Section 2.3 Problems 7, 9, 17, 18, 34 and Exercises from the Week 3 Supplement, Episode II

7. $\frac{\mathrm{d}y}{\mathrm{d}x} - y - e^{3x} = 0$ Solve.

No equilibria. The corresponding homogeneous differential equation is $\frac{\mathrm{d}y}{\mathrm{d}x} = y$, whose general solution is $y = \exp\left(\int_0^x 1\,\mathrm{d}t\right) = \exp(x)$. Then the solution to the above inhomogeneous differential equation is some $y = c(x)\exp(x)$. Then:

$$(c(x)\exp(x))' = (c(x)\exp(x)) + e^{3x} \to c'(x)\exp(x) + c(x)\exp(x) = c(x)\exp(x) + e^{3x} \to c'(x) = \frac{e^{3x}}{e^x}$$

Hence $c(x) = \frac{e^{2x}}{2}$ and so the solution to the above inhomogeneous differential equation is $y = c(x)\exp(x) = \frac{e^{3x}}{2}$.

9. $\frac{\mathrm{d}r}{\mathrm{d}\theta} + r\tan(\theta) = \sec(\theta)$ Solve.

No equilibria. The corresponding homogeneous differential equation is $\frac{\mathrm{d}r}{\mathrm{d}\theta} = -r\tan(\theta)$, whose general solution is $r = \exp\left(\int_{\frac{\pi}{2}}^{\theta} \frac{-\sin(t)}{\cos(t)}\,\mathrm{d}t\right) = \cos(\theta)$. Then the solution to the above inhomogeneous differential equation is some $r = c(\theta)\cos(\theta)$. Then:

$$(c(\theta)\cos(\theta))' = -\cos(\theta)\tan(\theta) + \sec(\theta) \to c'(\theta)\cos(\theta) - c(\theta)\sin(\theta) = -c(\theta)\sin(\theta) + \sec(\theta) \to c'(\theta) = \sec^2(\theta)$$

Hence $c(\theta) = \tan\theta$ and so the solution to the above inhomogeneous differential equation is $y = c(\theta)\cos(\theta) = \sin(\theta)$.

17. $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = xe^x$, $y(1) = e - 1$ Solve.

Multiply both sides of the differential equation by the generic integrating factor $\mu(x) = \exp\left(\int -x^{-1}\,\mathrm{d}x\right) = x^{-1}$ and simplify:

$$\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x^2} = e^x \to \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) = e^x \to \frac{y}{x} = \int 1\,\mathrm{d}\left(\frac{y}{x}\right) = \int e^x\,\mathrm{d}x \to y = xe^x + Cx$$

Then use the initial data to find C:

$$e - 1 = e + C \to C = -1$$

Hence the solution curve is $y = xe^x - x$.

18. $\frac{\mathrm{d}y}{\mathrm{d}x} + 4y - e^{-x} = 0$ Solve.

Multiply both sides of the differential equation by the generic integrating factor $\mu(x) = \exp\left(\int 4\,\mathrm{d}x\right) = e^{4x}$ and simplify:

$$e^{4x}\frac{\mathrm{d}y}{\mathrm{d}x} + 4e^{4x}y = e^{3x} \to \frac{\mathrm{d}}{\mathrm{d}x}\left(ye^{4x}\right) = e^{3x} \to \int 1\,\mathrm{d}\left(ye^{4x}\right) = \int e^{3x}\,\mathrm{d}x \to ye^{4x} = \frac{e^{3x}}{3} + C \to y = \frac{1}{3e^x} + Ce^{-4x}$$

Then use the initial data to find C:

$$\frac{4}{3} = \frac{1}{3} + C \to C = 1$$

Hence the solution curve is $y = \frac{1}{3e^x} + e^{-4x}$.

34.

(a) The integral of a continuous function is continuous, and compositions of continuous functions are also continuous. Hence the integrating factor, $\mu(x)$, is continuous on the same interval (a, b). Exponentiation produces positive numbers for real arguments.

(b) The derivative of y(x) as it is in Equation (8) is:

$$\frac{\mathrm{d}}{\mathrm{d}x}y(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\mu(x)}\left[\int \mu(x)Q(x)\,\mathrm{d}x + C\right] \to \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-P(x)}{\mu(x)}\left[\int \mu(x)Q(x)\,\mathrm{d}x + C\right] + \frac{1}{\mu(x)}\left[\mu(x)Q(x)\right]$$

Substitute back Equation (8):

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -P(x)y(x) + Q(x)$$

The above is just Equation (4) with the $P(x)y(x)$ term moved over to the right hand side.

(c) We will produce something of this form:

$$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)\,\mathrm{d}x + y_0\mu(x_0)\right]$$

Then at $x = x_0$ we have what we want:

$$y(x_0) = \frac{1}{\mu(x_0)}\left[0 + y_0\mu(x_0)\right] \to y(x_0) = y_0$$

(d) Since we're using nice functions $P(x)$ and $Q(x)$, the integrating factor is also nice (as outlined in (a)). Then we know by working backwards from the general solution in (b), we can show equation (8) is indeed a solution. Finally in part (c) we showed that with a suitable value for C, we can show that the modified solution curve satisfies the initial data. This means that the choice of C changes if the curve passes through the initial data point, and likewise this dependence should mean that the initial data informs our choice of C, that is unique.

Exercise 1 from the Week 3 Supplement:

a) Around $(0, 1)$, $\frac{\partial v}{\partial x} = \frac{2}{3x^{\frac{1}{3}}}$ (and the t partial derivative is 0) exists in any rectangle created that does not include $x = 0$. Thus, in some neighborhood around that point there is a unique solution curve that exists.

b) The solution curve cannot be guaranteed to exist nor be unique because when $x = 0$, $\frac{\partial v}{\partial x}$ does not exist.

c) Around $(1, 1)$, $\frac{\partial v}{\partial x} = t^{\frac{1}{2}}$ and $\frac{\partial v}{\partial t} = \frac{x}{2\sqrt{t}}$, which both exist in any rectangle created that does not include $t = 0$. Thus, in some neighborhood around that point there is a unique solution curve that exists.

d) The solution curve cannot be guaranteed to exist nor be unique because when $t = 0$, $\frac{\partial v}{\partial t}$ does not exist.

e) The partial derivative $\frac{\partial v}{\partial t}$ is continuous for all (t, x), while $\frac{\partial v}{\partial x}$ is not continuous at $x = 0$ which is the initial value given. Hence we have no guarantee that a solution curve is unique there or if it exists.

f) Within some rectangle around the point $(0,1)$ that does not contain $x=0$, the partial derivaties are continuous and hence a unique solution curve passes through that point in some neighborhood around it.

Exercise 2 from the Week 3 Supplement:

In light of Proposition 2.2, the blow-up equation satisfying the initial data $(t_0, x_0) = (0,2)$ will have a unique solution in the interval $t \in [t_0 - \gamma, t_0 + \delta]$. We found δ in the Week 3 Supplement, and using a similar method we can find γ. Let us choose the interval $x \in [a, 3]$ for example. Using Proposition 2.2, we have that γ can be found with the integral:

$$\gamma = \int_a^{x_0} \frac{1}{\xi^2} \, \mathrm{d}\xi$$

$$\gamma = \left. \frac{-1}{\xi} \right|_a^2 = -\frac{1}{2} - \frac{1}{a}$$

As a tends to 0 it is evident that γ tends to $-\infty$ and hence the unique solution curve can be continued backwards in time indefinitely. Combining the result from the Week 3 Supplement, and our observation here, we now know that the working domain of the solution curve passing through $(0,2)$ is $t \in \left(-\infty, \frac{1}{2}\right)$.