Part (d) of Hunter's Proof of Henkin's Completeness Theorem for PS

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The Lindenbaum Construction. We assume that we have an enumeration $\langle A_1, A_2, \ldots A_n, \ldots \rangle$ of all the formulas A_i of P. [This is part (c) of Hunter's proof. He does a pretty good job explaining parts (b) and (c), so I won't rehearse those parts here.] Now, let Γ be an arbitrary p-consistent set of formulas. Using our enumeration of P formulas, we construct (Lindenbaum-style) an infinite sequence of sets of formulas $\boldsymbol{\Gamma} = \langle \Gamma_0, \Gamma_1, \ldots \Gamma_n, \ldots \rangle$ — using Γ as the starting point of our construction — in the following way:

$$\Gamma_0 \stackrel{\text{def}}{=} \Gamma, \text{ and for } n \geq 1, \Gamma_n \stackrel{\text{def}}{=} \begin{cases} \Gamma_{n-1} \cup \{A_n\} \text{ if this set is } p\text{-consistent.} \\ \Gamma_{n-1} \text{ otherwise.} \end{cases}$$

This sequence $\boldsymbol{\Gamma}$ has several important properties, which I will now prove by induction.

For all $n \geq 0$, $\Gamma_n \in \boldsymbol{\Gamma}$ is p-consistent.

Basis Step. $\Gamma_0 = \Gamma$ is p-consistent, by assumption.

Inductive Step. Assume (IH) that Γ_i is p-consistent, for all i such that $0 \leq i < n$. And, use this to prove that Γ_n is p-consistent. So, the (IH) tells us that Γ_{n-1} is p-consistent. But,

$$\Gamma_n = \begin{cases} \Gamma_{n-1} \cup \{A_n\} \text{ if this set is } p\text{-consistent.} \\ \Gamma_{n-1} \text{ otherwise.} \end{cases}$$

So, *either* (*i*) $\Gamma_n = \Gamma_{n-1} \cup \{A_n\}$, which is p-consistent by construction, *or* (*ii*) $\Gamma_n = \Gamma_{n-1}$, which is p-consistent by the inductive hypothesis (IH). Either way, Γ_n is p-consistent. □

For all $n \geq 1$, $\Gamma'_n \stackrel{\text{def}}{=} \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_n = \Gamma_n$.

Basis Step. By the construction of $\boldsymbol{\Gamma}$ and the definition of Γ'_n, we have:

$$\begin{aligned}
\Gamma'_1 = \Gamma_0 \cup \Gamma_1 &= \Gamma_0 \cup \begin{cases} \Gamma_0 \cup \{A_1\} \text{ if this set is } p\text{-consistent.} \\ \Gamma_0 \text{ otherwise.} \end{cases} && [\text{definitions of } \Gamma'_1, \Gamma_1] \\
&= \begin{cases} \Gamma_0 \cup (\Gamma_0 \cup \{A_1\}) \text{ if this set is } p\text{-consistent.} \\ \Gamma_0 \cup \Gamma_0 \text{ otherwise.} \end{cases} && [\text{logic}] \\
&= \begin{cases} \Gamma_0 \cup \{A_1\} \text{ if this set is } p\text{-consistent.} \\ \Gamma_0 \text{ otherwise.} \end{cases} && [\text{set theory}] \\
&= \Gamma_1 && [\text{definition of } \Gamma_1]
\end{aligned}$$

Inductive Step. Assume (IH) $\Gamma'_i = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_i = \Gamma_i$, for all i such that $0 \leq i < n$. And, use this to prove that $\Gamma'_n = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_n = \Gamma_n$. We have:

$$\begin{aligned}
\Gamma'_n &= (\Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_{n-1}) \cup \Gamma_n && [\text{definition of } \Gamma'_n] \\
&= \Gamma_{n-1} \cup \Gamma_n && [\text{(IH)}] \\
&= \Gamma_{n-1} \cup \begin{cases} \Gamma_{n-1} \cup \{A_n\} \text{ if this set is } p\text{-consistent.} \\ \Gamma_{n-1} \text{ otherwise.} \end{cases} && [\text{definition of } \Gamma_n] \\
&= \begin{cases} \Gamma_{n-1} \cup (\Gamma_{n-1} \cup \{A_n\}) \text{ if this set is } p\text{-consistent.} \\ \Gamma_{n-1} \cup \Gamma_{n-1} \text{ otherwise.} \end{cases} && [\text{logic}] \\
&= \begin{cases} \Gamma_{n-1} \cup \{A_n\} \text{ if this set is } p\text{-consistent.} \\ \Gamma_{n-1} \text{ otherwise.} \end{cases} && [\text{set theory}] \\
&= \Gamma_n && [\text{definition of } \Gamma_n] \quad \square
\end{aligned}$$

For all $n \geq 0$, $\Gamma_n \subseteq \Gamma_{n+1}$.

> **Basis Step**. *Either* $\Gamma_1 = \Gamma_0 \cup \{A_1\}$ [$\therefore \Gamma_0 \subset \Gamma_1$] *or* $\Gamma_1 = \Gamma_0$. In either case, $\Gamma_0 \subseteq \Gamma_1$. □
>
> **Inductive Step**. Assume (IH) $\Gamma_i \subseteq \Gamma_{i+1}$, for all i such that $0 \leq i < n$. And, use this to prove that $\Gamma_n \subseteq \Gamma_{n+1}$. *Either* $\Gamma_{n+1} = \Gamma_n \cup \{A_{n+1}\} \supset \Gamma_n$ *or* $\Gamma_{n+1} = \Gamma_n$. So, $\Gamma_n \subseteq \Gamma_{n+1}$. [We didn't even need to *use* the inductive hypothesis here – just the definition of Γ_n.] □

For all m, n such that $0 \leq m \leq n$, $\Gamma_m \subseteq \Gamma_n$. When $m = n$, this is trivial. When $m < n$, this follows by a simple induction using and the transitivity of $\subseteq$. So, we have the following *chain* of sets:

$$\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots \subseteq \Gamma_n \subseteq \cdots \subseteq \Gamma'$$

Note: Using , we can provide a simpler inductive proof of . It is a basic fact about sets that if $x \subseteq y$, then $x \cup y = y$. So, by a simple induction on this set-theoretic fact, if $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots \subseteq \Gamma_n$, then $\Gamma_0 \cup \cdots \cup \Gamma_n = \Gamma_n$. I included the direct inductive proof of to give more exposure to induction.

Using – (plus some additional metatheoretic reasoning), we can now prove Lindenbaum's Lemma.

Lindenbaum's Lemma. Every p-consistent set Γ is a subset of some maximal p-consistent set Γ'.

Proof. Every p-consistent set Γ is a subset of its infinite Lindenbaum set $\Gamma' = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_n \cup \cdots$, which is maximal p-consistent. Obviously, $\Gamma \subseteq \Gamma'$. We just need to prove that Γ' is maximal p-consistent.

- First, we show that Γ' is p-consistent. Facts – imply that *all finite subsets* Γ_n *of* Γ' *are* p*-consistent*. But, just because all finite subsets of an infinite set have a property, this doesn't mean that the entire set must have that property (*e.g.*, finitude!). But, in this case, all is beer and skittles, because if all finite subsets of a set are p-consistent, then the entire set is p-consistent. Why? *Because all derivations are finite*. If no finite subset Γ_n of Γ' syntactically entails a contradiction, then neither does any infinite subset of Γ', including Γ'. So, Γ' is p-consistent. A rigorous *reductio* proof follows.

  > *Proof.* Assume, for *reductio*, that Γ' is p-*in*consistent. Then, there is some formula B such that $\Gamma' \vdash_{PS} B$ and $\Gamma' \vdash_{PS} \sim B$. Since *all derivations are finite*, there must be *finite subsets* $\Delta_1 \subseteq \Gamma'$ and $\Delta_2 \subseteq \Gamma'$ such that $\Delta_1 \vdash_{PS} B$ and $\Delta_2 \vdash_{PS} \sim B$. Of course, the union $\Delta_1 \cup \Delta_2$ is p-inconsistent, since $\Delta_1 \cup \Delta_2 \vdash_{PS} B$, and $\Delta_1 \cup \Delta_2 \vdash_{PS} \sim B$. Now, consider the formulas in the union $\Delta_1 \cup \Delta_2 \subseteq \Gamma'$. Order them using our enumeration. They will be $\langle A_m, \ldots, A_n \rangle$, for $0 \leq m \leq n$. And, because $\{A_m, \ldots, A_n\} \subseteq \Gamma'$, we know that $A_m \in \Gamma_m$, ..., $A_n \in \Gamma_n$. Why? Because, by construction, for any k, A_k can only appear in Γ' by appearing in Γ or Γ_k. And, since (by) for all k, $\Gamma \subseteq \Gamma_k$, this is equivalent to saying that the only way that A_k can appear in Γ' is by appearing in Γ_k. Moreover, (also from) we know that for all m, n such that $0 \leq m \leq n$, $\Gamma_m \subseteq \Gamma_n$. Thus, we have $\{A_m, \ldots, A_n\} \subseteq \Gamma_n$. Hence, Γ_n is p-inconsistent, which contradicts .[1] □

- Next, we will show that Γ' is *maximal* p-consistent. Recall, maximality means that adding any formula $A_n \notin \Gamma'$ to Γ' yields a set $\Gamma' \cup \{A_n\}$ that is p-inconsistent. So, for *reductio*, assume that there is some formula A_n such that $A_n \notin \Gamma'$, but $\Gamma' \cup \{A_n\}$ is p-consistent. Since, $A_n \notin \Gamma'$, we also know that $A_n \notin \Gamma_n \subseteq \Gamma'$. But, by construction, if $A_n \notin \Gamma_n$, then this must be because $\Gamma_{n-1} \cup \{A_n\}$ is p-inconsistent (otherwise, A_n would have been added to Γ_{n-1} to yield Γ_n). But, since $\Gamma_{n-1} \cup \{A_n\}$ is p-inconsistent, we must have $\Gamma_{n-1} \vdash_{PS} \sim A_n$ (by metatheorem 32.8). Then, because $\Gamma_{n-1} \subseteq \Gamma'$, we also know that $\Gamma' \vdash_{PS} \sim A_n$. Therefore, another application of metatheorem 32.8 shows that $\Gamma' \cup \{A_n\}$ is p-inconsistent. Contradiction. So, our assumption that Γ' is not maximal p-consistent is false. □

That completes our proof of Lindenbaum's Lemma, which sets us up for the rest of Henkin's proof. □

[1] Whichever way you prefer to prove that Γ' is p-consistent, you will have to make essential use of the crucial fact that *all derivations in PS are finite*. If we allow rules of inference that involve infinitely many premises, no such argument will be available to us. This is where the "finitely many premises" restriction in our definition of a rule of inference rears its head.