HW17:

The normal modes $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{(k+2k')/m}$ with the corresponding normalized eigenvectors are

$$|\omega_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad |\omega_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}.$$

(a) With $t = 0$, we can write

$$\begin{pmatrix}x_1(0)\\x_2(0)\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}|\omega_1\rangle + \frac{1}{\sqrt{2}}|\omega_2\rangle$$

so that $|x(t)\rangle = \frac{1}{\sqrt{2}}[|\omega_1\rangle\cos(\omega_1 t) + |\omega_2\rangle\cos(\omega_2 t)]$. Simplification yields

$$|x(t)\rangle = \frac{1}{2}\begin{pmatrix}\cos(\omega_1 t) + \cos(\omega_2 t)\\ \cos(\omega_1 t) - \cos(\omega_2 t)\end{pmatrix},$$

and of course taking $t \to 0$ we find that the first component tends to 1 and the second component tends to 0, which matches with $|x(0)\rangle$.

(b) The propogator $U(t)$ is found by taking $|\omega_1\rangle\langle\omega_1|\cos(\omega_1 t) + |\omega_2\rangle\langle\omega_2|\cos(\omega_2 t)$. Thus

$$\begin{aligned}U(t) &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\cos(\omega_1 t) + \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}\cos(\omega_2 t)\\ &= \frac{1}{2}\begin{pmatrix}\cos(\omega_1 t) + \cos(\omega_2 t) & \cos(\omega_1 t) - \cos(\omega_2 t)\\ \cos(\omega_1 t) - \cos(\omega_2 t) & \cos(\omega_1 t) + \cos(\omega_2 t)\end{pmatrix}.\end{aligned}$$

(c) Indeed $|x(t)\rangle = U(t)|x(0)\rangle$ (the left column is $|x(t)\rangle$):

$$U(t)|x(0)\rangle = \frac{1}{2}\begin{pmatrix}\cos(\omega_1 t) + \cos(\omega_2 t) & \cos(\omega_1 t) - \cos(\omega_2 t)\\ \cos(\omega_1 t) - \cos(\omega_2 t) & \cos(\omega_1 t) + \cos(\omega_2 t)\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\cos(\omega_1 t) + \cos(\omega_2 t)\\ \cos(\omega_1 t) - \cos(\omega_2 t)\end{pmatrix} = |x(t)\rangle.$$

HW18:

18.1 Prove that $\cos(\theta+\phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$ and $\sin(\theta+\phi) = \cos(\theta)\sin(\phi) + \sin(\theta)\cos(\phi)$.

Proof. Consider $\exp(i\theta)\exp(i\phi) = \exp(i(\theta+\phi))$, which has real part $\cos(\theta+\phi)$ and has imaginary part $\sin(\theta+\phi)$. Then

$$\begin{aligned}\exp(i\theta)\exp(i\phi) &= [\cos(\theta) + i\sin(\theta)][\cos(\phi) + i\sin(\phi)]\\ &= [\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)] + i[\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\phi)],\end{aligned}$$

but since complex numbers are equal if and only if their components are equal, we have that $\cos(\theta+\phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$ and $\sin(\theta+\phi) = \cos(\theta)\sin(\phi) + \sin(\theta)\cos(\phi)$. $\square$

18.2 (a) We have with algebra and trigonometry that

$$z = \frac{3+4i}{3-4i} = \frac{(3+4i)^2}{5} = \frac{1}{5}\left(\sqrt{5}\exp\left(i\arctan\left(\frac{4}{3}\right)\right)\right)^2 = \exp\left(2i\arctan\left(\frac{4}{3}\right)\right),$$

from which $z = \cos\bigl(2\arctan\bigl(\frac{4}{3}\bigr)\bigr) + i\sin\bigl(2\arctan\bigl(\frac{4}{3}\bigr)\bigr)$, and $z^* = \exp\bigl(-2i\arctan\bigl(\frac{4}{3}\bigr)\bigr) = \cos\bigl(2\arctan\bigl(\frac{4}{3}\bigr)\bigr) - i\sin\bigl(2\arctan\bigl(\frac{4}{3}\bigr)\bigr)$. Also $|z|$ we can extract as the coefficient of $\exp\bigl(2i\arctan\bigl(\frac{4}{3}\bigr)\bigr)$ since the exponential here we interpret as a rotated unit vector. Hence $|z| = 1$.

(b) We have $z_1 = 2\exp(i\pi/4) = 2(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = \sqrt{2}(1+i)$ and $z_2 = 6\exp(i\pi/3) = 6(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = 3(1+i\sqrt{3})$. Then $z_1 + z_2 = (\sqrt{2}+3) + i(\sqrt{2}+3\sqrt{3})$.

18.3 When $n \mapsto (n - i\alpha)$, we have

$$\exp[i\omega(t - (n-i\alpha))x/c] = \exp[i\omega(t - nx/c) - \omega\alpha x/c] = \exp(-\omega\alpha x/c)\exp[i\omega(t - nx/c)].$$

In effect the amplitude of the wave is reduced (when $\alpha > 0$; otherwise scaled up or unchanged).