My Rendition of Hunter's Strong Inductive Proof of The Deduction Theorem for PS
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Theorem. Let Γ be an arbitrary set of formulas of P, and let A and B be arbitrary formulas of P.

$$\textit{If } \Gamma \cup \{A\} \vdash_{PS} B, \textit{ then } \Gamma \vdash_{PS} A \supset B.$$

If there is a derivation (in PS) of B from $\Gamma \cup \{A\}$, then there is a derivation (in PS) of $A \supset B$ from Γ alone.

Proof. We will prove this by *strong induction* on the length of derivations establishing $\Gamma \cup \{A\} \vdash_{PS} B$. More precisely, we will prove by strong induction that the statement $S(n)$ is true for all $n \geq 1$, where $S(n)$ is:

> $S(n)$: *If* there is a derivation (in PS) of B from $\Gamma \cup \{A\}$ **of length n**,
> *then* there is a derivation (in PS) of $A \supset B$ from Γ alone.

To prove that $S(n)$ is true for all $n \geq 1$ by strong induction, we will proceed in two steps:

I. **Basis Step**: Prove that $S(1)$ is true.
II. **Inductive Step**: *Assume* as our **inductive hypothesis** that $S(i)$ is true for all i such that $1 < i < n$. Then, *use* this *inductive hypothesis* to show that $S(i)$ is true when $i = n$ (*i.e.*, that $S(n)$ is true).

Having accomplished both (I) and (II), we will have succeeded in showing that $S(n)$ is true for all $n \geq 1$.

In what follows, I will use $\mathcal{D}$ to stand for some derivation (in PS) of B from $\Gamma \cup \{A\}$, and I will use $\mathcal{D}'$ to stand for some derivation (in PS) of $A \supset B$ from Γ alone. Since we are concerned about the length of $\mathcal{D}$, I will use $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_m$ to stand for the m lines of a derivation $\mathcal{D}$ with length m. Now, the inductive proof.

I. Basis Step. Let $\mathcal{D}$ be a derivation (in PS) of B from $\Gamma \cup \{A\}$ that is exactly one formula long. Thus, $\mathcal{D}$ has exactly one term: B. So, $\mathcal{D} = \langle \mathcal{D}_1 \rangle = \langle B \rangle$. By the definition of a derivation in PS, we must have *either*:

1. B is an axiom, *or*
2. B is a member of the set Γ, *or*
3. B is A itself, *or*
4. B is an immediate consequence of two previous lines of $\mathcal{D}$ by Modus Ponens.

Case 4 is *impossible* here, since $\mathcal{D} = \langle \mathcal{D}_1 \rangle = \langle B \rangle$. So, we only need to look at Cases 1–3. For each of these three cases, we will show how to construct from $\mathcal{D}$ another derivation (in PS) $\mathcal{D}'$ of $A \supset B$ from Γ alone.

Case 1. B is an axiom. Then we can construct a 3-line derivation $\mathcal{D}'$ showing $\Gamma \vdash_{PS} A \supset B$:

[1]	B	[Axiom, by assumption of Case 1]
[2]	$B \supset (A \supset B)$	[Axiom, by PS1]
[3]	$A \supset B$	[MP, 1, 2]

Note: $\mathcal{D}'$ is a *proof* in this Case. So, $A \supset B$ is a *theorem* in this Case. Therefore, *trivially*, $A \supset B$ can be derived from *any* set Γ in this Case (23.5). Remember, these sorts of statements involving A, B, *etc.* are *metalinguistic forms*, *not* statements of P. What this gives you is a general *metatheoretic recipe* for taking *any* B falling into Case 1 and using it to generate a derivation of $A \supset B$ from Γ. □

Case 2. B is a member of the set Γ. Again, we have a 3-line derivation $\mathcal{D}'$ showing $\Gamma \vdash_{PS} A \supset B$:

[1]	B	[Given as a member of the set Γ]
[2]	$B \supset (A \supset B)$	[Axiom, by PS1]
[3]	$A \supset B$	[MP, 1, 2] □

Case 3. $B = A$. Here is a 5-line derivation $\mathcal{D}'$ showing $\varnothing \vdash_{PS} A \supset B$, *viz.*, $\varnothing \vdash_{PS} A \supset A$:

[1]	$A \supset ((A \supset A) \supset A)$	[Axiom, by PS1]
[2]	$(A \supset ((A \supset A) \supset A)) \supset ((A \supset (A \supset A)) \supset (A \supset A))$	[Axiom, by PS2]
[3]	$(A \supset (A \supset A)) \supset (A \supset A)$	[MP, 1, 2]
[4]	$A \supset (A \supset A)$	[Axiom, by PS1]
[5]	$A \supset A$	[MP, 3, 4]

That is to say, $A \supset B = A \supset A$ is a *theorem* in this Case. So, in this Case, $A \supset B$ can — *trivially* — be derived from *any* set Γ (23.5). That completes the Basis Step of the strong inductive proof. □

II. Inductive Step. *Assume* the following strong inductive hypothesis:

Inductive Hypothesis (IH): The Deduction Theorem $[\Gamma \cup \{A\} \vdash_{PS} B \Longrightarrow \Gamma \vdash_{PS} A \supset B]$ holds for every derivation $\mathcal{D}$ of B from $\Gamma \cup \{A\}$ **with length less than $\boldsymbol{n}$.**

And, *using this assumed inductive hypothesis* (IH), *prove*:

Inductive Conclusion: The Deduction Theorem $[\Gamma \cup \{A\} \vdash_{PS} B \Longrightarrow \Gamma \vdash_{PS} A \supset B]$ holds for every derivation $\mathcal{D}$ of B from $\Gamma \cup \{A\}$ **with length equal to $\boldsymbol{n}$.**

Let $\mathcal{D}$ be an arbitrary derivation (in PS) of B from $\Gamma \cup \{A\}$ with length *equal to n*. Our goal is to use the (IH) on $\mathcal{D}$ to show that there must be a derivation (in PS) of $A \supset B$ from Γ alone. Again, four Cases:

1. B is an axiom.
2. B is a member of the set Γ.
3. B is A itself.
4. B is an immediate consequence of two previous lines of $\mathcal{D}$ by Modus Ponens.

For Cases 1–3, we can generate a derivation $\mathcal{D}'$ just as we did in the Basis Step. This time, Case 4 is *not* impossible, since $\mathcal{D}$ *could* have *many* lines prior to the final line on which B occurs ($n \geq 3$ in Case 4).

Case 4. B is an immediate consequence of two lines $\mathcal{D}_i$ and $\mathcal{D}_j$ of $\mathcal{D}$, where i and j are both less than n. Therefore, because of the way (MP) works, we must have either $\mathcal{D}_i = \mathcal{D}_j \supset B$ or $\mathcal{D}_j = \mathcal{D}_i \supset B$. I will assume that $\mathcal{D}_j = \mathcal{D}_i \supset B$. [The proof could also be run on the other assumption $\mathcal{D}_i = \mathcal{D}_j \supset B$.]

There are derivations of each of $\mathcal{D}_i$ and $\mathcal{D}_j$ from $\Gamma \cup \{A\}$, each of which are *less than n* lines long. So, by (IH), we know that both $\Gamma \vdash_{PS} A \supset \mathcal{D}_i$, and $\Gamma \vdash_{PS} A \supset \mathcal{D}_j$ $[\Gamma \vdash_{PS} A \supset (\mathcal{D}_i \supset B)]$. Then, we reason:

1.	$\Gamma \vdash_{PS} A \supset \mathcal{D}_i$	$[\Gamma \cup \{A\} \vdash_{PS} \mathcal{D}_i$, $i < n$, (IH)]
2.	$\Gamma \vdash_{PS} A \supset (\mathcal{D}_i \supset B)$	$[\Gamma \cup \{A\} \vdash_{PS} \mathcal{D}_j$, $j < n$, (IH), $\mathcal{D}_j = \mathcal{D}_i \supset B]$
3.	$\vdash_{PS} (A \supset (\mathcal{D}_i \supset B)) \supset ((A \supset \mathcal{D}_i) \supset (A \supset B))$	[Axiom, by PS2]
4.	$\Gamma \vdash_{PS} (A \supset \mathcal{D}_i) \supset (A \supset B)$	[From 2 and 3, by 23.45, see below]
5.	$\Gamma \vdash_{PS} A \supset B$	[From 1 and 4, by Hunter's 23.4]

Metatheorem 23.45. If $\Gamma \vdash_{PS} A$ and $\vdash_{PS} A \supset B$, then $\Gamma \vdash_{PS} B$.

Proof. 23.45 follows from Hunter's 23.4 & 23.5. If $\Gamma \vdash_{PS} A$, then there is a derivation $\mathfrak{D}$ of A from Γ. And, if $\vdash_{PS} A \supset B$, then the (k-step) *proof* of $A \supset B$ can be inserted into $\mathfrak{D}$ (prior to its last line), yielding a new (k-step longer) derivation $\mathfrak{D}'$ of A from Γ. Then, we can apply (MP) to the lines in $\mathfrak{D}'$ containing A and $A \supset B$ to infer B, which yields a derivation $\mathfrak{D}''$ of B from Γ. □

Using (IH), we have shown that *if* there is a derivation $\mathcal{D}$ (of any length $n \geq 3$) of B from $\Gamma \cup \{A\}$ in which B is derived using (MP) from two previous lines of $\mathcal{D}$, *then* there is a derivation $\mathcal{D}'$ of $A \supset B$ from Γ alone (do we know how to *construct $\mathcal{D}'$ from $\mathcal{D}$*?). That's Case 4, and the Inductive Step. □

So, we have shown that (I) $S(1)$ is true, and that (II) *if* $S(i)$ is true for all $i < n$, *then* $S(i)$ is true when $i = n$. Therefore, by the principle of strong mathematical induction, $S(n)$ is true for all $n \geq 1$, as desired. □