Uncertainty Probability and Non-Classical Logic - Brian Weatherson - 12 May 2004

(Note to FEWers - this is very much a draft, and for that matter the written up version of a talk. The references are non-existent and two of the important proofs are incomplete. A more formal version of some parts of this paper is available here. I'll refer to that paper as Weatherson 2004 in what follows.)

1. Proportion Your Credences to the Evidence

This is a contribution to the debate about what are the constraints on rational credences. I'll start by introducing two of the parties to that debate, each of whom will be somewhat stylised and given a somewhat contentious label. The **classical Bayesian** says that it is a condition of coherence that one's credences be governed by the probability calculus. That is, if Cr is an agent's credence function, the function from propositions to the agent's credence in that proposition, the agent is coherent iff there is some probability function Pr such that $Cr(A) = Pr(A)$. The **heretic** objects to the classical Bayesian because she thinks the following is possible for a coherent agent: there is a proposition A such $Cr(A)$ and $Cr(\neg A)$ are both low. By 'low' she really means arbitrarily low, but to get the debate started she can just mean less than 0.5.

Why does the heretic think this? Well there are many arguments that heretics have given for their heresy. Heresy is not common amongst philosophers, but it is widespread in the economics and computer science literature, so we can look at arguments actual heretics have given. Some of these arguments seem to be philosophical howlers. For instance, the economist George Shackle was a heretic because he believed the following theses. When A is about the future, it is only reasonable to have positive credence in both A and $\neg A$ if realism about the future is true. Realism about the future is incompatible with libertarian free will. Humans have libertarian free will. So when A is about the future, it's unreasonable to have positive credence in both A and $\neg A$. There are some propositions about the future such that it's permissible to have credence less than 1 in both A and $\neg A$. From these claims heresy follows, but only the last seems philosophically plausible.

There are better arguments for heresy. The following argument is derived, very loosely, from some things the mathematician Glenn Shafer wrote. It sounds like a platitude that we should proportion our credences to the evidence. But sometimes we have no evidence, or next to no evidence, for either A or $\neg A$. Shafer suggested this was true when A is the proposition that there is intelligent life on other planets. Keynes famously said that many of the propositions we need to consider when making long-term investment decisions, like propositions about the position of wealth-holders in the social system in a generation's time, are like this, and I'm inclined to agree. This leads directly to heresy.

The heretics all believe that excluded middle is a law of logic, and that we should have credence 1 in all laws of logic, so they are forced to give up Addition.

> *Addition*: If A and B are known to not be both true, then $Cr(A \vee B) = Cr(A) + Cr(B)$

The classical Bayesian thinks Addition is a coherence constraint, the heretic does not. She thinks that when $B = \neg A$ it often fails, because the LHS = 1 while the RHS can be arbitrarily low.

The platitude that we should proportion our credences to the evidence leads to violation of Addition even when there is evidence available. In some cases, investigating into whether A is true often gives us evidence both for and against A. Someone who has investigated the Kennedy assassination in detail has better reasons both to believe that Oswald did it, and that he didn't, has better evidence for both these propositions, than I do. So she should, if the platitude is to be believed, have a higher credence in both *Oswald did it* and *Oswald didn't do it*, than I should. Obviously the heretic can accept this conclusion and the classical Bayesian cannot.

There are many interesting things to say about the formal models derived by heretics, but here I'm much more interested in their philosophical arguments, and about what we should do in response to them. As I can see, there are four possible responses to these kinds of arguments.

1. **Defend the classical Bayesian position by attacking the philosophical arguments offered.**
   I'd say that's the best response to the Shackle argument, but I'm less convinced that it's the best response to the argument from the platitude about evidence.
2. **Argue that the arguments can be accommodated with a small change to the classical Bayesian program.**
   For instance, one could argue that letting probabilities be vague (as a few people here have urged) lets us keep all we ever wanted from the Bayesian program while giving appropriate weight to these considerations. Officially this is the position I'd like to end up defending, but I think it's worthwhile to explore the other options.

3. **Accept the conclusions of these arguments and drop Addition.**

   While there is some merit to these, because the arguments the heretics have offered are worthy of consideration, I think Addition is pretty plausible, so we should investigate a fourth option.

4. **Accept the conclusions of these arguments and drop the rule that all classical tautologies get credence 1**.

   This is the approach I'll be investigating in this paper. Obviously the heretic's argument against Addition doesn't get off the ground unless we have the premise that the only correct credence for $A \vee \neg A$ is one. If we drop that premise, we don't have a problem for addition. Of course, we have *other* problems - but they may be solvable.

At this stage I need to introduce two new characters - the intuitionist and the trivalentist. Each of them thinks that the classical Bayesian is right that something like probability theory provides a set of coherence constraints on credences. But each of them decries the role classical logic plays in the classical Bayesian's formulation of those constraints. The intuitionist thinks that role should be played by intuitionist logic, and the trivalentist thinks it should be played by a 3 valued Łukasiewicz logic. In the next three sections I'll look at the intuitionist's position, and then I'll close by saying what the trivalentist's position is, without going into its pros and cons in any detail.

2. Probability and Logic

It is common for axiomatisations of probability theory to include a reference to an entailment function, or to a property of logical truth. For instance, here are two (classically) equivalent axiomatisations of the probability calculus. (I'm ignoring the axiom of countable additivity here, though in a fuller treatment it would be appropriate to put it back in. Indeed we'll ignore issues about infinity throughout here.)

Axiomatisation One

(A1) $0 \leq Pr(A) \leq 1$

(A2) If A is a logical truth, then $Pr(A) = 1$

(A3) If $\neg(A \wedge B)$ is a logical truth, then $Pr(A \vee B) = Pr(A) + Pr(B)$

Axiomatisation Two

(B1) If A is a logical truth, then $Pr(A) = 1$

(B2) If A is a logical falsehood, then $Pr(A) = 0$

(B3) If $A \vdash B$, then $Pr(A) \leq Pr(B)$

(B4) $Pr(A) + Pr(B) = Pr(A \vee B) + Pr(A \wedge B)$

The entailment relation here is usually interpreted as being *classical* entailment. And the concept of logical truth is usually interpreted as *classical* logical truth. But we can generate some philosophically interesting theories if we interpret them in other ways.

OK, first some house cleaning. Rather than have several different logical concepts lying around, I'll stipulate that I'm going to define logical truth and logical falsehood in the following way.

> T is a logical truth (relative to $\vdash$) iff for all sets S, $S \vdash T$
> T is a logical falsehood (relative to $\vdash$) iff for all sentences T', $T \vdash T'$

Now once we specify an entailment relation, we'll have specified what shall be the logical truths and the logical falsehoods, and the two axiomatisations above will give us a probability theory based on that entailment relation. So it looks like generating a probability theory based on, say, intuitionist logic should be a breeze. This is what the intuitionist in my debate wants to do. She thinks that probability theory mixed with intuitionist logic provides the right constraints on credences. There is a formal complication at this point though - it turns out that just how we stir the two together affects what happens when we mix probability theory and intuitionist logic.

Let's define an IPF_1 to be a function that satisfies axiomatisation one when the entailment relation is interpreted as intuitionist entailment. And let an IPF_2 be a function that satisfies axiomatisation two when the entailment relation is interpreted as intuitionist entailment. When the intuitionist says that she thinks intuitionist logic plus probability theory provides the right set of coherence constraints on credences, does she mean that a coherent credence function is an IPF_1 or that it is an IPF_2? This question is compulsory for the intuitionist because not all IPF_1s are IPF_2s. (The proof of this is in Weatherson 2004.)

The right answer presumably is that coherent credence functions are IPF_2s. Whatever we think of the heretic's arguments, they don't tell against at all against (B3). Assigning a higher credence to a logically stronger claim than a logically weaker claim is incoherent, whatever one's other views. But some IPF_1s assign higher probability to logically stronger claims than logically weaker claims. This seems to settle the issue in favour of IPF_2s over IPF_1s, but there are other possible functions to consider. (That is, there are other axiomatisations of the classical probability calculus that generate different types of probability theory when the entailment relations in them are interpreted intuitionistically.) For now I'll drop the subscript and call IPF_2s simply IPFs, with the argument for why this is permissible delayed until the next section.

Since $A \vee \neg A$ is not a logical truth, the intuitionist is free to agree with the heretic that it is possible to have a low credence in both A and $\neg A$. Indeed she can hold in a very strong sense that credences should be proportional to evidence. But she does so without giving up addition. In fact, Addition follows from (B2) and (B4). So if we thought the only, or even the primary, cost of the heretic's position was that it led to violations of Addition, the intuitionist's position seems rather attractive.

3. Intuitionist Probability Functions

As I said above, I'm going to take an intuitionist probability function (IPF) to be a function satisfying axiomatisation two, i.e.

(B1) If $\vdash_{\mathrm{IL}} A$ then $Pr(A) = 1$

(B2) If for all B, $A \vdash_{\mathrm{IL}} \mathrm{B}$ then $Pr(A) = 0$

(B3) If $A \vdash_{\mathrm{IL}} B$, then $Pr(A) \leq Pr(B)$

(B4) $Pr(A) + Pr(B) = Pr(A \vee B) + Pr(A \wedge B)$

where $\vdash_{\mathrm{IL}}$ represents the intuitionist entailment relation. I can prove that this set of axioms has one nice property, and I suspect it has one other nice property.

The first nice property is that anyone whose credences are an IPF is immune to a certain kind of Dutch Book. This might seem like an odd claim since not all intuitionist probability functions are classical probability functions, and it is widely reported that anyone whose credences do not form a classical probability function is vulnerable to a Dutch Book. But if we are taking intuitionist logic seriously, we should be a little wary of that claim, for reasons that become apparent as soon as we look at the kind of Dutch Book in question.

Consider an IPF such that $Pr(p \vee \neg p) = 0.8$, and assume that my credence function Cr matches that IPF. Say that a bet on A is a bet that pays \$1 iff A, and pays out nothing in any other circumstance. It might be thought I'm vulnerable to the following kind of Dutch Book. I'll sell you a bet on $p \vee \neg p$ for 90 cents. Since $p \vee \neg p$ is a logical truth, in all circumstances that bet pays \$1, so I'm down 10 cents. Or so goes the standard argument. But if we're intuitionists we shouldn't accept that $p \vee \neg p$ is a logical truth. Since there is no guarantee that $p \vee \neg p$ will be verified, it could be rational from this perspective to sell bets on $p \vee \neg p$ for less than \$1.

It is a little hard to say just what an intuitionist Dutch Book argument should look like. The intuition is that one doesn't want to engage in a series of trades that end in loss come what may. But the classical versions of Dutch Book arguments usually rely on the fact that we can say what may come by using a disjunction. In a logic like intuitionist logic with the disjunction property, that's a little tougher. In the above case, for instance, I lose money if p is true or if $\neg p$ is true, but that isn't sufficient to show that I lose money for sure.

Here's how I suggested getting out of that problem in Weatherson 2004, taking a cue from some of the work on 'depragmatised' Dutch Book arguments. What is wrong with agents who are vulnerable to Dutch Books is that their valuation of a bet, or a class of bets, is sensitive to how the bets are described rather than to the underlying facts of the bet. If $p \vee \neg p$ is really a logical truth, then a bet on it just *is* a bet on $p \rightarrow p$, or whatever we take to be the basic logical triviality. Indeed, assuming that I do value a bet on $p \rightarrow p$ the right way, we can restate the 'problem' by saying that I prefer a bet on $p \rightarrow p$ to a bet on $p \vee \neg p$, but those are just two ways of describing the same bet. Generalising a lot, the problem with an agent who is vulnerable to a Dutch Book can be put this way.

- There are two sequences of bets $C = \langle C_1, C_2, \ldots, C_k \rangle$ and $D = \langle D_1, D_2, \ldots, D_n \rangle$. (Each bet is assumed here to be a bet on a proposition in the sense above. We're ignoring issues about infinity here, so we assume each sequence is finite.)
- It is provable that there are at least as many winning bets in D as in C. (That is, for any $i \leq \max(k, n)$, it is provable that if there are i winning bets in C there are at least i winning bets in D.)
- The agent values C greater than D. (That is, if we assume that her value in dollars of a bet on A is her credence in A, and bets are neither complementary nor substitutes for each other so the value of the portfolio is the sum of the value of the component bets, she values the portfolio C above portfolio D.)

That way of putting the problem doesn't rely on any particular theory about what is and isn't possible, or indeed about any particular logical theory. What will count as 'provable' in the second bullet point will vary between different logical theories, but apart from that it looks like a reasonable way of stating what's wrong with most Dutch Bookable agents.

This is important because it can be proven that an agent whose credences are an IPF is not vulnerable to this kind of Dutch Book. That is, there are no sequences of bets C and D such that it is *intuitionistically* provable that D contains at least as many winning bets as C but such that she values C above D. (The proof is in the paper I linked in the introduction.) Conversely, if her credences do not form an IPF, there is such a Dutch Book. So that's one reason for thinking that the intuitionist should say that IPFs are the appropriate kinds of function to constrain credences.

It would be nice to be able to prove one more thing about IPFs, but I don't quite know how to do this. Say a Kripke Tree Function (KTF) is a function f from propositions to numbers generated as follows. Start with a Kripke tree, as in Kripke's well known semantics for intuitionist logic. Let m be any measure on the points in the tree, not necessarily one that assigns equal measure to sets of equal size. Then $f(A)$ is the measure of the set of points at which A is 'forced'. It is fairly clear that all KTFs are IPFs. If we could show that all IPFs are KTFs, then we would have a very nice semantics to go along with our axioms for intuitionist probability theory. But I don't know how to prove that, though I suspect it is true.

4. Pros and Cons

If we're at all moved by the thought that sometimes our credence in both A and ¬A should be low, I think there are two good reasons to prefer the intuitionist's position, that keeps Addition and drops excluded middle, over the heretic's position of keeping excluded middle and dropping Addition.

The first is that the addition principle itself is very plausible. On nonadditive theories, it is possible to get evidence for an exclusive disjunction (i.e. a disjunction where the disjuncts could not both be true) without getting evidence for either disjunct, which seems odd. Of course one can get *conclusive* evidence for a disjunction without getting *conclusive* evidence for either disjunct. But it seems that finding evidence in favour of the exclusive disjunction $p \vee q$ should make one somewhat more confident in each disjunct.

Moreover Addition follows given fairly weak assumptions from the two most plausible analyses of what it is to have a degree of belief. On one account to believe p to degree m/n just is to value a bet on p as worth m/n as much as a dollar. The well known Dutch Book arguments are often taken to show how Addition follows from that definition. I prefer an account due to Ramsey - to believe p to degree m/n just is to believe it to the same degree that you would believe $p_1 \vee \ldots \vee p_m$ if you believed $\{p_1, \ldots, p_n\}$ was an exclusive and exhaustive partition of the possibilities and you had the same credence in each of the p_i. It would take us too far from this topic to argue for this rigorously, but I believe a persuasive argument can be made that if this is the right definition of numerical degree of belief then Addition should hold.

If we want to hang onto Addition, but think it could be reasonable to have low credence in both A and $\neg A$, it follows that $A \vee \neg A$ must sometimes receive credence less than 1. And if we also want to hang on to the (plausible) principle that all logical truths must get credence 1, it follows that we mustn't regard $A \vee \neg A$ as a logical truth. At this point a move to intuitionist logic seems very natural. But of course there are many logics on which $A \vee \neg A$ is not a logical truth. Why intuitionist?

The main reason for this is that so many motivations for being unhappy with the classical Bayesian picture just are motivations that have traditionally been taken to support intuitionistism. In particular, it is hard to read through the work of Shafer and those sympathetic to him (or to similar views) and not be struck by how verificationist and anti-realist it all is. So they aren't a million miles from intuitionist approaches to start with.

It is amusing in this context to note, following a point Gil Harman made 20 years ago, that if we took all the talk about betting behaviour that's at the heart of arguments for classical Bayesianism seriously, we would be lead to something like intuitionist probability theory. Because in the real world, bets only pay out when the relevant propositions are *verified*. (As anyone who's tried to collect the minute their horse crosses the line in front will have found out.) We shouldn't make too much of this, because clearly the talk about betting is intended to be somewhat metaphorical, but it is also a way to motivate a distinctly *intuitionist* approach.

Obviously if you think the pure light of sweet reason reveals that classical logic is the one true logic, you won't be particularly moved by this approach. I don't plan to revisit all the debates about the correctness of one logic or another here, so I'll just take for granted that sweet reason reveals nothing quite so pure. This is not to beg *all* the relevant questions, because there are still costs of the intuitionist's approach. In particular, it might seem that it is *too* anti-realist. If we are to keep addition, then it turns out that whenever we want both $Cr(A)$ and $Cr(\neg A)$ to be less than ½, we must take $Cr(A \vee \neg A)$ to be less than one. But now we can't hold the following claims at once.

- Credences are governed by IPFs
- We should have a low credence in both A and $\neg A$ whenever we have little or no evidence for either.

- Whenever there is an effective procedure to decide whether A or $\neg A$ is true, then $A \vee \neg A$ is true, and this is knowable even if we don't know anything about the outcome of the procedure.
- There is a proposition p such that we have little or no evidence for either p or its negation, but there is an effective procedure for discovering whether p is true.

Yet the first three claims are essential to the intuitionist's theory, and it seems there may be values for p for which the fourth is true. To warm up to the problem, consider a proposition q about the distant past for which we have little evidence that it is true or that it is false, and no way of finding out whether q is true. It seems that in these cases we have to assign low credence to $q \vee \neg q$. That is to say, we have to assign high credence to anti-realism about q. So our approach here seems committed to at least taking very seriously anti-realism about the past.

Since there is no effective procedure for finding out whether q is true, this is not yet a troubling instance of the fourth bullet point. What about r, some proposition about the distant future? For concreteness, let r be the proposition that America will still have a predominantly market-based economy in 300 years time. I think it's true in quite a strong sense that we have little evidence either way about whether r is true. Now in one sense there's an effective procedure for working out whether r is true - just wait and see! So perhaps r will cause trouble for our theory of IPFs. But I think it's fair to restrict attention to effective procedures we could implement *now*, so this isn't really a problem. (That is, wait and see isn't really an effective procedure in the relevant sense, and the possibility of waiting and seeing doesn't undermine anti-realism about the future.)

The real problems arise when dealing with cases not about things that are far away in time, but things that are far away in space. Consider a proposition about the temperature at the South Pole right now. I know it's cold down there, but I really don't know how cold. Is it warmer or cooler than 40 degrees below zero? I don't know, and I don't really have much evidence one way or the other. (Maybe it's obvious which it is, but not to me!) Still, we wouldn't want to give any credibility to anti-realism about the temperature at the South Pole. There is, we can be sure, a fact of the matter. If s is the proposition that the temperature down there is lower than 40 below, our credence in $s \vee \neg s$ should be 1.Yet it seems possible that we could have next to no evidence one way or the other about s, so our theory suggests that we should assign low credence to both s and $\neg s$. Something has gone wrong, but what?

I think the only way out is to say that the existence of an effective procedure for determining whether s is true has to count as both evidence that s is true and evidence that $\neg s$ is true. That's consistent with the intuitionist's approach, so there's no worries about incoherence here. There is, however, a philosophical worry. If evidence for *s* can be so indirect, the effective procedure being evidence, then maybe we have much more indirect evidence for propositions like *q* than we thought. However that turns out, someone who thinks credences should be governed by IPFs has a stark choice here: in each area of investigation either assign credence less than 1 to some instances of excluded middle (i.e. take anti-realism seriously), or find some indirect form of evidence for either p or $\neg p$ for each p from that area. I think it's possible that there's an acceptable choice in every area of investigation, but there's obviously a lot of work to be done here.

5. Łukasiewicz and Probability

Intuitionist logic isn't the only logic that gives up on excluded middle because (in part) of concerns about realism. One of the major motivations behind Łukasiewicz's logics was anti-realism about the future, and that certainly meshes well with some of the reasons various writers, especially in economics, have given for abandoning classical Bayesianism. There's a lot one could do mixing Łukasiewicz-style logics with probability, but for now I'll focus on mixing probability with just one of these logics: the 3-valued Łukasiewicz logic with TRUE (or 1 if you prefer) as the only designated value. A Łukasiewicz Probability Function (LPF) is a function satisfying the following (by now familiar) constraints:

(L1) If $\vdash_Ł A$ then $Pr(A) = 1$

(L2) If for all B, $A \vdash_Ł \mathrm{B}$ then $Pr(A) = 0$

(L3) If $A \vdash_Ł B$, then $Pr(A) \leq Pr(B)$

(L4) $Pr(A) + Pr(B) = Pr(A \vee B) + Pr(A \wedge B)$

where $\vdash_Ł$ represents the entailment relation in the 3-valued Łukasiewicz logic with TRUE as the only designated value. When A and B are disjoint (they are never true together), $A \wedge B$ is a logical falsehood. (This doesn't mean that $\neg(A \wedge B)$ is a logical truth, just that $A \wedge B$ entails everything.) And by (L2) all logical falsehoods get probability zero. So when A and B are disjoint, $Pr(A \vee B) = Pr(A) + Pr(B)$. So one who says credences should be governed by LPFs holds on to Addition. This is the position of the trivalentist I introduced at the start.

If one accepts this Łukasiewicz logic as the correct logic, there is a Dutch Book argument that all rational credence functions should be LPFs. I won't labour the details here, but it turns out the argument I gave in the Weatherson 2004 generalises to show that there are no sequences of bets C and D such that the agent values C above D even though it can be proven that there at least as many winning bets in D as in C. (The argument I gave there was assuming intuitionist logic, but the only logical principles I appealed to were the equivalence of $A \vee B$ with $(A \vee B) \vee A$, and the equivalence of $A \wedge B$ with $(A \wedge B) \wedge A$. Each of these equivalences holds in the 3-valued Łukasiewicz logic.) And it isn't hard to show that any agent whose credences are not an LPF *is* vulnerable to a Dutch Book. So the trivalentist shouldn't feel threatened by classical Dutch Book arguments. (Unless, of course, she's threatened by non-probabilistic arguments for classical logic. Then she shouldn't be a trivalentist.)

Finally, it there is an intuitive semantics for LPFs. It is common to take classical probability functions to be measures on a possibility space. We can interpret each of the points in this possibility space as a classical assignment function - i.e. an assignment of either TRUE or FALSE to every sentence. Just like classical logic, Łukasiewicz logics are truth functional, so we can take truth assignments to represent the basic possibilities. So we can take LPFs to be measures on sets of 3-valued assignment functions. Then $Pr(A)$ is the measure of the set of assignment functions on which A is true. The axioms above are sound and complete with respect to this semantic model.

Soundness should be trivial to verify. For completeness, note that we can convert an arbitrary LPF, call it Pr, into a measure on the set of all 3-valued assignment functions. The measurable sets are the sets of assignment functions for which there is a proposition A that gets the same value on all assignments in that set. The measure of that set depends on the value A takes throughout. If A is true throughout, the measure is $Pr(A)$. If A is false throughout, the measure is $Pr(\neg A)$. If A is indeterminate throughout, the measure is $Pr((A \vee \neg A) \rightarrow (A \wedge \neg A))$. I'll stop here, but it isn't too hard to show that if Pr satisfies (L1) through (L4) this is a coherent definition, and it defines a measure, so the axioms are complete with respect to the intended semantics. We'll leave the philosophical costs and benefits of the trivalentist's position to another day.