

# **Comments on Eric Pacuit's "Understanding the Brandenburger-Keisler Paradox"**

## **I) Summary**

In his paper, Eric Pacuit takes a look at the Brandenburger-Keisler paradox from the perspective of modal logic. The paradox can be expressed by the following sentence:

*Ann believes that Bob assumes that Ann Believes that Bob's assumption is wrong.*

Upon closer analysis, it turns out that this sentence describes an impossible constellation of beliefs. Brandenburger and Keisler have used this paradox to argue that certain belief configurations cannot be represented by any type space. Brandenburger and Keisler (also) formulate their result in a particular modal logic that uses two modal operators, one for "agent i believes P" and one for "agent i assumes P", where an assumption is the strongest belief.

In my understanding, Eric's contributions in this paper are the following:

1. The paper gives a formalization of the paradox, but whereas Brandenburger & Keisler use a modal logic semantics based on Kripke models, Eric uses neighborhood structures.
2. Eric observes that a modal logic similar to the one at issue has been given a complete axiomatization by Humberstone. This axiomatization can be applied also to the case at hand, thereby answering a question posed by Brandenburger & Keisler in their paper.
3. Eric also gives a formalization of the paradox in a different formalism, namely hybrid logic.

## **II) Clarification Questions**

1. The informal argument for why the paradoxical sentence constitutes a paradox does not convince me. Consider again the sentence

*Ann believes that Bob assumes that Ann Believes that Bob's assumption is wrong.*

Consider we ask the question: Does Ann believe that Bob's assumption is wrong?

If we assume the answer is 'yes', then we obtain a contradiction, I agree. Now consider on the other hand that the answer is 'no'. The argument was [quote]:  
"Then Ann does not believe that Bob's assumption is wrong. Hence, Ann believes Bob's assumption is correct." But this seems to be an inference from

$$\neg \text{Bel}_A \text{WrongAss}_B \quad \text{to} \quad \text{Bel}_A \neg \text{WrongAss}_B$$

But couldn't it be that Ann is unsure about the correctness of Bob's assumption?  
Since I do believe the formal proof showing the impossibility of this constellation, I must be missing something. What is it?

2. Hybrid Logic: In your definition of the language of Hybrid Logic, I do not understand where the free variables come from. They seem to work like nominals, but in the grammar of the language, I only see propositional variables (i.e. constants) and nominals, but no variables from Var. So how does a formula of the form  $\forall x \phi(x)$  actually get formed?

### **III) General Questions**

1. METHODOLOGY: We are presented with a sequence of formalizations: We have the original Brandenburger & Keisler formalization using Kripke-style modal logic. Then Eric gives us a formalization in neighborhood-style modal logic, and finally also a hybrid logic formalization of the paradox. Also, the list of **possible** formalisms does not end here: why not consider standard first-order logic, for instance, or yet other systems? The central question is: what is gained through this progression through different formalisms? That is, we already had a formalization in modal logic, so even if I accept the virtues of formalizing a paradox in some formal logic, the question is what extra benefits Eric's formalizations provide. Putting it differently, Eric wants to study the paradox

“from the modal/hybrid logic point of view”, but what does that mean, i.e., what new insights would such a study hope to provide?

I did not find much in Eric’s paper to answer that question, but I have two guesses: First, let’s consider the transition from Kripke-style to neighborhood-style modal logic. In his paper, Eric mentions that the advantage of neighborhood semantics over Kripke semantics is that it tells us explicitly what properties of belief/assumption are needed in order to derive the paradox. Technically, beliefs are modeled as proper filters, and all the properties defining a proper filter are used. Actually, it would be interesting to look more closely what happens if some of these properties are dropped. Maybe one could still find analogous paradoxes.

Second, let’s consider the transition from neighborhood-style modal logic to hybrid logic. Here, we can compare the modal logic formulation with the hybrid logic formulation of the fact that there is no belief state satisfying the paradoxical sentence (lemmas 3 and 8). Is the hybrid logic version an improvement over the modal logic version? Possible guess: there is no formula of the modal language expressing the paradoxical sentence, we have to resort to defining a propositional atom with the right semantic content. Is that the point?

Still, in spite of these two arguments for moving from one formalism to another, I am left wondering whether we have to go through all these different formalisms just for these benefits. I would hope that there is more to be learned from the modal/hybrid logic point of view than this, and I would like to know from Eric what this might be.

2. THE PARADOX ITSELF: How surprising is this paradox? We knew to begin with that there are many belief constellations which are impossible, given a certain underlying conceptualization of beliefs. Hence, the fact that there are sentences like the Ann-Bob paradox should come as no surprise. To give a trivial example, we know that on the standard logical conception of belief there is no

belief state that makes both  $B_i\phi$  and  $B_i\neg\phi$  true. In principle, this is a result similar to the fact that there is no belief state that makes the paradoxical Ann-Bob sentence true. So why are we surprised by the latter but not by the former? Why does one seem paradoxical, but not the other? I think this is also where the analogy with the Russell paradox for set theory breaks down. In the set theory case, we had a certain naïve conception of what constitutes a set, and this conception is called into question through the paradox. In the case of belief, there are to begin with already different ideas about what constitutes a belief, and these often correspond to different epistemic logics. But no matter what notion of belief is adopted, it should not come as a surprise that certain constellations of belief are impossible. And concerning the particular Ann-Bob sentence, I certainly did not think to begin with that this is a sentence that describes a possible belief configuration.