Page 20: 8 (a,b,c,f), 10, 11, 13, 18, 20

Page 20:

8.

(a). The set W_1 is a subspace of $\mathbb{R}^3$.

We can rewrite vectors in this set as $(3a_2, a_2, -1a_2)$ for any $a_2 \in \mathbb{R}$. By choosing $a_2 = 0$, we can produce the zero vector $(0, 0, 0)$. So W_1 contains the zero vector.

To show that the set is closed under addition, take two vectors, $(3a_2, a_2, -1a_2)$ and $(3b_2, b_2, -1b_2)$, where $a_2, b_2 \in \mathbb{R}$. Then we may add them:
$(3a_2, a_2, -1a_2)+(3b_2, b_2, -1b_2) = (3a_2 + 3b_2, a_2 + b_2, -1a_2 + -1b_2) = (3(a_2 + b_2), (a_2 + b_2), -1(a_2 + b_2))$ Since the real numbers form a field, $a_2 + b_2 \in \mathbb{R}$, and so $(3a_2, a_2, -1a_2) + (3b_2, b_2, -1b_2) \in \mathsf{W}_1$.

Similarly show that the set is closed under scalar multiplication. Take the vector $(3a_2, a_2, -1a_2)$, where $a_2 \in \mathbb{R}$. Then for some $c \in \mathbb{R}$, $c\,(3a_2, a_2, -1a_2) = (3a_2c, a_2c, -1a_2c) = (3(a_2c), (a_2c), -1(a_2c))$. Again, since $a_2c \in \mathbb{R}$, $c\,(3a_2, a_2, -1a_2) \in \mathsf{W}_1$.

Hence W_1 is a subspace of $\mathbb{R}^3$.

(b). The set W_2 is not a subspace of $\mathbb{R}^3$.

Rewrite vectors in W_2 as $(a_3 + 3, a_2, a_3)$. We cannot produce the zero vector $(0, 0, 0)$, since there is no choice of a_3 that makes a_3 and $a_3 + 3$ simultaneously 0 (even if $a_2 = 0$). Therefore the set is not a subspace of $\mathbb{R}^3$.

(c). The set W_3 is a subspace of $\mathbb{R}^3$.

Choosing $a_1, a_2, a_3 = 0$ satisfies the equation $2a_1 - 7a_2 + a_3 = 0$ and produces the zero vector $(0, 0, 0)$. So W_3 contains the zero vector.

Take two vectors (a_1, a_2, a_3) and (b_1, b_2, b_3) (where all components of these vectors are real numbers) and add them to find $(a_1 + b_1, a_2 + b_2, a_3 + b_3)$. Then to check to see if W_3 contains this vector, the components must satisfy the contraining equation (which it does): $2\,(a_1 + b_1) - 7\,(a_2 + b_2) + (a_3 + b_3) = 2a_1 - 7a_2 + a_3 + 2b_1 - 7b_2 + b_3 = 0 + 0 = 0$. Therefore W_3 is closed under addition.

Take any real number c and a vector (a_1, a_2, a_3) (whose components are real numbers), and perform the following scalar multiplication: $c\,(a_1, a_2, a_3) = (ca_1, ca_2, ca_3)$. Then again, to check to see if W_3 contains this vector, the components must satisfy the contraining equation (which it does): $2(ca_1) - 7(ca_2) + (ca_3) = c\,(2a_1 - 7a_2 + a_3) = c\,(0) = 0$. Therefore W_3 is closed under scalar multiplication.

Hence W_3 is a subspace of $\mathbb{R}^3$.

(f). The set $\mathsf{W}_6 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$ is not a subspace of $\mathbb{R}^3$ because we fail to have closure under addition. Consider the vectors $\left(\sqrt{3}, \sqrt{7}, 1\right)$ and $\left(\sqrt{3}, \sqrt{7}, -1\right)$ which are both in W_6 (obtained via inspection). If we added these two together, we would find a third vector $\left(2\sqrt{3}, 2\sqrt{7}, 0\right)$. This vector is not in W_6 because $5\left(2\sqrt{3}\right)^2 - 3\left(2\sqrt{7}\right)^2 + 6\,(0)^2 = 60 - 84 \neq 0$.

10. The set $\mathsf{W}_1 = \{(a_1, a_2, \ldots, a_n) \in \mathbb{F}^n : a_1 + a_2 + \cdots + a_n = 0\}$ is a subspace.

Proof. The set W_1 contains the zero vector: For some vector $(a_1, a_2, \ldots, a_n)$, choose $a_1, a_2, \ldots, a_n \in \mathbb{F}$ to all be 0. Then the vector is rewritten as $(0, 0, \ldots, 0)$, and this vector satisfies the property that zero vectors should.

For $b_1, b_2, \ldots, b_n \in \mathbb{F}$, $(b_1, b_2, \ldots, b_n) + (0, 0, \ldots, 0) = (b_1 + 0, b_2 + 0, \ldots, b_n + 0) = (b_1, b_2, \ldots, b_n)$ (similarly if zero was added to the left). This zero vector also satisfies the condition imposed on vectors in the set W_1, that is the sum of the components is zero: $0 + 0 + \cdots + 0 = 0$. Hence there is a zero vector contained in W_1.

The set W_1 is closed under vector addition. Take two vectors in W_1, $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$, then $(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$. So to check if this vector is in W_1, it must satisfy the condition that the components must sum to zero, which it does. Observe that $(a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n) = a_1 + b_1 + a_2 + b_2 + \cdots + a_n + b_n = (a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n) = 0 + 0 = 0$. Hence W_1 is closed under addition.

The set W_1 is closed under scalar multiplication. Take some constant $c \in \mathbb{F}$ and then some vector in W_1, $(a_1, a_2, \ldots, a_n)$. Then $c\,(a_1, a_2, \ldots, a_n) = (ca_1, ca_2, \ldots, ca_n)$. To show that this resulting vector is in W_1, we must show that its components sum to zero, which it does: $ca_1 + ca_2 + \cdots + ca_n = c\,(a_1 + a_2 + \cdots + a_n) = c\,(0) = 0$. Hence W_1 is closed under scalar multiplication.

Therefore W_1 is a subspace of $\mathbb{F}^n$. □

The set $\mathsf{W}_2 = \{(a_1, a_2, \ldots, a_n) \in \mathbb{F}^n : a_1 + a_2 + \cdots + a_n = 1\}$ is not a subspace.

Proof. We immediately fail to produce a normal vector. Consider the same normal vector that I mentioned earlier, $(0, 0, \ldots, 0)$. This vector is not actually in W_2 because its components do not sum to $1 :\ 0 + 0 + \cdots + 0 \neq 1$. Therefore W_2 is not a subspace of $\mathbb{F}^n$. □

11. The set $\mathsf{W} = \{f(x) \in \mathsf{P}(\mathbb{F}) : f(x) = 0 \text{ or } f(x) \text{ has degree n}\}$ if $n \geq 1$ is not a subspace of $\mathsf{P}(\mathbb{F})$ because we fail to have closure under addition. Let $f(x)$ be a vector in W, and then let $p(x) = C + f(x)$, where $C \in \mathbb{F}$. The vector $p(x)$ is indeed in W because adding constants of degree zero to these polynomials will not change the degree of the polynomial. Similarly, consider another vector $-f(x)$ in W (because $-f(x)$ is an additive inverse of $f(x)$, this would be equivalent to taking $f(x)$ and negating all of the terms in the polynomial expansion, and so the degree of the polynomial is unchanged). Let us add $p(x)$ and $-f(x)$: $(p + (-f))\,(x) = p(x) + (-f(x)) = C + f(x) + (-f(x)) = C + \vec{0} = C$. So we added two vectors in W and found a vector C which is not in W because its degree is zero (≤ 1) but it is not the zero function (where $f(x) = 0$). Therefore W is not a subspace of $\mathsf{P}(\mathbb{F})$.

13. Let S be a nonempty set and $\mathbb{F}$ a field. Prove that for any $s_0 \in S$, $\mathcal{F}_0 = \{f \in \mathcal{F}\,(S, \mathbb{F}) : f(s_0) = 0\}$ is a subspace of $\mathcal{F}\,(S, \mathbb{F})$.

Proof. The zero vector is contained in $\mathcal{F}_0$, because there is a zero function $f = \vec{0}$ that maps any element of S to 0, and it behaves like a zero vector (that is, that $\vec{x} + \vec{0} = \vec{0}$ due to the definition of vector addition and vector/function equality). Furthermore we also know that the zero vector is unique, even in subspaces.

The set $\mathcal{F}_0$ is closed under scalar multiplication: for some $c \in \mathbb{F}$ and $x \in S$, $\left(c\vec{0}\right)(x) = c\,(0) = 0 = \left(\vec{0}\right)(x) \implies c\vec{0} = \vec{0}$ and since $\vec{0} \in \mathcal{F}_0$, we have closure under scalar multiplication.

The set $\mathcal{F}_0$ is closed under addition. Suppose we took $\vec{x}, \vec{y} \in \mathcal{F}_0$ (not necessarily distinct, and not even distinct since the zero vector is unique, so both are really $\vec{0}$). Then for $t \in S$, $(\vec{x} + \vec{y})(t) = (\vec{x})(t) + (\vec{y})(t) = \left(\vec{0}\right)(t) + \left(\vec{0}\right)(t) = 0 + 0 = 0 = \left(\vec{0}\right)(t) \implies \vec{x} + \vec{y} = \vec{0} \implies \vec{x} + \vec{y} \in \mathcal{F}_0$

Therefore $\mathcal{F}_0$ is a subspace of $\mathcal{F}(S, \mathbb{F})$. □

18. Prove that a subset W of a vector space V is a subspace of V if and only if $\vec{0} \in \mathsf{W}$ and $a\vec{x} + \vec{y} \in \mathsf{W}$ whenever $a \in \mathbb{F}$ and $\vec{x}, \vec{y} \in \mathsf{W}$.

Proof. Forwards direction: If W is a subspace of V, then it contains the zero vector and it is closed under addition and scalar multiplication. Then indeed, $\vec{0} \in \mathsf{W}$. Then if we choose $a = 1$, then $a\vec{x} + \vec{y} = \vec{x} + \vec{y}$, and since $\vec{x}, \vec{y} \in \mathsf{W}$, $\vec{x} + \vec{y} \in \mathsf{W}$. Similarly, if we choose $\vec{y} = \vec{0}$ (since the zero vector is contained in W), then $a\vec{x} + \vec{y} = a\vec{x}$, and since W is closed under scalar multiplication, $a\vec{x} \in \mathsf{W}$. So then the linear combination $a\vec{x} + \vec{y} \in \mathsf{W}$.

Reverse direction. If $\vec{0} \in \mathsf{W}$, then the set contains the zero vector. Similarly, if we choose $a = 1$, then $a\vec{x} + \vec{y} = \vec{x} + \vec{y}$, which is in W, meaning it is closed under addition. Again, if we choose $\vec{y} = \vec{0}$ (since the zero vector is contained in W), then $a\vec{x} + \vec{y} = a\vec{x}$, meaning that all scalar multiples of a vector are in W. Hence the set W is closed under scalar multiplication. Thus with the three qualities of having the zero vector, being closed under addition, and being closed under scalar multiplication, W is a subspace of V. □

20. Prove that if W is a subspace of a vector space V and $w_1, w_2, \ldots, w_n$ are in W, then $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in \mathsf{W}$ for any scalars $a_1, a_2, \ldots, a_n$.

Proof. Let us use induction.

Take the base case where $n = 2$ ($n = 0, 1$ are trivial cases, we want to know this for natural numbers upwards and equal to 2). When $n = 2$, we have $a_1w_1 + a_2w_2$, which is a linear combination of vectors in W, and since W is closed under addition and scalar multiplication (by definition of subspace), $a_1w_1 + a_2w_2 \in \mathsf{W}$.

Then suppose the $n-1$-th case is true. So this means that $a_1w_1 + a_2w_2 + \cdots + a_{n-1}w_{n-1} \in \mathsf{W}$, and then we use this to show that the n-th case holds. The n-th case is that $a_1w_1 + a_2w_2 + \cdots + a_{n-1}w_{n-1} + a_nw_n$, and we can see that the quantity a_nw_n is in W due to closure under scalar multiplication, and the quantity $a_1w_1 + a_2w_2 + \cdots + a_{n-1}w_{n-1}$ is as well due to the inductive hypothesis. So we can see the summation instead as a linear combination of two vectors in W, which is closed under scalar multiplication and addition. So $(a_1w_1 + a_2w_2 + \cdots + a_{n-1}w_{n-1}) + a_nw_n \in \mathsf{W}$, which means $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in \mathsf{W}$. □