HOMEWORK 4

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Let, for $n \in \mathbb{N}$,

$$b_n = \frac{2n^2+1}{n^2-n-3}.$$

Show (b_n) converges arguing directly from the definition of convergence.

Proof. Let $(b_n)_{n=1}^{\infty}$ be a sequence given by $b_n = (2n^2+1)/(n^2-n-3)$, for $n \in \mathbb{N}$. Observe that

$$n^2 - n - 3 - \frac{1}{3}n^2 = \frac{2}{3}n^2 - n - 3 = \frac{1}{3}(n-3)(2n+3) \geq 0$$

as long as $n \geq 3$. This implies that for $n \geq 3$, $n^2 - n - 3 \geq \frac{1}{3}n^2$.

Then for every $\varepsilon > 0$, choose, from an Archimedean property and the previous observation, $N \in \mathbb{N}$ such that $N > \max\left\{3, \frac{27}{\varepsilon}\right\}$. Then for all $n > N$,

$$\begin{aligned}
|b_n - 2| = \left|\frac{2n^2+1}{n^2-n-3} - 2\right| &= \frac{2n+7}{n^2-n-3} && \text{(positive, since } n > 3) \\
&< \frac{2n+7n}{n^2-n-3} && (9n > 2n+7, \text{ since } n > 3) \\
&< \frac{3 \cdot 9n}{n^2} && \left(n^2 - n - 3 > \frac{1}{3}n^2 \text{ for } n > 3\right) \\
&= \frac{27}{n} \\
&< \frac{27}{N} && (n > N) \\
&< \varepsilon
\end{aligned}$$

Hence (b_n) converges, and its limit is 2. $\square$