The “Short” Truth-Table Method: Three Examples

Philosophy 1115
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1 Example #1 — Page 66 #3

Answer. $A \to (C \vee E), B \to D \nvDash (A \vee B) \to (C \to (D \vee E))$

Explanation.[1] Assume that ‘$A \to (C \vee E)$’ is $\top$, ‘$B \to D$’ is $\top$, and ‘$(A \vee B) \to (C \to (D \vee E))$’ is $\bot$. In order for ‘$(A \vee B) \to (C \to (D \vee E))$’ to be $\bot$, both ‘$A \vee B$’ and ‘C’ must be $\top$, and both ‘D’ and ‘E’ must be $\bot$. This *guarantees* that the first premise is $\top$ (since ‘$A \to (C \vee E)$’ *must*, at this point, have a $\top$ consequent). We can also make the second premise $\top$, simply by making ‘B’ $\bot$. Finally, by making ‘A’ $\top$, we can ensure that the conclusion is $\bot$, which yields the following interpretation on which ‘$A \to (C \vee E)$’ and ‘$B \to D$’ are $\top$, but ‘$(A \vee B) \to (C \to (D \vee E))$’ is $\bot$ (*i.e.*, the following *counterexample* to validity).

A	B	C	D	E	$A \to (C \vee E)$	$B \to D$	$(A \vee B) \to (C \to (D \vee E))$
$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\top$ $\mathbf{\top}$ $\top$ $\top$ $\bot$	$\bot$ $\mathbf{\top}$ $\bot$	$\top$ $\top$ $\bot$ $\mathbf{\bot}$ $\top$ $\bot$ $\bot$ $\bot$ $\bot$

Therefore, by the definition of $\vDash$, $A \to (C \vee E), B \to D \nvDash (A \vee B) \to (C \to (D \vee E))$. ✦

2 Example #2 (not in the text)

Answer. $A \leftrightarrow (B \vee C), B \to D, D \leftrightarrow C \vDash A \leftrightarrow D$

Explanation. Assume ‘$A \leftrightarrow (B \vee C)$’ is true, ‘$B \to D$’ is true, ‘$D \leftrightarrow C$’ is true, and ‘$A \leftrightarrow D$’ is false. There are *exactly two* ways in which ‘$A \leftrightarrow D$’ can be false, and they are as follows:

1. ‘A’ is true, and ‘D’ is false. In this case, in order for ‘$D \leftrightarrow C$’ to be true, ‘C’ must be false. And, in order for ‘$B \to D$’ to be true, ‘B’ must be false. This means that the *disjunction* ‘$B \vee C$’ must be false. So, in order for the biconditional ‘$A \leftrightarrow (B \vee C)$’ to be true, we must have ‘A’ *false* as well, which contradicts our assumption. So, in this first case, we have been forced into a *contradiction*.[2]

2. ‘A’ is false, and ‘D’ is true. In this case, in order for ‘$D \leftrightarrow C$’ to be true, ‘C’ must be true. But, if ‘C’ is true, then so is ‘$B \vee C$’. Hence, if ‘$A \leftrightarrow (B \vee C)$’ is going to be true, then ‘A’ must be true, which contradicts our assumption. So, in this second (and *last*) case, we have been forced into a *contradiction*.

Therefore, it is *impossible* to make ‘$A \leftrightarrow (B \vee C)$’, ‘$B \to D$’, and ‘$D \leftrightarrow C$’ all true, but ‘$A \leftrightarrow D$’ false (at the same time). So, by the definition of $\vDash$, $A \leftrightarrow (B \vee C), B \to D, D \leftrightarrow C \vDash A \leftrightarrow D$. ✦

3 Example #3 (not in the text)

Answer. $A \to (B \,\&\, C) \vDash (A \to B) \,\&\, (A \to C)$

Explanation. Assume ‘$A \to (B \,\&\, C)$’ is true, and ‘$(A \to B) \,\&\, (A \to C)$’ is false. There are *exactly three* ways in which ‘$(A \to B) \,\&\, (A \to C)$’ can be false, and they are as follows:

1. ‘$A \to B$’ is true, and ‘$A \to C$’ is false. If ‘$A \to C$’ is false, then ‘A’ is true and ‘C’ is false. But, if ‘C’ is false, then so is ‘$B \,\&\, C$’. Thus, since ‘A’ is true and ‘$B \,\&\, C$’ is false, ‘$A \to (B \,\&\, C)$’ is false — *contradiction*.

2. ‘$A \to B$’ is false, and ‘$A \to C$’ is true. If ‘$A \to B$’ is false, then ‘A’ is true and ‘B’ is false. But, if ‘B’ is false, then so is ‘$B \,\&\, C$’. Thus, since ‘A’ is true and ‘$B \,\&\, C$’ is false, ‘$A \to (B \,\&\, C)$’ is false — *contradiction*.

3. ‘$A \to B$’ is false, and ‘$A \to C$’ is false. If ‘$A \to B$’ is false, then ‘A’ is true and ‘B’ is false. But, if ‘B’ is false, then so is ‘$B \,\&\, C$’. Thus, since ‘A’ is true and ‘$B \,\&\, C$’ is false, ‘$A \to (B \,\&\, C)$’ is false — *contradiction*.

Therefore, it is *impossible* to make ‘$A \to (B \,\&\, C)$’ true and ‘$(A \to B) \,\&\, (A \to C)$’ false (at the same time). So, by the definition of $\vDash$, $A \to (B \,\&\, C) \vDash (A \to B) \,\&\, (A \to C)$. ✦

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[1] You do *not* have to show *all* of your reasoning in cases like this one, where the argument is *in*valid (*i.e.*, where $\nvDash$). I am just showing you *all* of *my* reasoning to give you more information about how these kinds of problems are solved. All you *need* to do here is report an interpretation (*i.e.*, a single-row) which invalidates the inference. But, I do recommend filling-in all of the quasi-columns to make explicit all of the calculations required.

[2] We *cannot*, at this point in our reasoning, infer that $A \leftrightarrow (B \vee C), B \to D, D \leftrightarrow C \vDash A \leftrightarrow D$ (and, obviously, we cannot infer at this point that $A \leftrightarrow (B \vee C), B \to D, D \leftrightarrow C \nvDash A \leftrightarrow D$ either). We *must* examine *all possible cases* before we infer that an argument is *valid*.