15 Functions

A function is a *relation* that satisfies certain conditions. (A function is an abstract thing. It is not to be identified with any linguistic expression. There are, for example, uncountably many functions and only countably many actual or possible linguistic expressions.)

For simplicity, we confine ourselves for the moment to *functions of one argument* and *two-termed relations.*

The *domain* of a two-termed relation is the set of all things that have the relation to something or other. The *range* of a two-termed relation is the set of all things to which something or other has the relation.

Example 1. The domain of the relation of *being husband of* is the set of all husbands; the range is the set of all wives.

Definition. A *function of one argument* is a two-termed relation that assigns to each member of its domain *one and only one* member of its range.

Examples:

2. The two-termed relation *being husband of* is not a function, for in some countries a husband may have more than one wife. By contrast the relation *being monogamously married to* is a function.

3. The relation f that has for its domain the set of positive integers and for its range the set of even numbers and is defined by the rule

$$f(x) = 2x$$

is a function. (Each positive integer is assigned one and only one number that is the product of the given integer and 2.)

4. The relation of *having as father* (where father = human biological father) is a function. Each thing that has a father has one and only one father. Two or more things can have the same father, but this does not prevent the relation from being a function. What matters is that no member of the domain is assigned more than one member of the range.

We give examples to show what is meant by '*a function of n arguments*'.

5. The function g defined by the rule

$$g(x, y) = x + y$$

is a function of *two* arguments.

6. The function h defined by the rule

$$h(x, y, z) = (x \cdot y) + z$$

is a function of *three* arguments.

7. The function j defined by the rule

$$j(x, y) = (x^2 + y)^2$$

is a function of *two* arguments.

So a *function of n arguments* ($n > 1$) is a function whose domain is a set of *n-termed sequences* or *n-tuples.* By identifying a thing with the sequence of which it is the sole term, we can speak also, as we have been doing, of a function of *one* argument.

8. The function k defined by the rule

$$k(x) = x + 1$$

is a function of *one* argument.

'Arguments' and 'values' of a function

'Arguments' and 'values' are used as follows:
Consider the function m defined by the rule

$$m(x, y) = x \cdot y$$

and having for its domain the set of ordered pairs (two-termed sequences) of natural numbers and for its range the set of natural numbers. For the arguments $x = 3$, $y = 4$ this function has the value 12. For the arguments $x = 15$, $y = 0$ this function has the value 0. So the set of *values* of a function is simply the range of the function. An *argument* of a function is a term of a sequence that belongs to the domain of the function. So the set of arguments of a function does not coincide with the domain of the function except in the case of functions of one argument.

A function whose arguments and values are natural numbers is said to be a function from natural numbers to natural numbers.

Definition. f is the same function as g

Just as a set A is the same set as a set B if and only if it has exactly the same members as B, no matter how the members are specified or described, so *a function f is the same function as a function g* if and only if

(1) f and g have the same domain

and

(2) f and g have the same value for the same n-tuple of arguments, for each n-tuple in the domain.

What this amounts to is that a function f and a function g can be the same function even if described in wildly different terms. All that matters for the identity of f and g is (1) the identity of their domains, and (2) the identity of the things picked out by f and g for each member of the domain.

16 Truth functions

We call truth and falsity *truth values* (and in this book we allow nothing else to be a truth value). 'T' denotes the truth value truth; 'F' denotes the truth value falsity.

Definition. A *truth function* is a *function whose domain is a set of sequences of truth values and whose range is a subset of the set of truth values, i.e. the set {T, F}*. Or in other words:
A truth function is a function whose arguments and values are truth values.

Examples:

1. The function q that has as its domain the set of all two-termed sequences whose terms are from the set {T, F} and that has as its range the set {T, F}, and that is defined by the rule

$$\begin{cases} q(\mathrm{T},\mathrm{T}) = \mathrm{T} \\ q(\mathrm{F},\mathrm{T}) = \mathrm{F} \\ q(\mathrm{T},\mathrm{F}) = \mathrm{F} \\ q(\mathrm{F},\mathrm{F}) = \mathrm{F} \end{cases}$$

is a truth function, viz. *conjunction*.

2. The function r that has truth values as arguments and values and is defined by the rule

$$\begin{cases} r(\mathrm{T},\mathrm{T}) = \mathrm{T} \\ r(\mathrm{F},\mathrm{T}) = \mathrm{T} \\ r(\mathrm{T},\mathrm{F}) = \mathrm{F} \\ r(\mathrm{F},\mathrm{F}) = \mathrm{T} \end{cases}$$

is a truth function, viz. *material implication*.

3. The function s that has truth values as arguments and values and is defined by the rule

$$\begin{cases} s(\mathrm{T}) = \mathrm{F} \\ s(\mathrm{F}) = \mathrm{T} \end{cases}$$

is a truth function, viz. *negation*.

4. The function t that has truth values as its arguments and values and is defined by the rule

$$\begin{cases} t(\mathrm{T},\mathrm{T},\mathrm{T}) = \mathrm{T} \\ t(\mathrm{F},\mathrm{T},\mathrm{T}) = \mathrm{F} \\ t(\mathrm{T},\mathrm{F},\mathrm{T}) = \mathrm{F} \\ t(\mathrm{F},\mathrm{F},\mathrm{T}) = \mathrm{F} \\ t(\mathrm{T},\mathrm{T},\mathrm{F}) = \mathrm{F} \\ t(\mathrm{F},\mathrm{T},\mathrm{F}) = \mathrm{F} \\ t(\mathrm{T},\mathrm{F},\mathrm{F}) = \mathrm{F} \\ t(\mathrm{F},\mathrm{F},\mathrm{F}) = \mathrm{F} \end{cases}$$

is a truth function, viz. conjunction again, but this time the conjunction of *three* items.

The central point to grasp about truth functions is that they are relations between *sequences of truth values* and *truth values*. *Nothing else matters.* So, for instance, truth-functional relations between propositions are relations *simply between the truth values of the propositions concerned.* The meanings of the propositions enter in only in this respect, that a proposition has to have some meaning if it is to be true or false. But in order to determine whether a proposition stands in a certain truth-functional relation to another, you do not need to know *what* either proposition means: it is enough to know their truth values (and sometimes you do not even need to know what their truth values are: it may be enough to know *that* they have truth values, i.e. that they are propositions, in the sense defined).

Definition. A *truth-functional propositional connective* is a meaningful expression or symbol that can be combined with propositions (or formulas) to form propositions (or formulas) and that can be completely defined by a complete standard truth table (i.e. one in which for each row of the table the final column has a single definite truth value).[1]

Example: The symbol '$\supset$' is a truth-functional propositional connective. It can be completely defined by the following truth table:

A	B	$A \supset B$
T	T	T
F	T	T
T	F	F
F	F	T

(The order in which the four rows are written is unimportant.)

Each truth-functional propositional connective corresponds to a unique truth function, in the way now illustrated:

To the connective '$\supset$' there corresponds the function

$$\begin{cases} q(T, T) = T \\ q(F, T) = T \\ q(T, F) = F \\ q(F, F) = T \end{cases}$$

i.e. material implication.

(The order in which the four rows are written is unimportant.)

A monadic connective is a connective that combines with *one* proposition or formula to form a new one; a dyadic, or binary, connective combines with *two* propositions (formulas) to form a new one; and so on. '$\sim$' is a monadic connective; '$\supset$' and '$\wedge$' and '$\vee$' are dyadic (binary) connectives.[2] There are no familiar triadic connectives, since (as we shall show) everything that can be expressed by means of triadic or more complicated connectives can be expressed using only dyadic connectives.

There are $2^2 = 4$ total[3] truth functions of one argument, viz.

$$\begin{cases} f_1(T) = T \\ f_1(F) = T \end{cases} \quad \text{[No name]}$$

[1] See exercise 2 at the end of the section for examples of connectives that can and connectives that cannot be so defined.

[2] '$\wedge$' is a symbol for conjunction, '$\vee$' for [inclusive] disjunction.

[3] A total truth function of n arguments is a truth function whose domain is the set of all n-termed sequences whose terms are from the set $\{T, F\}$. In what follows 'truth function' is to be understood as short for 'total truth function'.

$$\begin{cases} f_2(T) = T \\ f_2(F) = F \end{cases} \quad \text{[Identity]}$$

$$\begin{cases} f_3(T) = F \\ f_3(F) = T \end{cases} \quad \text{[Negation]}$$

$$\begin{cases} f_4(T) = F \\ f_4(F) = F \end{cases} \quad \text{[No name]}$$

There are $(2^2)^2 = 16$ truth functions of two arguments, e.g.

$$\begin{cases} g_1(T, T) = T \\ g_1(F, T) = T \\ g_1(T, F) = T \\ g_1(F, F) = T \end{cases}$$

$$\begin{cases} g_2(T, T) = T \\ g_2(F, T) = T \\ g_2(T, F) = T \\ g_2(F, F) = F \end{cases} \quad \text{etc.}$$

There are $((2^2)^2)^2 = 16^2 = 256$ truth functions of three arguments. There are $2^{(2^m)}$ truth functions of m arguments.

For each positive integer n there is a finite number of distinct truth functions of n arguments. So the set of all truth functions is denumerable.

Do we need denumerably many connectives to express the denumerably many truth functions? We shall see later that the answer is 'No'. They can all be expressed by means of a single dyadic connective.

EXERCISES

1. Give a rule defining the truth function of three arguments that has the value truth when the second argument has the value truth, and the value falsity otherwise.

2. Which of the following are (in their context) truth-functional propositional connectives?
   - (*a*) 'It is not the case that' in the sentence 'It is not the case that Napoleon won the Battle of Waterloo'.
   - (*b*) 'And' in '$2 + 2 = 4$ and Napoleon won the Battle of Waterloo'.
   - (*c*) 'And then' in 'He took off his clothes and then he jumped into the water'.

M C

(*d*) 'Hunter believes that' in 'Hunter believes that Napoleon won the Battle of Waterloo'.
(*e*) 'If' in 'If Tom marries Mary, Susan will be unhappy'.
(*f*) 'Either . . . or . . .' in '2 + 2 = 4, so either 2 + 2 = 4 or there is life on Mars'.
(*g*) 'Either . . . or . . .' in 'Either he caught the bus or he had to walk'.
(*h*) 'If' in 'If he's a millionaire, I'm a Dutchman'.

ANSWERS

1. Let the function be g. Then the rule is

$$\begin{cases} g(T, T, T) = T \\ g(F, T, T) = T \\ g(T, F, T) = F \\ g(F, F, T) = F \\ g(T, T, F) = T \\ g(F, T, F) = T \\ g(T, F, F) = F \\ g(F, F, F) = F \end{cases}$$

2. The connectives in (*a*), (*b*) and (*f*) are truth-functional propositional connectives. The others are not.

(*a*) 'It is not the case that' can be completely defined by the truth table

A	It is not the case that A
T	F
F	T

(*b*) This 'and' can be completely defined by the truth table

A	B	A and B
T	T	T
F	T	F
T	F	F
F	F	F

(*c*) If we try to construct a truth table for 'and then' we find that the truth value for one row is left undetermined:

A	B	A and then B
T	T	?
F	T	F
T	F	F
F	F	F

(*d*) Here the values for both rows are left undetermined:

A	Hunter believes that A
T	?
F	?

(*e*) Values for three rows left undetermined:

A	B	If A then B
T	T	?
F	T	?
T	F	F
F	F	?

(*f*) This truth-functional (or 'extensional') use of 'either . . . or . . .' can be completely defined by the truth table

A	B	Either A or B
T	T	T
F	T	T
T	F	T
F	F	F

(*g*) This is a non-truth-functional (or 'intensional') use of 'either . . . or . . .'. The whole sentence is equivalent to 'If he did not catch the bus, then he had to walk', and, just as in Case (*e*), the values for three of the rows are left undetermined:

A	B	Either A or B	= If not A, then B
T	T	?	?
F	T	?	?
T	F	?	?
F	F	F	F

See further Strawson (1952, p. 90).

(*h*) As for (*e*). Cf. Strawson (1952, p. 89).

17 A formal language for truth-functional propositional logic: the formal language P

We define now a formal language that on its intended interpretation will be capable of expressing truths of truth-functional propositional logic. But our definition will make no essential reference to this or any interpretation. The terms 'propositional symbol', 'connective', 'bracket' used in describing the language *are therefore to be taken purely as handy labels*, devised with an eye on the intended interpretation, certainly, but replaceable in the context by arbitrary conglomerations of letters, such as 'schlumpf', 'snodgrass' or 'zbolg'. We shall call the language 'P' (for 'propositional logic').

The formal language P

Symbols of P

P has exactly six symbols, viz.:

p
$'$
$\sim$
$\supset$
$($
$)$

Names for these symbols:

The symbol p
The dash
The tilde
The hook
The left-hand bracket
The right-hand bracket

We shall call the tilde and the hook the *connectives* of P.

We shall say that the symbol p followed by one or more

dashes is a *propositional symbol* of P. So each of the following is a propositional symbol of P:

$$p'$$
$$p''$$
$$p'''$$
$$p''''$$

Formulas (wffs) of P

1. Any propositional symbol is a wff of P.
2. If A is a wff of P, then the string of symbols of P consisting of the tilde followed by the formula A is a wff of P. (We abbreviate this to: If A is a wff of P, then $\sim$A is a wff of P.)
3. If A and B are wffs of P, then the string of symbols of P consisting of the left-hand bracket, the formula A, the hook, the formula B, and the right-hand bracket, in that order, is a wff of P. (Abbreviated to: If A and B are wffs of P, then (A $\supset$ B) is a wff of P.)
4. Nothing else is a wff of P.

[In this description the letters 'A' and 'B' are metalinguistic variables.]

Examples: The following are wffs of P:

$$p'''''$$
$$\sim p''$$
$$(p''' \supset p')$$
$$\sim(p''' \supset p')$$
$$(\sim\sim(p''' \supset p') \supset \sim p'')$$

The following are *not* wffs of P:

p	[No dash]
$\sim(p'')$	[Superfluous brackets]
$(\sim p'')$	[Superfluous brackets]
$p''' \supset p'$	[No brackets]
q''	['q' is not a symbol of P]
$(p \supset q)$	[Obvious]
$(A \supset B)$	['A' and 'B' are not symbols of P]