HOMEWORK 7

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Suppose $E \subseteq \mathbb{R}$. Show, if there exist $U, V \subseteq \mathbb{R}$ such that

(i) U and V are open;
(ii) $U \cap E \neq \emptyset \neq V \cap E$;
(iii) $U \cap E \cap V = \emptyset$; and
(iv) $E \subseteq U \cup V$,

then E is disconnected.

Proof. Let $E \subseteq \mathbb{R}$ and suppose there exist $U, V \subseteq \mathbb{R}$ with the four properties listed above.

Let $A = U \cap E$ and $B = V \cap E$, and note that they are nonempty by (ii) and disjoint by (iii). Then $E = A \cup B$, since

$$\begin{aligned} A \cup B &= (U \cap E) \cup (V \cap E) \\ &= ((U \cap E) \cup V) \cap ((U \cap E) \cup E) \\ &= (U \cup V) \cap (E \cup V) \cap E \\ &= E, \end{aligned}$$

where in the fourth equality we used the fact that $E \subseteq U \cup V$, which was (iv). So E is the disjoint union of U and V. What remains is to show that A and B are separated.

With A, B forming a partition of E, we have that

$$E \setminus A = E \setminus (E \cap U) = V \cap E = B \subseteq U^c$$

and

$$E \setminus B = E \setminus (E \cap V) = U \cap E = A \subseteq V^c,$$

since every element of $E \setminus (E \cap U)$ is not in U and every element of $E \setminus (E \cap V) \subseteq V$ is not in V.

Because U and V are open, U^c and V^c are closed, from which it follows that $\overline{A} \subseteq V^c$ and $\overline{B} \subseteq U^c$. Then we have that $\overline{A} \cap B \subseteq V^c \cap V = \emptyset$ and $\overline{B} \cap A \subseteq U^c \cap U = \emptyset$, so that $\overline{A} \cap B = \emptyset = \overline{B} \cap A$. Therefore A and B are separated.

Hence E is disconnected. $\square$