1. Let A be an $n \times n$ rotation matrix whose entries are given by a_{ij}. Then for some vector $v = (v_1, v_2, \ldots, v_n)$, we demand $|v| = |Av|$. This means

$$\begin{aligned} |v|^2 = a_i a_i &\overset{!}{=} |Av|^2 \\ &= (Av)_i (Av)_i \\ &= (a_{ij} v_j)(a_{ik} v_k) \\ &= (a_{ij} a_{ik}) v_j v_k, \end{aligned}$$

and because $0 \leq i, j, k \leq n$, we can match terms when $j = k$ with the left hand side by enforcing $a_{ij} a_{ik} = 1$. All of the terms where $j \neq k$ must be zero. Hence $a_{ij} a_{ik} = \delta_{jk}$.

2. Observe that the condition $a_{ij} a_{ik} = \delta_{jk}$ is symmetric in j and k, so out of all n^2 ways to select j and k, for the cases when $j \neq k$ (there are n cases where $j = k$), there are $n^2 - n - \frac{n^2 - n}{2}$ redundant equations (due to commutativity of multiplication, we can pair each equation where $j \neq k$ with another one). Then by adding back on the number of equations where $j = k$, the number of equations which are not redundant is $n^2 - \frac{n^2-n}{2}$. Hence there are $n^2 - \left(n^2 - \frac{n^2-n}{2}\right) = \frac{n(n-1)}{2} = \binom{n}{2}$ degrees of freedom, same as the number of degrees of freedom of rotation in n dimensions.