Solution Manual

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34.6 Exercises

34.6.1

Since the solid region E is given by a rectangular prism, the bounds of the integral are easy to set up. The integral becomes

$$\int_0^2\int_1^2\int_0^1 (xy - 3z^2)dxdydz \to \int_0^2\int_1^2 \left(\frac{1}{2}y - 3z^2\right) dydz$$

$$\to \int_0^2 \left(\frac{3}{4} - 3z^2\right) dz = -\frac{13}{2}$$

34.6.4

The region is vertically simple (vertical meaning parallel to the $\vec{e}_3$ basis vector). We can find the bounds for z in the vertically simple manner quickly from the problem statement. So the following inequality holds: $0 \leq z \leq x + y + 1$.

The planar region in the xy plane that we wish to find bounds for in x and y is part of the interior side of a parabola. Note that since $x = 0$ and $y = 1$ are part of the bounds, it follows that the parabola given by $x = \sqrt{y}$ can only be traced out so long as $0 \leq y \leq 1$. It is also known that $0 \leq x \leq \sqrt{y}$. We may set up the triple integral as follows:

$$\int_0^1\int_0^{\sqrt{y}}\int_0^{x+y+1} (6xy)\, dzdxdy \to \int_0^1\int_0^{\sqrt{y}} (6x^2y + 6xy^2 + 6xy)dxdy$$

$$\to \int_0^1 \left(3y^2 + 2y^{\frac{5}{2}} + 3y^3\right) dy = \frac{65}{28}$$

34.6.7

We want to find out what the region of integration is. Imagine the positive portion of the cylinder $z = \sqrt{1 - y^2}$ is sliced by the planes $x = y$ and $x = 0$. we essentially have just a slice of the cylinder that lies in the first octant, much like the image in Figure **34.3 Right** (for Study Problem **34.1**).

So we know that for z we have $0 \leq z \leq \sqrt{1 - y^2}$. Then for the planar region we need to find bounds. We know that the cylinder intersects the line $x = y$

when $y = 1$. We can also deduce that y may only drop to 0 and nothing less. So bounds on y are $0 \leq y \leq 1$. As for x, we have a line $x = 0$ and another $x = y$ that give us bounds $0 \leq x \leq y$.

So the triple integral becomes

$$\int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} (zx)\, dzdxdy \to \frac{1}{2}\int_0^1 \int_0^y \left(x - xy^2\right) dxdy$$

$$\to \frac{1}{4}\int_0^1 \left(y^2 - y^4\right) dy = \frac{1}{30}$$

34.6.12

The plane $z = 1 - x$ intersects with the plane $z = 0$ when $x = 1$. This constitutes an upper bound for x, where y^2 is the lower bound. So $y^2 \leq x \leq 1$. Then we wish to find out how much y can vary. Notice that due to $x = 1 = y^2$, $-1 \leq y \leq 1$. It is also known from the start that $0 \leq z \leq 1 - x$. So we may set up the integral and evaluate it as follows:

$$\int_{-1}^1 \int_{y^2}^1 \int_0^{1-x} (1) dzdxdy \to \int_{-1}^1 \int_{y^2}^1 (1-x) dxdy \to \int_{-1}^1 \left(\frac{1}{2} - y^2 + \frac{y^4}{2}\right) dy$$

$$= \frac{8}{15}$$

34.6.13

To find the region of integration it is easiest to find it in the vertically simple way. The bounds in z are straightforward, they are $0 \leq z \leq 4 - y^2$. We may find the bounds for the x values by rearranging the equations for the lines given to find out the bounds. So $\frac{1}{2}y \leq x \leq y$. The parabolic sheet intersects the xy plane where $y = 2$ (omitting the negative root since we are in the first octant) and the lines $y = x$ and $y = 2x$ intersect at the origin so bounds on y are $0 \leq y \leq 2$.

We have the following triple integral:

$$\int_0^2 \int_{\frac{1}{2}y}^y \int_0^{4-y^2} dzdxdy \to \int_0^2 \int_{\frac{1}{2}y}^y (4-y^2) dxdy \to \frac{1}{2}\int_0^2 4y - y^3 dy = 2$$

34.6.16

34.6.19

34.6.22

The region E is symmetric axross the planes $z = 0$ and $x = 0$, and we may use this to our advantage. Notice that the integrand contains terms multiplied by

x or z^3, which is skew symmetric across those planes. Even when multiplied together, skew symmetry holds. So automatically the integral over the region E with the rectangular cavity not made vanishes.

Notice that by construction the integral we wish to find can be represented like so:

$$\iiint_E 24xy^2z^3 dA$$

$$= \iiint_{\text{E without cavity}} 24xy^2z^3 dA - \iiint_{[0,1]\times[-1,1]\times[0,1]} 24xy^2z^3 dA$$

$$= 0 - \iiint_{[0,1]\times[-1,1]\times[0,1]} 24xy^2z^3 dA$$

So really all we are tasked to do is to find out what the integral over the rectangular cavity (as a solid) would have been.

$$-\int_0^1 \int_{-1}^1 \int_0^1 (24xy^2z^3)dxdydz \to -\int_0^1 \int_{-1}^1 (12y^2z^3)dydz \to -\int_0^1 (8z^3)dz$$

$$= -2$$

34.6.21

The surface $z = 6 - x^2 - y^2$ is a paraboloid and the surface $z = \sqrt{x^2 + y^2}$ is a single cone opening upwards. The intersection of these surfaces happens where $z = 2$ (you may combine the equations for the surfaces into $z = 6 - z^2$ and solve for the positive root). At $z = 2$, the intersection is a circle of radius 2 centered at the origin. So the region of integration is given by $x^2 + y^2 \leq 4$ where $\sqrt{x^2 + y^2} \leq z \leq 6 - (x^2 + y^2)$.

We will opt to use polar coordinates to evaluate this integral. Knowing that $x^2 + y^2 \leq 4$, it is apparent that $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$. The triple integral becomes

$$\int_0^{2\pi} \int_0^2 \int_{\sqrt{x^2+y^2}}^{6-(x^2+y^2)} (1)dz(r)drd\theta \to \int_0^{2\pi} \int_0^2 (6r - r^3 - r^2)drd\theta \to \int_0^{2\pi} \frac{16}{3} d\theta$$

$$= \frac{32}{3}\pi$$

34.6.24

Rewite the integrand as $1 + \sin^2(xz) - \sin^2(xy)$. Apply linearity to find the two integrals given by

$$\iiint_E dV + \iiint_E (\sin^2(xz) - \sin^2(xy))dV$$

Notice that the integrand of the second integral is skew symmetric about the same plane $z = y$, because if we apply the transformation $(x, y, z) \to (x, z, y)$

the sign of the integrand is flipped. The region E itself is a region between two spheres, which is symmetric across the plane $z = y$, so we can conclude that the second integral will vanish.

We simply compute the first integral by geometry:

$$\iiint_E dV = \frac{4}{3}\pi(2^3 - 1^3) = \frac{28}{3}\pi$$

34.6.31

34.6.34