25.1 Consider a matrix operator

$$\Omega \equiv \begin{pmatrix} 1 & e^{i\theta} & 0 \\ e^{-i\theta} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The operator Ω satisfies the eigenvalue equation $\Omega \left|\lambda_i\right\rangle = \lambda_i \left|\lambda_i\right\rangle$, with eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 2$, and the corresponding set of orthonormal eigenvectors are

$$\left|\lambda_1\right\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\theta/2} \\ -e^{-i\theta/2} \\ 0 \end{pmatrix}, \quad \left|\lambda_2\right\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \left|\lambda_3\right\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\theta/2} \\ e^{-i\theta/2} \\ 0 \end{pmatrix},$$

such that $\langle\lambda_i|\lambda_j\rangle = \delta_{ij}$.

(a) Given a vector $\left|V\right\rangle \equiv \begin{pmatrix} e^{i\theta} \\ e^{-i\theta} \\ 1 \end{pmatrix}$, find the expectation value

$$\langle\Omega\rangle = \left\langle V\right| \Omega \left|V\right\rangle.$$

We have

$$\begin{aligned}
\langle\Omega\rangle = \left\langle V\right| \Omega \left|V\right\rangle &= \begin{pmatrix} e^{-\theta} & e^{i\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & e^{i\theta} & 0 \\ e^{-i\theta} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\theta} \\ e^{-i\theta} \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} e^{-\theta} & e^{i\theta} & 1 \end{pmatrix} \begin{pmatrix} e^{i\theta} + 1 \\ e^{-i\theta} + 1 \\ 1 \end{pmatrix} \\
&= (e^{-i\theta} + 1) + (e^{i\theta} + 1) + 1 = e^{i\theta} + e^{-i\theta} + 3.
\end{aligned}$$

(b) Expand the vector $\left|V\right\rangle$ defined in (a) in terms of the eigenvectors $\left|\lambda_i\right\rangle$; i.e. find the expansion parameters a_i in the expansion $\left|V\right\rangle = \sum_i a_i \left|\lambda_i\right\rangle$.

Taking inner products, we have

$$\begin{aligned}
\langle\lambda_1|V\rangle &= \frac{1}{\sqrt{2}}(e^{i\theta/2} - e^{-i\theta/2}) \\
\langle\lambda_2|V\rangle &= 1 \\
\langle\lambda_3|V\rangle &= \frac{1}{\sqrt{2}}(e^{i\theta/2} + e^{-i\theta/2})
\end{aligned}$$

so that

$$\left|V\right\rangle = \frac{1}{\sqrt{2}}(e^{i\theta/2} - e^{-i\theta/2}) \left|\lambda_1\right\rangle + \left|\lambda_2\right\rangle + \frac{1}{\sqrt{2}}(e^{i\theta/2} + e^{-i\theta/2}) \left|\lambda_3\right\rangle.$$

(c) Using the expansion $|V\rangle = \sum_i a_i |\lambda_i\rangle$, show that the expectation value $\langle\Omega\rangle$ van also be written as $\langle V|\,\Omega\,|V\rangle = \sum_i \lambda_i |a_i|^2$. Using a_i as obtained in (b), show that your result agrees with (a).

Since $|\lambda_i\rangle$ form an orthonormal eigenbasis, we have

$$\begin{aligned}
\langle\Omega\rangle = \langle V|\,\Omega\,|V\rangle &= \sum_i a_i \overline{a_i} \langle\lambda_i|\,\Omega\,|\lambda_i\rangle \\
&= \sum_i \lambda_i |a_i|^2 \langle\lambda_i|\lambda_i\rangle \\
&= \sum_i \lambda_i |a_i|^2 \\
&= (0)\cdot\frac{1}{2}(2-(e^{i\theta}+e^{-i\theta})) + (1)\cdot 1 + (2)\cdot\frac{1}{2}(2+e^{i\theta}+e^{-i\theta}) \\
&= e^{i\theta}+e^{-i\theta}+3,
\end{aligned}$$

which matches with the computation in (a).