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The Convex Configurations of “Sei Shonagon Chie no Ita,” Tangram, and Other Silhouette Puzzles with Seven Pieces

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SUMMARY The most famous silhouette puzzle is the tangram, which originated in China more than two centuries ago. From around the same time, there is a similar Japanese puzzle called *Sei Shonagon Chie no Ita*. Both are derived by cutting a square of material with straight incisions into seven pieces of varying shapes, and each can be decomposed into sixteen non-overlapping identical right isosceles triangles. It is known that the pieces of the tangram can form thirteen distinct convex polygons. We first show that the *Sei Shonagon Chie no Ita* can form sixteen. Therefore, in a sense, the *Sei Shonagon Chie no Ita* is more expressive than the tangram. We also propose more expressive patterns built from the same 16 identical right isosceles triangles that can form nineteen convex polygons. There exist exactly four sets of seven pieces that can form nineteen convex polygons. We show no set of seven pieces can form at least 20 convex polygons, and demonstrate that eleven pieces made from sixteen identical isosceles right triangles are necessary and sufficient to form 20 convex polygons. Moreover, no set of six pieces can form nineteen convex polygons.

key words: dissection puzzle, enumeration, *Sei Shonagon Chie no Ita*, silhouette puzzle, tangram

1. Introduction

A *silhouette puzzle* is a game where, given a set of polygons, one must decide whether they can be placed in the plane in such a way that their union is a target polygon. Rotation and reflection are allowed but scaling is not, and all polygons must be internally disjoint*. Formally, a set of polygons S can form a polygon P if there is an isomorphism up to rotation and reflection between a partition of P and the polygons of S (i.e. a bijection $f(\cdot)$ from a partition of P to S such that x and $f(x)$ are congruent for all x).

The *tangram* is the set of polygons illustrated in Fig. 1 (left side). Of anonymous origin, their first known reference in literature is from 1813 in China [2]. The tangram has grown to be extremely popular throughout the world; now, over 2000 silhouette and related puzzles exist for it [2], [3].

Much less famous is a quite similar Japanese puzzle called *Sei Shonagon Chie no Ita*. *Sei Shonagon* was a courtier and famous novelist in Japan, but there is no evidence that the puzzle existed a millennium ago when she was living (966?-1025?). *Chie no ita* means *wisdom plates*, which refers to this type of physical puzzle. It is said that the puzzle is named after *Sei Shonagon's* wisdom. Historically, the *Sei Shonagon Chie no Ita* first appeared in literature in

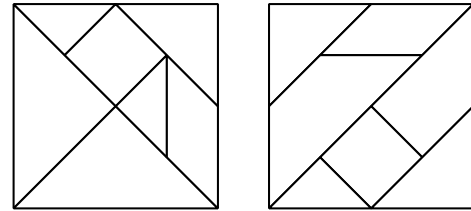


Fig. 1 Left: the tangram in square configuration. Right: *Sei Shonagon Chie no Ita* pieces in square configuration.



Fig. 2 A set of plates in the form of *Sei Shonagon Chie no Ita* pieces, crafted by Tomomi Takeda in Kanazawa, Japan.

1742 [2]. Even in Japan, the tangram is more popular than *Sei Shonagon Chie no Ita*, though *Sei Shonagon Chie no Ita* is common enough to have been made into ceramic dinner plates (see e.g. Fig. 2, [4]), and in puzzle communities, it is admired for being able to form some more interesting shapes that the tangram cannot, such as a square configuration with a hole missing (Fig. 3).

Wang and Hsiung considered the number of possible convex (filled) polygons formed by the tangram [5]. They first noted that, given sixteen identical isosceles right triangles, one can create the tangram pieces by gluing some edges together. So, clearly, the set of convex polygons one can cre-

*Sometimes this puzzle is also called “dissection puzzle.” However, dissection puzzle usually indicates the puzzles that focus on finding the cutting line itself. The most famous one is known as the *Haberdasher's Puzzle* by Henry Dudeney that asks to find cut lines of a regular triangle such that the resulting four pieces can be rearranged to form a square [1].

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ate from the tangram is a subset of those that sixteen identical isosceles right triangles can form. Embedded in the proof of their main theorem, Wang and Hsiung [5] demonstrate that sixteen identical isosceles right triangles can form exactly 20 convex polygons. These 20 are illustrated in Fig. 4. The tangram can realize thirteen of those 20.

It is quite natural to ask how many of these twenty convex polygons the Sei Shonagon Chie no Ita pieces can form. We first show that Sei Shonagon Chie no Ita achieves sixteen convex polygons out of twenty[†]. Therefore, in a sense, we can conclude Sei Shonagon Chie no Ita is more expressive than the tangram: while both the tangram and Sei Shonagon Chie no Ita contain seven pieces made from sixteen identical isosceles right triangles, Sei Shonagon Chie no Ita can form more convex polygons than the tangram.

One might next wonder if this can be improved with different shapes. We demonstrate a set of seven pieces that can form nineteen convex polygons among twenty candidates, and that to realize all twenty convex polygons, it is necessary and sufficient to have eleven pieces. We investigate all

possible cases and conclude that there are four sets of seven pieces that allow to form nineteen convex polygons as shown in Fig. 5. Based on this result, we also show that no set of six pieces can form nineteen convex polygons. That is, our results for general silhouette puzzles can be summarized as the following theorem:

Theorem 1: (1) There are only four patterns of seven pieces (Fig. 5) that can form nineteen convex polygons among twenty candidates in Fig. 4. (2) To form all twenty polygons in Fig. 4, eleven pieces are necessary and sufficient. (3)

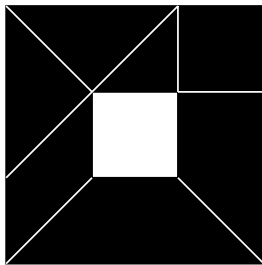


Fig. 3 A typical Sei Shonagon Chie no Ita layout as a square configuration with a hole missing.

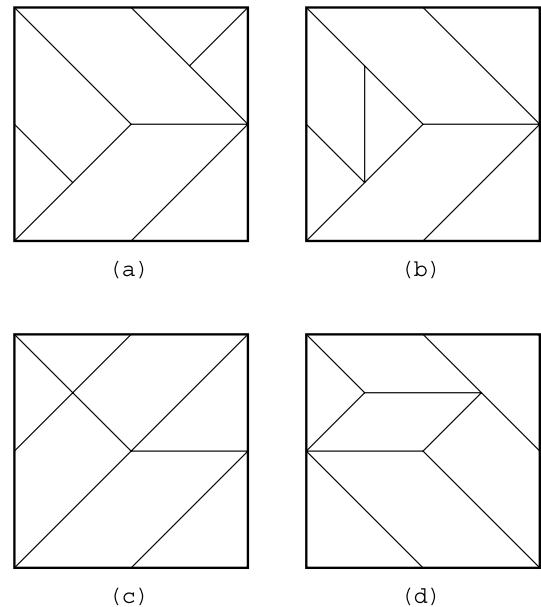


Fig. 5 Four patterns that can form nineteen convex polygons.

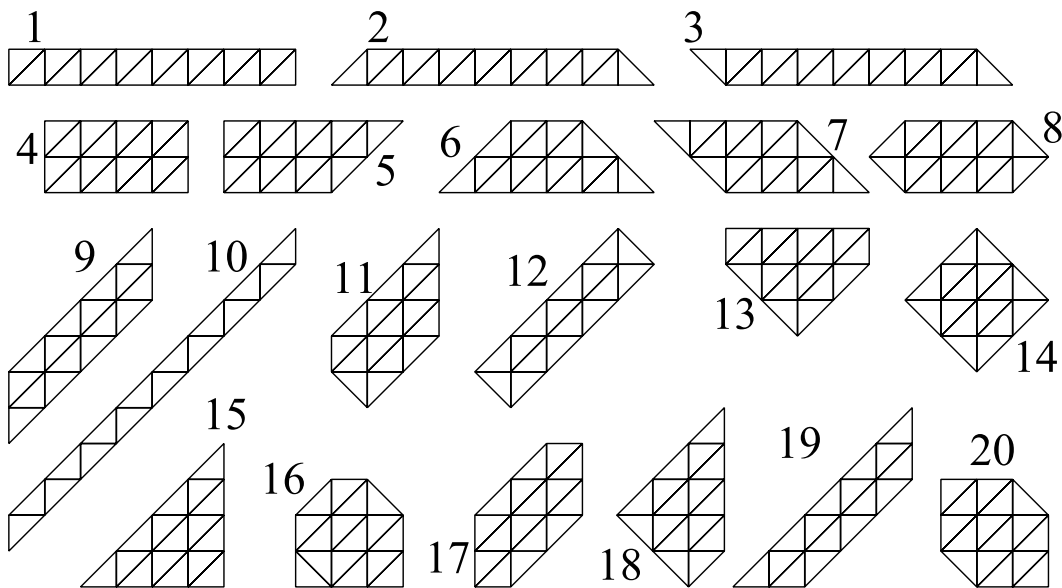


Fig. 4 All 20 potential convex polygons that can be formed from 16 identical isosceles right triangles.

[†]Later, we discovered this fact is folklore in the puzzle society in Japan [6].

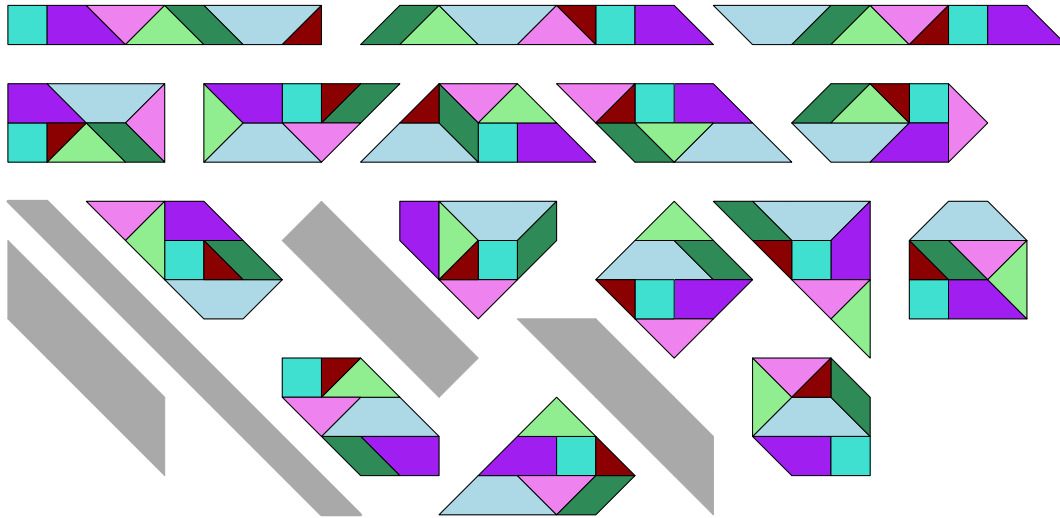


Fig. 6 All convex polygons that can be formed by Sei Shonagon Chie no Ita as well as the ones that can't be formed.

Any six pieces in the same manner cannot form nineteen convex polygons among twenty candidates.

Throughout, all triangles mentioned are identical isosceles right triangles with side lengths 1, 1, and $\sqrt{2}$. For simplicity, we refer to these triangles as *tiles*.

2. The Sei Shonagon Chie no Ita puzzle

Theorem 2: The Sei Shonagon Chie no Ita puzzle pieces can be rearranged into exactly sixteen distinct convex polygons up to reflection and rotation.

Proof. We first notice that the pieces of the puzzle can be decomposed into sixteen tiles, just like the tangram.

We make use of two important results from Wang and Hsiung [5]. First, there are only 20 candidate convex polygons in Fig. 4 that we need to consider, and second, in any convex polygon they can form, the bases of the sixteen triangles can be pairwise colinear, parallel, or perpendicular ([5, Lemma 1]). This means we only need to consider configurations that could be embedded with triangle and target polygon vertices on integer coordinates.

Sixteen convex polygons are filled as illustrated in Fig. 6. The remaining four polygons cannot be solved since they are too thin. More precisely, the largest trapezoid of area 2 has a base of length 3, and this length cannot be inside of the polygons. \square

3. Optimal Seven Piece Puzzles

Although Sei Shonagon Chie no Ita is more expressive than the tangram, Sei Shonagon Chie no Ita is not the optimal set of seven pieces if one wishes to form as many convex polygons as possible. We will prove Theorem 1(1); that is, we show that there exist four sets of seven pieces that allow us to form nineteen convex polygons. Our first lemma states that we can fix the nineteen convex polygons out of twenty

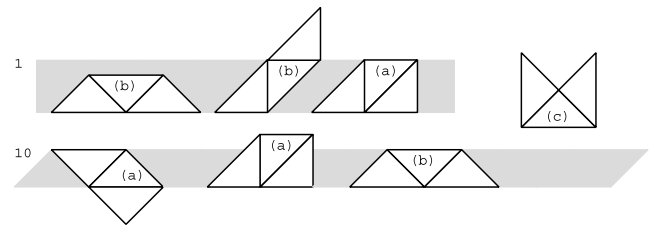


Fig. 7 Two skinny shapes and piece of three tiles.

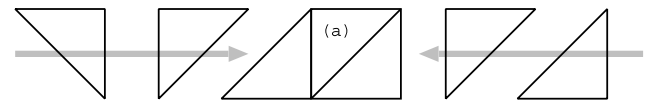


Fig. 8 Extensions of piece of size three.

that can be filled by our puzzle.

Lemma 3: Any set of seven pieces composed from sixteen tiles that can form nineteen of twenty convex polygons cannot form the convex shape 10 in Fig. 4.

Proof. We first observe that the average number of tiles in a piece is $16/7 = 2.285 \dots$. Therefore, any dissection pattern contains at least one piece containing at least three tiles. Then, there are three possible pieces that consists of three tiles (a), (b), and (c) as shown in Fig. 7. If we choose (a), we cannot fill the polygon 10. On the other hand, if we choose (b), we cannot fill the polygon 1. However, when we omit the polygon 1, we also have to omit the polygons 2 and 3 in Fig. 4. It is easy to see that (c) cannot fill these polygons. Therefore, to fill nineteen of them, we have to omit 10. \square

In the proof of Lemma 3, we choose polygons 1, 2, and 3 and omit the polygon 10. Then we can also say that any piece containing at least three tiles should be extended from the tile (a) in Fig. 7, and we cannot use the tile (b) and its extensions. To fill the shape 1, the possible tiles of size at least three are given in Fig. 8. However, if the number of tiles is greater

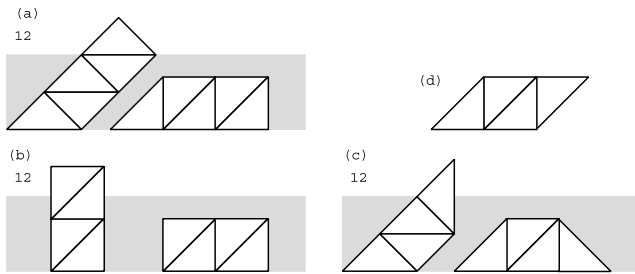


Fig. 9 Pieces of four/five tiles that cannot fill the polygon 12.

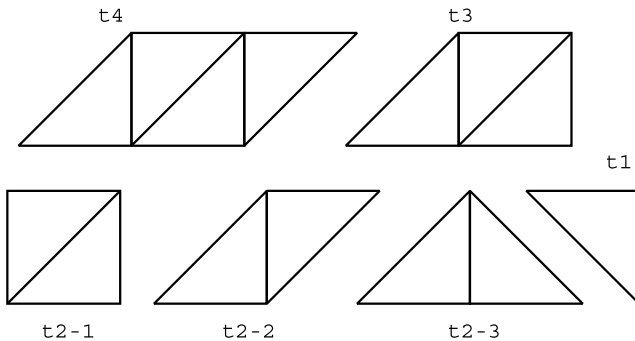


Fig. 10 Possible pieces to make nineteen convex polygons by seven pieces.

than 4, it cannot fill the polygon 12 (Fig. 9(a)). Moreover, even if the number of tiles is 4, two of three possible pieces cannot fill the polygon 12, either (Fig. 9(b)(c)). Therefore, the only possible piece of four tiles is one in Fig. 9(d). We summarize the above discussion:

Lemma 4: Any set of seven pieces that can form nineteen convex polygons contains only the copies of the pieces depicted in Fig. 10.

We name each tile as t_4 , t_3 , t_2-1 , t_2-2 , t_2-3 , and t_1 as in the figure. We also call any of t_2-1 , t_2-2 , and t_2-3 simply t_2 if we do not need to distinguish between the choices.

Let v, w, x, y be the number of tiles t_4 , t_3 , t_2 , and t_1 , respectively. Then they are integers, and we have $v + w + x + y = 7$ and $4v + 3w + 2x + y = 16$. The conditions are satisfied only when $(v, w, x, y) = (3, 0, 0, 4), (2, 1, 1, 3), (2, 0, 3, 2), (1, 3, 0, 3), (1, 2, 2, 2), (1, 1, 4, 1), (1, 0, 6, 0), (0, 4, 1, 2), (0, 3, 3, 1), (0, 2, 5, 0)$ by enumeration. For each of (v, w, x, y) with $x > 0$, we have three kinds of t_2 tiles. Considering the combinations, we obtain 98 candidates of the sets. We now proceed by case analysis.

We observe that the convex polygon 14 (or *square*) is a difficult polygon in this context. It contains four small squares made by two right isosceles triangle units at its central part as shown in Fig. 11. Big tiles like t_4 , t_3 , and t_2-1 should cover one of them, and this reduces the number of filling ways to be considered drastically.

Cases $(3, 0, 0, 4), (1, 3, 0, 3)$

We have one set in each case, and it cannot fill the convex polygon 14 (or square).

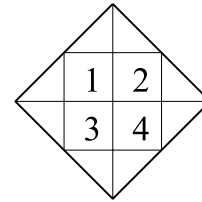


Fig. 11 Four small square in the square.

Case $(2, 1, 1, 3)$

We have three sets. When we choose t_2-3 , we have the set (a) in Fig. 5. Using the other two, we cannot fill the square.

Case $(2, 0, 3, 2)$

We have ten combinations for t_2 tiles. Among them, we can find the set (b) in Fig. 5. Two sets cannot fill the convex polygon 17, and the other sets cannot fill the square.

Case $(1, 2, 2, 2)$

We have six sets, and one is the set (c) in Fig. 5. The other sets cannot fill the square.

Case $(1, 1, 4, 1)$

We have 15 sets, and one of them is the set (d) in Fig. 5. The other sets cannot fill convex polygons 17 or 19, or the square.

Case $(1, 0, 6, 0)$

We have 28 sets, but none of them can fill the polygon 1.

Cases $(0, 4, 1, 2), (0, 3, 3, 1), (0, 2, 5, 0)$

We have 34 sets in total, but none of them can fill the polygon 19 or the square.

Therefore, we conclude that there are four possible sets that can fill nineteen out of twenty convex polygons shown in Fig. 4.

4. Beyond Seven Pieces

The next natural question is how many pieces might one need to form from sixteen tiles in order to form all 20 convex polygons. We turn to Theorem 1(2). That is, we prove that ten or fewer pieces formed from sixteen tiles cannot form 20 convex polygons, and eleven pieces can.

Proof of Theorem 1(2). In the negative direction, observe that to form the $1 \times 8\sqrt{2}$ parallelogram in Fig. 12(a) with ten pieces, there must be at least six t_2-2 pieces (larger pieces all contain it and do not fit within the shape of Fig. 12(b)).

Consider the square or polygon 14. The perimeter has 8 incident triangles, so the six t_2-2 pieces would have to cover at least four of those. Exhaustive case analysis, as seen in Fig. 14, shows that all arrangements that cover enough of the exterior triangles leave a square in the middle which cannot fit a single t_2-2 piece.

We observe that five t_2-2 pieces can fit inside each of the 20 shapes: see Fig. 13. So these with six single triangles can realize all 20 convex polygons. \square

5. Six Pieces are not Enough

We next turn to Theorem 1(3). Based on the result in Sect. 3, it is quite natural to ask if we can form 19 convex polygons with 6 pieces or less. In this section, we show that the answer is “no.”

Intuitively, this is proved by the following simple idea: Suppose that we have a six piece set. Then, by splitting each piece into two pieces, we obtain seven piece sets. However, we have already have all patterns in Fig. 5, which is too few.

To be precise, we fix some set of six pieces that form 19 convex polygons. Then, this set can be obtained by merging two pieces in one of four patterns in Fig. 5. If only one piece is built from more than one tile, a single tile can be cut off the large piece, resulting in a set of seven with a piece too large. If any piece contains at least 5 tiles, some other piece can be cut, giving a set of seven with a piece too large. If the set contains at least two $\mathbf{t4}$ pieces, a piece can be cut so that at least two intact $\mathbf{t4}$ pieces are in the set of seven. If the set

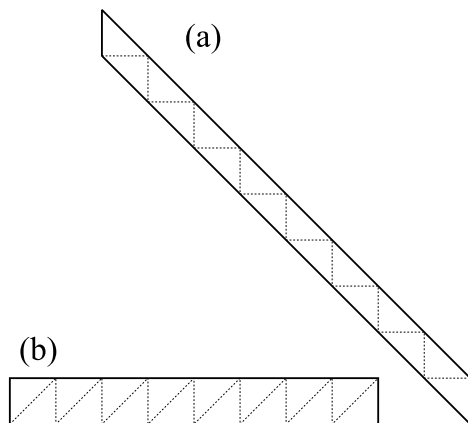


Fig. 12 Any set of 7 pieces covering shape (a) must have a piece that consists of at least 3 triangles, which cannot be covered by shape (b).

contains just one $\mathbf{t4}$, that piece can be cut; all valid sets of seven pieces have at least one $\mathbf{t4}$.

6. Concluding Remarks

Sixteen identical right isosceles triangles can form twenty convex polygons. We compare the power of expression of some classic silhouette puzzles constructed from these triangles. The “difficulty” of a silhouette puzzle for people to solve can be estimated by the number of ways in which one can solve it. Computing these numbers efficiently remains a compelling task for future work.

Another interesting direction of study is the number of convex polygons formed by different numbers of triangles. Let $f(n)$ be the number of convex polygons formed by n tiles. To analyze the tangram and Sei Shonagon Chie no Ita puzzles, the value $f(16) = 20$ plays an important role. If we design larger puzzles, it is natural to consider the number of composable polygons among them. The function $f(n)$ itself is also interesting to investigate. Although it is not monotone ($f(1) = 1$, $f(2) = 3$, and $f(3) = 2$), it is a generally increasing function (Fig. 16). Trivially, for all $x \geq 0$, we have $f(x) < f(2x)$ as one can subdivide every triangle into two to form all of the previous convex polygons, as well as a parallelogram with side lengths 1 and $x\sqrt{2}$.

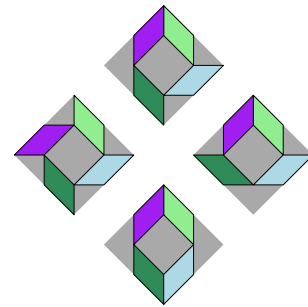


Fig. 14 Six parallelograms do not fit in a square.

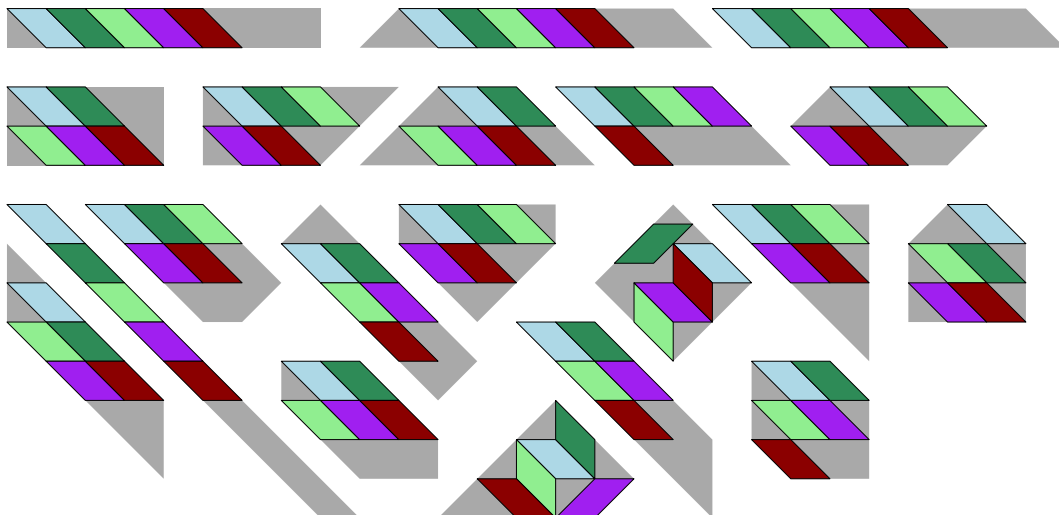


Fig. 13 11 pieces forming all 20 convex polygons (the 6 individual isosceles right triangles not shown).

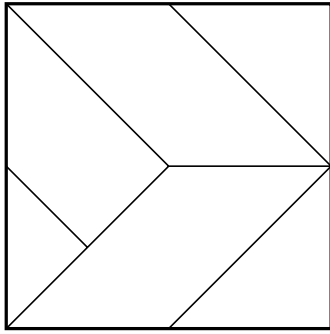


Fig. 15 Possible construction of six-piece puzzle.

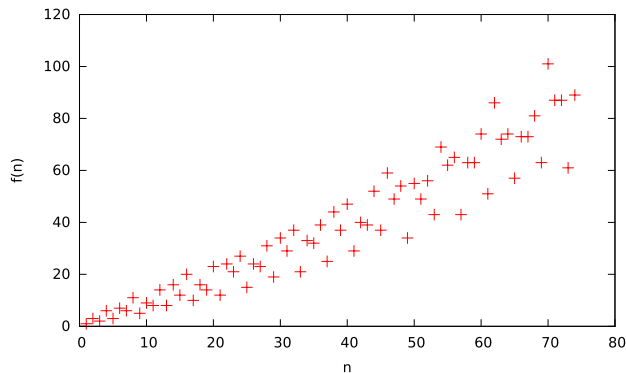


Fig. 16 The number $f(n)$ of convex polygons formed by n tiles.

Some puzzles with fewer pieces are also another topic. We need seven pieces to form 19 convex polygons, and eleven pieces to form 20 convex polygons. That is, letting $g(n)$ be the number of possible convex polygons with n pieces in this manner, we have $g(1) = 1$, $g(6) < 19$, $g(7) = g(8) = g(9) = g(10) = 19$, and $g(n) = 20$ for $11 \leq n \leq 16$. We also have $17 \leq g(6)$ by the pattern in Fig. 15, which can form all but polygons 10, 11, and 17 in Fig. 4. Thus we have $g(6)$ is either 17 or 18. Determining the values of $g(n)$ for $n = 2, 3, 4, 5, 6$ are future work[†].

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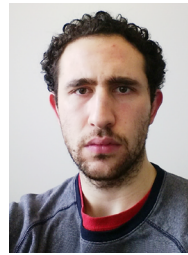
References

- [1] H.E. Dudeney, *The Canterbury Puzzles*, Dover Publications, 1958.
- [2] J. Slocum, *The Tangram Book: The Story of the Chinese Puzzle with Over 2000 Puzzle to Solve*, Sterling Publishing, 2004.
- [3] M. Gardner, *Time Travel and Other Mathematical Bewilderments*,

[†]Recently, Yasuhiko Takenaga told us that he concluded $g(6) = 17$ based on case analysis.

W.H. Freeman and Company, 1987.

- [4] T. Takeda, <http://www.matsusaki.jp/local-event/20140310/1024/>, (in Japanese; accessed 8 May 2014), March 2014.
- [5] F.T. Wang and C.-C. Hsiung, “A theorem on the tangram,” *The American Mathematical Monthly*, vol.49, no.9, pp.596–599, 1942.
- [6] H. Akiyama, “Personal communication,” 2014.



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