Homework 30. Sai Sivakumar

30.1 The Laguerre equation has the form

$$xy'' + (1-x)y' + qy = 0$$

where $q$ is a parameter.

(a) Use an infinite series solution to first obtain the indicial equation, and then the ratio of two consecutive terms in the series.

Write the differential equation as (intend to expand around $x=0$) $\quad y'' + (x^{-1}-1)y' + qx^{-1}y = 0$

ansatz: $y = \sum_{k=0}^{\infty} a_k x^{k+b}$ with $a_0 \neq 0$ and $b$ to be found later.

Then
$$0 = y'' + (x^{-1}-1)y' + qx^{-1}y = \sum_{k=0}^{\infty} a_k (k+b)(k+b-1)\, x^{k+b-2} + \sum_{k=0}^{\infty} a_k (k+b)\, x^{k+b-2} + \sum_{k=0}^{\infty} -a_k (k+b)\, x^{k+b-1} + \sum_{k=0}^{\infty} q a_k x^{k+b-1}$$

$$= a_0 (b)(b-1)\, x^{b-2} + a_0 (b)\, x^{b-2} + \sum_{k=0}^{\infty} a_{k+1}(k+b+1)(k+b)\, x^{k+b-1} + \sum_{k=0}^{\infty} a_{k+1}(k+b+1)\, x^{k+b-1} + \sum_{k=0}^{\infty} -a_k (k+b)\, x^{k+b-1} + \sum_{k=0}^{\infty} q a_k x^{k+b-1}.$$

From this we have the indicial equation

$$a_0(b)(b-1) + a_0(b) = a_0(b^2) = 0 \quad \text{so that } b=0. \ (a_0 \neq 0)$$

and then

$$a_{k+1}(k+b+1)(k+b) + a_{k+1}(k+b+1) - a_k(k+b) + q a_k$$
$$= a_{k+1}(k+1)^2 - a_k(q-k) = 0$$

so that $a_{k+1} = \dfrac{a_k(q-k)}{(k+1)^2}$, then $\boxed{\dfrac{a_{k+1}}{a_k} = \dfrac{(q-k)}{(k+1)^2}}$.

for $k \geq 0$

(b) Find the choice of $q$ such that the series terminates, giving a polynomial solution:

Any nonnegative integer $q$ will cause the solution to be in the form of a polynomial since eventually, $k = q$ and so

$$k_{q+1} = \frac{k_q (q-q)}{(q+1)^2} = 0,$$

and so $a_k$ for $k > q$ will all be $0$. Thus the series is in the form of a polynomial of degree at most $q+1$.