LMPL-Validity is Decidable

Branden Fitelson
June 18, 2010

Let p be a (closed) sentence of LMPL which contains k predicate symbols $P_1, \ldots, P_k$. And, let $\mathcal{I}$ be an interpretation of p with a (possibly infinite) domain $\mathcal{D}$, and which assigns extensions and references as follows:

- $\text{Ext}_{\mathcal{I}}(P_i) = E_i$.
- $\text{Ref}_{\mathcal{I}}(\tau) = r_\tau$.

Here is a recipe for constructing an interpretation $\mathcal{I}'$ of p with a *finite* domain $\mathcal{D}'$ (a domain of size 2^k).

First, note that there are 2^k *categories* (*viz.*, properties) that can be defined in terms of the k predicates $P_1, \ldots, P_k$. And, every object $d \in \mathcal{D}$ must fall under *exactly one* of these categories. Now, we can use this fact to construct $\mathcal{I}'$, as follows. Let the domain, extensions, and references of $\mathcal{I}'$ be given by the following.

- $\mathcal{D}' = \{d' \subseteq \mathcal{D} \mid d'$ is a category, definable in terms of $P_1, \ldots, P_k\}$. [Thus, $|\mathcal{D}'| = 2^k$.]
- $\text{Ext}_{\mathcal{I}'}(P_i) = E'_i = \{d' \in \mathcal{D}' \mid d' \subseteq E_i\}$. [That is, E'_i is the collection of categories that are subsets of E_i.]
- $\text{Ref}_{\mathcal{I}'}(\tau) = r'_\tau = \{d' \in \mathcal{D}' \mid r_\tau \in d'\}$. [That is, r'_τ is the collection of categories of which r_τ is a member.]

Theorem. p is true on $\mathcal{I}$ iff p is true on $\mathcal{I}'$.

Proof. The proof goes by induction on the number of connectives + quantifiers in p.

Basis Case. p contains zero connectives + quantifiers. Thus, p is an atomic sentence of the form $P_i\tau$. Then, by the semantics of LMPL, $P_i\tau$ is true on $\mathcal{I}$ iff $r_\tau \in E_i$. But, by our construction of $\mathcal{I}'$, $r_\tau \in E_i$ iff $r'_\tau \in E'_i$ (pause here to take some time to convince yourself that this is in fact the case!). Hence, $P_i\tau$ is true on $\mathcal{I}$ iff $P_i\tau$ is true on $\mathcal{I}'$. And, this establishes the basis case of the Theorem.

Inductive Case. Suppose (as inductive hypothesis) that the theorem holds for all p with fewer than n connectives + quantifiers. All that is left is to show that the theorem continues to hold for sentences p with exactly n connectives + quantifiers. The only interesting cases from an LMPL point of view are the *quantified* sentences p. There are two such cases. First, suppose p is of the form $\ulcorner(\forall v)\phi v\urcorner$. In this case, p will be true on $\mathcal{I}$ iff all of p's $\mathcal{D}$-instances are true on $\mathcal{I}$. But, all of p's $\mathcal{D}$-instances will have fewer than n quantifiers + connectives. So, by the inductive hypothesis, they will be true on $\mathcal{I}$ iff they are true on $\mathcal{I}'$. Similarly, if p is of the form $\ulcorner(\exists v)\phi v\urcorner$, then p will be true on $\mathcal{I}$ iff some of its $\mathcal{D}$-instances are true on $\mathcal{I}$. But, by the inductive hypothesis, this will be the case iff some of p's $\mathcal{D}$-instances are true on $\mathcal{I}'$. This shows that the theorem holds for all quantified sentences of LMPL. I leave the proofs for the cases involving the LSL connectives (which are quite straightforward) as (simple) exercises for the reader. □

Our theorem suggests a *decision procedure* for LMPL. Any LMPL argument $\mathscr{A}$ of the form $p_1, \ldots, p_n \therefore q$ (with k predicate symbols $P_1, \ldots, P_k$) will be valid iff its *corresponding conditional* $\mathfrak{c} = \ulcorner(p_1 \,\&\, \cdots \,\&\, p_n) \rightarrow q\urcorner$ is a logical truth of LMPL. That is, $\mathscr{A}$ will be invalid iff there exists an LMPL-interpretation $\mathcal{I}$ on which $\mathfrak{c}$ is false. By our Theorem, if there exists an LMPL interpretation $\mathcal{I}$ on which $\mathfrak{c}$ is false, then there exists an LMPL-interpretation $\mathcal{I}'$ **with a finite domain of size 2^k** on which $\mathfrak{c}$ is false. Hence, $\mathscr{A}$ will be invalid iff there exists an LMPL-interpretation $\mathcal{I}'$ **with a finite domain of size 2^k** on which $\mathfrak{c}$ is false. Thus, invalidity in LMPL is *decidable*. Specifically, in order to decide whether or not an LMPL argument $\mathscr{A}$ is valid, all we need to do (in the *worst* case) is compute the truth-value of $\mathfrak{c}$ on all of its LMPL interpretations that have domains of size 2^k. So, in the worst case, we will need to calculate the truth-value of $\mathfrak{c}$ on $2^{k \cdot 2^k}$ interpretations. This worst-case computation will not be feasible for large k (even for $k = 3$, this involves checking 16.7 million LMPL-interpretations!). But, in principle, validity in LMPL is decidable *via* such a method.[1]

[1] For further discussion concerning the decidability of LMPL-validity, see pages 212–215 of Hunter's text *Metalogic*.