Reflections on Skolem's Relativity of Set-Theoretical Concepts[†]

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From 1922 onwards Skolem maintained that set-theoretical concepts are relative (in a sense of 'relative' that we must discern). In 1958 he viewed all mathematical notions as relative. The main instrument he used in his argument for set-theoretical relativity is the Löwenheim-Skolem theorem, in the form that every consistent first-order countable set of sentences has a countable model. In 1938 he said of this theorem that 'its most important application is the critique of the set-theoretical concepts, and most especially that of the higher infinite powers' (Skolem [1941], p. 460).[1] My goal is to review Skolem's argument, chiefly as it appears in Skolem [1922], with the aim of gaining some insight into the ontology of set theory. Although I will often quote Skolem and try to be true to his words, what follows is not put forward as a faithful reconstruction of Skolem's actual view, but rather as an attempt to present it as a sensible one.

Skolem is commonly portrayed as arguing that certain otherwise well understood concepts are suspect simply because they cannot be characterized in a first-order language; in particular that, since all first-order formalizations of set theory (if consistent) have countable models, the concept of uncountability is flawed. I hope to show that Skolem's position is more solid than that. I see Skolem as arguing that all the evidence that has been given for the existence of uncountable sets is inconclusive, and the reason why he insists on considering countable models is that axiomatization was put forward at the time as the only way to secure set theory, and what sets are and which sets exist was claimed to be determined by the axioms and their models (much as what Euclidean geometry is about was claimed to be determined by Hilbert's axioms and their models). In this situation, bringing countable models into play was perfectly in order, all the more so as no other models could be supplied without set-theoretical means. Today

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[1] 'Son application la plus importante est, en effet, la critique des concepts de la théorie des ensembles, et plus spécialement celle des puissances infinies supérieures.'

we may no longer uphold this claim, but if we do believe that there are uncountable sets, we should be willing to comply with Skolem's requirement that their existence be substantiated by some means other than mere formal postulation.

Replics to Skolem usually take for granted that there are uncountable sets (and, of course, if there are, then our inability to characterize them with certain limited means is only a sign of the inadequacy of these means). They remind us that set theory is really about sets and then proceed to argue that Skolem's critique is misguided because it does not deal with the real thing, but rather has to do with a reinterpretation of the set-theoretical language, either by letting the variables range over objects other than sets or by limiting the domain of quantification to a fragment of the set-theoretical universe. At most, that there are uncountable sets is assumed to be conclusively established by Cantor's diagonal proof that the set of all sets of integers is uncountable. It must be pointed out, however, that Cantor's method does not reach that far; by diagonalizing we can prove that no countable set of sets of integers contains all sets of integers, but in order to conclude from this that there is an uncountable set, one needs to show that there is such thing as the set of all sets of integers.

These points are elaborated in sections 1 to 5 of this paper, where Skolem's argument is presented and discussed and some replies to it are considered. In the concluding section 6, we touch upon the existence of uncountable sets and sketch a view according to which a set-theoretical entity like the set of all sets of integers is to be seen as a kind of idealized completion of the open-ended plurality of pre-set-theoretically specifiable collections of integers. Some effort is made to render such a view plausible, and a fuller account of it will be given elsewhere.

1. Skolem's Starting Point

Skolem's [1922] was written as a critique of Zermelo's axiomatization of set theory in Zermelo [1908], against which eight points are leveled. Of these points, some are peculiar to Zermelo's axiomatization, which Skolem finds defective on several counts, and to some of which he offers a solution. Thus Skolem proposes replacing Zermelo's notion of *definite Aussage* by that of first-order formula in the language of set theory in the formulation of the separation axiom[2] (point 2), and adding the axiom of replacement in the form which is now customary, because without it, says Skolem, Zermelo's system is insufficient 'for the foundation of the usual set theory' (point 4).

[2] This is Zermelo's definition of the term *definit*: 'A question or assertion $\mathfrak{E}$ is said to be *definite* if the fundamental relations of the domain, by means of the axioms and the universally valid laws of logic, determine without arbitrariness whether it holds or not. Likewise a 'propositional function' [*Klassenaussage*] $\mathfrak{E}(x)$, in which the variable term x ranges over all individuals of a class $\mathfrak{K}$, is said to be definite if it is definite for *each single* individual x of the class $\mathfrak{K}$' (Zermelo [1908], p. 201).

But, most importantly, there is a particular issue, a certain relativity of set-theoretical concepts that, according to Skolem, affects not only Zermelo's system but every axiomatized set theory as well (point 3).

Skolem's charge of relativity depends crucially on his substitution of a first-order schema for Zermelo's original axiom of separation. He didn't see this substitution as a major change, but rather as a needed and uncontroversial correction, as can be gathered from the remarks preceding it:

> A very deficient point in Zermelo is the notion 'definite proposition'. Probably no one will find Zermelo's explanations of it satisfactory. So far as I know, no one has attempted to give a strict formulation of this notion; this is very strange, since it can be done quite easily and, moreover, in a very natural way that immediately suggests itself. (Skolem [1922], p. 292)

It must be stressed, however, that Zermelo strenuously rejected Skolem's revision of the separation axiom. In Zermelo [1929] he set out to clarify his notion of a definite proposition by axiomatizing it, and his attempt was again criticized by Skolem in Skolem [1930]. In any event, Zermelo never accepted the first-order formulation of set theory, which he saw as an aspect of what he called the *finitistic prejudice* [*das finitistische Vorurteil*] ([1932], p. 85). In his insightful analysis of the models of ZF (Zermelo [1930]), he used what amounts to a second-order version of both the separation and replacement axioms.[3]

Skolem argues against axiomatized set theory. But the question arises whether his intended target is just axiomatized set theory, or rather set theory in general. In his [1985], Benacerraf subjects Skolem's paper to a close scrutiny and concludes that in 1922 Skolem only argued against axiomatized set theory, and that the whole of Skolem [1922] was written as a *reductio ad absurdum* of the claim that axiomatized set theory is a foundation of mathematics.

The one-paragraph conclusion of Skolem [1922] begins with this sentence: 'The most important result above is that set-theoretical notions are relative' (p. 300). According to Benacerraf ([1985], p. 97), these words are 'certainly susceptible to misunderstanding', because (I fill in the missing argument, which is obvious from what Benacerraf has already said) it is not set-theoretical concepts that have been presumably proved to be relative, but only 'set-theoretical concepts [as] implicitly defined by the axioms' (*ibid.*, p. 91).

But is it so? Skolem argues against axiomatized set theory—but not as opposed to non-axiomatized set theory, *i.e.*, to what in the opening sentence of Skolem [1922] he calls 'the original set theory' [*die ursprüngliche Mengenlehre*], because, strictly speaking, for Skolem there is no such theory,

[3] A brief description of Zermelo's reaction to Skolem and of his views about logic can be found in sections 7 and 8 of Moore [1980].

being contradictory. Skolem says:

> Set theory in its original version led, as we know, to certain contradictions (antinomies), and no one has as yet succeeded in giving a clarification of them that has won general acceptance. In view of this threat to set theory, attempts have been made to develop that theory by means of certain fundamental assumptions, or axioms, in such a way that the part presumed to be correct and useful would remain provable while the contradictions would be avoided. (Skolem [1922], p. 291)

Thus, in Skolem's view, the original set theory is unacceptable and axiomatic set theory is what is left. Therefore, if axiomatic set theory is found to be faulty, set theory is thereby found faulty.

According to the axiomatic conception, no more can be asserted about sets than follows from the axioms. In his [1925] J. von Neumann gives us a clear statement of this conception:

> The methods of logic are not criticized to any extent, but are retained; only the (no doubt useless) naive notion of set is prohibited. To replace this notion the axiomatic method is employed; that is, one formulates a number of postulates in which, to be sure, the word 'set' occurs but without any meaning. Here (in the spirit of the axiomatic method) one understands by 'set' nothing but an object of which one knows no more and wants to know no more than what follows about it from the postulates. The postulates are to be formulated in such a way that all the desired theorems of Cantor's set theory follow from them, but not the antinomies. (von Neumann [1925], p. 395)

We may ascribe to Skolem this view of the axiomatic conception of set.[4]

There seems to be an obvious obstacle to maintaining that Skolem rejected non-axiomatized set theory, namely that throughout the paper Skolem deals with sets not only as axiomatically given but also in an intuitive, or informal way. In some special places the appeal to a presumed understanding of informal set theory is not difficult to dismiss as a conditional assumption. One such place is where Skolem argues 'that Zermelo's axiom system is not sufficient to provide a complete foundation for the usual theory of sets' (Skolem [1922], p. 296), meaning that some informal ways of obtaining sets are not provided for by Zermelo's axioms. There is no need to believe in the meaningfulness of intuitive set theory to make such a claim; it suffices to know what its accepted methods and its accepted results are. According to the last sentence of von Neumann's just quoted paragraph, showing that one (important) accepted result cannot be obtained from the axioms is enough to show that these are incomplete.

However, there are some places, and particularly in his argument for

[4] Referring to Zermelo's axiomatization of set theory, Skolem says: 'The entire content of this theory is after all as follows: for every domain in which the axioms hold, the further theorems of set theory also hold' ([1922], p. 292). Strictly speaking, this is either a vacuous description or a characterization of theoremhood; but it may be read as a suggestion that it is only the axioms and their models that matter.

relativity, where Skolem's use of intuitive set theory cannot be dismissed in this way. Skolem's most powerful weapon against axiomatic set theory, the Löwenheim-Skolem theorem, involves an intuitive handling of set-theoretical concepts, both in its formulation and in its proof, as it deals with sets and functions in a non-axiomatic context. The statement that axiomatic set theory, if consistent, has a countable model is not viewed as expressed by a formula of that theory, and its proof is not carried out from its axioms. Skolem seems to treat his theorem as having an absolute meaning. Is this not strong enough evidence that Skolem views as meaningful the original, informal theory?

Not necessarily—indeed, not at all. One can reject set theory without being forced to relinquish the use of some set-theoretical terms and means. We must distinguish between suitably *defined* sets (or collections) of unproblematic mathematical objects (such as natural numbers) and *arbitrary* sets of such objects. There is no need to call on set theory proper to countenance the set of even numbers, the set of primes or any set given as the range of a primitive recursive function. Arithmetical sets (*i.e.*, sets defined by a first-order formula involving sums and products and quantification over the natural numbers) can also be understood without having recourse to set theory (although Skolem would be ambivalent about their legitimacy). Insofar as we can define a set of natural numbers with the help of well understood concepts and quantification over natural numbers we need no set theory to deal with it, the notion of collection needed for this purpose being prior to set theory. We can even avoid talking about sets or collections by using instead their respective definitions. As far as sets of natural numbers are concerned, set theory is certainly involved when we view all such sets as forming a definite totality; the pre-set-theoretical notion of collection of natural numbers does not yield so much.

That Skolem did not view the use of suitably defined collections as a commitment to set theory can be substantiated in the paper under consideration. Skolem had already proved the theorem of Löwenheim-Skolem in 1920. However, in order to apply it in 1922 in his argument against axiomatized set theory he proves it anew. As Skolem says, in the old proof he employed two distinctly set-theoretical instruments: he took the intersection of an infinite family of sets (thereby defining a set by quantifying over sets) and he made use of the axiom of choice. And he continues:

> The formation of the intersection can be avoided immediately by use of a recursive definition; but here, where we are concerned with an investigation in the foundations of set theory, it will be desirable to avoid the principle of choice as well. Therefore I now indicate very briefly how this can be done. It will also appear from the proof that general set-theoretical notions are unnecessary for the understanding of the content of these theorems. (Skolem [1922], p. 293)

It is not simply the avoidance of the axiom of choice that is important, but the fact that the whole proof is carried with no use of set-theoretical machinery. Given an enumeration of a consistent set of sentences, Skolem's proof that they have a countable model involves the construction of a finitely branching infinite tree of finite sequences of positive integers and of an infinite path through it, two apparently set-theoretical objects. However, if the list of sentences is primitive recursive (as is the case for the axioms of set theory considered in the application), both the tree and the required path (giving the model) will be arithmetical.

To sum up, Skolem rejects non-axiomatic set theory, but his rejection does not prevent him from working with suitably defined sets and functions, in particular with primitive recursive sets and functions. He needs a little more than that in order to secure his theorem, but what he needs he can still define without quantification over sets. As a matter of fact, under the assumption of consistency, a model of ZFC can be proven to exist whose domain is the set of natural numbers and whose $\in$-relation is definable with only two numerical quantifiers preceding a primitive recursive matrix.[5]

2. Sets *versus* Collections

The axioms of set theory, as Skolem says paraphrasing Zermelo, are about a domain of objects among which a specific relation subsists.[6] To make matters smoother, we will pretend that the axioms Skolem considers are those of ZFC. Moreover, we may as well use present day terminology and say that the axioms are about some fixed but unspecified model $\mathcal{M} = (M, E)$. The elements of M, whose nature is a matter of no concern to us, are the *sets* and E is the *membership relation*. To emphasize once more, for Skolem, a set is just a member of the domain of this tacitly assumed model. But we want to compare these objects called 'sets' (as regards their behavior in $\mathcal{M}$) with what we usually understand by a set. In order to avoid misunderstandings, we shall follow Skolem in distinguishing (axiomatic) *sets* [*Mengen*] from intuitive *collections* [*Zusammenfassungen*]. Thus we are to compare sets with collections. We will also be careful to distinguish the membership relation E of $\mathcal{M}$ from the relation $\in$ which

[5] If Skolem were to admit only recursive, or even recursively enumerable sets, he could not guarantee that every consistent recursively axiomatized theory has a countable model. It may be worth remarking at this point that in order for Skolem to be able to wield his theorem against set theory, he cannot keep a strict constructivist course, for, as shown by McCarty and Tennant [1987], Löwenheim-Skolem's theorem cannot be proven intuitionistically. Thus, a constructivist is not entitled to appeal to this theorem to draw philosophical lessons the way Skolem appears to be doing. I thank an anonymous referee for pointing out this somewhat ironical situation to me and for drawing my attention to this reference.

[6] Thus Zermelo: 'Set theory is concerned with a *domain* $\mathfrak{B}$ of individuals, which we shall call simply *objects* and among which are the *sets* ... We say of an object a that it 'exists' if it belongs to the domain $\mathfrak{B}$ ... Certain *fundamental relations* of the form $a \in b$ obtain between the objects of the domain $\mathfrak{B}$'(Zermelo [1908], 201).

obtains between an object and a collection to which it belongs.

Suppose that $\mathcal{M}$ is the implicit model of the axioms of ZFC. To each set $a \in M$ a collection of elements of M corresponds, namely the collection

$$a^* = \{x \in M : x \, E \, a\},$$

so that for every $x \in M$, x is a member of a^* in the informal sense ($x \in a^*$) if and only if x is a member of a according to $\mathcal{M}$ ($x \, E \, a$).

Before going into Skolem's argument, it will prove useful to point out some more connections between what happens inside the implicit model and the informal set-theoretical notions. One consequence of the axioms is that there is a set which plays the role of the collection of natural numbers. We call this set 'ω' (thus $\omega \in M$) and we refer to the collection of natural numbers as $\mathbf{N}$. We also have to take into account relations and functions, both in the sense of $\mathcal{M}$ and in the informal sense. A relation is a set (respectively, a collection) of ordered pairs. If r is a relation in the sense of the model (so that $r \in M$), we assign to r the collection $r^\dagger$ of those ordered pairs (in the informal sense) $\langle x, y \rangle$ of elements of M such that, for some $u \in M$, $u \, E \, r$ and u is the ordered pair (in the sense of $\mathcal{M}$) of x and y. It is obvious that if r is a function in the sense of $\mathcal{M}$, then $r^\dagger$ is a function in the informal sense, and it is also obvious that if r is a bijection in the sense of $\mathcal{M}$ between the sets a and b, then $r^\dagger$ is a bijection in the informal sense between the collections a^* and b^*.

Skolem does not say 'in the sense of the model', but rather 'in the sense of the axiomatization'. Is Skolem here confused, as has sometimes been suggested, *e.g.*, by Hao Wang ([1970], p. 40)? It is not quite clear. However, for the discussion of Skolem's paper we may assume that there is no confusion, because the distinction between referring to the model and to the axiomatization can be easily explained away, since, as mentioned above, Skolem seems to assume that (implicitly) fixing a model is part of the axiomatization. At any rate, Skolem is elusive on this point.

3. Set-theoretical Relativism

We are ready to plunge into Skolem's argument. According to the Löwenheim-Skolem theorem, if the axioms of ZFC are consistent they have a countable model. Accordingly, for all we know the axioms could be about a countable model, that is to say, the domain M of the model $\mathcal{M}$ which is implicitly assumed in the axiomatization could be a countable collection. Let us, then, assume that it is. Since the axioms hold in $\mathcal{M}$, so do the theorems, *i.e.*, those propositions which can be derived from the axioms by means of logical inferences. One of the theorems of ZFC says that there are uncountable sets, *e.g.*, $\mathcal{P}(\omega)$, the power set of ω, whose elements are all subsets of ω. So, there is in M a set a which, according to $\mathcal{M}$, is uncountable. But it is clear that a—or, to be precise, a^*, the collection corresponding to a—is, in fact, countable, because all its members belong to M, which is a countable collection.

The reason for the discrepancy between $\mathcal{M}$'s assessment of a and a's (or rather, a^*'s) true status rests, in Skolem's own words, on the fact that, 'in the axiomatization, "set" does not mean an arbitrarily defined collection; the sets are nothing but objects that are connected with one another through certain relations expressed by the axioms' (Skolem [1922], p. 295). Although to every subset of a there corresponds a definable subcollection of a^* (definable in terms of the model, which is itself definable, as obtained by carrying out the construction of Skolem's outlined proof of the Löwenheim-Skolem theorem), not every definable subcollection of a^* corresponds to some subset of a, as can be seen by diagonalization.[7] Similarly, although to every bijection r in $\mathcal{M}$ between two sets a and b there corresponds an informal bijection $r^\dagger$ between the collections a^* and b^*, not every bijection between these two collections must correspond to some bijection in $\mathcal{M}$ between a and b. In particular, given our model $\mathcal{M}$ we can define a bijection between ω^* and $(\mathcal{P}(\omega))^*$, but no such bijection corresponds to a bijection in $\mathcal{M}$—because none exists: that is what it means for $\mathcal{P}(\omega)$ to be uncountable in $\mathcal{M}$. We can refer to this situation by saying that (although a^* is indeed countable) a is uncountable *relative* to the axiomatization, or to $\mathcal{M}$.

Skolem considers two more cases of relativity, but only as possibilities, since (contrary to the case of uncountability) he has no proof to offer, only plausibility arguments. The two cases in question are: (1) a set can be finite (in the sense of Dedekind) according to the axiomatization although the collection of its E-members is actually infinite; (2) ω may differ essentially from the collection of all natural numbers (ω^* and $\mathbf{N}$ may not be isomorphic).

Both cases of relativity are also due to the provable poverty of the model, more precisely, to the fact that definable collections of elements of the model and definable relations between elements of M can be exhibited that correspond to no set or relation in $\mathcal{M}$. Thus, a set a is finite according to $\mathcal{M}$ if there is no bijection in $\mathcal{M}$ between a and a proper subset of a. Now, it is possible that for some suitable such a we can define a bijection R between a^* and a proper subcollection B of a^*, thereby showing that a^* is actually infinite. In such a case, either (i) there is no subset b of a such that $B = b^*$, or (ii) there is no bijection r in $\mathcal{M}$ such that $R = r^\dagger$.

The possibility that ω be relative is also explained in similar terms. By

[7] With some detail: From a definable enumeration of M we first get a definable enumeration $\langle x_n : n \in \mathbf{N}\rangle$ of a^* and a definable enumeration $\langle y_n : n \in \mathbf{N}\rangle$ of all subsets of a (i.e., of all $x \in M$ such that for all $u \in M$, if $u\,E\,x$, then $u\,E\,a$). Then we see that $B = \{x_n : \neg x_n\,E\,y_n\}$ is a definable subcollection of a^* which corresponds to no subset of a, i.e., for all $n \in \mathbf{N}$, $B \neq y_n^*$.

definition, ω is the smallest inductive set, that is, the smallest set containing the number zero (the empty set) and containing the successor of each of its elements.[8] Technically, ω is the intersection of all inductive sets. The axioms guarantee that this intersection exists provided there are inductive sets at all—and that there are is what the axiom of infinity asserts. If a is an inductive set, ω will be the intersection of all inductive subsets of a. But again, not each subcollection of a^* corresponds to a subset of a. In particular, there can fail to be subsets of a corresponding to some inductive subcollection of a^*. It is, thus, possible that ω contain some objects which do not belong to all inductive subcollections of a^*. Indeed, Skolem suggests, $\mathcal{M}$ could contain more or fewer inductive sets, so that (adapting Skolem's words to our terminology), nothing can prevent the a *priori* possibility that there exist two different models $\mathcal{M}$ and $\mathcal{M}'$ for which different ω would result (Skolem [1922], p. 296).

How does Skolem conceive of this relativism of certain set-theoretical notions? According to what Skolem says about the possibility of two models having different versions of ω and taking into account that in order to speak of relativity there have to be at least two different points of view (and that points of view in an axiomatic context can only mean models of the axioms), we could conjecture that, for Skolem, whatever the relativism of set theoretical concepts is, it is due to the lack of categoricity of the axiomatic theory, *i.e.*, to the existence of non-isomorphic models of ZFC. This is indeed how Skolem's claim is usually dealt with. This is also how von Neumann viewed the relativity of cardinalities in axiomatic set theory.[9]

However, if we follow Skolem's arguments and reasons in the paper, we find no evidence that he viewed relativism as a by-product of lack of categoricity, at least not in Skolem [1922]. Rather we find some hints to the contrary. Thus

(1) Skolem establishes the relativity of set-theoretical notions in section 3 and he argues for the lack of categoricity of the axioms only in section 6. Moreover, he nowhere suggests that lack of categoricity is a source of relativism.

(2) According to Skolem, relativity would be present no matter which axioms for set theory are chosen, being a consequence of the Löwenheim-Skolem theorem, which guarantees the existence of a countable model (Skolem [1922], p. 296). But this theorem does not imply the existence of non-isomorphic models. For all Skolem knew in 1922, the theory could be extended so as to have a unique (countable, of course) model up to isomorphism. Relativity would still be present, but not lack of

[8] The successor of a set $x \in M$ is the set $x \cup \{x\}$; more precisely, it is the unique $y \in M$ such that for all $z \in M$, $z E y$ if and only if $z E x$ or $z = x$.

[9] See von Neumann [1925], II §3 (pp. 408–409) and the last half of VI (pp. 412–413).

categoricity.

Seven years later, Skolem related in an explicit way relativism and lack of categoricity. In his [1929a] he says:

> A very probable consequence of relativism is again that it may not be possible to characterize *completely* the mathematical concepts; this already holds for the concept of integer. Thus the question arises whether the usual idea of the unicity or categoricity of mathematics is not an illusion. (Skolem [1929a], p. 224)[10]

Observe that Skolem says that the lack of categoricity is very likely a consequence of relativism. Thus, for him, plurality of models may follow from relativism. He does not see relativism as following from lack of categoricity. Skolem's relativism, then, cannot be the result of set-theoretical concepts having different extensions in different models (say the large intended model and some non-intended, countable one). Rather, Skolem's relativism may be the source of different models, for, since $\mathcal{M}$, being countable, does not contain all definable subcollections of ω^*, it might be possible (though, as Skolem foresaw, very difficult) to extend $\mathcal{M}$ to a larger model containing a new subset of ω (Skolem [1922], pp. 298–299).[11]

In Skolem [1922], he does not say what we must understand by 'relativism', but he gives several clues, for example

> The relativity is due to the fact that to be an object in $[\mathcal{M}]$ means something different and far more restricted than merely to be in some way definable. That this relativity must be inseparably bound up with every thoroughgoing axiomatization is clear; for it rests upon the general theorems of mathematical logic mentioned above [the Löwenheim-Skolem theorem]. (Skolem [1922], p. 296)

We already know what this means: Because of the general applicability of the Löwenheim-Skolem theorem, nothing can prevent the axioms from having a countable model, thus nothing can prevent us from taking the model attached to the axiomatization to be countable, but then we see that not every definable subcollection of the collection corresponding to a set in $\mathcal{M}$ corresponds to a set in $\mathcal{M}$. Owing to the poverty of the model, some sets seem to have mathematical properties that the corresponding collections lack. Such sets, then, have these properties only relative to $\mathcal{M}$ (or, since $\mathcal{M}$ is the model attached to the axiomatization, they have these properties relative to the axiomatization).

[10] 'Eine sehr wahrscheinliche Konsequenz des Relativismus ist es wieder, daß es nicht möglich sein kann, die mathematischen Begriffe *vollständig* zu charakterisieren; dies gilt schon für den Begriff der ganzen Zahl. Dadurch entsteht die Frage, ob nicht die gewöhnliche Vorstellung von der Eindeutigkeit oder Kategorizität der Mathematik eine Illusion ist.'

[11] This was not done until 1963 by Paul Cohen in his proof of the independence of the continuum hypothesis.

About 20 years later, Skolem gave a definition of sorts of the notion of 'relativity' which accords with what he said in 1922. These are Skolem's words (with some innocuous terminological changes). Notice that now Skolem does not follow the practice of Skolem [1922] of distinguishing 'sets' from 'collections'.

> It is equally clear that the subsets of a given infinite set A which appear in a countable model cannot furnish all subsets in the absolute sense of the simple set theory [*i.e.*, of the original theory of Cantor], because these must build a set which is uncountable—and even in an absolute way. This is why in the axiomatic theory the set $\mathcal{P}(A)$ of all subsets of A comprises only those subsets of A *which appear in the model*, but this does not preclude there being as well a multitude of subsets of A that one can define or construct in some way; but these are not subsets of A in the sense of the axiomatization, and consequently they do not appear in the model as elements of $\mathcal{P}(A)$. All the concepts of set theory and therefore those of the whole of mathematics become thus *relativized*. The meaning of these concepts is not absolute; it is relative to the basic axiomatic model. (Skolem [1941], p. 467–468)[12]

Thus, the relativization of the concepts of set theory stems from the fact that the quantifiers appearing in their definition do not range over all collections of the relevant objects that can be specified by any means, but only over those corresponding to sets in the model. It is in this sense that the concepts become relativized to the model.

It is not easy to understand Skolem's position fully, mainly because he seems to speak of relativizing something that does not exist in an absolute sense. As he explicitly says in this paper (Skolem [1941]), there are only relativizations, there is no such thing as the uncountable totality of collections of natural numbers; more generally, there are no uncountable collections. Skolem is clear enough on this matter:

> Since within any axiomatization of set theory or any formal logical system one reasons in such a way that the absolutely uncountable does not exist, the statement that uncountable sets exist can only be considered as a play on words, hence this absolutely uncountable is just a fiction. The true scope of Löwenheim's theorem is precisely this critique of the absolutely uncountable. Briefly: this critique does not reduce the higher infinities of simple set theory

[12] 'Il est également clair que les sous-ensembles d'un ensemble infini M donné, qui se présentent dans un modèle dénombrable, ne peuvent pas fournir tous les sous-ensembles au sens absolu de la théorie simple des ensembles; car ceux-ci doivent former à eux tous un ensemble non-dénombrable et encore d'une façon absolue. C'est pourquoi dans la théorie axiomatique, l'ensemble UM de tous les sous-ensembles de M comprend seulement les sous-ensembles de M *qui se présentent dans le champ*, ce qui n'empêche pas qu'il y ait par ailleurs une multitude de sous-ensembles de M qu'on puisse définir ou construire d'une façon quelconque; mais alors ce ne sont pas des sous-ensembles de M au sens de l'axiomatique, et par conséquent ils ne rentrent pas dans le champ comme éléments de UM. Tous les concepts de la théorie des ensembles et par conséquent de la mathématique tout entière se trouvent ainsi *relativisés*. Le sens de ces concepts n'est pas absolu; il se rapporte au champ axiomatique basique.'

ad absurdum, it reduces them to non-objects. (Skolem [1941], p. 468)[13]

And he adds

> The fact that axiomatizing leads to relativism has been sometimes considered to be the weak spot of the axiomatic method. There is no reason for this. The analysis of mathematical thought, the fixation of fundamental hypotheses and ways of reasoning can only be an advantage for science. It is not a weakness of an axiomatic method that it cannot yield what is impossible. (Skolem [1941], p. 470)[14]

Skolem was faithful to this conception throughout his life. In *Abstract Set Theory*, a monograph published in 1962, one year before his death, he says:

> We must be content with a relativistic conception of set theory. Everything must be conceived in relation to D [the assumed model] as it is supposed to be by the axioms, and we must abandon the idea that the axioms shall yield an absolute notion of 'set' as in Cantor's theory ... Because of the general character of the theorem of Löwenheim and its generalization, it is clear that this set-theoretical relativism is unavoidable if we desire to have an exact reformulation of set theory at all. Of course, it shows the illusory character of the absolutist conceptions of Cantor's theory. ([1962], p. 47)

Do these remarks describe Skolem's attitude in 1922? At least they could. In 1922 Skolem argues that axiomatized set theory yields only relativized notions, but, owing to the alleged inconsistency of Cantorian set theory, there was no set theory but the axiomatized one. So, there were only relative set-theoretical concepts.

4. Skolem's View in a Favorable Light

Can we explain Skolem's notion of relativity in such a way that it is meaningful and reasonable? If we want to do that, we have to avoid considering some intended large model of set theory which yields the true versions of all concepts involved (in particular of uncountability) against which the countable model obtained from the proof of the Löwenheim-Skolem theorem is checked. If we feel comfortable with the idea that there is such a model, then Skolem's relativity, whatever it be, will not be seen as a threat to the legitimacy of the set-theoretical notions; we will rather see it as a

[13] 'Comme les raisonnements d'après toute axiomatique des ensembles ou d'après un système logico-formel se font de manière que l'absolu non-dénombrable n'existe pas, l'affirmation de l'existence des ensembles non-dénombrables ne doit être considerée que comme un jeu de mots, cet absolu non-dénombrable n'est donc qu'une fiction. La véritable portée du théorème de Löwenheim est justement cette critique du non-dénombrable absolu. Bref: cette critique ne réduit pas les infinis supérieurs de la théorie simple des ensembles *ad absurdum*, elle les réduit à des non-objets.'

[14] 'Que l'axiomatique conduise au relativisme, c'est un fait parfois considéré comme le point faible de la méthode axiomatique. Mais sans aucune raison. Une analyse de la pensée mathématique, une fixation des hypothèses fondamentales et des modes de raisonnements ne peut être qu'un avantage pour la science. Ce n'est pas une faiblesse d'une méthode scientifique, qu'elle ne puisse donner l'impossible.'

manifestation that some means (axiomatization in a first-order language) are unsuitable for some ends (characterization of the true set-theoretical concepts). At most, we would accord it epistemological significance, but it would leave our ontological assumptions untouched. Skolem's argument for relativity is only convincing as directed against the assumption of such a model—or even against the assumption of a fragment of such a model containing the alleged totality of all sets of natural numbers: this would be a genuinely uncountable totality.

Suppose one has doubts about the existence of such a totality. These doubts can be reasonable, owing to the fact that all efforts to specify its contents are bound to fail. The obstacle does not lie in the infinity of the alleged totality (for there are infinitely many natural numbers, but we feel we are able to single out any one of them), but rather in the lack of some canonical means to reach all of its members. If we could devise a procedure yielding such a totality, then no doubt the inability of axiomatization to pinpoint it should be considered a limitation of the axiomatic method and Skolem's argument for relativity would be weak indeed.

We may turn to axiomatization with at least two different goals, namely *systematization* of an existing theory or *characterization* of a new one. We may axiomatize an informal satisfactory theory in order to systematize it, reducing its multiple concepts to a small stock of primitives and the variety of propositions accepted to a well behaved list of axioms. If the original theory dealt with a definite domain of objects, then this will be the domain of the *intended model* of the axiomatized theory and the circumstance that the axioms have other, unintended models can have no effect on the definiteness of the original domain and its concepts. In no sensible way will the existence of these additional models make the concepts of the theory relative. It may even be that these unintended models are studied only by logicians, and not by the traditional practitioners of the theory. The paradigm here is number theory.

Systematization of its concepts and results was not the reason adduced for axiomatizing set theory. As we have already substantiated, it was claimed (in particular by Skolem and von Neumann) that there was no satisfactory intuitive theory to be faithfully systematized. There were only some concepts and methods that were fruitful if restrictively applied, but there was definitely no clear domain containing all sets. Moreover, not only was the theory's ontology unclear, but also the ways of reasoning employed led to contradictions if carried too far in particular directions. In this situation, axiomatized set theory was the only available theory of sets and no intended model for it was or could be proposed.

Nowadays the situation is apparently different. Thanks to the development of axiomatic set theory, we have been able to devise a motivating picture of the set-theoretical universe, namely the cumulative hierarchy of sets, according to which all sets are distributed in stages indexed by the ordinals in such a way that to each stage belong exactly those sets all of whose elements belong to some previous stage. There is a first, empty stage R_0, and for any ordinal α, if α has an immediate predecessor, the elements of R_α are all the subsets of the preceding stage, whereas if α has no immediate predecessor, R_α is just the union of all stages below the αth. Since we can figuratively describe this hierarchy as being obtained from the empty set by iterating the power-set operation along the ordinals, one refers to this view of the set-theoretical universe as the 'iterative conception of sets'.

However suggestive, this informal outline can by no means be taken as the description of a definite model, and this at least on two important counts regarding what is being iterated and for how long; that is, no account has been given (1) of how to get from a given set to its power set, *i.e.*, of what the contents of the power set of any given set are, and (2) of the length of the ordinal sequence, so to speak. Without clarifying these two points, nothing definite has been described, even though, and this is worth emphasizing, this sketchy picture with a few general hints about (1) and (2) is rich enough to motivate most of the axioms of ZFC. In sum, we can give a rough outline of a model and find some axioms appropriate to it[15]—but does it follow that there is a model?

Let us go back to Skolem. He could not compare the models of the axioms with an intended one, but some of the aspects of the informal theory were close to ordinary mathematical practice and some of its concepts had, at least partly, intuitive import. As we have already seen, restricted notions of collection, of relation and of function are pre-set-theoretical. Thus we can scrutinize the models of the axioms to see to what extent they are faithful to the insights developed in our pre-set-theoretical mathematical experience. By so doing, we may discover that a particular model is wanting.

We are talking about models of the set-theoretical axioms. What are they? We seem to be in a predicament here, since whenever we speak of models we presuppose a meaningful set theory to deal with them. Models are usually taken to be set-theoretical objects and it is plain that, if so, we cannot lean on them in this situation. For on the one hand we have to have models in order to put forward axiomatized set theory and thereby get hold of sets (as the objects in the model), but on the other hand we need sets and set theory in order to have models available. This is essentially the content of Skolem's first complaint against axiomatized set theory:

> If we adopt Zermelo's axiomatization, we must, strictly speaking, have a general notion of domains in order to be able to provide a foundation for set theory. ... But clearly it is somehow circular to reduce the notion of set to a

[15] We should keep in mind that the axioms of set theory were chosen before the iterative conception was propounded, not the other way around as the above description would suggest. That sets are distributed in stages as described is a theorem of ZF.

general notion of domain. (Skolem [1922], p. 292)

Without set theory, we don't have a general notion of model, so that characterizing sets via models is unsatisfactory. However, we have a restricted notion of model which might not suffice (it certainly does not) to found set theory axiomatically in Zermelo's or von Neumann's spirit, but which may be good enough to criticize the axiomatizer's claims. As we have maintained when discussing Skolem's proof of the Löwenheim-Skolem theorem, we can, under the assumption of its consistency, produce a countable model of a given list of sentences without calling on set theory.

Indeed, if we axiomatize set theory because we are skeptical of general informal set theory, we are not entitled to consider more models of the axioms than can be shown to exist (under the assumption of their consistency) with non-set-theoretical means. It is precisely in this context that Skolem's arguments have force, for without recourse to set theory no uncountable collection can be had.

If we do not presuppose the existence of uncountable collections, no uncountable model of the axioms will be shown to exist. According to any given model of ZF there will be uncountable sets, *i.e.*, there will be objects a in the model satisfying some formula that expresses that a is non-empty and there is no function on ω onto a. These objects will be 'uncountable' sets in a spurious sense, *relative* to the model, but they will be indeed countable, in the sense that the objects related to them by the membership relation of the model can be enumerated with the natural numbers.

With these considerations in mind, Skolem's conclusion that 'on an axiomatic basis higher infinities exist only in a relative sense' (Skolem [1922], p. 296) becomes cogent. It is not that there are *some* models in which countable objects will be mistakenly qualified as uncountable, but rather *any* model that can be shown to exist without calling on set theory will be such.

5. Replies to Skolem

The usual attempts to refute or to dismiss Skolem's set-theoretical relativism are wide of the mark, since they bring into play some alleged canonical interpretation of the set-theoretical terms and axioms. Skolem's main claim is taken to be that the unavoidable existence of countable models impairs the legitimacy of the original interpretation, and efforts are made to show him wrong by taking for granted the meaningfulness of such an interpretation. It is assumed that the reason why Skolem is skeptical about the existence of uncountable collections is that in some models some countable sets are wrongly reckoned to be uncountable, and the reply to Skolem consists in arguing (perhaps by comparing set theory to other theories) that the existence of non-canonical models does not affect the original meaning of the theory. Thus, W. D. Hart:

> We do not require the *expression* of any part of physics to pick out its subject matter uniquely; we are confident of forces without worrying about whether we have descriptions of them as their sole interpretations. What right, then, does Skolem have to raise doubts about the existence of uncountable collections simply because they cannot be uniquely specified by a formalism? (Hart [1970], p. 106)

We can easily answer this question by denying its presupposition. Skolem raises doubts about the existence of uncountable collections because he has been given no argument for their existence—either by means of a formalism or otherwise. On the other hand, he has been offered axiomatized set theory as the only reliable theory of sets, thus the one theory that could dispel his doubts, and he just makes it clear that, unaided, such a theory is unable to account for uncountable totalities.

In his [1951], J. R. Myhill also argues from the unquestioned assumption of a canonical interpretation:

> Let the interpretation 'x is a member of y' be given to the Greek letter epsilon. Then by the Löwenheim-Skolem theorem, denumerable models for set theory exist; that is, there exist relations having all the formal properties assigned to class-membership by the axioms of (any consistent) set theory, and also having denumerable fields. But *none of these relations is class-membership*; for class-membership certainly has a vastly non-denumerable field. Hence all the Skolem-Löwenheim models of set theory are non-standard, relative to the given interpretation. Indeed there is evidently only one standard model of set theory, because the predicate-letter epsilon, the only one which appears, is already preassigned on interpretation. (Myhill [1951], p. 68)

According to Myhill's view in this paper, what Skolem argues about is not set theory proper, but only its formalistic frame. Only with the right interpretation are the formalized axioms propositions of set theory.

> The Skolem 'paradox' thus proclaims our need never to forget completely our intuitions. We could shift to a formalism indistinguishable from set theory and it could be something other than set theory. It only remains set theory as long as the intuition of membership has not slipped away from us. It could be formally the same thing and have a grotesquely different meaning. (Myhill [1951], p. 69)

Myhill's paper was not meant to be a reply to Skolem's arguments, being rather a reflection on formalisms. Thus we cannot blame him for taking the 'vastly non-denumerable' standard model of set theory for granted. Now, in order to produce a rejoinder to Skolem, it is not necessary to bring up such a model. In his [1985], Benacerraf, who does argue against Skolem, depends on much weaker assumptions:

> One important reason for constructing a theory of sets is to represent the intuitive content of Cantor's theorem. That content is not preserved in a restricted model of some First Order Theory whose existence is guaranteed by one of the Löwenheim-Skolem theorems. Any interpretation I of ZF on which one can

define a function from $[(\omega)_{\mathrm{I}}$ onto $\mathcal{P}(\omega))_{\mathrm{I}}]$ is *eo ipso* an inadequate interpretation. There *should be* no such functions. Therefore, not every model of the axioms is an admissible interpretation—if one is doing set theory (although they might be for other purposes). (Benacerraf [1985], p. 104)

How could Skolem reply to these words? He could possibly say that if the informal validity of Cantor's theorem, *i.e.*, the absolute uncountability of $(\mathcal{P}(\omega))^*$, is a necessary condition for a model of ZFC to be admissible, then there may fail to be any admissible models. Cantor's diagonal argument shows that no countable list of collections of natural numbers contains all such collections by allowing one to define in terms of the list a collection not in the list. But that no countable list of number collections is complete does not imply that *the complete list* of all number collections is uncountable, since there may be no such list. If we believe that there are uncountable collections, we have only one sensible way to show Skolem wrong, namely to give an argument, even if a plausibility one, that uncountable collections do exist.

Suppose that, wishing to produce an uncountable collection, we leave to one side the sets of natural numbers and turn to the well-ordered sequence of the countable ordinals. In the last, seventh section of his [1929b], Skolem makes some remarks on the characterization of this sequence. First he looks at those ordinals which are the order-types of orderings $(A, <)$ definable by second-order means.[16] Since there are countably many second-order formulas, it is clear that 'it is not possible to characterize in this way all numbers of Cantor's second class—I speak as if this class were a reality.' (Skolem [1929b], p. 271)[17] The ordinals thus obtained are the ordinals of the *first level*. Allowing quantification over them, we get the ordinals of the *second level*. Again, these constitute a countable totality, with whose help the ordinals of the *third level* can be defined. This procedure can be carried on even to limit stages and it can be seen that there is no countable limit to the number of levels needed to define all countable ordinals. It could be even proved, Skolem adds, that there is a countable ordinal α which is of no level less than α. And he concludes:

In any case, there can be no derivation of the second number class [*i.e.*, of the class of all countable ordinals] from simpler concepts. Here we find again, as we found before during the discussion of Löwenheim's theorem, that there is no possibility of introducing something uncountable except as a pure dogma. (Skolem [1929b], p. 272)[18]

[16] As Skolem, says, he disregards the indeterminate nature of the second-order quantification, for the sake of the argument.

[17] 'Es ist aber klar, daß es nicht möglich sein kann, in dieser Weise alle Zahlen der Cantorschen zweiten Zahlenklasse zu charakterisieren—ich spreche als ob diese Zahlenklasse eine Realität wäre.'

[18] 'Jedenfalls kann gar keine Herleitung der zweiten Zahlenklasse aus einfacheren Begrif-

We see that Skolem's grounds for rejecting the uncountable in this paper of 1929 are similar to the ones he stated in 1922. There he objected that uncountable models can only be guaranteed for a theory with uncountably many axioms,[19] and now he makes it clear that ω_1 can only be constructed by means of an iteration of ω_1 steps. Briefly, uncountable collections can only be built from other uncountable collections. That is what he meant when he said that the introduction of the uncountable is *circular*[20] and that's what he means now when he labels it *dogmatic*.

As stated in the report of the discussion following a lecture by Skolem in 1958 (Skolem [1958], pp. 637–638), Tarski objected to Skolem's insistence on countable models. He remarked that by using the so-called upward Löwenheim-Skolem theorem, according to which every first-order theory with infinite models has a model of any specified uncountable cardinal κ, one could build an argument designed to shun the countable models in favor of uncountable ones. That is to say, if Skolem's argument is sound, then so is the one obtained by substituting 'of cardinality κ' for 'countable' in it. Benacerraf in [1985] develops such an argument serving 'to bring out of the closet The Skeptic bias for the finite, or at least the at-most-denumerable': Fix an uncountable cardinal, say $\aleph_{17}$, and consider a formal description of a set which the axioms prove to be infinite. By a suitable use of the upward Löwenheim-Skolem theorem for $\aleph_{17}$, we show that the theory has a model (M, E) in which the set a having that description has exactly $\aleph_{17}$ E-members (*i.e.*, the collection a^* has cardinality $\aleph_{17}$). Thus we seem entitled to conclude—if Skolem ever was—'that any infinite cardinality other than $\aleph_{17}$ is relative.' In particular, being countably infinite is relative, because there is a model whose ω has $\aleph_{17}$ members (Benacerraf [1985], pp. 107–108).

It is clear that Skolem's position as we have described it is immune to this counterattack. A preference for the countable against $\aleph_{17}$ is not a bias, since we can can give well understood rules for the generation of some countable sets, in particular for the generation of the natural numbers, but nothing of the kind is available for $\aleph_{17}$—or for $\aleph_1$ for that matter. What then are the grounds for its existence? We can let Skolem himself comment on the meaning of the upward Löwenheim-Skolem theorem:

> I may mention that some authors in connection with the Löwenheim theorem also set forth the inverse theorem that if a logical formula can be satisfied in a denumerable domain, it can also be satisfied in a non-denumerable one, even

fen möglich sein. Man erkennt hier wieder dasselbe wie früher bei der Besprechung des Löwenheimschen Satzes, daß es keine Möglichkeit gibt, etwas absolut Nicht-abzählbares anders einzuführen als durch ein reines Dogma.'

[19] Skolem [1922], p. 296.

[20] 'In order to obtain something absolutely nondenumerable, we would have to have either an absolutely nondenumerable number of axioms or an axiom that could yield an absolutely nondenumerable number of first-order propositions. But this would in all cases lead to a circular introduction of the higher infinities' (Skolem [1922], p. 296).

with an arbitrary cardinal number. What is meant by such a statement? What kind of set theory is used? Is Cantor's set theory still going strong in spite of the antinomies? Or are the text books in ordinary analysis the source of the knowledge that non-denumerable sets exist? Or is the word non-denumerable only meant in the relative sense, namely relative to a certain axiom system of set theory? (Skolem [1955], p. 583)

6. On the Status of Set-theoretical Notions

Skolem does not claim that uncountable sets are suspect simply because the notion of uncountability cannot be duly expressed in a first-order language. Any attempt to uphold this claim would be futile if there were cogent reasons for the existence of uncountable sets. Rather he concludes that there are no *bona fide* uncountable sets from the threefold assumption (advanced for a time, in particular by Zermelo and von Neumann) that

(1) the original Cantorian set theory is contradictory,[21]
(2) axiomatized set theory is the only acceptable version of set theory,
(3) the axioms determine (implicitly define) what sets are and which sets there are via their models.

From these premisses, I have argued, Skolem's conclusion follows; but we find the premisses hardly tenable, especially the first and the third.

With respect to (1), there have been many statements to the effect that Cantor's original set theory is contradictory, but no relevant evidence has been produced that it is. All alleged contradictions have been derived from the unrestricted comprehension axiom, which is assumed to be the content of Cantor's famous definition of set of 1895: 'By a "set" we are to understand any collection into a whole M of definite and separate objects m of our intuition or our thought ' (Cantor [1932], p. 282).[22] It should be obvious that this is not a mathematical proposition to which we can refer in order to justify the existence of a particular set in the course of a proof; Cantor's definition, which can be compared to Euclid's definition of a number as a multitude composed of units, is rather a suggestion of which role sets are to fulfill.

Nevertheless, it has been customary to draw from Cantor's words the consequence that, e.g., the ordinals can be collected into a set, in opposition to Cantor's claims since his introduction of the ordinals in 1883—thus long

[21] It is doubtful whether Zermelo believed that Cantor's original theory was contradictory. What he says in Zermelo [1908] is this: 'At present, however, the very existence of [set theory] seems to be threatened by certain contradictions or "antinomies" that can be derived from its principles—principles necessarily governing our thinking, it seems—and to which no entirely satisfactory solution has yet been found' (p. 200).

[22] 'Unter einer "Menge" verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten m unsrer Anschauung (welche die "Elemente" von M gennant werden) zu einem Ganzen.'

before there was any talk of paradoxes.[23] It is curious that Cantor's set theory has been considered inconsistent because of the strict reading of an imprecise, non-mathematical elucidation of the intuitive import of its central concept. That Cantor's set theory is inconsistent is a frequently made, albeit unsupported, charge. The main difficulty with Cantor's theory is not its alleged inconsistency, but rather its lack of a mathematically clear principle to sort those multiplicities which can be collected into a whole, namely sets, from those which cannot. It is because of this drawback that axioms are needed to guarantee the existence of enough particular sets.

In any event, the assumption that Cantor's theory is inconsistent is not essential in Skolem's argument, since he only used it to undermine the claim that there are good reasons for the existence of uncountable sets. Accordingly, the first premiss of Skolem's argument can be replaced by the assumption that without recourse to axiomatization we are unable to establish that there are uncountable sets (or collections).

As to premiss (3), we no longer maintain that sets and membership are to be determined by the models of some list of axioms.[24] For, on the one hand, the notion of the axioms implicitly defining the subject matter of a theory is rather obscure, except for the case where the axioms are categorical; when they are, the principle's import is that what matters about the objects with which the theory deals are only their mutual relations, not their individual nature. On the other hand, the proper discussion of models and isomorphism presupposes set theory, whereby these notions are out of place in a fitting characterization of sets.

If we do not take sets to be implicitly defined by some list of axioms, Skolem's argument cannot be carried out, but Skolem's source of discomfort remains. Recall that Skolem does not argue against the absolute meaning of certain set-theoretical notions because such notions take different forms in different models, but rather because he has been given no reason to believe that there is even one model in which these notions take their presumed right form; in all the models that he is able to build, sets deemed to be uncountable are indeed countable. Even granting that informal set theory is free from contradiction, we cannot conclude that there are, in any absolute sense, uncountably many sets of natural numbers. As already suggested, Cantor's diagonal argument is of no avail to us at this juncture: with its help we can show that if there is a maximal totality of sets of

[23] In 1883 he had defined a set as 'any many which can be thought of as one, that is any totality of definite elements which can be united into a whole through a law'. ('Unter einer "Mannigfaltigkeit" oder "Menge" verstehe ich nämlich allgemein jedes Viele, welches sich als Eines denken läßt, d. h. jeden Inbegriff bestimmter Elemente, welcher durch ein Gesetz zu einem Ganzen verbunden werden kann' (Cantor [1932], p. 204, footnote 1)). It is not said that any totality can be united into a whole and the phrasing may intimate that some totalities cannot be.

[24] One important reason why we don't is precisely the Löwenheim-Skolem theorem.

natural numbers, then such a totality is absolutely uncountable—but Cantor's argument yields no maximal totality; that such a totality exists is an independent assumption. This is precisely the assumption that, in the last analysis, Skolem questions.

It should be emphasized that no appeal to the iterative conception of sets can lend support to the existence of uncountable totalities. We may have a convincing picture of the iteration process behind the conception, but the basic ingredient to be iterated, namely the step from any set to its power set, is left unexplained (and justifiably so, since it is prior to the iteration; without a power set operation there is nothing to iterate). It is to be expected that the development of the theory describing the iteration will give us further hints about the contents and structure of power sets, but the existence of the full power set of any existing set must be assumed beforehand; in particular, the existence of the disputed totality of all sets of natural numbers must be independently justified.

Skolem's argument makes it clear that axiomatizing set theory will not secure the existence of uncountable sets, that in order to show that there are uncountable sets we have to argue explicitly for their existence; it is not enough to build a theory that proves them to exist. It may be objected that it does not show that much, since Skolem's considerations do not apply to higher-order axiomatizations; but this objection is off the mark, because the use of second-order logic to ensure enough categoricity to ban countable models presupposes the existence of the power set of some infinite set, thus of an uncountable set; and such an assumption cannot be accepted in an argument for the existence of uncountable sets, on pain of circularity.

What reasons can be given for the existence of a maximal collection of sets of integers? Since the existence of such a collection is equivalent to that of the complete field $\mathbf{R}$ of real numbers, we may as well deal with the reasons for the existence of $\mathbf{R}$. We certainly have one strong reason for wanting $\mathbf{R}$ to exist, namely its being complete, *i.e.*, its guaranteeing a limit to every sequence that should converge, in particular to every Cauchy sequence of rational numbers. We soon find two difficulties standing in the way of fulfilling our desire: (i) we cannot generate enough objects to insure completeness, and (ii) we do not know how to explain what we mean by '*all* Cauchy sequences', hence also '*all* real numbers'. We certainly can express with utmost precision what conditions a given sequence of rational numbers must satisfy in order to be Cauchy, and in what circumstances a given real number is the limit of a given sequence. The difficulty does not lie in explaining which sequences are Cauchy, but in saying which sequences there are: Given a definite totality of rational sequences, our definitions are enough to single out those that are Cauchy, but we know of no way to characterize the totality of all possible rational sequences; in other words, we do not know how to explain what an *arbitrary* rational sequence is.

But surely we do mathematics with them. So, it may prove fruitful to inspect how we work with arbitrary rational sequences or with arbitrary real numbers. To begin with, we deal with specific sequences, such as $\langle 1/2^n : n \in \mathbf{N}\rangle$, or with schematic ones, like $\langle p(n)/q(n) : n \in \mathbf{N}\rangle$, for unspecified polynomials $p(x)$ and $q(x)$, but these are not really arbitrary sequences and Skolem would object to none of them (indeed, Skolem would object to no particular sequence). When working with arbitrary sequences we operate in a purely formal manner; thus we may consider the sequence $\langle a_n + b_n : n \in \mathbf{N}\rangle$ obtained from the sequences $\langle a_n : n \in \mathbf{N}\rangle$ and $\langle b_n : n \in \mathbf{N}\rangle$, or we may form the subsequence $\langle a_{2^n} : n \in \mathbf{N}\rangle$ of $\langle a_n : n \in \mathbf{N}\rangle$. It is working thus formally that we show, for example, that if all Cauchy sequences of rational numbers have a real number as limit, then so do all Cauchy sequences of real numbers. We prove this without ever having to elucidate which real numbers there are. We only need to assume that every real number is the limit of some Cauchy sequence of rational numbers.

There is no mathematical difference between working *formally* and *generally*. In working with real numbers, we need have no knowledge of which real numbers there are. The only assumption we use about their totality is that it is a field containing the rational numbers and the limits of any Cauchy sequences we may ever encounter, in particular, any irrational numbers that we should ever be able to specify. What we do can be seen as drawing consequences from formal principles; and this can be done whether or not there is a well determined totality of numbers about which we speak.

But if so, why must we accept the existence of such a totality of objects? Of course, when we do mathematics we reason (or we can be described as reasoning) as if all these objects existed. But the soundness of our reasoning does not depend on this ontological assumption. Consider this situation: For some reason, we are interested in the models of a particular list of axioms; as we prove new theorems, we develop an increasingly vivid image of how the structures we are studying look, we attain ever clearer insights about the relations among the objects of some of these structures, and between these structures and some other, *etc.* Now suppose that we reach a contradiction arguing from our axioms. We conclude that there are no structures of the kind we were interested in, so that we were arguing about nothing—but even so, our reasoning was sound. Now, if we can argue flawlessly in proofs from contradiction, why must we assume that, when there is no contradiction involved, there must be objects over which our quantifiers range in order that our arguments be sound?

Without interfering with mathematical practice, the real numbers can be conceived as forming an ideal complete extension of the ordered field of rational numbers. Not that the field contains some ideal real numbers (although we may speak this way), but rather that the field itself is ideal, fictitious—which only means that we argue as if there were such a field.

Since completeness of $\mathbb{R}$ means that every Cauchy sequence of rational numbers converges in $\mathbb{R}$ and since we can construct some definite Cauchy sequences, we are able to show that some definite real numbers exist; indeed, that any irrational number ever considered exists, *i.e.*, belongs to $\mathbb{R}$.

This embedding of the rational-number field into an ideal order-complete field is analogous to the embedding of the totality of the arithmetical collections of natural numbers into an ideal power set of $\mathbf{N}$. We do not know what the exact contents of $\mathcal{P}(\mathbf{N})$ should be (if we could, there would be no need to idealize), as we do not know what all the real numbers are, but in both cases we have enough precise desiderata about them to be able to write down a list of axioms (both in ordinary mathematical vernacular and in a formalized language) on which to base our reasoning. Once we have taken the step from $\mathbf{N}$ to $\mathcal{P}(\mathbf{N})$, we see no compelling reason not to assign to any ideal set its ideal power set. No arbitrariness is allowed, since no more is assumed about $\mathcal{P}(A)$ than that its members are *all* subsets of A—and we know what it means for a set to be a subset of another set and how to reason formally with the quantifier *all*. We are now ready to give a sketch of a further idealization, the cumulative hierarchy, and to agree on a list of non-arbitrary axioms that it should satisfy.

The field $\mathbb{R}$ is a typical set-theoretical object of relatively low rank. As a set-theoretical object it is a fiction, as a low-rank one, it is strongly related to other basic mathematical objects, which, like integers and rationals, can be constructed in an absolute way. In general, basic mathematical objects can be made to correspond to some low-rank sets, that is, given some such basic object a, we can show from the axioms of set theory that there is (in the ideal set-theoretical universe) some low-rank set which behaves like a. Owing to such correspondences we are able to view, say, the natural numbers as sets. And owing to the fact that all set-theoretical objects of finite rank can be made to correspond to basic mathematical objects, we are able to conceive of the set-theoretical universe as anchored in the basic mathematical world.

Such a relation allows us to compare, as Skolem did, ideal sets with real definable collections. For ordinary mathematical purposes we can view the basic, pre-set-theoretical, constructible mathematical objects as belonging to the universe of sets, briefly, we can identify them with some sets; and this we do as a matter of course. Moreover, this identification (which can be very smooth, because mathematical objects seem to have no intrinsic nature) induces us to see high-rank sets as ordinary generable collections of well determined objects.

Let's come back to Skolem's critique. From a set-theoretical perspective we can prove that there exists a unique complete ordered field, thereby characterizing $\mathbb{R}$. But if, from his traditional, pre-set-theoretical standpoint, Skolem asks for $\mathbb{R}$'s credentials we can give none; none, that is, of the kind he is asking for: for Skolem only admits mathematical objects that can be suitably specified, and with such a restriction only incomplete, open-ended, countable totalities can be made. No proof of $\mathbb{R}$'s existence has been given outside of set theory, which is the theory of an ideal universe. Many reasons can be advanced for developing set theory, but none has been given for viewing all its objects as being at the same ontological level as more basic mathematical constructions.[25]

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ABSTRACT. In this paper an attempt is made to present Skolem's argument for the relativity of some set-theoretical notions as a sensible one. Skolem's critique of set theory is seen as part of a larger argument to the effect that no conclusive evidence has been given for the existence of uncountable sets. Some replies to Skolem are discussed and are shown not to affect Skolem's position, since they all presuppose the existence of uncountable sets. The paper ends with an assessment of the assumptions on which Skolem's argument rests from a present-day perspective.