PrSAT Tutorial (*Mathematica* Notebook)

June 16, 2011

■ First, load in the Prsat package

See my **PrSAT** website for instructions on downloading and installing **PrSAT** (assuming you have *Mathematica* installed).

<< Prsat

■ Example #1

The first example of a probability model that we saw was the following:

 $\left\{ \left\{ \text{X} \rightarrow \left\{ \text{a}_{2}\text{, a}_{4} \right\} \text{, Y} \rightarrow \left\{ \text{a}_{3}\text{, a}_{4} \right\} \text{, } \Omega \rightarrow \left\{ \text{a}_{1}\text{, a}_{2}\text{, a}_{3}\text{, a}_{4} \right\} \right\} \text{, } \left\{ \text{a}_{1} \rightarrow \frac{11}{24}\text{, a}_{2} \rightarrow \frac{1}{4}\text{, a}_{3} \rightarrow \frac{1}{8}\text{, a}_{4} \rightarrow \frac{1}{6} \right\} \right\}$

Prsat will show us an STT representation of **MODEL1**:

TruthTable[MODEL1]

X	Y	var	Pr
Т	Т	a ₄	$\frac{1}{6}$
Т	F	a ₂	$\frac{1}{4}$
F	Т	a ₃	1 8
F	F	a ₁	$\frac{11}{24}$

We can use **PrSAT** to calculate probability, using **MODEL1**:

EvaluateProbability[{Pr[X \neq Y], Pr[X], Pr[Y]}, MODEL1]

$$\left\{\frac{13}{24}, \frac{5}{12}, \frac{7}{24}\right\}$$

We can also check arbitrary claims to see if they are true on MODEL1:

EvaluateProbability[Pr[X | Y] > Pr[X], MODEL1]

True

■ Example #2

The second example we saw was an algebraic proof of the following theorem:

$$Pr(X \lor Y) = Pr(X) + Pr(Y) - Pr(X \land Y)$$

Prsat easily verifies this theorem (note: it does not present a readable proof).

```
PrSAT[{Pr[X \lor Y] \neq Pr[X] + Pr[Y] - Pr[X \land Y]}]
```

PrSAT::srchfail: Search phase failed; attempting FindInstance

{}

This output means there are no probability models on which $Pr[X \lor Y] \neq Pr[X] + Pr[Y] - Pr[X \land Y]$. That "proves" that the above statement is a theorem of probability calculus.

■ Example #3

The second example we saw was an algebraic proof of the following theorem:

$$Pr(X) = Pr(X \land Y) + Pr(X \land \neg Y)$$

$$Pr[X] = Pr[X \& \& Y] + Pr[X \& \& \neg Y]$$

Prsat easily verifies this theorem (note: it does not present a readable proof).

$$PrSAT[{Pr[X] \neq Pr[X \land Y] + Pr[X \land \neg Y]}]$$

PrSAT::srchfail: Search phase failed; attempting FindInstance

{}

This output means there are no probability models on which $Pr[X] \neq Pr[X \land Y] + Pr[X \land Y]$. That "proves" that the above statement is a theorem of probability calculus.

■ Example #4

The next example involves the following theorem:

$$\Pr(X \to Y) \ge \Pr(Y \mid X)$$

 $\Pr[X \to Y] \ge \Pr[Y \mid X]$

Prsat easily verifies this theorem (note: it does not present a readable proof). First, we need to define the conditional operator.

```
X_{\rightarrow} Y_{\rightarrow} := \neg X \lor Y;
PrSAT[\{Pr[X \rightarrow Y] < Pr[Y \mid X]\}]
```

PrSAT::srchfail: Search phase failed; attempting FindInstance

{}

This output means there are no probability models on which $Pr[X \to Y] \ge Pr[Y \mid X]$. That "proves" that the above statement is a theorem of probability calculus.

■ Example #5

The next example involves the following theorem:

```
d(X, Y) = d(X \lor Y, Y) + d(X \lor \neg Y, Y), where d(X, Y) = Pr(X | Y) - Pr(X).
```

Prsat easily verifies this theorem (note: it does not present a readable proof). First, we need to define **d(X, Y)**.

$$\texttt{PrSAT}[\{d[X, Y] \neq d[X \lor Y, Y] + d[X \lor \neg Y, Y]\}]$$

PrSAT::srchfail: Search phase failed; attempting FindInstance

{}

This output means there are no probability models on which $d[X, Y] \neq d[X \lor Y, Y] + d[X \lor Y, Y]$. That "proves" that the above statement is a theorem of probability calculus.

■ Example #6

The next example involves the fact that $Pr(X \mid Y \lor Z) = Pr(X \mid Y \land Z)$ is NOT a theorem:

Prsat easily finds a counter-model to this claim.

 $PrSAT[{Pr[X | Y \lor Z] \neq Pr[X | Y \land Z]}]$

$$\begin{split} \Big\{ \{ \mathbf{X} \to \{ \mathbf{a}_2, \, \mathbf{a}_5, \, \mathbf{a}_6, \, \mathbf{a}_8 \}, \, \mathbf{Y} \to \{ \mathbf{a}_3, \, \mathbf{a}_5, \, \mathbf{a}_7, \, \mathbf{a}_8 \}, \\ \mathbf{Z} \to \{ \mathbf{a}_4, \, \mathbf{a}_6, \, \mathbf{a}_7, \, \mathbf{a}_8 \}, \, \Omega \to \{ \mathbf{a}_1, \, \mathbf{a}_2, \, \mathbf{a}_3, \, \mathbf{a}_4, \, \mathbf{a}_5, \, \mathbf{a}_6, \, \mathbf{a}_7, \, \mathbf{a}_8 \} \big\}, \\ \Big\{ \mathbf{a}_1 \to \frac{63}{125528}, \, \mathbf{a}_2 \to 0, \, \mathbf{a}_3 \to \frac{1}{568}, \, \mathbf{a}_4 \to \frac{110}{221}, \, \mathbf{a}_5 \to 0, \, \mathbf{a}_6 \to 0, \, \mathbf{a}_7 \to 0, \, \mathbf{a}_8 \to \frac{1}{2} \Big\} \Big\} \end{split}$$

The model **PrSAT** finds by defualt is *non*-regular. We can force it to find a *regular* counter-model, as follows:

 $\texttt{MODEL2} = \texttt{PrSAT}[\{\texttt{Pr}[X \mid Y \lor Z] \neq \texttt{Pr}[X \mid Y \land Z]\}, \, \texttt{Probabilities} \rightarrow \texttt{Regular}]$

$$\begin{split} \Big\{ \{ X \to \{ \texttt{a}_2, \, \texttt{a}_5, \, \texttt{a}_6, \, \texttt{a}_8 \}, \, Y \to \{ \texttt{a}_3, \, \texttt{a}_5, \, \texttt{a}_7, \, \texttt{a}_8 \}, \\ Z \to \{ \texttt{a}_4, \, \texttt{a}_6, \, \texttt{a}_7, \, \texttt{a}_8 \}, \, \Omega \to \{ \texttt{a}_1, \, \texttt{a}_2, \, \texttt{a}_3, \, \texttt{a}_4, \, \texttt{a}_5, \, \texttt{a}_6, \, \texttt{a}_7, \, \texttt{a}_8 \} \big\}, \\ \Big\{ \texttt{a}_1 \to \frac{6053}{6111882}, \, \texttt{a}_2 \to \frac{1}{999}, \, \texttt{a}_3 \to \frac{1}{999}, \, \texttt{a}_4 \to \frac{1}{999}, \, \texttt{a}_5 \to \frac{1}{46}, \, \texttt{a}_6 \to \frac{10}{21}, \, \texttt{a}_7 \to \frac{85}{171}, \, \texttt{a}_8 \to \frac{1}{999} \Big\} \Big\} \end{split}$$

Here is an STT representation of **MODEL2**:

TruthTable[MODEL2]

X	Y	Z	var	Pr
Т	Т	Т	a ₈	<u>1</u> 999
Т	Т	F	a ₅	1 46
Т	F	Т	a ₆	$\frac{10}{21}$
Т	F	F	a ₂	999
F	Т	Т	a ₇	85 171
F	Т	F	a ₃	999
F	F	Т	a ₄	$\frac{1}{999}$
F	F	F	a ₁	6053 6111882

We can calculate the values of $Pr[X \mid Y \land Z]$, $Pr[X \mid Y \lor Z]$ on this model as follows:

EvaluateProbability[$\{Pr[X \mid Y \land Z], Pr[X \mid Y \lor Z]\}$, MODEL2]

$$\left\{\frac{19}{9454}, \frac{3049405}{6099711}\right\}$$

We can look at decimal representations of these exact real numbers, as follows:

% // N

We gave a different model in the lecture notes. We can enter that model in by hand, and then verify it has the desired properties, as follows:

MODEL3 =

$$\begin{aligned} & \text{PrSAT} \Big[\Big\{ \text{Pr} \big[X \land Y \land Z \big] = \frac{1}{6}, \, \text{Pr} \big[X \land Y \land \neg Z \big] = \frac{1}{6}, \, \text{Pr} \big[X \land \neg Y \land Z \big] = \frac{1}{4}, \, \text{Pr} \big[X \land \neg Y \land \neg Z \big] = \frac{1}{16}, \\ & \text{Pr} \big[\neg X \land Y \land Z \big] = \frac{1}{6}, \, \text{Pr} \big[\neg X \land Y \land \neg Z \big] = \frac{1}{12}, \, \text{Pr} \big[\neg X \land \neg Y \land Z \big] = \frac{1}{24}, \, \text{Pr} \big[\neg X \land \neg Y \land \neg Z \big] = \frac{1}{16} \Big\} \Big] \\ & \Big\{ \{ X \to \{ \text{al}_2, \, \text{al}_5, \, \text{al}_6, \, \text{al}_8 \}, \, Y \to \{ \text{al}_3, \, \text{al}_5, \, \text{al}_7, \, \text{al}_8 \}, \\ Z \to \{ \text{al}_4, \, \text{al}_6, \, \text{al}_7, \, \text{al}_8 \}, \, \Omega \to \{ \text{al}_1, \, \text{al}_2, \, \text{al}_3, \, \text{al}_4, \, \text{al}_5, \, \text{al}_6, \, \text{al}_7, \, \text{al}_8 \} \Big\}, \\ & \Big\{ \text{al}_1 \to \frac{1}{16}, \, \text{al}_2 \to \frac{1}{16}, \, \text{al}_3 \to \frac{1}{12}, \, \text{al}_4 \to \frac{1}{24}, \, \text{al}_5 \to \frac{1}{6}, \, \text{al}_6 \to \frac{1}{4}, \, \text{al}_7 \to \frac{1}{6}, \, \text{al}_8 \to \frac{1}{6} \Big\} \Big\} \end{aligned}$$

TruthTable[MODEL3]

X	Y	Z	var	Pr
Т	Т	Т	a ₈	$\frac{1}{6}$
Т	Т	F	a ₅	$\frac{1}{6}$
Т	F	Т	a ₆	$\frac{1}{4}$
Т	F	F	a ₂	$\frac{1}{16}$
F	Т	Т	a ₇	$\frac{1}{6}$
F	Т	F	a ₃	$\frac{1}{12}$
F	F	Т	a ₄	1 24
F	F	F	a ₁	$\frac{1}{16}$

 $\textbf{EvaluateProbability}[\{\texttt{Pr}[\texttt{X} \mid \texttt{Y} \land \texttt{Z}], \, \texttt{Pr}[\texttt{X} \mid \texttt{Y} \lor \texttt{Z}]\}, \, \texttt{MODEL3}]$

$$\left\{\frac{1}{2}, \frac{2}{3}\right\}$$

We can also see the algebraic form of an expression, as follows:

 $\texttt{AlgebraicForm}[\texttt{Pr}[\texttt{X} \mid \texttt{Y} \land \texttt{Z}] == \texttt{Pr}[\texttt{X} \mid \texttt{Y} \lor \texttt{Z}], \{\texttt{X}, \texttt{Y}, \texttt{Z}\}]$

Note that **PrSAT** uses different conventions (i.e., a different ordering in the truth-table) for the a_i than I use in the lecture notes.

Here is a verfifcation of some of our *independence* theorems:

```
PrSAT[{
    Pr[P \ Q] == Pr[P] Pr[Q],
    Pr[P | Q] \neq Pr[P],
}
```

PrSAT::srchfail: Search phase failed; attempting FindInstance

{}

```
PrSAT[{
    Pr[P ∧ ¬ Q] == Pr[P] Pr[¬ Q],
    Pr[P ∧ Q] ≠ Pr[P] Pr[Q]
}]
```

PrSAT::srchfail: Search phase failed; attempting FindInstance

{}

```
PrSAT[{
   Pr[P \land Q] = Pr[P] Pr[Q],
   Pr[P \land \neg Q] \neq Pr[P] Pr[\neg Q]
```

PrSAT::srchfail: Search phase failed; attempting FindInstance

Here is a (regular) model showing that pairwise independence does *not* entail mutual independence (for a set of 3 events)

```
PrSAT[{
     Pr[X \wedge Y] = Pr[X] Pr[Y],
      Pr[X \land Z] = Pr[X] Pr[Z],
     Pr[Y \land Z] = Pr[Y] Pr[Z],
      Pr[(X \land Y) \land Z] \neq Pr[X \land Y] Pr[Z],
     Pr[X] == Pr[Y] == Pr[Z] == 1 / 2
   }, Probabilities → Regular]
\{X \rightarrow \{a_2, a_5, a_6, a_8\}, Y \rightarrow \{a_3, a_5, a_7, a_8\},
      \mathbf{Z} \, \rightarrow \, \{ \texttt{a}_{4} \, , \, \texttt{a}_{6} \, , \, \texttt{a}_{7} \, , \, \texttt{a}_{8} \} \, , \, \, \Omega \, \rightarrow \, \{ \texttt{a}_{1} \, , \, \texttt{a}_{2} \, , \, \texttt{a}_{3} \, , \, \texttt{a}_{4} \, , \, \texttt{a}_{5} \, , \, \texttt{a}_{6} \, , \, \texttt{a}_{7} \, , \, \texttt{a}_{8} \} \, \} \, ,
  \left\{ a_1 \to \frac{1}{676} \text{, } a_2 \to \frac{42}{169} \text{, } a_3 \to \frac{42}{169} \text{, } a_4 \to \frac{42}{169} \text{, } a_5 \to \frac{1}{676} \text{, } a_6 \to \frac{1}{676} \text{, } a_7 \to \frac{1}{676} \text{, } a_8 \to \frac{42}{169} \right\} \right\}
```

TruthTable[%]

X	Y	Z	var	Pr
Т	Т	Т	a18	$\frac{42}{169}$
Т	Т	F	a ₅	$\frac{1}{676}$
Т	F	Т	a 6	$\frac{1}{676}$
Т	F	F	a ₂	$\frac{42}{169}$
F	Т	Т	a ₇	1 676
F	Т	F	a ₃	$\frac{42}{169}$
F	F	Т	a14	$\frac{42}{169}$
F	F	F	a ₁	1 676

Here is a (regular) model showing that independence is *not* transitive:

```
PrSAT[{
     Pr[X \land Y] = Pr[X] Pr[Y],
     Pr[Y \land Z] = Pr[Y] Pr[Z],
     Pr[X \land Z] \neq Pr[X] Pr[Z],
     Pr[X] = Pr[Y] = Pr[Z] = 1/2
  }, Probabilities → Regular]
\{X \rightarrow \{a_2, a_5, a_6, a_8\}, Y \rightarrow \{a_3, a_5, a_7, a_8\},
     \mathbf{Z} \rightarrow \{\texttt{a}_4,\, \texttt{a}_6,\, \texttt{a}_7,\, \texttt{a}_8\}\,,\, \boldsymbol{\Omega} \rightarrow \{\texttt{a}_1,\, \texttt{a}_2,\, \texttt{a}_3,\, \texttt{a}_4,\, \texttt{a}_5,\, \texttt{a}_6,\, \texttt{a}_7,\, \texttt{a}_8\}\}\,,
 \left\{ a_{1} \rightarrow \frac{1}{999}, \ a_{2} \rightarrow \frac{995}{3996}, \ a_{3} \rightarrow \frac{1}{999}, \ a_{4} \rightarrow \frac{995}{3996}, \ a_{5} \rightarrow \frac{995}{3996}, \ a_{6} \rightarrow \frac{1}{999}, \ a_{7} \rightarrow \frac{995}{3996}, \ a_{8} \rightarrow \frac{1}{999} \right\} \right\}
```

TruthTable[%]

X	Y	Z	var	Pr
Т	Т	Т	a ₈	<u>1</u> 999
Т	Т	F	a ₅	995 3996
Т	F	Т	a ₆	<u>1</u> 999
Т	F	F	a ₂	995 3996
F	Т	Т	a ₇	995 3996
F	Т	F	a ₃	<u>1</u> 999
F	F	Т	a ₄	995 3996
F	F	F	a ₁	$\frac{1}{999}$

Extra Example (not from my lecture #1 notes): Here's an example of a set of constraints that are satisfiable but only on probability models containing *irrational* numbers

$$\begin{split} & \text{PrSAT} \Big[\Big\{ \text{Pr} \big[\mathbf{Y} \mid \mathbf{X} \big] = \text{Pr} \big[\mathbf{Y} \vee \mathbf{X} \big] \text{, } \text{Pr} \big[\mathbf{X} \wedge \mathbf{Y} \big] = \frac{1}{4} \text{, } \text{Pr} \big[\neg \mathbf{X} \wedge \mathbf{Y} \big] = = \frac{1}{4} \Big\} \Big] \\ & \Big\{ \{ \mathbf{X} \to \{ \mathbf{a}_2, \, \mathbf{a}_4 \} \text{, } \mathbf{Y} \to \{ \mathbf{a}_3, \, \mathbf{a}_4 \} \text{, } \Omega \to \{ \mathbf{a}_1, \, \mathbf{a}_2, \, \mathbf{a}_3, \, \mathbf{a}_4 \} \big\} \text{,} \\ & \Big\{ \mathbf{a}_1 \to \frac{1}{8} \left(7 - \sqrt{17} \right) \text{, } \mathbf{a}_2 \to \frac{1}{8} \left(-3 + \sqrt{17} \right) \text{, } \mathbf{a}_3 \to \frac{1}{4} \text{, } \mathbf{a}_4 \to \frac{1}{4} \Big\} \Big\} \end{split}$$

TruthTable[%]

Х	Y	var	Pr
Т	Т	a14	$\frac{1}{4}$
Т	F	a ₂	$\frac{1}{8}\left(-3+\sqrt{17}\right)$
F	Т	a ₃	$\frac{1}{4}$
F	F	a ₁	$\frac{1}{8} \left(7 - \sqrt{17}\right)$

Can you prove that this is the *only* model for this set of constraints?

■ The first numerical probability model appearing in the lecture notes

MODEL1 = PrSAT [{
$$Pr[X \land Y] = \frac{1}{6},$$

$$Pr[X \land Y] = \frac{1}{4},$$

$$Pr[X \land Y] = \frac{1}{4},$$

$$Pr[-X \land Y] = \frac{1}{8}$$
 }] [{ $X \rightarrow \{a_2, a_4\}, Y \rightarrow \{a_3, a_4\}, \Omega \rightarrow \{a_1, a_2, a_3, a_4\}\}, \{a_1 \rightarrow \frac{11}{24}, a_2 \rightarrow \frac{1}{4}, a_3 \rightarrow \frac{1}{8}, a_4 \rightarrow \frac{1}{6}\}}$

X	Y	var	Pr
Т	Т	a4	$\frac{1}{6}$
Т	F	a ₂	$\frac{1}{4}$
F	Т	a3	1 8
F	F	a ₁	$\frac{11}{24}$

Using Prsat to translate probabilistic expressions into algebraic ones

$${\tt AlgebraicForm[Pr[\neg X \lor Y] < Pr[Y \mid X], \{X, Y\}]}$$

$$1 - a_2 < \frac{a_4}{a_2 + a_4}$$

DenialOfTheorem = AlgebraicForm[$Pr[\neg X \lor Y] < Pr[Y | X], \{X, Y\}$]

$$1 - a_2 < \frac{a_4}{a_2 + a_4}$$

We can then use *Mathematica*'s decision procedure for real algebra (**FindInstance**) to *refute* this claim, assuming that the underlying variables are constrained so as to constitute a *probability model*:

FindInstance[DenialOfTheorem &&
$$a_1 + a_2 + a_3 + a_4 = 1$$
 && $0 \le a_1 \le 1$ && $0 \le a_2 \le 1$ && $0 \le a_3 \le 1$ && $0 \le a_4 \le 1$, $\{a_1, a_2, a_3, a_4\}$]

■ Using Prsat to find a model on which Pr(X | Y ∧ Z) ≠ Pr(X | Y ∨ Z)

$$\begin{split} \Big\{ \{ X \to \{ \texttt{a}_2 \text{, a}_5 \text{, a}_6 \text{, a}_8 \} \text{, } Y \to \{ \texttt{a}_3 \text{, a}_5 \text{, a}_7 \text{, a}_8 \} \text{,} \\ Z \to \{ \texttt{a}_4 \text{, a}_6 \text{, a}_7 \text{, a}_8 \} \text{, } \Omega \to \{ \texttt{a}_1 \text{, a}_2 \text{, a}_3 \text{, a}_4 \text{, a}_5 \text{, a}_6 \text{, a}_7 \text{, a}_8 \} \} \text{,} \\ \Big\{ \texttt{a}_1 \to \frac{1}{16} \text{, a}_2 \to \frac{1}{16} \text{, a}_3 \to \frac{1}{4} \text{, a}_4 \to \frac{5}{16} \text{, a}_5 \to \frac{1}{32} \text{, a}_6 \to \frac{1}{8} \text{, a}_7 \to \frac{1}{8} \text{, a}_8 \to \frac{1}{32} \Big\} \Big\} \end{split}$$

TruthTable[MODEL2]

X	Y	Z	var	Pr
Т	Т	Т	a.8	$\frac{1}{32}$
Т	Т	F	a ₅	$\frac{1}{32}$
Т	F	Т	a16	1 8
Т	F	F	a ₂	$\frac{1}{16}$
F	Т	Т	a ₇	1 8
F	Т	F	a ₃	$\frac{1}{4}$
F	F	Т	a14	$\frac{5}{16}$
F	F	F	a ₁	$\frac{1}{16}$

 ${\tt EvaluateProbability[\{Pr[X \mid Y \land Z]\,,\, Pr[X \mid Y \lor Z]\}\,,\, MODEL2]}$

$$\left\{\frac{1}{5}, \frac{3}{14}\right\}$$

■ Using Prsat to show the ordinal *non*-equivalence of $\{d, r, \ell, s\}$

$$\begin{split} d\left[\mathbf{H}_{-},\,\mathbb{E}_{-}\right] &:= \Pr[\mathbb{H} \mid \mathbb{E}] - \Pr[\mathbb{H}]; \\ r\left[\mathbf{H}_{-},\,\mathbb{E}_{-}\right] &:= \frac{\Pr[\mathbb{H} \mid \mathbb{E}] - \Pr[\mathbb{H}]}{\Pr[\mathbb{H} \mid \mathbb{E}] + \Pr[\mathbb{H}]}; \\ \ell\left[\mathbf{H}_{-},\,\mathbb{E}_{-}\right] &:= \frac{\Pr[\mathbb{E} \mid \mathbb{H}] - \Pr[\mathbb{E} \mid \neg \mathbb{H}]}{\Pr[\mathbb{E} \mid \mathbb{H}] + \Pr[\mathbb{E} \mid \neg \mathbb{H}]}; \\ s\left[\mathbf{H}_{-},\,\mathbb{E}_{-}\right] &:= \Pr[\mathbb{H} \mid \mathbb{E}] - \Pr[\mathbb{H} \mid \neg \mathbb{E}]; \\ \\ J\left[\mathbf{H}_{-},\,\mathbb{E}_{-}\right] &:= \frac{1}{\Pr[\neg \mathbb{H} \mid \mathbb{E}]} - \frac{1}{\Pr[\neg \mathbb{H}]}; \end{split}$$

A model showing that d and r are not ordinally equivalent:

$$\begin{aligned} & \text{MODEL3} = \text{PrSAT}[\{ \\ & d\left[\mathbb{H}_{1}, \, \mathbb{E}_{1}\right] \geq d\left[\mathbb{H}_{2}, \, \mathbb{E}_{2}\right], \\ & r\left[\mathbb{H}_{1}, \, \mathbb{E}_{1}\right] < r\left[\mathbb{H}_{2}, \, \mathbb{E}_{2}\right] \\ \}, \\ & \text{Probabilities} \rightarrow \text{Regular}] \end{aligned} \\ & \left\{ \{\mathbb{E}_{1} \rightarrow \{a_{2}, \, a_{6}, \, a_{7}, \, a_{8}, \, a_{12}, \, a_{13}, \, a_{14}, \, a_{16}\}, \, \mathbb{E}_{2} \rightarrow \{a_{3}, \, a_{6}, \, a_{9}, \, a_{10}, \, a_{12}, \, a_{13}, \, a_{15}, \, a_{16}\}, \\ & \mathbb{H}_{1} \rightarrow \{a_{4}, \, a_{7}, \, a_{9}, \, a_{11}, \, a_{12}, \, a_{14}, \, a_{15}, \, a_{16}\}, \, \mathbb{H}_{2} \rightarrow \{a_{5}, \, a_{8}, \, a_{10}, \, a_{11}, \, a_{13}, \, a_{14}, \, a_{15}, \, a_{16}\}, \\ & \Omega \rightarrow \{a_{1}, \, a_{2}, \, a_{3}, \, a_{4}, \, a_{5}, \, a_{6}, \, a_{7}, \, a_{8}, \, a_{9}, \, a_{10}, \, a_{11}, \, a_{12}, \, a_{13}, \, a_{14}, \, a_{15}, \, a_{16}\}, \\ & \left\{a_{1} \rightarrow \frac{45 \, 122 \, 521 \, 985 \, 018 \, 694 \, 529}{395 \, 712 \, 164 \, 128 \, 624 \, 041 \, 600}, \, a_{2} \rightarrow \frac{1}{979}, \, a_{3} \rightarrow \frac{4}{27}, \, a_{4} \rightarrow \frac{6}{19}, \, a_{5} \rightarrow \frac{1}{999}, \\ & a_{6} \rightarrow \frac{1}{985}, \, a_{7} \rightarrow \frac{1}{998}, \, a_{8} \rightarrow \frac{2}{45}, \, a_{9} \rightarrow \frac{1}{116}, \, a_{10} \rightarrow \frac{28}{141}, \, a_{11} \rightarrow \frac{1}{384}, \\ & a_{12} \rightarrow \frac{11}{100}, \, a_{13} \rightarrow \frac{1}{447}, \, a_{14} \rightarrow \frac{1}{165}, \, a_{15} \rightarrow \frac{2}{45}, \, a_{16} \rightarrow \frac{1}{999} \right\} \end{aligned}$$

TruthTable[MODEL3]

\mathbb{E}_1	\mathbb{E}_2	\mathbb{H}_1	\mathbb{H}_{2}	var	Pr
Т	Т	Т	Т	a ₁₆	1 999
Т	Т	Т	F	a ₁₂	11 100
Т	Т	F	Т	a ₁₃	1 447
Т	Т	F	F	a16	<u>1</u> 985
Т	F	Т	Т	a ₁₄	1 165
Т	F	Т	F	a17	1 998
Т	F	F	Т	a18	2 45
Т	F	F	F	a ₂	1 979
F	Т	Т	Т	a ₁₅	2 45
F	Т	Т	F	al 9	1 116
F	Т	F	Т	a ₁₀	28 141
F	Т	F	F	a ₃	$\frac{4}{27}$
F	F	Т	Т	a ₁₁	384
F	F	Т	F	a ₄	$\frac{6}{19}$
F	F	F	Т	a ₅	1 999
F	F	F	F	a ₁	45 122 521 985 018 694 529 395 712 164 128 624 041 600

```
EvaluateProbability[\{d[\mathbb{H}_1, \mathbb{E}_1], d[\mathbb{H}_2, \mathbb{E}_2]\},
      {r[\mathbb{H}_1, \mathbb{E}_1], r[\mathbb{H}_2, \mathbb{E}_2]}, MODEL3] // N
\{\{0.21837, 0.178694\}, \{0.182368, 0.229258\}\}
```

A model showing that d and ℓ are not ordinally equivalent:

```
MODEL4 = PrSAT[{
                                         d\left[\mathbb{H}_1,\,\mathbb{E}_1\right]\geq d\left[\mathbb{H}_2,\,\mathbb{E}_2\right],
                                         \ell[\mathbb{H}_1, \mathbb{E}_1] < \ell[\mathbb{H}_2, \mathbb{E}_2]
                             Probabilities → Regular]
     \Big\{ \{ \mathbb{E}_1 
ightarrow \{ \mathbb{a}_2, \, \mathbb{a}_6, \, \mathbb{a}_7, \, \mathbb{a}_8, \, \mathbb{a}_{12}, \, \mathbb{a}_{13}, \, \mathbb{a}_{14}, \, \mathbb{a}_{16} \}, \, \mathbb{E}_2 
ightarrow \{ \mathbb{a}_3, \, \mathbb{a}_6, \, \mathbb{a}_9, \, \mathbb{a}_{10}, \, \mathbb{a}_{12}, \, \mathbb{a}_{13}, \, \mathbb{a}_{15}, \, \mathbb{a}_{16} \}, \Big\}
                           \begin{split} \mathbb{H}_1 &\to \{ \texttt{a4, a7, a9, a11, a12, a14, a15, a16} \}, \ \mathbb{H}_2 &\to \{ \texttt{a5, a8, a10, a11, a13, a14, a15, a16} \}, \\ \Omega &\to \{ \texttt{a1, a2, a3, a4, a5, a6, a7, a8, a9, a10, a11, a12, a13, a14, a15, a16} \}, \end{split}
            \left\{\text{al}_{1} \rightarrow \frac{1\,525\,917\,434\,359}{4\,455\,027\,404\,640}\,\text{, al}_{2} \rightarrow \frac{1}{113}\,\text{, al}_{3} \rightarrow \frac{1}{21}\,\text{, al}_{4} \rightarrow \frac{1}{129}\,\text{, al}_{5} \rightarrow \frac{1}{132}\,\text{, al}_{6} \rightarrow \frac{1}{38}\,\text{, al}_{7} \rightarrow \frac{1}{30}\,\text{, al}_{
                        \exists_{8} \rightarrow \frac{1}{32}, \exists_{9} \rightarrow \frac{1}{28}, \exists_{10} \rightarrow \frac{1}{40}, \exists_{11} \rightarrow \frac{1}{138}, \exists_{12} \rightarrow \frac{1}{20}, \exists_{13} \rightarrow \frac{1}{9}, \exists_{14} \rightarrow \frac{1}{47}, \exists_{15} \rightarrow \frac{1}{496}, \exists_{16} \rightarrow \frac{8}{33} \}
```

I leave the remaining pairs as exercises.

■ Selection of Properties (1)-(8)

Verification that *r satisfies commutativity* (4):

```
PrSAT[{
     r[\mathbb{H}, \mathbb{E}] \neq r[\mathbb{E}, \mathbb{H}]
   }, BypassSearch → True]
 {}
Model showing that s violates (4):
MODEL5 = PrSAT[{
         Pr[H \mid E] > Pr[H],
        s[H, E] \neq s[E, H]
      }, Probabilities → Regular]
\left\{ \{\mathbb{E} \to \{\mathbb{a}_2,\,\mathbb{a}_4\}\,,\,\mathbb{H} \to \{\mathbb{a}_3,\,\mathbb{a}_4\}\,,\,\Omega \to \{\mathbb{a}_1,\,\mathbb{a}_2,\,\mathbb{a}_3,\,\mathbb{a}_4\} \}\,,\,\left\{\mathbb{a}_1 \to \frac{16\,511}{27\,972}\,,\,\mathbb{a}_2 \to \frac{1}{4}\,,\,\mathbb{a}_3 \to \frac{1}{999}\,,\,\mathbb{a}_4 \to \frac{10}{63} \right\} \right\}
TruthTable[MODEL5]
Pr
т
      Т
               a_4
Т
      F
F
      Т
               aз
                          16511
      F
               a_1
EvaluateProbability[\{s[H, E], s[E, H]\}, MODEL5] // N
{0.386657, 0.696209}
Model showing that \( \ell \) violates the so-called "Law of Likelihood" (2):
MODEL6 = PrSAT[{
        Pr[E \mid H1] > Pr[E \mid H2],
        \ell[H1, E] < \ell[H2, E]
      }, Probabilities → Regular]
 \{E \rightarrow \{a_2, a_5, a_6, a_8\}, H1 \rightarrow \{a_3, a_5, a_7, a_8\},\
   \begin{split} &\mathbb{H}2 \to \{\mathbb{a}_4,\,\mathbb{a}_6,\,\mathbb{a}_7,\,\mathbb{a}_8\},\,\Omega \to \{\mathbb{a}_1,\,\mathbb{a}_2,\,\mathbb{a}_3,\,\mathbb{a}_4,\,\mathbb{a}_5,\,\mathbb{a}_6,\,\mathbb{a}_7,\,\mathbb{a}_8\}\}\,,\\ &\Big\{\mathbb{a}_1 \to \frac{71\,843\,009\,866}{96\,210\,749\,943},\,\mathbb{a}_2 \to \frac{3}{43},\,\mathbb{a}_3 \to \frac{10}{99},\,\mathbb{a}_4 \to \frac{1}{177},\,\mathbb{a}_5 \to \frac{1}{189},\,\mathbb{a}_6 \to \frac{1}{629},\,\mathbb{a}_7 \to \frac{2}{29},\,\mathbb{a}_8 \to \frac{1}{999}\Big\}\Big\} \end{split}
Model showing that s violates (3) — people were very surprised when I reported this one:
MODEL7 = PrSAT[{
         Pr[H \mid E1] > Pr[H \mid E2],
        s[H, E1] < s[H, E2]
      }, Probabilities → Regular]
\{E1 \rightarrow \{a_2, a_5, a_6, a_8\}, E2 \rightarrow \{a_3, a_5, a_7, a_8\},\
   \begin{array}{l} \mathbb{H} \to \{ \mathbb{a}_4 \text{, } \mathbb{a}_6 \text{, } \mathbb{a}_7 \text{, } \mathbb{a}_8 \} \text{, } \Omega \to \{ \mathbb{a}_1 \text{, } \mathbb{a}_2 \text{, } \mathbb{a}_3 \text{, } \mathbb{a}_4 \text{, } \mathbb{a}_5 \text{, } \mathbb{a}_6 \text{, } \mathbb{a}_7 \text{, } \mathbb{a}_8 \} \} \text{,} \\ \left\{ \mathbb{a}_1 \to \frac{211\,701\,818}{1\,118\,425\,347} \text{, } \mathbb{a}_2 \to \frac{2}{23} \text{, } \mathbb{a}_3 \to \frac{4}{27} \text{, } \mathbb{a}_4 \to \frac{3}{41} \text{, } \mathbb{a}_5 \to \frac{1}{31} \text{, } \mathbb{a}_6 \to \frac{5}{39} \text{, } \mathbb{a}_7 \to \frac{16}{69} \text{, } \mathbb{a}_8 \to \frac{12}{109} \right\} \right\} 
EvaluateProbability[\{Pr[H \mid E1], Pr[H \mid E2], s[H, E1], s[H, E2]\}, MODEL7] // N
```

{0.666543, 0.654647, 0.191741, 0.233022}

Here is a bizarre measure χ , which satisfies (\mathcal{R}) , but violates (8). This was discovered by Crupi *et al*. See their paper: "Towards a Grammar of Bayesian Confirmation", which can be downloaded from my Mathcamp webpage http://fitelson.org/mathcamp/

```
\gamma[\mathbb{H}_{-}, \mathbb{E}_{-}] := \Pr[\mathbb{H} \mid \mathbb{E}]^2 - \Pr[\mathbb{H}]^2;
This model shows that \gamma violates (8):
MODEL8 = PrSAT[{
         Pr[E \mid H1] > Pr[E \mid H2],
        Pr[E \mid \neg H1] == Pr[E \mid \neg H2],
         \gamma[H1, E] < \gamma[H2, E]
      }, SearchAttempts → 10, Probabilities → Regular]
 \Big\{ \{\mathbb{E} \to \{ \texttt{a}_{2} \text{, a}_{5} \text{, a}_{6} \text{, a}_{8} \} \text{, } \mathbb{H}1 \to \{ \texttt{a}_{3} \text{, a}_{5} \text{, a}_{7} \text{, a}_{8} \} \text{, } \mathbb{H}2 \to \{ \texttt{a}_{4} \text{, a}_{6} \text{, a}_{7} \text{, a}_{8} \} \text{, }
     \Omega \to \{\text{al}_1, \text{al}_2, \text{al}_3, \text{al}_4, \text{al}_5, \text{al}_6, \text{al}_7, \text{al}_8\}\}, \\ \left\{\text{al}_1 \to \frac{50\,999\,038\,403}{426\,712\,555\,305}, \text{al}_2 \to \frac{14}{37}, \text{al}_8\right\}
     \texttt{a}_3 \rightarrow \frac{118\,434\,037}{126\,860\,489\,415} \text{, } \texttt{a}_4 \rightarrow \frac{1}{27} \text{, } \texttt{a}_5 \rightarrow \frac{5}{33} \text{, } \texttt{a}_6 \rightarrow \frac{9}{29} \text{, } \texttt{a}_7 \rightarrow \frac{1}{999} \text{, } \texttt{a}_8 \rightarrow \frac{1}{785} \Big\} \Big\}
PrSAT[{
      Pr[E \mid H1] > Pr[E \mid H2],
      Pr[E \mid \neg H1] < Pr[E \mid \neg H2],
      J[H1, E] < J[H2, E],
      Pr[H1] = 2Pr[H2]
   }, SearchAttempts → 10, Probabilities → Regular]
\{\mathbb{E} \to \{a_2, a_5, a_6, a_8\}, \mathbb{H}1 \to \{a_3, a_5, a_7, a_8\},
      \mathbb{H}\mathbf{2} \rightarrow \{ \texttt{a_4, a_6, a_7, a_8} \} \text{, } \Omega \rightarrow \{ \texttt{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8} \} \} \text{,}
   \left\{ \text{a}_{1} \rightarrow \frac{28\,502}{177\,859} \text{, } \text{a}_{2} \rightarrow \frac{4}{37} \text{, } \text{a}_{3} \rightarrow \frac{280\,153}{865\,260} \text{, } \text{a}_{4} \rightarrow \frac{7}{46} \text{, } \text{a}_{5} \rightarrow \frac{13}{90} \text{, } \text{a}_{6} \rightarrow \frac{1}{19} \text{, } \text{a}_{7} \rightarrow \frac{1}{22} \text{, } \text{a}_{8} \rightarrow \frac{1}{76} \right\} \right\}
We can verify that \gamma satisfies (\mathcal{R}), as follows:
PrSAT[{
      Pr[H \mid E] > Pr[H] \&\& \gamma[H, E] \le 0 \mid \mid
        Pr[H \mid E] == Pr[H] \&\& \gamma[H, E] \neq 0 \mid | Pr[H \mid E] < Pr[H] \&\& \gamma[H, E] \ge 0
   }, BypassSearch → True]
```

My Hint about exercise (ii)

FullSimplify
$$\left[\frac{x-y}{x+y} = \operatorname{Tanh}\left[\frac{1}{2}\operatorname{Log}\left[\frac{x}{y}\right]\right]\right]$$

True

{}

Both **tanh** and **log** are increasing functions. And, *increasing functions preserve order*.

■ Best Probability Bounds

Prsat has a built-in function **PrRange** for computing best possible bounds on a probability expression, subject to (arbitrary) *numerical* constriants. Here are some examples:

Modus ponens:

Out[63]=
$$\left\{ \frac{1}{6}, \frac{5}{6} \right\}$$

Constraints don't have to be equalities:

$$ln[72] = PrRange \left[Pr[B], Pr[B \mid A] < \frac{1}{4} \&\& Pr[A] > \frac{1}{3} \right]$$

Out[72]=
$$\left\{0, \frac{3}{4}\right\}$$

Modus tollens:

$$ln[75]:= PrRange \left[Pr[\neg A], Pr[B \mid A] == \frac{1}{4} \&\& Pr[\neg B] == \frac{1}{3} \right]$$

Out[75]=
$$\left\{ \frac{5}{9}, 1 \right\}$$

Denying the antecedent:

$$ln[76]:=$$
 PrRange $\left[Pr[\neg B], Pr[B \mid A] == \frac{1}{4} \&\& Pr[\neg A] == \frac{1}{3}\right]$

Out[76]=
$$\left\{ \frac{1}{2}, \frac{5}{6} \right\}$$

We can also solve analytically for general symbolic best possible bounds. Here are the bounds for modus ponens:

$$In[77]:=$$
 MPCons = AlgebraicForm[Pr[B | A] == x && Pr[A] == y, {A, B}]

Out[77]=
$$\frac{a_4}{a_2 + a_4} = x \&\& a_2 + a_4 = y$$

Out[78]=
$$a_{3} + a_{4}$$

$$ln[91]:=$$
 BackCons = 0 < a_2 < 1 && 0 < a_3 < 1 && 0 < a_4 < 1 && a_2 + a_3 + a_4 ≤ 1;

$$ln[95]:=$$
 MPMax = Maximize[{PrB, MPCons && BackCons}, {a₂, a₃, a₄}, Reals][[1]]

$$\text{Out} [95] = \left\{ \begin{array}{ll} 1 - y + x \ y & 0 < y < 1 \ \&\& \ 0 < x < 1 \\ -\infty & \text{True} \end{array} \right.$$

$$ln[96]:=$$
 FullSimplify[MPMax, 0 < y < 1 && 0 < x < 1]

Out[96]=
$$1 + (-1 + x) y$$

$$[[97]] = MPMin = Minimize[{PrB, MPCons && BackCons}, {a_2, a_3, a_4}, Reals][[1]]$$

$$\text{Out} \texttt{[97]=} \quad \left\{ \begin{array}{ll} x \; y & 0 < y < 1 \; \&\& \; 0 < x < 1 \\ \infty & \text{True} \end{array} \right.$$

$$ln[98]:=$$
 FullSimplify[MPMin, 0 < y < 1 && 0 < x < 1]

And, here are the bounds for *modus tollens*:

In[99]:= MTCons = AlgebraicForm[Pr[B | A] ==
$$x & Pr[\neg B] == y$$
, {A, B}]

Out[99]=
$$\frac{a_4}{a_2 + a_4} = x \& \& 1 - a_3 - a_4 = y$$

Out[101]=
$$1 - a_2 - a_4$$

$$\text{Out[102]=} \ \left\{ \begin{array}{ll} 1 & 0 < y < 1 \, \&\& \, 0 < x < 1 \\ -\infty & True \end{array} \right.$$

ln[103]:= FullSimplify[MTMax, 0 < y < 1 && 0 < x < 1]

Out[103]= 1

 $[[1]] = MTMin = Minimize[{PrNotA, MTCons && BackCons}, {a_2, a_3, a_4}, Reals][[1]]$

$$\text{Out[104]=} \begin{array}{l} \left\{ \begin{array}{l} \frac{-1 + x + y}{-1 + x} & \text{$(0 < y < 1 \&\& x = 1 - y)$ } \mid \mid \; (0 < y < 1 \&\& 0 < x < 1 - y) \\ \frac{-1 + x + y}{x} & \text{$0 < y < 1 \&\& 1 - y < x < 1$} \\ \infty & \text{True} \end{array} \right.$$

ln[105] = FullSimplify[MTMin, 0 < y < 1 && 0 < x < 1]

$$\text{Out[105]=} \ \left\{ \begin{array}{ll} 1 + \frac{y}{-1+x} & x+y \leq 1 \\ \frac{-1+x+y}{x} & \text{True} \end{array} \right.$$

ln[106] = FullSimplify[MTMin, 0 < y < 1 && 0 < x < 1 && x + y ≤ 1]

Out[106]=
$$1 + \frac{y}{-1 + x}$$

 $\label{eq:local_local_local_local_local} $$ \ln[107] := \mbox{ FullSimplify[MTMin, 0 < y < 1 \&\& 0 < x < 1 \&\& x + y > 1] } $$$

$$Out[107] = \frac{-1 + x + y}{x}$$