Henkin's Model and Metatheorem 45.14
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Henkin's Model. Let T be a consistent, negation-complete, and closed first order theory. Henkin's model M is a a denumerable interpretation for T such that for each WFF A of T, A is true on M iff $\vdash_T A$. The existence of such a model M undergirds metatheorem 45.14. Characterizing M will involve doing five things: (1) specifying M's (denumerable) domain D, (2) saying for each constant symbol $\mathfrak{c}$ of T which object d in the domain M assigns to $\mathfrak{c}$, (3) saying for each n-place function symbol $\mathfrak{f}$ which n-ary function $\mathbf{f}$ is assigned to $\mathfrak{f}$ by M, (4) saying for each n-place predicate symbol $\mathfrak{F}$ which n-ary property $\mathbf{F}$ (*i.e.*, which set of ordered n-tuples of closed terms of T, since we identify properties with their *extensions*) is assigned by M to $\mathfrak{F}$, and (5) saying for each propositional symbol $\mathfrak{p}$ of T, what truth-value is assigned to $\mathfrak{p}$ by M. Here is Henkin's M, followed by a proof of 45.14 (arguably the most important metatheorem of the entire course).

1. The domain D of M is the set of closed terms of T. This set contains all the constant symbols $a', a'', a''', \ldots, b', b'', b''', \ldots, c', c'', c''', \ldots$ of T (the b's and c's are effectively enumerable sets of new constant symbols that may be added to Q for $Q+$ purposes and/or for the purpose of ensuring T is closed). D also contains all the closed terms with function symbols: $f^{*\prime}a', f^{**\prime}b'a'', \ldots$ of T.

   Important Digression on Symbols, Abstract Objects, Types, and Tokens. It is important to note that the symbols of T are *abstract objects*, and they are *types not tokens*. You should not confuse a token of a symbol with the symbol itself. For instance, when I write a token inscription "a'" (the physical inscription between the quotation marks preceding this parenthetical remark), I have not written down the symbol itself. It is not tokens of symbols of T that get assigned to objects by M, but rather the symbols themselves. For instance, when I say that the numeral "1" gets interpreted as the number one (which is also an abstract object), I do not mean that the token inscription that appears between the quotation marks on this sheet of paper (two lines up from this line) gets interpreted as the number one. Rather, I mean that the symbol itself (the numeral *type* of which the aforementioned physical inscription on this sheet of paper is a token) gets interpreted as the number one. So, interpretations assign objects (either abstract or concrete) to abstract objects which are types and not tokens. We understand the denotation of a token inscription (*e.g.*) "a'" assigned by M by (i) recognizing that "a'" is a token of a certain type, and then (ii) consulting M to see which object gets assigned to the type of which "a'" is a token.

2. To each constant symbol $\mathfrak{c}$ of T, M assigns to $\mathfrak{c}$ the constant symbol $\mathfrak{c}$ itself. For instance, the constant symbol (type!) of which the inscription "a'" is a token gets assigned by M the symbol (type!) a' itself.

3. To each n-place function symbol $\mathfrak{f}$ of T, M assigns the n-ary function $\mathbf{f}$ with arguments and values in D, which is defined by the following rule: The value of $\mathbf{f}(x_1, \ldots, x_n)$ for the arguments $x_1 = t_1$, $x_2 = t_2$, $\ldots$, $x_n = t_n$, where $t_1, \ldots, t_n$ are closed terms of T, is the closed term $\mathfrak{f}t_1, \ldots, t_n$ of T itself. For instance, M will assign to the 2-place function symbol (type!) $f^{**\prime}$ of T the two-place function $\mathbf{f}(x_1, x_2)$, which is such that $\mathbf{f}(t_1, t_2) = f^{**\prime}t_1t_2$, for all closed terms t_1 and t_2 of T.

4. To each n-place predicate symbol $\mathfrak{F}$ of T, M assigns the (or any) n-ary property $\mathbf{F}$ whose extension is the set of ordered n-tuples $\langle t_1, \ldots, t_n \rangle$ of closed terms of T such that $\vdash_T \mathfrak{F}t_1 \ldots t_n$. For instance, to the 2-place predicate symbol $F^{**\prime}$, M assigns the (or any) property whose extension is the set of ordered pairs $\langle t_1, t_2 \rangle$ of closed terms of T such that $\vdash_T F^{**\prime}t_1t_2$, *i.e.*, $F^{**\prime}t_1t_2$ is a theorem of T.

5. To each propositional symbol $\mathfrak{p}$ of T, M assigns T to $\mathfrak{p}$ if $\vdash_T \mathfrak{p}$, and F to $\mathfrak{p}$ if $\nvdash_T \mathfrak{p}$.

Metatheorem 45.14. Any consistent, closed, negation-complete first-order theory T has a denumerable model M, where M is defined in accordance with (1)–(5) above.

Proof. We will actually prove something *stronger* than 45.14: For each WFF A of T, A is true on M iff $\vdash_T A$. We only need to worry about the *closed* WFFs (*sentences*) of T, since A is true on M iff A^c is true on M (40.7), and $\vdash_T A$ iff $\vdash_H A^c$ (45.5). So, we'll show that all sentences A of T are such that A is true on M iff $\vdash_T A$. The proof will be by strong induction on n = the # of connectives + the # of quantifiers in A.

Basis Step. $n = 0$. In this case, A is either a propositional symbol $\mathfrak{p}$ or A is of the form $\mathfrak{F}t_1 \ldots t_n$, where $\mathfrak{F}$ is an n-place predicate symbol of T, and $t_1, \ldots, t_n$ are closed terms of T. In these cases, the desired result (that A is true on M iff $\vdash_T A$) follows directly from clauses (4) and (5) of the definition of M, respectively.

Inductive Step. $n > 0$. Here, we assume as our strong inductive hypothesis:

(IH) For each sentence A with fewer than n connectives + quantifiers, A is true on M iff $\vdash_T A$.

Using (IH), we'll prove that, for all sentences A of T with exactly n connectives and quantifiers, A is true on M iff $\vdash_T A$. There are only three cases that we need to consider, for the three kinds of sentences of T:

Case 1. $A = {\sim}B$, for some B with $n - 1$ connectives and quantifiers. Goal: A is true on $M \Leftrightarrow\ \vdash_T A$.

($\Rightarrow$) Suppose that A is true on M. Then, B is false on M. So, by (IH), $\nvdash_T B$ [(IH) applies to B, since it is a *sentence*]. Then, by the negation-completeness of T, $\vdash_T {\sim}B$. That is, $\vdash_T A$. □

($\Leftarrow$) Contrapositive: If A is not true on M, then $\nvdash_T A$. Suppose A is not true on M. Then, B is true on M. So, by (IH), $\vdash_T B$. Then, by the consistency of T, $\nvdash_T {\sim}B$. That is, $\nvdash_T A$. □

Case 2. $A = B \supset C$, for some B, C with $< n$ connectives + quantifiers. Goal: A is true on $M \Leftrightarrow\ \vdash_T A$.

($\Rightarrow$) Contrapositive: If $\nvdash_T A$, then A is not true on M. Suppose $\nvdash_T A$. Then, by the negation-completeness of T, $\vdash_T {\sim}A$. That is, $\vdash_T {\sim}(B \supset C)$. But, we have the tautological schema $\vdash_T {\sim}(B \supset C) \supset B$, and $\vdash_T {\sim}(B \supset C) \supset {\sim}C$. So, two applications of MP yield $\vdash_T B$ and $\vdash_T {\sim}C$. So, by (IH), B is true on M. And, by the consistency of T and (IH), $\nvdash_T C$, and C is not true on M. Since C is a *sentence*, we can conclude that C is *false* on M. Therefore, since B is true on M and C is false on M, $(B \supset C)$ is not true on M. That is, A is not true on M. □

($\Leftarrow$) Contrapositive: If A is not true on M, then $\nvdash_T A$. Suppose A is not true on M. Then, B is true on M and C is false (hence, not true) on M. So, by (IH), $\vdash_T B$ and $\nvdash_T C$. Then, by the negation-completeness of T, $\vdash_T {\sim}C$. Tautological schema: $\vdash_T B \supset ({\sim}C \supset {\sim}(B \supset C))$. So, two applications of MP yield: $\vdash_T {\sim}(B \supset C)$. That is, $\vdash_T {\sim}A$. By the consistency of T, $\nvdash_T A$. □

Case 3. $A = \bigwedge v_j B$, for some B with $n - 1$ connectives and quantifiers. This time, there are two cases: (3.1) B is closed, and (3.2) B is open. In both cases, our goal is to show that A is true on $M \Leftrightarrow\ \vdash_T A$.

(3.1) B is closed.

($\Rightarrow$) Suppose A [$\bigwedge v_j B$] is true on M. Then, by (40.6), B is also true on M. So, by (IH), $\vdash_T B$ [(IH) applies to B, since it is a *sentence*]. Hence, by (45.4), $\vdash_T \bigwedge v_j B$. That is, $\vdash_T A$. □

($\Leftarrow$) Suppose $\vdash_T A$ [$\vdash_T \bigwedge v_j B$]. By K4 (B is closed), $\vdash_T \bigwedge v_j B \supset Bt/v_j$ [$\vdash_T \bigwedge v_j B \supset B$]. Then, by MP, $\vdash_T B$. So, by (IH), B is true on M. And, by (40.6), $\bigwedge v_j B$ — *i.e.*, A — is true on M. □

(3.2) B is open. Since A is closed, the only free variable in B is v_j.

($\Rightarrow$) Suppose A [$\bigwedge v_j B$] is true on M. By (40.6), B is true on M. By (40.20), Bt/v_j is true on M for every closed term t of T. By (IH), $\vdash_T Bt/v_j$ for every closed term t of T (since every such Bt/v_j is a *sentence*). Then, by the closedness of T, $\vdash_T \bigwedge v_j B$. That is, $\vdash_T A$. □

($\Leftarrow$) Suppose $\vdash_T A$ [$\vdash_T \bigwedge v_j B$]. By K4, $\vdash_T \bigwedge v_j B \supset Bt/v_j$, for every closed term t of T. Hence, by MP, $\vdash_T Bt/v_j$, for every closed term t of T. So, by (IH), Bt/v_j is true on M, for every closed term t of T (since every such Bt/v_j is a *sentence*). By (40.21), $\bigwedge v_j B$ [*i.e.*, A] is true on M. □

This completes the inductive step, and with it the proof of metatheorem 45.14. The key lemmas we used here were: 40.6, 40.7, 40.20, 40.21, 45.4, 45.5. The least trivial of these are 40.20 and 40.21. Make sure you understand the proofs of these lemmas, as well as their rôles in this proof of metatheorem 45.14. □