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THE AXIOMS AND ALGEBRA OF INTUITIVE PROBABILITY

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1. The Idea of Probability

The intuitive thesis in probability holds that both in its meaning and in the laws which it obeys, probability derives directly from the intuition, and is prior to objective experience;[1] it holds that it is for experience to be interpreted in terms of probability and not for probability to be interpreted in terms of experience; and it holds that all the so-called objective definitions of probability depend for their effective application to concrete cases upon their translation into the terms of intuitive probability. In its purest form it stems from the aphorism: *knowledge is possible, while certainty is not*; for it sees in the resolution of this antinomy—it sees in the varying "probability" in which the articles of knowledge are held—its very conceptual germ.

Clearly the vindication of a thesis such as this falls outside the domain of mathematician or physicist, and belongs to the philosopher: the question at issue is one of epistemology.[2] And our function here is to undertake neither its defense nor its refutation, but solely to attempt the axiomatic formulation of the idea of intuitive probability, and the derivation of the classical mathematical theory of probability from this. That such a task is necessary at the present time appears from the fact that all the axiomatic treatments of intuitive probability current in the literature take as their starting point a number (usually between 0 and 1) corresponding to the "degree of rational belief" or "credibility" of the eventuality in question. Now we hold that such a number is in no wise a self-evident concomitant with or expression of the primordial intuition of probability, but rather a mathematical construct derived from the latter under very special conditions and as the result of a fairly complicated process implicitly based on many of the very intuitive assumptions which we are endeavouring to axiomatize: There is, in short, what appears to us to be a

[1] "(Objective) experience" is used here practically as the equivalent of "laboratory experiment" in the narrow objective sense of the word, and excludes introspective or "subjective experience." Moreover, it is a question here of *rational* derivation from experience, and the *a priori* view of probability is quite consistent with the idea that probability as well as logic may be derived by race experience through the process of evolution. In spite of terminological similarities, there is no question here of Kantean Transcendentalism.

[2] The philosophical controversy here involved is very old and is still unsettled. Without attempting to give general references to the literature on this point we may cite the Introduction and First Chapter of J. M. Keynes' *A Treatise on Probability* (London, 1921), in which the thesis of intuitive probability is eloquently urged.

serious rational lacuna between the primal intuition of probability, and that branch of the theory of measure which passes conventionally under the name of probability. Moreover, this assumption commits one to too great precision, leading either to absurdities, or else to the undue restriction of the field of applicability of the idea of probability.[3]

The fundamental viewpoint of the present work is that the primal intuition of probability expresses itself in a (partial) ordering of eventualities: A certain individual at a certain moment considers the propositions a, b, h, k, the meanings of which he apprehends, and which he considers to be determinate (i.e., either true or false) without (in general) knowing whether they are true or false (for the sharpening of these ideas, see §2). Then the phrase

> *"a on the presumption that h is true is equally or less probable than b on the presumption that k is true"*

conveys a precise meaning to his intuition—and this, utterly disregarding whether he accepts what it says, rejects it, or is non-committal. This is, as we see it, a first essential in the thesis of intuitive probability, and contains the ultimate answer to the question of the meaning of the notion of probability. Let us write the quoted phrase in either of the symbolic forms

$$a/h \prec b/k, \qquad b/k \succ a/h$$

and call such an expression a *comparison in probability*.[4] It will be convenient to call such symbols as a/h or b/k *eventualities*, the propositions a and b, the *contingencies* in the eventualities, and the propositions h and k, the *presumptions* in the eventualities. And we may read $a/h \prec b/k$: a on h *infraprobable* b on k ($\succ$ being read *supraprobable*).

The position of acceptance, rejection, or doubt taken with regard to $a/h \prec b/k$ may vary from individual to individual and from moment to moment with the same individual, depending as it will upon his entire state of mind. Never-

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[3] We have in mind, on the one hand, events of such elaborate and individual nature as to defy numerical evaluation of probability, yet which can often be definitely compared as to their likelihood of occurrence; and on the other hand, instances such as the following: Let 0 denote an impossible event, let A and B consist of a certain normally distributed variate x being found to have exactly the value a and b respectively, b being much closer to the central value than a, so that for the probability densities: $\varphi(a) < \varphi(b)$. Then as far as the numerical probabilities go, $p(0) = p(A) = p(B) = 0$, but it is rational to place them in the order of likelihood: $0 < A < B$.

But fundamentally our object is not so much the treatment of such cases, as to secure an adequate conceptual framework for the idea of probability.

[4] The notation a/h is suggested by that of Keynes (l.c.); yet we are using it in quite a different sense and in a manner much closer to the current notation for the quotient of a ring by an ideal (the ideal $(\sim h)$ cf. §2). For to begin with, Keynes regards a/h as a number, or at least susceptible of many of the rules of ordinary arithmetic, whereas for us it is but an ordered pair of propositions, or, as we shall see later, a remainder class in a Boolean ring. And secondly, the *logical type* of proposition which we denote by a, h, etc., is far different from what Keynes appears to understand by such letters.

theless, if a certain individual at a certain moment subscribes to both of the following: $a/h \prec b/k$, $b/k \prec c/l$, he *must* at the same moment subscribe to $a/h \prec c/l$, and also he must subscribe to each of $\sim a/h \succ \sim b/k$, $\sim b/k \succ \sim c/l$. For otherwise he will be in a condition of mental incoherence of the same nature as logical inconsistency. We thus come to a second essential in the thesis of intuitive probability, namely, that its comparisons obey fixed laws which, like the laws of intuitive logic to which they are akin, are axiomatic, and owe their authority to man's awareness of his own rational processes.[5]

The circumstance that notwithstanding the "subjectivity" with which comparisons in probability may be made, they are none the less fixed both as regards their meaning and their laws, is the very fact which makes a rational mathematical theory of probability possible. *It is the unique function of such a theory to develop the rules for the derivation of comparisons in probability from other comparisons in probability previously given.*

When the intuition exerts itself upon a situation involving uncertain events, it is capable of performing two tasks: firstly, it may assert comparisons in probability; and secondly, it may state laws of consistency governing such comparisons. There is between these two operations of the intuition an essential cleft; and the failure to realize this has been a most frequent cause of confusion. From the laws of consistency a theory may be constructed (and this is our present object) with the aid of which, comparisons in probability (and indeed, values of numerical probability) can be deduced from others which are hypothesized: but beyond this, the laws of consistency do not allow one to go. This naturally raises the question as to whether there are any general principles which determine (non-trivial) comparisons in probability *ab ovo*,—without taking any such comparisons as starting point.

One might consider a concrete case: Two wayfarers are on a road which, as they know, leads to their destination. After a while, to their surprize, the road divides in two, and as they are ignorant of their orientation, they do not know which way to turn. The first wayfarer asserts that the right-hand turn is equally or less probable as the way to their destination than the left; but the second disagrees with him. *Is there any general principle which will enable either one rationally to compel the other to accept his position?* We mean a principle which does not itself implicitly assume a comparison in probability (such as a statement of "irrelevancy") to have been granted and which can be applied to the present case without begging the question; a principle which, finally,

[5] There is a body of opinion at the present time which regards the laws of probability, and even the laws of logic, as subject to objective experimental proof or disproof, or at least as requiring possible alteration in the light of the evolution of physical theory. This school strengthens its case by the claim that certain facts in the quantum theory are at variance with the "laws of probability" (or indeed, of logic). While we shall treat this question in detail elsewhere, we may say here that we have examined with all logical minuteness every alleged case which we could discover, and have found in every case that the physical circumstances of the application of the laws of probability, etc., were the things that were altered, but never the laws themselves.

genuinely pertains to the general abstract theory of probability, and not to some specialized branch of experimental science.

While an authoritative answer to this question would go beyond the scope of the present paper, involving as it would a critique of the existing state of the foundations of probability *in toto*, we may nevertheless venture an expression of our personal view: We have never seen formulated nor been able ourselves to formulate such a principle. And we have come to the position that no such principle is in the nature of things susceptible of formulation, that we are indeed at the very threshold of the problem of the conditions of the possibility of knowledge, and that the difficulties here met with are those which must in the nature of things always be encountered when an attempt is made to give a mathematical or physical solution to a metaphysical problem.

2. Elementary Notions of Logic

In the theory of probability one is constantly occupied with what may be called determinate concrete propositions, that is to say, specific assertions of a physical or biological character, the truth or falsity of which is regarded as verifiable on the performance of an appropriate experiment. Such propositions and such only shall in the present paper be symbolized by small Latin letters or by finite clusters of such letters united with the aid of the logical constants $(\sim \cdot \vee)$.[6] For example, the proposition "it will rain (at stated time and place)" would be denoted by a letter such as a, "it will rain or snow (at stated time and place," by c or by $a \vee b$, etc. Now all propositions of this sort are conceived as *contemplated* propositions (cf. Frege and Russell) and not as mere symbols for 0 or 1:[7] we may talk about a and $a \vee b$ without automatically asserting them. Altogether different shall be our conventions regarding the use of the logical constants $(= \;\subset)$,[8] which shall be regarded as having predicative power. Thus "$c = (a \vee b)$" is an *asserted* proposition and shall never in the present text be symbolized by a single letter, nor shall assertions like it be united by means of logical constants: no formula shall contain more than one of the signs $(= \;\subset)$.

To sum up: We are dealing exclusively with intuitive logic, are employing the logical constants solely as abbreviations for notions which are regarded as intuitively evident and as obeying intuitively evident laws, and we are symbolizing by letters only propositions of the lowest "logical type."

[6] As usual, $\sim$ denotes negation, $\cdot$ (or mere juxtaposition) denotes conjunction or logical product, and $\vee$ denotes disjunction or logical sum.

[7] The identically false proposition is denoted by 0 and is regarded as coinciding with every $a \sim a$, the identically true proposition is denoted by 1 and is regarded as coinciding with every $a \vee \sim a$. Thus all the Boolean algebras here considered are regarded as having identical units.

[8] The symbol for identity $=$ and material implication $\subset$ ($a \subset b$ meaning "a implies b") are used here rather than $\equiv$ and $\supset$ in order to bring out the parallel with the mathematical theories of sets and algebras.

Parentheses shall be used in the self-evident manner of classical mathematics; but they are economized by the conventions concerning the binding strength of symbols, according to which: (i) $\sim$ is linked most strongly to the succeeding nearest letter, so that $\sim ab$ or $\sim a \cdot b$ each mean $(\sim a) \cdot b$, $\sim a \vee b$ means $(\sim a) \vee b$, and $\sim a = b$ and $\sim a \subset b$ mean $(\sim a) = b$ and $(\sim a) \subset b$ respectively; (ii) $\cdot$ (or the absence of symbol) binds the letters on either side of it more strongly than $\vee$ so that $a \cdot b \vee c$ or $ab \vee c$ means $(ab) \vee c$. In this connection it may be remarked that in accordance with our fundamental convention, such expressions as $a \vee (b \subset c)$ or $\sim(a \subset b)$ would have no use.

The knowledge of the laws which the symbols ($\sim \cdot \vee = \subset$) obey, or formal statement of the laws of intuitive logic, is here regarded as aquired and the reader is assumed to be familiar with the elementary facts of Boolean algebra.[9] We shall make explicit mention only of the following definitions and theorems, on account of the prominent rôle which they are to play in our theory of probability. The reader may easily furnish the proofs by analogy with the theorems on homomorphism of groups or rings, and the quotient group or remainder ring in classical algebra. Detailed proofs are furnished in the paper of M. H. Stone cited above.

Definition. *By a homomorphism* $\mathfrak{A} \to \mathfrak{A}'$ *from a Boolean ring* $\mathfrak{A}$ *to a Boolean ring* $\mathfrak{A}'$ *is meant a functional correspondence* $x' = f(x)$ *whose domain is* $\mathfrak{A}$ *and whose range is* $\mathfrak{A}'$, *such that, for all* $x, y \in \mathfrak{A}$,

$$f(\sim x) = \sim f(x), \qquad f(xy) = f(x)f(y), \qquad f(x \vee y) = f(x) \vee f(y).$$

In the case where $f(x) = f(y)$ *implies* $x = y$, *the correspondence* $\mathfrak{A} \to \mathfrak{A}'$ *can be inverted (the result* $\mathfrak{A}' \to \mathfrak{A}$ *being shown to be a homomorphism) and the correspondence is called an isomorphism* $\mathfrak{A} \leftrightarrow \mathfrak{A}'$.

Definition. *A non-empty subset* $\mathfrak{u}$ *of elements of a Boolean ring* $\mathfrak{A}$ *is said to be an ideal in* $\mathfrak{A}$ *when it satisfies the following conditions:*

1. *If* $a \in \mathfrak{u}$ *and* $b \in \mathfrak{u}$, *then* $a \vee b \in \mathfrak{u}$
2. *If* $a \in \mathfrak{u}$ *and* $b \in \mathfrak{A}$ *then* $ab \in \mathfrak{u}$

Theorem. *If* $\mathfrak{A} \to \mathfrak{A}'$, *then the aggregate* $\mathfrak{u}$ *of all elements* x *of* $\mathfrak{A}$ *such that* $x' = f(x) = 0'$ *(the zero in* $\mathfrak{A}'$*) constitutes an ideal in* $\mathfrak{A}$. *And* $f(a) = f(b)$ *if and only if simultaneously* $a \sim b \in \mathfrak{u}$, $\sim ab \in \mathfrak{u}$.

We shall now recall the definition of the quotient ideal $\mathfrak{A}/\mathfrak{u}$ of $\mathfrak{A}$ with respect to any one of its ideals $\mathfrak{u}$.

First, all the elements of $\mathfrak{A}$ are distributed into a system of mutually exclusive and exhaustive *remainder classes* of $\mathfrak{u}$ defined by the convention that any two elements a and b shall belong to the same remainder class of $\mathfrak{u}$ if and only if

[9] For general bibliographical references, see E. V. Huntington, Trans. Amer. Math. Soc., Vol. 35 (1933), pp. 274-304. For algebraic developments of Boolean rings, particularly in the light of their relation with the rings and ideals of classical abstract algebra, see M. H. Stone, *The Theory of Representations for Boolean Algebras*, Trans. Amer. Math. Soc., Vol. 40 (1936), pp. 37–111.

simultaneously $a \sim b \in \mathfrak{u}$, $\sim ab \in \mathfrak{u}$. Let the remainder class determined by the fact that it contains the element a be denoted by $a/\mathfrak{u}$.

Next, if $(\sim \cdot \vee)$ applied to classes be used to denote the classes resulting from the application of the respective operator to all elements or pairs of elements of the classes in question, and if $(= \subset)$ be used to denote class identity or inclusion, it is shown that

$$\sim a/\mathfrak{u} = \sim(a/\mathfrak{u}), \qquad ab/\mathfrak{u} = (a/\mathfrak{u})(b/\mathfrak{u}), \qquad a \vee b/\mathfrak{u} = (a/\mathfrak{u}) \vee (a/\mathfrak{u}).$$

Finally, $\mathfrak{A}/\mathfrak{u}$ is defined as the class of all remainder classes in $\mathfrak{A}$ of $\mathfrak{u}$, the $(\sim \cdot \vee)$ being defined as before, $\mathfrak{A}/\mathfrak{u}$ is seen to be a Boolean ring (with units $0/\mathfrak{u}$ and $1/\mathfrak{u}$) and the correspondence $x \to x/\mathfrak{u}$ $(x \in \mathfrak{A})$ is a homomorphism: $\mathfrak{A} \to \mathfrak{A}/\mathfrak{u}$. And the following theorem is an immediate consequence.

THEOREM. *If $\mathfrak{A} \to \mathfrak{A}'$, and if $\mathfrak{u}$ is the class of all elements of $\mathfrak{A}$ having $0'$ as their image in $\mathfrak{A}'$, then $\mathfrak{A}' \leftrightarrow \mathfrak{A}/\mathfrak{u}$.*

Turning now to the significance of these algebraic notions for the present theory of probability, suppose that an aggregate of comparisons in probability is being envisaged, involving a set of concrete propositions a, b, h, k, etc. After the possible adjunction to this set of all of the finite combinations of its propositions in terms of $(\sim \cdot \vee)$—in case such combinations are not already included in the set—it becomes closed, and forms a Boolean ring $\mathfrak{A}$, containing 0, 1.

In $\mathfrak{A}$ the element $\sim h$ (assuming $\sim h \neq 1$ i.e., $h \neq 0$) generates a *principal ideal* (i.e., ideal generated by an unique element $\neq 1$), which shall be denoted by $(\sim h)$, and which is composed of the class of all elements $\sim hx$, $x \in \mathfrak{A}$. If, now, a and b are two elements of $\mathfrak{A}$ for which simultaneously $a \sim b \in (\sim h)$, $\sim ab \in (\sim h)$, then *a and b are logically equivalent on the assumption that h is true* in the following intuitively self-explanatory sense: since $a \sim b = \sim hc (c \in \mathfrak{A})$, to assume h true is to assume $a \sim b = (a \sim b)h = h \sim hc = 0$, i.e., to rule out that a is true without b being true: $a \subset b$. And similarly, if $\sim ab = \sim hd$ $(d \in \mathfrak{A})$, then the assumption that h is true automatically makes $\sim ab = 0$, i.e., makes $b \subset a$.

It is convenient in this work to introduce the symbol a/h as simply an alternative notation for $a/(\sim h)$: in the light of the above, we become aware of the fact that the intuitive act of passing from the consideration of a on no presumption concerning any other of the propositions in $\mathfrak{A}$ (beyond 0, 1) to the consideration of a on the presumption that h is true, has its exact algebraic counterpart in the passage from a (which may be replaced by $a/1$) to $a/h = a/(\sim h)$. *Henceforth it is in this precise algebraic sense of a remainder class that the eventuality symbol a/h shall be employed*, its intuitive counterpart serving solely as a guide to the formulation of the axioms governing the $\prec$ sign, as well as for the interpretation and application of the theorems developed in the formal mathematical structure which we are about to erect.

In closing this section we note, firstly, that the symbol $a/0 = a/(1)$ is useless, since its intuitive content is the assertion of 0: *it shall be implicitly excluded throughout*; and, secondly, that the symbol $\prec$ is regarded as having predicative force, and hence all symbolic constructs into which it enters are asserted, not contemplated, propositions, and are never denoted by letters nor operated upon with logical constants.[10] Thus it, like $(= \ \subset)$, has a weaker binding force than $(\sim \cdot \vee)$, so that the parentheses in expressions such as $[(\ )/(\ )] \prec [(\ )/(\ )]$ are omitted. The binding strength of $(/)$ is greater than $(=, \subset, \prec)$ and less than $(\sim \cdot \vee)$.

Finally, it might be interesting to consider eventualities $a/\mathfrak{u}$ in which $\mathfrak{u}$ is not a principal ideal $(\mathfrak{u} \neq (\sim h))$. But a careful examination of this matter has shown us that such an extension is not a useful one at the present stage of the theory.

3. The Axioms of Probability

From the standpoint of intuition, the following axioms are the intuitively evident laws of consistency governing all comparisons in probability; in some cases, to be sure, the full understanding of their meaning may offer some difficulty: the only claim is that, their meaning once being apprehended, their truth will be granted.[11] From the standpoint of pure mathematics, they are the formal postulates governing the $\prec$ sign, and are accepted by convention. The axioms form a sufficient basis for what we envision as the legitimate rôle of a theory of intuitive probability. We offer no discussion of consistency, nor have we striven systematically to secure independence (Axiom S has indeed been stated for all integral n, whereas if assumed for prime n the general case will follow). In short, the situation is altogether comparable with the case of the laws of intuitive logic.

In each of the following axioms a definite Boolean ring $\mathfrak{A}$ (with units 0, 1) is presupposed in which the remainder classes are formed; and wherever in later discussion a theorem is stated or proved, a definite $\mathfrak{A}$ is again presupposed. But it has seemed unnecessary in general to make explicit mention of this fact in most of the following statements and discussion.

The Axioms

V. Axiom of Verified Contingency.
$a/h \prec k/k$.

[10] This circumstance shows itself throughout our whole work by the fact that we are never allowed to talk about the probability of a comparison in probability. This may seem to get us unto conflict with Bayes' Rule, which is often regarded as concerning itself with "the probability of a probability." A mere change in wording avoids this difficulty for us, and we regard Bayes' Rule to concern itself only with the probabilities of different values of a physical parameter θ, which, itself, determines the probability (in particular, by the equation $p = \theta$).

[11] A most useful auxiliary in the intuitive comprehension and interpretation of the axioms as well as later theorems and proofs is the logical diagram, in which plane regions represent the propositions, and $(\sim \cdot \vee = \subset)$ are given their set-theoretic rendering. One may go further and let $a/h \prec b/k$ be represented by having the ratio of the area of ah to that of $h \leqq$ that of bk to k.

I. AXIOM OF IMPLICATION.
If $a/h \succ k/k$, *then* $h \subset a$.

R. AXIOM OF REFLEXIVITY.
$a/h \prec a/h$.

T. AXIOM OF TRANSITIVITY.
If $a/h \prec b/k$ *and* $b/k \prec c/l$, *then* $a/h \prec c/l$.

A. AXIOM OF ANTISYMMETRY.
If $a/h \prec b/k$, *then* $\sim a/h \succ \sim b/k$.

C. AXIOMS OF COMPOSITION.
Let $a_1b_1h_1 \neq 0$ *and* $a_2b_2h_2 \neq 0$.
C_1. *If* $a_1/h_1 \prec a_2/h_2$ *and* $b_1/a_1h_1 \prec b_2/a_2h_2$, *then* $a_1b_1/h_1 \prec a_2b_2/h_2$.
C_2. *If* $a_1/h_1 \prec b_2/a_2h_2$ *and* $b_1/a_1h_1 \prec a_2/h_2$, *then* $a_1b_1/h_1 \prec a_2b_2/h_2$.

D. AXIOMS OF DECOMPOSITION (QUASI-CONVERSES OF C).
Let $a_1b_1h_1 \neq 0$, $a_2b_2h_2 \neq 0$, *and* $a_1b_1/h_1 \prec a_2b_2/h_2$. *Then if either of the eventualities*
(i) a_1/h_1, b_1/a_1h_1, *has the supraprobable relation* ($\succ$) *with either of*
(ii) a_2h_2, b_2/a_2h_2, *it will follow that the remaining eventuality of* (i) *will have the infraprobable relation* ($\prec$) *with the remaining one of* (ii).

P. AXIOM OF ALTERNATIVE PRESUMPTION.
If $a/bh \prec r/s$ *and* $a/\sim bh \prec r/s$, *then* $a/h \prec r/s$

S. AXIOM OF SUBDIVISION.
For any integer n *let the propositions* $a_1, \cdots, a_n, b_1, \cdots, b_n$ *be such that* $a_ia_j = b_ib_j = 0$ $(i \neq j)$ $i, j = 1, \cdots, n$; $a = a_1 \vee \cdots \vee a_n \neq 0$; $b = b_1 \vee \cdots \vee b_n \neq 0$; $a_1/a \prec \cdots \prec a_n/a$; $b_1/b \prec \cdots \prec b_n/b$; *then* $a_1/a \prec b_n/b$.

On fixing the attention upon a given Boolean ring $\mathfrak{A}$ of concrete propositions, in which the $\prec$ relation is supposed to have been introduced, it is seen that the formulation focuses attention upon the totality $\mathfrak{C}$ of remainder classes with respect to all the principal ideals of $\mathfrak{A}$ and that Axioms R and T show that $\prec$ defines a partial ordering of the elements of $\mathfrak{C}$.[12] We shall accordingly appropriate the conventional definitions and theorems of the theory of partially ordered sets.[13] In particular, we shall write in either of the two forms $a/h \prec$

[12] While it is shown at once that if $h \subset k$ (material implication) then $(h) \subset (k)$ (set inclusion), and conversely, we may state at once that there is no such simple relation between the $\prec$ and $\subset$ when they occur between remainder classes. Examples are most easily given after the subject of numerical probability has been developed.

[13] For a full account of this theory, see H. M. MacNeille, *Partially Ordered Sets*, Trans. Amer. Math. Soc., Vol. 42 (1937), pp. 416–460.

In most treatments of partially ordered sets, the greater part of the developments follow from the "Lattice Assumption," which in the present connection would take the following form: If $a/h \in \mathfrak{C}$ and $b/k \in \mathfrak{C}$ then $\mathfrak{C}$ contains c/l having the properties

(i) $c/l \prec a/h$, $c/l \prec b/k$ and

b/k, $b/k > a/h$, the abbreviation for the combined assertion of $a/h \prec b/k$ and denial of $a/h \succ b/k$, and we shall use the symbol $\approx$ for *equiprobability*: $a/h \approx b/k$ as the abbreviation for the joint assertion of the two following: $a/h \prec b/k$, $a/h \succ b/k$. (Contrast $\approx$ with $=$: $a/h = b/k$ signifying identity of remainder classes).

A process constantly to be employed in all later work is the application of a factor in the presumption to the contingency; for it is evident in view of the conventions of §2 that, for example $a/hk = ah/hk = ahk/hk$.

It will be remarked that Axiom I is in sharp contrast with the familiar fact that an eventuality which is not regarded as certain (as a/h when $h \not\subset a$) may nevertheless have the same *numerical* probability (unity) as one held as certain (as k/k). Our contention is that the truth of Axiom I resides in the very intuitions of comparison in probability and certainty, of which the numerical probability is but an unfaithful representation.

A final remark: If the contingency a is a determinate arithmetical statement, such as "the 100$^{\text{th}}$ decimal in 2^{π} is 2" our theory asserts that the eventuality $a/1$ either $\approx 0/1$ or $1/1$ (cf. Theorem 1 in §4). This is because the truth or falsity of a is determined by the Boolean ring $\mathfrak{A}$ within which all the propositions here envisaged find themselves: in this ring a may be computed by the formal laws of the algebra and found either to $= 0$ or 1. We hold that this is as it should be and that if the ideas of intuitive probability are to be applied non-trivially to arithmetic, the contingency must be stated in a logically undetermined form which reveals something of the "random process" by which the definition of the integer was settled upon.

4. Theorems on Comparison

THEOREM 1. *For all $a \neq 0$, $a/a \approx 1/1$ and $0/a \approx 0/1$ are true.*
If $ah \neq 0$ and $ah \neq h$, then $0/1 < a/h < 1/1$.

It follows from V that $a/a \prec 1/1$ and $1/1 \prec a/a$, hence $a/a \approx 1/1$. These with A yield $\sim a/a \succ \sim 1/1$ and $\sim 1/1 \succ \sim a/a$; but $\sim a/a = a \sim a/a = 0/a$ and $\sim 1/1 = 0/1$; whence the relation $0/a \approx 0/1$.

From V it follows that $a/h \prec 1/1$; if $a/h \succ 1/1$, I would lead to $ah = h$, contrary to hypothesis; hence $a/h < 1/1$. Now if $\sim ah = h$ it would follow that $0 = a(\sim ah) = ah$, contrary to hypothesis; hence we may replace a by $\sim a$ in the previous discussion and so obtain $\sim a/h \prec 1/1$ together with the denial of $\sim a/h \succ 1/1$; whereupon A shows that $a/h \succ 0/1$ is true and $a/h \prec 0/1$ false; hence $0/1 < a/h$. Thus our theorem is proved.

If in an assertion involving $\prec$ signs, *every* $\prec$ be replaced by $\approx$ a new statement is obtained which shall be called *the sharpening* of the original. If on the other hand *at least one* $\prec$ is replaced by $<$ the new statement shall be called *a strength-*

(ii) if $d/m \in \mathfrak{C}$ such that $d/m \prec a/h$ and $d/m \prec b/k$ then $d/m \prec c/l$.
The remaining Lattice Assumption is the ($\prec \to \succ$) dual of the above. It is perhaps not unworthy of note that these ideas play no rôle in the present theory of partially ordered sets.

ening of the original statement. If a deductive statement (axiom, theorem, corollary, etc.) has a $\prec$ statement for hypothesis and a $\prec$ statement for conclusion, the new deductive statement obtained by replacing both hypothesis and conclusion by their sharpening shall be called the sharpening of the original deductive statement, while the deductive statement obtained by replacing both hypothesis and the conclusion by one of their respective strengthenings shall be called a strengthening of the original deductive statement.

One of our tasks is the establishment of the truth of the sharpening and of every strengthening of every one of the axioms which involves the $\prec$ in both hypothesis and conclusion, to wit, of all except V, I and R. This is conveniently accomplished in two stages announced in Theorems 2 and 11.

THEOREM 2. *The sharpening and every strengthening of Axioms* T, A, C, D *are valid.*

The proofs for T and A are obvious and shall be omitted.

Turning to C_1, let the new hypothesis be $a_1b_1h_1 \neq 0$ $a_2b_2h_2 \neq 0$, and $a_1/h_1 \approx a_2/h_2$, $b_1/a_1h_1 \approx b_2/a_2h_2$. We are to prove $a_1b_1/h_1 \approx a_2b_2/h_2$. Now the hypothesis leads to the original hypothesis of C_1, and also to this latter in which the subscripts 1 and 2 have been interchanged; thus C_1 leads to the two conclusions $a_1b_1/h_1 \prec a_2b_2/h_2$, $a_1b_1/h_1 \succ a_2b_2/h_2$, which is equivalent to the conclusion sought.

As for the strengthening of C_1, let the new hypothesis be $a_1b_1h_1 \neq 0$, $a_2b_2h_2 \neq 0$, and $a_1/h_1 < a_2/h_2$, $b_1/a_1h_1 \prec b_2/a_2h_2$; we are to prove $b_1/a_1h_1 < a_2/h_2$. In view of C_1 all that is required is to show the falsity of $a_1b_1/h_1 \succ a_2b_2/h_2$. Let this relation be assumed true and apply D (with subscripts interchanged) to the two relations $b_1/a_1h_1 \prec b_2/a_2h_2$, $a_1b_1/h_1 \succ a_2b_2/h_2$, thus deriving the following contradiction of hypothesis: $a_2/h_2 \prec a_1/h_1$.

The remaining possibility (not reducible to the above) is the introduction into the hypothesis of C_1 of $b_1/a_1h_1 < b_2/a_2h_2$. The treatment of this case being quite similar to the other is left to the reader.

The discussion of C_2 is omitted, being altogether similar to the case of C_1. And as for D, we shall treat only the following case, leaving the rest to the reader:

Let $a_1b_1h_1 \neq 0$, $a_2b_2h_2 \neq 0$, and $a_1b_1/h_1 < a_2b_2/h_2$, $a_1/h_1 \succ b_2/a_2h_2$. We are to prove that $b_1/a_1h_1 < a_2/h_2$. It is required only to show the falsity of $b_1/a_1h_1 \succ a_2/h_2$. For this purpose we have but to apply C_2 (subscripts interchanged) to the relations $a_2/h_2 \prec b_1/a_1h_1$, $b_2/a_2h_2 \prec a_1/h_1$, whereupon there will follow the contradiction of hypothesis: $a_2b_2/h_2 \prec a_1b_1/h_1$.

In the succeeding theorems, when it is desired to indicate that sharpening and strengthenings are valid, a statement to this effect is appended to the statement of the theorem. But in those cases where the proof of the sharpened or strengthened theorem is furnished by simply paraphrasing that of the original theorem, with the replacement of axioms, theorems, etc., employed by their sharpening or strengthenings, the proof is omitted. And wherever in the future reference is made to an axiom or theorem, either the original or its sharpening or strengthening shall be meant: the reader will have no difficulty in seeing which one is needed.

THEOREM 3. *If* $ah \subset bh$ *then* $a/h \prec b/h$. *If in addition,* $bh \not\subset ah$, *then* $a/h < b/h$.

From R and V it follows that $b/h \prec b/h$, $a/bh \prec b/bh$. It is now possible to apply C_1 in which we put $h_1 = h_2 = h$, $a_1 = a_2 = b$, $b_1 = a$, $b_2 = b$, and so to derive the following relation, which furnishes the first conclusion desired, $ab/h \prec bb/h$. Here $a_1b_1h_1 = abh = ah$ and $a_2b_2h_2 = bh$, so that if $a_2b_2h_2 = 0$ we should have $ah = abh = 0$, in which case Theorem 1 would furnish the desired conclusion.

If, further, it is assumed that $bh \not\subset ah$, whereas $a/h \nless b/h$, i.e., $a/h \succ b/h$, we could apply D to the following relations $b/h \prec a/h$, $b/h \succ b/h$, where we put $h_1 = h_2 = h$, $a_1 = a_2 = b$, $b_1 = b$, $b_2 = a$, and so either $a_1b_1h_1(=bh)$ and $a_2b_2h_2(=abh=ah)$ are not both $\neq 0$ or else the relation $b/bh \prec a/bh$ is obtained. But we have $b/bh = bh/bh \approx 1/1$, hence this would give by Theorem 1 that $bh \subset a$, so that $bh \subset ah$, in contradiction with the hypothesis. If $ah = 0$ then in view of the relation we are trying to disprove, we should have $b/h \prec a/h = 0/h \approx 0/1$ and thus by Theorem 1, $bh = 0 \subset ah$. Finally, if $bh = 0$, the same contradiction with the hypothesis $bh \not\subset ah$ would obtain. Thus all parts of the theorem are proved.

The second part of this theorem may be regarded as a kind of strengthening of the first part. The sharpening in the sense that $ah = bh$ implies $ah/h \approx bh/h$ is a trivial consequence of R. Similar remarks apply to the following theorem.

THEOREM 4. *If* $a \subset k \subset h$, *then* $a/h \prec a/k$. *If in addition* $h \not\subset k$ *and* $a \neq 0$, *then* $a/h < a/k$.

From V and R it follows that $k/h \prec k/k$, $a/hk \prec a/hk$. It is possible to apply C_1 after setting $h_1 = h$, $h_2 = a_1 = a_2 = k$, $b_1 = b_2 = a$, from which the desired conclusion follows. Here $a_1b_1h_1 = ahk = a$, $a_2b_2h_2 = ak = a$, and if $a = 0$ the conclusion would follow from Theorem 1.

If, furthermore, $h \not\subset k$ and $a \neq 0$, we could apply D to the supposed relations $a/k \prec a/h$, $a/k \succ a/k$, where we set $h_1 = k$, $h_2 = h$, $a_1 = a_2 = k$, $b_1 = b_2 = a$, from which it would follow,—providing $a_1b_1h_1(=ak=a)$ and $a_2b_2h_2(=akh=a)$ are neither $= 0$, a case excluded by hypothesis,—that $1/1 \approx k/k \prec k/h$; thus $h \subset k$, and we have a contradiction.

THEOREM 5. *If* (i) $a_1b_1h_1 = a_2b_2h_2 = 0$, *and if* (ii) $a_1/h_1 \prec a_2/h_2$, (iii) $b_1/h_1 \prec b_2/h_2$, *then* $a_1 \vee b_1/h_1 \prec a_2 \vee b_2/h_2$.

The sharpening and all strengthenings are valid.

On applying A to (ii) one obtains

$$\sim a_1/h_1 \succ \sim a_2/h_2 . \tag{1}$$

From (i) it follows that $\sim a_1 \vee \sim b_1 \vee \sim h_1 = 1$, whence, multiplying through by b_1h_1: $\sim a_1b_1h_1 = b_1h_1$; thus we have $\sim a_1b_1/h_1 = \sim a_1b_1h_1/h_1 = b_1h_1/h_1 = b_1/h_1$, and similarly for b_2/h_2 . Hence (iii) may be written as

$$\sim a_1b_1/h_1 \prec \sim a_2b_2/h_2 . \tag{2}$$

It may be assumed that $\sim a_1b_1h_1 \neq 0$ and $\sim a_2b_2h_2 \neq 0$. For if $\sim a_1b_1h_1 = 0$, (i) would give us $b_1h_1 = \sim a_1b_1h_1 \vee a_1b_1h = 0$, and the conclusion would follow directly from the following relations made evident by (ii) and Theorem 3: $a_1 \vee b_1/h_1 = a_1h_1 \vee b_1h_1/h_1 = a_1h_1/h_1 = a_1/h_1$; $a_1/h_1 \prec a_2/h_2 \prec a_2 \vee b_2/h_2$. While if $\sim a_2b_2h_2 = 0$, we could derive $b_2h_2 = 0$ as before, whereupon (iii) would yield with the aid of Theorem 1 the already excluded case $\sim a_1b_1h_1 = 0$. We are, therefore, entitled to apply D to (1) and (2) ($\sim a_1$, $\sim a_2$ playing the rôle of "a_1, a_2") and so obtain

$$b_1/\sim a_1h_1 \prec b_2/\sim a_2h_2 . \tag{3}$$

This with A gives the relation $\sim b_1/\sim a_1h_1 \succ \sim b_2/\sim a_2h_2$. If $\sim a_1\sim b_1h_1 \neq 0$ and $\sim a_2\sim b_2h_2 \neq 0$, C_1 may be applied to this relation together with (1), ($\sim a_2$, $\sim b_2$, h_2, $\sim a_1$, $\sim b_1$, h_1 playing the rôle of "a_1, b_1, h_1, a_2, b_2, h_2"), whereupon $\sim a_1\sim b_1/h_1 \succ \sim a_2\sim b_2/h_2$, which, on application of A, yields the conclusion of our theorem.

If on the other hand $\sim a_2\sim b_2h_2 = 0$, it would follow that $a_2 \vee b_2 \vee \sim h_2 = 1$, whence $h_2 = h_2(a_2 \vee b_2 \vee \sim h_2) = (a_2 \vee b_2)h_2$, and thus $a_2 \vee b_2/h_2 = (a_2 \vee b_2) h_2/h_2 = h_2/h_2$ and our conclusion would be a consequence of V. Finally, if $\sim a_1\sim b_1h_1 = 0$ we should have (as $\sim a_1b_1h_1 \neq 0$ is assumed) $\sim a_1h_1 = \sim a_1\sim b_1h_1 \vee \sim a_1b_1h_1 = \sim a_1b_1h_1$, and thus by Theorem 1, $b_1/\sim a_1h_1 = \sim a_1b_1h_1/a_1h_1 \approx 1/1$, which in conjuction with (3) gives $b_2/\sim a_2h_2 \succ 1/1$ whence, again by Theorem 1, $\sim a_2b_2h_2 = \sim a_2h_2$ and multiplication of this by $\sim b_2$ gives $\sim a_2\sim b_2h = 0$, the case previously disposed of. Thus our proof is complete.

THEOREM 6. *If* $a_{1i}a_{1j}h_1 = a_{2i}a_{2j}h_2 = 0$ *for all* $i \neq j$, *and if* $a_{1i}/h_1 \prec a_{2i}/h_2$ *for all* $i = 1, 2, \cdots, n$, *then* $a_{11} \vee a_{12} \vee \cdots \vee a_{1n}/h_1 \prec a_{21} \vee a_{22} \vee \cdots \vee a_{2n}/h_2$.

The sharpening and all strengthenings are valid.

This follows from Theorem 5 by mathematical induction.

THEOREM 7. *If* (i) $b_1 \subset a_1$, $b_2 \subset a_2$ *and if* (ii) $a_1/h_1 \prec a_2/h_2$, (iii) $b_1/h_1 \succ b_2/h_2$ *then* $a_1 \sim b_1/h_1 \prec a_2 \sim b_2/h_2$.

The sharpening and all strengthenings are valid.

In virtue of (i), $a_1b_1 = b_1$ and $a_2b_2 = b_2$, so that (iii) becomes $a_1b_1/h_1 \succ a_2b_2/h_2$. Suppose firstly that $a_1b_1h_1 \neq 0$, $a_2b_2h_2 \neq 0$. Then the application of D (with subscripts interchanged) to this relation together with (ii) yields $b_1/a_1h_1 \succ b_2/a_2h_2$, which on application of A furnishes the relation

$$\sim b_1/a_1h_1 \prec \sim b_2/a_2h_2 . \tag{4}$$

It is now permitted to assume that $a_1 \sim b_1h_1 \neq 0$, $a_2 \sim b_2h_2 \neq 0$. For if $a_1 \sim b_1h_1 = 0$, then $a_1 \sim b_1/h_1 = a_1 \sim b_1h_1/h_1 = 0/h_1 \approx 0/1$, and the conclusion of our theorem would be a consequence of Theorem 1. And if $a_2 \sim b_2h_2 = 0$, relation (4) would give $\sim b_1/a_1h_1 \prec a_2 \sim b_2h_2/h_2 = 0/h_2 \approx 0/1$, whence Theorem 1 leads to the case $a_1 \sim b_1h_1 = 0$ just disposed of. We may, then, apply C_1 to (4) in conjunction with (ii) (with $\sim b_1$, $\sim b_2$ replacing "b_1, b_2"), and so obtain the conclusion of our theorem.

Turning lastly to the excluded cases, suppose that $a_2b_2h_2 = 0$. This gives in view of (i) $b_2h_2 = 0$, so that $\sim b_2 \vee \sim h_2 = 1$, $\sim b_2h_2 = h_2$; thus $a_2 \sim b_2/h_2 = a_2(\sim b_2h_2)/h_2 = a_2h_2/h_2 = a_2/h_2$. Now $a_1 \sim b_1 \subset a_1$, and we have by Theorem 3 $a_1 \sim b_1/h_1 \prec a_1/h_1$. The conclusion to our theorem then follows from (ii). Finally, if $a_1b_1h_1 = 0$, (i) gives that $b_1h_1 = 0$, and thus $b_1/h_1 = b_1h_1/h_1 = 0/h_1 \approx 0/1$; but from (iii) it follows that $b_2/h_2 \prec 0/1$ whence $b_2h_2 = 0$,—the case just disposed of.

THEOREM 8. *Let $a \subset k$, $b \subset k$, $k \subset h$. Then either of the following relations is a necessary and sufficient condition for the truth of the other*:

(i) $a/h \prec b/h$; (ii) $a/k \prec b/k$.

The sharpening and strengthening are valid.

Assume (i). Since $ka = a$ and $kb = b$, we have on account of (i) and R the relations $ka/h \prec kb/h$, $k/h \succ k/h$. Now $kah = a$ and $kbh = b$, and the supposition that either or both of these $= 0$ leads at once to (ii) (Theorem 1). We may accordingly apply D to the above pair of relations (with k, a, h, k, b, h, replacing $a_1, b_1, h_1, a_2, b_2, h_2$, respectively) and so obtain $a/kh \prec b/kh$, which, in view of $kh = k$, is the desired conclusion (ii).

Assume (ii). We may apply C_1 to the two relations $k/h \prec k/h$, $a/kh \prec b/kh$, (with k, a, h, k, b, h, replacing $a_1, b_1, h_1, a_2, b_2, h_2$, respectively) and so obtain a relation which, since $ka = a$ and $kb = b$, is identical with (i). Here again $kah = a$ and $kbh = b$, and if either $= 0$ the conclusion is trivial.

THEOREM 9. *Let $ah \neq 0$ and $bh \neq 0$. Then from*

$$\text{(i)} \qquad a/h \prec a/bh$$

it will follow that

$$\text{(ii)} \qquad b/h \prec b/ah.$$

The theorem remains valid if the $\prec$ be replaced by $\succ$ in both (i) *and* (ii).

The sharpening and strengthening are valid.

We may write (i) and a relation evident from R as the pair $ab/h \prec ab/h$, $a/bh \succ a/h$, and may then apply D (with b, a, h, a, b, h, replacing $a_1, b_1, h_1, a_2, b_2, h_2$, respectively), whereupon (ii) is obtained. That this application of D may be justified, we observe that $a_1b_1h_1 = abh = a_2b_2h_2$; suppose that $abh = 0$; then $b/ah = abh/ah = 0/ah \approx 0/1$, and (ii) becomes a trivial consequence of Theorem 1.

The proof of the remainder of the theorem follows precisely similar lines and may be omitted.

THEOREM 10. *Let $kh \neq 0$ and $\sim kh \neq 0$. Then either of the following relations is a necessary and sufficient condition for the truth of the other*:

$$\text{(i) } a/h \prec a/kh; \qquad \text{(ii) } a/h \succ a/\sim kh.$$

The sharpening and strengthening are valid.

It is only necessary for us to show that (i) implies (ii), in view of A. If $ah = 0$, then $a/\sim kh = ah/\sim kh = 0/\sim kh \approx 0/1$, and (ii) would be a consequence of Theorem 1. If $ah \neq 0$, Theorem 9 may be applied to (i) (with $b = k$) and thus a relation obtained which under the action of A yields $\sim k/h \succ \sim k/ah$. Again applying Theorem 9 (with $\sim k, a, \succ$, replacing $a, b, \prec$), we derive (ii).

THEOREM 11. *The sharpening and all strengthenings of Axioms* P *and* S *are valid.*

For the sharpening of P, the hypothesis $a/bh \approx r/s$, $a/\sim bh \approx r/s$ on the one hand contains the hypothesis of P and hence its conclusion, and on the other hand contains the relations $a/bh \succ r/s$, $a/\sim bh \succ r/s$, whence it follows by A that $\sim a/bh \prec \sim r/s$, $\sim a/\sim bh \prec \sim r/s$, whence again by P, $\sim a/b \prec \sim r/s$, and again by A, $a/b \succ r/s$, which in conjunction with the former conclusion of P gives the sharpened conclusion.

For the strengthening of P, let it be assumed that $a/bh < r/s$, $a/\sim bh \prec r/s$. All that is required is to prove the falsity of

$$a/h \succ r/s. \tag{5}$$

Assume this to be true; our hypothesis yields with the aid of T, $a/\sim bh \prec a/h$. Now apply Theorem 10,—the above relation being equivalent to (ii) with $k = b$,—and we have $a/h \prec a/bh$, which in conjunction with (5) gives the contradiction of hypothesis $a/bh \succ r/s$. The other case is automatically settled by the foregoing.

The sharpening of S is established at once on the double application of S to the $\prec$ and the $\succ$ hypotheses contained in the sharpened hypothesis.

For the strengthening of S, suppose the strengthened hypothesis to contain for some fixed $i(1 \leqq i \leqq n-1)$ $a_i/a < a_{i+1}/a$, we are to prove $a_1/a < b_n/b$. Assume on the contrary that $a_1/a \succ b_n/b$, it follows at once that

$$a_1/a \succ b_1/b, \quad \cdots, a_i/a > b_i/b,$$

$$a_{i+1}/a > b_{i+1}/b, \cdots, a_n/a > b_n/b;$$

whence in view of Theorem 6 (strengthened) the contradiction follows $1/1 \approx a/a > b/b \approx 1/1$. The remaining cases are left to the consideration of the reader.

THEOREM 12. *Let* $ah \neq 0$, $\sim ah \neq 0$, $bh \neq 0$, $\sim bh \neq 0$. *Then from*

$$a/bh \prec a/\sim bh \tag{i}$$

it follows that

$$b/ah \prec b/\sim ah \tag{ii}$$

$$a/bh \prec a/h. \tag{iii}$$

The theorem is valid if $\prec$ *be replaced by* $\succ$ *in every one of* (i), (ii), (iii).

The sharpening and strengthening are valid.

We have in view of R that $a/\sim bh \prec a/\sim bh$. Apply P to this in conjunction with (i) (with a, $\sim bh$, replacing r, s); we obtain $a/h \prec a/\sim bh$. Now we apply Theorem 10 (with $k = \sim b$), and so have (iii). The relation (ii) now follows by T.

The following extension of P may be given; we omit the proof which is immediate.

THEOREM 13. *Under the hypothesis that*

$$h_i h_j = 0: (i \neq j)\ i, j = 1, \cdots, m;$$

$$k_i k_j = 0: (i \neq j)\ i, j = 1, \cdots, n;$$

$$a/h_i \prec b/k_j; \quad i = 1, \cdots, m; \quad j = 1, \cdots, n.$$

It follows that

$$a/h_1 \vee \cdots \vee h_m \prec b/k_1 \vee \cdots \vee k_n .$$

The sharpening and strengthenings are valid.

5. Numerical Probability

The object of this section is to lay the foundations of the theory of numerical probability, or mathematical theory of probability in the conventional sense, upon the basis of the theory expounded in the preceding sections so that the classical developments can be made to follow from the present work. In addition to the axioms of §3 it is necessary to make a further assumption, the precise philosophical character of which we shall not undertake to examine, concerning the existence of what we shall call n-scales.

DEFINITION 1. *By an n-scale shall be meant any set of n propositions* $(u_1, \cdots, u_n)$ *for which the following is posited*:

1. $u = u_1 \vee \cdots \vee u_n \neq 0$.
2. $u_i u_j = 0\ (i \neq j)$ for all $i, j = 1, \cdots, n$.
3. $u_i/u \approx u_j/u$ for all $i, j = 1, \cdots, n$.

ASSUMPTION. *If n is any given positive integer, at least one n-scale may be regarded as existing.*

THEOREM 14. *If* $(u_1, \cdots, u_n)$ *is an n-scale and* $(v_1, \cdots, v_m)$ *an m-scale, and if* ρ *and* σ *are integers with* $0 \leqq \sigma \leqq n, 0 \leqq \rho \leqq m$, *then*

$$u_{i_1} \vee \cdots \vee u_{i_\sigma}/u <, \approx, > v_{j_1} \vee \cdots \vee v_{j_\rho}/v$$

according as $\sigma/n <, =, > \rho/m$.

Here it is understood that $i_1, \cdots, i_\sigma$ are each between 1 and n, and that no two are equal; similarly for $j_1, \cdots, j_\rho$ and m. And, finally, when $\sigma = 0$ the first member is replaced by $0/u$, and when $\rho = 0$, the second by $0/v$; the case where either or both of these are true is trivial, and shall be left out of consideration in our proof.

Since by hypothesis

$$u_{i_1}/u \approx u_1/u, \cdots, u_{i_\sigma}/u \approx u_\sigma/u,$$

$$v_{j_1}/v \approx v_1/v, \cdots, v_{j_\rho}/v \approx v_\rho/v,$$

we have by Theorem 6 (sharpened) that

$$u_{i_1} \vee \cdots \vee u_{i_\sigma}/u \approx u_1 \vee \cdots \vee u_\sigma/u,$$

$$v_{j_1} \vee \cdots \vee v_{j_\rho}/v \approx v_1 \vee \cdots \vee v_\rho/v,$$

and thus it suffices to prove that

$$u_1 \vee \cdots \vee u_\sigma/u <, \approx, > v_1 \vee \cdots \vee v_\rho/v$$

according as $\sigma/n <, \approx, > \rho/m$.

In virtue of our underlying assumption, we are permitted posit the existence of an mn-scale $(w_1, \cdots, w_{mn})$. Let U_i $(i = 1, \cdots, n)$ and V_j $(j = 1, \cdots, m)$ be defined as follows

$$U_i = w_{m(i-1)+1} \vee w_{m(i-1)+2} \vee \cdots \vee w_{mi},$$

$$V_j = w_{n(j-1)+1} \vee w_{n(j-1)+2} \vee \cdots \vee w_{nj}.$$

Clearly we have

$$w = w_1 \vee \cdots \vee w_{mn} = U_1 \vee \cdots \vee U_n = V_1 \vee \cdots \vee V_m;$$

$$U_iU_j = 0, \qquad V_iV_j = 0, \qquad (i \neq j);$$

$$U_i/w \approx U_j/w, \qquad V_i/w \approx V_j/w;$$

the last, again by Theorem 6. Thus we may apply Axiom S to $u_1, \cdots, u_n$ and $U_1, \cdots, U_n$, and so we obtain $u_i/u \approx U_j/w$, $(i, j = 1, \cdots, n$. Similarly, for $v_1, \cdots, v_m$ and $V_1, \cdots, V_m$, we have $v_i/v \approx V_j/w$, $i, j = 1, \cdots, m$. Applying Theorem 6 again, we derive

$$u_1 \vee \cdots \vee u_\sigma/u \approx U_1 \vee \cdots \vee U_\sigma/w$$

$$v_1 \vee \cdots \vee v_\rho/v \approx V_1 \vee \cdots \vee V_\rho/w.$$

Now

$$U_1 \vee \cdots \vee U_\sigma = w_1 \vee \cdots \vee w_{m\sigma},$$

$$V_1 \vee \cdots \vee V_\rho = w_1 \vee \cdots \vee w_{n\rho},$$

and thus by Theorem 3

$$u_1 \vee \cdots \vee u_\rho/u <, \approx, > v_1 \vee \cdots \vee v_\sigma/v$$

according as

$$w_1 \vee \cdots \vee w_{m\sigma} \subset \text{and} \not\supset, =, \supset \text{and} \not\subset w_1 \vee \cdots \vee w_{n\rho},$$

that is, as $m\sigma <, =, > n\rho$, from which the theorem follows.

Fixing the attention upon a given eventuality a/h, let n be an arbitrary positive integer, and $(u_1, \cdots, u_n)$ a certain n-scale. The relation $u_1 \vee \cdots \vee u_t/u \prec a/h$ will be true for at least one value of t $(0 \leqq t \leqq n)$, if we agree to include the case $0/u \prec a/h$ for $t = 0$. Furthermore, the maximum value of t for which this relation holds is independent of the particular n-scale $(u_1, \cdots, u_n)$ chosen, as an obvious application of Theorem 14 will show: This value of t is thus determined when n is given (for fixed a/h) and shall be denoted by $t(n)$.

Similarly, the minimum value of T for which $a/h \prec u_1 \vee \cdots \vee u_T/u$ is true is determined when n is given (for fixed a/h), and shall be denoted by $T(n)$. Evidently we have $0 \leqq t(n) \leqq T(n) \leqq n$.

THEOREM 15. *The following limits always exist*:

$$p_*(a, h) = \lim_{n \to \infty} \frac{t(n)}{n}; \qquad p^*(a, h) = \lim_{n \to \infty} \frac{T(n)}{n};$$

and they satisfy the relation

$$0 \leqq p_*(a, h) \leqq p^*(a, h) \leqq 1.$$

Let n, m be any positive integers, and $(u_1, \cdots, u_n)$ and $(v_1, \cdots, v_m)$ any n-scale and m-scale. By definition of $t(n)$, we can *assert*

$$u_1 \vee \cdots \vee u_{t(n)}/u \prec a/h$$

and *deny*

$$u_1 \vee \cdots \vee u_{t(n)+1}/u \prec a/h.$$

Hence we must have

$$\frac{t(m)}{m} \leqq \frac{t(n) + 1}{n};$$

for otherwise, by Theorem 14, we should have

$$u_1 \vee \cdots \vee u_{t(n)+1}/u \prec v_1 \vee \cdots \vee v_{t(m)}/v,$$

whereupon a contradiction is produced in view of the known relation

$$v_1 \vee \cdots \vee v_{t(m)}/v \prec a/h.$$

Similarly we prove that

$$\frac{t(n)}{n} \leqq \frac{t(m) + 1}{m}.$$

On combining these we have

$$-\frac{1}{m} \leqq \frac{t(m)}{m} - \frac{t(n)}{n} \leqq \frac{1}{n},$$

and thus $t(m)/m - t(n)/n \to 0$ as $n, m \to \infty$, and the existence of $p_*(a, h)$ is established. Similarly for $p^*(a, h)$. The relation $0 \leqq p_* \leqq p^* \leqq 1$ is an immediate consequence of $0 \leqq t(n) \leqq T(n) \leqq n$.

THEOREM 16. *The relation $a_1/h_1 \prec a_2/h_2$ implies both $p_*(a_1, h_1) \leqq p_*(a_2, h_2)$ and $p^*(a_1, h_1) \leqq p^*(a_2, h_2)$. And $p^*(a_1, h_1) < p_*(a_2, h_2)$ implies the relation $a_1/h_1 \prec a_2/h_2$.*

The first part of the theorem is obvious. To prove the second, we observe that since

$$\lim_{n \to \infty} \frac{T_1(n)}{n} = p^*(a_1, h_1) < p_*(a_2, h_2) = \lim_{n \to \infty} \frac{t_2(n)}{n},$$

we may take n such that $T_1(n)/n < t_2(n)/n$, whereupon

$$a_1/h_1 \prec u_1 \vee \cdots \vee u_{T_1(n)}/u < u_1 \vee \cdots \vee u_{t_2(n)}/u \prec a_2/h_2 ,$$

in virtue of Theorem 14.

THEOREM 17. $p_*(a, h) + p^*(\sim a, h) = 1$.

Let $(u_1, \cdots, u_n)$ be an n-scale, and observe that for any s $(0 \leqq s \leqq n)$,

$$\sim(u_1 \vee \cdots \vee u_s)/u = u_{s+1} \vee \cdots \vee u_n/u \approx u_1 \vee \cdots \vee u_{n-s}/u.$$

Thus with the aid of Axiom A and the relation

$$u_1 \vee \cdots \vee u_{t(n)}/u \prec a/h \prec u_1 \vee \cdots \vee u_{T(n)}/u$$

we derive the formula

$$u_1 \vee \cdots \vee u_{n-T(n)}/u \prec \sim a/h \prec u_1 \vee \cdots \vee u_{n-t(n)}/u.$$

Comparing this with the relation defining $t'(n)$, $T'(n)$ (for $\sim a/h$):

$$u_1 \vee \cdots \vee u_{t'(n)}/u \prec \sim a/h \prec u_1 \vee \cdots \vee u_{T'(n)}/u,$$

we have $t'(n) \geqq n - T(n)$ and $T'(n) \leqq n - t(n)$. Dividing by n and letting $n \to \infty$ we have

$$p_*(\sim a, h) + p^*(a, h) \geqq 1, \qquad p^*(\sim a, h) + p_*(a, h) \leqq 1.$$

Now the corresponding relations with a and $\sim a$ interchanged are evidently valid. Thus $1 \leqq p_*(a, h) + p^*(\sim a, h) \leqq 1$, and our theorem is proved.

THEOREM 18. (i) $p^*(a_1 \vee a_2, h) \leqq p^*(a_1, h) + p^*(a_2, h)$. (ii) *If* $a_1a_2h = 0$, *then* $p_*(a_1 \vee a_2, h) \geqq p_*(a_1, h) + p_*(a_2, h)$.

Let $t(n)$, $T(n)$ correspond to $a_1 \vee a_2/h$, and $t_1(n)$, $T_1(n)$ correspond to a_1/h, and, finally, $t_2(n)$, $T_2(n)$ correspond to a_2/h.

If for infinitely many values of n, $T_1(n) + T_2(n) \geqq n$, dividing by n and allowing $n \to \infty$ through such values would show that $p^*(a_1, h) + p^*(a_2, h) = 1$, and conclusion (i) becomes obvious. On the contrary assumption, for all sufficiently large n, $T_1(n) + T_2(n) < n$, and we should have

$$a_1/h \prec u_1 \vee \cdots \vee u_{T_1(n)}/u,$$

$$a_2/h \prec u_1 \vee \cdots \vee u_{T_2(n)}/u \approx u_{T_1(n)+1} \vee \cdots \vee u_{T_1(n)+T_2(n)}/u,$$

and hence, by Theorem 5

$$a_1 \vee a_2/h \prec u_1 \vee \cdots \vee u_{T_1(n)+T_2(n)}/u.$$

(In this application of Theorem 5, the requirement that $a_1a_2h = 0$ is dispensed with, inasmuch as we replace a_2 by $a_2 \sim a_1$, and can then replace a_2/h in the above relation by $a_2 \sim a_1/h$ which, by Theorem 3, $\prec a_2/h$.) Thus we obtain $T(n) \leqq T_1(n) + T_2(n)$, whence the conclusion (i).

Turning now to (ii), suppose that for some n, $t_1(n) + t_2(n) \geqq n$. We should have

$$u_1 \vee \cdots \vee u_{t_1(n)}/u \prec a_1/h,$$

$$u_{t_1(n)+1} \vee \cdots \vee u_n/u \prec u_1 \vee \cdots \vee u_{t_2(n)}/u \prec a_2/h,$$

and thus by Theorem 5, $u/u \prec a_1 \vee a_2/h$, hence by Theorems 1 and 16, $p_*(a_1 \vee a_2, h) = 1$, and (ii) is established. On the contrary assumption, $t_1(n) + t_2(n) < n$, and we have

$$u_{t_1(n)+1} \vee \cdots \vee u_{t_1(n)+t_2(n)}/u \approx u_1 \vee \cdots \vee u_{t_2(n)}/u \prec a_2/h$$

which, with the other relation, again by Theorem 5, yields

$$u_1 \vee \cdots \vee u_{t_1(n)+t_2(n)}/u \prec a_1 \vee a_2/h.$$

Hence $t_1(n) + t_2(n) \leqq t(n)$, and the conclusion (ii) follows.

THEOREM 19. *If a, b, c are any propositions for which $a \subset b \subset c$ $(b, c \neq 0)$, then*

(i) $$p_*(a, b)p_*(b, c) \leqq p_*(a, c) \leqq p_*(a, b)p^*(b, c)$$

(ii) $$p^*(a, b)p_*(b, c) \leqq p^*(a, c) \leqq p^*(a, b)p^*(b, c).$$

We shall assume in the proof that $a \neq 0$, the contrary case being trivial.

Let $t_1(n)$, $T_1(n)$ correspond to a/c, let $t_2(n)$, $T_2(n)$ correspond to b/c, and let $t_3(n)$, $T_3(n)$ correspond to a/b. .

From $a \subset b$ it follows by Theorem 3 that $a/c \prec b/c$, whence by Theorem 16 that

(1) $$p_*(a, c) \leqq p_*(b, c), \qquad p^*(a, c) \leqq p^*(b, c).$$

We shall first establish the inequality

(2) $$p_*(a, b)p_*(b, c) \leqq p_*(a, c).$$

We may evidently assume that $t_2(n) \to \infty$ as $n \to \infty$, as otherwise the conclusion $p_*(b, c) = 0$ could be drawn from Theorem 15, and (2) would be trivial. Clearly, for $t_2(n) \neq 0$, $(u_1, \cdots, u_{t_2(n)})$ forms a $t_2(n)$-scale: For if $u_1 \vee \cdots \vee u_{t_2(n)} = 0$ it would follow that $u_1 = 0$; hence that $u_i/u \approx u_1/u \approx 0/u$; whence by Theorem 6 that $u/u \approx 0/u$, contrary to Theorem 1. And the equiprobability of all expressions

$$u_i/u_1 \vee \cdots \vee u_{t_2(n)}, \qquad 1 \leqq i \leqq t_2(n),$$

is a consequence of that of all the u_i/u (Theorem 8, with a, b, k, h replaced by u_i, u_j, $u_1 \vee \cdots \vee u_{t_2(n)}$, u, respectively). Consequently we may write

$$u_1 \vee \cdots \vee u_{t_3[t_2(n)]}/u_1 \vee \cdots \vee u_{t_2(n)} \prec a/b.$$

This may be combined with

$$u_1 \vee \cdots \vee u_{t_2(n)}/u \prec b/c,$$

the application of Axiom C_1 giving, upon setting

$$a_1, b_1, h_1 = u_1 \vee \cdots \vee u_{t_2(n)}, u_1 \vee \cdots \vee u_{t_3[t_2(n)]}, u;$$

$$a_2, b_2, h_2 = b, a, c,$$

the relation

$$u_1 \vee \cdots \vee u_{t_3[t_2(n)]}/u \prec a/c;$$

from which by definition of $t_1(n)$ we conclude that

$$t_3[t_2(n)] \leqq t_1(n).$$

Since $t_2(n) \to \infty$ with n, we have for sufficiently large n

$$\frac{t_2(n)}{n} \frac{t_3[t_2(n)]}{t_2(n)} \leqq \frac{t_1(n)}{n},$$

and on letting $n \to \infty$ the inequality (2) is obtained.

In a precisely similar manner, we establish the inequality

$$(3) \qquad p^*(a, c) \leqq p^*(a, b)p^*(b, c).$$

In this case, the failure of the assumption $T_2(n) \to \infty$ with n would lead to $p^*(b, c) = 0$ and hence by inequality (1), to $p^*(a, c) = 0$, and thus trivially to (3). The remainder of the proof of (3) follows that of (2).

We turn now to the proof of the inequality

$$(4) \qquad p^*(a, b)p_*(b, c) \leqq p^*(a, c).$$

Suppose that for infinitely many values of n, $t_2(n) \leqq T_1(n)$: it will follow at once that $p_*(b, c) \leqq p^*(a, c)$, whence (4) results immediately, inasmuch as $p^*(a, b) \leqq 1$. Suppose on the contrary that $T_1(n) < t_2(n)$ for all sufficiently large n. We invoke Axiom D in which we set

$$a_1 = b, \qquad b_1 = a, \qquad h_1 = c, \qquad h_2 = u,$$

$$a_2 = u_1 \vee \cdots \vee u_{t_2(n)}, \qquad b_2 = u_1 \vee \cdots \vee u_{T_1(n)}.$$

In virtue of $a \subset b \subset c$ and the definition of $t_2(n)$, $T_1(n)$, we have

$$a_1b_1/h_1 \prec a_2b_2/h_2, \qquad a_1/h_1 \succ a_2/h_2;$$

furthermore, we have effectively excluded the possibility of either of the equations (since $T_1(n) = 0$ would imply $a/c \approx 0/1$, hence that $a = 0$)

$$a_1b_1h_1 \ (= a) = 0, \qquad a_2b_2h_2 \ (= u_1 \vee \cdots \vee u_{T_1(n)}) = 0;$$

and thus the conclusion of Axiom D permits us to write

$$a/b \prec u_1 \vee \cdots \vee u_{T_1(n)}/u_1 \vee \cdots \vee u_{t_2(n)}.$$

From this we conclude as in the proof of (2) that $T_3[t_2(n)] \leqq T_1(n)$. If $t_2(n)$ fails to become infinite with n, then $p_*(b, c) = 0$, and (4) is immediate. And when $t_2(n) \to \infty$ with n, we may write

$$\frac{t_2(n)}{n} \frac{T_3[t_2(n)]}{t_2(n)} \leqq \frac{T_1(n)}{n},$$

whence on letting $n \to \infty$ the inequality (4) is established.

Turning lastly to the inequality

$$p_*(a, c) \leqq p_*(a, b)p^*(b, c), \tag{5}$$

we may always assume $t_1(n) \leqq T_2(n)$. For if for some $n > 0$ $t_1(n) > T_2(n)$ we should have by Theorem 14

$$b/c \prec u_1 \vee \cdots \vee u_{T_2(n)}/u < u_1 \vee \cdots \vee u_{t_1(n)}/u \prec a/c$$

and $b/c < a/c$ contradicts $a \subset b$, in view of Theorem 3. We may then write

$$a_1 = u_1 \vee \cdots \vee u_{T_2(n)}, \qquad b_1 = u_1 \vee \cdots \vee u_{t_1(n)}, \qquad h_1 = u,$$

$$a_2 = b, \qquad b_2 = a, \qquad h_2 = c,$$

and exclude $a_1b_1h_1 = 0$ as leading to $t_1(n) = 0$ and thus to a triviality, and also observe that $a_2b_2h_2 = a \neq 0$. Now by definition of $t_1(n)$, $T_2(n)$, etc., we have the following relations

$$a_1b_1/h_1 \prec a_2b_2/h_2, \qquad a_1/h_1 \succ a_2/h_2,$$

and thus by Axiom D, a relation obtains which takes the form

$$u_1 \vee \cdots \vee u_{t_1(n)}/u_1 \vee \cdots \vee u_{T_2(n)} \prec a/b;$$

thus, as before, $t_1(n) \leqq t_3[T_2(n)]$. If $T_2(n)$ failed to become infinite with n, $p^*(b, c) = 0$, whence by (3) $p^*(a, c) = 0$, so that $p_*(a, c) = 0$, giving (5) trivially. Finally, in the case where $T_2(n) \to \infty$ with n, we have

$$\frac{t_1(n)}{n} \leqq \frac{T_2(n)}{n} \frac{t_3[t_2(n)]}{T_2(n)}$$

whence the inequality (5) when $n \to \infty$. Thus the proof of our theorem is complete.

DEFINITION 2. *The eventuality a/h shall be said to be appraisable in the case where $p_*(a, h) = p^*(a, h)$. When a/h is appraisable, the common limit*

$$p(a, h) = p_*(a, h) = p^*(a, h)$$

shall be called the (*numerical*) *probability of a/h* (*or of the contingency a on the presumption h*).

It may be remarked that if every $\mathfrak{A}$ were *completely* ordered by $\prec$, every eventuality in it would be appraisable.

If a_1/h and a_2/h are appraisable, then the inequality $p(a_1, h_1) < p(a_2, h_2)$

implies the relation $a_1/h_1 < a_2/h_2$ by Theorem 16. But the equation $p(a_1, h_1) = p(a_2, h_2)$ is perfectly consistant with any one of the relations $a_1/h_1 <, >, \approx, \parallel a_2/h_2$. Thus relations of numerical probabilities furnish but a blurred picture of the more fundamental comparisons in probability; a circumstance which is illustrated by the familiar fact that we may have $h(a, h) = 0$ even when $ah \neq 0$.

THEOREM 20. *If* $(a_1, \cdots, a_\nu)$ *is a* ν*-scale* $(\nu = 1, 2, \cdots)$ *and if* $b =$ *sum of* τ *distinct elements* a_i :

$$b = a_{i_1} \vee \cdots \vee a_{i_\tau} \; (b = 0 \text{ for } \tau = 0),$$

then b/a *is appraisable, and* $p(b, a) = \tau/\nu$. *In particular (whenever* $a \neq 0$*),* $0/a$ *and* a/a *are appraisable, and* $p(0, a) = 0$, $p(a, a) = 1$.

In virtue of Theorem 14, to say that $t(n)$ is the maximum t for which $u_1 \vee \cdots \vee u_t/u \prec b/a$ is to say that it is the maximum t such that $t/n \leqq \tau/\nu$, and for such $t(n)$, $\lim t(n)/n = \tau/\nu$. And likewise $\lim T(n)/n = \tau/\nu$, and our theorem is evident.

THEOREM 21. *In order that* a/h *be appraisable it is necessary and sufficient that, corresponding to any* $\epsilon > 0$, *there should exist appraisable* a'/h' *and* a''/h'' *such that*

$$a'/h' \prec a/h \prec a''/h'',$$

$$p(a'', h'') - p(a', h') < \epsilon.$$

The necessity of the condition appears at once on setting $a' = a'' = a$, $h' = h'' = h$.

To prove the sufficiency, it is only necessary to observe that $p(a', h') \leqq p_*(a, h)$: otherwise, by Theorem 16, $a'/h' > a/h$; and similarly, $p^*(a, h) \leqq p(a'', h'')$. Hence $p^*(a, h) - p_*(a, h) < \epsilon$ and this ϵ being arbitrarily small, $p^*(a, h) = p_*(a, h)$ follows.

An obvious theorem is the following:

THEOREM 22. *If* $p^*(a, h) = 0$, a/h *will be appraisable. If* $p_*(a, h) = 1$, a/h *will be appraisable.*

THEOREM 23. *If* a/h *is appraisable, then* $\sim a/h$ *will be appraisable, and* $p(a, h) + p(\sim a, h) = 1$ *will hold.*

For on applying Theorem 17, first directly, then with a and $\sim a$ interchanged, we obtain

$$p_*(a, h) + p^*(\sim a, h) = 1,$$

$$p^*(a, h) + p_*(\sim a, h) = 1$$

which in conjunction with the hypothesized relation $p_*(a, h) = p^*(a, h)$, make the theorem evident.

THEOREM 24. *If* $a_1a_2h = 0$ *and* a_1/h *and* a_2/h *are both appraisable, then* $a_1 \vee a_2/h$ *will be appraisable, and*

$$p(a_1 \vee a_2, h) = p(a_1, h) + p(a_2, h).$$

This is the *principle of total probability*; it may be extended to any *finite* number of disjoint summands. Its truth is an immediate consequence of Theorem 18.

THEOREM 25. *If* $a \subset b \subset c$ *and if* a/b *and* b/c *are appraisable, then* a/c *will be appraisable and* $p(a, c) = p(a, b)p(b, c)$ *will hold.*

This follows immediately from Theorem 19.

THEOREM 26. *If* $a \subset b \subset c$ *and* a/c *and* b/c *are appraisable, and lastly if* $p(b, c) \neq 0$, *then* a/b *will be appraisable, and* $p(a, c) = p(a, b)p(b, c)$ *will hold.*

Setting $p_*(b, c) = p^*(b, c) = p(b, c)$ and $p_*(a, c) = p^*(a, c) = p(a, c)$ in Theorem 19, we have

$$\begin{aligned} p_*(a, b)p(b, c) &\leqq p(a, c) \leqq p_*(a, b)p(b, c), \\ p^*(a, b)p(b, c) &\leqq p(a, c) \leqq p^*(a, b)p(b, c) \end{aligned} \tag{6}$$

whence $p^*(a, b)p(b, c) \leqq p_*(a, b)p(b, c)$, and the proof is completed by dividing by $p(b, c)$.

THEOREM 27. *If* $a \subset b \subset c$ *and if* a/b *and* b/c *are appraisable, then* $a \sim b/c$ *is appraisable, and* $p(a \sim b, c) = p(a, c) - p(b, c)$.

First suppose that $p(b, c) = 0$. The above proof of (6) may be applied, whence it follows that $p(a, c) = 0$. On the other hand $b \sim a \subset b$; hence $b \sim a/c \prec b/c$ (Theorem 3), so that by Theorem 16 $0 \leqq p_*(b \sim a, c) \leqq p^*(b \sim a, c) \leqq p(b, c) = 0$. Thus $b \sim a/c$ is appraisable, and $p(b \sim a, c) = 0 = p(a, c) - p(b, c)$.

Next, let $p(b, c) \neq 0$. Then by Theorem 26, a/b is appraisable, and hence by Theorem 23 $\sim a/b$ is appraisable and $p(\sim a, b) = 1 - p(a, b)$. Theorem 25 with $b \sim a \subset b \subset c$ replacing $a \subset b \subset c$ gives the appraisability of $b \sim a/c$ and we have

$$\begin{aligned} p(b \sim a, c) = p(b \sim a, b)p(b, c) = p(\sim a, b)p(b, c) &= [1 - p(a, b)]p(b, c) \\ = p(b, c) - p(a, b)p(b, c) &= p(b, c) - p(a, c). \end{aligned}$$

THEOREM 28. *Let* a_1, a_2, h *be any propositions such that* $a_1 \subset h$, $a_2 \subset h$, *and that* a_1/h *and* a_2/h *are appraisable. Then* $a_1 \vee a_2/h$ *will be appraisable if and only if* a_1a_2/h *is appraisable, and*

$$p(a_1, h) + p(a_2, h) = p(a_1 \vee a_2, h) + p(a_1a_2, h).$$

Firstly, let a_1a_2/h be appraisable. If $p(a_1, h) = 0$ and $p(a_2, h) = 0$, it would follow by Theorem 18 that $p^*(a_1 \vee a_2, h) = 0$, whence the appraisability of $a_1 \vee a_2/h$ (Theorem 22). In the contrary case, we should have, let us say, $p(a_1, h) > 0$. Then by Theorem 27 applied with $a = a_1a_2$, $b = a_1$, it would follow (since $a_1 \sim (a_1a_2) = a_1 \sim a_2$) that $a_1 \sim a_2/h$ is appraisable, and that $p(a_1 \sim a_2, h) = p(a_1, h) - p(a_1a_2, h)$. Now we have $a_1 \vee a_2/h = a_1 \sim a_2 \vee a_2/h$, $(a_1 \sim a_2)a_2h = 0$; and hence by Theorem 24, $a_1 \vee a_2/h$ is appraisable, and $p(a_1 \vee a_2, h) = p(a_1 \sim a_2, h) + p(a_2, h) = p(a_1, h) - p(a_1a_2, h) + p(a_2, h)$; and the first half of our theorem is established.

Secondly, let $a_1 \vee a_2/h$ be appraisable. Theorem 23 then establishes the appraisability of $\sim(a_1 \vee a_2)/h = \sim a_1 \sim a_2/h$, $\sim a_1/h$, and $\sim a_2/h$, and gives the relations

$$p(\sim a_1 \sim a_2, h) = 1 - p(a_1 \vee a_2, h),$$
$$p(\sim a_1, h) = 1 - p(a_1, h), \qquad p(\sim a_2, h) = 1 - p(a_2, h).$$

Now the established half of the present theorem yields the appraisability of $\sim a_1 \sim a_2/h$, and the equation

$$p(\sim a_1, h) + p(\sim a_1, h) = p(\sim a_1 \vee \sim a_2, h) + p(\sim a_1 \sim a_2, h).$$

A second application of Theorem 23 gives us the appraisability of $\sim(\sim a_1 \vee \sim a_2)/h = a_1a_2/h$, and also that $p(a_1a_2, h) = 1 - p(\sim a_1 \vee \sim a_2, h)$. On combining these equations, the desired conclusion is established.

We thus reach the conclusion of this section, and with it our final goal, having defined the numerical propability and established its conventional properties: In virtue of the Theorem of Representation of Boolean Algebras of Garrett Birkhoff and M. H. Stone[14] our underlying Boolean ring $\mathfrak{A}$ corresponds with a closed system of subclasses of a certain E, in a subsystem of which a (restrictedly) additive set function $p_H(A) = p(a, h)$ is defined which satisfies the Axioms of Kolmogoroff,[15] from which the classical theory of probability may be derived.

Our proposed task is thus accomplished.

The relation with frequency in sequences will be studied in a forthcoming paper.

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[14] For full references and discussion, see M. H. Stone, The Representation of Boolean Algebras, Bull. Amer. Math. Soc., Vol. 44, Dec. 1938, pp. 807–816.

[15] A. Kolmogoroff, Grundbegriffe der Wahrscheinlichkeits Rechnung. Ergebnisse der Mathematik, B2. (Berlin 1933.)