Evolutionary Game Theory in Economics

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Abstract

The main topic of this summary is Evolutionary Game Theory: briefly, we introduce the topic, the concept of equilibrium in this setting and finally we revise some "classical games" in game theory, such as Prisoner's Dilemma and Rock Scissor Paper, developing some numerical application in Matlab. The last part is related to the an application of Evolutionary Game Theory to an economic environment such as complex networks and the evolution of price competition game. The main reference of the summary is Friedman (1991), however a simple summary was restrictive for our purpose: after the introduction of the model and the concept of Evolutionary Stable State, the paper mainly focus on non-economic example and develop some mathematical result. The topic was of our interest so we decided to give an overview of the subject also using information that we were able to truly understand and replicate contained into other sources such as Weibull (1997), Tanimoto (2015),King (2019),Xie and Shi (2018).

1 Introduction

Evolutionary game theory has evolved from the classical game theoretic framework by merging it with the basic concept of **Darwinism** in order to take into account the idea of time evolution, which is partially lacking in the original Game Theory that primarily deals with equilibrium, instead, in this framework we can imagine that the game is played **over and over** again by biologically or socially conditioned players who are randomly drawn from large populations.

More specifically, each player is "pre-programmed" to some behavior in the game and one assumes that some **evolutionary selection** process operates over time on the **population distribution of behaviors**. The kind of questions Evolutionary Game Theory tries to give an answer are questions like: what are the connections between the long-run aggregate behavior in such an evolutionary process and solution concepts in non-cooperative game theory? More specifically: Are dominated strategies wiped out in the long run? Will aggregate behavior tend toward a **Nash Equilibrium** of the game? Are some **Nash Equilibrium** more likely to emerge in this setting than others? What is the nature of long-run aggregate behavior if it does not settle down on some equilibrium?

Recently these subjects has become of interest to **economists** and in general to **social scientists**. The interest of this theory is due to the following facts: first of all, the "evolution" treated by these kind of games do not need to be in a **biological sense**, but may be referred to the **cultural** evolution such as changes in beliefs and norms over time. Second, the usually **rationality assumption** of game theory is not enough in order to explain the complexity of the world, instead **evolutionary game theory** opens the possibility of modelling social system using more realistic sets of assumptions. Third, it is a **dynamic theory**, providing insights not only on "**how things works**" but also on "**how things works overtime**", in the sense that it studies the stability of the equilibrium, if one exists, overtime.

In an **evolutionary game** each individual chooses among alternative strategies whose payoff or fitness depends on the choices of others. The interesting fact is that, it not only allows the dependence of the payoff to other individuals but overtime the distribution of behaviour evolves, as fitter strategies become more prevalent. In general the dynamics can be quite complex and one can ask which behaviors become extinct and which survive overtime, whether the system approaches some steady-state and so forth. The next section introduces the general model for an evolutionary game.

2 The Model

We consider an **interacting population** indexed by $k = 1, ..., K \in \mathbb{N}$. For each member k there is a **finite number of strategies** indexed by $i = 1, ..., N \in \mathbb{N}$.

Given those strategies, it is natural to define the N-dimensional simplex

$$S^k := \{x = (x_1, ..., x_n) : x_i \geq 0 \; and \; \sum x_i = 1\}$$

where $r^k \in S^k$ represent a possible **mixed strategy** for an individual member of population. The use of the simplex is justified so that any point s^k represents the **fraction of population k** that uses each strategy.

The **cartesian product** $S := S^1 \times ... \times S^k$ is the set of **strategy profiles** and under the interpretation that interactions are anonymous it is also the **state space**.

The other two ingredients of the model are the **fitness function** and the **dynamic structure**.

Strategic interactions are summarized by using a **fitness function** which specifies for individuals in each population the evolutionary relevant payoff as a function of own strategy and current state. More formally, define $f : S \times S \to \mathbb{R}^K$ such that:

$$f(r, s) := (f^1(r^1, s), ..., f^k(r^k, s))$$

which, for any $f^i(r^i, s) \quad i = 1, ..., N$ we assume to be **linear** in $r^k \in S^k$ and C^K in the **population state** argument $s \in S$.

The **dynamical structure** that specifies how a state s evolves overtime is the time derivative $\dot{s} = (\dot{s}^1, .., \dot{s}^k)$ where $\dot{s}^k := ((\dot{s}_1)^k, .., (\dot{s}_N)^k)$, by means of some function $F : S \to \mathbb{R}^{NK}$. So from this we have that:

$$\dot{s} = F(s)$$

is an **autonomous system of ordinary differential equation** whose solution curve $s(t)$, given an initial value $s(0) \in S$ describes the evolution of all population beginning at any state of interest. Moreover, some **restriction** are needed for the dynamics. In general we say that the mapping $F: \ S \rightarrow \mathbb{R}^{NK}$ is **admissible** if:

1. $\sum_{i=1}^{N} F(s)_i^k = 0 \, \forall s \in S$ and $k = 1, ..., N$;
2. $s_i^k = 0$ implies $F_i^k = 0$
3. F is continuous and piecewise differentiable on S

The first two conditions ensure that s^k do not leave the simplex, that the population fractions will sum to 1 without being negative also preventing the revival of extinct strategies. The third condition is technical in order to have "well-behaved" solutions. The next section deals with a refinement of the concept of **Nash Equilibrium** called **Evolutionary Stable State**

3 Evolutionary Stable State

A refinement of the concept of **Nash Equilibrium** seems to be necessary in this framework and is the so-called **Evolutionary Stable State**. An intuition of Evolutionary Stable State was given by Maynard Smith: suppose a quantity of mutant added to the population. There is the possibility that this mutation could not only invade the population but also absorb it and replace the original population. If the strategy persist both to the **invasion** and to the **replacement**, then it is a **ESS**.

In general, given a fitness function f, s is an Evolutionary Stable State if: for each $k = 1, ..., K$ and $x \neq s^k \in S^k$, we have either $f^k(x, s) < f^k(s^k, s)$ or $f^k(x, s) = f^k(s^k, s)$ and $f^k(x, s|_k x_\epsilon) < f^k(s^k, s|_k x_\epsilon)$ **for** $\epsilon > 0$ **sufficiently small**.

The notation $s|_k x$ means $(s^1, ..., s^{k-1}, x, s^{k+1}, ..., s^K)$ and $s|_k x_\epsilon$ is a slight perturbation around x. At this point, since we can define a Nash Equilibrium point for the fitness function f as $s \in NE(f)$ such that $f^k(x, s) \leq f^k(s^k, s)$ for all $x \in S^k$, then it seems clear that Evolutionary Stable State is a refinement of the concept Nash Equilibrium.

Let's analyse the dynamic equilibrium concepts and characterize steady states. These equilibrium are defined in terms of the function F that specifies the dynamical system S. s is a **fixed point** for F, $s \in FP(F)$, if $F(s) = 0$. Moreover, s is an **evolutionary equilibrium** for F, $s \in EE(F)$, if s is a locally asymptotically stable fixed point.

To reinforce the stability of the Evolutionary Stable State we have to specify the relationship between the four kind of equilibrium:

$$ESS(f) \subset EE(F) \subset NE(f) \subset FP(F)$$

Recalling that all Nash Equilibria are fixed points and all interior fixed points are Nash Equilibria and that the evolutionary equilibria are always NE for a corresponding fitness function. The following section analyzes the classical **Prisoner's Dilemma** game, translating it into an evolutionary context.

4 Prisoner's Dilemma - 2x2 Games

Prisoner's Dilemma is one of the most important example in Game Theory and describes a scenario involving cooperation and defection: ff both individuals cooperate, they both end up winners, if they defect, they both end up in losing. Even if the optimal strategy for both seems to cooperate, in reality what happens is that, since defection yields the higher payoff regardless of what the opponent choose, then both players assume that their opponent will defect and they they both choose to defect. With this in mind, one can use the **evolutionary framework** in order to study the dynamics of the game in two precise population: **collaborator** and **defectors**. First of all, we define the payoff matrix:

$$A = \begin{array}{cc} \begin{array}{cc} C & D \end{array} & \\ \begin{pmatrix} R & S \\ T & P \end{pmatrix} & \begin{array}{c} C \\ D \end{array} \end{array}$$

With the following restriction: $T > R > P > S$ and $R > (T + P)/2$, necessary in order to ensure the feasibility of a purely cooperative strategy.. This is just a generalization of the payoffs that can be classified by four different outcomes:

1. **Reward** for mutual cooperation - R
2. **Temptation** to defect - T
3. **Punishment** for mutual defection - P
4. **Sucker's** payoff - S

If we denote with e_1 and e_2 the usual basis vector (by convention they are column vector, so when transpose we refer to row vector), then we have that:

$$e_2^T U e_2 = P > S = e_1^T U e_2$$

which implies that **defection** is a **unique strict Nash Equilibrium** and so an ESS.

As anticipated in previous sections we can characterize the dynamics of the evolution of the population of **cooperators** and **defectors** by defining the **fitness** or **replicator** function. Here we have K = 2 so:

$$\dot{x_1} = x_1((A\mathbf{x})_1 - \mathbf{x}^T A\mathbf{x})$$

$$\dot{x_2} = x_2((A\mathbf{x})_2 - \mathbf{x}^T A\mathbf{x})$$

Where $(A\mathbf{x})_i$ for $i = 1, 2$ is the position of the used element in the dot product result. After some algebraic manipulation we end up with:

$$\dot{x_1} = -x_1(Rx_1^2 - Rx_1 + (T + S)x_1x_2 + Px_2^2 - Sx_2)$$

$$\dot{x_2} = -x_2(Rx_1^2 - Tx_1 + (T + S)x_1x_2 + Px_2^2 - Px_2)$$

which is a **system of nonlinear differential equation**. The system can be further simplified noticing that x_1 and x_2 are on the simplex and $x_2 = 1 - x_1$, hence we can derive an unique equation:

$$\dot{x_1} = x_1(1 - x_1)((R - S - T + P)x_1 + S - P)$$

This equation have two **fixed points**, $x_1 = 0$ and $x_1 = 1$. In general, given a dynamical system for which the state equations are non-linear function, it is useful to study the stability of the system as it approach to the equilibrium and an important result is given by **Lyapunov Theorem**[1]. Suppose that f and g are C^1 functions and let (a, b) be an **equilibrium point** for the system:

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases} \tag{1}$$

Denote with **A** the **Jacobian Matrix**:

$$A = \begin{pmatrix} \frac{\partial f(a,b)}{\partial x} & \frac{\partial f(a,b)}{\partial y} \\ \frac{\partial g(a,b)}{\partial x} & \frac{\partial g(a,b)}{\partial y} \end{pmatrix}$$

If

$$tr(A) = \frac{\partial f(a, b)}{\partial x} + \frac{\partial g(a, b)}{\partial y} < 0$$

and

$$|A| = \frac{\partial f(a, b)}{\partial x}\frac{\partial g(a, b)}{\partial y} - \frac{\partial f(a, b)}{\partial y}\frac{\partial g(a, b)}{\partial x} > 0$$

i.e. if both **eigenvalues** of **A** have negative real parts, then (a, b) is **locally asymptotically stable**.

So, given this result, we can inspect the **Jacobian matrix** of the system: in $x_1 = 0$, the Jacobian have a single **eigenvalue** given by $\lambda = S - P$ and since $P > S$, this means that $\lambda < 0$ implying $x_1 = 0$ is a **sink**. All interior initial conditions converge to it asymptotically in forwards time, i.e., x_1 has a **basin of attraction** given by $x_1 \in (0, 1)$.

In $x_1 = 1$, the Jacobian has a single eigenvalue given by $\lambda = T - R$ and since $T > R$ this implies $\lambda > 0$, making $x_1 = 1$ a **source**: all interior initial conditions converge to it asymptotically in backwards time.

In order to solve the system, we used a numerical method with Matlab for deriving a plot of the dynamics of the populations under different initial conditions.

[1] A good treatment of the stability for nonlinear systems is given by Sydsæter, Hammond, Seierstad, and Strom (2008) in "Further Mathematics for Economic Analysis"

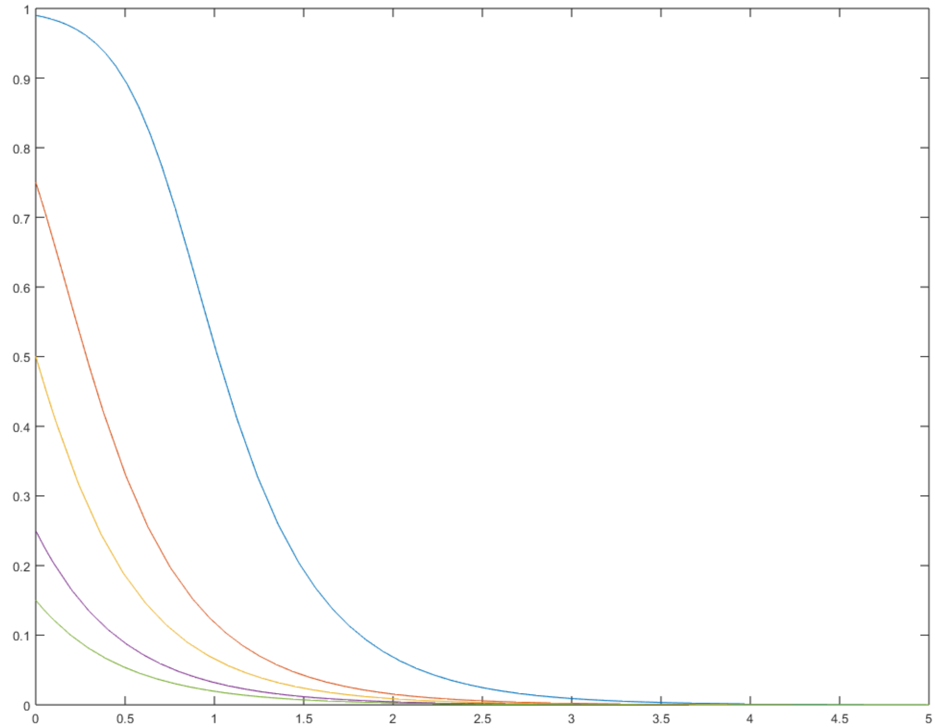


Figure 1: Trajectories for different initial condition of x_1 (0.99, 0.75, 0.5, 0.25, 0.15)

As we can notice from the picture regardless of the starting point which determines only the **speed** at which the system reach the **steady state**, the population of **collaborator** becomes always zero. Overtime the system will exhibit always the defective behaviour of the players.

5 Rock Scissor Paper - 3 Strategy Games

Rock Scissor and Paper is an interesting example because we do not end up with a stable equilibrium (under certain condition) while we have a cyclic behaviour of the players moves. We can define the payoff matrix as:

$$A = \begin{array}{c} \begin{array}{ccc} R & S & P \end{array} \\ \begin{pmatrix} \varepsilon & 1 & -1 \\ -1 & \varepsilon & 1 \\ 1 & -1 & \varepsilon \end{pmatrix} \end{array} \begin{array}{c} \\ R \\ S \\ P \end{array}$$

where the three populations are **Rock players**, **Scissor player** and **Paper player**. From the payoff structure this is a **zero-sum game** and with $\varepsilon = 0$

$$
A = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cc} & \begin{array}{ccc} R & S & P \end{array} \\ & \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{array}{c} R \\ S \\ P \end{array} \end{array}
$$

we do not have any **Pure Nash Equilibrium** or even **Evolutionary Stable State**, we have only a **Mixed Strategy Nash Equilibrium** at $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \hat{p}$ for each type of player. Again, the system can be derived as the one in the case of Prisoner's Dilemma:

$$
\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -x_1(\varepsilon x_1^2 - \varepsilon x_1 + \varepsilon x_2^2 - x_2 + \varepsilon x_3^2 + x_3) \\ -x_2(\varepsilon x_1^2 + x_1 + \varepsilon x_2^2 - \varepsilon x_2 + \varepsilon x_3^2 - x_3) \\ -x_3(\varepsilon x_1^2 - x_1 + \varepsilon x_2^2 + x_2 + \varepsilon x_3^2 - \varepsilon x_3) \end{bmatrix} = f \tag{2}
$$

Now, for any $p \neq \hat{p}$ we have:

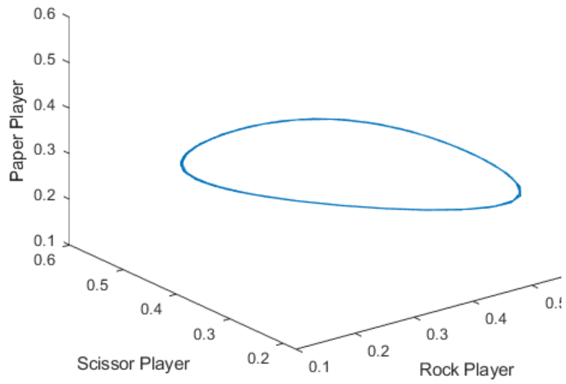
$$
\hat{p}Ap - pAp = -\varepsilon(p_1^2 + p_2^2 + p_3^2 - 1)
$$

If $-1 < \varepsilon < 0$, the equation above is positive, then $\hat{p}$ is an Evolutionary Stable State. For $0 \leq \varepsilon < 1$ the equation above is non-positive, then $\hat{p}$ is not an Evolutionary Stable State. Moreover the system of ordinary differential equations has **fixed points** given by the set of points $x^* \in \mathbb{R}^3$ such that $\dot{x} = 0$, i.e, the set:

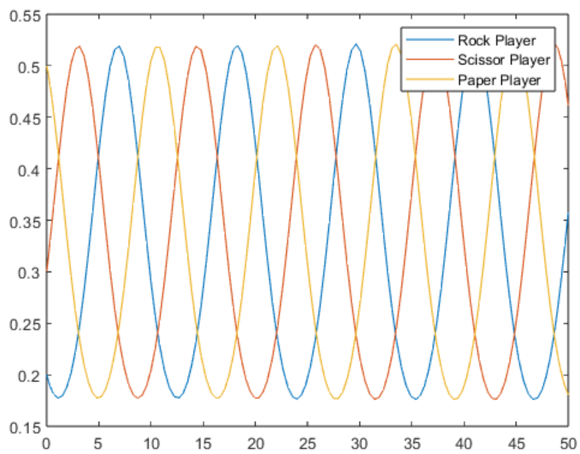
$$
\{e_1, e_2, e_3, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{\varepsilon - 1}{2\varepsilon}, \frac{\varepsilon + 1}{2\varepsilon}, 0), (0, \frac{\varepsilon - 1}{2\varepsilon}, \frac{\varepsilon + 1}{2\varepsilon}), (\frac{\varepsilon + 1}{2\varepsilon}, 0, \frac{\varepsilon - 1}{2\varepsilon})\}
$$

and we proceed to solve it numerically using Matlab and investigating what happens as ε changes. The code for reproduce the model and the plots are at the end of the summary.

In this first case, with $\varepsilon = 0$ we end up without a steady state of the system, it can be seen that the evolution of the population is particular, they seems to cyclically switch in playing a certain strategy whenever the winning strategy is dominating the other, this leads to the behaviour of the right picture. Also the phase portrait depicts this situation.



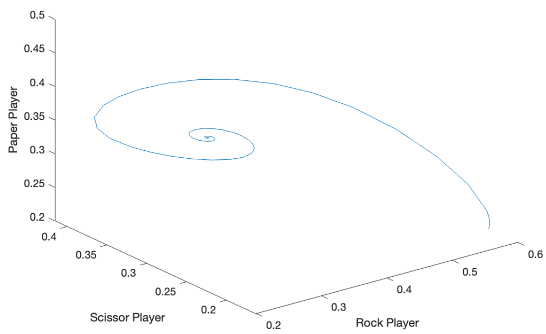
(a) 3-Dimensional Phase Portrait with $\varepsilon = 0$



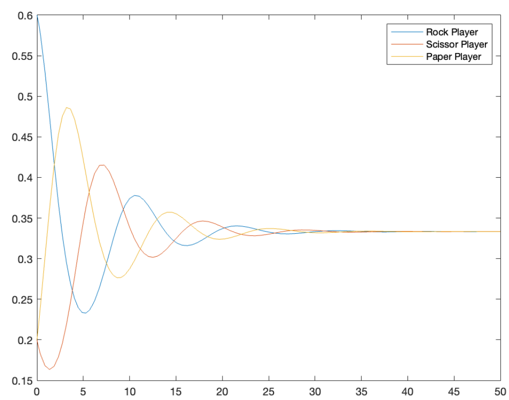
(b) Trajectories for $\varepsilon = 0$ and $x_1(0) = 0.2$, $x_2(0) = 0.3$, $x_3(0) = 0.5$

Figure 2

The most **interesting** case is with $\varepsilon = -0.5$: as we can see from the phase portrait and from the trajectory plots, what happens is that the population seems to "**learn**" the equilibrium playing the game a number of times until the system eventually reach a steady state on which the proportion of population becomes $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The same kind of behaviour is reproduces faster if one set $\varepsilon = -1$ or slower as long as we set ε closer and closer to 0 (where the system exhibit the behaviour seen before).



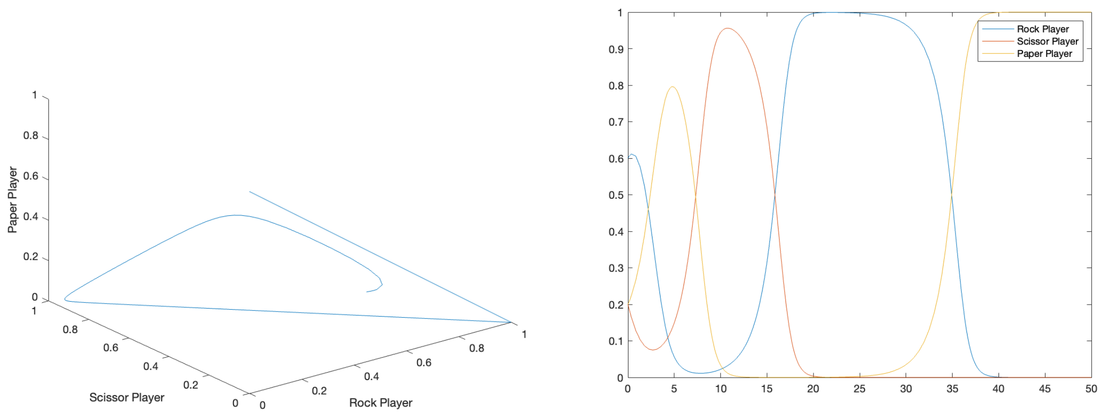
(a) 3-Dimensional Phase Portrait with $\varepsilon = -0.5$



(b) Trajectories for $\varepsilon = -0.5$ and $x_1(0) = 0.2$, $x_2(0) = 0.3$, $x_3(0) = 0.5$

Figure 3

As long as we go further than $\varepsilon = 0$, then the system exhibit a strange behaviour: from the pictures, with a $\varepsilon = 0.5$ it seems that the system reach a point in which only one population is dominating the others. From a mathematical point of view the stability of the systems can be evaluated by studying the **eigenvalues** of the system. A more formal mathematical treatment of these two games and an analysis of the mathematical property of these system is contained in King (2019).



(a) 3-Dimensional Phase Portrait with $\varepsilon = 0.5$

(b) Trajectories for $\varepsilon = 0.5$ and $x_1(0) = 0.2$, $x_2(0) = 0.3$, $x_3(0) = 0.5$

Figure 4

6 Networks and Price Competition

Following the work of Xie and Shi (2018) we report an interesting study about a revisitation of the **Bertrand** model under an evolutionary framework. First we briefly recall the classical model: we have two firms that compete on prices selling an homogeneous good and have identical costs. The outcome is that both charge a price equal to the marginal cost, and is defined a **paradox** since they are in an oligopolistic market in which they could collude and get easy higher profits. Denote with $D(p_i) = a - bp_i$ the demand function with $b > 0$ and for simplicity set the marginal cost $c = 0$, then the **payoff function** $\pi_i(p_i, p_j)$ is:

$$\begin{cases} \frac{p_i(a-bp_i)}{2}, \quad if \quad p_i = p_j \\ p_i(a-bp_i), \quad if \quad p_i < p_j \\ 0, \quad if \quad p_i > p_j \end{cases} \tag{3}$$

Under evolutionary game theory framework firms are **bounded rational** and thus have no capability of making the perfect decision of setting the price at marginal cost, in this framework they make a simple decision:

- They keep the **original price** $p > 0$
- They **cut the price** to $p^* = \lambda p$ with $0 < \lambda < 1$ being the degree of **price cutting**

All firms simultaneously decide what prices they should offer and uses the same price for all of its competitive relations, that is, for all of its neighbor firms. Price evolution is carried out implementing the rule of imitate best and is due to the fact that empirical evidence highlight the fact that firms imitate the most successful behavior that is the best one.

In each stage, each firm offers a price $p_{i,t}$ and the payoff is calculated by accumulated payoff or average payoff:

- Under the **accumulated payoff scheme**: $U_{i,t} = \sum_{j \in \Omega_i} \pi_{i,t}(p_{i,t}, p_{j,t})$ with ω_i the set of neighbor firms of i;
- Under the **average payoff scheme**: $U_{i,t} = \frac{\sum_{j \in \Omega_i} \pi_{i,t}(p_{i,t}, p_{j,t})}{n_{\omega i}}$ where $n_{\omega i}$ is the number of neighbor firms.

After all firms obtain payoffs, the **update prices simultaneously** and compares the payoffs between itself and all of its neighbor firms and adopts the price that **yields the highest payoff** at time t.

According to the payoff function described before, we can describe a payoff matrix:

$$M_1 = \begin{pmatrix} \frac{p(a-bp)}{2} & 0 \\ p^*(a-bp^*) & \frac{p^*(a-bp^*)}{2} \end{pmatrix}$$

According to the evolutionary game theory, the relative order of four elements of payoff matrix $M1$ can lead to different evolutionary equilibriums. Denoting four payoffs in the matrix M1, (Same notation as the one used into the prisoner's dilemma example) respectively, as:

- $R = p(a - bp)/2$;
- $T = p^*(a - bp^*)$;
- $P = p^*(a - bp^*)/2$;
- $S = 0$

And we can proceed analyzing the relationship between the four payoffs, but first we have to give some conditions about the parameters included in the expressions: $p > 0$, $b > 0$, $a > bp$, $p* = \lambda p$ and $0 < \lambda < 1$. Highlighting the former conditions, we can deduct the following relationships: $R, T, P > S$, $P = \frac{1}{2}T$.

So the order $R > S, T > P$ is supported. Now let's analyze the relationship between R with T and P: suppose $R > T$ then $p(a - bp)/2 > p^*(a - bp^*)$ (1), since $a > bp$ we can set $a = \beta bp^2$ with $\beta > 1$. It follows that $(\beta - 1)bp^2 > 2\lambda(\beta - \lambda)bp^2$ (2), dividing by bp^2 we get $(\beta - 1) > 2\lambda(\beta - \lambda)$ (3). Rearranging the last expression we have $\beta(1 - 2\lambda) > (1 - 2\lambda^2)$ (4).

Now let's see if the inequality $R > T$ is supported when λ varies:

- $0.5 < \lambda < 1$: $(1-2\lambda) < 0$ and $(1-2\lambda) < 1-2\lambda^2$ (5) are supported, but this contradicts the inequality (4). So we can deduct that for $0.5 < \lambda < 1$: $(1 - 2\lambda) < 0$, $T > R$ is supported;

- $\lambda = 0.5$:$(\beta - 1) < 2\lambda(\beta - \lambda)$ is supported which contradicts inequality (3), indicating that also in this case $T > R$ is supported;
- $0 < \lambda < 0.5$: $(1 - 2\lambda) > 0$ holds and from (4) we get $\beta > \frac{(1-2\lambda^2)}{(1-2\lambda)}$, so if the last inequality holds $R > T$, otherwise $T > R$ is supported.

Given $R > P$, $p(a - bp)/2 > p^*(a - bp^*)/2$ (6) is supported and applying the initial conditions it becomes $\beta(1 - 2\lambda) > (1 - \lambda^2)$. Thus, while $\beta > (1 + \lambda)$, $R > T$ is supported, otherwise $P > R$ holds.

From these analysis results we can obtain the order of R, S, T and P when the parameter λ varies and the rage of value that β can assume.

Range of λ	Range of β	Oder of R, S, T and P	Evolutionary Equilibrium
$0.5 \leq \lambda \leq 1$	$\beta > 1 + \lambda$	$T > R > P > S$	Full price cutting
	$1 + \lambda > \beta > 1$	$T > P > R > S$	Full price cutting
$0 \leq \lambda \leq 0.5$	$\frac{(1-2\lambda^2)}{(1-2\lambda)} > \beta > 1 + \lambda$	$T > R > P > S$	Full price cutting
	$\beta > \frac{(1-2\lambda^2)}{(1-2\lambda)}$	$R > T > P > S$	Full price cutting or full price keeping

As regard the last row of the table, depending on the initial fraction of firms offering price p, the price competition system will converge to different evolutionary equilibria. If the initial fraction $f_p(0)$ of firms offering price p is greater than $x^*(x^* = \frac{(P-S)}{(R-T-S+P)})$, the system will converge to full price keeping. If $f_p(0) < x^*$, the system will converge to full price cutting.

So the values of x^* and $f_p(0)$ are crucial for the evolutionary results. Recalling the definition of x^* just mentioned we can plug the values of the payoffs and get $x^* = \frac{(P-S)}{(R-T-S+P)} = (\frac{\lambda(\beta-\lambda)bp^2}{(\beta-1)bp^2-\lambda(\beta-\lambda)bp^2}$, so the value of x^2 depends on λ and β, $x^* = (\frac{\lambda(\beta-\lambda)}{(\beta-1)-\lambda(\beta-\lambda)}$. x^2 increases with λ, but hardly changes with β, while p^* increases also λ increases, and x^* increases accordingly.

7 Conclusions

To conclude, Evolutionary Game Theory seems to be a nice framework to adopt in order to analyze and characterize complex environments such as economical one giving the opportunity to add more realism to models. To sum up, we introduced the topic, the description of a general model and the evolution of the concept of Nash Equilibrium, called Evolutionary Stable State. Moreover, we presented some example that could clarify how models are developed under this framework and how they are characterized by certain solutions under certain conditions: we examined a 2x2 game, a 3 strategy game and an application in Economics related to complex networks and price competition. A lot of research has been done in this topic and one could go even further by looking at the stochastic evolution, for instance in Foster and Young (1990).

8 Matlab Codes

```
%% Prisoner Dilemma
T = 10;
R = 5;
P = 3;
S = 1;
x1_init = [0.99,0.75,0.5,0.25,0.15];
x2_init = [1-x1_init(1), 1-x1_init(2), 1-x1_init(3), 1-x1_init(4), ...
    1-x1_init(5)];



dXdt = @(t,X) [X(1)*( R*X(1)+S*X(2)-R*X(1)^2-T*X(1)*X(2)- ...
    S*X(1)*X(2)-P*X(2)^2);
                 X(2)*(T*X(1)+P*X(2)-R*X(1)^2-T*X(1)*X(2)- ...
                     S*X(1)*X(2)-P*X(2)^2)];



for i=1:5
    [t,X] = ode45(dXdt, [0,5], [x1_init(i),x2_init(i)]);
    plot(X(:,1),X(:,1))
    hold on
end
```

```
%% Rock Scissor Paper
eps = 0;
x1_init = 0.6;
x2_init = 0.2;
x3_init = 1 - x1_init - x2_init;

dXdt = @(t,X) [X(1)*(eps*X(1)+X(2)-X(3)-eps*(X(1)^2+X(2)^2+X(3)^2));
                 X(2)*(-X(1)+eps*X(2)+X(3)-eps*(X(1)^2+X(2)^2+X(3)^2));
                 X(3)*(X(1)-X(2)+eps*X(3)-eps*(X(1)^2+X(2)^2+X(3)^2))];

[t,X] = ode45(dXdt, [0,50], [x1_init,x2_init,x3_init]);
```

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