1. Evaluate $\oint_C \vec{a} \cdot \mathrm{d}\vec{r}$ where C is given by a single loop of the curve $(x-2)^2 + (y-3)^2 = R^2$, where $R \in \mathbb{R}$, and $\vec{a} = \hat{\mathbf{i}}(x^2 + yz^2) + \hat{\mathbf{j}}(2x - y^3)$.

   We could parameterize the curve C by $\hat{\mathbf{i}}(2 + R\cos(t)) + \hat{\mathbf{j}}(3 + R\sin(t))$ for $t \in [0, 2\pi]$. This will make the integral annoying to compute so instead we will use Stokes' theorem. Let D be the closed disc which C is the boundary of. Then

$$\begin{aligned}\oint_C \vec{a} \cdot \mathrm{d}\vec{r} &= \int_D (\vec{\nabla} \times \vec{a}) \cdot \hat{n} \, \mathrm{d}A \\ &= \int_D \hat{\mathbf{k}} \left(2 - z^2\right) \cdot \hat{n} \, \mathrm{d}A \\ &= 2\int_D 1 \, \mathrm{d}A \\ &= 2\pi R^2.\end{aligned}$$

2. Prove that

$$\begin{aligned}\delta(ax) &= \frac{1}{a}\delta(x) \\ \delta(g(x)) &= \frac{\delta(x)}{|g'(0)|} && \text{if } g(0) = 0 \text{ and } g(x) \neq 0 \text{ otherwise} \\ \delta(g(x)) &= \sum_i \frac{\delta(x - a_i)}{|g'(a_i)|} && \text{where } g(x) \text{ is a general function and } g(a_i) = 0\end{aligned}$$

   Proof. To see the first equality, we know that both integrals over $\mathbb{R}$ must be 1. So

$$1 = \int_{-\infty}^{\infty} \delta(ax) \, \mathrm{d}x \underset{a\mathrm{d}x \to \mathrm{d}x}{\overset{ax \to x}{=}} \int_{-\infty}^{\infty} a^{-1}\delta(x) \, \mathrm{d}x = 1,$$

   and because delta functions are everywhere 0 except for the origin, the integrands are identical. Hence $\delta(ax) = a^{-1}\delta(x)$.

   Similarly, see that

$$\begin{aligned}1 = \int_{-\infty}^{\infty} \delta(g(x)) \, \mathrm{d}x &\underset{g'(x)\mathrm{d}x \to \mathrm{d}x}{\overset{g(x) \to x}{=}} \int_{g(-\infty)}^{g(\infty)} \frac{\delta(x)}{g'(g^{-1}(x))} \, \mathrm{d}x \\ &= \int_{\min\{g(\infty), g(-\infty)\}}^{\max\{g(\infty), g(-\infty)\}} \frac{\delta(x)}{|g'(g^{-1}(x))|} \, \mathrm{d}x \\ &= \frac{1}{|g'(g^{-1}(0))|} = \frac{1}{|g'(0)|},\end{aligned}$$

   because $g(x) = 0$ if and only if $x = 0$ implies that one of $g(\infty), g(-\infty)$ will be positive and the other will be negative. Therefore at $x = 0$ the sign of the derivative $g'(0)$ is determined entirely by whether $g(\infty)$ was less than $g(-\infty)$ (or vice versa). That is to say, if g "increased" then the derivative $g'(0)$ is positive, and otherwise, negative. Therefore the integral is simplified as above, into the convolution of the delta function with the reciprocal of $|g'(0)|$, and the result follows.

Then since $1 = \int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x$, and delta functions agree everywhere except zero, we must have that

$$\delta(g(x)) = \frac{\delta(x)}{|g'(0)|},$$

which is the second equality.

The third equality follows as an extension of the second equality. Fix $n \in \mathbb{N}$, and let $\{a_1, a_2, \ldots, a_n\}$ be zeroes of g, with $a_1 < a_2 < \cdots < a_n$. Furthermore, let $a_0 < a_1$, and $a_{n+1} > a_n$ (they are not zeroes but we need them in the following formulation). Using the previous result, where instead of the zeroes occurring at $x = 0$, they occur on $x = a_i$, we have

$$\begin{aligned} n = \int_{-\infty}^{\infty} \delta(g(x))\,\mathrm{d}x &= \sum_i \int_{\frac{1}{2}(a_{i-1}+a_i)}^{\frac{1}{2}(a_i+a_{i+1})} \delta(g(x-a_i))\,\mathrm{d}x \\ &= \sum_i \frac{1}{|g'(a_i)|}. \end{aligned}$$

Similarly, we can attach on each numerator a delta function (each shifted by a_i so that each term is still equal to 1). Then we may equate the integrand $\delta(g(x))$ to this new summation. We find that

$$\delta(g(x)) = \sum_i \frac{\delta(x-a_i)}{|g'(a_i)|},$$

and so we have proven all three parts. $\square$