a + e = a$, which implies that $e = 0$. And, if $e = 0$, then

$$\frac{a+b}{a+b+e+f} = \frac{a+b}{a+b+f} \text{ and } \frac{a+e}{a+b+e+f} = \frac{a}{a+b+f}$$

Finally, it is easy to see that

$$\frac{a+b}{a+b+f} \geq \frac{a}{a+b+f}$$

because $a + b \geq a$. And, this establishes the algebraic rendition of **Theorem 2**.