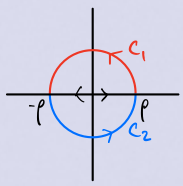
Homework 22 — Sai Sivakumar

22.1 Complete the calculation in Example 3 for both I_+' and I_-' by choosing appropriate contours where the semicircles can be neglected, and find I.

$I = \int_{\mathbb{R}} \frac{\cos(x)}{1+x^2}\,dx$. Then compute $\int_{\mathbb{R}} \frac{(e^{iz}+e^{-iz})\,dz}{2(1+z^2)}\,dz$ by splitting it up into two integrals: $\frac{1}{2}\int_{\mathbb{R}} \frac{e^{iz}}{(1+z^2)}\,dz + \frac{1}{2}\int_{\mathbb{R}} \frac{e^{-iz}}{(1+z^2)}\,dz$ and in the first integral close the contour with a semicircle in the upper half plane, and in the second integral close the contour with a semicircle in the lower half plane:



$$I_1 = \frac{1}{2}\int_{\mathbb{R}} \frac{e^{iz}}{(1+z^2)}\,dz = \lim_{\rho\to\infty}\left[\frac{1}{2}\int_{-\rho}^{\rho} \frac{e^{iz}}{(1+z^2)}\,dz + \frac{1}{2}\int_{C_1} \frac{e^{iz}}{(1+z^2)}\,dz\right]$$ since as $\rho\to\infty$, $|e^{iz}| \to 0$. ($= |e^{ix}||e^{-y}|,\ y\to\infty$)

$$I_2 = \frac{1}{2}\int_{\mathbb{R}} \frac{e^{-iz}}{(1+z^2)}\,dz = \lim_{\rho\to\infty}\left[-\frac{1}{2}\int_{\rho}^{-\rho} \frac{e^{-iz}}{(1+z^2)}\,dz - \frac{1}{2}\int_{C_2} \frac{e^{-iz}}{(1+z^2)}\,dz\right]$$ since as $\rho\to\infty$, $|e^{-iz}| \to 0$. ($= |e^{i(-x)}||e^{y}|,\ y\to-\infty$)

Then by the Residue theorem (poles $\pm i$),

$I_1 = \frac{2\pi i}{2}\left(\frac{e^{-1}}{2i}\right)$, $I_2 = \frac{-2\pi i}{2}\left(\frac{e^{-1}}{-2i}\right)$ so that $I = I_1 + I_2 = \pi/e$.

22.2 Use contour integration to evaluate the integral

$$\int_0^\infty \frac{\cos(kx)}{4x^4+5x^2+1}\,dx;\quad k>0.$$

Write the integral as

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(kx)}{(4x^2+1)(x^2+1)}\,dx = \frac{1}{4}\int_{\mathbb{R}} \frac{e^{ikz}\,dz}{(4x^2+1)(x^2+1)} + \frac{1}{4}\int_{\mathbb{R}} \frac{e^{-ikz}\,dz}{(4x^2+1)(x^2+1)}.$$

so that it is easy to see the poles are $z = \pm \frac{i}{2}, \pm i$, so that we close the contour(s) with a semicircular arc in the upper and lower half planes, similar to 22.1.

Hence

$$I = \frac{1}{4}\oint \frac{e^{ikz}\,dz}{(4z^2+1)(z^2+1)} - \frac{1}{4}\oint \frac{e^{-ikz}\,dz}{(4z^2+1)(z^2+1)} = \frac{2\pi i}{4}\left[\frac{e^{-k}}{-6i} - \frac{e^{-k}}{6i} + \frac{e^{-k/2}}{3i} - \frac{e^{-k/2}}{-3i}\right]$$

$$= \frac{\pi}{3e^{k/2}} - \frac{\pi}{6e^{k}}$$

22.3 Choose C_2 counterclockwise in Example 4. By the residue theorem,

$$\lim_{\substack{R\to\infty\\ r\to 0}} \operatorname{Im}\left[\oint \frac{e^{iz}dz}{z}\right] = \operatorname{Im}\left[\lim_{R\to\infty}\int_{C_1+C_3} z^{-1}e^{iz}dz + \lim_{R\to\infty}\int_{C_4} z^{-1}e^{iz}dz + \lim_{r\to 0}\int_{C_2} z^{-1}e^{iz}dz\right]$$

$$= \operatorname{Im}\left[\int_{\mathbb{R}} z^{-1}e^{iz}dz + (0) + \lim_{r\to 0}\int_{\pi}^{2\pi} ie^{re^{i\theta}}\,d\theta\right]$$

$$= \operatorname{Im}\left[\int_{\mathbb{R}} z^{-1}e^{iz}dz + \int_{\pi}^{2\pi} i\,d\theta\right] = \operatorname{Im}(2\pi i(1))$$

Thus $\operatorname{Im}\left[\int_{\mathbb{R}} z^{-1}e^{iz}dz\right] = \int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx = \pi = \operatorname{Im}(2\pi i - \pi i)$. The result is the same.