8. Let $n \geq 0$ and $x \in [0, \pi]$. Prove that $|\sin(nx)| \leq n\sin(x)$.

Proof. We will show this by induction. Observe for $n = 0$, $|\sin(0x)| = 0 \leq 0 = 0\sin(x)$. Then suppose that $|\sin(nx)| \leq n\sin(x)$, and we will show that $|\sin((n+1)x)| \leq (n+1)\sin(x)$. So

$$\begin{aligned} |\sin((n+1)x)| = |\sin(nx + x)| &= |\sin(nx)\cos(x) + \sin(x)\cos(nx)| \\ &\leq |\sin(nx)\cos(x)| + |\sin(x)\cos(nx)| \\ &= |\sin(nx)||\cos(x)| + \sin(x)|\cos(nx)|, \end{aligned}$$

where the triangle inequality was used as well as omitting the absolute value bars for $\sin(x)$ since on $[0, \pi]$, $\sin(x)$ is nonnegative. Then from the inductive hypothesis,

$$\begin{aligned} &\leq n\sin(x)|\cos(x)| + \sin(x)|\cos(nx)| \\ &\leq n\sin(x) + \sin(x) = (n+1)\sin(x), \end{aligned}$$

because $\cos(x)$ and $\cos(nx)$ are bounded functions so we may bound them above by 1.

Hence $|\sin((n+1)x)| \leq (n+1)\sin(x)$, and by induction we have that for all $n \geq 0$ and $x \in [0, \pi]$, $|\sin(nx)| \leq n\sin(x)$. $\square$

13. Find the 100-th derivative of $f(x) = 1/(5 - x^2)$.

The 100-th derivative of $f(x)$ is $\frac{\sqrt{5}}{10}\left(\frac{100!}{(x+\sqrt{5})^{101}} - \frac{100!}{(x-\sqrt{5})^{101}}\right)$.

We can find a closed form for the n-th derivative of $f(x)$ (where $n = 0$ means to take no derivatives). First we can rewrite $f(x)$ by partial fraction decomposition, so that

$$f(x) = \frac{1}{5 - x^2} = \frac{\sqrt{5}}{10}\left(\frac{1}{x + \sqrt{5}} - \frac{1}{x - \sqrt{5}}\right).$$

Then by taking successive derivatives, we find that

$$f^{(n)}(x) = \frac{\sqrt{5}}{10}\left(\frac{(-1)^n n!}{(x + \sqrt{5})^{n+1}} - \frac{(-1)^n n!}{(x - \sqrt{5})^{n+1}}\right),$$

and we prove that this is the closed form for the n-th derivative of $f(x)$.

Proof. Let f be given as above. Then for $n = 0$, we saw earlier that

$$f(x) = \frac{1}{5 - x^2} = \frac{\sqrt{5}}{10}\left(\frac{1}{x + \sqrt{5}} - \frac{1}{x - \sqrt{5}}\right),$$

and for $n = 1$ we also have

$$f'(x) = \frac{2x}{(5 - x^2)^2} = \frac{\sqrt{5}}{10}\left(\frac{(-1)}{(x + \sqrt{5})^2} - \frac{(-1)}{(x - \sqrt{5})^2}\right).$$

Then suppose that the n-th derivative of f is given as above, so that $f^{(n)}(x) = \frac{\sqrt{5}}{10}\left(\frac{(-1)^n n!}{(x+\sqrt{5})^{n+1}} - \frac{(-1)^n n!}{(x-\sqrt{5})^{n+1}}\right)$. Then to show that this formula holds for the $n+1$-th derivative, see that

$$\begin{aligned} f^{(n+1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x} f^{(n)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sqrt{5}}{10}\left(\frac{(-1)^n n!}{(x+\sqrt{5})^{n+1}} - \frac{(-1)^n n!}{(x-\sqrt{5})^{n+1}}\right) \\ &= \frac{\sqrt{5}}{10}\left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{(-1)^n n!}{(x+\sqrt{5})^{n+1}} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{(-1)^n n!}{(x-\sqrt{5})^{n+1}}\right) \\ &= \frac{\sqrt{5}}{10}\left(\frac{(-1)^n n! \cdot (-(n+1))}{(x+\sqrt{5})^{(n+1)+1}} - \frac{(-1)^n n! \cdot (-(n+1))}{(x-\sqrt{5})^{(n+1)+1}}\right) \\ &= \frac{\sqrt{5}}{10}\left(\frac{(-1)^{n+1}(n+1)!}{(x+\sqrt{5})^{n+2}} - \frac{(-1)^{n+1}(n+1)!}{(x-\sqrt{5})^{n+2}}\right). \end{aligned}$$

Therefore, by induction, the formula given is the closed form for the n-th derivative of f. □

Then we may take $n = 100$ to find that

$$f^{(100)}(x) = \frac{\sqrt{5}}{10}\left(\frac{100!}{(x+\sqrt{5})^{101}} - \frac{100!}{(x-\sqrt{5})^{101}}\right).$$