1. (4.1) The central force problem.

   (a) Generalized momenta conjugate to generalized coordinates r and ϕ, where $\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r)$ are

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}$$
$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\dot{\phi}.$$

   (b) Apply the Legendre transformation to find

$$\mathcal{H} = \dot{r}\left(m\dot{r}\right) + \dot{\phi}\left(mr^2\dot{\phi}\right) - \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + V(r) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + V(r) = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + V(r),$$

   which by inspection is equivalent to the total energy. The quantity $\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2$ is the sum of the "radial" kinetic energy and the angular kinetic energy, and $V(r)$ is the potential energy.

   (c) Then from Hamilton's equations,

$$\dot{r} = \frac{\partial \mathcal{H}}{\partial p_r} = \frac{p_r}{m} \quad \text{(no information)}$$
$$-m\ddot{r} = -\dot{p_r} = \frac{\partial \mathcal{H}}{\partial r} = mr\dot{\phi}^2 + \frac{\partial V(r)}{\partial r} \quad \left(m\ddot{r} + mr\dot{\phi}^2 = f(r)\right)$$
$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{mr^2} \quad \text{(no information)}$$
$$-\dot{p_\phi} = \frac{\partial \mathcal{H}}{\partial \phi} = 0 \quad \left(p_\phi = \text{constant}\right).$$

   I have a sign error for some reason in the second equation, and I am unsure how to fix it. It should be equivalent to the equations obtained via the Lagrangian. The time derivative of p_ϕ is consistent with the result from the Lagrangian.

2. (4.2) Free particle.

   (a) Let $\mathcal{L} = -mc^2\sqrt{1 - \dot{x}^2/c^2}$, where c is the speed of light and define $\gamma = 1/\sqrt{1 - \dot{x}^2/c^2}$. Then the momentum p conjugate to x in terms of γ is

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}\left(1 - \frac{\dot{x}^2}{c^2}\right)^{-\frac{1}{2}} = m\dot{x}\gamma.$$

   (b) Then the Hamiltonian is given by

$$\mathcal{H} = \dot{x}\left(m\dot{x}\gamma\right) + mc^2\sqrt{1 - \dot{x}^2/c^2} = \left[m\dot{x} + mc^2\left(1 - \frac{\dot{x}}{c^2}\right)\right]\gamma = mc^2\gamma.$$

   (c) With some algebra, see that from the momentum equation that we have $p^2/(m^2c^2 + p^2) = \dot{x}^2/c^2$ and so in $mc^2\gamma$, we simplify

$$\mathcal{H} = \frac{mc^2}{\sqrt{1 - \dot{x}^2/c^2}} = \frac{mc^2}{\sqrt{1 - p^2/(m^2c^2 + p^2)}} = \frac{mc^2}{\sqrt{m^2c^2/(m^2c^2 + p^2)}} = mc^2\sqrt{\frac{p^2}{m^2c^2} + 1},$$

   and this last term is the total energy in terms of p alone (how the Hamiltonian should be written as there is no explicit dependence on $\dot{x}$).