Stalnaker and Lance on Import-Export (If-And)

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Stalnaker and Lance (S&L) give what they claim to be a counterexample to the probabilistic import-export law (what Bennett calls "If-And"). This is the principle which says that

$$\Pr(A \to (B \to C)) = \Pr((A \,\&\, B) \to C)$$

Note: given The Equation (which is *assumed* in this context), this is equivalent to

$$\Pr(C \mid A \,\&\, B) = \Pr(A \to C \mid B)$$

Bennett discusses the example of Stalnaker and Lance in §40. I think his response is inadequate, and I also think the analyses of S&L and Bennett are incorrect. However, I think there is a way to bolster the argument of S&L (for a slightly different conclusion). Below, I will present their analyses, along with my commentary/alternative analysis.

The Example: There are 2 urns containing 100 balls each with the following constitutions

Urn 1: 90 red iron balls and 10 green copper balls
Urn 2: 90 green iron balls and 10 red copper balls

Agnes picks an urn at random, draws a ball (a) out, and returns the ball without noting its color or composition. Then she draws a second ball (b) from the same urn. Let

Ra: the first ball she picked was red.
Gb: the second ball she picked was green.
Cb: the second ball she picked was copper.

Bennett says that (assuming The Equation) our probability for $(Ra \,\&\, Gb) \to Cb$ should be 0.1 "because adding the supposition of $Ra \,\&\, Gb$ to our stock of beliefs doesnt make either urn likelier than the other to be the one Agnes drew both balls from. And 1/10 of all the balls are copper (in both urns combined)." I think this reasoning (also made by S&L) is incorrect. More rigorously, letting U_1 = the urn she drew was urn 1, and U_2 = the urn she drew was urn 2, we have the following, by The Equation (Bennettian Ramsey Test)

$$\Pr((Ra \,\&\, Gb) \to Cb) = \Pr(Cb \mid Ra \,\&\, Gb)$$

which (since U_1 and U_2 are m.e.e) by the Law of Total Probability expands to

$$\Pr(Cb \mid Ra \,\&\, Gb \,\&\, U_1) \cdot \Pr(U_1 \mid Ra \,\&\, Gb) + \Pr(Cb \mid Ra \,\&\, Gb \,\&\, U_2) \cdot \Pr(U_2 \mid Ra \,\&\, Gb)$$

Now, we know that $\Pr(Cb \mid Ra \,\&\, Gb \,\&\, U_1) = 1$ and $\Pr(Cb \mid Ra \,\&\, Gb \,\&\, U_2) = 0$, since U_1 entails $(\forall x)(Gx \supset Cx)$, and U_2 entails $(\forall x)(Gx \supset \sim Cx)$. Combining these results yields

$$\Pr((Ra \,\&\, Gb) \to Cb) = \Pr(U_1 \mid Ra \,\&\, Gb)$$

Moreover, given standard assumptions about random sampling (with replacement) and Bayes's Theorem, we can calculate $\Pr(U_1 \mid Ra \,\&\, Gb)$, as follows

$$\Pr(U_1 \mid Ra \& Gb) = \frac{\Pr(Ra \,\&\, Gb \mid U_1) \cdot \Pr(U_1)}{\Pr(Ra \,\&\, Gb \mid U_1) \cdot \Pr(U_1) + \Pr(Ra \,\&\, Gb \mid U_2) \cdot \Pr(U_2)} = \frac{(\frac{9}{10} \cdot \frac{1}{10}) \cdot \frac{1}{2}}{(\frac{9}{10} \cdot \frac{1}{10}) \cdot \frac{1}{2} + (\frac{1}{10} \cdot \frac{9}{10}) \cdot \frac{1}{2}}$$

Note: this equals $\frac{1}{2}$ (*not* $\frac{1}{10}$). Next, let's see what S&L and B say about $\Pr(Ra \to (Gb \to Cb))$.

S&L argue that the probability of $Ra \to (Gb \to Cb)$ should be 0.9. "Because when we add Ra to our stock of beliefs, our probability that she drew from Urn 1 becomes 0.9. Under the supposition she picked Urn 1, our probability for $Gb \to Cb$ becomes 1, since all green balls are copper in Urn 1. Under the supposition she picked Urn 2, our probability for $Gb \to Cb$ becomes 0, because no green copper balls there. Our probability for $Ra \to (Gb \to Cb)$ should then be $0.9 \cdot 1 + 0.1 \cdot 0$, which equals 0.9." Again, this is rather sloppy probability reasoning. But, there is some truth to their conclusions here. Let me explain.

First, by The Equation (alone), what we know is

$$\Pr(Ra \to (Gb \to Cb)) = \Pr(Gb \to Cb \mid Ra)$$

which, by the Law of Total Probability, expands to

$$\Pr(Gb \to Cb \mid Ra \,\&\, U_1) \cdot \Pr(U_1 \mid Ra) + \Pr(Gb \to Cb \mid Ra \,\&\, U_2) \cdot \Pr(U_2 \mid Ra)$$

At this stage, S&L claim that $\Pr(Gb \to Cb \mid Ra \& U_1) = 1$ and $\Pr(Gb \to Cb \mid Ra \& U_2) = 0$. But, this does not follow from the probability calculus + The Equation. Here, S&L assume that $\Pr(Gb \to Cb \mid Ra \,\&\, (\forall x)(Gx \supset Cx)) = 1$, and $\Pr(Gb \to Cb \mid Ra \,\&\, (\forall x)(Gx \supset {\sim}Cx)) = 0$. But, surely, *they* don't think this is true! What's true (as above) is that $\Pr(Cb \mid Gb \& Ra \& (\forall x)(Gx \supset Cx)) = 1$, and $\Pr(Cb \mid Gb \,\&\, Ra \,\&\, (\forall x)(Gx \supset {\sim}Cx)) = 0$. So, what do they have in mind?

[Note: Bennett's "response" to S&L seems very weak to me (I don't really even follow it).]

Note: S&L's claim that $\Pr(U_1 \mid Ra) = 0.9$ [$\Pr(U_2 \mid Ra) = 0.1$] is correct. Bayes's Theorem:

$$\Pr(U_1 \mid Ra) = \frac{\Pr(Ra \mid U_1) \cdot \Pr(U_1)}{\Pr(Ra \mid U_1) \cdot \Pr(U_1) + \Pr(Ra \mid U_2) \cdot \Pr(U_2)} = \frac{\frac{9}{10} \cdot \frac{1}{2}}{\frac{9}{10} \cdot \frac{1}{2} + \frac{1}{10} \cdot \frac{1}{2}} = \frac{9}{10}$$

But, their argument for $\Pr(Ra \to (Gb \to Cb)) = 0.9 \neq \Pr((Ra \,\&\, Gb) \to Cb) = 0.1$ is incorrect. They *should* argue for $\Pr(Ra \to (Gb \to Cb)) = 0.9 \neq \Pr((Ra \,\&\, Gb) \to Cb) = 0.5$. But, *even that* just doesn't follow from The Equation (plus the probability calculus) *alone*. They never establish $\Pr(Gb \to Cb \mid Ra \,\&\, U_1) = 1$ and $\Pr(Gb \to Cb \mid Ra \,\&\, U_2) = 0$, which they seem to need for their argument to go through (as they and Bennett construe the argument).

Alternative S&L-Style Argument: Assume, for *reductio*, that *both* The Equation *and* "If-And" are true. Then, we *can* (rigorously!) infer *both* that $\Pr((Ra \,\&\, Gb) \to Cb) = \frac{1}{2}$ (first derivation above), *and* that $\Pr((Ra \,\&\, Gb) \to Cb) = \frac{9}{10}$.[1] This is absurd. So, *at least one of The Equation and "If-And" must be false*. I think this is a knock-down argument against The Equation + "If-And". And, it seems to me, this gets S&L exactly what they wanted. □

[1] "If-And" and The Equation jointly entail $\Pr(Gb \to Cb \mid Ra \,\&\, U_1) = \Pr(Cb \mid Gb \,\&\, Ra \,\&\, U_1) = 1$, and, hence, that $\Pr((Ra \,\&\, Gb) \to Cb) = \Pr(Ra \to (Gb \to Cb)) = \frac{9}{10}$. This fills the gap in the second derivation, above.