Branden Fitelson

Philosophy 12A Notes

Announcements & Such

- · Steel Pulse.
- Administrative Stuff
 - HW #4 resubs are still being graded. Stay tuned...
 - HW #5 resubmission is due today (follow models on handout).
 - HW #6 is posted. Final HW assignment! LMPL Proofs.
 - From now on, my office hours are: 4-6pm Tuesdays.
- Today: Chapter 6 Natural Deductions in LMPL
 - Introduction and Elimination rules for the quantifiers.
 - Sequents and Theorems (SI/TI) for the quantifiers.
 - Lots of proofs in LMPL!
- **Next**: Two-Place predicates (*i.e.*, *binary relations*) "L2PL".

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The Rule of ∃-Introduction

Rule of \exists **-Introduction**: For any sentence $\phi \tau$, if $\phi \tau$ has been inferred at line j in a proof, then at line k we may infer $(\exists v) \phi v$, labeling the line j ∃I' and writing on its left the numbers that occur on the left of j.

$$a_1, \dots, a_n$$
 (j) $\phi \tau$

$$\vdots$$

$$a_1, \dots, a_n$$
 (k) $(\exists v) \phi v$ j $\exists I$

Where $\lceil (\exists v) \phi v \rceil$ is obtained syntactically from $\phi \tau$ by:

- Replacing *one or more occurrences* of τ in $\phi \tau$ by a *single* variable ν .
- Note: the variable ν must not already occur in the expression $\phi \tau$. [This prevents double-binding, e.g., $(\exists x)(\exists x)(\exists x)(Fx \& Gx)$ '.]
- And, finally, prefixing the quantifier $(\exists v)$ in front of the resulting expression (which may now have both τ 's and τ 's occurring in it).

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Natural Deduction Proofs in LMPL

- The natural deduction rules for LMPL will *include* the rules for LSL that we already know (viz., Ass., &E, &I, \neg E, \neg I, \sim E, \sim I, DN, \vee E, \vee I, Df.).
- Plus, we will be *adding* 4 new rules. We will need both introduction and elimination rules for each of the two quantifiers ($\exists I, \exists E, \forall I, \forall E$).
- As in LSL, the system will be sound and complete (140A!). That is, \vdash will apply to the same sequents that \models does in our semantics for LMPL.
- We begin with the simplest: the introduction rule for \exists (\exists I). Intuitively, if we have proved $\phi \tau$ for some individual constant τ , then we may infer that ϕ is true of *something* (e.g., that $(\exists x)\phi x$).
- E.g., if we've proved 'Pa & Qa', we may validly infer ' $(\exists x)(Px \& Qx)$ '.
- We may also infer ' $(\exists x)(Pa \& Qx)$ ' and ' $(\exists x)(Px \& Qa)$ ' from 'Pa & Qa'.
- These (and similar) considerations lead us to the ∃I rule ...

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The Rule of ∀-Elimination

Rule of \forall -Elimination: For any sentence $\lceil (\forall v) \phi v \rceil$ and constant τ , if $(\forall v) \phi v$ has been inferred at a line j, then at line k we may infer $\phi \tau$, labeling the line 'j ∀E' and writing on its left the numbers that appear on the left of j.

$$a_1,...,a_n$$
 (j) $(\forall \nu)\phi \nu$

$$\vdots$$

$$a_1,...,a_n$$
 (k) $\phi \tau$ j $\forall E$

Where $\phi \tau$ is obtained syntactically from $(\forall v) \phi v^{\dagger}$ by:

- Deleting the quantifier prefix $(\forall v)$.
- Replacing *every occurrence* of ν in the open sentence $\phi\nu$ by *one and the same* constant τ . [This prevents fallacies, e.g., $(\forall x)(Fx \& Gx) \not\vdash Fa \& Gb$.]
- Note: since '∀' means *everything*, there are *no* restrictions on *which* individual constant may be used in an application of $\forall E$.

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An Example Proof Involving Both ∃I and ∀E

Let's prove that $(\forall x)(Fx \to Gx), Fa \vdash (\exists x)(\sim Gx \to Hx).$

1	(1)	$(\forall x)(Fx\rightarrow Gx)$	Premise
2	(2)	Fa	Premise
3	(3)	~Ga	Assumption
4	(4)	~Ha	Assumption
1	(5)	Fa→Ga	1 ∀E
1,2	(6)	Ga	5,2 →E
1,2,3	(7)	Λ	3,6 ~E
1,2,3	(8)	~~Ha	4,7 ~I
1,2,3	(9)	На	8 DN
1,2	(10)	~Ga→Ha	3,9 →I
1,2	(11)	$(\exists x)(\sim Gx\rightarrow Hx)$	10 JI

• This example illustrates a typical pattern in quantificational proofs: quantifiers are removed from the premises using elimination rules, sentential (*viz.*, LSL) rules are applied, and then quantifiers are reintroduced using introduction rules to obtain the conclusion.

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The Rule of ∀-Introduction: II

- We can legitimately infer ' $(\forall x)Gx$ ' at line 6 of this proof, because our inference to 'Gb' is *generalizable i.e.*, we could have deduced ${}^{r}G\tau$, for *any* individual constant τ using *exactly parallel* reasoning.
- However, consider the following *il*legitimate "∀-Introduction" step:

1	(1)	$(\forall x)(Fx\rightarrow Gx)$	Premise	
2	(2)	Fb	Premise	
1	(3)	Fb→Gb	1 ∀ E	
1,2	(4)	Gb	2,3 →E	
1,2	(5)	(∀x)Gx	4 VI	NO!!

- This is *not* a valid inference, since $(\forall x)(Fx \rightarrow Gx), Fb \not\models (\forall x)Gx!$
- So, what went wrong? The problem is that the inference to 'Gb' at (4) is *not* generalizable. We can *not* deduce " $G\tau$ " for *any* τ from the premises ' $(\forall x)(Fx \rightarrow Gx)$ ' and 'Fb'. We can *only* infer 'Gb'.

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The Rule of ∀-Introduction: Some Background

- It is useful to think of a universal claim $\lceil (\forall v) \phi v \rceil$ as a *conjunction* which asserts that the predicate expression ϕ is satisfied by *all objects* in the domain of discourse (*i.e.*, the conjunction $\lceil \phi a \& (\phi b \& (\phi c \& ...)) \rceil$ is true).
- So, in order to be able to *introduce* the universal quantifier (*i.e.*, to *legitimately infer* $(\forall v)\phi v$ in a proof), we must be in a position to prove $\phi \tau$, for *anv* individual constant τ . This is called *generalizable reasoning*.
- Consider the following *legitimate* introduction of a universal claim:

Problem is: $(\forall x)(Fx \rightarrow Gx)$, $(\forall x)Fx \vdash (\forall x)Gx$

1	(1)	$(\forall x)(Fx\rightarrow Gx)$	Premise
2	(2)	(∀x)Fx	Premise
1	(3)	Fa→Ga	1 ∀E
2	(4)	Fa	2 ∀ E
1,2	(5)	Ga	3,4 →E
1,2	(6)	(∀x)Gx	2 AI

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The Rule of ∀-Introduction: III

Rule of \forall **-Introduction**: For any sentence $\phi\tau$, if $\phi\tau$ has been inferred at a line j, then *provided that* τ *does not occur in any premise or assumption whose line number is on the left at line* j, we may infer $\lceil(\forall v)\phi v\rceil$ at line k, labeling the line 'j \forall l' and writing on its left the same numbers as occur on the left at line j.

$$a_1,..., a_n$$
 (j) $\phi \tau$
 \vdots
 $a_1,..., a_n$ (k) $(\forall v)\phi v$ j $\forall I$

Where $\lceil (\forall v) \phi v \rceil$ is obtained by:

- Replacing *every* occurrence of τ in $\phi \tau$ with ν and prefixing $\lceil (\forall \nu) \rceil$. [Again, 'every' prevents *fallacies*, *e.g.*, $(\forall x)(Fx \to Gx) \not\vdash (\forall x)(\forall y)(Fx \to Gy)$.]
- τ *does not occur in* any of the formulae a_1, \ldots, a_n . [ensures *generalizability*]
- ν *does not occur in* $\phi \tau$. [prevents *double-binding*]

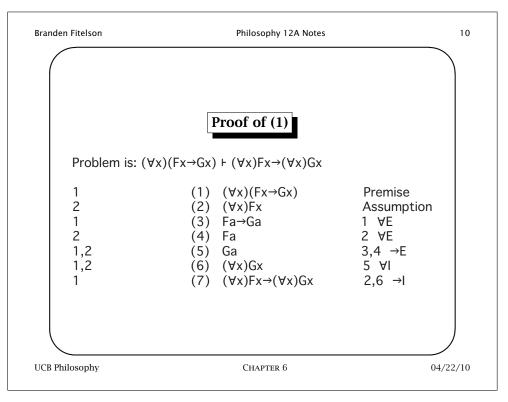
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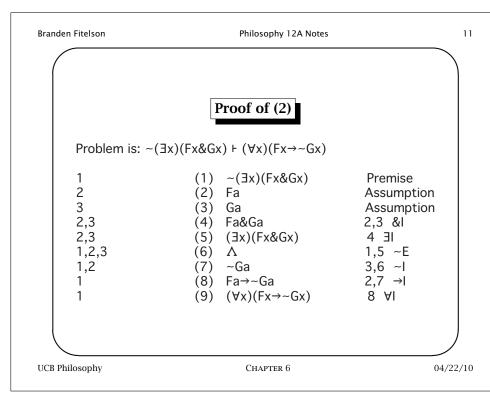
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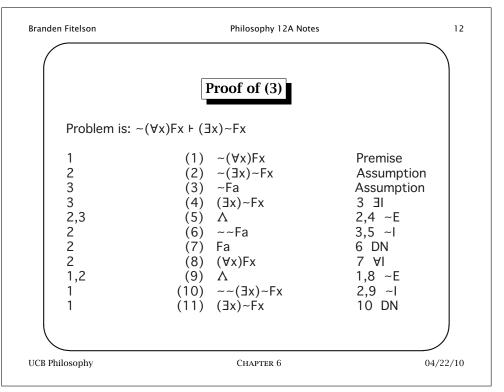
The Rule of ∀-Introduction: Four Examples

- Here are four examples of LMPL sequents involving the three quantifier rules we've learned so far (∃I, ∀E, and ∀I).
- (1) $(\forall x)(Fx \rightarrow Gx) \vdash (\forall x)Fx \rightarrow (\forall x)Gx$
- $(2) \sim (\exists x) (Fx \& Gx) \vdash (\forall x) (Fx \to \sim Gx)$
- (3) $\sim (\forall x)Fx \vdash (\exists x) \sim Fx$
- $(4) \ (\forall x)[Fx \to (\forall y)Gy] \vdash (\forall x)(\forall y)(Fx \to Gy)$

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Proof of (4)

Problem is: $(\forall x)(\mathsf{F}x \rightarrow (\forall y)\mathsf{G}y) \vdash (\forall x)(\forall y)(\mathsf{F}x \rightarrow \mathsf{G}y)$

(1) $(\forall x)(Fx \rightarrow (\forall v)Gv)$ Premise 2 (2) Fa Assumption (3) $Fa \rightarrow (\forall y)Gy$ 1 ∀E 1,2 3,2 →E (4) (∀y)Gy 1,2 (5) Gb 4 ∀E 2.5 → (6) Fa→Gb (7) (∀y)(Fa→Gy) 6 AI (8) $(\forall x)(\forall y)(Fx \rightarrow Gy)$ 7 VI

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The Rule of ∃-Elimination: Some Background

- It is useful to think of an existential claim $\lceil (\exists v) \phi v \rceil$ as a *disjunction* which asserts that the predicate expression ϕ is satisfied by *at least one* object in the domain (*i.e.*, that the disjunction $\lceil \phi a \lor (\phi b \lor (\phi c \lor ...)) \rceil$ is true).
- In this way, we would expect the elimination rule for ∃ to be similar to the elimination rule for ∨. That is, we'd expect the ∃E rule to be similar to the ∨E rule. Indeed, this is the case. It's best to start with a simple example.
- Consider the following *legitimate* elimination of an existential claim:

Problem is: $(\exists x)(Fx\&Gx) + (\exists x)Fx$

(1) (∃x)(Fx&Gx) Premise
(2) Fa&G	a Assumption
(3) Fa	2 &E
(4) (∃x)F	E C x
(5) (3x)F	x 1,2,4 ∃E
	(2) Fa&G (3) Fa (4) (3x)F

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The Rule of ∃-Elimination: II

- To derive a sentence using the $\exists E$ rule (with some existential sentence $\lceil (\exists v) \phi v \rceil$), we must first *assume* an *instance* $\phi \tau$ of $\lceil (\exists v) \phi v \rceil$.
- If we can deduce from this assumed instance $\phi \tau$ *using generalizable reasoning* then we may infer *outright*.
- It is because our reasoning from the *instance* $\phi \tau$ of $\lceil (\exists v) \phi v \rceil$ to *does not depend on our choice of constant* τ (*i.e.*, that our reasoning from $\phi \tau$ to is *generalizable*) that makes this inference valid.
- When our reasoning is generalizable in this sense, it's as if we are showing that can be deduced from *any* instance $\phi \tau$ of $(\exists v) \phi v$.
- As such, this is just like showing that can be deduced from *any disjunct* of the disjunction ${}^{r}\phi a \vee (\phi b \vee (\phi c \vee \ldots))^{r}$. And, this is just like \vee E reasoning (except that \exists E only requires *one* assumption).

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The Rule of ∃-Elimination: III

• Here's an *il*legitimate "∃-Elimination" step:

(1) $(\exists x)Fx$ Premise (2) Ga Premise (3) Fa Assumption 2,3 (4) Fa&Ga 2,3 &1 2,3 $(\exists x)(Fx\&Gx)$ 4 3I $(\exists x)(Fx\&Gx)$ 1,3,5 3E NOII

- This is *not* a valid inference: $(\exists x)Fx$, $Ga \not\models (\exists x)(Fx \& Gx)!$
- So, what went wrong here? The problem is that the inference to $(\exists x)(Fx \& Gx)$ at line (5) does *not* use *generalizable* reasoning.
- We can *not* legitimately infer ' $(\exists x)(Fx \& Gx)$ ' at line (5) from an *arbitrary instance* $\ulcorner F\tau \urcorner$ of ' $(\exists x)Fx$ '. We *must* assume 'Fa' in *particular* at line (3) in order to deduce ' $(\exists x)(Fx \& Gx)$ ' at line (5).

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The Rule of ∃-Elimination: Official Definition

 \exists -Elimination: If $\lceil (\exists v) \phi v \rceil$ occurs at i depending on a_1, \ldots, a_n , an instance $\phi \tau$ of $\lceil (\exists v) \phi v \rceil$ is *assumed* at j, and is inferred at k depending on b_1, \ldots, b_u , then at line m we may infer , with label 'i, j, k \exists E' and dependencies $\{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_u\}/j$:

Provided that *all four* of the following conditions are met:

- τ (in $\phi \tau$) replaces *every* occurrence of ν in $\phi \nu$. [avoids fallacies]
- τ *does not occur in* $(\exists v) \phi v$. [generalizability]
- τ *does not occur in* . [generalizability]
- τ does not occur in any of b_1, \ldots, b_u , except (possibly) $\phi \tau$ itself. [generalizability]

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The Rule of ∃-Elimination: Nine Examples

• Here are 9 examples of proofs involving all four quantifier rules.

1. $(\exists x) \sim Fx \vdash \sim (\forall x)Fx$ [p. 200, example 5]

2. $(\exists x)(Fx \to A) \vdash (\forall x)Fx \to A$ [p. 201, example 6]

3. $(\forall x)(\forall y)(Gy \rightarrow Fx) \vdash (\forall x)[(\exists y)Gy \rightarrow Fx]$ [p. 203, I. # 19 \(\Rightarrow\)]

4. $(\exists x)[Fx \rightarrow (\forall y)Gy] \vdash (\exists x)(\forall y)(Fx \rightarrow Gy)$ [p. 203, I. # 20 \Leftarrow]

5. $A \vee (\exists x)Fx \vdash (\exists x)(A \vee Fx)$ [p. 203, II. # 2 \Leftarrow]

6. $(\exists x)(Fx \& \sim Fx) \vdash (\forall x)(Gx \& \sim Gx)$ [p. 203, I. # 12 \Rightarrow]

7. $(\forall x)[Fx \rightarrow (\forall y) \sim Fy] \vdash \sim (\exists x)Fx$ [p. 203, I. # 5]

8. $(\forall x)(\exists y)(Fx \& Gy) \vdash (\exists y)(\forall x)(Fx \& Gy)$ [p. 201, example 7]

9. $(\exists y)(\forall x)(Fx \& Gy) \vdash (\forall x)(\exists y)(Fx \& Gy)$ [other direction]

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Proof of (1)

Problem is: $(\exists x) \sim Fx \vdash \sim (\forall x)Fx$

 $(1) (\exists x) \sim Fx$ Premise $(2) (\forall x)Fx$ Assumption 3 (3) ~Fa Assumption 2 2 **YE** (4) Fa (5) A 3.4 ~E (6) A 1.3.5 JE $(7) \sim (\forall x) Fx$ 2,6 ~1

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Proof of (2) Problem is: $(\exists x)(Fx \rightarrow A) \vdash (\forall x)Fx \rightarrow A$ Premise $(1) (\exists x)(Fx \rightarrow A)$ 2 (∀x)Fx Assumption 3 (3) Fa→A Assumption 2 (4) Fa 2 AE 3.4 →E (5) A (6) A 1.3.5 3E $(7) (\forall x)Fx \rightarrow A$ 2,6 → UCB Philosophy CHAPTER 6 04/22/10

