14.1 Show that

(a) Product of two orthogonal matrices is orthogonal.

*Proof.* Let $A, B$ be two orthogonal matrices; that is, $A^T A = I$ and $B^T B = B$. Then $(AB)^T(AB) = B^T A^T AB = B^T IB = B^T B = I$ and the same is true for $BA$ by symmetry. $\square$

(b) Trace of a matrix remains invariant under a similarity transformation.

*Proof.* Let $A, S$ be matrices and let $S$ be invertible. Then $\mathrm{Tr}(S^{-1}AS) = \mathrm{Tr}(S^{-1}(AS)) = \mathrm{Tr}((AS)S^{-1}) = \mathrm{Tr}(A(SS^{-1})) = \mathrm{Tr}(A)$. $\square$

(c) A Hermitian matrix remains Hermitian under unitary transformation.

*Proof.* Let $H$ be a Hermitian matrix and $U$ be a unitary matrix. Then $U^{-1}HU = U^\dagger HU$, so that $(U^\dagger HU)^\dagger = U^\dagger H^\dagger (U^\dagger)^\dagger = U^\dagger HU$. Hence $H$ under a unitary transformation is still Hermitian. $\square$

14.2 (a) Show that if $|v'\rangle = U|v\rangle$ where $|v\rangle$ is complex and $\langle v|v\rangle = \langle v'|v'\rangle$, then $U$ must be unitary.

*Proof.* Let $U$, $|v\rangle$ be as given, with $|v'\rangle = U|v\rangle$. Suppose that $\langle v|v\rangle = \langle v'|v'\rangle$. Then by definition,

$$\begin{aligned}\langle v|v\rangle = v_i^* v_i &= \langle v'|v'\rangle \\ &= (U_{ij}v_j)^*(U_{ik}v_k) \\ &= U_{ij}^* U_{ik} v_j v_k \\ &= U_{ji}^\dagger U_{ik} v_j v_k \\ &= (U^\dagger U)_{jk} v_j v_k.\end{aligned}$$

So by enforcing the equality we must have that $(U^\dagger U)_{jk} = \delta_{jk}$, that is $U^\dagger U = I$. Hence $U$ is unitary (its adjoint is its inverse). $\square$

(b) Two matrices $U$ and $H$ are related by $U = e^{i\alpha H}$ where $\alpha$ is real and $H$ is independent of $\alpha$. Show that if $H$ is Hermitian, then $U$ is unitary.

*Proof.* Let $U$, $H$ be given with $U = e^{i\alpha H}$ and $H$ independent of $\alpha$. For any positive integer power, $(H^n)^\dagger = (H^\dagger)^n$, as $H$ commutes with itself. Then with the definition of the exponential function, the fact that the adjoint is a linear transformation, and the above fact, we find that $U^\dagger = (\exp(i\alpha H))^\dagger = \exp(-i\alpha H^\dagger)$:

$$U^\dagger = (\exp(i\alpha H))^\dagger = \left(\sum_{n=0}^\infty \frac{(i\alpha H)^n}{n!}\right)^\dagger = \sum_{n=0}^\infty \frac{(-i\alpha H^\dagger)^n}{n!} = \exp\left(-i\alpha H^\dagger\right)$$

But because $H$ is Hermitian, $U^\dagger = \exp(-i\alpha H)$. Again using the fact that $H$ commutes with itself, $U^\dagger U = \exp(i\alpha H)\exp(-i\alpha H) = \exp(0H) = I$. Hence $U$ is unitary. $\square$

14.3 Consider the matrix $L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(a) Evaluate $L^2$.

$$L^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I.$$

(b) Show that $e^{\theta L}$ can be written as a two-dimensional rotation matrix.

Using the fact that $L^2 = -I$, it follows that

$$\begin{aligned} \exp(\theta L) = \sum_{n=0}^{\infty} \frac{\theta^n L^n}{n!} &= L \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} + I \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} \\ &= \begin{pmatrix} 0 & -\sin(\theta) \\ \sin(\theta) & 0 \end{pmatrix} + \begin{pmatrix} \cos(\theta) & 0 \\ 0 & \cos(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \end{aligned}$$

which is a rotation of vectors through $\theta$ degrees counterclockwise. (This is consistent with $e^{i\theta} \cdot z$ rotating $z$ through $\theta$ degrees counterclockwise in the complex plane; the algebra is the same due to an isomorphism where $i \mapsto L$.)