HOMEWORK 8

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Given $\emptyset \neq A \subseteq \mathbb{R}$, define $f : \mathbb{R} \to [0, \infty)$ by $f(x) = \inf(S_x)$, where

$$S_x = \{|x - a| : a \in A\}.$$

(i) Show f is continuous;
(ii) Show $\{x \in \mathbb{R} : f(x) = 0\} = \overline{A}$.

Proof. (i) Let A be a nonempty subset of $\mathbb{R}$ and define $f \colon \mathbb{R} \to [0, \infty)$ by $f(x) = \inf(S_x)$, where

$$S_x = \{|x - a| \colon a \in A\}$$

as given. Then observe that for any $a' \in A$, because $f(x) = \inf(S_x)$, we have that $f(x) \leq |x - a'|$.

Then for any $\eta > 0$, we can select an $a \in A$ such that $f(x) \leq |x - a| < f(x) + \eta$. If instead we suppose by way of contradiction that there exists an $\eta' > 0$ so that there does not exist an $a \in A$ that satisfies $f(x) \leq |x - a| < f(x) + \eta'$, then we arrive at a contradiction with the assumption that $f(x)$ was the infimum of S_x. In this scenario either $f(x) > |x - b|$ for some $b \in A$, or $f(x) + \eta'$ is a lower bound for S_x which is greater than $f(x)$, both of which are impossible. Therefore such an $a \in A$ exists.

For some given $\eta > 0$, let $|y - x| < \eta$ and choose $a \in A$ such that $f(x) \leq |x - a| < f(x) + \eta$. It is also true that $f(y) \leq |y - a|$ holds, since $f(y) \leq |y - a'|$ for any $a' \in A$. Note that x is a limit point of R. Without loss of generality, let $f(y) \geq f(x)$ (if $f(y) \leq f(x)$, then interchange the positions of x and y).

Then

$$\begin{aligned} f(y) - f(x) = f(y) - (f(x) + \eta) + \eta &< |y - a| - |x - a| + \eta \\ &= |(y - x) + (x - a)| - |x - a| + \eta \\ &\leq |y - x| + |x - a| - |x - a| + \eta \\ &< \eta + \eta = 2\eta, \end{aligned}$$

and by taking the absolute value on both sides, we find that $|f(y) - f(x)| < 2\eta$ whenever $|y - x| < \eta$. Hence f is continuous on $\mathbb{R}$.

(ii) We will show that $\overline{A} \subseteq \{x \in \mathbb{R} : f(x) = 0\}$, and that for $x \notin \overline{A}$, that $f(x) \neq 0$ so that $x \notin \{x \in \mathbb{R} : f(x) = 0\}$, so by the contrapositive, that $\{x \in \mathbb{R} : f(x) = 0\} \subseteq \overline{A}$.

Observe that whenever $x \in A$, we automatically have that $f(x) = 0$ since S_x will contain an element of the form $|x - x| = 0$, and since S_x contains only nonnegative real numbers, it follows that $f(x) = \inf(S_x) = 0$. This means that $A \subseteq \{x \in \mathbb{R} \colon f(x) = 0\}$.

Now consider the case where $x \in A'$. For every $\varepsilon > 0$, the ε-neighborhood of x contains some $a_\varepsilon \in A$ such that $|x - a_\varepsilon| < \varepsilon$. But for each ε, we have that $|x - a_\varepsilon| \in S_x$. Because $\varepsilon > 0$ can be made arbitrarily small, the only value $f(x) = \inf(S_x)$ may take on is 0.

Hence $\overline{A} = A \cup A' \subseteq \{x \in \mathbb{R} : f(x) = 0\}$. Then suppose that $x \notin \overline{A}$; that is, $x \in (\overline{A})^c$. Then because $\overline{A}$ is a closed set, $(\overline{A})^c$ is an open set. Thus there exists $\varepsilon > 0$ such that the ε-neighborhood of x does not contain any point of $\overline{A}$, so that the quantity $|x - a|$ for every $a \in A$ is strictly greater than ε. Thus $f(x) = \inf(S_x) > \varepsilon > 0$, which means that $x \notin \{x \in \mathbb{R} : f(x) = 0\}$. Thus by the contrapositive, it follows that if $x \in \{x \in \mathbb{R} : f(x) = 0\}$, then $x \in \overline{A}$, and so $\{x \in \mathbb{R} : f(x) = 0\} \subseteq \overline{A}$.

Hence $\{x \in \mathbb{R} : f(x) = 0\} = \overline{A}$. $\square$