Branden Fitelson Philosophy 12A Notes

Announcements & Such

- Black Roots.
- Administrative Stuff
 - HW #4 resubs are still being graded. Stay tuned...
 - HW #6 is due Thurs. Final HW assignment! LMPL Proofs.
 - Next week, I *will* be holding lectures. I will use them for both review, and for some interesting "logic beyond 12A" topics.
- Today: Chapter 6 Natural Deductions in LMPL
 - Introduction and Elimination rules for the quantifiers.
 - Sequents and Theorems (SI/TI) for the quantifiers.
 - Lots of proofs in LMPL!
- Next: Two-Place predicates (i.e., binary relations) "L2PL".

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The Rule of ∃-Elimination: Some Background

- It is useful to think of an existential claim $\lceil (\exists v) \phi v \rceil$ as a *disjunction* which asserts that the predicate expression ϕ is satisfied by *at least one* object in the domain (*i.e.*, that the disjunction $\lceil \phi a \lor (\phi b \lor (\phi c \lor ...)) \rceil$ is true).
- In this way, we would expect the elimination rule for \exists to be similar to the elimination rule for \lor . That is, we'd expect the \exists E rule to be similar to the \lor E rule. Indeed, this is the case. It's best to start with a simple example.
- Consider the following *legitimate* elimination of an existential claim:

Problem is: $(\exists x)(Fx\&Gx) + (\exists x)Fx$

(1) (∃x)(Fx&Gx)	Premise
(2) Fa&Ga	Assumption
(3) Fa	2 &E
(4) (3x)Fx	3 3I
(5) (3x)Fx	1,2,4 ∃E
	(2) Fa&Ga (3) Fa (4) (3x)Fx

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The Rule of ∃-Elimination: II

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- To derive a sentence \mathscr{P} using the $\exists E$ rule (with some existential sentence $\lceil (\exists v) \phi v \rceil$), we must first *assume* an *instance* $\phi \tau$ of $\lceil (\exists v) \phi v \rceil$.
- If we can deduce \mathscr{P} from this assumed instance $\phi \tau using$ *generalizable reasoning* then we may infer \mathscr{P} *outright*.
- It is because our reasoning from the *instance* $\phi \tau$ of $\lceil (\exists v) \phi v \rceil$ to \mathscr{P} does not depend on our choice of constant τ (i.e., that our reasoning from $\phi \tau$ to \mathscr{P} is *generalizable*) that makes this inference valid.
- When our reasoning is generalizable in this sense, it's as if we are showing that \mathscr{P} can be deduced from *any* instance $\phi\tau$ of $\lceil (\exists v)\phi v \rceil$.
- As such, this is just like showing that \mathscr{P} can be deduced from *any disjunct* of the disjunction ${}^{r}\phi a \vee (\phi b \vee (\phi c \vee \ldots))^{\neg}$. And, this is just like $\vee E$ reasoning (except that $\exists E$ only requires *one* assumption).

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The Rule of ∃-Elimination: III

• Here's an *il*legitimate "∃-Elimination" step:

1 (1) (3x)Fx Premise
2 (2) Ga Premise
3 (3) Fa Assumption
2,3 (4) Fa&Ga 2,3 &I
2,3 (5) (3x)(Fx&Gx) 4 3I
1,2 (6) (3x)(Fx&Gx) 1,3,5 3E NO!!

- This is *not* a valid inference: $(\exists x)Fx$, $Ga \not\models (\exists x)(Fx \& Gx)!$
- So, what went wrong here? The problem is that the inference to $(\exists x)(Fx \& Gx)$ ' at line (5) does *not* use *generalizable* reasoning.
- We can *not* legitimately infer ' $(\exists x)(Fx \& Gx)$ ' at line (5) from an *arbitrary instance* $\ulcorner F\tau \urcorner$ of ' $(\exists x)Fx$ '. We *must* assume 'Fa' in *particular* at line (3) in order to deduce ' $(\exists x)(Fx \& Gx)$ ' at line (5).

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The Rule of ∃-Elimination: Official Definition

 \exists -Elimination: If $\lceil (\exists v) \phi v \rceil$ occurs at i depending on a_1, \ldots, a_n , an instance $\phi \tau$ of $\lceil (\exists v) \phi v \rceil$ is *assumed* at j, and \mathscr{P} is inferred at k depending on b_1, \ldots, b_u , then at line m we may infer \mathscr{P} , with label 'i, j, k \exists E' and dependencies $\{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_u\}/j$:

Provided that *all four* of the following conditions are met:

- τ (in $\phi \tau$) replaces *every* occurrence of ν in $\phi \nu$. [avoids fallacies]
- τ *does not occur in* $(\exists v) \phi v$. [generalizability]
- τ *does not occur in* \mathscr{P} . [generalizability]
- τ *does not occur in any* of b_1, \ldots, b_u , except (possibly) $\phi \tau$ itself. [generalizability]

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The Rule of ∃-Elimination: Nine Examples

• Here are 9 examples of proofs involving all four quantifier rules.

1. $(\exists x) \sim Fx \vdash \sim (\forall x) Fx$

[p. 200, example 5]

2. $(\exists x)(Fx \rightarrow A) \vdash (\forall x)Fx \rightarrow A$

[p. 201, example 6]

3. $(\forall x)(\forall y)(Gy \rightarrow Fx) \vdash (\forall x)[(\exists y)Gy \rightarrow Fx]$

 $[p. 203, I. # 19 \Rightarrow]$

4. $(\exists x)[Fx \to (\forall y)Gy] \vdash (\exists x)(\forall y)(Fx \to Gy)$

[p. 203, I. # 20 \Leftarrow]

5. $A \vee (\exists x) Fx \vdash (\exists x) (A \vee Fx)$

[p. 203, II. # 2 \Leftarrow]

6. $(\exists x)(Fx \& \sim Fx) \vdash (\forall x)(Gx \& \sim Gx)$

 $[p. 203, I. # 12 \Rightarrow]$

7. $(\forall x)[Fx \rightarrow (\forall y) \sim Fy] \vdash \sim (\exists x)Fx$

[p. 203, I. # 5]

8. $(\forall x)(\exists y)(Fx \& Gy) \vdash (\exists y)(\forall x)(Fx \& Gy)$

[p. 201, example 7]

9. $(\exists y)(\forall x)(Fx \& Gy) \vdash (\forall x)(\exists y)(Fx \& Gy)$

[other direction]

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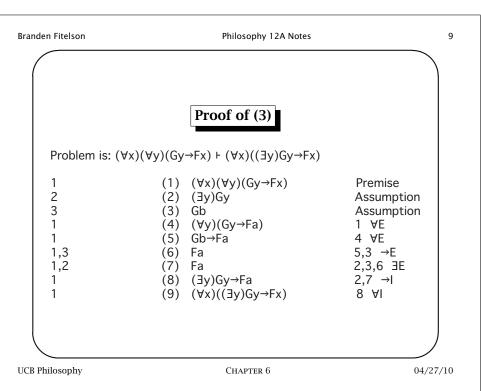
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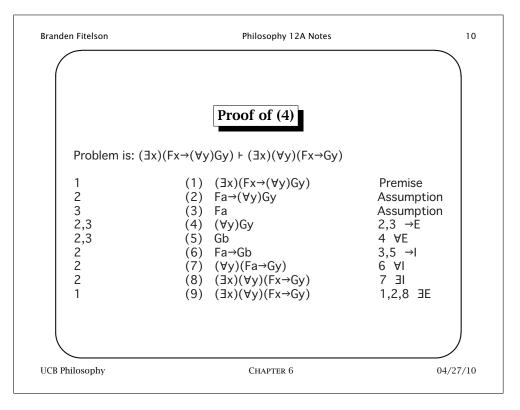
Proof of (1)

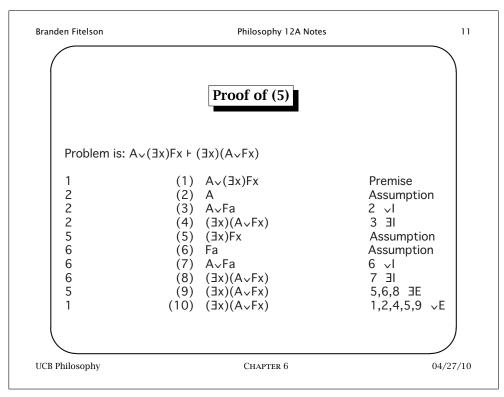
Problem is: $(\exists x) \sim Fx \vdash \sim (\forall x)Fx$

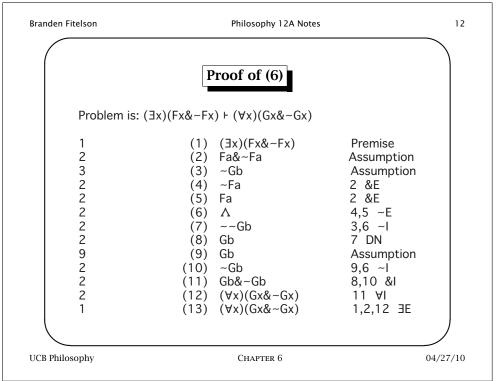
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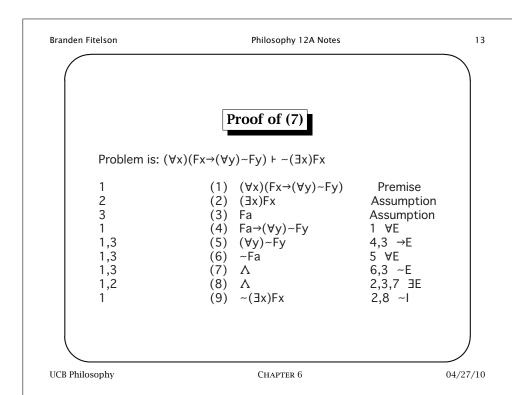
Proof of (2) Problem is: $(\exists x)(Fx \rightarrow A) \vdash (\forall x)Fx \rightarrow A$ Premise $(1) (\exists x)(Fx \rightarrow A)$ 2 (∀x)Fx Assumption 3 (3) Fa→A Assumption 2 (4) Fa 2 AE 3.4 →E (5) A (6) A 1.3.5 3E $(7) (\forall x)Fx \rightarrow A$ 2,6 → UCB Philosophy CHAPTER 6 04/27/10

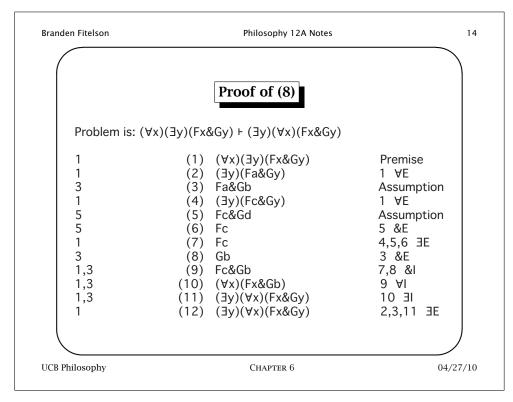












Branden Fitelson Philosophy 12A Notes 15 Proof of (9) Problem is: $(\exists y)(\forall x)(Fx\&Gy) + (\forall x)(\exists y)(Fx\&Gy)$ $(1) (\exists y)(\forall x)(Fx\&Gy)$ Premise 2 $(2) (\forall x)(Fx\&Gb)$ Assumption 2 (3) Fa&Gb 2 ∀E IE 8 $(4) (\exists y)(Fa\&Gy)$ (5) $(\exists y)(Fa\&Gy)$ 1,2,4 3E (6) $(\forall x)(\exists y)(Fx\&Gy)$ 5 ¥I UCB Philosophy CHAPTER 6 04/27/10

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Two LMPL Extensions of Sequent Introduction

- Here are two additions to our list of SI sequents:
- (QS) One can infer $\lceil (\forall x) \sim \phi x \rceil$ from (the *logically equivalent* sentence) $\lceil \sim (\exists x) \phi x \rceil$, and *vice versa*; and, that one can infer $\lceil (\exists x) \sim \phi x \rceil$ from (the *logically equivalent*) $\lceil \sim (\forall x) \phi x \rceil$, and *vice versa*.

$$(\forall x) \sim \phi x \dashv \vdash \sim (\exists x) \phi x; \text{ and, } (\exists x) \sim \phi x \dashv \vdash \sim (\forall x) \phi x \tag{QS}$$

(AV) One can infer a *closed* LMPL sentence ψ from (the *logically equivalent* sentence) ψ' , and *vice versa*, where ψ and ψ' are *alphabetic variants*. Two formulas are *alphabetic variants* if and only if they differ *only* in a (conventional) choice of individual *variable* letters (*not* kosher for constants!). *E.g.*, ' $(\forall x)Fx$ ' and ' $(\forall y)Fy$ ' are (closed) *alphabetic variants*, because they differ *only* in which individual variable ('x' or 'y') is used, but they have the same *logical* (*i.e.*, *syntactical*) *structure*.

$$\psi \dashv \vdash \psi'$$
 (AV)

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Our (New) Official List of Sequents and Theorems (see pp. 123, 204, and 206)

$$A \lor B, \sim A \vdash B; \text{ or; } A \lor B, \sim B \vdash A \quad \text{ (DS)} \qquad \qquad A \to B \dashv \vdash \sim A \lor B \quad \text{ (Imp)}$$

$$A \to B, \sim B \vdash \sim A \quad \text{ (MT)} \qquad \sim (A \to B) \dashv \vdash A \& \sim B \quad \text{ (Neg-Imp)}$$

$$A \vdash B \to A \quad \text{ (PMI)} \qquad A \& (B \lor C) \dashv \vdash (A \& B) \lor (A \& C) \quad \text{ (Dist)}$$

$$\sim A \vdash A \to B \quad \text{ (PMI)} \qquad A \lor (B \& C) \dashv \vdash (A \lor B) \& (A \lor C) \quad \text{ (Dist)}$$

$$A \vdash \sim \sim A \quad \text{ (DN^+)} \qquad \land \vdash A \qquad \text{ (EFQ, or } \land E)$$

$$\sim (A \& B) \dashv \vdash \sim A \lor \sim B \quad \text{ (DEM)} \qquad A \ast B \vdash B \ast A \quad \text{ (Com)}$$

$$\sim (A \lor B) \dashv \vdash \sim A \& \sim B \quad \text{ (DEM)} \qquad A \ast B \dashv \vdash A \ast B \quad \text{ (SDN)}$$

$$\sim (\sim A \lor \sim B) \dashv \vdash A \lor B \quad \text{ (DEM)} \qquad A \ast B \dashv \vdash \sim \sim A \ast B \dashv \vdash A \ast \sim \sim B \quad \text{ (SDN)}$$

$$\sim (\sim A \& \sim B) \dashv \vdash A \lor B \quad \text{ (DEM)} \qquad A \ast B \dashv \vdash \sim \sim A \ast B \dashv \vdash A \ast \sim \sim B \quad \text{ (LEM)}$$

$$(\forall X) \sim \phi X \dashv \vdash \sim (\exists X) \phi X \quad \text{ (QS)} \qquad (\exists X) \sim \phi X \dashv \vdash \sim (\forall X) \phi X \quad \text{ (QS)}$$

In (Com), '*' can be any binary connective *except* ' \rightarrow '. In (SDN), '*' can be *any* binary connective. In (AV), ψ must be *closed*, and ψ' must be an *alphabetic variant* of ψ .

 $\psi \dashv \vdash \psi'$

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The Value of (QS) — Its Four Simplest Instances

	(∀	'x)~Fx + ~(∃	x)Fx		~(x∀) + x7(xE))~Fx
1 2		(∀x)~Fx (xE)	Premise Ass	1 2	(1) (2)	~(∃x)Fx Fa	Premise Ass
3	(3)	Fa	Ass	2	(3)	x∃(xE)	2 JI
1	(4)	~Fa	1 ∀E	1,2	(4)	Λ	1,3 ~E
1,3	(5)	Λ	4,3 ~E	1	(5)	~Fa	2,4 ~I
1,2	(6) (7)	Λ ~(∃x)Fx	2,3,5 ∃E 2,6 ~I	1	(6)	(∀x)~Fx	5 AI
1 '	(.,	(=11). 11	_, .				

	(∃x)~Fx ⊦ ~(∀	/x)Fx	~(∀x)Fx + (∃x)~Fx
1 2 3 2 2,3 1,2	(1) (∃x)~Fx (2) (∀x)Fx (3) ~Fa (4) Fa (5) Λ (6) Λ (7) ~(∀x)Fx	Premise Ass Ass 2 ∀E 3,4 ~E 1,3,5 ∃E 2,6 ~I	1 (1) -(\forall x)Fx Premise 2 (2) -(\forall x)-Fx Ass 3 (3) -Fa Ass 3 (4) (\forall x)-Fx 3 3 II 2,3 (5) \(\Lambda -Fa \) 3,5 -I 2 (7) Fa 6 DN 2 (8) (\forall x)Fx 7 \(\text{VI} \) 1,2 (9) \(\Lambda \) 1,8 -E 1 (10)(\forall x)-Fx 2,9 -I 1 (11) (\forall x)-Fx 10 DN

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Three Examples Involving the LMPL SI Extension (QS)

- Here are three examples of proofs involving SI (QS):
 - 1. $\sim (\forall x) \sim Fx \vdash (\exists x) Fx$

 $[p, 207, #7 \Leftarrow]$

(AV)

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2. $\sim (\exists x)(Fx \& Gx) \lor (\exists x) \sim Gx, (\forall y)Gy \vdash (\forall z)(Fz \rightarrow \sim Gz) [p. 205, ex. 1]$

3. $(\forall x)Fx \rightarrow A \vdash (\exists x)(Fx \rightarrow A)$ [p. 205, ex. 2]

Proof of (1) (1) $\sim (\forall x) \sim Fx$ Premise (2) $\sim (\exists x) Fx$ Assumption (3) $(\forall x) \sim Fx$ 2 SI (QS) 1,2 (4) 人 1, 3 ∼E (5) $\sim \sim (\exists x) F x$ 2, 4 \sim I (6) $(\exists x)Fx$ 5 DN UCB Philosophy CHAPTER 6 04/27/10

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Proof of (2)

1	(1)	$\sim (\exists x)(Fx \& Gx) \lor (\exists x) \sim Gx$	Premise
2	(2)	$(\forall y)Gy$	Premise
3	(3)	$\sim (\exists x)(Fx \& Gx)$	Assumption
3	(4)	$(\forall x) \sim (Fx \& Gx)$	3 SI (QS)
3	(5)	\sim (Fa & Ga)	4 ∀E
3	(6)	$\sim Fa \vee \sim Ga$	5 SI (DeM)
3	(7)	$Fa \rightarrow \sim Ga$	6 SI (Imp)
3	(8)	$(\forall z)(Fz \rightarrow \sim Gz)$	7 ∀I
9	(9)	$(\exists x) \sim Gx$	Assumption
10	(10)	$\sim Ga$	Assumption
2	(11)	Ga	2 ∀E
2,10	(12)	A	10, 11 ∼E
2,10	(13)	$(\forall z)(Fz \rightarrow \sim Gz)$	12 SI (EFQ)
2,9	(14)	$(\forall z)(Fz \rightarrow \sim Gz)$	9, 10, 13 ∃E
1,2	(15)	$(\forall z)(Fz \rightarrow \sim Gz)$	$1, 3, 8, 9, 14 \lor E$

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The Value of (AV)

• Here are the two simplest instances of (AV):

	(∀x)Fx ⊦ (∀ <u>;</u>	y)Fy		(∃x)Fx ⊦ (∃	ly)Fy
1	(1) (∀x)Fx (2) Fa	Premise 1 ∀E	1 2	(1) (∃x)Fx (2) Fa	Premise Ass
1	(3) (∀y)Fy	2 ¥I	2	(3) (3y)Fy	2 3I
				(4) (∃y)Fy	1,2,3 ∃E

• Here's an (AV)-aided proof of the following sequent

$$(\forall x)Fx, (\forall y)Fy \rightarrow (\forall y)Gy \vdash (\forall z)Gz$$

1	(1)	$(\forall x)Fx$	Premise
2	(2)	$(\forall y) Fy \to (\forall y) Gy$	Premise
1	(3)	$(\forall y)Fy$	1 SI (AV)
1,2	(4)	$(\forall y)Gy$	2,3 →E
1,2	(5)	$(\forall z)Gz$	4 SI (AV)

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	Proof of (3)	
Problem is: ($(\forall x) Fx \rightarrow A \vdash (\exists x) (Fx \rightarrow A)$	
1 1 3 3 5 5 5 3 9 9	(1) (∀x)Fx→A (2) ~(∀x)Fx∨A (3) ~(∀x)Fx (4) (∃x)~Fx (5) ~Fa (6) Fa→A (7) (∃x)(Fx→A) (8) (∃x)(Fx→A) (9) A (10) Fa→A (11) (∃x)(Fx→A) (12) (∃x)(Fx→A)	Premise 1 SI (Imp) Assumption 3 SI (QS) Assumption 5 SI (PMI) 6 3I 4,5,7 3E Assumption 9 SI (PMI) 10 3I 2,3,8,9,11 VE
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