The simplest Lewis-style triviality proof yet?

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In his celebrated 'Probabilities of conditionals and conditional probabilities' David Lewis showed that the identification of the probability of conditionals $a \Rightarrow c$ with the conditional probability of consequent given antecedent when that antecedent has non-zero probability,

$$\text{i.e. for all } a \text{ and } c,\ P(a \Rightarrow c) = P(c \mid a), \text{ when } P(a) > 0,$$

trivializes the probability distribution in question. Lewis presented three triviality results:

LEWIS 1 If $P(a \& c) > 0$ and $P(a \& {\sim}c) > 0$ then $P(c \mid a) = P(c)$;
LEWIS 2 P assigns non-zero probabilities to at most two of any set of pairwise inconsistent propositions;
LEWIS 3 P takes at most four values.

The premisses for Lewis's results are

(i) the probability expansion rules,

$$P(a \Rightarrow c) = P((a \Rightarrow c)\ \& b) + P((a \Rightarrow c)\ \& {\sim}b)$$

$$= P(a \Rightarrow c \mid b)P(b) + P(a \Rightarrow c \mid {\sim}b)P({\sim}b),$$

the first holding generally, the second when $0 < P(b) < 1$;

(ii) the family of probability distributions satisfying the identification of conditional probability and probability of conditional is closed under conditionalization, so that

$$P(a \Rightarrow c \mid b) = P(c \mid a \& b) \text{ when } P(a \& b) > 0.$$

Lewis used the second expansion rule, expanding the probability of the conditional with respect to its consequent, i.e. setting $b = c$ in (i), then applied (ii) to obtain LEWIS 1, from which he derived LEWIS 2 and, from it in turn, LEWIS 3.

It's more fun to expand with respect to the corresponding material conditional. To make that pay off we need two facts.

(1) It was, it seems, Karl Popper – see Dorn 1992/93 – who first remarked that the probability of the material conditional, $a \supset c$, is never less than the conditional probability $P(c \mid a)$ and but for exceptional cases to be noted exceeds it.

When $P(a) > 0$,

$$P(a \supset c) = P({\sim}a \vee (a \& c)) = P({\sim}a) + P(a \& c)$$

$$= P({\sim}a) + P(c \mid a)P(a)$$

$$\geq P(c|a)P(\sim a) + P(c|a)P(a) = P(c|a),$$

with equality if, and only if, $P(\sim a) = 0$ or $P(c|a) = 1$.

(2) $a \Rightarrow c$ being a conditional, an obvious question to ask is, how probable is $a \Rightarrow c$ given the corresponding material conditional? The surprising answer:

$$\text{when } P(a \,\&\, c) > 0, P(a \Rightarrow c|a \supset c) = P(c|a \,\&\, (a \supset c))$$
$$= P(c|a \,\&\, c) = 1.$$

Expanding $P(a \Rightarrow c)$ we find:

$$\begin{aligned} P(a \Rightarrow c) &= P((a \Rightarrow c) \,\&\, (a \supset c)) + P((a \Rightarrow c) \,\&\, \sim(a \supset c)) \\ &\geq P((a \Rightarrow c) \,\&\, (a \supset c)) \\ &= P(a \Rightarrow c|a \supset c)P(a \supset c) = P(a \supset c), \end{aligned}$$

when $P(a \,\&\, c) > 0$.

Putting (1) and (2) together yields a new variation on the triviality theme:

$$P(a \Rightarrow c) = P(a \supset c) \text{ when } P(a \,\&\, c) > 0.$$

From (1) we know that with $P(a) > 0$, $P(a \Rightarrow c) = P(a \supset c)$ when, and only when, $P(a) = 1$ or $P(c|a) = 1$. So,

$$P(a) = 1 \text{ or } P(c|a) = 1 \text{ when } P(a \,\&\, c) > 0.$$

Put another way, if $0 < P(a) < 1$ and $0 < P(a \,\&\, c)$ then $P(c|a) = 1$. Since $P(c|a) = 0$ when $P(a) > 0$ and $P(a \,\&\, c) = 0$, we have our *Basic Triviality Result* (*BTR*):

> the function $P(.|a)$ is two-valued, i.e. takes only the values 0 and 1, when $0 < P(a) < 1$.

BTR tells us that if one learns/comes to believe something – anything – about which one is not currently certain one way or the other, i.e. $0 < P(a) < 1$, then, updating by conditionalization, one will be certain, one way or the other, about everything!

BTR affords easy proofs of LEWIS 1, LEWIS 2 and LEWIS 3:

LEWIS 1 If $P(a \,\&\, c) > 0$ and $P(a \,\&\, \sim c) > 0$, then $P(a) > 0$ and $0 < P(c|a) = P(a \,\&\, c)/[P(a \,\&\, c) + P(a \,\&\, \sim c)] < 1$.

As $P(.|a)$ is not two-valued, $P(a) = 1$ so $P(c|a) = P(c)$.

This proof gives us extra information about the independence under P of a and c when $P(a \,\&\, c) > 0$ and $P(a \,\&\, \sim c) > 0$. It shows it to be essentially trivial, resulting from the fact that

$$P(a) = 1 \text{ when } P(a \,\&\, c) > 0 \text{ and } P(a \,\&\, \sim c) > 0.$$

What is in essentials the same proof works for

LEWIS 2 If $P(a) > 0$, $P(b) > 0$, and a and b are inconsistent, $P(a \vee b) = P(a) + P(b) > 0$ and
$0 < P(a|a \vee b) = P(a)/[P(a) + P(b)] < 1$.

As $P(.|a \vee b)$ is not two-valued, $P(a \vee b) = 1$. Consequently, if c is inconsistent with both a and b and so inconsistent with $a \vee b$, $P(c) = 0$.

LEWIS 3 If, for some a, $0 < P(a) < 1$ then $P(.|a)$ and $P(.|\sim a)$ are both two-valued. But then, for any b, as

$$P(b) = P(b|a)P(a) + P(b|\sim a)P(\sim a),$$

$P(b) \in \{0, P(a), P(\sim a), 1\}$.

BTR, LEWIS 1 and LEWIS 2 are equivalent. As mentioned above, Lewis derived LEWIS 2 from LEWIS 1. It remains to show that LEWIS 2 entails *BTR*:

Suppose that $0 < P(a) < 1$ and that $0 < P(c|a) < 1$. Then $P(a \& c) > 0$, $P(a \& \sim c) > 0$, and $P(\sim a) > 0$. But $a \& c$, $a \& \sim c$, and $\sim a$ are pairwise inconsistent.

LEWIS 3, perhaps the most eye-catching of the original triviality results, is weaker. LEWIS 3 does not entail LEWIS 2: consider a fair, three-ticket lottery.

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