What Is Graded Membership?

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Abstract

It has seemed natural to model phenomena related to vagueness in terms of graded membership. However, so far no satisfactory answer has been given to the question of what graded membership is nor has any attempt been made to describe in detail a procedure for determining degrees of membership. We seek to remedy these lacunae by building on recent work on typicality and graded membership in cognitive science and combining some of the results obtained there with a version of the conceptual spaces framework.

We classify some shades as red readily and unhesitatingly and other shades as orange just as readily and unhesitatingly. But some shades strike us as being neither quite red nor quite orange but rather—as we might say—as being "reddish–orangish." It is not that such shades are clearly outside the category of red in the way that, for instance, any shade of green is clearly outside that category. But neither are they clearly within the category. It is as if they are within the category to some extent, but also outside the category to some extent.

What goes for "red" and "orange" goes for many other terms in our language. Indeed, Braun and Sider [2007] go so far as to suggest that there is little that we say that is not to some extent infected by vagueness. Familiar though this type of phenomenon may be, it has turned out to be far from easy to make sense of it. Various authors have tried to model phenomena of vagueness in terms of a notion of graded or partial membership.[1] However, it seems that such models must leave something to be desired so long as the notion of graded membership has not itself been clarified. To be sure, logicians have developed so-called fuzzy set theories that formalize a graded membership relation (e.g., Zadeh [1965], Hájek [1998]). But while these theories, and in particular the accompanying fuzzy logics, have been successfully applied in a variety of practical domains—ranging from air traffic control to consumer electronics—theorists have remained skeptical about the notion of graded membership that lies at their root. As the statistician Lindley [2004:877] notes: "One of the difficulties many people have with the ideas of fuzzy logic lies in the interpretation of the membership function: What does it mean to say that $m(x) = .2$ [where m is a graded membership function]?"

[1] See, e.g., Lakoff [1973], Machina [1976], Kamp and Partee [1995], and Hampton [2007].

In this paper, we address the question of what to make of graded membership from a psychological rather than from a set-theoretic or logical perspective. In aiming to answer this question, we look at recent psychological literature on vagueness, typicality, and graded membership, and we take our cue from proposals by Kamp and Partee and by Hampton, which account for graded membership in terms of similarity to prototype. We intend to add both predictive power and explanatory depth to these proposals by describing a precise way of measuring degrees of membership. The upshot will be an operational definition of graded membership that is based on findings in the cognitive sciences and which ultimately explains graded membership by reference to how the mind conceptualizes reality.

Some will say that our proposal comes at a price. Whereas neither Kamp and Partee nor Hampton make any specific assumptions about the similarity relation, we assume a geometrical conception of similarity, and more broadly the conceptual spaces framework, as developed by a number of authors in cognitive science and as presented in its clearest form in Gärdenfors [2000]. Although the conceptual spaces framework is enjoying increasing popularity, we are aware that it is not universally accepted. Moreover, it is presently unclear what the scope of the framework is. While it seems eminently suitable for modelling perceptual concepts—such as color concepts and auditory concepts—it remains to be seen how far it can be extended to model other concepts as well. That being said, within the conceptual spaces framework, a perfectly natural answer can be given to the question of what graded membership is, and we hope that this fact is a reason for critics of the framework to reconsider.

Section 1 sets the stage by recounting a recent discussion in cognitive science about the relation between graded membership and similarity. Section 2 summarizes the conceptual spaces approach and describes a recent add-on to that approach that has been explicitly introduced to accommodate various phenomena of vagueness. Section 3 discusses Kamp and Partee's [1995] proposal for defining graded membership and also points out certain inadequacies of this proposal. And Section 4, finally, combines Kamp and Partee's proposal with the version of the conceptual spaces approach summarized in Section 3 to come to a new account of graded membership.

1. Prototype theory and graded membership. To say that a shade falls under a color concept to some non-extreme extent is to say that the concept lacks a sharply defined boundary; otherwise the shade would be either to one side of the boundary or to the other, and accordingly would either fall fully under the given concept or not fall under the concept at all. Thus, any theory of the type of phenomenon we aim to elucidate will have to account for what goes on at concept boundaries and in particular for how such boundaries can lack sharpness. Ideally, such a theory starts with a precise characterization of what concepts are. Unfortunately, however, there is no universally agreed upon theory of concepts; some authors even doubt that such a theory is forthcoming (Machery [2009]). According to the traditional Aristotelian–Fregean conception, concepts are characterized by necessary and sufficient conditions for membership. But this conception sits badly with much experimental work carried out by cognitive psychologists over the past forty years or so (Murphy [2004:40]). In light of this work, most psychologists have abandoned the traditional account of concepts, many of them in favor of some version of prototype theory, which will also serve as our background theory in the following.[2,3]

Prototype theory starts from the observation that some instances of a concept are more representative or typical of it than others, and that some instance or instances are more representative than *any* other instance. These most representative instances are the prototypical instances, or prototypes, of the concepts. A further essential element of prototype theory is a similarity function which is supposed to underlie concept formation in that whether or not something falls under a given concept depends on how similar the entity is to that concept's prototype.

This still leaves wide open the question of whether entities can fall under a concept to only some extent. Here it is relevant how we understand the notion of similarity. There are at least two competing accounts of similarity to be found in the literature, a set-theoretic one and a geometric one (see Goldstone and Sun [2005]). The set-theoretic account represents objects as sets of properties and explains similarity between objects in terms of sets of shared and non-shared properties. On a standard version of this view, an entity must share at least a certain number of properties with the concept's prototype in order to fall under the concept (where these properties may be weighted, in the manner of Tversky [1977]), and otherwise it does *not* fall under the concept—which makes falling under a concept a categorical matter. Basically the same is true for the standard version of the geometric account, which represents concepts geometrically and measures similarity between objects in much the same way we measure distances between different locations. We will have more to say about this account in the next section, in which we will also look at a refined version of it. But for now the crucial point to note is that on the common version of the geometric account of similarity, it is again a yes–no matter whether something falls under a concept.

Still, the basic tenets of prototype theory as summarized above do not commit one to a categorical notion of falling under a concept. To the contrary, one would think that once it is assumed that things can be more or less similar to a prototype, it should not be so hard to make sense of a graded notion of membership that permits things to fall under a concept to different degrees.

Osherson and Smith [1981] discuss a version of prototype theory that exploits precisely the fact that similarity is a graded notion to arrive at a graded membership relation. They take concepts to be represented by means of quadruples $\langle A, d, p, c\rangle$, with A the conceptual domain; d a distance function defined on that domain, indicating for each pair of elements of A how dissimilar they are to each other; p the

[2]Something like the Aristotelian–Fregean position still survives in psychology in the form of theory theory, which does not seem to enjoy much popularity nowadays, however. A more popular alternative is exemplar theory, in which instances that are or were central in acquiring a concept play much the same role that prototypes play in prototype theory. We assume prototype theory here largely because, first, it underlay the discussion—to be summarized in the text—which inspired our own view of graded membership, and second, it is a basic assumption of the account of borderline vagueness that we will use in this paper. Our reliance on prototype theory may further limit the scope of our proposal (next to our reliance on the conceptual spaces framework). It has been argued, even by authors sympathetic to prototype theory, that not all concepts can be assumed to have prototypical instances. For instance, Kamp and Partee [1995:143, 176] claim that TALL, HOT, HEAVY, BIG, and WIDE, among other concepts, do not possess prototypical instances. This claim has been contested, however; see Tribushinina [2008, Ch. 9] and references given there.

[3]Of course, philosophers have abandoned the Aristotelian–Fregean conception of concepts as well, though mostly under the influence of the criticisms to be found in Putnam [1975] and Kripke [1980].

concept's prototype; and c a function that indicates to what degree an object falls under the concept. The function c takes values in the real number interval between 0 and 1 inclusive and is a monotonically decreasing function of distance from prototype, meaning that for any prototype p and related distance function d, c satisfies this condition:

$$\forall x, y \in A\colon\ d(x,p) \leqslant d(y,p)\ \rightarrow\ c(y) \leqslant c(x).$$

Osherson and Smith present this version of prototype theory only to argue that it is inadequate; in particular, that it makes the wrong predictions about combined concepts.

The issue of concept combination is of central importance to the project of devising an adequate semantics for fuzzy logics, but it is orthogonal to our present goal, which is to clarify the nature of the relation that is likely to form the backbone of any such semantics, to wit, graded membership.[4] For our present dialectical purposes, Osherson and Smith's paper is important mainly for raising the question of how prototype theorists are to account for graded membership.

In a partly constructive, partly critical response to Osherson and Smith's paper, Kamp and Partee [1995] argue that Osherson and Smith run two functions together that should be kept separate: the graded membership function M and the typicality or goodness of example function T.[5,6] As Kamp and Partee [1995:133] note, a pelican unarguably falls under the concept BIRD, even though it is a rather atypical bird. (For our concerns, the chief constructive contribution of Kamp and Partee's paper consists in their use of a supervaluationist semantics for modelling graded membership. This will be discussed in Section 3.)

In their reply to Kamp and Partee, Osherson and Smith [1997] acknowledge the need to distinguish between M and T, and argue that the two functions differ even more than Kamp and Partee seem to realize. According to the latter authors, the two functions have at least the same scale, but not even that much is true, Osherson and Smith contend. While it is natural to take $[0,1]$ as the range of M, for some concepts T may go to infinity. For example, however typical an instance of the concept BULLY a given person may be, we have no difficulty imagining one that is still more typical of it (e.g., by being still a bit more vicious). Osherson and Smith further argue that the two functions also have different bases, in that T, but not M, is a function of similarity to prototype. After all, it is patently because of their greater similarity to the prototype of BIRD that robins receive higher T values than pelicans receive. Yet, equally patently, this greater similarity does not manifest itself in a difference in M values.

As Hampton [2007] points out, however, this last conclusion is rash. That M does not differentiate among the clear instances of a concept between the more and

[4] For an argument to the effect that combining concepts poses no special problems to prototype theory when this is embedded in the conceptual spaces approach, see Gärdenfors [2000:114–122]. See Prinz [2012] for a more general defense of prototype theory against objections from concept combination.

[5] See, in the same vein, Fuhrmann [1991]. For earlier critiques of Osherson and Smith [1981], see Jones [1982] and Zadeh [1982]. See Osherson and Smith [1982] for a response to Jones and Zadeh.

[6] Kamp and Partee actually use c^e to designate the graded membership function and c^p to designate the typicality function. We follow Hampton [2007] in using the more transparent notations M and T for these functions. Also, it is misleading to speak of *the* graded membership function and *the* typicality function: each concept has its own such functions. So, to be entirely exact, M and T should always be indexed. However, again following Hampton, we omit indexes when speaking about such functions generically, and when no confusion can arise.

less typical ones does not mean that M is not a function of similarity to prototype. Indeed—Hampton notes—it may still hold that $T = t(\mathrm{sim})$ and $M = m(\mathrm{sim})$, where t and m are nondecreasing functions of the same similarity measure sim. For the aforementioned reasons, t may then be assumed to have the range $[0, \infty\rangle$ while m takes values in the interval $[0, 1]$ only. Furthermore, we may set some threshold value of similarity to prototype to define concept boundaries, in the sense that anything which is above the threshold belongs to the category. That makes it easy to explain the robin–pelican contrast: robins are more typical birds than pelicans are, but both robins and pelicans are above the similarity threshold for determinate membership in the category BIRD.

Hampton [2007:365] then goes on to give a precise definition of M as a function of the similarity measure. In doing so, he assumes that for each concept there is a determinate boundary region for membership, and that the values S_L and S_H of the similarity function are respectively the lower and upper bound of the boundary region.[7] The value S_T is the value where $M = .5$, and it is assumed that this is midway between S_L and S_H. Then, where $S(x)$ measures the similarity of x to the relevant prototype, the function M is defined as follows:[8]

$$M(x) = \begin{cases} 0 & \text{if } S_L \geqslant S(x); \\ 2\left(\frac{S(x)-S_L}{S_H-S_L}\right)^2 & \text{if } S_T \geqslant S(x) > S_L; \\ 1 - 2\left(\frac{S_H-S(x)}{S_H-S_L}\right)^2 & \text{if } S_H \geqslant S(x) > S_T; \\ 1 & \text{if } S(x) > S_H. \end{cases}$$

According to Hampton, the membership function, thus defined, has several attractive features. First, there are regions of determinate membership and non-membership where M takes the values 1 and 0, respectively, which is in accordance with pretheoretical intuitions. Second, the definition is consistent with T assuming different values for instances whose M value is 1, which is again pretheoretically as it should be. Third, there are smooth transitions at the endpoints of the boundary region, which may be enough to handle the issue of second-order vagueness, that is, the fact that there do not seem to be sharp dividing lines between, on the one hand, the clear and not-so-clear members of a category and, on the other hand, the not-so-clear members and clear non-members (see, e.g., Sainsbury [1991] and Keefe and Smith [1997:15 f]). And fourth, M's S-shaped form (see Figure 1, upper right graph) fits rather well the data concerning categorization probabilities based on empirical findings reported in McCloskey and Glucksberg [1978].[9]

We believe that Hampton's threshold model of graded membership is working along the right lines, and that the aforementioned features are attractive indeed. At the same time, the proposal raises issues that Hampton does not address. First, the proposal is very specific regarding the form of the graded membership function.

[7] These values may be different for different concepts, and so, like M and T, should actually be indexed. Again, we take the indices to be read.

[8] Hampton's definition of M has $>$ everywhere, also where we have $\geqslant$. That is presumably just an oversight, as it would make M undefined at S_L, S_T, and S_H, which cannot be intended.

[9] Or rather—as Hampton notes—it fits those data rather well if it is assumed that categorization probability is linearly related to graded membership and similarity is linearly related to typicality.

Consider the following generalization of M:

$$M(x,n) = \begin{cases} 0 & \text{if } S_L \geqslant S(x); \\ 2^n \left(\frac{S(x)-S_L}{S_H-S_L}\right)^{n+1} & \text{if } S_T \geqslant S(x) > S_L; \\ 1 - 2^n \left(\frac{S_H-S(x)}{S_H-S_L}\right)^{n+1} & \text{if } S_H \geqslant S(x) > S_T; \\ 1 & \text{if } S(x) > S_H. \end{cases}$$

Hampton's function M is the special case where $n = 1$. All instances of $M(x,n)$ produce sigmoid curves, where the higher the value of n, the steeper the curve. (For an illustration, see Figure 1, which gives the graphs of $M(x,n)$ for $n \in \{.5, 1, 4, 8\}$.) Why

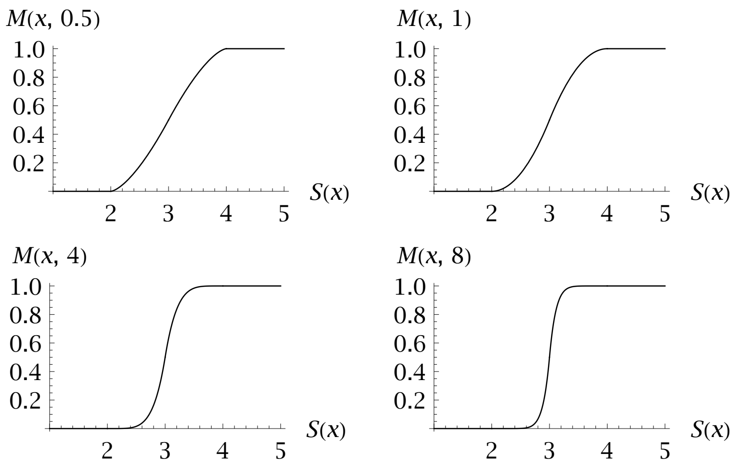


Figure 1: Graphs of $M(x,n)$ for different values of n, with in each case $S_L = 2$ and $S_H = 4$

should graded membership go by $M(x,1)$ rather than by $M(x,2)$ or $M(x,7)$ or any other instance of $M(x,n)$? Indeed, why should graded membership not rather go by one of the infinitely many S-shaped functions that are *not* instances of $M(x,n)$? Besides, could not the membership of, say, RED be given by one S-shaped function while that of BITTER is given by another? In a note, Hampton distances himself from $M(x)$, saying that it serves merely illustrative purposes (Hampton [2007:381 n]). But then the problem is rather that his proposal is very *un*specific. For theoretical reasons, and certainly for empirical ones, one would like to be able to at least approximate the membership function of a given concept, and thus to be told more than that it has a certain general shape.

Second, Hampton's goal is not just to develop an empirically adequate model of graded membership; he also wants to elucidate that notion. And, by itself, the model he proposes fails to shed any light on the question of why there are at all cases that fall between the clear instances and the clear non-instances of a vague concept. Hampton is aware of this challenge, and he tries to answer it in terms of people's felt uncertainty about how an instance is to be classified (Hampton [2007, Sect. 4]). Specifically, our hesitancy in classifying certain instances as belonging to a given category is related to our uncertainty about how other members of our language community would classify those instances. However, this seems to get the explanatory order backwards. It is precisely our hesitancy in classifying items that we would like to explain in terms of graded membership. We would like to say that it is *because* these items are borderline cases, cases that partially fall under one concept and partially fall under another, that we and possibly most other members of our language community feel uncertain about how to classify them. Of course, this invites the further question of what it is to fall only partially under a concept. And, on pain of circularity, this further question cannot be answered in terms of our felt uncertainty.[10]

A final critical point relates to the lower and upper bounds of the boundary region, the region in which we find the instances that only partially fall under the relevant concept. In Hampton's model, these are taken as given—not *sharply* given, but given nonetheless. But what determines these bounds? Even if Hampton's model is not disqualified because it fails to answer this question, it should be uncontroversial that if a model *is* able to predict these bounds, then, all else being equal, that model is to be preferred.

In the remainder of this paper, we want to propose a model that could be regarded as an expansion of Hampton's threshold model. Rather than postulating S-shaped membership functions, we present a model that is backed by independently plausible considerations and show that it gives rise to membership functions that, in a clear sense, can be said to be S-shaped. Moreover, the model suggests a natural interpretation of what underlies graded membership, not in terms of our uncertainty concerning particular classification tasks, but by pointing at what recent literature has argued to be some general features of our representational system. Finally, the model capitalizes on recent work on vagueness that *predicts* the locations of the upper and lower bounds of boundary regions on the basis of information about prototypical instances.

While some passages in Hampton [2007] suggest that he has a set-theoretic conception of similarity in mind, we will rely on the geometric conception, which we prefer on grounds unrelated to the issue of graded membership; see Decock and Douven [2011]. It should be noted, however, that nothing we will have to say excludes that an approach to graded membership very similar to the one to be presented might not yield a measure of graded membership that is based on a set-theoretic conception of similarity. Whether it really does is something we leave to aficionados of the set-theoretic conception to explore.

2. Conceptual spaces and vagueness. Douven et al. [2012] propose an account of vagueness within the conceptual spaces approach. That account is focussed on a central question concerning vagueness, to wit, the question of what a borderline case of a concept is; the issue of graded membership, which basically is the issue of how to explain that some borderline cases seem to fall to a greater (non-extreme) degree under the given concept than other borderline cases, was not addressed in that paper.[11]

[10] Égré and Barberousse [2012, p. 14 of manuscript] make a parallel point in their comments on an early statistical approach to the sorites paradox by the French mathematician Émile Borel: "... Borel, in his discussion of lexical vagueness, outlines a frequentist solution to the question whether an object belongs to a category, namely by assuming that the answer is a probability coefficient determined by the percentage of Yes answers in a given population. This solution is not particularly illuminating, however, for it gives no indication of how individual decisions are taken concerning category membership."

[11] See the opening pages of Smith [2008] for a useful overview of the various aspects of vagueness—including but not limited to the existence of borderline cases—that have been discussed in the literature.

Before summarizing Douven et al.'s account of vagueness, we briefly describe the conceptual spaces approach to categorization that serves as the account's underlying theory. In the last section of our paper, we show how a model of graded membership can be obtained from this account that goes some way toward vindicating Hampton's intuitions about the graded membership function.

The basic idea of the conceptual spaces approach is that concepts can be represented geometrically, as regions in conceptual spaces, that is, one- or multidimensional spaces with a metric defined on them.[12] Objects are mapped onto points in these spaces, whose dimensions represent fundamental qualities in terms of which objects may be compared with one another. How similar two objects are in a given respect is supposed to be a monotonically decreasing function of the distance metric defined on the space associated with that respect: the closer their representations in the space, the more similar they are in the corresponding respect. It is generally agreed that the Euclidean metric familiar from high school geometry is the appropriate metric for spaces whose dimensions are (what is called) *integral*, that is, whose dimensions correspond to qualities that are always experienced jointly. However, some have argued that the so-called Manhattan metric is more suited for spaces whose dimensions are *separable* (i.e., not integral).[13] Be this as it may, because the better known conceptual spaces are all of the former variety, we shall assume Euclidean geometry in the account of graded membership to be proposed in the following. We do not foresee any obstacles preventing a generalization of this proposal to a wider class of metrical spaces.

Among the examples of conceptual spaces that authors have discussed in the literature are temporal space, auditory space, taste space, spaces corresponding to physical parameters such as weight or acidity, and various shape spaces and action spaces.[14] One of the most thoroughly investigated conceptual spaces is color space. This space is generally assumed to be three-dimensional, with one dimension representing hue—think of a color circle with yellow, green, blue, violet, red, and orange (neighboring yellow again) lying in that order on the circle—one dimension representing saturation—the intensity of the color—and one representing brightness, which ranges from white to black, through all shades of gray.

In his book, Smith defends a graded notion of truth using what he calls "fuzzy plurivaluationism." In the present paper, we defend a graded membership relation using a version of supervaluationist semantics. While the aims are different, there are clear affinities between the two endeavors. Whether fuzzy plurivaluationism could equally serve to define graded membership is a topic we leave for another occasion.

[12]A function f defined on a space S is a *metric* for S iff for all points $a, b, c \in S$: (i) $f(a,b) \geqslant 0$, with $f(a,b) = 0$ iff $a = b$; (ii) $f(a,b) = f(b,a)$; and (iii) $f(a,b) \leqslant f(a,c) + f(c,b)$.

[13]See, e.g., Nosofsky [1988]. The Euclidean and Manhattan metrics both belong to the family of Minkowski metrics, where a Minkowski metric d_k is defined as

$$d_k(x,y) \quad = \quad \sqrt[k]{\textstyle\sum_i |x_i - y_i|^k}.$$

The Euclidean metric is the instance with $k = 2$, and the Manhattan metric is the instance with $k = 1$. It is to be noted that these metrics are only unique up to the choice of a unit of measurement. For the purposes of the conceptual spaces approach, it is immaterial which unit of measurement is chosen, given that any constant rescaling of a Minkowski metric will preserve all geometrical and topological properties of the space on which the metric is defined. Nor does the choice of unit of measurement matter to the specific purposes of this paper: the mathematical constructions that are key to our project are easily shown to be invariant under constant metric rescaling.

[14]See, e.g., Clark [1993], Gärdenfors [2000], [2007], and Gärdenfors and Warglien [2011].

It was already briefly mentioned that the conceptual spaces approach represents concepts as regions of conceptual spaces. For example, PINK is a certain region in color space. While, strictly speaking, any set of points in a space is a region of that space, not all regions in this sense correspond to concepts that are actually used by humans. As Gärdenfors [2000:70–77] points out, the concepts that are in use tend to have a specific topological feature, namely, they correspond to *convex* regions in the appropriate spaces, where a region is convex iff, for any pair of points in the region, the line segment connecting the points lies in the region as well.

A further essential element of the conceptual spaces approach (at least of the version that we will be assuming) is the mathematical technique of Voronoi diagrams. This is a very general technique which can be applied to any metrical space in order to divide the space into cells such that each cell has a center and also contains all and only those points that lie no closer to the center of any other cell than to its own center. Formally, where S is an m-dimensional space with associated metric δ_S and $\vec{p} = \langle p_1, \ldots, p_n \rangle$ a sequence of pairwise distinct points in S, the region

$$v(p_i) \ := \ \{p \mid \delta_S(p, p_i) \leqslant \delta_S(p, p_j), \text{ for all } j \in \{1, \ldots, n\} \text{ with } j \neq i\}$$

is called the *Voronoi polygon/polyhedron associated with p_i*. Jointly the elements of $\{v(p_i)\}_{1 \leqslant i \leqslant n}$ constitute $V(\vec{p})$, the *Voronoi diagram generated by $\vec{p}$*.[15] Figure 2 gives an example of a Voronoi diagram of a bounded two-dimensional Euclidean space. It

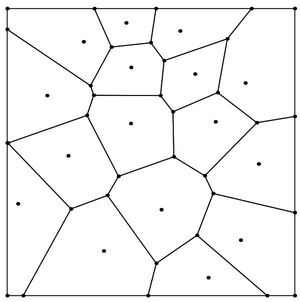


Figure 2: A two-dimensional Voronoi diagram

is readily seen that the Voronoi polygons of the depicted diagram are convex. This is no coincidence: Okabe et al. [2000:58] prove that all Voronoi polygons/polyhedrons of Voronoi diagrams based on a Euclidean metric are convex.

Gärdenfors [2000:88 f] argues that the combination of conceptual spaces with prototypes and the technique of Voronoi diagrams yields a classification system that is highly economical in cognitive terms. Rather than store the categorization of each single point of a given space in memory, on this view it suffices to know the locations of the prototypes in the space. Supposing the space is carved up into concepts by the Voronoi diagram generated by those prototypes, one can simply determine the categorization of any given point by locating the prototypical point closest to it.

Not all points have a *unique* closest prototypical point. The points that do not are of special interest to the topic of vagueness, given that they suggest a straightforward

[15] For a more detailed formal presentation of the technique of Voronoi diagrams, see Okabe et al. [2000, Ch. 2].

answer to the question of what a borderline case is, to wit, a case that is equally far from at least two prototypes. On closer inspection, however, this answer is not entirely satisfactory. The reason is that in this picture, boundaries between concepts are too thin: we find non-borderline cases immediately adjacent to every borderline case, whereas it would seem that clear borderline cases may be surrounded by equally clear borderline cases. For instance, it will not be difficult to find a clear borderline case of both RED and ORANGE such that a shade that is just *slightly* more reddish than it does not yield a clear instance of RED, nor even a case that is borderline borderline RED, but is another clear borderline case of RED and ORANGE.

Motivated by this problem—the "thickness problem," as they call it—Douven et al. [2012] propose an extension of the conceptual spaces approach that also carves up conceptual spaces into convex regions, yet with the possibility of yielding thick boundaries between those regions. The proposed extension begins by noticing that concepts need not have unique prototypes. For example, there appear to be many shades of blue that people regard as being typically blue. This is not only introspectively clear but has also been confirmed in Berlin and Kay's [1969/1999] famous color-naming experiments, in which participants were asked to designate the most typical instances of various color terms on a board with all the Munsell chips glued on it. In their responses, participants frequently designated more than one chip as being typical of a given color term. Accordingly, Douven et al. assume that conceptual spaces may contain prototypical *areas* rather than isolated prototypical *points*.[16]

Once the possibility of concepts having multiple prototypes is recognized, it is no longer clear how to apply the technique of Voronoi diagrams. Picking a representant from each prototypical area and letting those generate a Voronoi diagram would lead to arbitrary determination of concept boundaries. Moreover, it would not help with the problem that, given standard Voronoi diagrams, boundary lines between concepts come out as being too thin to account for the above-mentioned fact that borderline cases need not be immediately adjacent to non-borderline cases. To overcome these difficulties, Douven et al. [2012, Sect. 4] propose a new type of Voronoi diagram which they call a "collated Voronoi diagram."

Informally speaking, the construction of this type of diagram goes as follows: Consider all sets of prototypical points such that each prototypical area in a given space is represented in the set by exactly one prototypical point. Each such set generates a Voronoi diagram of the given space. Then jointly the set of all those Voronoi diagrams make up the collated Voronoi diagram of the space, by simply being projected onto each other.

To state this in more precise terms, let $R = \{r_1, \ldots, r_n\}$ be a set of prototypical areas in a given space S. Then

$$\Pi(R) \ := \ \prod_{i=1}^{n} r_i \ = \ \{\langle p_1, \ldots, p_n \rangle \,|\, p_i \in r_i\}$$

is the set of all ordered sequences $\vec{p} = \langle p_1, \ldots, p_n \rangle$ such that $p_i \in r_i \in R$ for $1 \leqslant i \leqslant n$, and

$$\mathcal{V}(R) \ := \ \{V(\vec{p}) \,|\, \vec{p} \in \Pi(R)\}$$

[16] See, in the same vein, Regier et al. [2005:8390].

is the set of all Voronoi diagrams of S generated by elements of $\Pi(R)$. Furthermore, where

$$\underline{v}(p_i) \;:=\; \{p \,|\, \delta_S(p, p_i) < \delta_S(p, p_j), \text{ for all } j \in \{1, \ldots, n\} \text{ with } j \neq i\}$$

defines the *restricted* Voronoi polygon/polyhedron associated with p_i (i.e., the Voronoi polygon/polyhedron associated with that point minus its edges/faces), the region

$$\underline{u}(r_i) \;:=\; \bigcap\{\underline{v}(p) \,|\, p \in r_i \,\wedge\, v(p) \in V(\vec{p}) \in \mathcal{V}(R)\}$$

is the *collated Voronoi polygon/polyhedron associated with* r_i. The *collated Voronoi diagram generated by* R is then the set

$$\underline{U}(R) \;:=\; \{\underline{u}(r_i) \,|\, 1 \leqslant i \leqslant n\},$$

with $S \setminus \bigcup \underline{U}(R)$ being the *boundary region* of $\underline{U}(R)$. Finally, the region

$$\overline{u}(r_i) \;:=\; \bigcup\{v(p)\}_{r_i \in R}$$

is the *expanded collated Voronoi polygon/polyhedron associated with* r_i, and $\overline{u}(r_i) \setminus \underline{u}(r_i)$ the *boundary region associated with* r_i.

In this picture, concepts may be identified with expanded collated Voronoi polygons/polyhedrons, which, as just seen, can be decomposed into a restricted collated Voronoi polygon/polyhedron part and a boundary region part. Clear instances of a concept are represented by points lying in the former, borderline cases by points lying in the latter. It will sometimes be convenient to refer to the restricted collated Voronoi polygon/polyhedron part of a concept as "the concept in the narrow sense" and to the concept in that sense together with its boundary region as "the concept in the broad sense."

Douven et al. show that, thus interpreted, collated Voronoi diagrams have a number of properties which seem desirable for the purposes of providing an account of vagueness. The three most important ones can to some extent already be gleaned from the example of a collated Voronoi diagram of a two-dimensional Euclidean space shown in Figure 3. They are, first, that concepts still come out as being con-

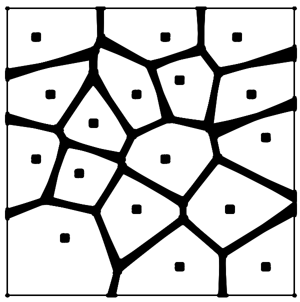


Figure 3: A two-dimensional collated Voronoi diagram

vex; second—and this one is particularly crucial in view of the thickness problem mentioned above—that borderline cases may find themselves surrounded solely by borderline cases; and third, that if prototypical areas are assumed to be *connected*—meaning that between each two points in the area there is a path (not necessarily a straight line) that lies in its entirety in the area as well—then there are no "gaps" in the boundary region of the collated Voronoi diagram. Gaps in a boundary region might raise questions as to the status of points lying in them, but Douven et al. argue that connectedness of prototypical areas is an independently supported assumption, so that the boundary regions of conceptual spaces may be reasonably taken to be non-gappy.[17]

In spite of these advantages, the proposal may still seem unsatisfactory as an account of the nature of borderline cases. After all, in the present picture there are abrupt transitions from concepts to their boundaries, so that the phenomena of second-order vagueness seem to go unaccounted for. Douven et al. are aware of this problem, however, and offer a number of different responses to it. Their main response is premised on Gärdenfors's [2000:89] claim that psychological measures, such as the metrics defined on conceptual spaces, are typically imprecise so that "the borderlines [of the Voronoi diagram of a conceptual space] will not be exactly determined." Given that collated Voronoi diagrams are composed out of simple Voronoi diagrams, any indeterminacy affecting the borderlines of the latter will affect the boundary regions of the former as well. Consequently, it will be indeterminate to some degree where a collated polygon/polyhedron ends and its boundary region begins.[18] As will be seen, the work to be presented in the final section of this paper offers a possible refinement of this response to the higher-order vagueness problem.

To end this section, it is worth briefly digressing on the provenance of conceptual spaces. The idea of representing concepts as regions in a space had already surfaced in the philosophical literature in the 1960s and 1970s, first in van Fraassen's [1967], [1969a] attempts to provide non-metaphysically-laden accounts of various intensional notions, and then in Stalnaker's [1979] anti-essentialist semantics for modal logic (which explicitly builds on van Fraassen's earlier work). While there are clear commonalities between van Fraassen's and Stalnaker's work on the one hand and the conceptual spaces approach on the other, there are also some notable differences. One in particular is crucial to the purposes of this paper. In their work, van Fraassen and Stalnaker refer throughout to *logical* space. As Stalnaker [1979:82] remarks, one would ultimately like to know more about the geometrical and topological properties of that space, but neither he nor van Fraassen has made any concrete proposal as to what these properties might be. Not that it would be so difficult to *impose* a geometrical and topological structure on logical space. But how to do this in a motivated, non-arbitrary way is not immediately obvious.

If the geometrical and topological structure of conceptual spaces were arbitrary, then that would pose an immediate obstacle to our project of defining graded membership within the conceptual spaces approach. Specifically, it would give rise to the worry that our account of graded membership makes degrees of membership seem to be arbitrary, as a result of which a given color shade may seem to have a degree of

[17] There is nothing in the construction of a collated Voronoi diagram to preclude that points in a prototypical area do not also lie in the boundary region. That might raise questions too. However, in practice, in which the prototypical instances of a concept form only a small subset of all instances of the concept (as they do in the abstract example of Figure 3), no overlap between prototypical areas and boundary regions will occur.

[18] Douven et al. [2012, Sect. 4] note that the imprecision of the distance function is not in itself enough to explain borderline vagueness, contrary to what one might at first suspect.

membership of, say, red of 1 on one choice of metric for color space and as having some intermediate degree of membership of red on a different but equally legitimate choice of metric for color space. We have convergent intuitions about the degrees to which various items fall into specific categories, and any account that implied that degrees of membership are arbitrary would have difficulty explaining that convergence.

This worry is unfounded, however, precisely because there is nothing arbitrary about the geometrical and topological structure of the type of spaces that figure in the conceptual spaces approach. On the contrary, these spaces are derived from empirical data by means of a number of standard statistical techniques, the main ones being a set of techniques known as "multidimensional scaling."[19] The data typically concern similarity judgments, on the basis of which a similarity matrix is constructed, indicating for each pair of stimuli (e.g., color patches, sounds, specific tastes) how similar they are to each other.[20] Multidimensional scaling (or less frequently, a different technique[21]) is then applied to this similarity matrix to determine, first, the number of dimensions needed to adequately represent the data, and then, once the dimensionality has been determined, to locate the various items in a space of the right dimensionality in such a way that their positions in that space reflect the relative similarities. Given data on which items are judged as being prototypical, we thereby automatically come to know the locations of the prototypical points in the space. The technique of Voronoi diagrams then does the rest, in the way explained above. Naturally, the result of this procedure—a conceptual space—can be easily put to the test, by using the space to generate predictions about how people will rate the similarities among items representable in the space that were not used in its construction, and by comparing these predictions with actual similarity judgments. If need be, the conceptual space can be further refined by applying the just-described technique again, now to the extended data set. Cognitive scientists work with various spaces that have been constructed and tested in precisely this way. Those are the spaces that form the backbone of our definition of graded membership.[22]

3. Kamp and Partee on graded membership. Given how we characterized concepts and their boundary regions, there are two adequacy constraints on membership functions that immediately suggest themselves: that anything that is represented by a point inside the concept in the narrow sense be assigned a degree of membership of 1, and that anything that is represented by a point outside the concept in the broad sense be assigned a degree of membership of 0. What to say about things represented by points inside the concept's boundary region is less obvious. A first thought might

[19] See Gärdenfors [2000:21–30]. See also the Appendix of Clark [1993] for a detailed yet accessible introduction to multidimensional scaling.

[20] Such data are usually obtained by asking people to rate the similarity between the stimuli on a Likert scale or on a ratio scale. The data may also be obtained in a more indirect way, for instance, by determining the likelihood of mistaking for each other the elements of pairs of stimuli.

[21] See Gärdenfors [2000:142 ff, 221 ff].

[22] Might different procedures than the one described in the text not yield conceptual spaces different from those currently employed by cognitive scientists, spaces with different structures and different associated metrics, which are nonetheless empirically adequate? Of course, cognitive science may face the problem of underdetermination no less than other sciences do. But just as few take this problem to discredit research in physics (say) that is based on currently accepted theories in that field, so it should not be deemed to discredit our account of graded membership, which is based on what is arguably solid work in cognitive science.

be that their degrees of membership should be fixed by the distance between the point representing them and the nearest point in the concept's prototypical area.[23] On closer examination, however, this turns out not to be such a good idea. As is clear from Figure 3, collated Voronoi polygons/polyhedrons may be quite irregularly shaped. Due to this, a borderline point only lying *just* outside the concept in the narrow sense may be at a greater distance from the concept's prototypical area than a point that only *just* made it into the concept in the broader sense. Here one might think that the former falls under the concept to a greater degree than the latter, despite the fact that the latter is closer to the prototypical area. To arrive at a more plausible graded membership function, we build on a construction of a measure proposed by Kamp and Partee [1995:148–155]. It will be seen here that, as it stands, the construction is not quite satisfactory. However, in the next section it will be shown how the construction can be turned into a new and adequate measure of graded membership by embedding it in the version of the conceptual spaces approach discussed above.

Kamp and Partee's proposal applies van Fraassen's [1969b] supervaluationist semantics, which was originally developed as a semantics for free logics (i.e., logics for languages with non-referring terms), to vague terms. This had been done before (cf. Fine [1975] and Kamp [1975]), but Kamp and Partee go further than previous authors in that they use the semantics to model the graded membership relation. The relevant part of their paper begins by considering a two-valued partial model $\mathfrak{M}$ for a language containing what they call "simple predicates," by which they seem to mean predicates that have monolexemic expressions in English (like "apple," "fish," "red," and "smart"). The model consists of a universe of discourse $U_{\mathfrak{M}}$ and, for each simple predicate P in the language, a partial function $P_{\mathfrak{M}}$, which assigns the value 1 to all clear instances of P, the value 0 to all clear non-instances of P, and is undefined for the remaining objects. Correspondingly, sentences expressing concept membership, such as "a falls under the concept RED in $\mathfrak{M}$," are assigned the value true if $\text{red}_{\mathfrak{M}}(a) = 1$ and the value false if $\text{red}_{\mathfrak{M}}(a) = 0$; in all other cases, such sentences lack a truth value.

For each partial model $\mathfrak{M}$, there is a set of *completions*, where a completion $\mathfrak{M}'$ of $\mathfrak{M}$ is a valuation that eliminates all truth value gaps that $\mathfrak{M}$ gives rise to by extending the positive and negative extensions of each predicate. In more exact terms, for all simple predicates P, $\mathfrak{M}'$ determines a total function on $U_{\mathfrak{M}}$ that agrees with the partial function $P_{\mathfrak{M}}$ but assigns one of the values 0 and 1 to each of the objects for which $P_{\mathfrak{M}}$ is undefined. As Kamp and Partee point out, in the context of prototype theory it is natural to consider not all logically possible extensions but only those that respect typicality rankings. What this means is that if objects a and b are such that both $P_{\mathfrak{M}}(a)$ and $P_{\mathfrak{M}}(b)$ are undefined, and if $P_{\mathfrak{M}'}(a) = 1$ and $P_{\mathfrak{M}'}(b) = 0$ for some completion $\mathfrak{M}'$ of $\mathfrak{M}$, then it should hold that $T_P(a) > T_P(b)$. Suppose, for instance, that the boundary region of TALL, as applied to people, is the interval $\langle 170\,\text{cm}, 180\,\text{cm}\rangle$. Then a completion of a model that designated people 172 cm in height as being tall and people 177 cm in height as being not tall would not correspond to a psychologically realistic way of classifying people with regard to tallness. Kamp and Partee use $\mathfrak{M}^*$ to designate the set of all completions $\mathfrak{M}'$ of $\mathfrak{M}$ together with $\mathfrak{M}$ itself; they refer to $\mathfrak{M}^*$ as the "supermodel" of $\mathfrak{M}$.

Kamp and Partee use the machinery of completions to describe a graded membership function:

[23] This amounts to the so-called Hausdorff distance; see Douven et al. [2012, Sect. 5].

> [S]uppose that in $\mathfrak{M}$ Bob and Alma are both in the truth value gap of the concept ADULT, but Bob is *closer* to the positive extension of ADULT than Alma, perhaps because he is a little older or a little more grown up or both. It is reasonable to suppose that this comparison is reflected by the set of completions in $\mathfrak{M}^*$: more of these will have Bob in the extension of ADULT than Alma. More precisely: the set of completions in which Alma belongs to the extension of ADULT will be a proper subset of those in which Bob belongs to the extension.
>
> This suggests that we might take the set of completions in which Bob belongs to the extension of ADULT as a *degree* to which he is an adult. More generally, the set of completions in which an object a belongs to the extension of a concept A indicates the degree to which a falls under A. (Kamp and Partee [1995:153]; typographic changes made for uniformity of reading)

Kamp and Partee argue for a number of formal constraints to be imposed on this function of graded membership. First, its range is to be the interval $[0, 1]$. Second, it is to reflect the size of the sets of completions which make an object come out as falling under a given concept; the larger that set, the higher the degree should be. Third, if an object a belongs to a concept in all completions, then the membership function for that concept should assign the value 1 to a; conversely, it should assign a the value 0 if a belongs to the concept in no completion. And fourth, the numerical value associated with a set of completions and its complement should add up to 1, or more generally, if A is a subset of the set of completions B, then the numerical values associated with the sets of completions A and $B - A$ should add up to the value associated with B. Together these constraints ensure that the resulting function—which Kamp and Partee designate by μ—is a normalized measure in the technical sense of measure theory.[24]

Elegant though the proposal is, it does not quite offer a well-defined function of graded membership, as Kamp and Partee admit:

> [T]he constraints do not determine the function μ completely. Indeed, it is far from clear on what sorts of criteria a particular μ could or should be selected.
>
> But let us ignore this difficulty for the moment and assume that with each supermodel $\mathfrak{M}^*$ comes a measure function that assigns numbers in $[0, 1]$ to a sufficiently rich subfield of the set of completions of $\mathfrak{M}^*$. (Kamp and Partee [1995:153 f])

Commenting on this, Hampton [2007:367] notes that, for some concepts, it would seem rather straightforward to fully determine a corresponding μ function. For example, in the case of TALL considered above, which was supposed to have as a boundary region the interval $\langle 170\,\text{cm}, 180\,\text{cm}\rangle$, it is natural to think that the relevant sets of completions are related to the ratios of this height scale, so that μ takes the value $.x$ for any person who is $17x$ cm tall, with $x \in \langle 0, 10\rangle$. But, as Hampton also notes, this proposal may already be problematic in that it yields a linear rather than an S-shaped graded membership function, as one would expect on empirical grounds. More troublesome still appears the fact that nothing Kamp and Partee (or Hampton, for that matter) say gives an indication of how to generalize the proposal to concepts that, unlike TALL, do not impose a linear ordering on their domain of application. As the next section shows, however, these difficulties can be overcome.

[24] A function $\mu : \wp(S) \to [0, \infty\rangle$ is a *measure* over set S iff (i) $\mu(\emptyset) = 0$ and (ii) for all $A, B \subseteq S$ such that $A \cap B = \emptyset$, $\mu(A \cup B) = \mu(A) + \mu(B)$. If, in addition, $\mu(S) = 1$, then μ is a *normalized* measure.

4. A geometrical model of graded membership. To obtain a graded membership function that is both fully specific and general (at least as general as the conceptual spaces approach is), we combine Kamp and Partee's proposal with the conceptual spaces framework. As a result, both partial models and completions can be construed as precisely defined geometrical objects. The extra constraints needed to turn Kamp and Partee's proposal into a well-defined graded membership function, constraints that Kamp and Partee themselves were unable to provide, will be seen to flow directly from the geometry of partial models *cum* completions, and they will allow us to determine degrees of membership with mathematical exactitude.

Note that a collated Voronoi diagram of a given conceptual space S with set R of prototypical areas can be regarded as defining a two-valued partial model S', consisting of the domain of S (the set of points of S) and the smallest set containing, for any concept C in S, a partial function $C_{S'}$ that assigns the value 1 to every point that lies in the restricted collated Voronoi polygon/polyhedron associated with C's prototypical area; that assigns the value 0 to every point that lies outside the expanded collated Voronoi polygon/polyhedron associated with the same area; and that is undefined for any point in the boundary region associated with that area.

As to possible completions of this model, we also stick closely to Kamp and Partee's proposal in not considering *all* logically possible ways to extend S' to a full model. Instead, as completions we only consider the elements of the set $\Pi(R)$, which determine the Voronoi diagrams that jointly make up the collated Voronoi diagram generated by the prototypical areas in S. Douven et al. [2012, Sect. 4] prove that, given the earlier-mentioned assumption that each $r_i \in R$ is connected (in the topological sense of the word), the boundary region associated with r_i is the union of the Voronoi edges/faces of the elements of $\{v(p) \,|\, p \in r_i \wedge v(p) \in V(\vec{p}) \in \mathcal{V}(R)\}$. Consequently, the boundary line of any Voronoi polygon/polyhedron associated with a given point $p \in r_i$ lies entirely within the boundary region associated with r_i, and it divides that region into two parts (see Figure 4). Thus, we can think of the elements of $\Pi(R)$ as offering a precisification of each concept in S, in the sense that, for every r_i, they assign 1 to all points lying inside the Voronoi polygon/polyhedron associated with a point $p \in r_i$ and 0 to all other points. This proposal is along the lines of Kamp and Partee's proposal insofar as completions do not assign the values 0 and 1 arbitrarily to the points in the boundary region of S'. Rather, they respect the structure imposed by the Voronoi diagrams generated by the elements of $\Pi(R)$. To see that this is a psychologically plausible way of assigning values to points in the boundary, notice that any single Voronoi diagram $V(\vec{p})$ corresponds to a way we would carve up the relevant conceptual space if the space contained as prototypical points only those in $\vec{p} \in \Pi(R)$.

Given the assumption that prototypical areas are connected and that they consist of more than a single prototypical point, such areas will contain infinitely many points. Nevertheless, to get an intuitive feel for the measure to be developed, it is worthwhile first to imagine that there exist finite cases. For this hypothetical type of case, the basic idea of the measure to be proposed can be expressed as follows: the degree to which an entity falls under a concept modelled by a space S with set R of prototypical areas equals the ratio of, on the one hand, the number of elements of $\Pi(R)$ that generate Voronoi diagrams which make the entity come out as falling under the concept, and on the other hand, the total number of elements of $\Pi(R)$. More formally, given a space S with set R of prototypical areas, let $r_C \in R$ be the area of prototypes of concept C.

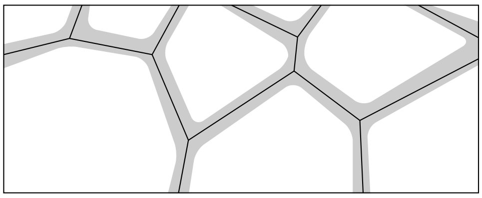


Figure 4: Fragment of the collated Voronoi diagram shown in Figure 3 with the boundary region given in gray and one of the diagrams constituting the boundary highlighted

Then, assuming each $r \in R$ to consist of finitely many prototypical points, the degree to which x is a member of C, $M_C(x)$, is given by this function:

$$\text{(M)} \qquad M_C(x) := \frac{\left|\{\vec{p} \mid x \in v(p) \in V(\vec{p}) \in \mathcal{V}(R) \wedge p \in r_C\}\right|}{|\Pi(R)|}.$$

For example, consider the collated Voronoi diagram depicted in Figure 5, which is generated by a set of prototypical areas consisting of only finitely many points. In

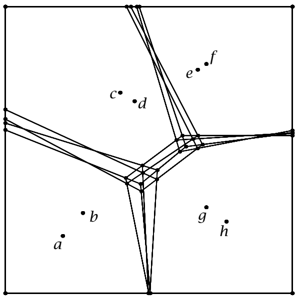


Figure 5: A collated Voronoi diagram of a space with prototypical areas each consisting of only two points

this diagram, $R = \{\{a,b\},\{c,d\},\{e,f\},\{g,h\}\}$; so, $|\Pi(R)| = 2^4 = 16$. Now consider the points i and j lying in this space. Given the locations of the prototypical points, each $\vec{p} \in \Pi(R)$ generates a *unique* Voronoi diagram. (This need not be the case; see Douven et al. [2012, Sect. 4].) The sixteen resulting diagrams can be seen in Figure 6. To determine the degree to which i and j belong to a given concept, we just need to count the number of Voronoi diagrams that locate them in a Voronoi polygon associated with a point lying in the prototypical area of the given concept and then divide this number by 16. Letting $M_{\{a,b\}}(x)$ denote the degree to which x falls under the concept with prototypical area $\{a,b\}$, and similarly for the other prototypical areas, we find that $M_{\{a,b\}}(i) = M_{\{c,d\}}(i) = 1/2$, $M_{\{a,b\}}(j) = 1/4$, and $M_{\{g,h\}}(j) = 3/4$. Furthermore, we have $M_{\{g,h\}}(i) = M_{\{e,f\}}(i) = M_{\{c,d\}}(j) = M_{\{e,f\}}(j) = 0$.

Clearly, though, for the kind of spaces we are interested in, which have prototypical areas containing infinitely many points, the measure defined by (M) will not do. To construct a measure for such spaces, we must proceed differently. Let S

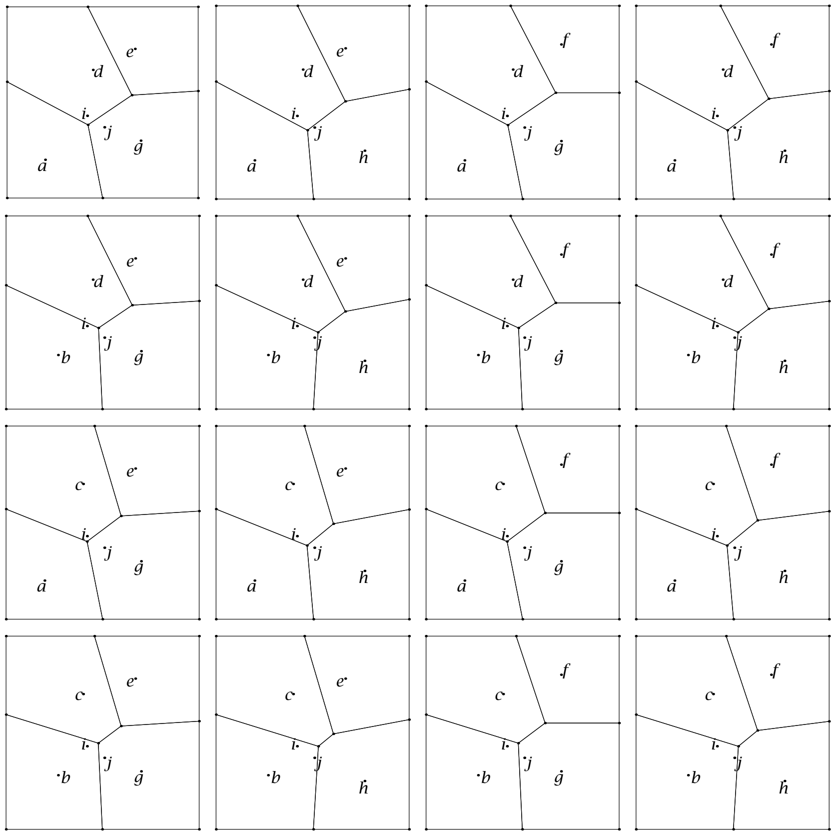


Figure 6: The sixteen Voronoi diagrams that make up the collated Voronoi diagram shown in Figure 5

be an m-dimensional space, with $R = \{r_1, \ldots, r_n\}$ being the set of prototypical areas in S. As said, each $\vec{p} = \langle p_1, \ldots, p_n \rangle \in \Pi(R)$ can be conceived as a completion of the partial model S' determined by the collated Voronoi diagram on S. Given that each prototypical point $p_i \in \vec{p}$ can be represented by an m-tuple $\langle x_{i_1}, \ldots, x_{i_m} \rangle$ of spatial coordinates, we can represent each completion by means of an $m \times n$-tuple $\langle x_{1_1}, \ldots, x_{1_m}, \ldots, x_{n_1}, \ldots, x_{n_m} \rangle$ of real numbers. This makes it possible to define a measure over a set of completions in terms of the volume occupied by the related set of coordinates in the space $\mathbb{R}^{m \times n}$. Here is how: Let $\mathcal{I}_T$ be the indicator function for a subset T of this space.[25] The measure of this set is standardly defined as[26]

$$\mu(T) \;:=\; \int \mathcal{I}_T(\langle x_{1_1},\ldots,x_{1_m},\ldots,x_{m_1},\ldots x_{m_n}\rangle)\, dx_{1_1}\ldots dx_{1_m}\ldots dx_{n_1}\ldots dx_{n_m},$$

with the complete space $\mathbb{R}^{m\times n}$ being the domain of integration. In particular, the measure of the entire set of completions is determined through indicator function $\mathcal{I}_{\Pi(R)}$:

$$\begin{aligned}
\mu(\Pi(R)) &= \int \mathcal{I}_{\Pi(R)}(\langle x_{1_1},\ldots,x_{1_m},\ldots,x_{m_1},\ldots x_{m_n}\rangle)\, dx_{1_1}\ldots dx_{1_m}\ldots dx_{n_1}\ldots dx_{n_m}\\
&= \int_{V_{r_1}}\cdots\int_{V_{rn}} dx_{1_1}\ldots dx_{1_m}\ldots dx_{n_1}\ldots dx_{n_m}\\
&= \prod_{i=1}^{n} V_{r_i}.
\end{aligned}$$

Given that all prototypical areas occupy a bounded region in conceptual space, the value of $\mu(\Pi(R))$ is always finite.[27]

To arrive at a graded membership function, note that, for any point a and any concept C_i with prototypical area r_i, where δ is the distance function defined on the relevant space, the set

$$\begin{aligned}
S_{a,i} \;:=\; \{\langle x_{1_1},\ldots,x_{1_m},\ldots,x_{n_1},\ldots,x_{n_m}\rangle \mid \delta(a,\langle x_{i_1},\ldots,x_{i_m}\rangle) < \delta(a,\langle x_{j_1},\ldots,x_{j_m}\rangle),\\
\text{for all } \langle x_{j_1},\ldots,x_{j_m}\rangle \in r_j \text{ such that } i \neq j\}
\end{aligned}$$

is the set of elements of $\Pi(R)$ that determine Voronoi diagrams which locate a in the Voronoi polygon/polyhedron associated with $\langle x_{i_1},\ldots,x_{i_m}\rangle \in r_i$. The measure of this set is given by $\mu(S_{a,i})$. To obtain a measure of the proportion of completions in $S_{a,i}$ relative to the set $\Pi(R)$, we define

$$\mu^*(S_{a,i}) \;:=\; \frac{\mu(S_{a,i})}{\mu(\Pi(R))},$$

which is a normalized measure.[28] Finally, the degree to which a belongs to C_i, $M_{C_i}(a)$, is set equal to $\mu^*(S_{a,i})$.

This is all still very schematic and gives us little information about the form of the graded membership function. To be more specific, we start by considering the simplest type of case: a one-dimensional space S with an associated Euclidean metric δ and with $R = \{r_1, r_2\}$, where $r_1 = [p_1, p_2]$ and $r_2 = [p_3, p_4]$. To determine the graded

[25] Given sets S and S' such that $S \subseteq S'$, a function $\mathcal{I}_S : S' \to \{0,1\}$ is an *indicator function* for S iff, for all $x \in S'$, $\mathcal{I}_S(x) = 1$ if $x \in S$ and $\mathcal{I}_S(x) = 0$ if $x \notin S$. As an aside, it may be noted that graded membership functions can be thought of as generalizations of indicator functions.

[26] This is the so-called Lebesgue measure. For more on this, see, e.g., Temple [1971]. Lebesgue integration and measure theory were conceived to deal with "wild" functions containing many discontinuities. Because the sets we consider in the context of collated Voronoi diagrams hardly contain any discontinuities, it is safe to assume that they are measurable.

[27] We are assuming here that all prototypical areas are non-degenerate in that the prototypical points belonging to a given area do not all lie on an $m-1$-dimensional surface. This assumption makes the mathematics simpler but is otherwise dispensable.

[28] In note 13, we remarked that the mathematical constructions that are crucial to our account of graded membership are invariant under constant metric rescaling. Here in particular it is worth noting that whereas μ depends on a particular choice of unit of measurement, μ^* does not.

membership functions for the corresponding concepts C_1 and C_2, we must look at $\mathbb{R}^2$. In this space, we find the elements of $\Pi(R)$ in the rectangle $ABCD$ with vertices $A = \langle p_1, p_3\rangle$, $B = \langle p_2, p_3\rangle$, $C = \langle p_1, p_4\rangle$, and $D = \langle p_2, p_4\rangle$, these elements being pairs of coordinates $\langle x, y\rangle$ such that $x \in r_1$ and $y \in r_2$. The indicator function $\mathcal{I}_{S_{a,1}}(\langle x, y\rangle)$ for C_1 for a point with coordinate a takes the value 1 if the Voronoi diagram generated by $\langle x, y\rangle$ locates a and r_1 on the same Voronoi line segment, which is the case if $\delta(a, x) \leqslant \delta(a, y)$, or equivalently, if $a - x \leqslant y - a$; the indicator function takes the value 0 otherwise. Notice that if $a - x = y - a$, then a coincides with the Voronoi point of the diagram generated by $\langle x, y\rangle \in \Pi(R)$. The set of all such elements of $\Pi(R)$ forms a line l in $ABCD$ that is defined by the equation $y = 2a - x$. Thus, set $S_{a,1}(\langle x, y\rangle)$ consists of all and only points in $ABCD$ that lie above that line. The function $\mu(S_{a,1})$ measures the area of that part of the rectangle. So,

$$M_{C_1}(a) \;=\; \mu^*(S_{a,1}) \;=\; \frac{\mu(S_{a,1})}{\mu(\Pi(R))} \;=\; \frac{\mu(S_{a,1})}{V_{r_1} \times V_{r_2}} \;=\; \frac{\mu(S_{a,1})}{(p_2 - p_1)(p_4 - p_3)}.$$

Calculations of this measure are straightforward, albeit that we have three different cases to consider, depending on whether (a) $p_2 - p_1 < p_4 - p_3$, (b) $p_2 - p_1 > p_4 - p_3$, or (c) $p_2 - p_1 = p_4 - p_3$.

For case (a), we have to consider five different subcases, depending on the value of a. Cases (i) and (ii) are trivial: if a does not lie in the boundary region between concepts C_1 and C_2—the interval $\left[{}^1\!/\!{}_2(p_1 + p_3), {}^1\!/\!{}_2(p_2 + p_4)\right]$, as one easily verifies—then (i) $M_{C_1}(a) = 1$ if $a < {}^1\!/\!{}_2(p_1 + p_3)$, or (ii) $M_{C_1}(a) = 0$ if $a > {}^1\!/\!{}_2(p_2 + p_4)$. The remaining cases are these:

(iii) $a \in \left[{}^1\!/\!{}_2(p_1 + p_3), {}^1\!/\!{}_2(p_2 + p_3)\right]$, which corresponds to the situation in which l intersects the line segment AB in point $p_{AB} = \langle 2a - p_3, p_3\rangle$ and the line segment AC in the point $p_{AC} = \langle p_1, 2a - p_1\rangle$;

(iv) $a \in \left[{}^1\!/\!{}_2(p_2 + p_3), {}^1\!/\!{}_2(p_1 + p_4)\right]$, which corresponds to the situation in which l intersects AC in p_{AC} and BD in point $p_{BD} = \langle p_2, 2a - p_2\rangle$; and

(v) $a \in \left[{}^1\!/\!{}_2(p_1 + p_4), {}^1\!/\!{}_2(p_2 + p_4)\right]$, which corresponds to the situation in which l intersects BD and CD in points p_{BD} and $p_{CD} = \langle 2a - p_4, p_4\rangle$.

(See Figure 7 for a graphical representation of these situations.) In case (iii), the area of $S_{a,1}$ is the area of the rectangle $ABCD$ minus the area of the triangle $Ap_{AB}p_{AC}$. Hence,

$$M_{C_1}(a) \;=\; \frac{(p_2 - p_1)(p_4 - p_3) - \frac{1}{2}(2a - p_1 - p_3)^2}{(p_2 - p_1)(p_4 - p_3)}. \tag{1}$$

In case (iv), we have to consider the area above the line segment $p_{AC}p_{BD}$. This area is composed of the triangle CDE and a parallelogram $CEp_{AC}p_{BD}$, where $E = \langle p_2, p_1 + p_4 - p_2\rangle$, and the line CE is the line l for $a = {}^1\!/\!{}_2(p_1 + p_4)$. The area of the triangle CDE is ${}^1\!/\!{}_2(p_2 - p_1)^2$, and the area of the parallelogram is $\left(p_4 - (2a - p_1)\right)(p_2 - p_1)$. Hence,

$$M_{C_1}(a) \;=\; \frac{p_1 + p_2 + 2p_4 - 4a}{2(p_4 - p_3)}. \tag{2}$$

And in case (v), the area above line l is the area of the triangle $Dp_{BD}p_{CD}$, so that

$$M_{C_1}(a) \;=\; \frac{(2a - p_2 - p_4)^2}{2(p_4 - p_3)(p_2 - p_1)}. \tag{3}$$

In case (b), everything stays the same except that, for the subcase in which $a \in [1/2(p_2 + p_3), 1/2(p_1 + p_4)]$, we have

$$M_{C_1}(a) \;=\; \frac{2p_2 + p_3 + p_4 - 4a}{2(p_2 - p_1)} \tag{4}$$

instead of (2). And in the special case (c), in which $ABCD$ is a square, the parallelogram $CEp_{AC}p_{BD}$ collapses into the diagonal BC, so that the graded membership function is composed only of (1) and (3) and, of course, the constant functions for the trivial subcases.

By way of example, consider a one-dimensional space $S = [1, 11]$ with a Euclidean metric defined on it and with $R = \{r_1 = [1, 3], r_2 = [8, 11]\}$, which is a type (a) case. Figure 7 illustrates, for objects mapped onto points 5, 5.75, and 6.5 in this space, the degree to which they fall under the concept C_1: in the gray areas, we find the elements of $\Pi(R)$ that locate the various points on the Voronoi line segment associated with a point in r_1; the whole rectangle represents $\Pi(R)$. Making suitable substitutions in

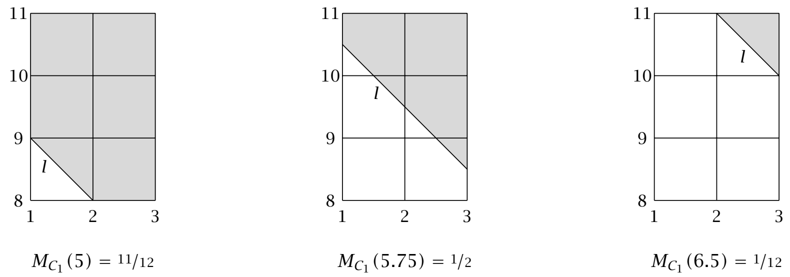


Figure 7: Degrees of membership of C_1 for objects represented by points 5, 5.75, and 6.5; the gray areas above the lines l consist of points $p = \langle x, y\rangle$ which represent Voronoi diagrams that locate 5, 5.75, and 6.5, respectively, on the Voronoi line segment on which the point represented by the x-coordinate of p lies

(1)–(3) gives

$$M_{C_1}(a) \;=\; \begin{cases} 1 & \text{if } a < 4.5; \\ \frac{1}{3}(a-7)^2 & \text{if } 4.5 \leqslant a \leqslant 5.5; \\ \frac{1}{3}(13-2a) & \text{if } 5.5 \leqslant a \leqslant 6; \\ \frac{1}{3}(-a^2+4a-1)^2 & \text{if } 6 \leqslant a \leqslant 7; \\ 0 & \text{if } a > 7, \end{cases}$$

which, as Figure 8 shows, has an S-shaped graph.

In fact, it is obvious from Equations (1)–(4) that the membership function of any concept represented in a space of the kind at issue produces an S-shaped graph. One also easily verifies that the membership function of any such concept is smooth in that

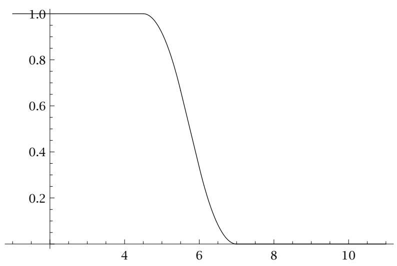


Figure 8: Graph of the membership function M_{C_1} for the space $[1, 11]$ with prototypical areas $r_1 = [1, 3]$ and $r_2 = [8, 11]$

there are no discontinuities in it nor its first derivative. Further note that, while the form of the graded membership function of the type of concept at issue is S-shaped, in general there is a straight line segment that is part of the S. The length of this line segment is determined by the difference in lengths of the prototypical areas: the line segment has length 0 if the prototypical areas are of equal length, and otherwise the line segment is proportional to the difference in length between the prototypical areas. As a final comment on the one-dimensional case, we mention that the method for deriving graded membership functions described above is not really limited to one-dimensional spaces with only two prototypical areas. In one-dimensional spaces with more than two prototypical areas, the part of the graded membership function that concerns points lying to one side of a prototypical area depends only on that area together with the adjacent prototypical area lying to the same side. Therefore, we can, for concepts represented in such spaces, derive their membership functions by applying the above method to each pair of adjacent concepts and then collating the thus obtained partial functions in the obvious way.

For multi-dimensional conceptual spaces, it is much more difficult to get analytical expressions for membership functions. Recall that we are making no assumptions about the topological properties of prototypical areas other than that they are connected. For one-dimensional spaces, that is still very informative: connected areas are simply line segments, so that the set $\Pi(R)$ of a one-dimensional space with set R of prototypical areas will always be represented by a rectangle or (in case R contains more than two prototypical areas) a hyperrectangle in $\mathbb{R}^{1\times n}$. But, for any m-dimensional space with set $R = \{r_1, \ldots, r_n\}$ of prototypical areas, if $m > 2$, then the assumption of connectedness still allows $\Pi(R)$ to have an endless variety of shapes in $\mathbb{R}^{m\times n}$.

Nevertheless, by taking large random samples of elements of $\Pi(R)$ of a multi-dimensional space S with set R of prototypical areas, and then applying the measure (M), we can *approximate* the various membership functions for the concepts represented in S. To start with a very simple example, let S be the bounded two-dimensional space with $R = \{r_1, r_2\}$ that is depicted in Figure 9. To approximate the membership function of concept C_1 with prototypical area r_1 for the points to be found in the indicated area G, we first divide each 1×1-square in G into 100 $.1 \times .1$-squares. We then randomly sample 2500 elements of $\Pi(R)$ and we register, for each of them and for each of the 4800 $.1\times.1$-squares in G, whether or not the square falls in the Voronoi polygon associated with a point in r_1 that is part of the Voronoi diagram generated by the given element of $\Pi(R)$. Figure 10 presents graphs that show, for each of the

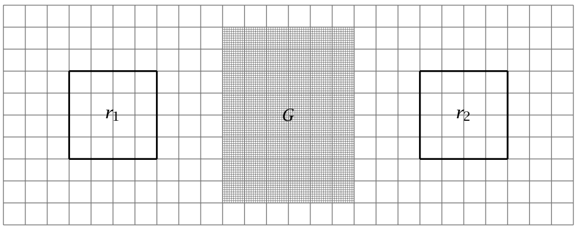
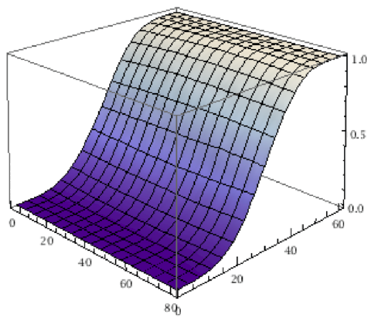


Figure 9: A two-dimensional space S with $R = \{r_1, r_2\}$ and grid G

$.1 \times .1$-squares in G, the proportion of elements of $\Pi(R)$ that locate the square in the same Voronoi polygon as r_1; the left graph is more or less a front view of the function values in G, the right graph a top view. These graphs give a visual impression of what the membership function M_{C_1} looks like in the area G.[29]



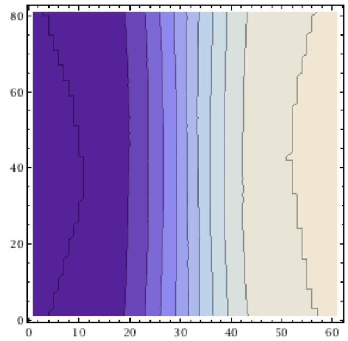


Figure 10: Approximation of the membership function M_{C_1}; color and (in the left graph) position on the z-axis indicate the proportion of elements of $\Pi(R)$ that locate a given square in the Voronoi polygon associated with some point in r_1

The graphs in the upper row of Figure 12 do the same for the space depicted in the left panel of Figure 11. This space has four prototypical areas of various shapes; again, call the set of these R. Here, we considered a grid with the same granularity as G but now spanning the whole space. The graphs give the results of picking randomly 500 elements of $\Pi(R)$ and registering for each of those and for each square in the grid whether or not the Voronoi diagram generated by the element of $\Pi(R)$ locates the square in the Voronoi polygon associated with a point in the prototypical area represented by the black rectangle.[30]

Approximating membership functions via simulations is by no means restricted to one- and two-dimensional spaces but can be done for spaces with any number of

[29]This simulation as well as the simulations to be presented in the text below were carried out using *Mathematica* 7.

[30]We used fewer samplings in this simulation than in the previous one (and used fewer still in the simulation to be presented shortly) because of computer memory limitations.

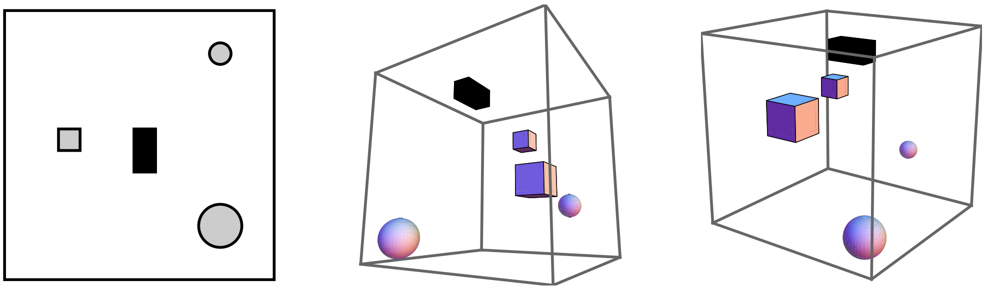


Figure 11: Two-dimensional space with four differently shaped prototypical areas (left); three-dimensional space with five differently shaped prototypical areas, seen from different angles (middle and right)

dimensions. Naturally, visualization of the outcomes of the simulations *is* limited. To see that we can still get *some* visual representation of the results for three-dimensional spaces, consider the $250 \times 250 \times 250$ space with five differently shaped prototypical areas shown in the middle and right panels of Figure 11. For the purpose of doing simulations, we divided this into $15,625 \times 10^6$ equal cubes, and we randomly selected 250 sequences of points, one from each prototypical area. For each of the cubes and each of the sequences, we recorded whether or not the cube was located in the Voronoi polyhedron associated with a point in the black prototypical area that was part of the Voronoi diagram generated by the sequence. The graphs in the lower row of Figure 12 give the results of this simulation for the plane transecting the space at $z = 150$. By doing the same for other z-values, one can get a fairly good impression of the shape of the membership function of the concept with the dark gray prototypical area. For spaces with more than three dimensions, not even this more involved procedure will work. Note, though, that visualization is not a requirement for calculating the values of instances of (M), which approximate the membership functions of the concepts corresponding to those instances.

On our account, membership functions cannot simply be reduced to similarity to prototypes. The shape of the membership function of a given concept depends not only on the location of *its* prototypical area but also on the locations of the prototypical areas of adjacent concepts. Thus, in general, our membership functions will not be instances of the schema $M(x, n)$ defined in Section 1. Yet, the results of the various simulations we carried out—of which those presented here only form a small subset—all suggest that in an unambiguous sense these functions are still S-shaped: as is readily appreciated from the graphs, any straight path starting in a given concept and ending in an adjacent concept corresponds to function values that yield a sigmoid plot. To this extent, our proposal is in accordance with Hampton's intuitions about the membership function.

As another comment, we note that our model of graded membership, which builds on Douven et al.'s account of vagueness, may provide another solution to the problem of higher-order vagueness that was raised for that account. We noted earlier that the problem arises from the fact that a concept's being vague is not just a matter of its

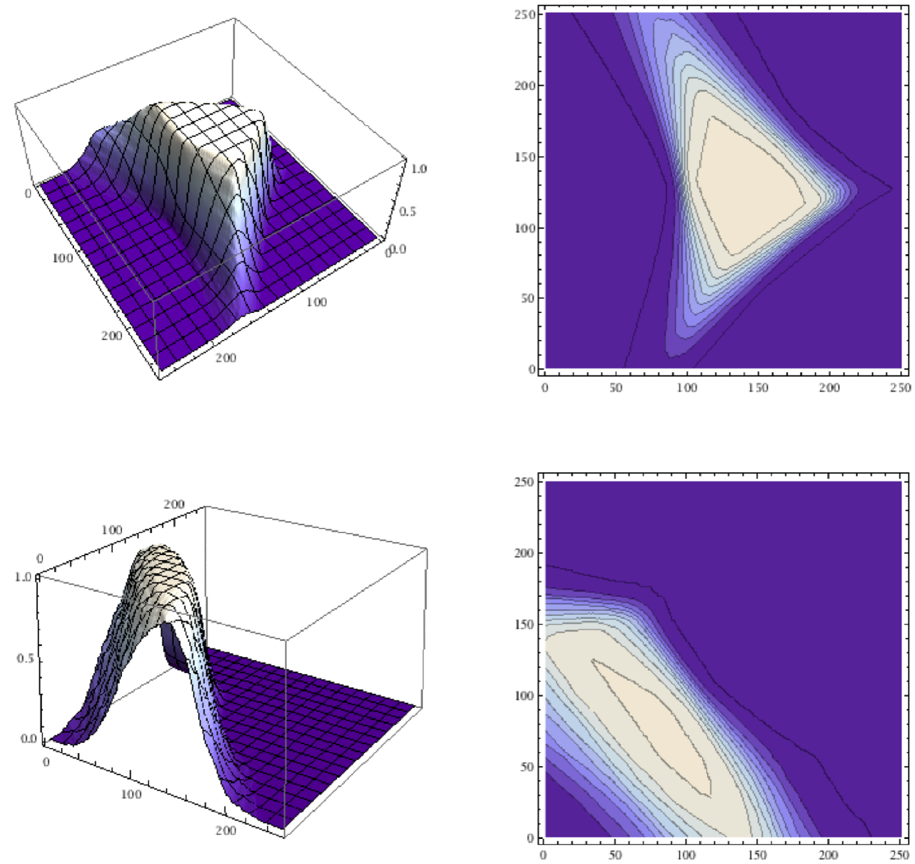


Figure 12: Upper row: front view (left) and top view (right) of approximation of the membership function of the concept with the black prototypical area in the two-dimensional space of Figure 11; bottom row: front view (left) and top view (right) of the result of the simulation described in the text for a two-dimensional cross-section at $z = 150$ of the three-dimensional space of Figure 11

having borderline cases, but also a matter of its having *blurred* boundaries. As mentioned, Douven et al. try to accommodate this fact within their approach by invoking the imprecision of psychological metrics. But if our experiences of vagueness are due not to our being confronted with borderline cases *per se* but rather to the different degrees to which such cases fall under a given concept, then the present model of graded membership suggests another, possibly simpler, explanation of higher-order vagueness experiences, along the lines of Hampton's elegant explanation (see Section 1). For while, in Douven et al.'s picture of concepts, there are abrupt transitions from clear cases to borderline cases, it is here seen that these transitions need not correspond to abrupt changes in degree of membership.

5. Conclusion. We conclude by summing up a number of distinctive advantages that our model of graded membership has compared to previous such models, notably Kamp and Partee's and Hampton's. First and foremost, where Kamp and Partee were able to only outline a definition of graded membership, use of the conceptual spaces framework and geometric intuition provided enough guidance to fill in the missing details. The result is a definition that is computationally tractable to the extent that its values can always be approximated; sometimes they can even be calculated exactly.

And whereas Hampton had to take the locations of concept boundaries as given, in our model they follow from the geometrical structure of the relevant conceptual space, where this structure is fixed by the global shape of the space, the associated distance metric, and the location of the prototypical areas. Of course, this is something we get for free by capitalizing on Douven et al.'s account of vagueness.

In addition to this, we saw that although Hampton proposed a very specific membership function, he was in the end largely non-committal on the exact form of that function. As a result, his proposal does not make precise predictions about the degree of membership of a given borderline case. Our account does: even where analytic solutions for the membership function are hard to come by, by means of simulations we can approximate the value for the membership function of a given concept for any given item that is in its domain. Indeed, given enough computing power and time, we can approximate this value as closely as we like by increasing the level of granularity of the simulation as well as the sample size.

Whether our account is also predictively *accurate* is an open question. It is not hard to think of experiments that may yield at least partial answers to this question. For example, while the three-dimensional space of Figure 12 is purely abstract, for the purposes of conducting simulations the only relevant difference between that space and the standard color spaces (like CIELab or CIELuv) is the number of prototypical areas to be considered. Again, given enough computing power and time, it is possible to determine the membership functions of the various color concepts in the same way in which we determined the membership function for one of the concepts in the three-dimensional space of Figure 11. Once those functions have been determined, they can be tested against data from actual color classification tasks, provided degree of membership and prototypicality can be suitably operationalized. If, for instance, response times can be used for these purposes (cf. Jraissati et al. [2012]), then judged degrees of membership can be checked for correlations with degrees of membership as issued by an application of our model to the appropriate color space.

As a final advantage, we mention that our model is explanatorily quite powerful in that it accounts for graded membership largely in terms of general and independently motivated considerations from cognitive science. In this model, membership functions have the form they do because conceptual spaces play a pivotal role in how humans categorize the world; these spaces are carved up by means of collated Voronoi diagrams; and the things that make up these diagrams work much like Kamp and Partee's completions to determine degree of membership.

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