1. Prove that every subgroup of the quaternion group Q_8 is normal.

Proof. In the quaternion group $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$, observe that the subgroups we investigate (due to Lagrange's theorem) are the trivial subgroup, $\{1, -1\}$, $\langle i\rangle$, $\langle j\rangle$, $\langle k\rangle$, and Q_8 itself. A subgroup is normal in Q_8 if for every element a in the subgroup, and for every element q in the quaternion group, qaq^{-1} is an element of the subgroup (so for a subgroup H, $qHq^{-1} \subseteq H$).

Clearly, the trivial subgroup and Q_8 itself are normal in Q_8. For the trivial group, we know that 1 commutes with every element in Q_8, so for any element $q \in Q_8$, $q1q^{-1} = 1qq^{-1} = 1 \cdot 1 = 1$. Then for the whole group, observe that for any elements a, b, $aba^{-1} \in Q_8$, since Q_8 is a group. Similarly, we saw earlier that $\{1, -1\}$ was the center of Q_8, so for the same reason as the trivial group (we also already know that the center of a group is a normal subgroup), this subgroup is also normal in G.

The remaining three subgroups, $\langle i\rangle$, $\langle j\rangle$, and $\langle k\rangle$, are symmetric in that they only differ by two elements each:

$$\langle i\rangle = \{1, -1, i, -i\} \quad (= \langle -i\rangle)$$
$$\langle j\rangle = \{1, -1, j, -j\} \quad (= \langle -j\rangle)$$
$$\langle k\rangle = \{1, -1, k, -k\} \quad (= \langle -k\rangle)$$

For each of these groups, again observe that because 1 and -1 are in the center of Q_8, for elements q in the quaternion group we have that $q(\pm 1)q^{-1} = (\pm 1)qq^{-1} = (\pm 1)$. Recall that multiplication of elements not in the center of Q_8 is anticommutative; that is, for $x \in Q_8 \setminus Z(Q_8), y \in Q_8$, $xy = y^{-1}x$. So let $a \in Q_8 \setminus Z(Q_8) = \{i, -i, j, -j, k, -k\}$, and see that for $\langle a\rangle = \{1, -1, a, -a\}$ and $q \in Q_8$, $qaq^{-1} = qqa = \pm a$, and $q(-a)q^{-1} = qq(-a) = \mp a$, since q^2 is 1 or -1 depending on the choice of q. Hence for every element r in the subgroup $\langle a\rangle$, $qrq^{-1} \in \langle a\rangle$, and because a took on the elements not in the center of Q_8, we have that $\langle i\rangle$, $\langle j\rangle$, and $\langle k\rangle$ are each normal in Q_8.

Alternatively, because these last three subgroups are of index 2, we already know that they are normal in Q_8 because subgroups of index 2 are normal subgroups (example from the textbook).

Hence all subgroups of the quaternion group are normal subgroups. $\square$

2. (DF3.2.6) Let $H \leq G$ and let $g \in G$. Prove that if the right coset Hg equals *some* left coset of H in G then it equals the left coset gH and g must be in $N_G(H)$.

Proof. Let H, G be groups with $H \leq G$ as given, and let $g \in G$. Then suppose that the right coset Hg is equal to some left coset of H in G; that is, there exists a representative $r \in G$ such that $Hg = rH$.

Then for each $h_2 \in H$ there exists $h_1 \in H$ such that $h_1 g = rh_2$. So for the case when $h_2 = 1$ (as $1 \in H$), there exists an h_1 such that $h_1 g = r1 = r$, so that $Hg = rH = h_1 gH$. Then by multiplying on the left by h_1^{-1}, we have that $h_1^{-1}Hg = gH$. But since $h_1 \in H$, we have that $h_1^{-1} \in H$, so that $h_1^{-1}H = H$. Thus $h_1 Hg = Hg = gH$.

Then by multiplying by g^{-1} on the right, we have that $gHg^{-1} = H$, which by definition implies that g is in $N_G(H)$. $\square$

3. (DF3.2.12) Let $H \leq G$. Prove that the map $x \mapsto x^{-1}$ sends each left coset of H in G onto a right coset of H and gives a bijection between the set of left cosets and the set of right cosets of H in G.

   Proof. Let $H \leq G$ as given. Then let $g \in G$ so that gH is an arbitrary left coset of H in G. Then $gH = \{gh \mid h \in H\}$, and then under the mapping $x \mapsto x^{-1}$, we find that every element gh of gH maps to $(gh)^{-1} = h^{-1}g^{-1}$. But because h is any element in H, and H is a group, it follows that the set $\left\{h^{-1}g^{-1} \mid h \in H\right\}$ is equal to Hg^{-1}. So under the mapping $x \mapsto x^{-1}$, left cosets of H in G are sent to right cosets of H in G. In a sense, there is an induced map between the set of left cosets and right cosets given by $gH \mapsto Hg^{-1}$.

   We should check if the induced mapping makes sense, or is well defined. If we have some left coset of H in G written in two ways, say for representatives $u, v \in G$ we have $uH = vH$, we should have that after mapping them to their corresponding right cosets, $Hu^{-1} = Hv^{-1}$. So from $uH = vH$, we have for every element $uh_1 \in uH$ that there exists a unique (since the cosets are equal) element $vh_2 \in vH$ such that $uh_1 = vh_2$. Then under the inversion map, we have that $h_1^{-1}u^{-1} = h_2^{-1}v^{-1}$. Then since $h_1^{-1}, h_2^{-1} \in H$ and h_1^{-1}, h_2^{-1} each can take on every element of H, we have that $Hu^{-1} = Hv^{-1}$. So in this sense we may use any element in a coset to represent the coset in our notation, and the induced mapping still makes sense.

   To see that this induced mapping is surjective, consider any right coset of H in G, say Hg. Then the preimage of this coset under the mapping is the left coset $g^{-1}H$, since under the induced map, $g^{-1}H \mapsto H(g^{-1})^{-1} = Hg$.

   Then to show that this mapping is injective, we show that for two distinct left cosets, they are mapped into two distinct right cosets. So let $u, v \in G$, and let $uH \neq vH$. Then for some element $uh_1 \in uH$, this element is not equal to *any* element in vH, so $uh_1 \neq vh$ for every $h \in H$. Then by inversion, $h_1^{-1}u^{-1} \neq h^{-1}v^{-1}$, and since h takes on every element of H, h^{-1} will take on every element of H as well. Then this means in the right cosets Hu^{-1}, Hv^{-1}, there is an element $h_1^{-1}u^{-1} \in Hu^{-1}$ which is not equal to any element in Hv^{-1}. It follows that $Hu^{-1} \neq Hv^{-1}$. Since these right cosets are the images of the distinct left cosets uH, vH under the induced mapping, we have that this mapping is injective.

   Thus the induced mapping is bijective, and so there is a bijection between the set of left cosets of H in G and the set of right cosets of H in G. $\square$

4. (DF3.3.9) Let p be a prime and let G be a group of order $p^a m$ where p does not divide m. Assume P is a subgroup of G of order p^a and N is a normal subgroup of G of order $p^b n$, where p does not divide n. Prove that $|P \cap N| = p^b$ and $|PN/N| = p^{a-b}$.

   Proof. Let G, $P \leq G$, and $N \trianglelefteq G$ be given, with $|G| = p^a m$ with $p \nmid m$, $|P| = p^a$, $|N| = p^b n$ with $p \nmid n$. Then because $P \leq G = N_G(N)$ (N is a normal subgroup of G), we have that PN is a subgroup of G (from the textbook). So by Lagrange's theorem, $|PN|$ divides $|G| = p^a m$.

We can factorize $|PN|$ into $p^r s$, with $p \nmid s$ for integers r, s. But $P \leq PN$, since $1 \in N$, so again we have by Lagrange's theorem that $|P| = p^a \mid p^r s$, so $r \geq a$. Then since $p^r \mid p^r s$ and $p^r s \mid p^a m$ imply that $p^r \mid p^a m$, and p is coprime to m, we have that $p^r \mid p^a$ and so $r \leq a$. Hence $r = a$.

Then since $N \trianglelefteq PN$, we have that $|N| = p^b n \mid p^a s = |PN|$. From $p^b \mid p^b n$ and $p^b n \mid p^a s$, we have that $n \mid p^a s$ and since p and n are coprime, we find that $n \mid s$. Then because P and N are finite subgroups of G we may use the formula

$$|PN| = p^a s = \frac{p^a \cdot p^b n}{|P \cap N|} = \frac{|P||N|}{|P \cap N|} \implies |P \cap N| = \frac{p^b n}{s}$$

to deduce that $s \mid n$, because p and s are coprime and that the order of $|P \cap N|$ is an integer. Since $n \mid s$ and $s \mid n$, we have that $s = n$. Then $|PN| = p^a n$, and using the same formula above, we arrive at $|P \cap N| = p^b$.

Then we invoke the Second Isomorphism Theorem to find that

$$\frac{PN}{N} \cong \frac{P}{P \cap N}$$

because $P \leq G = N_G(N)$ and $P, N \leq G$, with $N \trianglelefteq PN$ and $P \cap N \trianglelefteq P$. Then

$$\left|\frac{PN}{N}\right| = \left|\frac{P}{P \cap N}\right| = \frac{|P|}{|P \cap N|} = \frac{p^a}{p^b} = p^{a-b}.$$

Furthermore, since $N \trianglelefteq PN$, we have that $|N| = p^b n \mid p^a n = |PN|$, and since p is coprime to n, $p^b \mid p^a$ and so $b \leq a$, which again confirms that the order of PN/N is finite.

Therefore $|P \cap N| = p^b$ and $|PN/N| = p^{a-b}$. $\square$