1.5: 13 (b) , and 1.6: 22

1.5 13 (b). Let V be a vector space over a field of characteristic not equal to two.

Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be distinct vectors in V. Prove that $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent if and only if $\{\vec{u}+\vec{v}, \vec{u}+\vec{w}, \vec{v}+\vec{w}\}$ is linearly independent.

Proof. Suppose $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent. Then by way of contradiction suppose that $\{\vec{u}+\vec{v}, \vec{u}+\vec{w}, \vec{v}+\vec{w}\}$ is linearly dependent, so that there exist scalars $a, b, c \in \mathbb{F}$, not all zero, such that $a(\vec{u}+\vec{v})+b(\vec{u}+\vec{w})+c(\vec{v}+\vec{w}) = 0$. Then by some algebra $(a+b)\,\vec{u}+(a+c)\,\vec{v}+(b+c)\,\vec{w} = 0$. At least one of $(a+b), (a+c), (b+c)$ is nonzero (since one of a, b, c is nonzero). This is in contradiction with the assumption that $\{\vec{u}, \vec{v}, \vec{w}\}$ was linearly independent. Hence $\{\vec{u}+\vec{v}, \vec{u}+\vec{w}, \vec{v}+\vec{w}\}$ is linearly independent.

For the converse, use the contrapositive. Suppose $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent. Then there exist scalars $a, b, c \in \mathbb{F}$ not all zero such that $a\vec{u}+b\vec{v}+c\vec{w} = 2a\vec{u}+2b\vec{v}+2c\vec{w} = 0$. Notice that

$$
\begin{aligned}
2\vec{u} &= (\vec{u}+\vec{v}) - (\vec{v}+\vec{w}) + (\vec{u}+\vec{w}) \\
2\vec{v} &= (\vec{u}+\vec{v}) + (\vec{v}+\vec{w}) - (\vec{u}+\vec{w}) \\
2\vec{w} &= -(\vec{u}+\vec{v}) + (\vec{v}+\vec{w}) + (\vec{u}+\vec{w}),
\end{aligned}
$$

so that we also have

$$
\begin{aligned}
2a\vec{u}+2b\vec{v}+2c\vec{w} &= a\left((\vec{u}+\vec{v}) - (\vec{v}+\vec{w}) + (\vec{u}+\vec{w})\right) \\
&\qquad + b\left((\vec{u}+\vec{v}) + (\vec{v}+\vec{w}) - (\vec{u}+\vec{w})\right) + c\left(-(\vec{u}+\vec{v}) + (\vec{v}+\vec{w}) + (\vec{u}+\vec{w})\right) \\
&= (a+b-c)(\vec{u}+\vec{v}) + (-a+b+c)(\vec{v}+\vec{w}) + (a-b+c)(\vec{u}+\vec{w}) \\
&= 0
\end{aligned}
$$

Revision: At least one of $(a+b-c), (-a+b+c), (a-b+c)$ is not zero (since at least one of a, b, c is not zero). Suppose by way of contradiction that each quantity $(a+b-c), (-a+b+c), (a-b+c)$ was equal to zero, which forms the following system of equations:

$$
\begin{aligned}
a+b-c &= 0 & a+0b+0c &= 0 \\
-a+b+c &= 0 \rightarrow & 0a+\ \ b+0c &= 0 \\
a-b+c &= 0 & 0a+0b+\ \ c &= 0
\end{aligned}
$$

This is a contradiction with the fact that at least one of a, b, c is not zero. Deduce then that $\{\vec{u}+\vec{v}, \vec{u}+\vec{w}, \vec{v}+\vec{w}\}$ is linearly dependent.

Thus $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent if and only if $\{\vec{u}+\vec{v}, \vec{u}+\vec{w}, \vec{v}+\vec{w}\}$ is linearly independent. $\square$

1.6 22. Let W_1 and W_2 be subspaces of a finite-dimensional vector space V. Determine necessary and sufficient conditions on W_1 and W_2 so that $\dim(\mathsf{W}_1 \cap \mathsf{W}_2) = \dim(\mathsf{W}_1)$.

We have $\dim(\mathsf{W}_1 \cap \mathsf{W}_2) = \dim(\mathsf{W}_1)$ if and only if W_1 is a subspace of W_2.

Proof. Forwards direction. Suppose $\dim(\mathsf{W}_1 \cap \mathsf{W}_2) = \dim(\mathsf{W}_1)$. Then assume by way of contradiction that W_1 is not a subspace of W_2.

Then we can find some vector $\vec{v}$ in W_1 that is not in W_2. Let $\alpha = \{a_1, a_2, \ldots, a_n\}$ be a basis for W_1 and $\beta = \{b_1, b_2, \ldots, b_m\}$ be a basis for W_2. Then we can also say that $\vec{v} \notin \operatorname{span}(\beta)$.

Then let γ be a basis for $\mathsf{W}_1 \cap \mathsf{W}_2$. It is also true that $\vec{v} \notin \operatorname{span}(\gamma)$, since $\vec{v} \notin \operatorname{span}(\beta)$ - thus $\gamma \cup \{\vec{v}\}$ is linearly independent, furthermore, it is a linearly independent subset of W_1.

We may deduce the following: $\dim(\mathsf{W}_1) \geq |\gamma \cup \{\vec{v}\}|$, but $|\gamma \cup \{\vec{v}\}| = |\gamma| + 1 = \dim(\mathsf{W}_1 \cap \mathsf{W}_2) + 1$. So $\dim(\mathsf{W}_1) \geq \dim(\mathsf{W}_1 \cap \mathsf{W}_2) + 1$ implies that $\dim(\mathsf{W}_1) \neq \dim(\mathsf{W}_1 \cap \mathsf{W}_2)$, which is in contradiction to the assumption that $\dim(\mathsf{W}_1 \cap \mathsf{W}_2) = \dim(\mathsf{W}_1)$. Thus we must have that W_1 is a subspace of W_2.

To prove the converse we may do so directly. Suppose W_1 is a subspace (and therefore a subset) of W_2. Then we may simplify $\dim(\mathsf{W}_1 \cap \mathsf{W}_2)$ into $\dim(\mathsf{W}_1)$ because $\mathsf{W}_1 \subseteq \mathsf{W}_2$. Therefore we automatically have that $\dim(\mathsf{W}_1 \cap \mathsf{W}_2) = \dim(\mathsf{W}_1)$.

Hence $\dim(\mathsf{W}_1 \cap \mathsf{W}_2) = \dim(\mathsf{W}_1)$ if and only if W_1 is a subspace of W_2. $\square$