Show that the formula in Green's theorem is invariant under coordinate changes, in the sense that if the theorem holds for a bounded domain U with piecewise smooth boundary, and if $F(x, y)$ is a smooth function that maps U one-to-one onto another such domain V and that maps the boundary of U one-to-one smoothly onto the boundary of V, then Green's theorem holds for V. *Hint.* First note the change of variable formulae for line and area integrals, given by

$$\int_{\partial V} P \,\mathrm{d}\xi = \int_{\partial U} (P \circ F) \left(\frac{\partial \xi}{\partial x} \,\mathrm{d}x + \frac{\partial \xi}{\partial y} \,\mathrm{d}y \right)$$
$$\iint_V R \,\mathrm{d}\xi \,\mathrm{d}\eta = \iint_U (R \circ F) \det J_F \,\mathrm{d}x \,\mathrm{d}y$$

where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is the transformation $(x, y) \to (\xi(x, y), \eta(x, y))$, and where J_F is the Jacobian matrix of F (take the absolute value of the determinant). Use these formulae, with $R = -\frac{\partial P}{\partial \eta}$. The summand $\int_{\partial V} Q \,\mathrm{d}\eta$ is treated similarly.

All the conditions are met for Green's theorem. Then generically the formula we know for Green's theorem is:

$$\int_{\partial D} P \,\mathrm{d}x + Q \,\mathrm{d}y = \iint_D \left(Q'_x - P'_y \right) \mathrm{d}x \,\mathrm{d}y$$

Start with the following change of variables (after which ξ and η are read as functions of (x, y)):

$$\begin{aligned}
\int_{\partial V} P \,\mathrm{d}\xi + Q \,\mathrm{d}\eta &= \int_{\partial U} (P \circ F) \left(\xi'_x \,\mathrm{d}x + \xi'_y \,\mathrm{d}y \right) + (Q \circ F) \left(\eta'_x \,\mathrm{d}x + \eta'_y \,\mathrm{d}y \right) \\
&= \int_{\partial U} \left((P \circ F)\, \xi'_x + (Q \circ F)\, \eta'_x \right) \mathrm{d}x + \left((P \circ F)\, \xi'_y + (Q \circ F)\, \eta'_y \right) \mathrm{d}y
\end{aligned}$$

Then apply Green's theorem:

$$\begin{aligned}
&= \iint_U \left[\frac{\mathrm{d}}{\mathrm{d}x} \left((P \circ F)\, \xi'_y + (Q \circ F)\, \eta'_y \right) - \frac{\mathrm{d}}{\mathrm{d}y} \left((P \circ F)\, \xi'_x + (Q \circ F)\, \eta'_x \right) \right] \mathrm{d}x \,\mathrm{d}y \\
&= \iint_U \left[\frac{\mathrm{d}\,(P \circ F)}{\mathrm{d}x} \xi'_y + (P \circ F)\, \xi''_{yx} + \frac{\mathrm{d}\,(Q \circ F)}{\mathrm{d}x} \eta'_y + (Q \circ F)\, \eta''_{yx} \right. \\
&\qquad \left. - \frac{\mathrm{d}\,(P \circ F)}{\mathrm{d}y} \xi'_x - (P \circ F)\, \xi''_{xy} - \frac{\mathrm{d}\,(Q \circ F)}{\mathrm{d}y} \eta'_x - (Q \circ F)\, \eta''_{xy} \right] \mathrm{d}x \,\mathrm{d}y
\end{aligned}$$

Cancel out the terms containing mixed second partial derivatives via Clairaut's theorem. Continue by using the chain rule, and for brevity, let $\mathcal{P} = P \circ F$ and $\mathcal{Q} = Q \circ F$:

$$\begin{aligned}
&= \iint_U \left[\frac{\mathrm{d}\,(P \circ F)}{\mathrm{d}x} \xi'_y + \frac{\mathrm{d}\,(Q \circ F)}{\mathrm{d}x} \eta'_y - \frac{\mathrm{d}\,(P \circ F)}{\mathrm{d}y} \xi'_x - \frac{\mathrm{d}\,(Q \circ F)}{\mathrm{d}y} \eta'_x \right] \mathrm{d}x \,\mathrm{d}y \\
&= \iint_U \left[\mathcal{P}'_\xi \xi'_x \xi'_y + \mathcal{P}'_\eta \eta'_x \xi'_y + \mathcal{Q}'_\xi \xi'_x \eta'_y + \mathcal{Q}'_\eta \eta'_x \eta'_y - \mathcal{P}'_\xi \xi'_y \xi'_x - \mathcal{P}'_\eta \eta'_y \xi'_x - \mathcal{Q}'_\xi \xi'_y \eta'_x - \mathcal{Q}'_\eta \eta'_y \eta'_x \right] \mathrm{d}x \,\mathrm{d}y \\
&= \iint_U \left[\mathcal{P}'_\eta \eta'_x \xi'_y + \mathcal{Q}'_\xi \xi'_x \eta'_y - \mathcal{P}'_\eta \eta'_y \xi'_x - \mathcal{Q}'_\xi \xi'_y \eta'_x \right] \mathrm{d}x \,\mathrm{d}y \\
&= \iint_U \left(\mathcal{Q}'_\xi - \mathcal{P}'_\eta \right) \det J_F \,\mathrm{d}x \,\mathrm{d}y
\end{aligned}$$

Carry out the change of variables for area integrals to find the result we want:

$$\iint_U \left(\mathcal{Q}'_\xi - \mathcal{P}'_\eta\right) \det J_F \,\mathrm{d}x\,\mathrm{d}y = \iint_V \left(Q'_\xi - P'_\eta\right) \mathrm{d}\xi\,\mathrm{d}\eta$$

Hence Green's theorem holds for V:

$$\int_{\partial V} P\,\mathrm{d}\xi + Q\,\mathrm{d}\eta = \iint_V \left(Q'_\xi - P'_\eta\right) \mathrm{d}\xi\,\mathrm{d}\eta$$