1. Prove the following product rules

$$\vec{\nabla}(\psi\phi) = \phi\vec{\nabla}\psi + \psi\vec{\nabla}\phi \tag{1}$$
$$\vec{\nabla}\cdot(\psi\vec{A}) = \psi\vec{\nabla}\cdot\vec{A} + \vec{\nabla}\psi\cdot\vec{A} \tag{2}$$
$$\vec{\nabla}\times(\psi\vec{A}) = \psi\vec{\nabla}\times\vec{A} + \vec{\nabla}\psi\times\vec{A} \tag{3}$$

   (1) $(\vec{\nabla}(\psi\phi))_i = \partial_i(\psi\phi) = (\partial_i\psi)\phi + \psi(\partial_i\phi) \implies \vec{\nabla}(\psi\phi) = \phi\vec{\nabla}\psi + \psi\vec{\nabla}\phi$

   (2) $\vec{\nabla}\cdot(\psi\vec{A}) = \partial_i(\psi\vec{A})_i = \partial_i(\psi A_i) = (\partial_i A_i)\psi + (\partial_i\psi)A_i \implies \vec{\nabla}\cdot(\psi\vec{A}) = \psi\vec{\nabla}\cdot\vec{A} + \vec{\nabla}\psi\cdot\vec{A}$

   (3) $(\vec{\nabla}\times(\psi\vec{A}))_i = \epsilon_{ijk}\partial_j(\psi\vec{A})_k = \epsilon_{ijk}\partial_j(\psi A_k) = \epsilon_{ijk}[(\partial_j A_k)\psi + (\partial_j\psi)A_k] = \epsilon_{ijk}(\partial_j A_k)\psi + \epsilon_{ijk}(\partial_j\psi)A_k$ $\implies \vec{\nabla}\times(\psi\vec{A}) = \psi\vec{\nabla}\times\vec{A} + \vec{\nabla}\psi\times\vec{A}$

2. Quantum mechanical orbital angular momentum(?) is given by $\vec{L} = \vec{r}\times -i\vec{\nabla}$.

   (a) Show that $\vec{L}$ has components $L_x = -i(y\partial_z - z\partial_y), L_y = -i(z\partial_x - x\partial_z), L_z = -i(x\partial_y - y\partial_x)$.

   Keep in mind that the subscript i is different from the complex number i. We have that $\vec{r} = (x, y, z)$. Thus $L_i = (\vec{r}\times -i\vec{\nabla})_i = -i\epsilon_{ijk}r_j\partial_k = -i(r_j\partial_k - r_k\partial_j)$. Thus by selecting i, j, k cyclically in x, y, z we can obtain the components as stated above.

   (b) Show that these components satisfy the "commutation relation" $[L_x, L_y] \equiv L_xL_y - L_yL_x = iL_z$.

   Directly, we have that

$$\begin{aligned}
[L_x, L_y] \equiv L_xL_y - L_yL_x &= -i(y\partial_z - z\partial_y)\cdot -i(z\partial_x - x\partial_z) - -i(z\partial_x - x\partial_z)\cdot -i(y\partial_z - z\partial_y) \\
&= -(y\partial_z z\partial_x - z\partial_y z\partial_x - y\partial_z x\partial_z + z\partial_y x\partial_z) + (z\partial_x y\partial_z - x\partial_z y\partial_z - z\partial_x z\partial_y + x\partial_z z\partial_y) \\
&= -(y\partial_x + yz\partial_z\partial_x - z^2\partial_y\partial_x - yx\partial_z\partial_z + zx\partial_y\partial_z) + (zy\partial_x\partial_z - xy\partial_z\partial_z - z^2\partial_x\partial_y + x\partial_y + xz\partial_z\partial_y) \\
&= -y\partial_x + x\partial_y = i(-i(x\partial_y - y\partial_x)) = iL_z,
\end{aligned}$$

   where mixed second partial derivative operators are equivalent when they act on twice continuously differentiable functions.