Two Triviality Results for the Indicative Conditional: An Algebraic Approach

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1 Warm-Up: Triviality for the Truth-Functional Conditional

If the indicative conditional $P \to Q$ is *truth-functional* (i.e., if the truth-value of $P \to Q$ is a function of the truth-values of its antecedent P and its consequent Q), then it must be equivalent to the material conditional $P \supset Q$. This is because $\supset$ is the only binary truth-function $\star$ that satisfies the following three logical constraints (which are basic constraints that must be satisfied by any conditional connective).

(1) *Modus Ponens* in valid (*i.e.*, P and $P \star Q$ always jointly entail Q).

(2) Affirming the consequent is *not* valid (*i.e.*, Q and $P \star Q$ do *not* always jointly entail P).

(3) $P \star P$ is a logical truth.

It is a good exercise in truth-functional logic to prove the claim that the only binary truth-function $\star$ that satisfies (1)–(3) is the material conditional connective $\supset$. Here are all of the 16 possible binary truth-functions. I've given them all names or descriptions. [Only a few of these names were made up by me.]

P	Q	$\top$	NAND	$\supset$	$\sim P$	FI ($\subset$)	$\sim Q$	$\equiv$	NOR	$\vee$	NIFF	Q	NFI	P	NIF	&	$\bot$
T	**T**	**T**	**F**	**T**	**F**	**T**	**F**	**T**	**F**	**T**	**F**	**T**	**F**	**T**	**F**	**T**	**F**
T	**F**	**T**	**T**	**F**	**F**	**T**	**T**	**F**	**F**	**T**	**T**	**F**	**F**	**T**	**T**	**F**	**F**
F	**T**	**T**	**T**	**T**	**T**	**F**	**F**	**F**	**F**	**T**	**T**	**T**	**T**	**F**	**F**	**F**	**F**
F	**F**	**T**	**T**	**T**	**T**	**T**	**T**	**T**	**T**	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**
(1)?				Yes													
(2)?				Yes													
(3)?				Yes													

So, if $P \to Q$ is truth-functional, then we have the following constraint on any rational credence function:

Truth-Functionality. $\mathrm{cr}(P \to Q) = \mathrm{cr}(P \supset Q) = \mathrm{cr}(\sim P \vee Q)$.

One intuitive proposal for the way we should assign credence/probability to an indicative conditional is: *the probability of an indicative conditional is equal to the conditional probability of its consequent given its antecedent*. Formally, I will refer to this proposal as **The Equation**.

The Equation. $\mathrm{cr}(P \to Q) = \mathrm{cr}(Q \mid P)$.

Our first triviality result involves the combination of **Truth-Functionality** + **The Equation**. If we combine these two ideas about the indicative conditional, then we are led to the following (simple) triviality result.

Triviality #1. **Truth-Functionality** + **The Equation** $\Longrightarrow$ *either* $\mathrm{cr}(P) = 1$ *or* $\mathrm{cr}(P \to Q) = 1$.

I will now give an algebraic proof of **Triviality #1**. The generic stochastic truth-table representation of the class of probabilistic credence functions $\mathrm{cr}(\cdot)$ over the four states determined by P, Q is as follows.

P	Q	$\mathrm{cr}(\cdot)$
T	**T**	a
T	**F**	b
F	**T**	c
F	**F**	d

Given this setup, the conjunction of **Truth-Functionality** and **The Equation** jointly entail

$$\mathrm{cr}(P \to Q) = \mathrm{cr}(Q \mid P) = \frac{a}{a+b} = 1 - b = \mathrm{cr}({\sim}P \vee Q) = \mathrm{cr}(P \supset Q)$$

Cross-multiplying (and expanding and simplifying) this equation yields

$$0 = b \cdot (1 - (a + b)) = \mathrm{cr}({\sim}(P \supset Q)) \cdot \mathrm{cr}({\sim}P) = \mathrm{cr}({\sim}(P \to Q)) \cdot \mathrm{cr}({\sim}P)$$

There are only two ways this equation can hold: *either* $\mathrm{cr}({\sim}(P \to Q)) = 0$ *or* $\mathrm{cr}({\sim}P) = 0$. In other words, **Truth-Functionality** and **The Equation** jointly entail that *either* $\mathrm{cr}(P \to Q) = 1$ *or* $\mathrm{cr}(P) = 1$. *QED*

It goes without saying that **Triviality #1** is a *very strong* triviality result. It implies that if **Truth-Functionality** and **The Equation** are both true, then, *for every* P and Q that feature as the antecedent and consequent of some indicative conditional $P \to Q$, one should *either* (a) be certain that the indicative conditional $P \to Q$ is true *or* (b) be certain that P is true. This is clearly not a rational requirement (in general). So, this means that **Truth-Functionality** and **The Equation** *cannot both be true*. The natural thing to do at this point is to give up **Truth-Functionality**, and try to hold on to **The Equation** for a *non*-truth-functional interpretation of the indicative conditional. Unfortunately, as we'll see in the next section, this response runs into a triviality result of its own.

2 Trviality for *Non*-Truth-Functional Indicative Conditionals

If **The Equation** is true *in virtue of the meaning* of the indicative conditional, then it seems plausible that **The Equation** should *continue to hold, even when* $cr(\cdot)$ *is conditionalized on some third proposition* X — provided that X is compatible with the antecedent P of the indicative conditional $P \to Q$. That is, it seems plausible that the following, more general ("resilient") form of **The Equation** should also hold.

The Resilient Equation. $\mathrm{cr}(P \to Q \mid X) = \mathrm{cr}(Q \mid P \,\&\, X)$, provided that $\mathrm{cr}(P \,\&\, X) > 0$.

Unfortunately, **The Resilient Equation** (all by itself) leads to the following triviality result.

Triviality #2. **The Resilient Equation** $\Longleftrightarrow$ $\mathrm{cr}\,(P \,\&\, (Q \equiv (P \to Q))) = 1$.[1]

Here is an algebraic proof of **Triviality #2**. The generic stochastic truth-table representation of the class of probabilistic credence functions $\mathrm{cr}(\cdot)$ over the eight states determined by $P, Q, P \to Q$ is as follows.

P	Q	$P \to Q$	$\mathrm{cr}(\cdot)$
T	**T**	**T**	a
T	**T**	**F**	b
T	**F**	**T**	c
T	**F**	**F**	d
F	**T**	**T**	e
F	**T**	**F**	f
F	**F**	**T**	g
F	**F**	**F**	h

We will now prove a *stronger* claim than **Triviality #2**. It turns out that one does not need the full strength of **The Resilient Equation** here. That is, one does not need to conditionalize on *all* X's such that $\mathrm{cr}(P \,\&\, X) > 0$ in order to derive this (strongest) triviality result from **The Resilient Equation**. In fact, all we need are *three instances* of **The Resilient Equation**. I will now work my way up to **Triviality #2** — in three stages.

[1] Strictly speaking, the $\Longleftarrow$ direction of **Triviality #2** also requires $\mathrm{cr}(P \,\&\, Q) > 0$ and $\mathrm{cr}(P \,\&\, {\sim}Q) > 0$. This is clarified in the proof.

2.1 Stage 1: The $\sim Q$–instance of The Resilient Equation

Consider the following instance of **The Resilient Equation**, where $X := \sim Q$.

The Resilient Equation$_{\sim Q}$. $\mathrm{cr}(P \to Q \mid \sim Q) = \mathrm{cr}(Q \mid P \,\&\, \sim Q)$, provided that $\mathrm{cr}(P \,\&\, \sim Q) > 0$.

Algebraically, **The Resilient Equation**$_{\sim Q}$ is equivalent to the following (assuming $\mathrm{cr}(P \,\&\, \sim Q) > 0$).

$$\mathrm{cr}(P \to Q \mid \sim Q) = \frac{\mathrm{cr}((P \to Q) \,\&\, \sim Q)}{\mathrm{cr}(\sim Q)} = \frac{c+g}{c+d+g+h} = 0 = \mathrm{cr}(Q \mid P \,\&\, \sim Q)$$

This equation will be true iff $c + g = 0$, which implies that c and g *must both be equal to zero*. The effect of **The Resilient Equation**$_{\sim Q}$ is therefore reflected in the following revised stochastic truth-table.

P	Q	$P \to Q$	$\mathrm{cr}(\cdot)$
T	**T**	**T**	a
T	**T**	**F**	b
T	**F**	**T**	0
T	**F**	**F**	d
F	**T**	**T**	e
F	**T**	**F**	f
F	**F**	**T**	0
F	**F**	**F**	h

2.2 Stage 2: The $P \supset Q$–instance of The Resilient Equation

Consider the following instance of **The Resilient Equation**, where $X := P \supset Q$.

The Resilient Equation$_{P \supset Q}$. $\mathrm{cr}(P \to Q \mid P \supset Q) = \mathrm{cr}(Q \mid P \,\&\, (P \supset Q))$, provided that $\mathrm{cr}(P \,\&\, (P \supset Q)) > 0$.

Algebraically, **The Resilient Equation**$_{P \supset Q}$ is equivalent to the following (assuming $\mathrm{cr}(P \,\&\, (P \supset Q)) > 0$).

$$\mathrm{cr}(P \to Q \mid P \supset Q) = \frac{\mathrm{cr}((P \to Q) \,\&\, (P \supset Q))}{\mathrm{cr}(P \supset Q)} = \frac{a+e}{a+b+e+f+h} = 1 = \mathrm{cr}(Q \mid P \,\&\, (P \supset Q))$$

Cross-multiplying (and expanding and simplifying) this equation yields

$$0 = b + f + h$$

This equation will be true iff b, f and h *are all equal to zero*. The effects of **The Resilient Equation**$_{\sim Q}$ + **The Resilient Equation**$_{P \supset Q}$ are reflected in the following revised stochastic truth-table.

P	Q	$P \to Q$	$\mathrm{cr}(\cdot)$
T	**T**	**T**	a
T	**T**	**F**	0
T	**F**	**T**	0
T	**F**	**F**	d
F	**T**	**T**	e
F	**T**	**F**	0
F	**F**	**T**	0
F	**F**	**F**	0

2.3 Stage 3: The $\top$–instance of The Resilient Equation — *i.e.*, The Equation *Itself*

Consider the following instance of **The Resilient Equation**, where $X := \top$.

The Resilient Equation$_{\top}$. $\mathrm{cr}(P \to Q \mid \top) = \mathrm{cr}(Q \mid P \,\&\, \top)$, provided that $\mathrm{cr}(P \,\&\, \top) > 0$.

Of course, **The Resilient Equation**$_\top$ *is just* **The Equation** *itself*. Algebraically, **The Equation** is now

$$\mathrm{cr}(P \to Q) = a + e = \frac{a}{a+d} = \mathrm{cr}(Q \mid P)$$

Cross-multiplying (and expanding and simplifying) this equation yields the following quadratic equation

$$a^2 + ad + ae + de - a = 0$$

Recall, we are assuming (from Stage 1) that $\mathrm{cr}(P \,\&\, {\sim}Q) > 0$. That is, we are assuming that $d > 0$. As it happens, when $d > 0$, the quadratic equation above is satisfied *iff* $e = 0$, $d = 1 - a$, and $a, d \in (0, 1)$.[2] The effects of **The Resilient Equation**$_{\sim Q}$ + **The Resilient Equation**$_{P \supset Q}$ + **The Equation** are reflected in the following (final) *single-parameter* stochastic truth-table, where $a \in (0, 1)$.

P	Q	$P \to Q$	$\mathrm{cr}(\cdot)$
T	**T**	**T**	a
T	**T**	**F**	0
T	**F**	**T**	0
T	**F**	**F**	$1 - a$
F	**T**	**T**	0
F	**T**	**F**	0
F	**F**	**T**	0
F	**F**	**F**	0

In other words, **The Resilient Equation**$_{\sim Q}$ + **The Resilient Equation**$_{P \supset Q}$ + **The Equation** jointly entail that *the only two states which can be assigned non-zero credence* are $P \,\&\, Q \,\&\, (P \to Q)$ and $P \,\&\, {\sim}Q \,\&\, {\sim}(P \to Q)$. This is equivalent to saying that the proposition $P \,\&\, (Q \equiv (P \to Q))$ *must receive maximal credence. QED*

Triviality #2 is quite strong. It implies that, for *every* P and Q that feature as the antecedent and consequent of some indicative conditional $P \to Q$, *both* P *and* the material biconditional $Q \equiv (P \to Q)$ must receive maximal credence. This is clearly not (generally) a rational requirement. Therefore, *at least one of* our three assumptions **The Resilient Equation**$_{\sim Q}$, **The Resilient Equation**$_{P \supset Q}$, and **The Equation** *must be false*. It seems very difficult to rationalize rejecting any of these assumptions, *if* **The Equation** really does capture an essential feature of the way we ought to assign credences to indicative conditionals. So, it seems that — although it sounds plausible, initially — **The Equation** cannot ultimately be correct.

3 Epilogue: Why Triviality #2 is *The Strongest* (Lewisian) Triviality Result

I mentioned in passing that **Triviality #2** is *the strongest* triviality result of its kind. Here's what I mean. If one assumes *all* of the instances of **The Resilient Equation**, then this *still* (*only*) implies **Triviality #2**. That is, adding further instances of **The Resilient Equation** to the three we used above *does not add any additional constraints to* $\mathrm{cr}(\cdot)$. This can be shown algebraically by proving that the conjunction of *all* (191) instances of **The Resilient Equation** (where X ranges over the 256 propositions in the Boolean algebra generated by $P, Q, P \to Q$) *is equivalent to* the conjunction of the *three* instances of **The Resilient Equation** that we used above (and this also secures the $\Leftarrow$ direction of **Triviality #2**).[3]

[2] This is a good exercise in high-school algebra. But, it is easily verified using *Mathematica* (see *fn.* 2).

[3] This can easily be verified using *Mathematica*. I have a *Mathematica* (version 10) notebook which verifies that **Triviality #2** is — in this sense — *the strongest* (Lewisian) triviality result for the indicative conditional.