1. Say $A \subset X$ is dense if $\overline{A} = X$.

   (a) Show that A is dense in X if and only if every nonempty open subset V in X satisfies $V \cap A \neq \emptyset$.

   Proof. Suppose that every nonempty open set V in X satisfies $V \cap A \neq \emptyset$. Then for any $x \in X$, any neighborhood containing x intersects nontrivially with A so that $x \in \overline{A}$, and because x was arbitrary $X \subset \overline{A}$. It is clear that $\overline{A} \subset X$ (since X is closed) so that $\overline{A} = X$ as a result.

   Suppose that $\overline{A} = X$. Then any nonempty open set V in X contains at least one point $x \in X = \overline{A}$ so that necessarily V (an open neighborhood of x) must intersect nontrivially with A.

   Hence A is dense in X if and only if every nonempty open subset V in X satisfies $V \cap A \neq \emptyset$. □

   (b) Assume that X and Y are topological spaces with Y Hausdorff and A is dense in X. Suppose that $f\colon X \to Y$ and $g\colon X \to Y$ are continuous functions with $f(a) = g(a)$ for all $a \in A$. Prove that $f(x) = g(x)$ for all $x \in X$.

   Proof. Suppose by way of contradiction that there is an $x \in X$ such that $f(x) \neq g(x)$. Since Y is Hausdorff, choose neighborhoods U of $f(x)$ and V of $g(x)$ which intersect trivially. Then $x \in f^{-1}(U)$ and $x \in g^{-1}(V)$ with both $f^{-1}(U), g^{-1}(V)$ open in X since f, g are continuous. Because $\overline{A} = X$, the open set $f^{-1}(U) \cap g^{-1}(V)$ intersects nontrivially with A; that is, there exists $a \in A$ with $a \in f^{-1}(U) \cap g^{-1}(V)$. Then by hypothesis $f(a) = g(a)$, but $f(a) \in U$ and $f(a) = g(a) \in V$, which is a contradiction since U and V were chosen to be disjoint.

   Hence $f(x) = g(x)$ for all $x \in X$. □

2. A is a subset of the topological space X.

   (a) Show that $x \in \operatorname{Int}(A)$ if and only if there is an open set U with $x \in U \subset A$.

   Proof. For $x \in X$, suppose that there is an open set U with $x \in U \subset A$. Then by definition of $\operatorname{Int}(A)$ as the union of all open sets contained in A, we have that U is one such open set contained in A and so $x \in U \subset \operatorname{Int}(A)$.

   Conversely, suppose that $x \in \operatorname{Int}(A)$. Then by definition of $\operatorname{Int}(A)$, it follows that x is contained in some open set contained in A. □

   (b) Let the boundary of A be $\operatorname{Bd}(A) = \overline{A} \cap \overline{(X - A)}$. Show that $x \in \operatorname{Bd}(A)$ if and only if every open set V with $x \in V$ contains points of both A and $X - A$.

   Proof. For $x \in X$, suppose that every open set V containing x contains points of both A and $X - A$. Then every open set containing x intersects nontrivially with A, so it follows that $x \in \overline{A}$; similarly every open set containing x intersects nontrivially with $X - A$ so that $x \in \overline{(X - A)}$. Hence $x \in \overline{A} \cap \overline{(X - A)} = \operatorname{Bd}(A)$.

   Conversely, suppose that $x \in \overline{A} \cap \overline{(X - A)} = \operatorname{Bd}(A)$. Then $x \in \overline{A}$ so that every open neighborhood of x intersects nontrivially with A; similarly $x \in \overline{(X - A)}$, from which we have that every open neighborhood of x intersects nontrivially with $X - A$. Then any neighborhood V of x intersects nontrivially with A and also intersects nontrivially with $X - A$ so that V contains points of both A and $X - A$. $\square$

(c) Prove that $\mathrm{Bd}(A) \cap \mathrm{Int}(A) = \emptyset$ and that $\overline{A} = \mathrm{Int}(A) \cup \mathrm{Bd}(A)$.

Proof. Suppose $x \in \mathrm{Bd}(A) \cap \mathrm{Int}(A)$. Then *every* neighborhood of x contains points of $X - A$ since $x \in \mathrm{Bd}(A)$. This is in contradiction with the requirement that $x \in \mathrm{Int}(A)$, which stipulates the existence of a neighborhood of x completely contained in A. Therefore there cannot be any elements x in $\mathrm{Bd}(A) \cap \mathrm{Int}(A)$, meaning $\mathrm{Bd}(A) \cap \mathrm{Int}(A) = \emptyset$.

Suppose $x \in \overline{A}$. Then every neighborhood of x intersects A nontrivially; that is, for any open neighborhood V of x, V contains points of A. What remains is whether or not some V contains points of $X - A$ or not: If some V does not contain points of $X - A$, then V only contains points of A so that $V \subset A$ and so $x \in \mathrm{Int}(A)$. Otherwise *every* V contains both points of A and $X - A$ so that $x \in \mathrm{Bd}(A)$. Hence $x \in \mathrm{Int}(A) \cup \mathrm{Bd}(A)$.

Conversely, suppose that $x \in \mathrm{Int}(A) \cup \mathrm{Bd}(A)$, so that either $x \in \mathrm{Int}(A)$ or $x \in \mathrm{Bd}(A)$ (but not both). If $x \in \mathrm{Int}(A)$ then there exists a neighborhood of x contained in A, from which it follows that $x \in A$ and so every neighborhood containing x necessarily intersects nontrivially with A. In this case $x \in \overline{A}$. In the other case, $x \in \mathrm{Bd}(A)$ so that every neighborhood of x contains points in A as well as points in $X - A$; this is enough to see that every neighborhood of x intersects nontrivially with A so that $x \in \overline{A}$. Hence $\overline{A} = \mathrm{Int}(A) \cup \mathrm{Bd}(A)$. $\square$

3. Consider $\mathbb{Z}_+$ with the finite complement topology. Determine if the following sequences converge, and if so, to which point or points.

(a) $x_n = 2n + 3$ Converges to every number in the set $\mathbb{Z}_+$.

Proof. Every open set in $\mathbb{Z}_+$ is of the form $\mathbb{Z}_+ - A$ where A is a finite nonempty set of positive integers. To specify a neighborhood $\mathbb{Z}_+ - A$ of some integer m, demand that $m \notin A$.

Take any neighborhood $\mathbb{Z}_+ - A$ of $m \in \mathbb{Z}_+$ (so $m \notin A$). Because the positive integers are well-ordered and $x_{n+1} > x_n$, we can choose N large enough so that x_N is larger than the maximal element of A (one such choice for N is the maximal element of A). Then all but finitely many x_n is in any neighborhood of m for every $m \in \mathbb{Z}_+$. Hence x_n converges to every positive integer. $\square$

(b) $x_n = 3 + (-1)^n$ Does not converge.

Proof. Take the neighborhood of any positive integer $m \neq 2, 4$ of the form $\mathbb{Z}_+ - A$ (with A being a finite nonempty set of positive integers) where $m \notin A$ and $2, 4 \in A$. This neighborhood does not contain x_n for every $n \in \mathbb{Z}_+$, so there is no way for this sequence to converge to m.

Then if $m = 2$ or $m = 4$ consider the neighborhood $\mathbb{Z}_+ - A$ with $m \notin A$ and 2 or 4 in A depending on whichever m is not equal to (so if $m = 2$, then $4 \in A$). This neighborhood does not contain all but finitely many x_n since we can choose n to be even or odd depending on if 2 or 4 is in A and find that an infinite number of elements x_n is not contained in the neighborhood. So in these cases the sequence also cannot converge.

Hence x_n does not converge. $\square$

4. Recall that two topological spaces X and Y are homeomorphic if and only if there is a homeomorphism $h\colon X \to Y$. Suppose that $\{X_\lambda\colon \lambda \in \Lambda\}$ and $\{Y_\lambda\colon \lambda \in \Lambda\}$ are indexed families of topological spaces with X_λ homeomorphic to Y_λ for each $\lambda \in \Lambda$. Prove that $\prod_{\lambda\in\Lambda} X_\lambda$ and $\prod_{\lambda\in\Lambda} Y_\lambda$ are homeomorphic. Use the product topology on the product spaces.

Proof. Let $f_\lambda\colon X_\lambda \to Y_\lambda$ be given homeomorphisms for each $\lambda \in \Lambda$. Then let $h\colon \prod_{\lambda\in\Lambda} X_\lambda \to \prod_{\lambda\in\Lambda} Y_\lambda$ be given by the formula

$$h((x_\lambda)_{\lambda\in\Lambda}) = (f_\lambda(x_\lambda))_{\lambda\in\Lambda};$$

that is, h is just f_λ for every coordinate. It is clear that h is a bijection since each f_λ is a bijection. Define h^{-1} in the natural way by the formula

$$h^{-1}((y_\lambda)_{\lambda\in\Lambda}) = (f_\lambda^{-1}(y_\lambda))_{\lambda\in\Lambda}.$$

We show that h and h^{-1} map open sets to open sets, by showing that they map basis elements to basis elements.

A basis element of $\prod_{\lambda\in\Lambda} X_\lambda$ with the product topology is a product of open sets $\prod_{\lambda\in\Lambda} U_\lambda$ where $U_\lambda = X_\lambda$ for all but finitely many $\lambda \in \Lambda$. Then

$$h\left(\prod_{\lambda\in\Lambda} U_\lambda\right) = (f_\lambda(U_\lambda))_{\lambda\in\Lambda},$$

and since each f_λ is a homeomorphism, it follows that each $f_\lambda(U_\lambda)$ is open (all but finitely many of them will be Y_λ) so that the resulting set is a product of open sets $\prod_{\lambda\in\Lambda} V_\lambda$ where all but finitely many $V\lambda$ are Y_λ. This set is a basis element of $\prod_{\lambda\in\Lambda} Y_\lambda$.

Any basis element of $\prod_{\lambda\in\Lambda}$ is a product of open sets $\prod_{\lambda\in\Lambda} V_\lambda$ where all but finitely many V_λ are Y_λ. We have

$$h^{-1}\left(\prod_{\lambda\in\Lambda} V_\lambda\right) = (f_\lambda^{-1}(V_\lambda))_{\lambda\in\Lambda}.$$

Since each f_λ^{-1} is also a homeomorphism, we have that each $f_\lambda^{-1}(V_\lambda)$ is open (all but finitely many of them will be X_λ), so that the resulting set is a product of open sets $\prod_{\lambda\in\Lambda} U_\lambda$. This set is a basis element of $\prod_{\lambda\in\Lambda} X_\lambda$.

Hence h is a homeomorphism as desired, so that $\prod_{\lambda\in\Lambda} X_\lambda$ and $\prod_{\lambda\in\Lambda} Y_\lambda$ are homeomorphic. $\square$

5. Assume that d and d' are metrics on X and that there are positive constants c_1, c_2 with

$$c_1 d(x,y) \le d'(x,y) \le c_2 d(x,y)$$

for all $x, y \in X$ Show that d and d' induce the same topology.

6. We showed in class that on $\mathbb{R}^{\mathbb{Z}_+}$ the box topology is finer than the uniform topology which in turn is finer than the product topology. Give examples that show that the box topology is *strictly* finer than the uniform topology which in turn is *strictly* finer than the product topology. You can use the fact that the product topology is induced by the metric D.

7. Give $X^{\mathbb{Z}_+}$ the product topology and let $\{\underline{x}_n\}$ be a sequence in $x^{\mathbb{Z}_+}$.
   (a) Show that $\underline{x}_n \to \underline{x}$ if and only if for each $i \in \mathbb{Z}_+$, $\pi_i(\underline{x_n}) \to \pi_i(\underline{x})$. In other words, a sequence converges if and only if all its components converge.
   (b) Is this result true when we give $X^{\mathbb{Z}_+}$ the box topology?

8. Let (X, d) be a metric space.
   (a) Show that $d\colon X \times X \to \mathbb{R}$ is continuous where $X \times X$ is given the product topology.
   (b) If the sequences $x_n \to x$ and $y_n \to y$ converge in X show that the sequence of real numbers $d(x_n, y_n) \to d(x, y)$.

9. Given metric spaces (X_i, d_i) for $i = 1, \ldots, n$ show that

$$\rho(x, y) = \max\{d_1(x, y), \ldots, d_n(x, y)\}$$

   is a metric on $\prod_{i=1}^{n} X_i$.