1. Consider two successive two-dimensional rotations given by $r'' = R_{\phi'} r' = R_{\phi'} R_\phi r$. Show that $R_{\phi'} R_\phi = R_{\phi'+\phi}$.
   Directly, we have

$$R_{\phi'} R_\phi = \begin{pmatrix} \cos(\phi') & \sin(\phi') \\ -\sin(\phi') & \cos(\phi') \end{pmatrix} \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\phi')\cos(\phi) - \sin(\phi')\sin(\phi) & \cos(\phi')\sin(\phi) + \sin(\phi')\cos(\phi) \\ -(\cos(\phi')\sin(\phi) + \sin(\phi')\cos(\phi)) & \cos(\phi')\cos(\phi) - \sin(\phi')\sin(\phi) \end{pmatrix}$$

   which by trigonometric angle sum formulas, is equal to

$$\begin{pmatrix} \cos(\phi'+\phi) & \sin(\phi'+\phi) \\ -\sin(\phi'+\phi) & \cos(\phi'+\phi) \end{pmatrix} = R_{\phi'+\phi}.$$

2. Given

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 0 & 3 & 2 \\ 1 & -3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

   evaluate the commutator $[A, B] \equiv AB - BA$. The commutator is

$$[A, B] \equiv AB - BA = \begin{pmatrix} 3 & 2 & -1 \\ 0 & 3 & 2 \\ 1 & -3 & 4 \end{pmatrix} \begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} - \begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & -1 \\ 0 & 3 & 2 \\ 1 & -3 & 4 \end{pmatrix}$$
$$= \begin{pmatrix} 5 & -6 & 8 \\ 9 & 7 & 2 \\ 11 & 3 & 7 \end{pmatrix} - \begin{pmatrix} 9 & -11 & 6 \\ 3 & 5 & 1 \\ 10 & 9 & 5 \end{pmatrix}$$
$$= \begin{pmatrix} -4 & 5 & 2 \\ 6 & 2 & 1 \\ 1 & -6 & 2 \end{pmatrix}.$$

3. Write the set of simultaneous equations

$$\begin{cases} 2x_1 + 4x_2 + 3x_3 = 4 \\ x_1 - 2x_2 - 2x_3 = 0 \\ -3x_1 + 3x_2 + 2x_3 = -7 \end{cases}$$

   in matrix form and obtain the solutions by inverting the matrix.

   The system in matrix form is

$$\begin{pmatrix} 2 & 4 & 3 \\ 1 & -2 & -2 \\ -3 & 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -7 \end{pmatrix},$$

and by using an augmented matrix we can form the inverse matrix:

$$\left(\begin{array}{ccc|ccc} 2 & 4 & 3 & 1 & 0 & 0 \\ 1 & -2 & -2 & 0 & 1 & 0 \\ -3 & 3 & 2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{7}{8} & \frac{1}{8} & \frac{-1}{4} & 0 \\ 0 & 0 & 1 & \frac{-3}{11} & \frac{-18}{11} & \frac{-8}{11} \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{11} & \frac{1}{11} & \frac{-2}{11} \\ 0 & 1 & 0 & \frac{4}{11} & \frac{13}{11} & \frac{7}{11} \\ 0 & 0 & 1 & \frac{-3}{11} & \frac{-18}{11} & \frac{-8}{11} \end{array}\right),$$

so that the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{11} \begin{pmatrix} 2 & 1 & -2 \\ 4 & 13 & 7 \\ -3 & -18 & -8 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ -7 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}.$$