What Is Graded Membership?

Lieven Decock Igor Douven

Overview

What is graded membership?

Prototype theory and graded membership

Conceptual spaces and vagueness

Kamp and Partee on graded membership

A geometrical model of graded membership

Graded membership (I)

Familiar observation:

- Some shades strike us as being neither quite red nor quite orange but rather as being "reddish–orangish."
- They are neither clearly outside the category of red nor clearly within it.
- They are within the category to some extent, but also outside the category to some extent.

Various authors have tried to make sense of this type of phenomenon in terms of a notion of graded or partial membership.

Graded membership (II)

- Models leave something to be desired so long as the notion of graded membership has not been clarified.
- Fuzzy set theories formalize a graded membership relation.
- These theories + accompanying fuzzy logics have numerous successful applications.
- Many remain skeptical about the notion of graded membership that lies at their root.
- Lindley: "One of the difficulties many people have with the ideas of fuzzy logic lies in the interpretation of the membership function: What does it mean to say that $m(x) = .2$?"

Plan

- Address question of what to make of graded membership from a psychological perspective.
- Look at recent psychological literature on vagueness, typicality, and graded membership which account for graded membership in terms of similarity to prototype (Kamp and Partee, Hampton).
- Add predictive power and explanatory depth to previous proposals (reliance on conceptual spaces approach).
- Upshot: operational definition of graded membership based on findings in the cognitive sciences.

Prototype theory

- Among the instances of a property or concept, some are more representative of that property/concept than others.
- The ones that are most representative are the prototypes of the property/concept.
- The original advocates of prototype theory had clearly universalist aspirations.
- For our purposes one can stay noncommittal on this issue: a prototype may just be a case that ordinary speakers/observers in a given community classify or would classify as typical.

Similarity to prototype

Similarity function underlies concept formation: whether or not something falls under a given concept depends on how similar the entity is to that concept's prototype.

Leaves open the question of whether entities can fall under a concept to only some extent.

Depends on how the notion of similarity is understood (on some popular views, falling under a concept is a categorical matter).

Given that things can be more or less similar to a prototype, it should be possible to make sense of a graded notion of membership that permits things to fall under a concept to different degrees.

Osherson and Smith

Osherson and Smith discuss a version of prototype theory that exploits the fact that similarity is a graded notion to arrive at a graded membership relation.

Concepts are represented as quadruples $\langle A, d, p, c \rangle$.

Function $c : A \to [0, 1]$ indicates to what degree an object falls under the concept.

c is a monotonically decreasing function of distance from prototype:

$$\forall x, y \in A\colon\ d(x, p) \leqslant d(y, p) \ \supset\ c(y) \leqslant c(x).$$

Kamp and Partee vs Osherson and Smith (I)

Kamp and Partee's response: O&S run two functions together that should be kept separate: the graded membership function M and the typicality or goodness of example function T.

E.g., a pelican unarguably falls under the concept BIRD, even though it is a rather atypical bird.

Kamp and Partee vs Osherson and Smith (II)

O&S's reply: correct; M and T differ even more than K&P seem to realize.

- According to K&P, M and T have the same scale.
- That's false: natural to take $[0, 1]$ as the range of M; for some concepts T may go to infinity.
- Also, M and T have different bases in that T, but not M, is a function of similarity to prototype.
- E.g., because of greater similarity to prototype of BIRD, robins receive higher T values than pelicans, but this greater similarity does not manifest itself in a difference in M values.

Hampton's response

- That M does not differentiate among the clear instances of a concept between the more and less typical ones does not mean that M is not a function of similarity to prototype.
- It may still hold that $T = t(\mathrm{sim})$ and $M = m(\mathrm{sim})$, where t and m are nondecreasing functions of the same similarity measure sim.
- We may assume that t has range $[0, \infty\rangle$, m has range $[0, 1]$.
- We may set some threshold value of similarity to prototype to define concept boundaries: anything which is above the threshold belongs to the category.
- Explains robin–pelican contrast: robins are more typical birds than pelicans are, but both robins and pelicans are above the similarity threshold for determinate membership in the category BIRD.

Hampton's proposal (I)

Hampton proposes precise definition of M as a function of similarity measure.

Assumption: for each concept there is a determinate boundary region for membership.

The values S_L and S_H of the similarity function are lower and upper bound of the boundary region; S_T is the value where $M = .5$ (further assumption: S_T is midway between S_L and S_H).

$S(x)$ measures similarity of x to relevant prototype.

$$M(x) = \begin{cases} 0 & \text{if } S_L \geqslant S(x); \\ 2\left(\frac{S(x)-S_L}{S_H-S_L}\right)^2 & \text{if } S_T \geqslant S(x) > S_L; \\ 1 - 2\left(\frac{S_H-S(x)}{S_H-S_L}\right)^2 & \text{if } S_H \geqslant S(x) > S_T; \\ 1 & \text{if } S(x) > S_H. \end{cases}$$

Hampton's proposal (II)

According to Hampton, this function has several attractive features:

1. There are regions of determinate membership and non-membership where M takes the values 1 and 0 (pre-theoretically right).
2. Definition is consistent with T assuming different values for instances whose M value is 1 (again pre-theoretically right)
3. Smooth transitions at the endpoints of the boundary region, which may be enough to handle the issue of second-order vagueness.
4. M's S-shaped form fits rather well the data concerning categorization probabilities based on empirical findings reported in work by McCloskey and Glucksberg.

Hampton's proposal: critique (I)

Hampton's proposal is very specific regarding the form of the graded membership function.

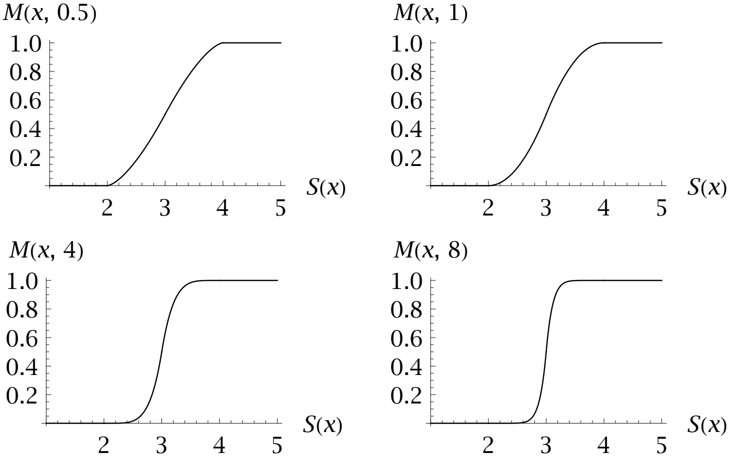
Generalization of M:

$$M(x,n) = \begin{cases} 0 & \text{if } S_L \geqslant S(x); \\ 2^n \left(\frac{S(x)-S_L}{S_H-S_L}\right)^{n+1} & \text{if } S_T \geqslant S(x) > S_L; \\ 1 - 2^n \left(\frac{S_H-S(x)}{S_H-S_L}\right)^{n+1} & \text{if } S_H \geqslant S(x) > S_T; \\ 1 & \text{if } S(x) > S_H. \end{cases}$$

Hampton's function M is the special case where $n = 1$; all instances of $M(x,n)$ produce sigmoid curves, where the higher the value of n, the steeper the curve.

Hampton's proposal: critique (II)

This gives the graphs of $M(x, n)$ for $n \in \{.5, 1, 4, 8\}$ (in each case $S_L = 2$ and $S_H = 4$):



Hampton's proposal: critique (III)

- Why should graded membership go by $M(x, 1)$ rather than by $M(x, 2)$ or $M(x, 7)$ or any other instance of $M(x, n)$?
- Why should graded membership not go by one of the infinitely many S-shaped functions that are *not* instances of $M(x, n)$?
- Could not the membership of (say) RED be given by one S-shaped function while that of BITTER is given by another?
- Proposal does not explain why graded membership should be S-shaped to begin with.
- Hampton takes lower and upper bounds of boundary as given – it would be nice if one could derive them.

Aim

- We propose a model that could be regarded as an expansion of Hampton's threshold model.
- Rather than postulating S-shaped membership functions, the model gives rise to membership functions that, in a clear sense, are S-shaped.
- The model suggests a natural interpretation of what underlies graded membership, by pointing at what recent literature has argued to be some general features of our representational system.
- The model capitalizes on recent work on vagueness that predicts the locations of the upper and lower bounds of boundary regions on the basis of information about prototypical instances.

The conceptual spaces approach

- Currently one of the main approaches to categorization.
- According to it, properties and concepts (categories) can be represented geometrically, as regions in conceptual spaces.
- Conceptual spaces: one-dimensional or multidimensional structures with a metric defined on them. (We assume throughout a Euclidean metric.)
- Objects are mapped onto points in these spaces, and the dimensions of these spaces correspond to "qualities" objects may have.
- Metrics measure similarity between objects: the closer they are, the more similar they are in the respect corresponding to the space.

Conceptual spaces: examples

- A three-dimensional Euclidean space: represents proximity relations between objects in the world.
- Temporal space, with one dimension: time.
- Auditory space, with two dimensions: pitch and loudness.
- Color space, with three dimensions:
  1. hue: think of color circle;
  2. saturation: intensity of color;
  3. brightness: amount of white mixed with color.
- More controversial examples: multidimensional shape spaces, action spaces, ...

Categorization

Gärdenfors connects the idea of conceptual spaces with two further ideas:

- prototypes;
- Voronoi diagrams.

Jointly these ideas yield an arguably plausible account of categorization, that is, of how people carve up conceptual spaces into regions corresponding to “natural” properties or concepts.

Voronoi diagrams: example

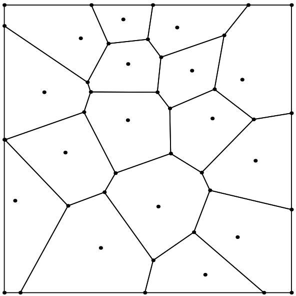


Figure: A two-dimensional Voronoi diagram

Borderline cases

- On the conceptual spaces approach, properties and concepts clearly have borderlines.
- Obvious (?) account of borderline cases: a borderline case is a case that falls on the borderline between different natural properties, a case for which there is no *unique* prototype to which it is closest.
- Cannot be quite right: there are (e.g.) red/orange borderline cases such that small changes in their color – any small change – will still result in something that is borderline red/orange.
- Borderlines are too “thin.”

First amendment: prototypical areas

- It is probably a simplification to suppose that every property and concept has a unique prototype. (Is there only one typical shade of red?)
- This is certainly true if prototypes are understood in a metaphysically lightweight sense, as the typical instances of a property or concept.
- Empirical evidence: Berlin and Kay's studies on color categorization.
- So: replace idea of unique prototypes by that of prototypical areas.

Second amendment: collated Voronoi diagrams

Basic idea: each choice function that picks from all prototypical areas exactly one point determines a single Voronoi diagram, and these diagrams can be collated, or projected onto each other.

The result is a collated Voronoi diagram, which in general will have "thick" boundaries.

Collated Voronoi diagrams – definition (I)

Let $R = \{r_1, \ldots, r_n\}$ be a set of pairwise disjoint regions of a space S. Then

$$\Pi(R) \;:=\; \prod_{i=1}^{n} r_i \;=\; \{\langle p_1, \ldots, p_n \rangle \mid p_i \in r_i\}$$

is the set of all sequences that contain exactly one point p_i of each region $r_i \in R$.

Given a set R of disjoint regions of a space S, we can consider the set of all Voronoi diagrams of S generated by elements of $\Pi(R)$:

$$\mathcal{V}(R) \;\;:=\;\; \{V(P) \mid P \in \Pi(R)\}.$$

Collated Voronoi diagrams – definition (II)

Given a set $\mathcal{V}(R)$, we can from its elements construct a collated Voronoi diagram, as follows: Let $R = \{r_1, \ldots, r_n\}$ be a set of pairwise disjoint regions of a space S. Then the region

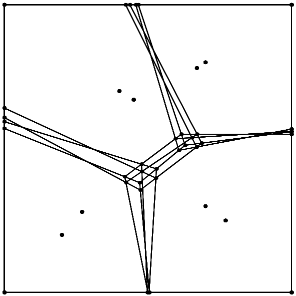
$$u(r_i) \quad := \quad \bigcap\{v(p) \mid v(p) \in V(P) \in \mathcal{V}(R),\ p \in r_i\}$$

is the *collated polygon/polyhedron associated with* r_i, and the set

$$U(R) \quad := \quad \{u(r_i) \mid r_i \in R\}$$

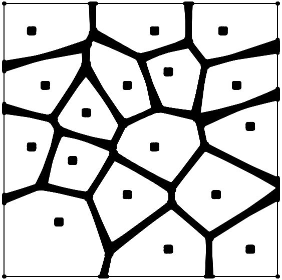
the *collated diagram generated by* R. The r_i are called the *generator regions* of $U(R)$. The set $S \setminus \bigcup U(R)$ is called the *boundary region* of $U(R)$.

Collated diagram – example (I)



Collated Voronoi diagram generated by four prototypical regions, each consisting of two points.

Collated diagram – example (II)



If prototypical regions are connected, boundary region is “full.”

Solution to earlier problem: borderline cases may find themselves surrounded by borderline cases.

Aside: the provenance of conceptual spaces (I)

- The idea of representing concepts as regions in a space surfaced in the philosophical literature in the 1960s and 1970s: van Fraassen's attempts to provide non-metaphysically-laden accounts of various intensional notions; Stalnaker's anti-essentialist semantics for modal logic.
- Van Fraassen and Stalnaker refer throughout to logical space.
- Stalnaker: one would ultimately like to know more about the geometrical and topological properties of that space – but no one had a clue.

Aside: the provenance of conceptual spaces (II)

- Conceptual spaces are derived from empirical data by means of a number of standard statistical techniques, the main ones being a set of techniques known as "multidimensional scaling."
- The data typically concern similarity judgments, on the basis of which a similarity matrix is constructed, indicating for each pair of stimuli how similar they are to each other.
- Multidimensional scaling then gives representation that best reflects the relative similarities.

Constraints on membership function

- Collated Voronoi polygon/polyhedron associated with prototypical region r: intersection of "simple" Voronoi polygons/polyhedrons associated with p, for all $p \in r$.
- Anything that is represented by a point inside this intersection should be assigned a degree of membership of 1 for the concept whose prototypes are represented by the points in r.
- Expanded collated Voronoi polygon/polyhedron associated with prototypical region r: union of "simple" Voronoi polygons/polyhedrons associated with p, for all $p \in r$.
- Anything that is represented by a point outside this union should be assigned a degree of membership of 0 for the concept whose prototypes are represented by the points in r.
- What to say about other points (points inside the concept's boundary region)?

Hausdorff metric?

- First thought: their degrees of membership should be fixed by the distance between the point representing them and the nearest point in the concept's prototypical area.
- This amounts to the Hausdorff distance.
- Not a good idea, given that collated Voronoi polygons/polyhedrons may be quite irregularly shaped.
- Better proposal: build on a construction of a measure by Kamp and Partee.
- The construction is not quite satisfactory, but it can be turned into a new and adequate measure of graded membership by embedding it in the present version of the conceptual spaces approach.

Kamp and Partee's supervaluations (I)

- K&P use van Fraassen's supervaluationist semantics to model the graded membership relation.
- K&P begin by considering a two-valued partial model $\mathfrak{M}$ for a language containing "simple predicates" (like "apple," "fish," "red," and "smart").
- The model consists of universe of discourse $U_{\mathfrak{M}}$ and, for each simple predicate P in the language, a partial function $P_{\mathfrak{M}}$, which assigns 1 to all clear instances of P, 0 to all clear non-instances of P, and is undefined for the remaining objects.
- Sentences expressing concept membership (e.g., "a falls under the concept RED in $\mathfrak{M}$") are assigned the value true if $\text{red}_{\mathfrak{M}}(a) = 1$ and the value false if $\text{red}_{\mathfrak{M}}(a) = 0$; in all other cases, such sentences lack a truth value.

Kamp and Partee's supervaluations (II)

- For each partial model $\mathfrak{M}$, there is a set of completions.
- A completion $\mathfrak{M}'$ of $\mathfrak{M}$ is a valuation that eliminates all truth value gaps that $\mathfrak{M}$ gives rise to by extending the positive and negative extensions of each predicate.
- They do not consider all ways of eliminating truth value gaps but only those that respect typicality rankings.

Graded membership function? (I)

[S]uppose that in $\mathfrak{M}$ Bob and Alma are both in the truth value gap of the concept ADULT, but Bob is *closer* to the positive extension of ADULT than Alma, perhaps because he is a little older or a little more grown up or both. It is reasonable to suppose that this comparison is reflected by the set of completions in $\mathfrak{M}^*$ [= supermodel, set of all completions together with $\mathfrak{M}$ itself]: more of these will have Bob in the extension of ADULT than Alma. More precisely: the set of completions in which Alma belongs to the extension of ADULT will be a proper subset of those in which Bob belongs to the extension.

This suggests that we might take the set of completions in which Bob belongs to the extension of ADULT as a *degree* to which he is an adult. More generally, the set of completions in which an object a belongs to the extension of a concept A indicates the degree to which a falls under A.

Graded membership function? (II)

- Kamp and Partee argue for a number of formal constraints to be imposed on this function of graded membership.
- Together these constraints ensure that the resulting function – designated by μ – is a normalized measure in the sense of measure theory.
- But "the constraints do not determine the function μ completely. Indeed, it is far from clear on what sorts of criteria a particular μ could or should be selected. But let us ignore this difficulty for the moment and ..."

A geometrical model of graded membership

Approach:

- Combine K&P's proposal with conceptual spaces framework.
- Then partial models and completions can be construed as precisely defined geometrical objects.
- The extra constraints needed to turn K&P's proposal into a well-defined graded membership function flow directly from the geometry of partial models *cum* completions.

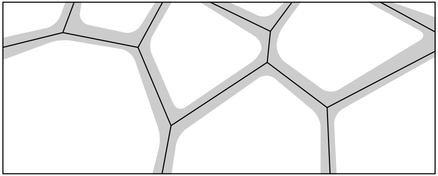
Collated Voronoi diagrams as partial models

A collated Voronoi diagram of a conceptual space S with set R of prototypical areas can be regarded as defining a two-valued partial model S', consisting of

- the domain of S (the set of points of S);
- for any concept C in S, a partial function $C_{S'}$ that
  1. assigns 1 to every point that lies in the collated Voronoi polygon/polyhedron associated with C's prototypical area;
  2. assigns 0 to every point that lies outside the expanded collated Voronoi polygon/polyhedron associated with the same area;
  3. is undefined for any point in the boundary region associated with that area.

Completions (I)

- Like K&P, we do not consider all logically possible ways to extend S' to a full model.
- As completions we only consider the elements of the set $\Pi(R)$.
- The boundary line of any Voronoi polygon/polyhedron associated with a given point $p \in r_i$ lies entirely within the boundary region associated with r_i, and it divides that region into two parts.



Completions (II)

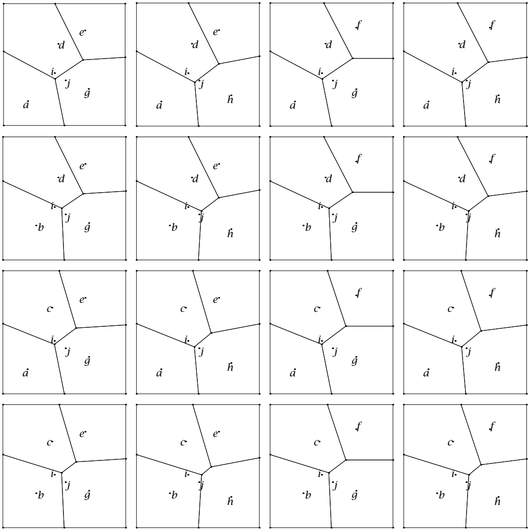
- The elements of $\Pi(R)$ offer a precisification of each concept in S: for every r_i, they assign 1 to all points lying inside the Voronoi polygon/polyhedron associated with a point $p \in r_i$ and 0 to all other points.
- This proposal is along the lines of Kamp and Partee's proposal: completions do not assign the values 0 and 1 arbitrarily to the points in the boundary region of S', but respect the structure imposed by the Voronoi diagrams generated by the elements of $\Pi(R)$.

The finite case (probably non-existent)

- Imagine there exist spaces whose prototypical regions consists of only finitely many points.
- The basic idea of the measure then is this: the degree to which an entity falls under a concept modelled by a space S with set R of prototypical areas equals the ratio of
  1. the number of elements of $\Pi(R)$ that generate Voronoi diagrams which make the entity come out as falling under the concept;
  2. the total number of elements of $\Pi(R)$.
- Formally, the degree to which x is a member of C, $M_C(x)$, is given by

$$M_C(x) \;:=\; \frac{\left|\left\{\vec{p}|x \in v(p) \in V(\vec{p}) \in \mathcal{V}(R) \wedge p \in r_C\right\}\right|}{|\Pi(R)|}.$$

The finite case: illustrated

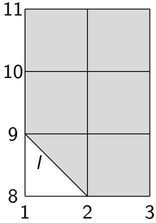


E.g., $M_{\{a,b\}}(i) = M_{\{c,d\}}(i) = 1/2$.

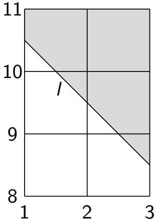
The infinite case

- Let S be an m-dimensional space, with $R = \{r_1, \ldots, r_n\}$ being the set of prototypical areas in S.
- Each $\vec{p} = \langle p_1, \ldots, p_n \rangle \in \Pi(R)$ can be conceived as a completion of the partial model S' determined by the collated Voronoi diagram on S.
- Given that each prototypical point $p_i \in \vec{p}$ can be represented by an m-tuple $\langle x_{i_1}, \ldots, x_{i_m} \rangle$ of spatial coordinates, we can represent each completion by means of an $m \times n$-tuple $\langle x_{1_1}, \ldots, x_{1_m}, \ldots, x_{n_1}, \ldots, x_{n_m} \rangle$ of real numbers.
- This makes it possible to define a measure over a set of completions in terms of the volume occupied by the related set of coordinates in the space $\mathbb{R}^{m \times n}$.

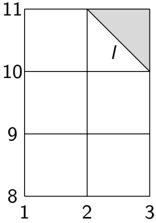
Illustration of the measure



$M_{C_1}(5) = {}^{11}\!/_{12}$



$M_{C_1}(5.75) = {}^{1}\!/_{2}$

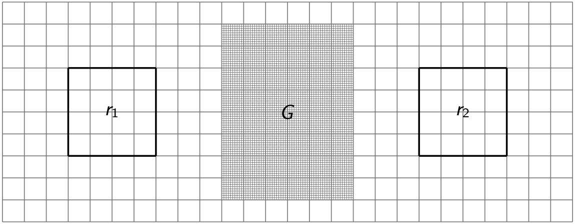


$M_{C_1}(6.5) = {}^{1}\!/_{12}$

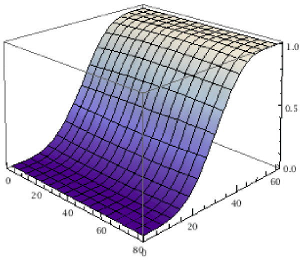
Membership functions: S-shaped?

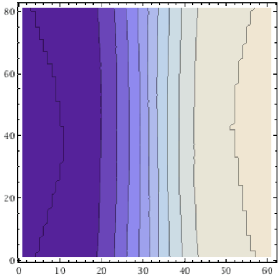
- For concepts representable in one-dimensional spaces, it can be proved that their membership functions are S-shaped.
- For multi-dimensional conceptual spaces, it is much more difficult to get analytical expressions for membership functions (we are making no assumptions about the topological properties of prototypical areas other than that they are connected).
- Computer simulations help with this.

Two dimensional space: simulation (I)



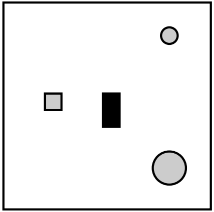
Two dimensional space: simulation (II)

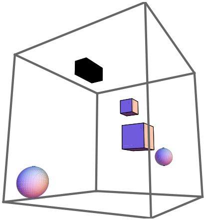


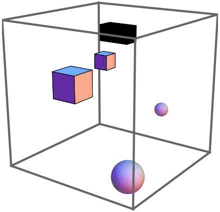


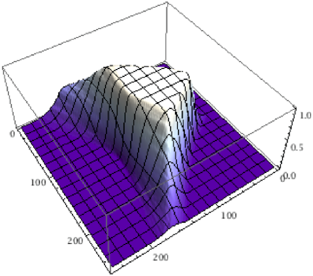
Approximation of the membership function M_{C_1}; color and (in the left graph) position on the z-axis indicate the proportion of elements of $\Pi(R)$ that locate a given square in the Voronoi polygon associated with some point in r_1.

Two/three dimensional space





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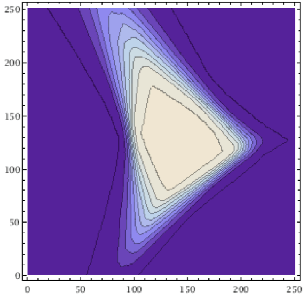
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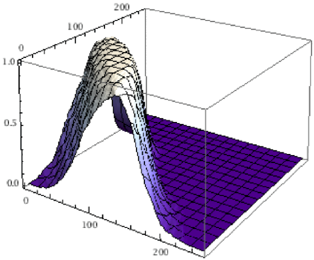
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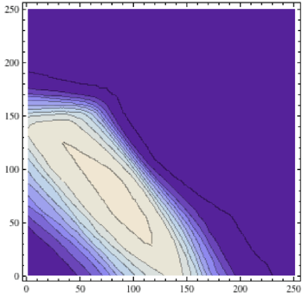
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Conclusions

- K&P only outlined a definition of graded membership – use of the conceptual spaces framework and geometric intuition provided enough guidance to fill in the missing details.
- Result is a definition that is computationally tractable: its values can always be approximated; sometimes they can even be calculated exactly.
- Hampton had to take the locations of concept boundaries as given – in our model they follow from the geometrical structure of the relevant conceptual space.
- Whether our account is also predictively accurate is an open question.
- It is not hard to think of experiments that may yield at least partial answers to this question.