24.1 Let $|V\rangle = (3-4i)\,|1\rangle + (5-6i)\,|2\rangle$ and $|W\rangle = (1-i)\,|1\rangle + (2-3i)\,|2\rangle$ where $|1\rangle$ and $|2\rangle$ form an orthonormal basis. Find $\langle V|V\rangle$, $\langle W|W\rangle$, and $\langle V|W\rangle$.

We have

$$\begin{aligned}
\langle V|V\rangle &= (3+4i)(3-4i)\,\langle 1|1\rangle + (5+6i)(5-6i)\,\langle 2|2\rangle \\
&= 25+61 = \boxed{86}, \\
\langle W|W\rangle &= (1+i)(1-i)\,\langle 1|1\rangle + (2+3i)(2-3i)\,\langle 2|2\rangle \\
&= 2+13 = \boxed{15}, \\
\langle V|W\rangle &= (3+4i)(1-i)\,\langle 1|1\rangle + (5+6i)(2-3i)\,\langle 2|2\rangle \\
&= 7+i+28+-3i = \boxed{35-2i}.
\end{aligned}$$

24.2 (a) Show that $|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$ and $|2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$ form an orthonormal basis.

Proof. Directly computing gives

$$\begin{aligned}
\langle 1|1\rangle &= \frac{1}{2}[1\cdot 1 + (-i)\cdot i] = 1 \\
\langle 2|2\rangle &= \frac{1}{2}[1\cdot 1 + i\cdot(-i)] = 1 \\
\langle 1|2\rangle &= \frac{1}{2}[1\cdot 1 + (-i)\cdot(-i)] = 0,
\end{aligned}$$

which implies that the vectors given form an orthonormal basis. □

(b) Expand $|V\rangle = \begin{pmatrix}1+i\\ \sqrt{3}+i\end{pmatrix}$ in this basis. What is $\langle V|V\rangle$ in this basis?

Using the inner product,

$$\begin{aligned}
\langle 1|V\rangle &= \frac{1}{\sqrt{2}}[(1+i)(1) + (\sqrt{3}+i)(-i)] = \frac{1}{\sqrt{2}}(2+i(1-\sqrt{3})) \\
\langle 2|V\rangle &= \frac{1}{\sqrt{2}}[(1+i)(1) + (\sqrt{3}+i)(i)] = \frac{1}{\sqrt{2}}i(1+\sqrt{3}),
\end{aligned}$$

which implies

$$|V\rangle = \frac{1}{\sqrt{2}}(2+i(1-\sqrt{3}))\,|1\rangle + \frac{1}{\sqrt{2}}i(1+\sqrt{3})\,|2\rangle\,.$$

Then

$$\begin{aligned}
\langle V|V\rangle &= \frac{1}{2}(2+i(1-\sqrt{3}))(2-i(1-\sqrt{3}))\,\langle 1|1\rangle - \frac{1}{2}i(1+\sqrt{3})i(1+\sqrt{3})\,\langle 2|2\rangle \\
&= \frac{1}{2}[4+(\sqrt{3}-1)^2+(\sqrt{3}+1)^2] = 6,
\end{aligned}$$

which matches with the computation in (6).