Quine "Methods of Logic" Appendix

Case 1: All of 'z_1', 'z_2', ... without end turn up in sequents of S. It is evident from our general method of generating sequents that, given any sequent Q (of S) whose first quantifier is universal, an instantiation of Q with respect to each of 'z_1', 'z_2', ... will eventually occur. So, given the universe of positive integers and the stated interpretation of 'z_1', 'z_2', ..., it follows that Q will count as true if all sequents with fewer quantifiers than Q count as true. But also any sequent Q' whose first quantifier is existential will be true if all sequents with fewer quantifiers are true, for these latter will include an instance of Q'. So, to sum up, *any* sequent with quantifiers will count as true if all sequents with fewer quantifiers do. But $\mathfrak{I}$ makes all unquantified sequents true. Hence it also makes all singly quantified sequents true; hence also all doubly quantified sequents; and so on. Hence finally, S itself; q.e.d.

Case 2: 'z_1', 'z_2', ... up to only some finite number n turn up in sequents of S. Then take the universe as comprising only the integers up to n, and argue as before.

This ends the proof of (VI) and, therewith, of the completeness of our deductive method in quantification theory.

It was Gödel who, in 1930, first proved the completeness of a deductive method in quantification theory. The deductive method which he proved complete was very different from ours and more like the one on p. 191. But this difference is of little moment, since a completeness proof for one method of quantification theory can be adapted fairly easily to others. In the above adaptation, actually, I have depended partly on Gödel's original argument and partly on a variant due to Dreben.

(VI) has as corollary a celebrated theorem which, antedating Gödel, goes back to Löwenheim: *Any consistent quantificational schema comes out true under some interpretation in the universe of positive integers.* For, consider any consistent quantificational schema S. Let S' be its prenex equivalent, closed by existential quantification of any free variables. Then S', like S, is consistent. Then certainly, in view of the soundness of EI and UI (§28), no truth-functional inconsistencies can be got by EI and UI from S'. Then, by (VI), S' is true under some interpretation in a non-empty universe of positive integers.

But then, by the reasoning of p. 97n, S' will be true also under some interpretation in the full universe of positive integers. Then so will S.

The notion of consistency admits of a natural extension from schemata to classes of schemata. A class of schemata is consistent if, under some interpretation in a non-empty universe, all its members come out true together. (If some of the schemata of the class contain 'F' monadically, say, and others contain 'F' dyadically, what sense is there in speaking of a joint interpretation? Let us settle this point by treating the monadic 'F' and the dyadic 'F' as if they were different letters.) Now Löwenheim's theorem admits immediately of this superficial extension: If a finite class of quantificational schemata is consistent, all its members come out true together under some interpretation in the universe of positive integers. For, we have merely to take the schema in Löwenheim's theorem as a conjunction of all the schemata in the finite class.

Actually this limitation to finite classes can be lifted, as Skolem showed in 1920. The result is the Löwenheim-Skolem theorem: *If a class of quantificational schemata is consistent, all its members come out true together under some interpretation in the universe of positive integers.* The proof is omitted here.[1]

Consider any non-empty universe U and any assortment of predicates, all interpreted in that universe. Consider, further, the whole infinite totality of truths, known and unknown, that are expressible with help of those predicates together with the truth functions and quantification over U. Then the Löwenheim-Skolem theorem assures us that there is a reinterpretation of the predicates, in the universe of positive integers, that preserves the whole body of truths.

E.g., taking U as the universe of real numbers, we are told that the truths about real numbers can by a reinterpretation be carried over into truths about positive integers. This consequence has been viewed as paradoxical, in the light of Cantor's proof that the real numbers cannot be exhaustively correlated with integers. But the air of paradox may be dispelled by this reflection: whatever disparities between real numbers and integers may be guaranteed in

[1] For a version of the proof, see my "Interpretations of sets of conditions." *Journal of Symbolic Logic*, vol. 19 (1954), pp. 97–102.

those original truths about real numbers, the guarantees are themselves revised in the reinterpretation.

In a word and in general, the force of the Löwenheim-Skolem theorem is that the narrowly logical structure of a theory—the structure reflected in quantification and truth functions, in abstraction from any special predicates—is insufficient to distinguish its objects from the positive integers.

Bibliography

This list includes only such logical and nearly logical works as happen to have been alluded to, by title or otherwise, in the course of the book. For a comprehensive register of the literature of mathematical logic to the end of 1935 see Church's *Bibliography*. This work, which is helpfully annotated and thoroughly indexed by subjects, is invaluable to logicians. Subsequent literature is covered by the Reviews section of the *Journal of Symbolic Logic*, which is indexed by subjects every five years and by authors every two.

ACKERMANN, Wilhelm. "Mengentheoretische Begründung der Logik." *Mathematische Annalen*, vol. 115 (1937), pp. 1–22.

———. *Grundzüge*. See Hilbert.

BEHMANN, Heinrich. "Beiträge zur Algebra der Logik, insbesondere zum Entscheidungsproblem." *Mathematische Annalen*, vol. 86 (1922), pp. 163–229.

BERNAYS, Paul. "A system of axiomatic set-theory." *Journal of Symbolic Logic*, vol. 2 (1937), pp. 65–77; vol. 6 (1941), pp. 1–17; vol. 7 (1942), pp. 65–89, 133–145; vol. 8 (1943), pp. 89–106; vol. 13 (1948), pp. 65–79.

———. *Grundlagen*. See Hilbert.

BOOLE, George. *The Mathematical Analysis of Logic*. London and Cambridge, England, 1847. Reprinted in *Collected Logical Works*, Chicago and London, 1916.

———. *An Investigation of the Laws of Thought*. London, 1854. Reprinted ibid.

CANTOR, Georg. "Ueber eine elementare Frage der Mannigfältigkeitslehre." *Jahresberichte* der deutschen Mathematiker-Vereinigungen, vol. 1 (1890–1), pp. 75–78. Reprinted in *Gesammelte Abhandlungen*, Berlin, 1932.

CARNAP, Rudolf. *The Logical Syntax of Language*. London and New York, 1937.