Some Remarks and Extra-Credit Exercises Concerning the Deductive System of Hiż

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Hiż's system (H) for propositional logic consists of the following axiom and inference rule *schemata* for P:

Axiom schemata for H:

(HA1) $\sim(A \supset B) \supset A$

(HA2) $\sim(A \supset B) \supset \sim B$

Inference Rule schemata for H:

(HR1) From $\vdash_H A \supset B$ and $\vdash_H B \supset C$, infer $\vdash_H A \supset C$.

(HR2) From $\vdash_H A \supset (B \supset C)$ and $\vdash_H A \supset B$, infer $\vdash_H A \supset C$.

(HR3) From $\vdash_H \sim A \supset B$ and $\vdash_H \sim A \supset \sim B$, infer $\vdash_H A$.

Note: the premises of these rules must all be *theorems* of H! This is different than MP in PS!

Some interesting facts about H, and some extra-credit exercises concerning H:

1. **H *is weakly* semantically sound**. For all formulae A of P, if $\vdash_H A$, then $\vDash_P A$. Prove this!
2. **Question: Is H *strongly* semantically sound**? That is, are there sets of formulae Γ and formulae A of P such that $\Gamma \vdash_H A$, but $\Gamma \nvDash_P A$? Settle this question with a proof!
3. **H *is weakly* semantically complete**. For all formulae A of P, if $\vDash_P A$, then $\vdash_H A$. Note: One cannot use Henkin's method to prove this! This proof is worth a *serious* amount of extra credit![1] You may *assume* (3) for the other exercises below. [Note: (1) and (3) imply: $\vdash_H A \Leftrightarrow \vDash_P A$ ($\Leftrightarrow \vdash_{PS} A$).]
4. **H is *not strongly* semantically complete**. That is, there are sets of formulae Γ and formulae A of P such that $\Gamma \vDash_P A$, but $\Gamma \nvdash_H A$. Why? One way to see why is to note that:
5. **Modus Ponens is *not* (in at least one sense) an inference rule schemata of H**. That is, there exist formulae A and B of P such that $\{A \supset B, A\} \nvdash_H B$. Exercise: prove this! [Hint: choose a B that is not a tautology!] If you can prove this, you'll nearly have a proof of (4) as well. Do you see why?
6. **The deduction theorem *fails* for H**. That is to say, there exist sets of formulae Γ of P and formulae A and B of P such that $\Gamma \cup \{A\} \vdash_H B$, but $\Gamma \nvdash_H A \supset B$. Exercise: prove this!
7. **Question: Does the interpolation theorem hold for H**? That is, if A and B are formulae of P such that (*i*) they share at least one propositional symbol in common, and (*ii*) $\vdash_H A \supset B$, then must there exist a formula C of P such that (*iii*) $\vdash_H A \supset C$, (*iv*) $\vdash_H C \supset B$, and (*v*) C has only propositional symbols that are shared by A and B? Like (3), this one is also a very difficult problem!
8. **H *is* syntactically complete**. That is, **one may *not* add an unprovable schema to H without rendering it *simply* inconsistent**. In this way, H is similar to PS. This follows from (3). [33.1]
9. **One *may* add an unprovable schema to H *without* rendering it *absolutely* inconsistent**. In this way, H is *dis*similar to PS. See the back of this handout for some insight into this problem.
10. **Proofs of (PS1)–(PS3) in H**. Here is a proof sketch of (PS1) and (PS3). Exercise: fill-in this sketch!

    1. $\vdash_H (\sim A \supset (B \supset C)) \supset \sim C$ [HR1, HA2, HA2]
    2. $\vdash_H (\sim A \supset (B \supset C)) \supset B$ [HR1, HA2, HA1]
    3. $\vdash_H \boldsymbol{A \supset (B \supset A)}$ **[HR3, 1, HA1]**
    4. $\vdash_H (\sim(\sim A \supset B) \supset (C \supset A)) \supset B$ [HR2, 1, HA1]
    5. $\vdash_H \boldsymbol{(\sim A \supset \sim B) \supset (B \supset A)}$ **[HR3, 4, 2]**

    Exercise #2: give a proof of (PS2) in H. This is also a very difficult problem!

[1] Feel free to read Hiż's paper, and to explain his weak completeness proof if you can. It's linked from the course website.

Independence proofs for the axiom schemata and rule schemata of H

1. **Axiom (HA1) is independent of axiom (HA2) in H**. Consider the following class of models $\mathcal{M}$:

   The set of *values* is $V = \{0, 1, 2\}$. The set of *designated values* is $D = \{0, 2\}$.

   The table for $\sim$ is:

A	0	1	2
$\sim A$	2	2	0

   . The table for $\supset$ is:

$\supset$	0	1	2
0	0	0	2
1	0	0	2
2	1	1	2

   In this class of models $\mathcal{M}$, (*i*) axiom (HA2) is valid, (*ii*) axiom (HA1) is *not* valid, and (*iii*) rules (HR1), (HR2), and (HR3) all preserve validity-in-$\mathcal{M}$. Exercise: prove these claims about $\mathcal{M}$! I'll do (*ii*) for you. The following single row of the "truth-table" of (HA1) in $\mathcal{M}$ suffices. Why?

A	B	$\sim$	$(A$	$\supset$	$B)$	$\supset$	A
0	0	2	0	0	0	**1**	0

2. **Axiom (HA2) is independent of axiom (HA1) in H**. Exercise: Give a class of models that shows this. That is, give a class of models in which (*i*) axiom (HA1) is valid, (*ii*) axiom (HA2) is *not* valid, and (*iii*) rules (HR1), (HR2), and (HR3) all preserve validity-in-$\mathcal{M}$. [You'll need at least 3 values.]

3. **Rule (HR1) is independent of the rest of H**. To prove this, we need a class of models in which (*i*) axiom (HA1) is valid, (*ii*) axiom (HA2) is valid, (*iii*) rules (HR2) and (HR3) are both validity preserving, but (*iv*) (HR1) is *not* validity preserving. Here's one such class of models:

   The set of *values* is $V = \{0, 1, 2, 3\}$. The set of *designated values* is $D = \{0, 1, 2\}$.

   The table for $\sim$ is:

A	0	1	2	3
$\sim A$	2	2	0	1

   . The table for $\supset$ is:

$\supset$	0	1	2	3
0	0	0	2	2
1	0	0	3	3
2	0	0	0	0
3	0	0	0	0

   Indeed, this one requires at least 4 values. Exercise: verify that (*i*)–(*iv*) hold in this class.

4. **Rule (HR2) is independent of the rest of H**. To prove this, we need a class of models $\mathcal{M}$ in which (*i*) axiom (HA1) is valid, (*ii*) axiom (HA2) is valid, (*iii*) rules (HR1) and (HR3) are both validity preserving, but (*iv*) (HR2) is *not* validity preserving. Exercise: give such an $\mathcal{M}$! [You'll need 4 values.]

5. **Rule (HR3) is independent of the rest of H**. To prove this, we need a class of models $\mathcal{M}$ in which (*i*) axiom (HA1) is valid, (*ii*) axiom (HA2) is valid, (*iii*) rules (HR1) and (HR2) are both validity preserving, but (*iv*) (HR3) is *not* validity preserving. Exercise: give such an $\mathcal{M}$! [You'll need 2 values.]

Proving that Modus Ponens is independent of H

To prove that MP is independent of H, you just need to give a class of models in which (*i*) axiom (HA1) is valid, (*ii*) axiom (HA2) is valid, (*iii*) rules (HR1), (HR2), and (HR3) are all validity preserving, but (*iv*) MP is *not* validity preserving. [You'll need at least 3 values.] Question: How does the existence of such classes of models bear on question (5) on the front of this handout?

An interesting class of models for H. Let $\mathcal{M}$ be the following class of models:

The set of *values* is $V = \{0, 1, 2\}$. The set of *designated values* is $D = \{1, 2\}$.

The table for $\sim$ is:

A	0	1	2
$\sim A$	2	2	1

. The table for $\supset$ is:

$\supset$	0	1	2
0	2	2	2
1	2	2	2
2	0	0	2

In $\mathcal{M}$, (*i*) axiom (HA1) is valid, (*ii*) axiom (HA2) is valid, (*iii*) rules (HR1), (HR2), and (HR3) are all validity preserving, (*iv*) the schema $\sim A$ is valid, but (*v*) the schema A is not valid. Prove this! Question: How does the existence of such an $\mathcal{M}$ bear on question (9) on the front of this handout?