Philosophy 201 Homework Assignment #6 Solutions

April 27, 2016

1 Problem #1

For this problem, please use the following stochastic truth table to determine all of your algebraic translations for probabilistic claims involving $\{X, Y, Z\}$.

State (s_i)	X	Y	Z	$\Pr(s_i)$
s_1	$\top$	$\top$	$\top$	$\Pr(s_1) = a_1$
s_2	$\top$	$\top$	$\bot$	$\Pr(s_2) = a_2$
s_3	$\top$	$\bot$	$\top$	$\Pr(s_3) = a_3$
s_4	$\top$	$\bot$	$\bot$	$\Pr(s_4) = a_4$
s_5	$\bot$	$\top$	$\top$	$\Pr(s_5) = a_5$
s_6	$\bot$	$\top$	$\bot$	$\Pr(s_6) = a_6$
s_7	$\bot$	$\bot$	$\top$	$\Pr(s_7) = a_7$
s_8	$\bot$	$\bot$	$\bot$	$\Pr(s_8) = a_8$

The goal is to prove the following general claim (the Unconditional Sure Thing Principle holds for Factor #2):

$$[\Pr(Z \mid X \,\&\, Y) > \Pr(Z) \text{ and } \Pr(Z \mid X \,\&\, {\sim}Y) > \Pr(Z)] \Longrightarrow \Pr(Z \mid X) > \Pr(Z).$$

That is, the goal is the prove that the following two assumptions:

(1) $\Pr(Z \mid X \,\&\, Y) > \Pr(Z)$

(2) $\Pr(Z \mid X \,\&\, {\sim}Y) > \Pr(Z)$

generally entail this third claim:

(3) $\Pr(Z \mid X) > \Pr(Z)$

In order to do this, you should follow these two steps:

Step 1. Translate claims (1)–(3) into their algebraic counterparts, using our definitions of unconditional and conditional probability (and the above table for the salient variables). That is, using:

$$\Pr(p) \overset{\text{def}}{=} \sum_{s_i \vDash p} \Pr(s_i) = \sum_{s_i \vDash p} a_i$$

$$\Pr(p \mid q) \overset{\text{def}}{=} \frac{\Pr(p \,\&\, q)}{\Pr(q)}, \text{ provided that } \Pr(q) > 0.$$

Step 2. Use our two general assumptions about the a_i's:

(i) Each of the a_i's are on $[0, 1]$. That is: $a_1, \ldots, a_8 \in [0, 1]$.

(ii) The a_i's must sum to 1. That is: $\sum_{i=1}^{8} a_i = 1$.

to show (in algebraic terms) that *whenever (1) and (2) are both true, (3) must also be true.*

Answer: First, we rewrite (1), (2), and (3) in algebraic form, as follows:

(1) $\dfrac{a_1}{a_1 + a_2} > a_1 + a_3 + a_5 + a_7$

(2) $\dfrac{a_3}{a_3 + a_4} > a_1 + a_3 + a_5 + a_7$

(3) $\dfrac{a_1 + a_3}{a_1 + a_2 + a_3 + a_4} > a_1 + a_3 + a_5 + a_7$

Cross-multiplying (1) and (2), yields (note, because the denominators of (1) and (2) are non-negative, cross-multiplying these inequalities will not change their direction).

(1) $a_1 > (a_1 + a_2) \cdot (a_1 + a_3 + a_5 + a_7)$

(2) $a_3 > (a_3 + a_4) \cdot (a_1 + a_3 + a_5 + a_7)$

Adding the left and right hand sides of these renditions of (1) and (2), yields [note, because all terms involved in these two inequalities are non-negative, adding these inequalities yields the new inequality (4)].

(4) $a_1 + a_3 > ((a_1 + a_2) \cdot (a_1 + a_3 + a_5 + a_7)) + ((a_3 + a_4) \cdot (a_1 + a_3 + a_5 + a_7))$

Factoring out the $(a_1 + a_3 + a_5 + a_7)$ term on the right hand side of (4) and simplifying the result, yields:

(5) $a_1 + a_3 > (a_1 + a_3 + a_5 + a_7) \cdot (a_1 + a_2 + a_3 + a_4)$

Dividing both sides of (5) by the non-negative term $(a_1 + a_2 + a_3 + a_4)$ yields:

(6) $\dfrac{a_1 + a_3}{a_1 + a_2 + a_3 + a_4} > a_1 + a_3 + a_5 + a_7$

Finally, translating (6) back into the language of probability calculus yields:

(3) $\Pr(Z \mid X) > \Pr(Z)$,

which was precisely the claim we were trying to prove from (1) and (2).

2 Problem #2

Suppose we have an urn containing 320 objects. We are going to sample a single object o at random from the urn (each individual object is equally likely to be chosen). Consider the following three statements:

- $B = o$ is black ($\sim B = o$ is white).
- $M = o$ is metal ($\sim M = o$ is plastic).
- $S = o$ is a sphere ($\sim S = o$ is a cube).

Assume that these three properties are distributed according to the following *probabilistic truth-table*:

World (s_i)	B	M	S	$\Pr(s_i)$
s_1	$\top$	$\top$	$\top$	$\Pr(s_1) = \frac{24}{320}$
s_2	$\top$	$\top$	$\bot$	$\Pr(s_2) = \frac{6}{320}$
s_3	$\top$	$\bot$	$\top$	$\Pr(s_3) = \frac{24}{320}$
s_4	$\top$	$\bot$	$\bot$	$\Pr(s_4) = \frac{42}{320}$
s_5	$\bot$	$\top$	$\top$	$\Pr(s_5) = \frac{33}{320}$
s_6	$\bot$	$\top$	$\bot$	$\Pr(s_6) = \frac{33}{320}$
s_7	$\bot$	$\bot$	$\top$	$\Pr(s_7) = \frac{47}{320}$
s_8	$\bot$	$\bot$	$\bot$	$\Pr(s_8) = \frac{111}{320}$

That is, 24 of the 320 objects are black metallic spheres; 47 of the 320 objects are white plastic spheres *etc.* With these basic probabilities in mind, we can use our definitions of unconditional and conditional probability (on page 1) to calculate *any* probability in this example.

The HW is to answer the following eleven (11) questions. [Note: once you've answered questions (1)–(5), you'll have everything you need to answer questions (6)–(11). See my 11/10/15 lecture for the 3 Proposals.]

1. What is $\Pr(S)$?

   - **Answer**: $\Pr(S) = \Pr(s_1) + \Pr(s_3) + \Pr(s_5) + \Pr(s_7) = \frac{24}{320} + \frac{24}{320} + \frac{33}{320} + \frac{47}{320} = \frac{128}{320} = \mathbf{\frac{2}{5}}$.

2. What is $\Pr(S \mid B)$? [That is, what is $\frac{\Pr(S\&B)}{\Pr(B)}$?]

   - **Answer**:

$$\begin{aligned}\Pr(S \mid B) = \frac{\Pr(S \,\&\, B)}{\Pr(B)} &= \frac{\Pr(s_1) + \Pr(s_3)}{\Pr(s_1) + \Pr(s_2) + \Pr(s_3) + \Pr(s_4)} \\ &= \frac{\frac{24}{320} + \frac{24}{320}}{\frac{24}{320} + \frac{6}{320} + \frac{24}{320} + \frac{42}{320}} \\ &= \frac{48/320}{96/320} \\ &= \frac{48}{96} = \mathbf{\frac{1}{2}}.\end{aligned}$$

3. What is $\Pr(S \mid B \,\&\, M)$? [That is, what is $\frac{\Pr(S\&(B\&M))}{\Pr(B\&M)}$?]

   - **Answer**:

$$\begin{aligned}\Pr(S \mid B \,\&\, M) = \frac{\Pr(S \,\&\, (B \,\&\, M))}{\Pr(B \,\&\, M)} &= \frac{\Pr(s_1)}{\Pr(s_1) + \Pr(s_2)} \\ &= \frac{\frac{24}{320}}{\frac{24}{320} + \frac{6}{320}} \\ &= \frac{24/320}{30/320} \\ &= \frac{24}{30} = \mathbf{\frac{4}{5}}.\end{aligned}$$

4. What is $\Pr(B \to S)$? [Hint: do the truth-table for $B \to S$ to see in which of the 8 worlds $B \to S$ is true.]

   - **Answer**:

$$\begin{aligned}\Pr(B \to S) &= \Pr(s_1) + \Pr(s_3) + \Pr(s_5) + \Pr(s_6) + \Pr(s_7) + \Pr(s_8) \\ &= \frac{24}{320} + \frac{24}{320} + \frac{33}{320} + \frac{33}{320} + \frac{47}{320} + \frac{111}{320} \\ &= \frac{272}{320} = \mathbf{\frac{17}{20}}.\end{aligned}$$ [1]

5. What is $\Pr((B \,\&\, M) \to S)$? [Hint: do the truth-table for $(B \,\&\, M) \to S$ to see in which worlds it is true.]

[1] Alternatively, you could calculate $\Pr(B \to S) = 1 - \Pr(\sim(B \to S)) = 1 - (\Pr(s_2) + \Pr(s_4)) = 1 - \left(\frac{6}{320} + \frac{42}{320}\right) = 1 - \frac{48}{320} = \frac{272}{320}$.

- **Answer**:

$$\begin{aligned}
\Pr((B \,\&\, M) \to S) &= \Pr(s_1) + \Pr(s_3) + \Pr(s_4) + \Pr(s_5) + \Pr(s_6) + \Pr(s_7) + \Pr(s_8) \\
&= \frac{24}{320} + \frac{24}{320} + \frac{42}{320} + \frac{33}{320} + \frac{33}{320} + \frac{47}{320} + \frac{111}{320} \\
&= \frac{314}{320} = \mathbf{\frac{157}{160}}.^{2}
\end{aligned}$$

6. Is the argument ⌜$B \therefore S$⌝ inductively strong, according to Proposal #1? [Hint: use your answer to (4).]

   - **Answer**: According to proposal #1, the argument ⌜$B \therefore S$⌝ is strong iff $\Pr(B \to S) > \frac{1}{2}$. From (4), we know that $\Pr(B \to S) = \frac{17}{20} > \frac{1}{2}$. So, according to proposal #1, the argument ⌜$B \therefore S$⌝ ***is*** strong.

7. Is ⌜$B \therefore S$⌝ inductively strong, according to Proposal #2 (Skyrms's proposal)? [Hint: use (2).]

   - **Answer**: According to proposal #2, the argument ⌜$B \therefore S$⌝ is strong iff $\Pr(S \mid B) > \frac{1}{2}$. From (2), we know that $\Pr(S \mid B) = \frac{1}{2} \not> \frac{1}{2}$. So, according to proposal #2, the argument ⌜$B \therefore S$⌝ is ***not*** strong.

8. Is ⌜$B \therefore S$⌝ inductively strong, according to Proposal #3 (my proposal)? [Hint: use (2) and (1).]

   - **Answer**: According to proposal #3, the argument ⌜$B \therefore S$⌝ is strong iff *both* $\Pr(S \mid B) > \frac{1}{2}$, *and* $\Pr(S \mid B) > \Pr(S)$. From (2), we know that $\Pr(S \mid B) = \frac{1}{2} \not> \frac{1}{2}$. So, according to proposal #2, the argument ⌜$B \therefore S$⌝ is ***not*** strong.[3]

9. Is the argument ⌜$B \,\&\, M \therefore S$⌝ inductively strong, according to Proposal #1? [Hint: use (5).]

   - **Answer**: According to proposal #1, the argument ⌜$B \,\&\, M \therefore S$⌝ is strong iff $\Pr((B \,\&\, M) \to S) > \frac{1}{2}$. From (5), we know that $\Pr((B \,\&\, M) \to S) = \frac{157}{160} > \frac{1}{2}$. So, according to proposal #1, the argument ⌜$B \,\&\, M \therefore S$⌝ ***is*** strong.

10. Is ⌜$B \,\&\, M \therefore S$⌝ inductively strong, according to Proposal #2 (Skyrms's proposal)? [Hint: use (3).]

    - **Answer**: According to proposal #2, the argument ⌜$B \,\&\, M \therefore S$⌝ is strong iff $\Pr(S \mid B \,\&\, M) > \frac{1}{2}$. From (3), we know that $\Pr(S \mid B \,\&\, M) = \frac{4}{5} > \frac{1}{2}$. So, according to proposal #2, the argument ⌜$B \,\&\, M \therefore S$⌝ ***is*** strong.

11. Is ⌜$B \,\&\, M \therefore S$⌝ inductively strong, according to Proposal #3 (my proposal)? [Hint: use (3) and (1).]

    - **Answer**: According to proposal #3, the argument ⌜$B \,\&\, M \therefore S$⌝ is strong iff *both* $\Pr(S \mid B \,\&\, M) > \frac{1}{2}$, *and* $\Pr(S \mid B \,\&\, M) > \Pr(S)$. From (3) and (1), we know that $\Pr(S \mid B \,\&\, M) = \frac{4}{5} > \frac{1}{2} > \frac{2}{5} = \Pr(S)$. So, according to proposal #3, the argument ⌜$B \,\&\, M \therefore S$⌝ ***is*** strong.

[2] Alternatively, you could calculate $\Pr((B \,\&\, M) \to S) = 1 - \Pr(\sim((B \,\&\, M) \to S)) = 1 - \Pr(s_2) = 1 - \frac{6}{320} = \frac{314}{320}$.

[3] Note: the premise B *is positively relevant to* the conclusion S, since $\Pr(S \mid B) = \frac{1}{2} > \frac{2}{5} = \Pr(S)$.