Gödel's Metatheorem (45.17) and the Strong Completeness Theorem for FOTs (46.2)

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Before getting to the salient proofs, it's important to understand Hunter's terminology "consistent set Γ of WFFs of a first order theory K". For Hunter, Γ is a consistent set of WFFs of K iff there is no WFF A of K such that $\Gamma \vdash_K A$ and $\Gamma \vdash_K \sim A$. As a result, this definition of "consistent set of WFFs Γ of K" implies that K is itself a consistent first-order theory! That is, an inconsistent first order theory K does not have any consistent sets of WFFs on this definition. This sounds a bit odd, but it's crucial for the proofs below. In this handout, I will go through the proper proofs of 45.17 and 46.2. To this end, I will begin with the background ingredients of the proof of 45.17: metatheorem 45.16, and Lemmas 1 and 2.

45.16. If Γ is a consistent set of *closed* WFFs of a first order theory K, then Γ has a denumerable model.

Proof. Assume Γ is a consistent set of *closed* WFFs of a first order theory K. Then, by Hunter's definition (above), there is no WFF A of K such that $\Gamma \vdash_K A$ and $\Gamma \vdash_K \sim A$. Therefore, it follows that the first order theory $K+\Gamma$ is a consistent first order theory. If $K+\Gamma$ were inconsistent, then there would have to be a WFF A of $K+\Gamma$ such that both A and $\sim A$ were theorems of $K+\Gamma$. That would imply the existence of an A such that $\Gamma \vdash_K A$ and $\Gamma \vdash_K \sim A$, which contradicts Hunter's definition of "consistent set of WFFs Γ of K." Since $K+\Gamma$ is a consistent first order theory, it must have a denumerable model [this is implied by theorems 45.10–45.14]. Thus, Γ itself has a denumerable model (Γ is a subset of the set of theorems of $K+\Gamma$). □

Lemma 1 for 45.17. If Γ is a consistent set of WFFs of a first order theory K, then Γ is also a consistent set of WFFs of of the first order theory K', where K' is the first order theory one gets when one adds denumerably many new constant symbols with an effective enumeration $\langle c_1, \ldots c_n, \ldots\rangle$ to K.

Proof. Assume Γ is a consistent set of WFFs of a first order theory K, and assume that K' is K with the new constant symbols $\langle c_1, \ldots c_n, \ldots\rangle$ added to it. Now, assume, for *reductio*, that Γ is an inconsistent set of K'. Then, by definition, this means that there is a WFF B of K' such that $\Gamma \vdash_{K'} B$ and $\Gamma \vdash_{K'} \sim B$. So, since derivations are finite, there is a finite subset $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{K'} B$ and $\Delta \vdash_{K'} \sim B$. These derivations in K' can be converted into derivations *in* K, as follows. Let $X' = Xv_i/c_i$, where v_i is the ith variable in our enumeration that does not occur in either of the derivations $\Delta \vdash_{K'} B$ or $\Delta \vdash_{K'} \sim B$, and c_i is the ith constant symbol in our enumeration of new symbols added to K to yield K'. Then, $\Delta' = \Delta$, since $\Delta \subseteq \Gamma$ and Γ is a set of WFFs of K (and so do not contain any c_i's). Moreover, $\Delta \vdash_K B'$ and $\Delta \vdash_K \sim B'$. Why? Think about $\Delta \vdash_{K'} B$ (parallel argument for $\sim B$). There are four possibilities: (i) B is a logical axiom of K', (ii) B is a proper axiom of K', (iii) B follows by MP from two WFFs in Δ, or (iv) $B \in \Delta$. If B is a logical axiom of K' [case (i)], then B' is a logical axiom of K. There are five sub-cases here (this is just like part I of 45.12):

(i.1) B is a logical axiom by K1–K3. Then, B is an instance of a propositional axiom schemata. In this case, $B' = Bv_i/c_i$ is also an instance of a propositional axiom schemata, since merely substituting v_i for c_i in B cannot change the propositional logical form of any formula B.

(i.2) B is a logical axiom by K4. So, B is of the form $\wedge uC \supset Ct/u$, where t is free for u in C. Thus, $B' = (\wedge uC \supset Ct/u)v_i/c_i = (\wedge uCv_i/c_i \supset (Ct/u)v_i/c_i) = (\wedge uCv_i/c_i \supset (Cv_i/c_i)t/u)$, where t is free for u in C. This is because v_i does not occur in B (hence, v_i does not occur in C, Ct/u, or t), and so $(Ct/u)v_i/c_i = (Cv_i/c_i)t/u$. Therefore, B' is of the form $\wedge uD \supset Dt/u$ (with $D = Cv_i/c_i$), where t is free for u in D. So, B' is also a logical axiom by K4.

(i.3) B is a logical axiom by K5. So, B is of the form $C \supset \wedge uC$, where u does not occur free in C. Thus, $B' = (C \supset \wedge uC)v_i/c_i = (Cv_i/c_i \supset \wedge uCv_i/c_i)$. Therefore, B' is of the form $D \supset \wedge uD$ (with $D = Cv_i/c_i$), where u does not occur free in D. So, B' is also a logical axiom by K5.

(i.4) B is a logical axiom by K6. So, B is of the form $\wedge u(C \supset D) \supset (\wedge uC \supset \wedge uD)$. As a result, $B' = (\wedge u(C \supset D) \supset (\wedge uC \supset \wedge uD))v_i/c_i = \wedge u(Cv_i/c_i \supset Dv_i/c_i) \supset (\wedge uCv_i/c_i \supset \wedge uDv_i/c_i)$, which is also a logical axiom by K6.

(i.5) B is a logical axiom by K7. So, B is of the form $\wedge uC$, where C is a logical axiom by K1–K6. Thus, $B' = \wedge uCv_i/c_i$, where Cv_i/c_i is a logical axiom by the above six arguments (which show that if C is an axiom by K1–K6, then so is $C' = Cv_i/c_i$). Hence, by K7, B' is also a logical axiom.

In the other three cases (ii–iv), $B' = B$, and this is why $\Delta \vdash_{K'} B \Rightarrow \Delta \vdash_K B'$ in those cases:

(ii) If B is a proper axiom of K', then B' is a proper axiom of K. For, if B is a proper axiom of K', then $B' = Bv_i/c_i = B$, since the new c_i's cannot occur in any proper axiom of K (so "replacing" c_i with v_i in B does nothing to B). So, in this case, $\Delta \vdash_{K'} B \Rightarrow \Delta \vdash_K B'$.

(iii) If B follows by MP from two WFFs in Δ, then so does B', since (again) $B' = B$. This is because the WFFs in Δ (WFFs of K) do not contain any of the new c_i, and therefore anything that follows by MP from formulas in Δ cannot contain any of the new c_i either. So, $B' = B$, and $\Delta \vdash_{K'} B \Rightarrow \Delta \vdash_K B'$.

(iv) Finally, if $B \in \Delta$, then B is already a WFF of K, and so (again) B cannot contain any of the new c_i, and once again $B' = B$, which ensures that $\Delta \vdash_{K'} B \Rightarrow \Delta \vdash_K B'$ in this case.

A parallel argument shows that $\Delta \vdash_{K'} {\sim}B \Rightarrow \Delta \vdash_K {\sim}B'$. So, if Γ is an inconsistent set of K', then Γ is an inconsistent set of K. That is, if Γ is a consistent set of K, then Γ is also a consistemt set of K'. □

Lemma 2 for 45.17. Let A be a WFF of K in which v occurs free, and let c be a constant not occurring in A or in any proper axiom of K. Then, if $\vdash_K Ac/v$, then $\vdash_K A$.

Proof. We already proved this lemma (basically) in the course of proving 45.12. Here goes. Since $\vdash_K Ac/v$, we know there is a proof $\langle B_1, \dots, B_m\rangle$ of Ac/v in K. Let u be any variable that does not occur in this proof. As a result, note that u does not occur in A, unless $u = v$, since u does not occur in Ac/v. Let B_i' be the result of substituting u for c in B_i. That is, $B_i' = B_iu/c$. Now, $\langle B_1', \dots, B_m'\rangle$ is also proof of in K. To see this, note that each B_i is either a logical axiom, a proper axiom, or an immediate consequence by modus ponens (MP) from two previous lines, and that this is also true for each B_i'. There are three cases:

I. If B_i is a logical axiom, then so is B_i'. There are five sub-cases here [similar to (i.1)–(i.5), above]:

(a) B_i is a logical axiom by K1–K3. Then, B_i is an instance of a propositional axiom schemata. In this case, $B_i' = B_iu/c$ is also an instance of a propositional axiom schemata, since merely substituting u for c in B_i cannot change the propositional logical form of any formula B_i.

(b) B_i is a logical axiom by K4. So, B_i is of the form $\wedge vC \supset Ct/v$, where t is free for v in C. Thus, $B_i' = (\wedge vC \supset Ct/v)u/c = (\wedge vCu/c \supset (Ct/v)u/c) = (\wedge vCu/c \supset (Cu/c)t/v)$, where t is free for v in C. This is because u does not occur in B_i (hence, u does not occur in C, Ct/v, or t), and so $(Ct/v)u/c = (Cu/c)t/v$. Therefore, B_i' is of the form $\wedge vD \supset Dt/v$ (with $D = Cu/c$), where t is free for v in D. So, B_i' is also a logical axiom by K4.

(c) B_i is a logical axiom by K5. So, B_i is of the form $C \supset \wedge vC$, where v does not occur free in C. Thus, $B_i' = (C \supset \wedge vC)u/c = (Cu/c \supset \wedge vCu/c)$. Therefore, B_i' is of the form $D \supset \wedge vD$ (with $D = Cu/c$), where v does not occur free in D. So, B_i' is also a logical axiom by K5.

(d) B_i is a logical axiom by K6. So, B_i is of the form $\wedge v(C \supset D) \supset (\wedge vC \supset \wedge vD)$. Thus, $B_i' = (\wedge v(C \supset D) \supset (\wedge vC \supset \wedge vD))u/c = \wedge v(Cu/c \supset Du/c) \supset (\wedge vCu/c \supset \wedge vDu/c)$, which is also a logical axiom by K6.

(e) B_i is a logical axiom by K7. So, B_i is of the form $\wedge vC$, where C is a logical axiom by K1–K6. Thus, $B_i' = \wedge vCu/c$, where Cu/c is a logical axiom by the above six arguments (which show that if C is an axiom by K1–K6, then so is $C' = Cu/c$). Hence, by K7, B_i' is also a logical axiom.

II. If B_i is a proper axiom of K, then so is B_i'. For, if B_i is a proper axiom, then $B_i' = B_i u/c = B_i$, since c does not occur in any proper axiom of K (so "replacing" c with u in B_i does nothing to B_i).

III. If B_i is an immediate consequence by MP of two previous lines in $\langle B_1, \ldots, B_m \rangle$, then B_i' is an immediate consequence by MP of two previous lines in $\langle B_1', \ldots, B_m' \rangle$. Assume B_i is an immediate consequence by MP of two previous lines B_j and $B_j \supset B_i$. Then, $(B_j \supset B_i)' = (B_j' \supset B_i')$, and B_i' will be an immediate consequence by MP of B_j' and $(B_j' \supset B_i') = (B_j \supset B_i)'$.

1. $\vdash_K Ac/v$ [Assumption of Lemma 2]
2. $\vdash_K (Ac/v)u/c$ [(1), our proof above, and $(Ac/v)' = (Ac/v)u/c$]
3. $\vdash_K Au/v$ [(2), c does not occur in A]
4. $\vdash_K \wedge u Au/v$ [(3), metatheorem 45.4]
5. $\vdash_K \wedge u Au/v \supset (Au/v)v/u$ [Axiom K4, v is free for u in Au/v since either $u \notin A$ or $u = v$]
6. $\vdash_K \wedge u Au/v \supset A$ [(5), $(Au/v)v/u = A$]
7. $\vdash_K A$ [(4), (6), MP]

That completes the proof of Lemma 2. Now, we're ready to prove Gödel's metatheorem 45.17. □

45.17 (Gödel, 1930). Let Γ be a consistent set of WFFs of a first order theory K. Then, Γ is simultaneously satisfiable in a denumerable domain. That is, there is a denumerable interpretation I and a denumerable sequence s of elements of the domain of I such that s satisfies all members of Γ.

Proof. Let $\Gamma = \{A_1, A_2, \ldots\}$ be a consistent set of WFFs of a first order theory K. Let $\langle v_1, v_2, \ldots \rangle$ be an enumeration of the variables that occur free in Γ. Let K' be K plus denumerably many new constant symbols with an effective enumeration $\langle c_1, c_2, \ldots \rangle$. And, let $\Gamma' = \Gamma c_i/v_i = \{A_1', A_2', \ldots\} = \{A_1 c_i/v_i, A_2 c_i/v_i, \ldots\}$. Now, we'll construct a denumerable interpretation I and a denumerable sequence s of elements of I's domain such that s satisfies all members of Γ' and all members of Γ. The interpretation will be a denumerable *model M of* Γ'. That Γ' *has* a denumerable model is guaranteed by 45.16 (proved above), and the fact that:

- Γ' is a *consistent* set of *closed* WFFs of a first order theory K'.

  Proof. It is obvious that Γ' is a set of *closed* formulas of K'. What we need to prove that Γ' is a *consistent* set of WFFs of K'. Assume, for *reductio*, that Γ' is an inconsistent set of WFFs of K'. Then, there exists a finite subset $\{A_1', \ldots, A_n'\} \subseteq \Gamma'$ such that $\{A_1', \ldots, A_n'\} \vdash_{K'} B$ and $\{A_1', \ldots, A_n'\} \vdash_{K'} \sim B$, for some WFF B of K'. So, applying the Deduction Theorem for K' n times (twice) yields:

  (8) $$\vdash_{K'} (A_1' \supset (\ldots (A_n' \supset B) \ldots))$$
  (9) $$\vdash_{K'} (A_1' \supset (\ldots (A_n' \supset \sim B) \ldots))$$

  Let $X^* = X v_i/c_i$, and recall $X' = X c_i/v_i$. So, $X'^* = (X c_i/v_i) v_i/c_i = X^{*\prime} = (X v_i/c_i) c_i/v_i = X$. Thus, we can re-write (8) and (9) in the following form, by re-writing B as $B^{*\prime}$:

  (8) $$\vdash_{K'} (A_1' \supset (\ldots (A_n' \supset B^{*\prime}) \ldots))$$
  (9) $$\vdash_{K'} (A_1' \supset (\ldots (A_n' \supset \sim B^{*\prime}) \ldots))$$

  Now, letting $C = B^*$, we see that (8) and (9) are really of the following form:

  (8) $$\vdash_{K'} (A_1' \supset (\ldots (A_n' \supset C') \ldots))$$
  (9) $$\vdash_{K'} (A_1' \supset (\ldots (A_n' \supset \sim C') \ldots))$$

But, since $X' \supset Y' = (X \supset Y)' = (X \supset Y)c_i/v_i$, we can write (8), (9) in even more perspicuous form:

$$\text{(8)} \qquad \vdash_{K'} (A_1 \supset (\ldots (A_n \supset C)\ldots))c_i/v_i$$
$$\text{(9)} \qquad \vdash_{K'} (A_1 \supset (\ldots (A_n \supset {\sim}C)\ldots))c_i/v_i$$

Now, we can apply Lemma 2 to (8) and (9) in this perspicuous form, and infer:

$$\text{(10)} \qquad \vdash_{K'} (A_1 \supset (\ldots (A_n \supset C)\ldots))$$
$$\text{(11)} \qquad \vdash_{K'} (A_1 \supset (\ldots (A_n \supset {\sim}C)\ldots))$$

Then, applying the converse Deduction Theorem to (10) and (11) n times (twice) yields:

$$\text{(12)} \qquad \{A_1, \ldots, A_n\} \vdash_{K'} C$$
$$\text{(13)} \qquad \{A_1, \ldots, A_n\} \vdash_{K'} {\sim}C$$

So, $\{A_1, \ldots, A_n\}$ is an inconsistent set of K'. And, since $\{A_1, \ldots, A_n\} \subseteq \Gamma$, it follows that Γ is an inconsistent set of WFFs of K'. Hence, by the contrapositive of Lemma 1, we may infer that Γ is an inconsistent set of WFFs of K. Contradiction. So, Γ' is a *consistent* set of closed WFFs of K'. □

Therefore, Γ' has a denumerable model M, by 45.16. Now, we construct a denumerable sequence $s = \langle s_1, \ldots, s_m, \ldots\rangle$ of elements of the domain D of M, as follows:

$s_m =$ the object assigned to c_i by M, if v_i is the mth variable in our enumeration of variables of K

If we run out of v_i's and c_i's occurring in the formulas of Γ and Γ' (or if there are gaps in the v_i's and c_i's occurring in the formulas of Γ and Γ' – see below for an example), then we will complete the construction of s (or fill-in the gaps in s) by repeating an arbitrary element d of the domain of M as often as necessary.

Since the elements of Γ' are all closed and M is a model of Γ', s will satisfy all members of Γ' on M (*trivially*, since with closed formulas, we don't even need to look at the structure of s to realize that it will satisfy – on M – any closed formula that is true on M). And, *by design*, s will also satisfy all members of Γ on M. Why? Because, the members of Γ are just the members of Γ', but with the c_i's replaced by the v_i's. And, this ensures that s will satisfy all members of Γ as well. To see this, think about the following example:

- $\Gamma = \{Fv_1, Gv_6v_7\}$. Then, $s = \langle c_1, d, d, d, d, c_6, c_7, d, d, \ldots\rangle$, where d is some arbitrary member of the domain of interpretation M, which is a model of the set $\Gamma' = \{Fc_1, Gc_6c_7\}$. The sequence s satisfies all members of Γ. Why? Precisely because M is a model of Γ'. So, Fc_1 and Gc_6c_7 are both true on M. Hence, the object assigned to c_1 by M is in the extension of F, and the pair of objects assigned to $\langle c_6, c_7\rangle$ by M is in the extension of G. So, the sequence s satisfies both $\{Fv_1, Gv_6v_7\}$ on M.

This s will always satisfy all members of Γ on M. Hence, any consistent set Γ of WFFs of K is simultaneously satisfiable in a denumerable domain (*i.e.*, each member of Γ will be satisfied by the sequence s, as defined above, in the domain of the model M of the set Γ', whose existence is guaranteed by 45.16). □

46.2. If $\Gamma \vDash_Q A$, then $\Gamma \vdash_{QS} A$. [Strong Completeness of QS]

Proof. Assume $\Gamma \vDash_Q A$. Then, by the definition of $\vDash_Q$, $\Gamma \cup \{{\sim}A\}$ is not simultaneously satisfiable. Thus, as a special case of Gödel's metatheorem (45.17) which we just proved above, $\Gamma \cup \{{\sim}A\}$ is not a consistent set of WFFs of QS. Therefore, by metatheorem (43.9), $\Gamma \vdash_{QS} A$. That completes the proof of the strong completeness of QS, which is the main metatheorem of quantifier logic. [As a corollary of strong completeness and strong soundness, we now have an exact correspondence between $\vDash_Q$ and $\vdash_{QS}$, which was our aim.] □