Consider the matrix

$$A = \begin{pmatrix} \alpha & -\alpha & 0 \\ -\beta & 2\beta & -\beta \\ 0 & -\alpha & \alpha \end{pmatrix}$$

where α and β are real numbers.

(a) Find the eigenvalues.

The solutions to the secular equation

$$\det(A - \lambda I) = \det \begin{pmatrix} \alpha-\lambda & -\alpha & 0 \\ -\beta & 2\beta-\lambda & -\beta \\ 0 & -\alpha & \alpha-\lambda \end{pmatrix} = (\alpha-\lambda)\left[(2\beta-\lambda)(\alpha-\lambda) - \alpha\beta\right] - \alpha\beta(\alpha-\lambda)$$

$$= (\alpha-\lambda)\left[(2\beta-\lambda)(\alpha-\lambda) - 2\alpha\beta\right]$$

are $\lambda = \alpha, 0, \alpha + 2\beta$, which are the eigenvalues.

(b) Find the corresponding normalized eigenvectors. (By inspection; a guess is enough because eigenvectors can be scaled by nonzero coefficients before normalization. Furthermore, we have three eigenvalues which makes it impossible for any of the matrices below to have rank less than two.)

With $\lambda = \alpha$,

$$\begin{pmatrix} 0 & -\alpha & 0 \\ -\beta & 2\beta-\alpha & -\beta \\ 0 & -\alpha & 0 \end{pmatrix} \left| v^{(1)} \right\rangle = \vec{0} \implies \left| v^{(1)} \right\rangle = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

With $\lambda = 0$,

$$\begin{pmatrix} \alpha & -\alpha & 0 \\ -\beta & 2\beta & -\beta \\ 0 & -\alpha & \alpha \end{pmatrix} \left| v^{(2)} \right\rangle = \vec{0} \implies \left| v^{(2)} \right\rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

With $\lambda = \alpha + 2\beta$,

$$\begin{pmatrix} -2\beta & -\alpha & 0 \\ -\beta & -\alpha & -\beta \\ 0 & -\alpha & -2\beta \end{pmatrix} \left| v^{(3)} \right\rangle = \vec{0} \implies \left| v^{(3)} \right\rangle = \begin{pmatrix} 1 \\ \frac{-2\beta}{\alpha} \\ 1 \end{pmatrix}.$$

Normalizing we have

$$\left| v'^{(1)} \right\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \left| v'^{(2)} \right\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \left| v'^{(3)} \right\rangle = \frac{1}{\sqrt{2 + 4\frac{\beta^2}{\alpha^2}}} \begin{pmatrix} 1 \\ \frac{-2\beta}{\alpha} \\ 1 \end{pmatrix}.$$

(c) The matrix becomes symmetric if $\alpha = \beta$. Show that in this case the eigenvectors are orthogonal to each other.

With $\alpha = \beta$,

$$\left|v'^{(3)}\right\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

We compute three dot products which all evaluate to zero:

$$\begin{aligned}
\left\langle v'^{(1)} \middle| v'^{(2)} \right\rangle &= \frac{1}{\sqrt{6}}(1+0-1) = 0 \\
\left\langle v'^{(1)} \middle| v'^{(3)} \right\rangle &= \frac{1}{2\sqrt{3}}(1+0-1) = 0 \\
\left\langle v'^{(2)} \middle| v'^{(3)} \right\rangle &= \frac{1}{3\sqrt{2}}(1-2+1) = 0
\end{aligned}$$

Therefore the eigenvectors are orthogonal to each other.