Announcements & Overview

- Administrative Stuff
  - **HW #2 grades & solutions will be posted tonight**
  - **The mid-term is *next Friday* — March 4**
    - ∗ I've posted a practice mid-term (same structure as actual mid-term)
    - ∗ We will go over the practice mid-term on Tuesday (March 1)
    - ∗ I've also posted a handout with some rules/definitions you'll be given at the mid-term (otherwise, it'll be a closed-book exam).
  - **HW #3 has been posted**
    - ∗ 5 truth-table exercises — due next Friday (same day as mid-term)
  - **I have posted 25 additional truth-table problems (with solutions)**
- Today: Unit #3, Continued
  - Truth-tables and their applications (continued)

p	$\sim p$
$\top$	$\bot$
$\bot$	$\top$

p	q	$p \,\&\, q$
$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$

p	q	$p \vee q$
$\top$	$\top$	$\top$
$\top$	$\bot$	$\top$
$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\bot$

p	q	$p \to q$
$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\top$

p	q	$p \leftrightarrow q$
$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\bot$
$\bot$	$\bot$	$\top$

Chapter 3 — Semantics of LSL: Additional Remarks on →

- Last time, I explained *why* our conditional → behaves "like a disjunction."
  1. We want a *truth-functional* semantics for →. This is a simplifying *idealization*. Truth-functional semantics are the simplest compositional semantics for sentential logic. [A "Newtonian" semantic model.]
  2. Given (1), the *only* way to define → is *our* way, since it's the *only* binary truth-function that has the following three essential *logical* properties:
     - (i) *Modus Ponens* [p and $\ulcorner p \to q \urcorner \therefore q$] is a valid sentential form.
     - (ii) Affirming the consequent [q and $\ulcorner p \to q \urcorner \therefore p$] is *not* a valid form.
     - (iii) All sentences of the form $\ulcorner p \to p \urcorner$ are logical truths.
- There are *non*-truth-functional semantics for the English conditional.
- These may be "closer" to the English *meaning* of "if". But, they agree with our semantics for →, when it comes to the crucial *logical* properties (i)–(iii). Indeed, our → captures *most* of the (intuitive) *logical* properties of "if".

Constructing Truth-Tables for LSL Sentences

- With the truth-table definitions of the five connectives in hand, we can now construct truth tables for arbitrary compound LSL statements.
- The procedure for constructing the truth-table of p is as follows:
  1. Determine the number of rows in the truth-table. This is 2^n, where n is the number of atomic sentences in the compound statement p.
  2. The table will have $n + 1$ main columns: n columns for the atomic sentences in p, and one for the truth-values of p itself.
  3. The table will also have some "quasi-columns" — one for each atom and each connective occurring in p — which needn't be drawn explicitly, but which go into the determination of p's truth values.
  4. Place the atomic letters in the left most columns, in alphabetical order from left to right. And, place p in the right most column.
  5. Write in all possible combinations of truth-values for the atomic statements. There are 2^n of these — one for each row of the table.

6. Convention: start on the nth column (farthest down the alphabet) with the pattern $\top\bot\top\bot\ldots$ repeated until the column is filled. Then, write $\top\top\bot\bot\ldots$ in the $(n-1)$st column, $\top\top\top\top\bot\bot\bot\bot\ldots$ in the $(n-2)$nd column, … alternations of 2^{n-m} $\top$s + 2^{n-m} $\bot$s in the mth column … until the first ($m = 1$) column has been completed.
7. Finally, we compute the truth-values of p in each row of the table. Here, we start from the inside-out. We first copy the truth-values of the atoms, then we compute the negations, conjunctions, etc. which compose p. Finally, we will be in a position to compute the value of the main connective of p, at which point we'll be done with the table.

- Example: Step-By-Step Truth-Table Construction of '$A \leftrightarrow (B \,\&\, A)$.'

A	B	A	$\leftrightarrow$	$(B$	$\&$	$A)$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\top$
$\bot$	$\top$	$\bot$	$\top$	$\top$	$\bot$	$\bot$
$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$

Interpretations and the Relation of Logical Consequence

- An *interpretation* of an LSL formula p is an assignment of truth-values to all of the sentence letters in p — *i.e.*, a row in p's truth-table.
- A formula p is a *logical consequence* of a set of formulae $\mathcal{S}$ [written $\mathcal{S} \vDash p$] just in case there is no interpretation (*i.e.*, no row in the joint truth-table of $\mathcal{S}$ and p) on which all the members of $\mathcal{S}$ are $\top$ but p is $\bot$.
- $\mathcal{S} \vDash p$ is another way of saying that the argument from $\mathcal{S}$ to p is *valid*.
- Two LSL sentences p and q are said to be *logically equivalent* [written $p \mathrel{=\!\!|\!\!\vDash} q$] iff they have the same truth-value on all (joint) interpretations.
- That is, p and q are logically equivalent iff *both* $p \vDash q$ *and* $q \vDash p$.
- I will often express $\ulcorner p \vDash q \urcorner$ by saying that $\ulcorner p$ *entails* $q\urcorner$. This is easier than saying that $\ulcorner q$ is a logical consequence of $p\urcorner$.
- The logical consequence relation $\vDash$ is our central theoretical relation.

Logical Truth, Logical Falsity, and Contingency: Definitions

- A statement is said to be logically true (or tautologous) if it is $\top$ on all interpretations. *E.g.*, any statement of the form $p \leftrightarrow p$ is tautological.

p	p	$\leftrightarrow$	p
$\top$	$\top$	$\top$	$\top$
$\bot$	$\bot$	$\top$	$\bot$

- A statement is logically false (or self-contradictory) if it is $\bot$ on all interpretations. *E.g.*, any statement of the form $p \,\&\, {\sim}p$ is logically false:

p	p	$\&$	$\sim$	p
$\top$	$\top$	$\bot$	$\bot$	$\top$
$\bot$	$\bot$	$\bot$	$\top$	$\bot$

- A statement is contingent if it is *neither* tautological *nor* self-contradictory. Example: 'A' (or *any* basic sentence) is contingent.

A	A
$\top$	$\top$
$\bot$	$\bot$

Logical Truth, Logical Falsity, and Contingency: Problems

- Classify the following statements as logically true (tautologous), logically false (self-contradictory), or contingent:

1. $N \to (N \to N)$
2. $(G \to G) \to G$
3. $(S \to R) \,\&\, (S \,\&\, {\sim}R)$
4. $((E \to F) \to F) \to E$
6. $(M \to P) \vee (P \to M)$
11. $[(Q \to P) \,\&\, ({\sim}Q \to R)] \,\&\, {\sim}(P \vee R)$
12. $[(H \to N) \,\&\, (T \to N)] \to [(H \vee T) \to N]$
15. $[(F \vee E) \,\&\, (G \vee H)] \leftrightarrow [(G \,\&\, E) \vee (F \,\&\, H)]$

Equivalence, Contradictoriness, Consistency, and Inconsistency

- Statements p and q are equivalent [$p \dashv\vDash q$] if they have the same truth-value on all interpretations. For instance, '$A \to B$' and '$\sim A \vee B$'.

A	B	A	$\to$	B	$\sim$	A	$\vee$	B
$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\bot$

- Statements p and q are contradictory [$p \dashv\vDash \sim q$] if they have opposite truth-values on all interpretations. For instance, '$A \to B$' and '$A \,\&\, \sim B$'.

A	B	A	$\to$	B	A	$\&$	$\sim$	B
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$	$\bot$	$\top$
$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\top$	$\bot$
$\bot$	$\top$	$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\bot$	$\top$
$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$

- Statements p and q are inconsistent [$p \vDash \sim q$] if there is no interpretation on which they are both true. For instance, '$A \leftrightarrow B$' and '$A \,\&\, \sim B$' are inconsistent [Note: they are *not* contradictory!].

A	B	A	$\leftrightarrow$	B	A	$\&$	$\sim$	B
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$	$\bot$	$\top$
$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\top$	$\bot$
$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\top$
$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$

- Statements p and q are consistent [$p \nvDash \sim q$] if there's an interpretation on which they are both true. *E.g.*, '$A \,\&\, B$' and '$A \vee B$' are consistent:

A	B	A	$\&$	B	A	$\vee$	B
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\bot$
$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

Semantic Equivalence, Contradictoriness, *etc.*: Relationships

- What are the logical relationships between $\ulcorner p$ and q are equivalent$\urcorner$, $\ulcorner p$ and q are consistent$\urcorner$, $\ulcorner p$ and q are contradictory$\urcorner$, and $\ulcorner p$ and q are inconsistent$\urcorner$? That is, which of these entails which (and which don't)?

Equivalent Contradictory

$\Downarrow$? $\Uparrow$ $\Downarrow$? $\Uparrow$

Consistent Inconsistent

- Answers:

1. Equivalent $\not\Rightarrow$ Consistent (*example*?)
2. Consistent $\not\Rightarrow$ Equivalent (*example*?)
3. Contradictory $\Rightarrow$ Inconsistent (*why*?)
4. Inconsistent $\not\Rightarrow$ Contradictory (*example*?)

Semantic Equivalence: Example #1

- Recall that $\ulcorner p$ unless $q\urcorner$ translates in LSL as $\ulcorner \sim q \to p\urcorner$.
- We've said that we can also translate $\ulcorner p$ unless $q\urcorner$ as $\ulcorner p \vee q\urcorner$.
- This is because $\ulcorner \sim q \to p\urcorner$ is *semantically equivalent to* $\ulcorner p \vee q\urcorner$. We may demonstrate this, using the following joint truth-table.

p	q	$\sim q$	$\to$	p	$p \vee q$
$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$
$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$

- The truth-tables of $\ulcorner p \vee q\urcorner$ and $\ulcorner \sim q \to p\urcorner$ are the same.
- Thus, $\sim q \to p \dashv\vDash p \vee q$.

Semantic Equivalence: Example #2

- ⌜$p \leftrightarrow q$⌝ is an *abbreviation* for ⌜$(p \to q) \,\&\, (q \to p)$⌝.
- The following truth-table shows it is a *legitimate* abbreviation:

p	q	$(p \to q)$	$\&$	$(q \to p)$	$p \leftrightarrow q$
⊤	⊤	⊤	⊤	⊤	⊤
⊤	⊥	⊥	⊥	⊤	⊥
⊥	⊤	⊤	⊥	⊥	⊥
⊥	⊥	⊤	⊤	⊤	⊤

- ⌜$p \leftrightarrow q$⌝ and ⌜$(p \to q) \,\&\, (q \to p)$⌝ have the same truth-table.
- Thus, $p \leftrightarrow q \dashv\vDash (p \to q) \,\&\, (q \to p)$.

Semantic Equivalence: Example #3

- Intuitively, the truth-conditions for *exclusive or* ($\oplus$) are such that ⌜$p \oplus q$⌝ is true if and only if *exactly* one of p or q is true.
- I said that we could say something equivalent to this using our $\vee$, &, and $\sim$. Specifically, I said $p \oplus q \dashv\vDash (p \vee q) \,\&\, {\sim}(p \,\&\, q)$.
- The following truth-table shows that this is correct:

p	q	$(p \vee q)$	$\&$	${\sim}(p \,\&\, q)$	$p \oplus q$
⊤	⊤	⊤	⊥	⊥	⊥
⊤	⊥	⊤	⊤	⊤	⊤
⊥	⊤	⊤	⊤	⊤	⊤
⊥	⊥	⊥	⊥	⊤	⊥

- ⌜$p \oplus q$⌝ and ⌜$(p \vee q) \,\&\, {\sim}(p \,\&\, q)$⌝ have the same truth-table.

Equivalence, Contradictoriness, *etc.*: Some Problems

- Use truth-tables to determine whether the following pairs of statements are semantically equivalent, contradictory, consistent, or inconsistent.
  1. ‘$F \,\&\, M$’ and ‘${\sim}(F \vee M)$’
  2. ‘$R \vee {\sim}S$’ and ‘$S \,\&\, {\sim}R$’
  3. ‘$H \leftrightarrow {\sim}G$’ and ‘$(G \,\&\, H) \vee ({\sim}G \,\&\, {\sim}H)$’
  4. ‘$N \,\&\, (A \vee {\sim}E)$’ and ‘${\sim}A \,\&\, (E \vee {\sim}N)$’
  5. ‘$W \leftrightarrow (B \,\&\, T)$’ and ‘$W \,\&\, (T \to {\sim}B)$’
  6. ‘$R \,\&\, (Q \vee S)$’ and ‘$(S \vee R) \,\&\, (Q \vee R)$’
  7. ‘$Z \,\&\, (C \leftrightarrow P)$’ and ‘$C \leftrightarrow (Z \,\&\, {\sim}P)$’
  8. ‘$Q \to {\sim}(K \vee F)$’ and ‘$(K \,\&\, Q) \vee (F \,\&\, Q)$’

Some More Semantic Equivalences

- Here is a simultaneous truth-table which establishes that

$$A \leftrightarrow B \dashv\vDash (A \,\&\, B) \vee ({\sim}A \,\&\, {\sim}B)$$

A	B	A	$\leftrightarrow$	B	$(A$	$\&$	$B)$	$\vee$	$({\sim}$	A	$\&$	${\sim}$	$B)$
⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊥	⊤	⊥	⊥	⊤
⊤	⊥	⊤	⊥	⊥	⊤	⊥	⊥	⊥	⊥	⊤	⊥	⊤	⊥
⊥	⊤	⊥	⊥	⊤	⊥	⊥	⊤	⊥	⊤	⊥	⊥	⊥	⊤
⊥	⊥	⊥	⊤	⊥	⊥	⊥	⊥	⊤	⊤	⊥	⊤	⊤	⊥

- Can you prove the following equivalences with truth-tables?
  - ${\sim}(A \,\&\, B) \dashv\vDash {\sim}A \vee {\sim}B$
  - ${\sim}(A \vee B) \dashv\vDash {\sim}A \,\&\, {\sim}B$
  - $A \dashv\vDash (A \,\&\, B) \vee (A \,\&\, {\sim}B)$
  - $A \dashv\vDash A \,\&\, (B \to B)$
  - $A \dashv\vDash A \vee (B \,\&\, {\sim}B)$

A More Complicated Equivalence (Distributivity)

- The following simultaneous truth-table establishes that

$$p \,\&\, (q \vee r) \mathrel{=\!\!|\!\!\models} (p \,\&\, q) \vee (p \,\&\, r)$$

p	q	r	p	$\&$	$(q \vee r)$	$(p \,\&\, q)$	$\vee$	$(p \,\&\, r)$
⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤
⊤	⊤	⊥	⊤	⊤	⊤	⊤	⊤	⊥
⊤	⊥	⊤	⊤	⊤	⊤	⊥	⊤	⊤
⊤	⊥	⊥	⊤	⊥	⊥	⊥	⊥	⊥
⊥	⊤	⊤	⊥	⊥	⊤	⊥	⊥	⊥
⊥	⊤	⊥	⊥	⊥	⊤	⊥	⊥	⊥
⊥	⊥	⊤	⊥	⊥	⊤	⊥	⊥	⊥
⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥

- This is *distributivity* of & over ∨. It also works for ∨ over &.

The Exhaustive Truth-Table Method for Testing Validity

- Remember, an argument is valid if it is *impossible* for its premises to be true while its conclusion is false. Let $p_1, \dots, p_n$ be the premises of a LSL argument, and let q be the conclusion of the argument. Then, we have:

$$\begin{array}{l} p_1 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$ is valid if and only if there is no row in the simultaneous truth-table of $p_1, \dots, p_n$, and q which looks like the following:

atoms	premises			conclusion
…	p_1	…	p_n	q
…	⊤	⊤	⊤	⊥

- We will use simultaneous truth-tables to prove validities and invalidities. For example, consider the following valid argument:

$$\begin{array}{l} A \\ A \to B \\ \hline \therefore B \end{array}$$

atoms		premises				conclusion
A	B	A	A	$\to$	B	B
⊤	⊤	⊤	⊤	⊤	⊤	⊤
⊤	⊥	⊤	⊤	⊥	⊥	⊥
⊥	⊤	⊥	⊥	⊤	⊤	⊤
⊥	⊥	⊥	⊥	⊤	⊥	⊥

☞ VALID — there is no row in which A and $A \to B$ are both ⊤, but B is ⊥.

- In general, we'll use the following procedure for evaluating arguments:
  1. Translate and symbolize the the argument (if given in English).
  2. Write out the symbolized argument (as above).
  3. Draw a simultaneous truth-table for the symbolized argument, outlining the columns representing the premises and conclusion.
  4. Is there a row of the table in which all premises are ⊤ but the conclusion is ⊥? If so, the argument is invalid; if not, it's valid.
- We will practice this on examples. But, first, a "short-cut" method.

The "Short" Truth Table Method for Validity Testing I

- Consider the following LSL argument:

  $A \to (B \,\&\, E)$
  $D \to (A \vee F)$
  $\sim E$
  $\therefore D \to B$

- This argument has 3 premises and contains 5 atomic sentences. This would lead to a complete truth-table with 32 rows and 8 columns (this will be far more than 256 distinct computations).
- As such, the exhaustive truth-table method does not seem practical in this case. So, instead, let's try to construct or "reverse engineer" an invalidating interpretation.
- To do this, we "work backward" from the *assumption* that the conclusion is ⊥ and all the premises are ⊤ on some row.

- Step 1: Assume there is an interpretation on which all three premises are $\top$ and the conclusion is $\bot$. This leads to:

A	B	D	E	F	A	$\to$	$(B \& E)$	D	$\to$	$(A \vee F)$	$\sim E$	$D \to B$
						$\top$			$\top$		$\top$	$\bot$

- Step 2: From the assumption that $\sim E$ is $\top$, we may infer that both E and $B \& E$ are $\bot$. This fills-in two more cells:

A	B	D	E	F	A	$\to$	$(B \& E)$	D	$\to$	$(A \vee F)$	$\sim E$	$D \to B$
			$\bot$			$\top$	$\bot$		$\top$		$\top$	$\bot$

- Step 3: Now, the only way that $A \to (B \& E)$ can be $\top$ (as we've assumed) is if its antecedent A is $\bot$. This yields the following:

A	B	D	E	F	A	$\to$	$(B \& E)$	D	$\to$	$(A \vee F)$	$\sim E$	$D \to B$
$\bot$			$\bot$		$\bot$	$\top$	$\bot$		$\top$		$\top$	$\bot$

- Step 4: Now, $D \to B$ can be $\bot$ (as we've been assuming) if and only if D is $\top$ and B is $\bot$ (just by the definition of $\to$). So:

A	B	D	E	F	A	$\to$	$(B \& E)$	D	$\to$	$(A \vee F)$	$\sim E$	$D \to B$
$\bot$	$\bot$	$\top$	$\bot$		$\bot$	$\top$	$\bot$	$\top$	$\top$		$\top$	$\bot$

- Step 5: Then, $D \to (A \vee F)$ can be $\top$ (as we've assumed) only if its consequent $A \vee F$ is $\top$, which gives the following:

A	B	D	E	F	A	$\to$	$(B \& E)$	D	$\to$	$(A \vee F)$	$\sim E$	$D \to B$
$\bot$	$\bot$	$\top$	$\bot$		$\bot$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\bot$

- Step 6: Finally, since A is $\bot$, the only way that $A \vee F$ can be $\top$ is if F is $\top$, which completes our construction!

A	B	D	E	F	A	$\to$	$(B \& E)$	D	$\to$	$(A \vee F)$	$\sim E$	$D \to B$
$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\bot$