Solution Manual

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39.5 Exercises

39.5.1

Give the equation of this plane as $abz + acy + bcx = abc$, or otherwise $z = c - \frac{c}{b}y - \frac{c}{a}x$. Then find the surface area element dS by finding

$$dS = \sqrt{1 + \left(-\frac{c}{a}\right)^2 + \left(-\frac{c}{b}\right)^2} dA = \sqrt{1 + \frac{c^2}{a^2} + \frac{c^2}{b^2}} dA$$

where dA is the rectangular area element within the triangle whose vertices are $(0, 0)$, $(0, b)$, and $(a, 0)$. We can give the integration region here by bounding x from 0 to a and giving the curves in y to be from 0 to $b - \frac{b}{a}x$. The double integral becomes

$$\int_0^a \int_0^{b - \frac{b}{a}x} \sqrt{1 + \frac{c^2}{a^2} + \frac{c^2}{b^2}} dydx \rightarrow \int_0^a (b - \frac{b}{a}x)\sqrt{1 + \frac{c^2}{a^2} + \frac{c^2}{b^2}} dx$$

$$= \frac{1}{2}\sqrt{a^2b^2 + b^2c^2 + a^2c^2}$$

39.5.4

We are already given the equation for the surface. To find the area of integration it is sufficient to equate the two values of z given for the planes with the paraboloid expression and find that the region is an annulus of inner radius 1 and outer radius 3, where θ takes on the natural range (we will be using polar coordinates).

To find the surface area elements dS find

$$dS = \sqrt{1 + (-2x)^2 + (2y)^2} dA = \sqrt{1 + 4r^2} dA$$

Then the double integral is given in the polar form:

$$\int_0^{2\pi} \int_1^3 \sqrt{1 - 4r^2} r dr d\theta \rightarrow \frac{\pi}{4} \int_5^{37} \sqrt{u} du = \frac{\pi}{6} \left(37^{\frac{3}{2}} - 5^{\frac{3}{2}}\right)$$

39.5.7

Like before to find the region of integration simply equate the given values of z for the planes to the cone expression to find that the region of integration is the annulus of inner radius 1 and outer radius 2. It is also seen that θ ranges from 0 to 2π. We will be using polar coordinates to evaluate this surface integral.

Then the surface area element dS is found as:

$$dS = \sqrt{1 + \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2} dA$$

$$= \sqrt{1 + \left(\frac{r\cos(\theta)}{r}\right)^2 + \left(\frac{r\sin(\theta)}{r}\right)^2} dA = \sqrt{2} dA$$

After writing everything in polar coordinates, noting that $z = r$ due to S being a part of a cone, the double integral becomes:

$$\sqrt{2}\int_0^{2\pi}\int_1^2 (r) r^4 \cos^2(\theta) dr d\theta \to \frac{2}{\sqrt{2}}\left(\int_0^{2\pi} \cos^2(\theta) d\theta\right)\left(\int_1^2 r^5 dr\right)$$

$$= \frac{21\pi}{\sqrt{2}}$$

39.5.10

Find the surface area element dS as:

$$dS = \sqrt{1 + 4(x^2+y^2)dA}$$

where dA is an area element over the portion of the unit disk in the first octant (since the paraboloid forms the unit circle as the boundary of the disk at $z = 0$). Then rewrite the double integral as

$$\iint_S (1 - x^2 - y^2)(\sin(x^2) - \sin(y^2))\sqrt{1 - 4(x^2+y^2)} dA$$

Notice that the integral exhibits skew symmetry as the integrand switches sign completely (due to the trigonometric terms) if we apply the transformation $(x, y) \to (y, x)$, otherwise a reflection over $x = y$. Note that the region of integration is symmetric across this line as well and hence the integral vanishes.