1. Let a, b be integers which are relatively prime. Prove that $\gcd(a^2 - ab + b^2, a + b) \leq 3$.

Proof. Let a, b be coprime integers as given. Write $a^2 - ab + b^2$ equivalently as $(a + b)^2 - 3ab$. Then $\gcd(a^2 - ab + b^2, a + b) = \gcd((a + b)^2 - 3ab, a + b) = \gcd(-(a + b)^2 + 3ab, a + b)$, and apply one step of the Euclidean algorithm to find

$$-(a + b)^2 + 3ab = -(a + b) \cdot (a + b) + 3ab,$$

so that $\gcd(-(a + b)^2 + 3ab, a + b) = \gcd(a + b, 3ab)$. But because a, b were coprime, any divisor of $a + b$ will not divide a or b, and similarly, any divisor of a or b will not divide $a + b$. This is the same as saying that $\gcd(a + b, a) = \gcd(a + b, b) = \gcd(a + b, ab) = 1$, which makes sense from considering prime factorizations for a and b. Thus any common divisor of $a + b$ and $3ab$ is actually a common divisor of $a + b$ and 3.

So we may simplify $\gcd(a + b, 3ab)$ into $\gcd(a + b, 3)$, and because 3 is prime, the greatest common factor can either be 1 or 3. Hence $\gcd(a^2 - ab + b^2, a + b) \leq 3$. $\square$

2. Prove that two successive Fibonacci numbers are relatively prime.

Proof. Let the Fibonacci numbers start with $f_0 = 1, f_1 = 1$, and we have that $f_{n+2} = f_{n+1} + f_n$ for all $n \geq 0$. We have that $\gcd(1, 1) = \gcd(2, 1) = 1$, so the first couple pairs of successive Fibonacci numbers are relatively prime. Then suppose that $\gcd(f_j, f_{j-1}) = 1$ for all $0 \leq j \leq n$. We show that $\gcd(f_{n+1}, f_n) = 1$.

Writing $f_{n+1} = f_n + f_{n-1}$, we have that

$$\gcd(f_{n+1}, f_n) = \gcd(f_n + f_{n-1}, f_n),$$

which by the Euclidean algorithm we have that

$$\gcd(f_n + f_{n-1}, f_n) = \gcd(f_n, f_{n-1}).$$

But by the inductive hypothesis, the greatest common factor is 1, so $\gcd(f_{n+1}, f_n) = 1$.

Hence, by induction, any two successive Fibonacci numbers are relatively prime. $\square$