11. Let m, n be integers, and set $d = \gcd(m, n)$. Prove that there are integers x, y such that $mx + ny = d$.

Proof. It suffices to prove the theorem for m, n being nonnegative integers since we can pass any negative signs into the coefficients x, y (or pass them back into $|m|, |n|$); that is, if m is negative and we have coefficients x, y that satisfy $mx + ny = d$, then we also have that $|m|(-x) + ny = d$, so that $d = \gcd(m, n) = \gcd(|m|, n)$. We can do the same if n was negative, or if both m and n were negative.

Let m, n be nonnegative integers and $d = \gcd(m, n)$ as given, where without loss of generality, let $n \geq m$. By strong induction on $h \geq 0$, suppose that for nonnegative integers a, b where $a + b < h$ there exist integers s, t such that $d = \gcd(a, b) = as + bt$. When $h = n + m = 0$, then both m, n are zero and so x, y are both zero as well (where the greatest common of 0 and 0 is 0 because the divisor should be less than or equal to 0 in the nonnegative integers). If m (the lesser of the two) is 0 and n is nonzero (so that $h = n$), then the greatest common divisor is n, and we find that $n = n(1) + m(0)$. Then let $m \geq 1$, with $h = n + m$.

Consider $\gcd(m, n - m)$. Observe that $m + (n - m) = n = h - m < h$, since $m \geq 1$. Then by the inductive hypothesis, there exist integers a, b such that $\gcd(m, n - m) = ma + (n - m)b = m(a - b) + nb$. This quantity is actually the greatest common divisor of m and n. Any common divisor of m and n will divide the quantity $m(a - b) + nb$ since this is a linear combination of m, n. Hence there are integers $x = (a - b), y = b$ such that $d = \gcd(m, n) = mx + ny$.

Therefore, by mathematical induction, for any integers m, n, there exist integers x, y such that $d = \gcd(m, n) = mx + ny$. $\square$

14. Let x be a real number such that $x + x^{-1}$ is an integer. Prove that $x^n + x^{-n}$ is an integer for all positive integers n.

Proof. Let $x \in \mathbb{R}$ be given so that $x + x^{-1} \in \mathbb{Z}$ as given. Then by strong induction on n, suppose that for $1 \leq k < n$, $x^k + x^{-k} \in \mathbb{Z}$.

Since $x + x^{-1} \in \mathbb{Z}$, we have that $(x + x^{-1})^n \in \mathbb{Z}$. Then by binomial expansion,

$$(x + x^{-1})^n = x^n + x^{-n} + \sum_{k=1}^{n-1} \binom{n}{k} x^k (x^{-1})^{n-k},$$

and we can further simplify this using the symmetry of binomial coefficients, where $\binom{n}{k} = \binom{n}{n-k}$. However, we must handle an "extra" constant term that forms when n is even (where $k = n - k$ for some k). So in the first case when n is even, write $n = 2a$ for some positive integer a. Then substitute and simplify using the symmetry of binomial coefficients to find

$$x^n + x^{-n} + \sum_{k=1}^{2a-1} \binom{2a}{k} x^k (x^{-1})^{2a-k} = x^n + x^{-n} + \binom{2a}{a} + \sum_{k=1}^{a-1} \binom{2a}{k} \left[x^{2(a-k)} + (x^{-1})^{2(a-k)} \right],$$

and observe that $\binom{2a}{a}$ is an integer. By the inductive hypothesis, all of the terms $\binom{2a}{k}\left[x^{2(a-k)}+(x^{-1})^{2(a-k)}\right]$ for $1 \leq k \leq a-1$ are integers as well since $2(a-k) < n$. Then it follows that $x^n + x^{-n} = (x+x^{-1})^n - \binom{2a}{a} - \sum_{k=1}^{a-1}\binom{2a}{k}\left[x^{2(a-k)}+(x^{-1})^{2(a-k)}\right]$ is an integer since a sum of integers is an integer.

Similarly, when n is odd, write $n = 2b+1$ for some positive integer b, and we have that

$$x^n + x^{-n} + \sum_{k=1}^{2b}\binom{2b+1}{k}x^k(x^{-1})^{2b+1-k} = x^n + x^{-n} + \sum_{k=1}^{b}\binom{2b+1}{k}\left[x^{2(b-k)+1}+(x^{-1})^{2(b-k)+1}\right],$$

where because $1 \leq k \leq b$, $2(b-k)+1 < n$, all of the terms $\binom{2b+1}{k}\left[x^{2(b-k)+1}+(x^{-1})^{2(b-k)+1}\right]$ are integers. Again, we have that $x^n + x^{-n} = (x+x^{-1})^n - \sum_{k=1}^{b}\binom{2b+1}{k}\left[x^{2(b-k)+1}+(x^{-1})^{2(b-k)+1}\right]$ is an integer.

Hence in both cases $x^n + x^{-n}$ is an integer, and by mathematical induction, $x^n + x^{-n}$ is an integer for all positive integers n. $\square$