Comment on Brian Weatherson's "Uncertainty, Probability, and Non-Classical Logic"

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1 Points that bear repeating

In addition to the sets A and B of axioms that Weatherson gives, there are multiple other potential axiomatizations of probability that are inequivalent for intuitionist logic.

(A1),(A2),(A3) imply (B1),(B2), together with a weaker version of (B3) asserting that "If $A \vdash B$ then $Pr(\neg B) \leq 1 - Pr(A)$". This is because if $A \vdash B$ then $A \wedge \neg B$ is a logical falsehood, then $\neg(A \wedge \neg B)$ is a logical truth. Thus $1 \geq Pr(A \vee \neg B) = Pr(A) + Pr(\neg B)$.

(A1),(A2),(A3) don't imply (B4), by the long argument Weatherson gives in section 3 of "From Classical to Intuitionistic Probability". In addition, (A1),(A2),(B4) don't imply (B3), as can be seen from the counterexample given at the end of the same section of that paper.

However, (B1),(B2),(B3) imply (A1),(A2), and (B2),(B4) imply (A3), because $\neg(A \wedge B)$ is an intuitionist truth iff $A \wedge B$ is an intuitionist falsehood. Thus, in addition to the two axiomatizations given, we see that (A1),(A2),(B4) is another potential axiomatization that is strictly weaker than B and stronger than A. Thus, there is a real multiplicity of potential axiom systems for intuitionist logic that one must consider.

Weatherson suggests in "Uncertainty, Probability, and Non-Classical Logic" that there are strong arguments in favor of (B3) just as a simple fact about rationality. However, his real support for axiomatization B is a nice Dutch book argument.

In fact, the Dutch book shows that (B1)-(B4) are axioms that should be believed for probability as long as the background logic satisfies $A \vdash A \vee B$ and $A, B \vdash A \wedge B$ and $A \wedge B \vdash A$, and such that just as many bets from the set $\{A, B\}$ can be collected as from the set $\{A \vee B, A \wedge B\}$. This is easily seen to be true for intuitionist and classical logic, and it turns out to be true for a three-valued Łukasiewicz logic as well, defining $\wedge$ and $\vee$ as taking the min and max respectively of their arguments' truth-values, rather than the other possible interpretations.[1]

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[1] See the section on Łukasiewicz logic in the Stanford Encyclopedia of Philosophy's article on many-valued logic, at http://plato.stanford.edu/entries/logic-manyvalued/

If neither A nor B has the designated value 1, then the greatest value for $A \vee B$ comes when both have value $1/2$, which gives $\max\{1/2, 1/2\} = 1/2$ and thus $A \vee B$ can't be collected unless one of A and B can be. Similarly, for $A \wedge B$ to take the designated value 1, both A and B must as well, since $A \wedge B$ takes the minimum of the two.

This Dutch book argument is fairly convincing, but still relies on the fact that the value of a pair of bets is the sum of the values of the individual bets. This fact seems to prejudice any argument in favor of additivity from the beginning, although it seems like a fairly plausible fact to use. At any rate, it is more plausible than the axiom (B4).

However, while all these arguments tell in favor of a particular axiomatization, they give no guide as to what underlying logic we should use. And even if we choose to use an underlying intuitionist logic, any classical probability function will satisfy all these axioms as well, and thus the arguments in favor of letting $Pr(A \vee \neg A) < 1$ are essential to the program as well.

2 Potential extensions of these results

Weatherson mentions that he hasn't yet found a completeness theorem for his intuitionist probability functions with the semantics of measures over a Kripke tree yet, but other semantics seem like they may work as well. He indicated that such a completeness theorem holds for models of Łukasiewicz probability based on a measurable set together with a three-valued truth-function over formulas for each point. (While Weatherson only insists that the set of points assigning truth-value 1 to any particular sentence be measurable, it seems necessary to insist that the set of points assigning 0 and $1/2$ each be measurable as well.) While it is known that no finite set of truth-values is possible for intuitionist logic, perhaps it's worth looking into Jaskowski (1936)'s infinite-valued logic for intuitionist logic.[2] However, in discussion after my comments, Weatherson indicated that his completeness theorem for Łukasiewicz logic relied on the fact that there are three formulas that are true iff A has respectively truth-value 1, $1/2$, or 0. I doubt that such formulas exist for all the truth-values that arise in Jaskowski's system.

In personal communication, Branden Fitelson suggested that while Weatherson's project involves choosing Kolmogorov-style probabilistic axioms to fix while varying the underlying logic, a system in which all axioms are chosen on intuitionistic grounds would be more satisfying. He pointed out Popper's axiomatization of classical probability in appendix (iv) of *The Logic of Scientific Discovery*, where the axioms are given as follows:

1. There are countably many sentences.

2a. For any two sentences, a, b, $P(a, b)$ is a real number.

2b. For every a, b there are c, d such that $P(a, b) \neq P(c, d)$.

2c. For all a, b, if $P(a, c) = P(b, c)$ for all c, then for all d, $P(d, a) = P(d, b)$.

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[2] Cited in Stanford Encyclopedia of Philosophy entry on logic-manyvalued. English translation: Studia Logica 34 (1975), 117-120.

2d. For every a, b, $P(a, a) = P(b, b)$.

3a. For every a, b, there is a sentence $a \wedge b$.

3b. For all a, b, c, $P(a \wedge b, c) \leq P(a, c)$.

3c. For all a, b, c, $P(a \wedge b, c) = P(a, b \wedge c)P(b, c)$.

4a. For all a, there is a sentence $\neg a$.

4b. For all a, b if there is c such that $P(b, b) \neq P(c, b)$ then $P(a, b) + P(\neg a, b) = P(b, b)$.

Together, these axioms prove that the relation $\forall c((P(a, c) = P(b, c)) \wedge (P(c, a) = P(c, b)))$ is an equivalence relation. The set of equivalence classes of formulas under this relation then forms a boolean algebra, and for any particular a, the function $P(x, a)$ satisfies the Kolmogorov axioms, when x varies over elements of the boolean algebra.

For the intuitionist, these axioms are all fine except 4b, which must be weakened at least to an inequality rather than an equality. In addition, since disjunction is not definable in terms of negation and conjunction for the intuitionist, a new set of axioms must be added to deal with disjunction. Ideally, such a set of axioms would prove that equivalence classes of formulas form a Heyting algebra and would satisfy the B axioms given in Weatherson's paper, while being more directly motivated.

However, an additional issue may arise for this sort of axiomatization of intuitionist logic. Equalities and inequalities about real numbers are slightly trickier to work with for intuitionists, and therefore some facts about the equivalence relation may not be fully provable. Much of this can be solved by working with rationals instead of reals, but this again would count as some sort of reduction in scope of the project.

Such a program is pursued in part in Roeper and Leblanc's *Probability Theory and Probability Semantics*, although it doesn't seem to be as prior to the underlying logic as Popper's approach is. However, they show that their probability additionally satisfies the desideratum that $Pr(A|B) = Pr(A \wedge B)/Pr(B)$, although on intuitionist grounds, this again needs to be remotivated. After all, classically this is equivalent to $Pr(\neg(\neg A \vee \neg B))/Pr(B)$, which would often give quite different numbers intuitionistically.

3   Intuitionist use of negation

The largest issue that isn't fully dealt with in Weatherson's paper is that of the meaning of negations in probability formulas. To justify the use of intuitionistic logic and probabilities, we must justify the potential use of different probabilities for $\neg\neg p$ and p, since they are intuitionistically non-equivalent. If $\neg\neg p$ and p had the same probabilities for all p, then $1 = Pr(\neg\neg(A \vee \neg A)) = Pr(A \vee \neg A)$ and the probability would be classical.

There are cases where one can *prove* $\neg\neg A$ without proving A - as for instance when one gives a (classical) non-constructive proof of an existential claim. But there is no obvious sort of evidence that would be less than conclusive that would count towards $\neg\neg A$ without counting towards A as well. If Weatherson

is serious about considering $Pr(p)$ as the amount of evidence that one has for p, then such potential evidence must be found in order to justify non-classical probabilities.

$(\neg\neg\neg p \leftrightarrow \neg p)$ is an intuitionistic truth, and thus the same problem doesn't arise for negated formulas. But it means that for any atomic formula A, the three formulas $A, \neg A, \neg\neg A$ are inequivalent, and thus the question of which propositions should be represented atomically becomes more important. Classically, it doesn't matter if "This brick is colored" or "This brick is colorless" is taken as atomic, but intuitionistically it becomes iportant.

Evidence for $\neg A$ isn't just evidence that Oswald didn't do it, but rather evidence that we will never know or believe that Oswald did it, taking an appropriately intuitionist account. $\neg\neg A$ says that we will never find evidence that we will never find evidence that Oswald did it. And so on. But if I am ignorant of the whole situation, then it seems I lack evidence for all of these statements, not just the first two.

Thus, for the intuitionist, $Pr(\neg A)$ isn't the same as my credence that Oswald didn't do it, as Weatherson would have it, but rather my credence that I will never (or perhaps, can never) find evidence that Oswald *did* do it. The arguments for supposing that my credence that Oswald didn't do it is low are different from those suggesting that I will never find evidence that Oswald did it. Thus, Weatherson needs to be a bit more careful when describing the cases in which $Pr(A) + Pr(\neg A) < 1$ may be considered plausible.

The problem of anti-realism that he discusses towards the end seems connected to this fact. If $p \vee \neg p$ is taken to be true whenever there is an effective procedure for determining whether or not p is true, then we can't interpret $Pr(p)$ and $Pr(\neg p)$ as being the amount of evidence available right now for the proposition and its negation respectively, if we want to be intuitionists. It seems plausible to reject the former, if truth is taken to be the existence of evidence for a statement, and truth of a negation is taken to be the existence of evidence that there is no evidence for the statement. Just because there is an effective procedure for determining whether or not a particular proposition holds doesn't seem to be sufficient to guarantee evidence either for p or for the lack of evidence for p. Thus, one can hold onto a realist position about p without being forced to intuitionistically adopt the formula $p \vee \neg p$. Weatherson seems to attempt the other possible solution, suggesting that the existence of an effective procedure for determining whether or not p is true can be taken as evidence both for p and for $\neg p$. This may seem reasonable as well, but in both cases, one must be careful just what negation means when things are being done intuitionistically.

If a Popper-style axiomatization for intuitionist probabilities can be found, then these worries will at least partially disappear. Under such an axiomatization, it becomes less clear that the connectives have any meaning at all, and at any rate, their meaning is separated from any logical meaning they may have. As long as their intended meaning is reflected in the chosen axioms, the function will be correct, but we must remain careful not to be tempted by our standard ways of formalizing natural-language statements.