Some History Surrounding our Deductive Apparatus for P

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In the *Begriffsschrift*, Frege gives the following deductive apparatus (PS′) for P:

- Six (6) Axiom Schemata:

  (PS1′) $A \supset (B \supset A)$

  (PS2′) $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$

  (PS3′) $(A \supset (B \supset C)) \supset (B \supset (A \supset C))$

  (PS4′) $(A \supset B) \supset (\sim B \supset \sim A)$

  (PS5′) $\sim\sim A \supset A$

  (PS6′) $A \supset \sim\sim A$

- One (1) Rule of Inference (Schemata): Modus Ponens (MP). From A and $A \supset B$, infer B.

A couple of decades later, Łukasiewicz discovered simpler sets of axioms that would suffice for P (including the three-axiom basis of Hunter's system PS – see below.). Along the way, he also proved that Frege's axiom (PS3′) is *redundant*, since it can be deduced from (PS1′) and (PS2′) by (MP). The reason Frege didn't notice this is that it is *not obvious*! Here's Łukasiewicz's proof of (PS3′) from (PS1′) and (PS2′), in schematic form. This is (in a sense) the *shortest possible* proof of (PS3′) from (PS1′) and (PS2′)! Moral: these ain't easy!

[1] $((X \supset (Y \supset Z)) \supset ((X \supset Y) \supset (X \supset Z))) \supset ((Y \supset Z) \supset ((X \supset (Y \supset Z)) \supset ((X \supset Y) \supset (X \supset Z))))$ [PS1′]
[2] $(Y \supset Z) \supset ((X \supset (Y \supset Z)) \supset ((X \supset Y) \supset (X \supset Z)))$ [MP, 1, PS2′]
[3] $((Y \supset Z) \supset ((X \supset (Y \supset Z)) \supset ((X \supset Y) \supset (X \supset Z)))) \supset (((Y \supset Z) \supset (X \supset (Y \supset Z))) \supset ((Y \supset Z) \supset ((X \supset Y) \supset (X \supset Z))))$ [??]
[4] $((Y \supset Z) \supset (X \supset (Y \supset Z))) \supset ((Y \supset Z) \supset ((X \supset Y) \supset (X \supset Z)))$ [??]
[5] $(Y \supset Z) \supset (X \supset (Y \supset Z))$ [??]
[6] $(Y \supset Z) \supset ((X \supset Y) \supset (X \supset Z))$ [??]
[7] $((Y \supset Z) \supset ((X \supset Y) \supset (X \supset Z))) \supset (((Y \supset Z) \supset (X \supset Y)) \supset ((Y \supset Z) \supset (X \supset Z)))$ [??]
[8] $((Y \supset Z) \supset (X \supset Y)) \supset ((Y \supset Z) \supset (X \supset Z))$ [??]
[9] $(((X \supset Y) \supset Z) \supset (Y \supset (X \supset Y))) \supset (((X \supset Y) \supset Z) \supset (Y \supset Z))$ [??]
[10] $(Z \supset (Y \supset Z)) \supset (X \supset (Z \supset (Y \supset Z)))$ [??]
[11] $X \supset (Z \supset (Y \supset Z))$ [??]
[12] $((X \supset Y) \supset Z) \supset (Y \supset (X \supset Y))$ [??]
[13] $((X \supset Y) \supset Z) \supset (Y \supset Z)$ [??]
[14] $(((X \supset Y) \supset Z) \supset (Y \supset Z)) \supset ((U \supset ((X \supset Y) \supset Z)) \supset (U \supset (Y \supset Z)))$ [??]
[15] $(U \supset ((X \supset Y) \supset Z)) \supset (U \supset (Y \supset Z))$ [??]
[16] $((X \supset (Y \supset Z)) \supset ((X \supset Y) \supset (X \supset Z))) \supset ((X \supset (Y \supset Z)) \supset (Y \supset (X \supset Z)))$ [??]
[17] $((X \supset (Y \supset Z)) \supset (Y \supset (X \supset Z)))$ [??]

Exercise #1: Fill-in the "??"s (*i.e.*, the justifications of these steps) on the right!

Exercise #2: Prove the three axioms (PS1)–(PS3) of Hunter's (PS) in Frege's (PS′).

Exercise #3: Prove Frege's six axioms (PS1′)–(PS6′) in Hunter's (PS).

Note: These are *hard* exercises (in increasing order of difficulty!). Don't spend too much time on them. It's not crucial that you be able to prove lots of *theorems in* (PS). Our focus is on *meta*theorems *about* (PS).

Postscript: The system (PS) in Hunter was discovered by Łukasiewicz in the 1920's. Since the 1920's many other interesting axiomatizations equivalent to (PS) have been discovered. Meredith proved that the following *single* axiom is sufficient (assuming only Modus Ponens)! This requires a very deep proof.

(PSM) $((((A \supset B) \supset (\sim C \supset \sim D)) \supset C) \supset E) \supset ((E \supset A) \supset (D \supset A))$

Exercise #4: Prove *anything* (non-trivially) from (PSM)! The *very first step* will be a killer!

An Annotated Version of Hunter's Proof of $p' \supset p'$ in (PS)

[1] $p' \supset ((p' \supset p') \supset p')$ [PS1]

This is an instance of $A \supset (B \supset A)$, with $A = p'$, $B = p' \supset p'$.

[2] $(p' \supset ((p' \supset p') \supset p')) \supset ((p' \supset (p' \supset p')) \supset (p' \supset p'))$ [PS2]

This is an instance of $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$, with $A = p'$, $B = p' \supset p'$, $C = p'$.

[3] $(p' \supset (p' \supset p')) \supset (p' \supset p')$ [MP, 1, 2]

This is the consequent of [2], and [1] is the antecedent of [2].

[4] $p' \supset (p' \supset p')$ [PS1]

This is an instance of $A \supset (B \supset A)$, with $A = p'$, $B = p'$.

[5] $p' \supset p'$ [MP, 3, 4]

This is the consequent of [3], and [4] is the antecedent of [3].

A Simple Example of an Inductive Proof of a Metathrorem of Propositional Logic (Semantics)

Theorem. Let A be any statement of propositional logic containing either no connectives at all (*i.e.*, a propositional symbol) or *only* the connective '$\vee$'. A is true (on an interpretation I) if and only if at least one of its propositional symbols is true (on I).

Proof. We will prove this theorem by induction on the number (n) of occurrences of '$\vee$' in A. The first step is to express the theorem as a statement about the natural numbers. This can be done as follows:

> $S(n) \stackrel{\text{def}}{=}$ Any statement A (of P) containing n occurrences of '$\vee$' (and no other connectives) will be true (on an interpretation I) iff at least one of the propositional symbols occurring in A is true (on I).

Now, we will prove — *by induction on* n — that $S(n)$ is true *for all* $n \geq 0$. This involves two steps:

1. **Basis Step**. *Prove* $S(0)$. When $n = 0$, A is just a propositional symbol p. Of course, A is true (on I) just in case p is true (on I). That is, A is true iff at least one of its propositional symbols is true.

2. **Inductive Step**. The inductive step is a *conditional proof*. We will *assume* as our **inductive hypothesis** that $S(m)$ is true for all m such that $0 < m < n$. Then, we will *use* this assumption to *prove* that $S(n)$ is true. This will complete the inductive proof that $S(n)$ is true *for all* $n \geq 0$ (*i.e.*, the theorem).

   (a) $S(m)$ is true for all m such that $0 < m < n$. Assumption (inductive hypothesis)

   (b) $S(n-1)$ is true. Follows from (a)

   (c) Let B be any statement containing $n-1$ occurrences of '$\vee$'. Then, it follows from (b) that B will be true iff at least one of its propositional symbols is true.

   (d) Now, define $A \stackrel{\text{def}}{=} B \vee C$ such that A has n occurrences of '$\vee$'. Since A has n occurrences of '$\vee$', C must be a propositional symbol p (because B has $n-1$ occurrences of '$\vee$'). Thus, $A = B \vee p$. Hence, A will be true iff either B is true or p is true. But, since (c) B is true iff at least one of its propositional symbols is true, we know that A is true iff either at least one of the propositional symbols in B is true or p is true. This just means that A is true iff at least one of its propositional symbols is true. Since B was an *arbitrary* statement with $n-1$ occurrences of '$\vee$', this constitutes a proof that $S(n)$ is true, from our inductive hypothesis (a) that $S(m)$ is true for all m such that $0 < m < n$. In other words, we've just completed the inductive step.

Having completed the Basis and Inductive steps, we have proven — by induction on n — that $S(n)$ is true *for all* $n \geq 0$. Since this is equivalent to the metatheorem we aimed to prove, we are done. □