that the lemma holds for any wff A_b in which b occurs. And if b does not occur in A_b, we know that the lemma holds for A_b, which then falls under the special case. This completes the proof of the substitution lemma. It follows as a corollary of the substitution lemma that substitution preserves validity (details are left to the reader).

28.1. Independence of Axioms and Rules

Other things being equal, elegance is preferable to inelegance. Accordingly, in logic and mathematics some emphasis is put on the creation of elegant primitive bases for axiomatic systems. One feature that contributes to the elegance of a primitive basis is the independence of its axioms and rules of inference. An axiom or rule of inference of a system L is *dependent in* L if the system that results by dropping the given axiom or rule from the primitive basis of L has the same set of theorems (actual output) that L has. So, if an axiom of a logistic system L is independent in L, then its omission from the primitive basis would diminish the set of theorems of L. The same holds for an independent rule of inference.

We will show that all the axioms and rules of system P are independent (in system P). Hence none of the axioms or rules of system P could be dropped from its primitive basis except on pain of denying theoremhood to some of the theorems (the valid wffs) of system P. First, the rules of inference. If the rule of substitution were dropped from the primitive basis of system P, no theorem of the resulting system would be longer than its longest axiom, axiom 2. The reason is that the conclusion of an application of *modus ponens* is always shorter than the longer premiss. But there are theorems of system P longer than axiom 2. So it follows from the definition of dependence that substitution is an independent rule in system P.

Just as substitution is needed in system P to obtain theorems longer than the longest axiom, so *modus ponens* is needed to obtain theorems shorter than the shortest axiom. (Full details are left to the reader.) There are theorems of system P shorter than its shortest axiom, for example, '$[p \supset p]$', so it follows that *modus ponens* is an independent rule in system P.

To prove that an axiom of a logistic system L is independent

(in L), it suffices to find some property which the given axiom lacks but which is possessed by the remaining axioms and preserved by the rules of inference. If there is such a property ϕ, then the theorems of the system that results on dropping the given axiom all have ϕ. But this entails that the given axiom is not a theorem of the resulting system, which means that the axiom is independent in L. We will employ the strategy just described to prove that the axioms of system P are all independent (in P). To get the requisite properties ϕ, we introduce several *secondary interpretations* of system P.

Disregard for the moment the principal or intended semantics of system P and consider the following secondary interpretation. We let the range of values of the variables of system P be the three numbers 0, 1, and 2. That is, we let the variables take on any of the three "truth values" 0, 1, and 2.[1] And we introduce these fundamental "truth tables" for the tilde and horseshoe.

A	$\sim A$
0	1
1	1
2	1

A	B	$A \supset B$
0	0	0
0	1	2
0	2	2
1	0	2
1	1	2
1	2	0
2	0	0
2	1	0
2	2	0

We call 0 the *designated* truth value, and we refer to 1 and 2 as *undesignated* truth values. By a *valid wff* we now understand a wff that has a designated truth value under every assignment of values to its variables. For example, as the tables below show, axiom 2 and axiom 3 are valid in the sense just defined.

[1] Any three objects would have done just as well as the numbers 0, 1, and 2.

p	q	$[\sim p \supset \sim q] \supset [q \supset p]$				
0	0	$\cancel{1}$	$\cancel{2}$	$\cancel{1}$	0	$\cancel{0}$
0	1	$\cancel{1}$	$\cancel{2}$	$\cancel{1}$	0	$\cancel{2}$
0	2	$\cancel{1}$	$\cancel{2}$	$\cancel{1}$	0	$\cancel{0}$
1	0	$\cancel{1}$	$\cancel{2}$	$\cancel{1}$	0	$\cancel{2}$
1	1	$\cancel{1}$	$\cancel{2}$	$\cancel{1}$	0	$\cancel{2}$
1	2	$\cancel{1}$	$\cancel{2}$	$\cancel{1}$	0	$\cancel{0}$
2	0	$\cancel{1}$	$\cancel{2}$	$\cancel{1}$	0	$\cancel{2}$
2	1	$\cancel{1}$	$\cancel{2}$	$\cancel{1}$	0	$\cancel{0}$
2	2	$\cancel{1}$	$\cancel{2}$	$\cancel{1}$	0	$\cancel{0}$

p	q	r	$[p \supset [q \supset r]] \supset [[p \supset q] \supset [p \supset r]]$		
0	0	0	$\cancel{0}$	0	$\cancel{0}$
0	0	1	$\cancel{2}$	0	$\cancel{2}$
0	0	2	$\cancel{2}$	0	$\cancel{2}$
0	1	0	$\cancel{2}$	0	$\cancel{0}$
0	1	1	$\cancel{2}$	0	$\cancel{0}$
0	1	2	$\cancel{0}$	0	$\cancel{0}$
0	2	0	$\cancel{0}$	0	$\cancel{0}$
0	2	1	$\cancel{0}$	0	$\cancel{0}$
0	2	2	$\cancel{0}$	0	$\cancel{0}$
1	0	0	$\cancel{2}$	0	$\cancel{0}$
1	0	1	$\cancel{0}$	0	$\cancel{0}$
1	0	2	$\cancel{0}$	0	$\cancel{0}$
1	1	0	$\cancel{0}$	0	$\cancel{0}$
1	1	1	$\cancel{0}$	0	$\cancel{0}$
1	1	2	$\cancel{2}$	0	$\cancel{0}$
1	2	0	$\cancel{2}$	0	$\cancel{2}$
1	2	1	$\cancel{2}$	0	$\cancel{2}$
1	2	2	$\cancel{2}$	0	$\cancel{0}$
2	0	0	$\cancel{0}$	0	$\cancel{0}$
2	0	1	$\cancel{0}$	0	$\cancel{0}$
2	0	2	$\cancel{0}$	0	$\cancel{0}$
2	1	0	$\cancel{0}$	0	$\cancel{0}$
2	1	1	$\cancel{0}$	0	$\cancel{0}$
2	1	2	$\cancel{0}$	0	$\cancel{0}$
2	2	0	$\cancel{0}$	0	$\cancel{0}$
2	2	1	$\cancel{0}$	0	$\cancel{0}$
2	2	2	$\cancel{0}$	0	$\cancel{0}$

Axiom 1, however, is not valid in this sense, as shown by the table below.

p	q	$p \supset [q \supset p]$		
0	0	$\cancel{0}$	0	$\cancel{0}$
0	1	$\cancel{0}$	2	$\cancel{2}$
0	2	$\cancel{0}$	0	$\cancel{0}$
1	0	$\cancel{1}$	0	$\cancel{2}$
1	1	$\cancel{1}$	0	$\cancel{2}$
1	2	$\cancel{1}$	2	$\cancel{0}$
2	0	$\cancel{2}$	0	$\cancel{2}$
2	1	$\cancel{2}$	0	$\cancel{0}$
2	2	$\cancel{2}$	0	$\cancel{0}$

That *modus ponens* preserves validity (in the present sense) is evident from the truth table for the horseshoe. For suppose that A and $\ulcorner A \supset B \urcorner$ are valid and let Σ be a value assignment to the variables of $\ulcorner A \supset B \urcorner$. Clearly, both A and $\ulcorner A \supset B \urcorner$ have the value 0 under Σ. But, referring to the fundamental table for the horseshoe, we see that A and $\ulcorner A \supset B \urcorner$ can both have the value 0 only when B also has the value 0. Σ was an arbitrary value assignment, so it follows that B comes out 0 under every value assignment to its variables, that is, that B is valid. We conclude, therefore, that *modus ponens* preserves validity. As the proof that substitution preserves validity (in the sense just defined) parallels exactly the proof given in Section 28.0 that substitution preserves validity (in the familiar sense), details of this proof are left to the reader. Validity (defined with reference to the above secondary interpretation), therefore, is a property preserved by the two rules of inference, possessed by the second and third axioms but not possessed by axiom 1. It follows that axiom 1 is an independent axiom in system P.

To show that axiom 2 is independent, we use another three-valued secondary interpretation of system P, with 0 again as the sole designated truth value, in which the following are taken as the fundamental truth tables for the tilde and horseshoe.

A	$\sim A$
0	2
1	1
2	0

A	B	$A \supset B$
0	0	0
0	1	1
0	2	2
1	0	0
1	1	0
1	2	1
2	0	0
2	1	0
2	2	0

It is left to the reader to verify that under the interpretation of system P determined by the above tables axiom 1 and axiom 3 are valid, axiom 2 is not, and both *modus ponens* and substitution preserve validity. Such verification establishes the independence of axiom 2 in system P.

Similarly, it is left to the reader to prove that axiom 3 is independent by appealing to still another secondary three-valued interpretation, with 0 again as sole designated truth value, determined by the following fundamental truth tables for the tilde and horseshoe:

A	$\sim A$
0	2
1	2
2	2

A	B	$A \supset B$
0	0	0
0	1	1
0	2	2
1	0	0
1	1	0
1	2	2
2	0	0
2	1	0
2	2	0

Although we have used secondary three-valued interpretations to establish the independence of the three axioms of system P, the reader should not infer that all independence proofs make use of

secondary three-valued interpretations. There are many ways to show independence. For example, we could have shown that axiom 3 is independent in system P by appealing to the ordinary truth table for the horseshoe coupled with the following table for the tilde:

A	$\sim A$
t	t
f	f

With respect to these tables, all the axioms except axiom 3 are obviously valid (because we have not changed the table for the horseshoe), axiom 3 is not valid, and the rules of inference preserve validity. It follows, then, that axiom 3 is independent in system P.

That independence is only one factor which contributes to the elegance of a logistic system is strikingly illustrated by the system P_1, which has the following primitive basis:

Vocabulary: same as system P.
Formation rules: same as system P.
Axioms: all and only valid wffs are axioms.
Rules of inference: none.

Both system P and system P_1 have the same set of theorems, the valid wffs. Furthermore, every axiom of system P_1 is independent in system P_1. Nevertheless, system P_1 is exceedingly inelegant because it is wholly deficient in *systematicity*. Unlike the axioms and rules of system P, those of system P_1 do not organize or interrelate its theorems in an economical or interesting way. It would be a mistake to attribute this lack of systematicity, with its resulting inelegance, to the fact that system P_1 contains infinitely many axioms. There are logistic systems of sentential logic with infinitely many axioms which are just as economical, systematic, and elegant as system P. For example, there is the system P_2 with the following primitive basis:

Vocabulary: same as system P.
Formation rules: same as system P.
Axioms: any wff that has any of these three forms is an axiom:
Schema 1: $A \supset [B \supset A]$.

Schema 2: $[A \supset [B \supset C]] \supset [[A \supset B] \supset [A \supset C]]$.
Schema 3: $[\sim A \supset \sim B] \supset [B \supset A]$.
Rules of inference: *modus ponens.*

The schemata 1, 2, and 3 are known as *axiom schemata.* An axiom schema singles out infinitely many wffs as axioms. For example, schema 1 confers axiomhood on '$p \supset [q \supset p]$', '$r \supset [p \supset r]$', '$r \supset [[p \supset q] \supset r]$', and so on. System P_2, therefore, has infinitely many axioms. Moreover, it can be proved that system P and system P_2 have exactly the same theorems, the tautologies. Yet, with respect to economy, systematicity, and elegance, the two systems seem quite comparable.

28.2. Independence and Consistency

But, after all, elegance is only the icing on the systematic cake. If independence were relevant only to elegance, it would surely not merit the considerable attention paid it by logicians and mathematicians. But independence is germane to, among other things, problems of consistency. Of a host of consistency concepts, we have so far dealt with just one, consistency with respect to a property. Another fundamental consistency concept is absolute consistency. A logistic system is said to be *absolutely consistent* if at least one of its wffs is not a theorem. Clearly, the three systems we have just examined are all absolutely consistent. But suppose we formed a new logistic system by adding '$p \supset q$' as an axiom to the primitive basis of system P. This new system would not only be inconsistent with respect to validity, which is obvious, but would also be absolutely inconsistent, which is not quite so evident.

Independence bears a close relationship to absolute consistency. A logistic system L is absolutely consistent if and only if it contains at least one nonaxiom A which is independent in the system L^+ that results on adding A as an axiom to the primitive basis of L. The principal goal of the famous Hilbert program in mathematics was to prove the absolute consistency of a logistic system of ordinary arithmetic. To realize this goal, it would have sufficed to adduce a single wff A such that A was independent in the system that resulted on adding A as an axiom to the given logistic system of arithmetic. (Hilbert put rather severe restrictions on the methods of proof permissible in proving that a given wff was independent in a system.) Because of the intimate connection between independence and consistency, methods for establishing independence become valuable techniques for proving consistency, and conversely. It is this relationship that accounts for much of the emphasis placed on independence in logic and mathematics.

29. EXERCISES

(1) Prove that each axiom of system P_2 is valid.

(2) Prove that, if *modus tollens* were added to system P as a rule of inference, the resulting system would still be consistent with respect to validity, by showing that *modus tollens* preserves validity. *Modus tollens* is the rule that authorizes the inference from the premisses $\ulcorner A \supset B \urcorner$ and $\ulcorner \sim B \urcorner$ to the conclusion $\ulcorner \sim A \urcorner$.

(3) Prove that the rule of simultaneous substitution preserves validity. This rule authorizes the inference from a wff A to any wff B that results from A by the simultaneous substitution of wffs $B_1, B_2, \ldots, B_n$ for *distinct* variables $b_1, b_2, \ldots, b_n$ throughout A. [See Exercises (11) and (12) of Chapter 27.]

{4} What factors besides independence contribute to the elegance of a formal axiomatic or logistic system?

(5) Formulate a precise definition of the concept of an independent rule of inference.

{6} Rather than use the pseudo truth values 0, 1, 2 in our secondary interpretations, might we have not used instead truth, falsehood, and a third bonafide truth value on a par with truth and falsehood? That is, is there such a third truth value?

(7) Prove that the system which results from system P on adding '$p \supset q$' as an axiom is absolutely inconsistent.

(8) Prove that a logistic system L is absolutely consistent if and only if L contains at least one nonaxiom A which is independent in the system L^+ which results on adding A as an axiom to the primitive basis of L.