HOMEWORK 10

SAI SIVAKUMAR

This homework consists of two problems.

(A) Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Show, if there is an M such that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$, then f is uniformly continuous.

(B) A point $p \in \mathbb{R}$ is a *fixed point* of a function $g : \mathbb{R} \to \mathbb{R}$ if $g(p) = p$. Show, if g is differentiable and $|g'(x)| < 1$ for all $x \in \mathbb{R}$, then g has at most one fixed point.

Proof (A). Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function as given and suppose that there exists $M \geq 0$ such that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$.

Let $x, y \in \mathbb{R}$ with $x < y$. By the mean value theorem, there exists $c \in (x, y)$ such that $f(x) - f(y) = f'(c)(x - y)$. By taking the absolute value, we have that

$$\begin{aligned} |f(x) - f(y)| &= |f'(c)(x-y)| \\ &= |f'(c)||x-y| \\ &\leq M|x-y|. \end{aligned}$$

Given $\varepsilon > 0$, choose $\delta = \varepsilon/M$. When $|x - y| < \delta$, we have that $|f(x) - f(y)| < M\delta = \varepsilon$. Hence f is uniformly continuous. □

Proof (B). Let g be a differentiable function as given with $|g'(x)| < 1$ for all $x \in \mathbb{R}$.

Suppose by way of contradiction that g has more than one fixed point; that is, there exist $p, q \in \mathbb{R}$ with $p < q$ such that $g(p) = p$ and $g(q) = q$.

Then by the mean value theorem, there exists $c \in (p, q)$ such that

$$\begin{aligned} |q-p| = |g(q) - g(p)| &= |g'(c)(q-p)| \\ &= |g'(c)||q-p| \\ &< |q-p| \end{aligned}$$

which is a contradiction. Hence g has at most one fixed point. □