Notes on Isomorphism, Elementary Equivalence, the Generalized Löwenheim-Skolem-Tarski Theorem, Categoricity, Non-Standard Models, *etc.*

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1 Preliminaries and Setup

I will use the notation "$=$", rather than "$F^{**\prime}$", for the two-place predicate whose intended (normal) interpretation is the identity relation. And, I will use the notation "$x \neq y$", rather than "$\sim(x = y)$," to express the claim that x and y are non-identical.

A *theory* is just a set of formulas in a particular language. A *model* for a theory is a interpretation on which every formula in the set is true (and thus every consequence of the set is true on such an interpretation as well). We will only consider normal models here, *i.e.*, models where the symbol "$=$" is interpreted as the identity relation. The *size of a model* is the cardinality of the domain of the interpretation. Thus, the standard model of any theory of the natural numbers is countable, while any model of the theory consisting just of $\bigwedge x \bigwedge y(x = y)$ has size 1.

A *consistent* theory is one from which no contradictions can be derived. A *maximal consistent* theory is a theory T in a language L such that for every sentence $S \in L$, if $S \notin T$ then $T \cup \{S\}$ is inconsistent. Clearly, any maximal consistent theory is *negation-complete*, meaning that for any sentence $S \in L$, either $S \in T$ or $\sim S \in T$. T is said to be *Henkinized* if, for every formula A of one free variable, some sentence H – called its Henkin sentence – given as $(Ac/v \supset \bigwedge v A)$ is in T, where v is the one free variable in A and c is some constant. A Henkinized theory is clearly one where if $T \vdash_{QS^=} Ad/v$ for all d in the language, then $T \vdash_{QS^=} \bigwedge v A$, because some d will be identical to the c in the Henkin sentence.

2 Isomorphism and elementary equivalence

Let M_1 and M_2 be two interpretations for a language L. That is, each M_i will specify a domain D_i, together with an object in that domain for each constant symbol in the language, a function from the n-tuples of the domain to the domain for each n-place function symbol, and a set of n-tuples from the domain for each n-place relation symbol. In particular, since we're only considering normal models, both must specify that the symbol "$=$" gets interpreted as the actual identity relation on the domain.

We will say that M_1 is *isomorphic* to M_2 if there is a way to pair up the elements of the two domains that preserves all the structure mentioned in the language. That is, there should be a function f from D_1 to D_2 such that no two elements of D_1 get sent to the same element of D_2 and every element of D_2 is the image of some element of D_1 under f. In addition, if x is the object assigned to constant c by M_1, then $f(x)$ must be the object assigned to c by M_2. Similarly, $\langle x_1, \ldots, x_n\rangle$ is in the interpretation of some n-place relation R under M_1 iff $\langle f(x_1), \ldots, f(x_n)\rangle$ is in the interpretation of R under M_2. A similar consideration applies to the interpretations of function symbols.

We say that M_1 is *elementarily equivalent* to M_2 if they make true exactly the same sentences of L (hence, if M_1 is *elementarily equivalent* to M_2, then any set that is modeled by M_2 must also be modeled by M_1, and *vice versa*).

Notice that if two models are isomorphic, then they must have the same size, because otherwise there is no function to pair up their domains, let alone a function that does it while preserving structure. It is not too hard to see that two isomorphic models must also be elementarily equivalent. (It is clear that a sequence $s = \langle x_1, x_2, \ldots \rangle$ from model M_1 will satisfy an atomic formula ϕ iff the sequence $s_f = \langle f(x_1), f(x_2), \ldots \rangle$ from model M_2 satisfies ϕ because of the structure preserving properties. This provides a base case for an induction on the number of connectives and quantifiers in a formula, proving that every sequence from one model satisfies an arbitrary formula iff the corresponding sequence from the other model satisfies the same formula. Thus, all the same sentences must be true.)

However, just because two models are elementarily equivalent doesn't mean that they are isomorphic. For instance, let M be an infinite interpretation for some language L and let $\mathrm{Th}(M)$ be the set of all sentences of L true in M. We can easily see that $\mathrm{Th}(M)$ is a negation-complete theory (because every sentence is either true or false on any given model). Thus, any two models of $\mathrm{Th}(M)$ are going to be elementarily equivalent, since the theory specifies the truth value of every sentence. By the Löwenheim-Skolem-Tarski theorem (discussed below), we see that $\mathrm{Th}(M)$ has models of every cardinality larger than that of L. But since isomorphic models must have the same cardinality, we see that two of these models of different sizes will be elementarily equivalent but not isomorphic.

3 Logic with uncountable languages:

In the system Q, we have required so far that there be exactly countably many constants, variables, functions, and predicates/relations. Hunter mentions relaxing the rule so there can be finitely many or zero members of one of these categories. To talk about the LST theorem in full generality, we'll need to allow there to be κ for any cardinal κ, countable, finite, zero, or uncountable. (Remember from before that for any cardinality, we can find a larger cardinality by considering powersets. Thus, there are infinitely many uncountable cardinals.)

An important fact to note is that for any infinite cardinal, $\kappa \times \kappa = \kappa$, and thus if there are κ symbols in a language, there are $\kappa \times \kappa = \kappa$ pairs of symbols, and then $(\kappa \times \kappa) \times \kappa = \kappa \times \kappa = \kappa$ triples of symbols, and in general κ strings of every finite length. Since there are only countably many finite lengths, there are thus only κ many finite strings from the language, and thus κ wffs. On the other hand, if there are only finitely many symbols, then there are countably many wffs. To be specific, we will say that the *size of a language* is the number of formulas in the language, rather than the number of symbols in the language. Thus, every language is infinite, though some are countable (namely, those with finitely many or countably many symbols) and some are uncountable.

Another very useful fact about infinite sets is the Well-Ordering Principle (also known as the Axiom of Choice or Zorn's Lemma, if you want to look it up – there is a lot written about this principle because it used to be very controversial in the world of mathematics). Basically, it generalizes the fact that you can number the elements of a countable set. However, instead of using the natural numbers to do the numbering, you have to use what are called "ordinals". Basically, what it says is that if you've got a set of cardinality κ, then you can number its elements in some way so that (1) every non-empty subset has a first element, and (2) every element of the set has fewer than κ predecessors. Standard numberings of countable sets clearly have both these properties, because every element has only finitely many predecessors, which is less than $\aleph_0$, which is the cardinality of the whole set. The most important thing about well-orderings is that you can use them to do "transfinite induction", which is basically like normal induction.

4 Proving the Downwards Löwenheim-Skolem-Tarski theorem: Any consistent theory in a language of size κ has a model of size at most κ.

4.1 Generalized Henkinization

Let T be a consistent theory in a language L of size κ. Let L' be the result of adding κ new constant symbols to L and otherwise leaving it unchanged. L' is also a language of size κ. Thus, we can use the Well-Ordering Principle to number the wffs of L' with exactly one free variable as A_α. Because only finitely many of the new constant symbols appear in each of these formulas, and there are fewer than κ formulas before A_α, we can name some of the new constant symbols c_α, where c_α is a new constant symbol that doesn't appear in any formula before A_α. Then let us name the formula $(A_\alpha c_\alpha / v \supset \bigwedge v A_\alpha)$ as H_α (the Henkin sentence for A_α), where v is the one variable that occurs free in A_α.

Now let us consider the theory $T_\alpha = T \cup \{H_0, H_1, \ldots, H_\alpha\}$ which is the result of adding the first α of these Henkin sentences to the theory T. In the countable case we proved by induction that each of these theories was consistent, and in the general case we can prove it by transfinite induction. This is because adding just one of these Henkin sentences can never render a consistent theory inconsistent (because the constant involved is always one that doesn't appear in the previous set, given how we chose the constants). If we then let T' be the union of all the T_α then we see just as before that T' is consistent, since every finite subset is.

Thus, if T is a consistent theory in a language L of size κ, then we can extend it to a consistent Henkinized theory T' in a language L' of size κ.

4.2 Generalized Lindenbaum Lemma

If T' is a consistent Henkinized theory in a language L' of size κ, then we can extend it to a maximal consistent theory in the same language, which will then be consistent, Henkinized, and negation-complete.

The proof is as in the countable case. We use well-ordering to number all the sentences A'_α of L'. Then we let $T''_0 = T'$ and $T''_{\alpha+1} = T''_\alpha \cup \{A'_\alpha\}$ if that is consistent, and just T''_α otherwise. Then we let T'' be the union of all the T''_α.

A simple transfinite induction shows that each T''_α is consistent, just as in the countable case, and as a result the whole T'' is consistent, because proofs are finitely long.

T'' is clearly Henkinized because it extends a Henkinized theory and we haven't added any new formulas to the language. To show that it is maximal consistent, we note that every sentence appears as some A'_α, because we have numbered them all. Therefore, if the sentence doesn't show up in T'', then it must not show up in $T''_{\alpha+1}$, which means that $T''_\alpha \cup \{A'_\alpha\}$ is inconsistent. Thus, $T'' \cup \{A'_\alpha\}$ is inconsistent, which means that T'' must be maximal consistent, and we have finished proving the generalized Lindenbaum lemma.

4.3 Completeness

Using Henkinization and the Lindenbaum lemma, we can extend every theory T in some language L of size κ to a maximal consistent Henkinized theory T'' in some language L' also of size κ. If we can show that T'' has a model of size at most κ, then this means that T has a model also of this size, and this will complete the downwards Löwenheim-Skolem-Tarski theorem.

But we can construct such a model exactly as we did in the countable case (no transfinite induction or anything else needed differently this time, unlike when we generalized Henkinization and the Lindenbaum lemma). Just let the set of terms of the language be the domain (which will have size at most κ, because the language does) and say that an atomic sentence is true in the interpretation just in case it is in T''. By the same (finite) induction that we had before, we see that every sentence is true in the interpretation iff it is in T'', and thus we have constructed a model for T''. In particular, it is a model with domain of size κ, QED.

5 The Full (Upward and Downward) Löwenheim-Skolem-Tarski theorem: Any theory in a language of size κ with an infinite model has models of every size $\lambda \geqslant \kappa$

In particular, any theory in a countable language (like most of the theories we consider in this class) with an infinite model has models of every infinite cardinality, and is therefore not totally categorical (though it may be categorical in some cardinalities). Here is the proof.

Let T be a theory in language L of size κ. Let λ be any cardinal at least as large as κ. Let L' be L with λ new constants added. Let U be the set of all sentences $c \neq d$ where c and d are two of the new constants added in L', so that U is the theory that says that any two of these new constants stand for distinct objects.

Now consider the theory $T \cup U$. Any model of this theory is also a model of T, so if we can show that it has a model of size exactly λ, then we will be done, because we can construct such a model for each cardinality at least as large as κ. But by the downwards LST theorem, we see that if $T \cup U$ is consistent, then it has a model at most as large as λ. And because U says that each of the λ new constants refers to a different object, then we see that any model must be at least as large as λ. So we just need to show that $T \cup U$ has at least one model, and we will be done.

But by the completeness theorem (and in particular the compactness theorem that is its corollary), we see that if every finite subset of $T \cup U$ has a model, then so does $T \cup U$. So let T_0 and U_0 be arbitrary finite subsets of T and U respectively. If we can show for each of these finite subsets that $T_0 \cup U_0$ has a model, then we will be done. But we already know that T_0 has an infinite model M (namely, some infinite model of T). And there are only finitely many constants that appear in U_0, so if we just assign each of these constants to a different element of the domain of M, then it becomes a model of U_0 also, and thus of $T_0 \cup U_0$. Thus, each finite subset of $T \cup U$ has a model, and so $T \cup U$ as a whole does as well, and we have shown above that if it has a model, then it has a model of size exactly λ.

Thus, T has a model of each cardinality at least κ, QED.

6 Categoricity

We say that a theory is *totally categorical* if all of its models are isomorphic. We say that it is *categorical in cardinality* κ if all of its models of size κ are isomorphic. By the Löwenheim-Skolem-Tarski theorem, we see that no first-order theory with an infinite model is totally categorical, because it will have models of many different cardinalities, which are thus not isomorphic.

However, there are some theories that are totally categorical. For instance, let L be a language with no constants or function symbols, and whose only relation symbol is "$=$", and with one 1-place predicate symbol P. Let T be the theory that consists of just the single sentence $\bigvee x \bigvee y[Px \wedge \sim Py \wedge \bigwedge z(x = z \vee y = z)]$. Without much work, one can see that any model of this theory must have exactly two objects in the domain, one with property P and one without it. Thus, this theory is categorical. It turns out that similarly, for any finite model M, the set $\mathrm{Th}(M)$ of all sentences true in M is categorical, and these are basically all the totally categorical theories.

Now, let's consider a theory T in a language L of size κ that only has infinite models. Let's assume that T is not negation complete, so that there is some sentence ϕ such that $T \cup \{\phi\}$ and $T \cup \{\sim\phi\}$ are both consistent. Now each of these theories has an infinite model, since they are both consistent, and any model of T must be infinite. By the full LST theorem, we see that both $T \cup \{\phi\}$ and $T \cup \{\sim\phi\}$ must have models of every size at least κ. These models can't be isomorphic because they're not even elementarily equivalent (ϕ is true in one model and not the other). So, T is not categorical in any cardinality larger than κ. Hence, by contraposition, if a theory (with only infinite models) is categorical in some cardinality larger than the size of its language, then it must be negation-complete. (This is called the Łoś-Vaught test.)

Let L be the language with no symbols at all other than "$=$". Let T be the theory consisting of the following infinitely many sentences:

$$\bigvee x \bigvee y (x \neq y) \tag{1}$$

$$\bigvee x \bigvee y \bigvee z (x \neq y \wedge x \neq z \wedge y \neq z) \tag{2}$$

$$\bigvee x \bigvee y \bigvee z \bigvee w (x \neq y \wedge x \neq z \wedge x \neq w \wedge y \neq z \wedge y \neq w \wedge z \neq w) \tag{3}$$

$$\vdots$$

Each sentence i says "There are more than i objects in the domain". Thus, T has no finite models, because if the model has only n objects, then sentence n would be false. This theory is in addition categorical in every infinite cardinality, because there is no structure to preserve other than the cardinality of the domain.

7 Non-Standard Models

So far I have only given examples of theories that are either totally categorical or categorical in all infinite cardinalities. Now I will show that some common theories are not categorical.

Let L be the language with constant "0", relation "$=$", 1-place function "s", and 2-place functions "$+$" and "$\times$". Let R be the theory of L consisting of the following sentences, known as Robinson's arithmetic ("$p \equiv q$" means "$(p \supset q) \wedge (q \supset p)$"):

$$\bigwedge x \bigwedge y [s(x) = s(y) \supset x = y] \tag{4}$$

$$\bigwedge x [\bigvee y (s(y) = x) \equiv x \neq 0] \tag{5}$$

$$\bigwedge x (x + 0 = x) \tag{6}$$

$$\bigwedge x \bigwedge y [x + s(y) = s(x + y)] \tag{7}$$

$$\bigwedge x (x \times 0 = 0) \tag{8}$$

$$\bigwedge x \bigwedge y [x \times s(y) = (x \times y) + x] \tag{9}$$

It should be clear that $\mathbb{N}$ (with 0 as zero, $+$ as addition, $\times$ as multiplication, and s as successor: $s(x) = x + 1$) form a model for R, and we call this the *standard model*.

Now let L' be the language L with a new constant symbol "c" added. Let R' be the set of sentences $\{c \neq 0, c \neq s(0), c \neq s(s(0)), \ldots\}$. Every finite subset of $R \cup R'$ can be made true in an extension of the standard model for R where c is interpreted as some number not mentioned in the finitely many axioms from R' that appear. Thus, every finite subset of $R \cup R'$ is consistent, so by the compactness theorem, $R \cup R'$ is as well.

Now let M' be a model of $R \cup R'$. If we reduce M' to a model M of the language L (just by forgetting which element c was assigned to), then we can see that M must be a model of R that is not isomorphic to $\mathbb{N}$ (because whatever element c was assigned to can't be paired with any element of $\mathbb{N}$), and is thus called a *non-standard model*. By the downwards LST theorem, we know there are countable models with this property, in addition to the obvious uncountable ones. Thus, this theory is not categorical in cardinality $\aleph_0$.

We can use a similar construction to show that the theory of the real numbers $\mathbb{R}$ is not categorical in any cardinality.