11.1 The relativistic Lagrangian of a particle of mass m in an electromagnetic field is given by

$$\mathcal{L} = -mc^2\sqrt{1-v^2/c^2} + \frac{e}{c}\vec{A}\cdot\vec{v} - e\phi; \quad \vec{v} = \dot{\vec{r}}.$$

(a) Calculate $\partial\mathcal{L}/\partial\vec{r} = \vec{\nabla}\mathcal{L}$, keeping $\vec{v}$ constant. Then

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\vec{r}} = \vec{\nabla}\mathcal{L} = \frac{e}{c}\vec{\nabla}\left(\vec{A}\cdot\vec{v}\right) - e\vec{\nabla}\phi &= \frac{e}{c}\left((\vec{v}\cdot\vec{\nabla})\vec{A} + (\vec{A}\cdot\vec{\nabla})\vec{v} + \vec{A}\times(\vec{\nabla}\times\vec{v}) + \vec{v}\times(\vec{\nabla}\times\vec{A})\right) - e\vec{\nabla}\phi \\ &= \frac{e}{c}\left((\vec{v}\cdot\vec{\nabla})\vec{A} + \vec{v}\times(\vec{\nabla}\times\vec{A})\right) - e\vec{\nabla}\phi \\ &= \frac{e}{c}(\vec{v}\cdot\vec{\nabla})\vec{A} + \frac{e}{c}\vec{v}\times(\vec{\nabla}\times\vec{A}) - e\vec{\nabla}\phi,\end{aligned}$$

where the first term in the Lagrangian was constant with respect to position (so the spatial derivative vanished), and all spatial derivatives of $\vec{v}$ vanished.

(b) Define the generalized momentum $\vec{P} = \frac{\partial\mathcal{L}}{\partial\vec{v}}$ and show that $\vec{P}$ can be written as $\vec{P} = \vec{p} + \frac{e}{c}\vec{A}$ where $\vec{p} = \gamma m\vec{v}$ is the relativistic free particle momentum as obtained in HW 4.2.

We have

$$\begin{aligned}\vec{P} &= \frac{\partial}{\partial\vec{v}}\left(-mc^2\sqrt{1-(\vec{v}\cdot\vec{v})/c^2} + \frac{e}{c}\vec{A}\cdot\vec{v} - e\phi\right) \\ &= \frac{-mc^2\cdot -2\frac{\partial}{\partial\vec{v}}\vec{v}\cdot\vec{v}/c^2}{2\sqrt{1-(\vec{v}\cdot\vec{v})/c^2}} + \frac{e}{c}\vec{A} \\ &= \frac{m\vec{v}}{\sqrt{1-v^2/c^2}} + \frac{e}{c}\vec{A} = \gamma m\vec{v} + \frac{e}{c}\vec{A} = \vec{p} + \frac{e}{c}\vec{A}.\end{aligned}$$

(c) Use the previous result as well as equations (7.12) to show that the E-L equation gives

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = e\vec{E} + \frac{e}{c}(\vec{v}\times\vec{B}).$$

So by the Euler-Lagrange equations, we have $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}} = \frac{\partial\mathcal{L}}{\partial\vec{r}}$. But $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\vec{v}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{p} + \frac{e}{c}\vec{A}\right) = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} + \frac{e}{c}\frac{\mathrm{d}\vec{A}}{\mathrm{d}t}$. Then

$$\begin{aligned}\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \frac{\partial\mathcal{L}}{\partial\vec{r}} - \frac{e}{c}\frac{\partial\vec{A}}{\partial t} &= \frac{e}{c}(\vec{v}\cdot\vec{\nabla})\vec{A} + \frac{e}{c}\vec{v}\times(\vec{\nabla}\times\vec{A}) - e\vec{\nabla}\phi - \frac{e}{c}\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} \\ &= \frac{e}{c}(\vec{v}\cdot\vec{\nabla})\vec{A} + \frac{e}{c}\vec{v}\times\vec{B} + e\left(\vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) - \frac{e}{c}\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} \\ &= \frac{e}{c}\left(\left(\frac{\partial\vec{r}}{\partial t}\cdot\vec{\nabla}\right)\vec{A} + \frac{\partial\vec{A}}{\partial t}\right) + \frac{e}{c}\vec{v}\times\vec{B} + e\vec{E} - \frac{e}{c}\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} \\ &= \frac{e}{c}\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} + \frac{e}{c}\vec{v}\times\vec{B} + e\vec{E} - \frac{e}{c}\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} \\ &= \frac{e}{c}\vec{v}\times\vec{B} + e\vec{E},\end{aligned}$$

where we used the vector(?) chain rule.