1. A particle of mass m moves under the influence of gravity on the inner surface of a frictionless paraboloid of revolution $x^2 + y^2 = az$.

   (a) In cylindrical coordinates the constraint is written as $r^2 - az = 0$. The coefficient for the Lagrange multipliers will be $(r^2 - az)$.

   (b) To find the kinetic energy of this mass we must differentiate position, but to do so we can take time derivatives of $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Then the kinetic energy is given by $\frac{1}{2}m\dot{z}^2 + \frac{1}{2}m(\dot{r}\cos(\theta) - r\sin(\theta)\dot{\theta})^2 + \frac{1}{2}m(\dot{r}\sin(\theta) + r\cos(\theta)\dot{\theta})^2 = \frac{1}{2}m\dot{z}^2 + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2$ (this is like using angular velocity and radial velocity and extracting kinetic energy from these coordinates) and the potential energy of this mass is given by mgz.

   (c) Then $\mathcal{L} = \frac{1}{2}m\dot{z}^2 + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgz + \lambda(r^2 - az)$, so

$$S = \int_{t_i}^{t_f} \left(\frac{1}{2}m\dot{z}^2 + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgz + \lambda(r^2 - az)\right) \mathrm{d}t$$

$$\vdots$$

$$\begin{aligned}
\implies \frac{\partial \mathcal{L}}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{r}} &= mr\dot{\theta}^2 + 2\lambda r - m\ddot{r} = 0 \\
\frac{\partial \mathcal{L}}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= -2mr\dot{r}\dot{\theta} - mr^2\ddot{\theta} = 0 \\
\frac{\partial \mathcal{L}}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{z}} &= -\lambda a - mg - m\ddot{z} = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\lambda}} &= r^2 - az = 0
\end{aligned}$$

2. The frictional force is given by f, and so the work done by this force is given by the negative of displacement times f. So say an object with mass m moves along the x axis, where its displacement is given by ϕ (the potential energy does not change so along this axis so we can say it is 0). Then to "retrofit" the Lagrangian, we take off from the kinetic energy the energy lost to friction at a given position, so

$$\mathcal{L}(\phi, \dot{\phi}; t) = \frac{1}{2}m\dot{\phi}^2 - f\phi.$$

   Then in taking derivatives, see that

$$\frac{\partial \mathcal{L}}{\partial \phi} = -f \text{ and } \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m\ddot{\phi},$$

   so the equation of motion matches with what we would expect from Newtonian mechanics,

$$m\ddot{\phi} = -f.$$

   This does not work if the mass changes direction.