HOMEWORK 11

SAI SIVAKUMAR

Suppose $f : [0,1] \to \mathbb{R}$ is bounded, $f(x) \geq 0$ for all $x \in [0,1]$ and there is point $a \in (0,1)$ such that $f(a) > 0$. Show, if f is continuous at a, then the lower integral of f is positive. Conclude, if f is Riemann integrable, then

$$\int_0^1 f\,dx > 0.$$

I spoke with Jude Flynn about a few details of the proof.

Proof. Let f be as given. Because f is continuous at a, there exists $0 < \eta < \min\{a, 1-a\}$ such that if $|x-a| < \eta$, then $|f(x) - f(a)| < f(a)/2$; that is, $0 < f(a)/2 < f(x) < 3f(a)/2$.

With the partition $P = \{0 = x_0 < a - \eta = x_1 < a + \eta = x_2 < 1 = x_3\}$,

$$\begin{aligned} 0 < \eta f(a) &\leq (a-\eta)\inf\{f(x)\colon x \in [x_0,x_1]\} + 2\eta\inf\{f(x)\colon x \in [x_1,x_2]\} \\ &\qquad + (1-a-\eta)\inf\{f(x)\colon x \in [x_2,x_3]\} \\ &= L(f,P) \\ &\leq \underline{\int_0^1} f\,\mathrm{d}x\,. \end{aligned}$$

If f is Riemann integrable, it follows that

$$0 < \underline{\int_0^1} f\,\mathrm{d}x \leq \int_0^1 f\,\mathrm{d}x$$

as desired. $\square$