HOMEWORK 6

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Show, if (a_n) is a sequence from $\mathbb{R}$ that converges with limit $L \in \mathbb{R}$, that

(a) the complement of $S = \{a_n : n \in \mathbb{N}\} \cup \{L\}$ is open arguing directly from the definition of open set;
(b) if L is not in $T = \{a_n : n \in \mathbb{N}\}$, then T^c is not open.

Proof. (a) Let (a_n) be a sequence from $\mathbb{R}$ that converges to $L \in \mathbb{R}$, and let $S = \{a_n : n \in \mathbb{N}\} \cup \{L\}$ as given. Then consider any element $p \in S^c$; that is, a real number p which is not any value the sequence (a_n) takes on and is not the limit L.

The set $C_p = \{n \in \mathbb{N} : |a_n - p| < \frac{|L-p|}{2}\}$ enumerates natural numbers n for which a_n lies in the $\frac{|L-p|}{2}$-neighborhood of p, $N_{\frac{|L-p|}{2}}(p)$. Note that L is not in this neighborhood. Because L is the limit of the sequence (a_n), the $\frac{|L-p|}{2}$-neighborhood of L, $N_{\frac{|L-p|}{2}}(L)$, contains all but finitely many terms a_n. This is because there exists $M \in \mathbb{N}$ such that for all $n > M$, $|a_n - L| < \frac{|L-p|}{2}$.

There are less than or equal to M terms in the sequence (a_n) for which the inequality fails to hold. So the complement of $N_{\frac{|L-p|}{2}}(L)$ contains finitely many terms a_n. Moreover, $N_{\frac{|L-p|}{2}}(p)$ is a subset of $(N_{\frac{|L-p|}{2}}(L))^c$, and as a result, $N_{\frac{|L-p|}{2}}(p)$ contains finitely many terms a_n. Hence C_p is finite.

We may take $\varepsilon = \min\{|a_n - p| : n \in C_p\}$ whenever C_p is nonempty due to the finiteness of C_p. If C_p is the empty set, then take $\varepsilon = \frac{|L-p|}{2}$. Then $N_\varepsilon(p)$ is an open ball containing p which contains only elements of S^c; that is, only containing real numbers which are not any of the values the sequence (a_n) takes on or the limit L. Since $p \in S^c$ was arbitrary, it follows by definition that S^c is an open set.

(b) Suppose that L is not in $T = \{a_n : n \in \mathbb{N}\}$ as given. Then L is in T^c, and we may consider any open set U containing L. There exists $\varepsilon > 0$ such that $N_\varepsilon(L) \subseteq U$, but because the sequence (a_n) converges to L, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < \varepsilon$.

In particular, $a_N \in T$ satisfies $|a_N - L| < \varepsilon$, which means that a_N is an element of $N_\varepsilon(L)$, and by inclusion, a_N is an element of U. But because U was an arbitrary open set containing L, all open sets containing L contain at least one element of T. And because L was an element of T^c, it follows that T^c is not open. $\square$