My Rendition of Hunter's Proof of Metatheorem 45.12

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Theorem. Let K be a consistent first order theory. And, let $K' = K + \{\alpha\}$, where α is an arbitrary (particular) well-formed formula of the following form (added as a new proper axiom to K to form K'):

$$(\alpha) \qquad Ac/v \supset \bigwedge vA$$

where (i) A is a well-formed formula of K, (ii) c does not occur in any proper axiom of K, (iii) c does not occur in A, and (iv) $\bigwedge vA$ is a closed formula of K (as a result, v is the only variable that possibly occurs free in A). Then, K' is also a consistent first order theory, and one which is an extension of K.

Proof. First, we will show that K' is a first order theory that extends K. Then, we will show that K' is consistent. To see that K' is a first order theory that extends K, note that any WFF of form α will be a *closed* formula. This is because (a) its consequent $\bigwedge vA$ is closed by hypothesis (iv) of the theorem, and (b) its antecedent Ac/v is closed (since, again by hypothesis (iv) of the theorem, v is the only variable that possibly occurs free in A, and we are replacing all of its free occurrences in A with a constant c). And, any conditional with a closed antecedent and a closed consequent is itself closed. Thus, since K' is just K plus a closed formula α, K' is just K with one additional proper axiom. So, since K is a first order theory, K' is a first order theory which extends K. That was the easy part. Now, we have to show that K' is *consistent*.

Assume, for *reductio*, that K' is inconsistent. That is, $K + \{\alpha\}$ is inconsistent. Then, we can reason:

1. $K + \{\alpha\}$ is inconsistent (*i.e.*, K' is inconsistent) [*reductio* assumption]
2. $\vdash_K \sim(Ac/v \supset \bigwedge vA)$ (*i.e.*, $\vdash_K \sim\alpha$) [(1), metatheorem 45.6(b)]
3. $\vdash_K \sim(Ac/v \supset \bigwedge vA) \supset Ac/v$ [45.3, instance of tautological schema]
4. $\vdash_K \sim(Ac/v \supset \bigwedge vA) \supset \sim\bigwedge vA$ [45.3, instance of tautological schema]
5. $\vdash_K Ac/v$ [MP, (2), (3)]
6. $\vdash_K \sim\bigwedge vA$ [MP, (2), (4)]

Now, the strategy will be to use (5) to show that $\vdash_K \bigwedge vA$. This, together with (6), will imply that K is inconsistent. That will contradict the hypothesis of the theorem, hence refuting our *reductio* assumption, and establishing that K' is in fact consistent. So, the goal is to prove $\vdash_K \bigwedge vA$ from $\vdash_K Ac/v$, using (i)–(iv). Since $\vdash_K Ac/v$, we know there is a proof $\langle B_1, \ldots, B_m\rangle$ of Ac/v in K. Let u be any variable that does not occur in this proof. As a result, note that u does not occur in A, since this proof is a proof of $\bigwedge vA$. Let B_i' be the result of substituting u for c in B_i. That is, $B_i' = B_i u/c$.[1] Now, $\langle B_1', \ldots, B_m'\rangle$ is also proof of in K. To see this, note that each B_i is either a logical axiom, a proper axiom, or an immediate consequence by modus ponens (MP) from two previous lines, and that this is also true for each B_i'. There are three cases:

I. If B_i is a logical axiom, then so is B_i'. There are five sub-cases here:

   (a) B_i is a logical axiom by K1–K3. Then, B_i is an instance of a propositional axiom schemata. In this case, $B_i' = B_i u/c$ is also an instance of a propositional axiom schemata, since merely substituting u for c in B_i cannot change the propositional logical form of any formula B_i.

[1] I am using the notation Au/c to stand for the formula obtained by substituting the variable u for *every* occurrence of the constant c in A. Hunter only uses this notation when variables appear on the right hand side of the slash. Recall, on this more standard usage of Hunter's, At/v stands for the formula obtained by substituting the term t for every *free* occurrence of the variable v in A. In my usage, this restriction will apply when a variable appears on the right hand side of the slash.

(b) B_i is a logical axiom by K4. So, B_i is of the form $\bigwedge vC \supset Ct/v$, where t is free for v in C. Thus, $B_i' = (\bigwedge vC \supset Ct/v)u/c = (\bigwedge vCu/c \supset (Ct/v)u/c) = (\bigwedge vCu/c \supset (Cu/c)t/v)$, where t is free for v in C. This is because u does not occur in B_i (hence, u does not occur in C, Ct/v, or t), and so $(Ct/v)u/c = (Cu/c)t/v$. Therefore, B_i' is of the form $\bigwedge vD \supset Dt/v$ (with $D = Cu/c$), where t is free for v in D. So, B_i' is also a logical axiom by K4.

(c) B_i is a logical axiom by K5. So, B_i is of the form $C \supset \bigwedge vC$, where v does not occur free in C. Thus, $B_i' = (C \supset \bigwedge vC)u/c = (Cu/c \supset \bigwedge vCu/c)$. Therefore, B_i' is of the form $D \supset \bigwedge vD$ (with $D = Cu/c$), where v does not occur free in D. So, B_i' is also a logical axiom by K5.

(d) B_i is a logical axiom by K6. So, B_i is of the form $\bigwedge v(C \supset D) \supset (\bigwedge vC \supset \bigwedge vD)$. Thus, $B_i' = (\bigwedge v(C \supset D) \supset (\bigwedge vC \supset \bigwedge vD))u/c = \bigwedge v(Cu/c \supset Du/c) \supset (\bigwedge vCu/c \supset \bigwedge vDu/c)$, which is also a logical axiom by K6.

(e) B_i is a logical axiom by K7. So, B_i is of the form $\bigwedge vC$, where C is a logical axiom by K1–K6. Thus, $B_i' = \bigwedge vCu/c$, where Cu/c is a logical axiom by the above six arguments (which show that if C is an axiom by K1–K6, then so is $C' = Cu/c$). Hence, by K7, B_i' is also a logical axiom.

II. If B_i is a proper axiom of K, then so is B_i'. For, if B_i is a proper axiom, then $B_i' = B_iu/c = B_i$, since c does not occur in any proper axiom of K (so "replacing" c with u in B_i does nothing to B_i).

III. If B_i is an immediate consequence by MP of two previous lines in $\langle B_1, \ldots, B_m\rangle$, then B_i' is an immediate consequence by MP of two previous lines in $\langle B_1', \ldots, B_m'\rangle$. Assume B_i is an immediate consequence by MP of two previous lines B_j and $B_j \supset B_i$. Then, $(B_j \supset B_i)' = (B_j' \supset B_i')$, and B_i' will be an immediate consequence by MP of B_j' and $(B_j' \supset B_i') = (B_j \supset B_i)'$.

Therefore, a proof of $\vdash_K Ac/v$ can be turned into a proof of $\vdash_K (Ac/v)'$, *i.e.*, $\vdash_K (Ac/v)u/c$. And, we reason:

5. $\vdash_K Ac/v$ [established above]
7. $\vdash_K (Ac/v)u/c$ [(5), our proof above, and $(Ac/v)' = (Ac/v)u/c$]
8. $\vdash_K Au/v$ [(7), c does not occur in A]
9. $\vdash_K \bigwedge uAu/v$ [(8), metatheorem 45.4]
10. $\vdash_K \bigwedge uAu/v \supset (Au/v)v/u$ [axiom K4, v is free for u in Au/v since u does not occur in A]
11. $\vdash_K \bigwedge uAu/v \supset A$ [(10), $(Au/v)v/u = A$]
12. $\vdash_K A$ [(9), (11), MP]
13. $\vdash_K \bigwedge vA$ [(12), metatheorem 45.4]
14. $\vdash_K \bigwedge vA$ and $\vdash_K \sim\bigwedge vA$ [(13) and (6)]
15. K is inconsistent. [(14), definition of inconsistency of K]

Therefore, our assumption that $K' = K+\{\alpha\}$ is inconsistent has led to the conclusion that K is inconsistent. But, K is consistent by the hypothesis of the theorem. Contradiction. And, by *reductio ad absurdum*, we conclude that $K' = K + \{\alpha\}$ is, in fact, consistent. This completes the proof of metatheorem 45.12. □