Section 7.4: 27, 38, Section 7.5: 15, 31, Section 7.6: 19

Section 7.4

27. Find $\mathcal{L}^{-1}\{F\}$ for $s^2F(s) - 4F(s) = 5(s+1)^{-1}$

Rearrange the equation to find

$$F(s) = \frac{5}{(s+1)(s-2)(s+2)}$$

which we can expand using the cover-up method into

$$F(s) = \frac{-\frac{5}{3}}{s+1} + \frac{\frac{5}{12}}{s-2} + \frac{\frac{5}{4}}{s+2}$$

Using our handy Laplace transforms table the inverse is:

$$-\frac{5}{3}e^{-t} + \frac{5}{12}e^{2t} + \frac{5}{4}e^{-2t}$$

38. First we can give the partial fraction expansion of the rational function like so:

$$\frac{P(s)}{Q(s)} = \frac{A}{s-r} + E(s)$$

The quantity $E(s)$ represents extra terms that are in a form such that each term's denominator is not a multiple of $(s-r)$, since $(s-r)$ is a nonrepeated linear factor of $Q(s)$. Then multiply through by $s-r$ to find:

$$\frac{(s-r)P(s)}{Q(s)} = A + (s-r)E(s)$$

Then take the limit as $s$ tends towards $r$, which immediately causes the extra terms $E(s)$ to vanish:

$$\lim_{s\to r} \frac{(s-r)P(s)}{Q(s)} = A$$

Notice that we may rewrite the left hand side as (using the definition of the derivative):

$$\lim_{s\to r} \frac{1}{\frac{Q(s)-0}{s-r}} P(s) \to \frac{1}{Q'(r)}P(r)$$

This is valid so long as the derivative $Q'(r)$ is nonzero.

Thus

$$A = \frac{P(r)}{Q'(s)}$$

Section 7.5

15. Use the table along with the initial data:

$$y'' - 3y' + 2y = \cos(t) \to s^2Y(s) + 1 - 3sY(s) + 2Y(s) = \frac{s}{s^2+1} \to Y(s) = \frac{-(s^2-s+1)}{(s^2+1)(s-1)(s-2)}$$

31. Use the table along with the initial data:

$$y'' + 2y' + 2y = 5 \to s^2Y(s) - sa - b + 2sY(s) - 2a + 2Y(s) = \frac{5}{s} \to Y(s) = \frac{5}{s((s+1)^2+1)} + \frac{a(s+1)+b+a}{(s+1)^2+1}$$

Use the cover-up method to expand part of the first fraction out. Then the following is true, which we can use to deduce the other fraction (whose numerator is $Bs+C$):

$$5 = \frac{5}{2}((s+1)^2+1) + Bs^2 + Cs \implies B = -\frac{5}{2}, C = -5$$

$$\to Y(s) = \frac{\frac{5}{2}}{s} + \frac{-\frac{5}{2}s - 5}{(s+1)^2+1} + \frac{a(s+1)+b+a}{(s+1)^2+1}$$

$$\to y(t) = \frac{5}{2} - \frac{5}{2}e^{-t}\sin(t) - \frac{5}{2}e^{-t}\cos(t) + ae^{-t}\cos(t) + be^{-t}\sin(t) + ae^{-t}\sin(t)$$

$$\to y(t) = \frac{5}{2} + \left(a - \frac{5}{2}\right)e^{-t}\cos(t) + \left(a + b - \frac{5}{2}\right)e^{-t}\sin(t)$$

Section 7.6

19. First give $g(t) = 20 - \Pi_{3\pi,4\pi}(t)$. I will call the Laplace transform of $I(t)$ as $Y(s)$ just because a capital letter was already in use. Apply the Laplace transform to both sides:

$$I''(t) + 2I'(t) + 2I = g(t) \to s^2Y(s) - 10s + 2sY(s) - 20 + 2Y(s) = \frac{20}{s} - e^{-3\pi s}\frac{20}{s} + e^{-4\pi s}\frac{20}{s}$$

$$\to Y(s) = 20\left(1 - e^{-3\pi s} + e^{-4\pi s}\right)\left(\frac{1}{s((s+1)^2+1)}\right) + \frac{10(s+1)+10}{(s+1)^2+1}$$

$$\to Y(s) = 20\left(1 - e^{-3\pi s} + e^{-4\pi s}\right)\left(\frac{\frac{1}{2}}{s} + \frac{-\frac{1}{2}s - 1}{(s+1)^2+1}\right) + \frac{10(s+1)+10}{(s+1)^2+1} \text{ using a similar process as \#31}$$

$$\to I(t) = 20\left(\frac{1}{2} - \frac{1}{2}e^{-t}\sin(t) - \frac{1}{2}e^{-t}\cos(t)\right) - 20u(t-3\pi)\left(\frac{1}{2} - \frac{1}{2}e^{-(t-3\pi)}\sin(t-3\pi) - \frac{1}{2}e^{-(t-3\pi)}\cos(t-3\pi)\right)$$

$$+20u(t-4\pi)\left(\frac{1}{2} - \frac{1}{2}e^{-(t-4\pi)}\sin(t-4\pi) - \frac{1}{2}e^{-(t-4\pi)}\cos(t-4\pi)\right) + 10\left(e^{-t}\sin(t) + e^{-t}\cos(t)\right)$$

$$\to I(t) = 10 - 10u(t-3\pi)\left(1 + e^{-(t-3\pi)}\sin(t) + e^{-(t-3\pi)}\cos(t)\right) + 10u(t-4\pi)\left(1 - e^{-(t-4\pi)}\sin(t) - e^{-(t-4\pi)}\cos(t)\right)$$

A sketch of the current looks like: