Announcements and Such

- Administrative Stuff
  - **HW #5 has been posted. It's due on April 8**
    * This HW consists of two sets of exercises from Skyrms's Chapter 2 (which you should have read by now).
  - The times & locations are now known for our Final Exam
    * Morning Section: **8-10am, April 28 @ Dodge Hall 150**
    * Afternoon Section: **8-10am, April 29 @ Dodge Hall 119**
- Unit #4 — *Probability & Inductive Logic, Continued*
  - Two Probabilistic Proposals Regarding Argument Strength
  - "Irrelevance Objections" to both of these proposals
  - Our Third "Two Factor" Proposal
  - The Two Factors Compared (theoretically)
- We will also have a short ResponseWare quiz after the break

Skyrms's Two Accounts of Inductive Argument Strength

- Skyrms considers the following proposal for "inductive strength":

  Proposal #1. An argument $P \therefore C$ is strong just in case the claim $P \to C$ (the argument's corresponding conditional) is *probable*.

- This first proposal is inadequate, since an argument will be judged as strong if either P is improbable or C is probable.
- As a result, P can *fail to be positively relevant to* C, *even if* the argument is "very strong" according to this proposal.
- The most extreme case of this involves any argument of the form $\sim X \therefore X$. As long as $\Pr(X)$ is very high, this argument will be deemed "very strong" by Proposal #1. *Why?* This *refutes* Proposal #1.
- Skyrms (rightly) abandons Proposal #1. Instead, he goes for:

  Proposal #2. An argument $P \therefore C$ is inductively strong just in case C is probable, *given that* (*i.e.*, *on the supposition that*) P is true.

Generalizing Skyrms's Objection to Proposal #1

- The probability of C ***given that*** P is a *better* guide to the inductive strength of "$P \therefore C$" than the probability of $P \rightarrow C$. This can be seen by noting that $\sim X \therefore X$ will be deemed *weak* by Proposal #2. *Why?*
- But, a more general "irrelevance objection" also applies to Proposal #2.

  (P) Fred Fox (who is a man) is on birth control pills.

  Therefore, (C) Fred Fox (who is a man) will not get pregnant.

- The probability of C *given that* P is very high (as is the probability that $P \rightarrow C$). So, proposal #2 (and proposal #1) says "$P \therefore C$" is *strong*.
- But, intuitively, P is *irrelevant* to C, and so (intuitively) P *does not provide evidence in favor of* C. This suggests a third proposal.

  Proposal #3. "$P \therefore C$" is strong just in case (1) the probability of C *given that* P is *high*, *and* (2) P is *positively relevant to* C — *i.e.*, the probability of C *given that* P is high***er*** than the probability of C.

Our "Two Factor" Approach For Inductive Strength

- With our $\Pr(\cdot)$ notations in hand, we can formally state proposal #3.

  Proposal #3. An argument $P \therefore C$ is *inductively strong* iff

  (1) C is probable, *given* P, *i.e.*, $\Pr(C \mid P) > \frac{1}{2}$, *and*

  (2) P is *positively relevant* to C, *i.e.*, $\Pr(C \mid P) > \Pr(C)$.

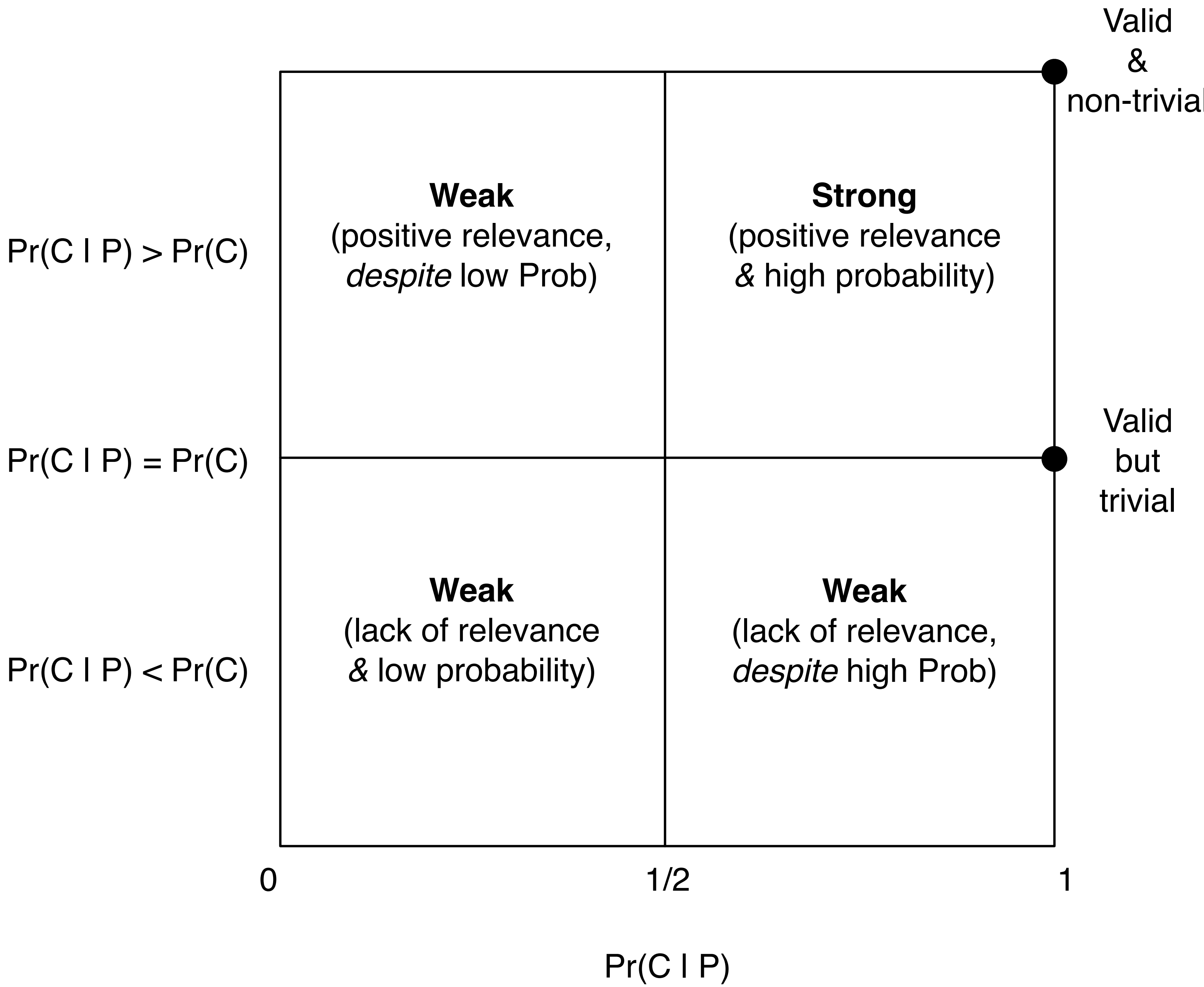
- This proposal is superior to Skyrms's, as it requires *both* that $\Pr(C \mid P)$ be *high* ($> \frac{1}{2}$), *and* that $\Pr(C \mid P)$ be high*er* than $\Pr(C)$. This means P has to *raise the probability of C to a number that is greater than* $\frac{1}{2}$.

- I won't offer a measure of *degree* of inductive strength [$\mathfrak{c}(C, P)$], but, presumably, $\mathfrak{c}(C, P)$ would be some function of $\Pr(C \mid P)$ and $\Pr(C)$.

- The important thing is that one must think about *two factors* when assessing whether an argument $\ulcorner P \therefore C \urcorner$ is strong.

  - **Factor #1.** $\Pr(C \mid P)$ must be *high* ($> \frac{1}{2}$).

  - **Factor #2.** $\Pr(C \mid P)$ must be high***er*** than $\Pr(C)$.


Valid
&
non-trivial
Valid
but
trivial
Pr(C | P) > Pr(C)
Pr(C | P) = Pr(C)
Pr(C | P) < Pr(C)
Weak
(positive relevance,
despite low Prob)
Strong
(positive relevance
& high probability)
Weak
(lack of relevance
& low probability)
Weak
(lack of relevance,
despite high Prob)
0
1/2
1
Pr(C | P)

Theoretical Comparison of Our "Two Factors" I

- The two factors that go into determining whether an inductive argument is *strong* are different in some crucial ways.

  The Conjunction Condition. If a claim (X) constitutes a strong argument for a conjunction ($Y \,\&\, Z$), then X also constitutes a strong argument for each of its conjuncts (Y, Z).

- Factor #1 *satisfies* The Conjunction Condition, because, in general

$$\Pr(Y \,\&\, Z \mid X) > \tfrac{1}{2} \Longrightarrow \Pr(Y \mid X) > \tfrac{1}{2} \text{ and } \Pr(Z \mid X) > \tfrac{1}{2}.$$

  – Because: $\Pr(Y \mid E) \geq \Pr(Y \,\&\, Z \mid E)$ and $\Pr(Z \mid E) \geq \Pr(Y \,\&\, Z \mid E)$.

- Factor #2 can *violate* The Conjunction Condition. That is:

$$\Pr(P \,\&\, Q \mid E) > \Pr(P \,\&\, Q) \not\Rightarrow \Pr(P \mid E) > \Pr(P).$$

  – Let E = card is black, P = card is an ace, and Q = card is a spade.

Theoretical Comparison of Our "Two Factors" II

- Another crucial difference between our Two Factors involves
- **The Disjunction Condition** (DC). If $P \therefore X$ is a strong argument, and $P \therefore Y$ is a strong argument, then $P \therefore X \vee Y$ is a strong argument.
- If we measure strength using *only* Factor 1, then (DC) is true. This is because of the following fact (let's prove it *via* our algebraic method).

$$\text{If } \Pr(X \mid P) > \tfrac{1}{2} \text{ and } \Pr(Y \mid P) > \tfrac{1}{2}\text{ , then } \Pr(X \vee Y \mid P) > \tfrac{1}{2}.$$

- But, if we think about the Factor 2 component of strength, then (DC) can *fail*. That is to say, there are examples (see next slide) in which
  - $\Pr(X \mid P) > \Pr(X)$ [$^{13}/_{22} > {}^{1}/_{2}$]
  - $\Pr(Y \mid P) > \Pr(Y)$ [$^{6}/_{11} > {}^{1}/_{2}$]
  - $\Pr(X \vee Y \mid P) < \Pr(X \vee Y)$ [$^{9}/_{11} < {}^{7}/_{8}$]

State (s_i)	P	X	Y	$\Pr(s_i) = a_i$
s_1	$\top$	$\top$	$\top$	$\Pr(s_1) = a_1 = \frac{7}{64}$
s_2	$\top$	$\top$	$\bot$	$\Pr(s_2) = a_2 = \frac{6}{64}$
s_3	$\top$	$\bot$	$\top$	$\Pr(s_3) = a_3 = \frac{5}{64}$
s_4	$\top$	$\bot$	$\bot$	$\Pr(s_4) = a_4 = \frac{4}{64}$
s_5	$\bot$	$\top$	$\top$	$\Pr(s_5) = a_5 = \frac{1}{64}$
s_6	$\bot$	$\top$	$\bot$	$\Pr(s_6) = a_6 = \frac{18}{64}$
s_7	$\bot$	$\bot$	$\top$	$\Pr(s_7) = a_7 = \frac{19}{64}$
s_8	$\bot$	$\bot$	$\bot$	$\Pr(s_8) = a_8 = \frac{4}{64}$

Theoretical Comparison of Our "Two Factors" III

- Here is another property satisfied by Factor 1, but not Factor 2.

  The Sure Thing Principle. If X consitutes a strong argument for Z *given* Y and X consitutes a strong argument for Z *given* $\sim Y$, then X constitutes a strong argument for Z (*unconditionally*).

- The reason Factor 1 satisfies The Sure Thing Principle is that, in general

$$\left[\Pr(Z \mid X \,\&\, Y) > \tfrac{1}{2} \text{ and } \Pr(Z \mid X \,\&\, {\sim}Y) > \tfrac{1}{2}\right] \Longrightarrow \Pr(Z \mid X) > \tfrac{1}{2}.$$

- Let's prove this claim using our algebraic method.

- Factor 2 can *violate* The Sure Thing Principle. In other words,

$$[\Pr(Z \mid X \,\&\, Y) > \Pr(Z \mid Y) \text{ and } \Pr(Z \mid X \,\&\, {\sim}Y) > \Pr(Z \mid {\sim}Y)] \not\Rightarrow \Pr(Z \mid X) > \Pr(Z).$$

- See the next slide for an "urn-style" counterexample.

World (w_i)	X	Y	Z	$\Pr(w_i)$
w_1	$\top$	$\top$	$\top$	$\Pr(w_1) = \frac{31}{192}$
w_2	$\top$	$\top$	$\bot$	$\Pr(w_2) = \frac{59}{192}$
w_3	$\top$	$\bot$	$\top$	$\Pr(w_3) = \frac{40}{192}$
w_4	$\top$	$\bot$	$\bot$	$\Pr(w_4) = \frac{14}{192}$
w_5	$\bot$	$\top$	$\top$	$\Pr(w_5) = \frac{1}{192}$
w_6	$\bot$	$\top$	$\bot$	$\Pr(w_6) = \frac{5}{192}$
w_7	$\bot$	$\bot$	$\top$	$\Pr(w_7) = \frac{24}{192}$
w_8	$\bot$	$\bot$	$\bot$	$\Pr(w_8) = \frac{18}{192}$

Theoretical Comparison of Our "Two Factors" IV

- The fact that Factor 2 can violate The Sure Thing Principle is known as "Simpson's Paradox". Here is a real-life example from a medical study comparing the success rates of two treatments for kidney stones.
- We can interpret the STT above (with X, Y, Z), as follows. Let X be the claim that a patient is given a treatment t for disease d. Let Z be the claim that a patient recovers from d. And, let Y be the claim that a patient is male. If we calculate the salient probabilities, we get:

(1) $\Pr(Z \mid X \,\&\, Y) > \Pr(Z \mid Y)$. [$^{31}/_{90} > {}^{1}/_{3}$]

(2) $\Pr(Z \mid X \,\&\, {\sim}Y) > \Pr(Z \mid {\sim}Y)$. [$^{20}/_{27} > {}^{2}/_{3}$]

(3) $\Pr(Z \mid X) < \Pr(Z)$. [$^{71}/_{144} < {}^{1}/_{2}$]

☞ (1) implies that the treatment is (somewhat) effective *for males*, and (2) implies that the treatment is (somewhat) effective *for females*. But, (3) implies that the treatment is *counter-productive for humans*!

Theoretical Comparison of Our "Two Factors" V

- Although Simpson's Paradox implies that Factor #2 can violate The Sure Thing Principle, there is a related principle that *both* Factors *do* satisfy.

  The Unconditional Sure Thing Principle. If $X \,\&\, Y$ constitutes a strong argument for Z (unconditionally) and $X \,\&\, {\sim}Y$ constitutes a strong argument for Z (unconditionally), then X *alone* constitutes a strong argument for Z (unconditionally).

- In terms of Factor 1, The Unconditional Sure Thing Principle *is equiavlent to* The Sure Thing Principle (thus it satisfies both).

- From the point of view of Factor 2, these principles are *not* equivalent. Indeed, The Unconditional Sure Thing Principle *holds* for Factor 2, since

$$[\Pr(Z \mid X \,\&\, Y) > \Pr(Z) \text{ and } \Pr(Z \mid X \,\&\, {\sim}Y) > \Pr(Z)] \Longrightarrow \Pr(Z \mid X) > \Pr(Z).$$

- So, this disagreement trades *essentially* on the "*given*"s in the STP.

	Does Factor satisfy property?	
Property	Factor 1?	Factor 2?
The Conjunction Condition	Yes	No
The Disjunction Condition	Yes	No
The Sure Thing Principle	Yes	No
$\dfrac{P}{\therefore Q \vee {\sim}Q}$ is *weak*.	No	Yes
$\dfrac{P \,\&\, {\sim}P}{\therefore Q}$ is *weak*.	Yes	Yes
$\dfrac{{\sim}X}{\therefore X}$ is *weak*.	Yes	Yes
$\dfrac{P \vee Q}{\therefore P}$ is (generally) strong*er* than $\dfrac{P \vee {\sim}P}{\therefore P}$	Yes	Yes
The Unconditional Sure Thing Principle	Yes	Yes

A Peculiar Probability Distribution

- All of the numerical probability distributions we've been looking at so far have involved *rational numbers*. Not all examples are like this.
- Consider the following three constraints:
  1. $\Pr(Y \mid X) = \Pr(X \vee Y)$.
  2. $\Pr(Y) = \Pr(\sim Y)$.
  3. $\Pr(X \,\&\, Y) = \Pr(\sim X \,\&\, Y)$.

  Fact. (1)–(3) are satisfied by a *unique* numerical probability distribution, and this distribution assigns some *irrational* numbers to some states.

X	Y	$\Pr(s_i)$
$\top$	$\top$	a_1
$\top$	$\bot$	a_2
$\bot$	$\top$	a_3
$\bot$	$\bot$	a_4

Inverse Probability and Bayes's Theorem I

- $\Pr(H \mid E)$ is called the *posterior* H (on E). $\Pr(H)$ is called the *prior* of H. $\Pr(E \mid H)$ is called the *likelihood* of H (on E).
- By the definition of $\Pr(\bullet \mid \bullet)$, we can write the posterior and likelihood as:

$$\Pr(H \mid E) = \frac{\Pr(H \,\&\, E)}{\Pr(E)} \qquad \text{and} \qquad \Pr(E \mid H) = \frac{\Pr(H \,\&\, E)}{\Pr(H)}$$

- So, the posterior and the likelihood are related by *Bayes's Theorem*:

$$\Pr(H \mid E) = \frac{\Pr(E \mid H) \cdot \Pr(H)}{\Pr(E)}$$

- **Law of Total Probability**. If $\Pr(H)$ is non-extreme, then:

$$\begin{aligned}\Pr(E) &= \Pr((E \,\&\, H) \vee (E \,\&\, {\sim}H)) \\ &= \Pr(E \,\&\, H) + \Pr(E \,\&\, {\sim}H) \\ &= \Pr(E \mid H) \cdot \Pr(H) + \Pr(E \mid {\sim}H) \cdot \Pr({\sim}H)\end{aligned}$$

- This allows us to write a more perspicuous form of *Bayes's Theorem*:

$$\Pr(H \mid E) = \frac{\Pr(E \mid H) \cdot \Pr(H)}{\Pr(E \mid H) \cdot \Pr(H) + \Pr(E \mid {\sim}H) \cdot \Pr({\sim}H)}$$

Inverse Probability and Bayes's Theorem II

- Here's a famous example, illustrating the subtlety of Bayes's Theorem:

  The (unconditional) probability of breast cancer is 1% for a woman at age forty who participates in routine screening. The probability of such a woman having a positive mammogram, given that she has breast cancer, is 80%. The probability of such a woman having a positive mammogram, given that she does not have breast cancer, is 10%. What is the probability that such a woman has breast cancer, given that she has had a positive mammogram in routine screening?

- We can formalize this, as follows. Let H = such a woman (age 40 who participates in routine screening) has breast cancer, and E = such a woman has had a positive mammogram in routine screening. Then:

$$\Pr(E \mid H) = 0.8, \Pr(E \mid {\sim}H) = 0.1, \text{ and } \Pr(H) = 0.01.$$

- **Question**: What is $\Pr(H \mid E)$? What would you guess? Most experts guess a pretty high number (near 0.8, usually).

- If we apply Bayes's Theorem, we get the following answer:

$$\begin{aligned}\Pr(H \mid E) &= \frac{\Pr(E \mid H) \cdot \Pr(H)}{\Pr(E \mid H) \cdot \Pr(H) + \Pr(E \mid \sim H) \cdot \Pr(\sim H)} \\ &= \frac{0.8 \cdot 0.01}{0.8 \cdot 0.01 + 0.1 \cdot 0.99} \approx 0.075\end{aligned}$$

- We can also use our algebraic technique to compute an answer.

E	H	$\Pr(w_i)$
T	T	$a_1 = 0.008$
T	⊥	$a_2 = 0.099$
⊥	T	$a_3 = 0.002$
⊥	⊥	0.891

$$\Pr(E \mid H) = \frac{\Pr(E \,\&\, H)}{\Pr(H)} = \frac{a_1}{a_1 + a_3} = 0.8$$

$$\Pr(E \mid \sim H) = \frac{\Pr(E \,\&\! \sim H)}{\Pr(\sim H)} = \frac{a_2}{1 - (a_1 + a_3)} = 0.1$$

$$\Pr(H) = a_1 + a_3 = 0.01$$

- Note: The posterior is about eight times the prior in this case, but since the prior is *so* low to begin with, the posterior is still pretty low.

- This mistake is usually called the *base rate fallacy*. People tend to neglect base rates in their estimates of probability — *when E is strongly relevant to H*. Here, our Two Factors *pull in opposite directions*.