PHIL 424: HW #3 Solutions

October 26, 2014

2 Irrational Numbers

One might think that real humans only assign credences that are rational numbers—and perhaps only rational numbers involving relatively small whole-number numerators and denominators. But we can write down simple conditions that require *an irrational-valued credence function. For example, these three conditions:*

1. $\mathrm{cr}(Y \mid X) = \mathrm{cr}(X \vee Y)$
2. $\mathrm{cr}(X \,\&\, Y) = 1/4$
3. $\mathrm{cr}({\sim}X \,\&\, Y) = 1/4$

Show that there is exactly one credence distribution over language $\mathcal{L}$ with atomic propositions X and Y that satisfies all three of these conditions, and that that distribution contains irrational-valued credences

Let's construct the stochastic truth-table and see what happens.

X	Y	cr
T	**T**	$1/4$
T	**F**	a
F	**T**	$1/4$
F	**F**	$1 - 1/4 - 1/4 - a = 1/2 - a$

By the ratio formula, $\mathrm{cr}(Y|X) = \frac{\mathrm{cr}(Y\&X)}{\mathrm{cr}(X)} = \frac{1/4}{1/4+a}$. From the stochastic truth table, we also know that $\mathrm{cr}(X \vee Y) = 1/2 + a$. So, by the first constraint, $\frac{1/4}{1/4+a} = 1/2 + a$. Doing some algebra:

$$\begin{aligned}
\frac{1/4}{1/4 + a} &= 1/2 + a \\
&\Downarrow \\
1/4 &= (1/4 + a)(1/2 + a) \\
&\Downarrow \\
1/4 &= a^2 + \frac{3}{4}a + 1/8 \\
&\Downarrow \\
0 &= a^2 + \frac{3}{4}a - 1/8 \\
&\Downarrow \\
0 &= 8a^2 + 6a - 1 \\
&\Downarrow \\
a &= \frac{1}{8}(\sqrt{17} - 3) \qquad \text{(The other solution is negative, so cannot be a credence)}
\end{aligned}$$

If a were rational, then $\sqrt{17} = 8a - 3$ would be. But $\sqrt{17}$ is irrational, so a is, too.

Point Values

This question was worth 25 points. Partial credit was awarded.

6 Stochastic Truth-Table

Once more, consider the probabilistic credence distribution specified by this stochastic truth-table (from Exercise 2.5):

P	Q	R	cr
T	**T**	**T**	0.1
T	**T**	**F**	0.2
T	**F**	**T**	0
T	**F**	**F**	0.3
F	**T**	**T**	0.1
F	**T**	**F**	0.2
F	**F**	**T**	0
F	**F**	**F**	0.1

Answer the following questions about this distribution:

What is cr$(P|Q)$?

By the Ratio Formula, $\mathrm{cr}(P \mid Q) = \frac{\mathrm{cr}(P\&Q)}{\mathrm{cr}(Q)}$. $\mathrm{cr}(P \& Q) = 0.1 + 0.2 = 0.3$. $\mathrm{cr}(Q) = 0.1 + 0.2 + 0.1 + 0.2 = 0.6$. So $\mathrm{cr}(P \mid Q) = 1/2$.

Is Q positively relevant to P, negatively relevant to P, or probabilistically independent of P?

$\mathrm{cr}(P) = 0.1 + 0.2 + 0 + 0.3 = 0.6$. So $\mathrm{cr}(P \mid Q) = 1/2 < 0.6 = \mathrm{cr}(P)$. So Q is negatively relevant to P.

What is cr$(P \mid R)$?

By the Ratio Formula, $\mathrm{cr}(P \mid R) = \frac{\mathrm{cr}(P\&R)}{\mathrm{cr}(R)}$. $\mathrm{cr}(P \& R) = 0.1 + 0 = 0.1$. $\mathrm{cr}(R) = 0.1 + 0 + 0.1 + 0 = 0.2$. So $\mathrm{cr}(P \mid R) = 1/2$.

What is cr$(P \mid Q \& R)$?

By the Ratio Formula, $\mathrm{cr}(P \mid Q \& R) = \frac{\mathrm{cr}(P\&Q\&R)}{\mathrm{cr}(Q\&R)}$. $\mathrm{cr}(P \& Q \& R) = 0.1$.
$\mathrm{cr}(Q \& R) = 0.1 + 0.1 = 0.2$. So $\mathrm{cr}(P \mid Q \& R) = 1/2$.

Conditional on R, is Q positively relevant to P, negatively relevant to P, or probabilistically independent of P?

$\mathrm{cr}(P \mid Q \& R) = 1/2 = \mathrm{cr}(P \mid R)$. So, conditional on R, Q is probabilistically independent of P.

Does R screen off P from Q? Explain why or why not.

Yes. P is unconditionally dependent on Q because $\mathrm{cr}(P \mid Q) \neq \mathrm{cr}(P)$, but, as in part (e), P is probabilistically independent of Q conditional on R. But this is just to say that R screens off P from Q.

Point Values

This question was worth 25 points, with each of the six parts worth an equal number of points.

8 Transitivity

Show that probabilistic independence is not transitive. That is, provide a single probability distribution on which all of the following are true: X is independent of Y, and Y is independent of Z, but X is not independent of Z. Show that your distribution satisfies all three conditions. (For an added challenge, have your distribution assign every state-description a nonzero unconditional credence.)

Many answers are acceptable, here is the simplest:

P	cr
T	1/4
F	3/4

Letting $X := P$, $Y := P \vee {\sim}P$, and $Z := {\sim}P$, we see $\text{cr}(X \mid Y) = \text{cr}(P \mid P \vee {\sim}P) = 1/4 = \text{cr}(P) = \text{cr}(X)$, so X and Y are independent. $\text{cr}(Y \mid Z) = \text{cr}(P \vee {\sim}P \mid {\sim}P) = 1 = \text{cr}(P \vee {\sim}P) = \text{cr}(Y)$, so Y and Z are independent. However, $\text{cr}(X \mid Z) = \text{cr}(P \mid {\sim}P) = 0 \neq 1/4 = \text{cr}(P) = \text{cr}(X)$. So X and Z are not independent.

Point Values

This question was worth 25 points. Partial credit was awarded.

9 Collective Independence

In the text we discussed what makes a pair *of propositions probabilistically independent. If we have a larger collection of propositions, what does it take to make them all independent of each other? You might think all that's necessary is* pairwise independence*?for each pair within the set of propositions to be independent. But pairwise independence doesn't guarantee that each proposition will be independent of* combinations *of the others.*

To demonstrate this fact, describe a real-world example (spelling out the propositions represented by X, Y, and Z) in which it would be rational for an agent to assign credences meeting all four of the following conditions:

1. $\text{cr}(X \mid Y) = \text{cr}(X)$
2. $\text{cr}(X \mid Z) = \text{cr}(X)$
3. $\text{cr}(Y \mid Z) = \text{cr}(Y)$
4. $\text{cr}(X \mid Y \,\&\, Z) \neq \text{cr}(X)$

Show that your example satisfies all four conditions.

Many answers are acceptable.

Coin flips are usually good for making examples involving probabilistic independence, since they are independent of each other and it's easy to think about the probabilities involved. Here is one example:

Suppose we are going to flip two fair coins. Let Y be the proposition that the first coin lands heads. Let Z be the proposition that the second coin lands heads. We can see that Y and Z are probabilistically independent: $\text{cr}(Y \mid Z) = \frac{\text{cr}(Y\&Z)}{\text{cr}(Z)}$. $\text{cr}(Y \,\&\, Z) = 1/4$ because exactly 1 of the four equally-likely outcomes (HH, HT, TH, TT) makes $Y \,\&\, Z$ true. $\text{cr}(Z) = 1/2$ because the coins are fair. So we get $\text{cr}(Y \mid Z) = 1/4 \cdot 2/1 = 1/2$. On the other hand, $\text{cr}(Y) = 1/2$ because the coins are fair. So Y and Z are probabilistically independent. Condition 3 is satisfied

Finally, let X be the proposition "The first coin lands heads iff the second coin does" (equivalently, both land heads or both land tails). $\text{cr}(X \mid Y) = \text{cr}(Y \equiv Z \mid Y) = \frac{\text{cr}((Y\equiv Z)\&Y)}{\text{cr}(Y)} = \frac{\text{cr}(Y\&Z)}{\text{cr}(Y)} = 1/4 \cdot 2/1 = 1/2$. On the other hand, $\text{cr}(X) = 1/2$ because exactly 2 of the 4 equally likely outcomes make $Y \equiv Z$ true. So X and Y are probabilistically independent. A similar proof shows that X and Z are probabilistically independent. So the first and second conditions are satisfied.

However, $\text{cr}(X \mid Y \,\&\, Z) = \frac{\text{cr}(X\&Y\&Z)}{\text{cr}(Y\&Z)} = \frac{\text{cr}((Y\equiv Z)\&Y\&Z)}{\text{cr}(Y\&Z)} = \frac{\text{cr}(Y\&Z)}{\text{cr}(Y\&Z)} = 1 \neq 1/2 = \text{cr}(X)$. So the fourth condition is satisfied. This works because, by propositional logic $(Y \equiv Z) \,\&\, Y \,\&\, Z$ is equivalent to $Y \,\&\, Z$.

Point Values

This question was worth 25 points. Partial credit was awarded.

10 Extra Credit: Using PrSAT

Using the program PrSAT referenced in the Further Readings for Chapter 2, find a probability distribution satisfying all the condition in Exercise 3.9, plus the following additional *condition: Every state-description expressible in terms of X, Y, and Z must have a non-zero unconditional probability.*

I have created a *Mathematica* (v10) notebook which contains a solution to this extra-credit problem. It can be downloaded from the following URL:

`http://fitelson.org/bayes/hw3.nb`

Here is the salient input and output from that notebook.

MODEL = PrSAT[{

$\mathbf{Pr}[X|Y] == \mathbf{Pr}[X]$,

$\mathbf{Pr}[X|Z] == \mathbf{Pr}[X]$,

$\mathbf{Pr}[Y|Z] == \mathbf{Pr}[Y]$,

$\mathbf{Pr}[X|Y \wedge Z] \neq \mathbf{Pr}[X]$,

$\mathbf{Pr}[X] == \frac{1}{2}, \mathbf{Pr}[Y] == \frac{1}{4}, \mathbf{Pr}[Z] == \frac{1}{2}$

}, Probabilities → Regular, BypassSearch → True];

TruthTable[MODEL]

X	Y	Z	var	Pr
T	T	T	$\mathbf{a}_8$	$\frac{4}{77}$
T	T	F	$\mathbf{a}_5$	$\frac{45}{616}$
T	F	T	$\mathbf{a}_6$	$\frac{61}{308}$
T	F	F	$\mathbf{a}_2$	$\frac{109}{616}$
F	T	T	$\mathbf{a}_7$	$\frac{45}{616}$
F	T	F	$\mathbf{a}_3$	$\frac{4}{77}$
F	F	T	$\mathbf{a}_4$	$\frac{109}{616}$
F	F	F	$\mathbf{a}_1$	$\frac{61}{308}$

Point Values

This question was worth 10 points of extra credit.