PHIL 424: HW #4 Solutions

11/25/14

Point Values

Each part of each question was worth 10 points. Partial credit was awarded.

4 Consistency and Entailment

4.a Consistency

Suppose we are flipping a fair coin twice. Let H be the proposition that the coin lands heads at least once. Let E be the proposition that the coin doesn't lands tails on the first toss. Let $\mathrm{cr}_1(\cdot)$ represent the agent's credences before either toss, and $\mathrm{cr}_2(\cdot)$ is the agent's credences after seeing the first toss land tails.

H and E are consistent - even if E is true, the second toss could land heads and that would make H true. $\mathrm{cr}_1(H) = \frac{3}{4}$ - there are four equally probable possibilities, 3 of which involve at least one toss coming up heads. $\mathrm{cr}_1(E) = \frac{1}{2}$, the coin is fair. $\sim E \vDash H$ so $\frac{1}{2} = \mathrm{cr}_1(\sim E) = \mathrm{cr}_1(\sim E \,\&\, H)$. This also implies that $cr_1(H\&E) = \frac{1}{4}$

$\mathrm{cr}_2(H) = \mathrm{cr}_1(H|E) = \frac{\mathrm{cr}_1(H \,\&\, E)}{\mathrm{cr}_1(E)} = \frac{1}{4} \cdot \frac{2}{1} = \frac{1}{2}$. And $\frac{1}{2} < \frac{3}{4}$

4.b Entailment

$\mathrm{cr}_2(H) \geq \mathrm{cr}_1(H)$ just in case $\frac{\mathrm{cr}_2(H)}{\mathrm{cr}_1(H)} \geq 1$.

Consider $\mathrm{cr}_2(H)$. We know $\mathrm{cr}_2(H) = \mathrm{cr}_1(H|E) = \frac{cr_1(H \,\&\, E)}{cr_1(E)}$. So $\frac{cr_2(H)}{cr_1(H)} = \frac{cr_1(H \,\&\, E)}{cr_1(H) \cdot cr_1(E)}$. It suffices to show that $\mathrm{cr}_1(H \,\&\, E) \geq \mathrm{cr}_1(H) \cdot \mathrm{cr}(E)$. $\mathrm{cr}_1(H \,\&\, E) = \mathrm{cr}_1(H)$ since $H \vDash E$. Since $\mathrm{cr}_1(E) \leq 1$, $\mathrm{cr}_1(H \,\&\, E) = \mathrm{cr}_1(H) \geq \mathrm{cr}_1(H) \cdot \mathrm{cr}_1(E)$. And we are done.

4.c Entailment and Non-Maximality

Investigating the above proof, we see that if $\mathrm{cr}_1(E) < 1$ then $\mathrm{cr}_1(H) > \mathrm{cr}_1(H) \cdot \mathrm{cr}_1(E)$. The conclusion follows.

5 Base Rate Fallacy

5.a Repeat Tests

Letting P_2 be the proposition that the second test is positive, we want to know about $\mathrm{cr}(D|P \,\&\, P_2)$, the credence that the patient has the disease, conditional on two positive tests.

$$\begin{aligned}
\mathrm{cr}(D|P \,\&\, P_2) &= \frac{\mathrm{cr}(P \,\&\, P_2|D) \cdot \mathrm{cr}(D)}{\mathrm{cr}(P \,\&\, P_2)} && \text{Bayes' Theorem} \\
&= \frac{\mathrm{cr}(P \,\&\, P_2|D) \cdot \mathrm{cr}(D)}{\mathrm{cr}(P \,\&\, P_2 \,\&\, D) + \mathrm{cr}(P \,\&\, P_2 \,\&\, {\sim}D)} && \text{Additivity} \\
&= \frac{\mathrm{cr}(P \,\&\, P_2|D) \cdot \mathrm{cr}(D)}{\mathrm{cr}(P \,\&\, P_2|D) \cdot \mathrm{cr}(D) + \mathrm{cr}(P \,\&\, P_2| \sim D) \cdot \mathrm{cr}({\sim}D)} && \text{Ratio Formula}
\end{aligned}$$

So, this depends on $\mathrm{cr}(P \,\&\, P_2|D)$ and $\mathrm{cr}(P \,\&\, P_2| \sim D)$. We can see that

$$\begin{aligned}
\mathrm{cr}(P \,\&\, P_2|D) &= \frac{\mathrm{cr}(P \,\&\, P_2 \,\&\, D)}{\mathrm{cr}(D)} && \text{Ratio Formula} \\
&= \frac{\mathrm{cr}(P \,\&\, P_2 \,\&\, D)}{\mathrm{cr}(P_2 \,\&\, D)} \cdot \frac{\mathrm{cr}(P_2 \,\&\, D)}{\mathrm{cr}(D)} && \text{Algebra} \\
&= \mathrm{cr}(P|P_2 \,\&\, D) \cdot \mathrm{cr}(P_2|D) && \text{Ratio Formula} \\
&= \mathrm{cr}(P|D) \cdot \mathrm{cr}(P_2|D) && D \text{ screens off } P_2 \text{ from } P \\
&= 0.81 && \text{Test is 90\% accurate}
\end{aligned}$$

A similar proof shows $\mathrm{cr}(P \,\&\, P_2| \sim D) = \mathrm{cr}(P| \sim D) \cdot \mathrm{cr}(P_2| \sim D) = 0.01$
Putting everything together, we get

$$\begin{aligned}
\mathrm{cr}(D|P \,\&\, P_2) &= \frac{\mathrm{cr}(P \,\&\, P_2|D) \cdot \mathrm{cr}(D)}{\mathrm{cr}(P \,\&\, P_2|D) \cdot \mathrm{cr}(D) + \mathrm{cr}(P \,\&\, P_2| \sim D) \cdot \mathrm{cr}({\sim}D)} \\
&= \frac{0.81 \cdot 0.001}{0.81 \cdot 0.001 + 0.01 \cdot 0.999} \\
&= 0.075
\end{aligned}$$

5.b 50% Confidence

What we are looking for is smallest natural number n such that $\mathrm{cr}(D|P \,\&\, P_2 \,\&\, \ldots \,\&\, P_n) \geq 0.5$. Looking at the above answer, we can see that for n tests,

$$\begin{aligned}
\mathrm{cr}(D|P \,\&\, P_2 \,\&\, \ldots \,\&\, P_n) &= \frac{\mathrm{cr}(P \,\&\, P_2 \,\&\, \ldots \,\&\, P_n|D) \cdot \mathrm{cr}(D)}{\mathrm{cr}(P \,\&\, P_2 \,\&\, \ldots \,\&\, P_n|D) \cdot \mathrm{cr}(D) + \mathrm{cr}(P \,\&\, P_2 \,\&\, \ldots \,\&\, P_n| \sim D) \cdot \mathrm{cr}({\sim}D)} \\
&= \frac{\mathrm{cr}(P|D)^n \cdot \mathrm{cr}(D)}{\mathrm{cr}(P|D)^n \cdot \mathrm{cr}(D) + \mathrm{cr}(P| \sim D)^n \cdot \mathrm{cr}({\sim}D)} \\
&= \frac{0.9^n \cdot 0.001}{0.9^n \cdot 0.001 + 0.1^n \cdot 0.999}
\end{aligned}$$

This is greater than 0.5 just in case

$$\begin{aligned}
0.9^n \cdot 0.001 &\geq 0.5(0.9^n \cdot 0.001 + 0.1^n \cdot 0.999) \\
0.9^n \cdot 0.001 \cdot 0.5 &\geq 0.1^n \cdot 0.999 \cdot 0.5 \\
0.9^n \cdot 0.001 &\geq 0.1^n \cdot 0.999 \\
\left(\frac{0.9}{0.1}\right)^n &\geq 999 \\
9^n &\geq 999 \\
n &\gtrapprox 3.143
\end{aligned}$$

Since you can only do a natural number of tests, the answer is 4.

5.c Second Opinions

Many answers are acceptable, but the short answer is yes.

7 Hypothetical Priors

7.a t_2 and t_3

By condition 2, $\mathrm{cr}_2(P) = 1$ so $\mathrm{cr}_2(\sim P \,\&\, Q) = \mathrm{cr}_2(\sim P \,\&\, \sim Q) = 0$. Since Jane updates by conditionalization between t_1 and t_2, $\mathrm{cr}_2(Q) = \mathrm{cr}_1(Q|P) = 2/3$ by condition 3. So $2/3 = \mathrm{cr}_2(Q) = \mathrm{cr}_2(Q \,\&\, P) + \mathrm{cr}_2(Q \,\&\, \sim P) = \mathrm{cr}_2(Q\&P) + 0 = \mathrm{cr}_2(P\&Q)$. Summarizing, $\mathrm{cr}_2(P\&Q) = 2/3$, $\mathrm{cr}_2(\sim P\&Q) = \mathrm{cr}_2(\sim P\&\sim Q) = 0$, $\mathrm{cr}_2(P\&\sim Q) = 1/3$

By condition 6, $\mathrm{cr}_3(P \,\&\, Q) = 0$. $\mathrm{cr}_3(P \,\&\, \sim Q) = 1 - \mathrm{cr}_3(P \supset Q)$ and $\mathrm{cr}_2(P \,\&\, \sim Q) = 1 - \mathrm{cr}_2(P \supset Q)$. By condition 6, $\mathrm{cr}_3(P \supset Q) = \mathrm{cr}_2(P \supset Q)$ so $\mathrm{cr}_3(P \,\&\, \sim Q) = \mathrm{cr}_2(P \,\&\, \sim Q) = 1/3$. This implies that $\mathrm{cr}_3(\sim P) = 2/3$. By condition 4, $1/2 = \mathrm{cr}_3(Q| \sim P) = \dfrac{\mathrm{cr}_3(\sim P \,\&\, Q)}{\mathrm{cr}_3(\sim P)} = \dfrac{\mathrm{cr}_3(\sim P \,\&\, Q)}{2/3}$. So $\mathrm{cr}_3(\sim P \,\&\, Q) = 1/3$. We conclude also that $\mathrm{cr}_3(\sim P \,\&\, \sim Q) = 1/3$

P	Q	cr_2	cr_3
T	T	2/3	0
T	F	1/3	1/3
F	T	0	1/3
F	F	0	1/3

7.b Hypothetical Priors

At t_3 the agent is certain only of $\sim(P \,\&\, Q)$ and its consequences. So the hypothetical prior, whatever it is, must assign equal credence (let it be x) to the other three state-descriptions. At t_2, the agent is certain only of P and its consequences. We also know that the hypothetical prior, whatever it is, assigns twice as much credence to $P \,\&\, Q$ as to $P \,\&\, \sim Q$. So $\mathrm{cr}_H(P \,\&\, Q) = 2x$ and $\mathrm{cr}_H(P \,\&\, \sim Q) = \mathrm{cr}_H(\sim P \,\&\, Q) = \mathrm{cr}_H(\sim P \,\&\, \sim Q) = x$. It follows that $x = 1/5$.

This leads to the table below. It is easy to check that this is a hypothetical prior distribution.

P	Q	c_H	cr_1	cr_2	cr_3
T	T	2/5	1/2	2/3	0
T	F	1/5	1/4	1/3	1/3
F	T	1/5	0	0	1/3
F	F	1/5	1/4	0	1/3

7.c Does Jane Conditionalize?

No. There are many ways to see this. One is that Conditionalization preserves certainties, but $\mathrm{cr}_2(P) = 1$ and $\mathrm{cr}_3(P) < 1$, so Jane has lost a certainty.

7.d Converse of Hypothetical Priors

Apparently not - though Jane's credences can be represented by a hypothetical prior, she does not conditionalize.