p.88: 2, 8, 9, 10, 11, 14

2. Prove that the vector space $\ell^2(\mathbb{Z})$ over $\mathbb{C}$ is complete.

Proof. Suppose $A_k = \{a_{k,n}\}_{n\in\mathbb{Z}}$ with $k = 1, 2, \ldots$ is a Cauchy sequence.

For each $n \in \mathbb{Z}$, observe that the collection of n-th terms in each of the sequences A_k for $k = 1, 2, \ldots$ form a Cauchy sequence $\{a_{k,n}\}_{k=1}^{\infty}$. This is because for each $n \in \mathbb{Z}$ and $\varepsilon > 0$, there exists $K \in \mathbb{Z}^+$ such that for $k, k' > K$,

$$\left|a_{k,n} - a_{k',n}\right|^2 \leq \|A_k - A_{k'}\|^2 = \sum_{n\in\mathbb{Z}} \left|a_{k,n} - a_{k',n}\right|^2 < \varepsilon^2,$$

implying that $\left|a_{k,n} - a_{k',n}\right| < \varepsilon$. Then because $\{a_{k,n}\}_{k=1}^{\infty}$ is Cauchy, it converges to some $b_n \in \mathbb{C}$. In particular, this means that the sequence A_k converges to some sequence $B = \{b_n\}_{n\in\mathbb{Z}}$. To see this, observe that for any given $\varepsilon > 0$, we may choose a positive integer K large enough so that for all $k, k' > K$, the partial sums $\sum_{n=-N}^{N} \left|a_{k,n} - a_{k',n}\right|^2$ of $\|A_k - A_{k'}\|^2$ satisfy

$$\sum_{n=-N}^{N} \left|a_{k,n} - a_{k',n}\right|^2 \leq \|A_k - A_{k'}\|^2 < \varepsilon^2,$$

for any positive integer N. Then let k' tend to positive infinity, so that

$$\sum_{n=-N}^{N} |a_{k,n} - b_n|^2 < \varepsilon^2.$$

Because N was arbitrary, it follows that $\|A_k - B\| < \varepsilon$ for $k > K$, so A_k does converge to B.

The vector space $\ell^2(\mathbb{Z})$ is complete if we show that $B \in \ell^2(\mathbb{Z})$; that is, $\|B\|^2$ is finite. By the triangle inequality, $\|B\| = \|B - A_k + A_k\| \leq \|B - A_k\| + \|A_k\|$. But $\|A_k\|$ and $\|B - A_k\|$ are finite, so $B \in \ell^2(\mathbb{Z})$.

Hence $\ell^2(\mathbb{Z})$ over $\mathbb{C}$ is complete. □

8.

(a) The Fourier coefficients of $f(\theta) = |\theta|$ are $a_0 = \pi/2$ and $a_n = ((-1)^n + 1)/(\pi n^2)$ for $n \neq 0$. Then because a_n vanishes for nonzero even n, we have by Parseval's identity that

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} |\theta|^2 \,\mathrm{d}\theta = \frac{\pi^2}{3} &= \left(\frac{\pi}{2}\right)^2 + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \left(\frac{(-1)^n - 1}{\pi n^2}\right)^2 \\ &= \frac{\pi^2}{4} + \frac{8}{\pi^2}\sum_{n \text{ odd } \geq 1} \frac{1}{n^4},\end{aligned}$$

so that

$$\sum_{n \text{ odd } \geq 1} \frac{1}{n^4} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.$$

Then observe that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^4} &= \sum_{n \text{ even } \geq 2}\frac{1}{n^4} + \sum_{n \text{ odd } \geq 1}\frac{1}{n^4}\\ &= \frac{1}{16}\sum_{n=1}^{\infty}\frac{1}{n^4} + \frac{\pi^4}{96},\end{aligned}$$

which implies that

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

(b) The Fourier coefficients of the odd function defined on $[0,\pi]$ by $f(\theta) = \theta(\pi-\theta)$ are given by $-4i/(\pi n^3)$ for odd $n \in \mathbb{Z}$. Then

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|^2\,\mathrm{d}\theta &= \frac{1}{2\pi}\int_{-\pi}^{0}\theta^2(\pi+\theta)^2\,\mathrm{d}\theta + \frac{1}{2\pi}\int_{0}^{\pi}\theta^2(\pi-\theta)^2\,\mathrm{d}\theta\\ &= \frac{1}{\pi}\int_0^{\pi}\theta^2(\pi-\theta)^2\,\mathrm{d}\theta\\ &= \frac{\pi^4}{30},\end{aligned}$$

so that

$$\begin{aligned}\frac{\pi^4}{30} = \sum_{n \text{ odd } \geq 1}\left|\frac{-4i}{\pi n^3}\right|^2 &= 2\sum_{n \text{ odd } \geq 1}\left(\frac{4}{\pi n^3}\right)^2\\ &= \frac{32}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^6}.\end{aligned}$$

Hence

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^6} = \frac{\pi^6}{960}.$$

Then observe

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^6} &= \sum_{n \text{ even } \geq 2}\frac{1}{n^6} + \sum_{n \text{ odd } \geq 1}\frac{1}{n^6}\\ &= \frac{1}{64}\sum_{n=1}^{\infty}\frac{1}{n^6} + \frac{\pi^6}{960},\end{aligned}$$

which implies that

$$\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945}.$$

9. The Fourier coefficients of $\frac{\pi}{\sin(\pi\alpha)}e^{i(\pi-x)\alpha}$ are

$$\begin{aligned} a_n = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\pi}{\sin(\pi\alpha)}e^{i(\pi-x)\alpha}\right)e^{-inx}\,\mathrm{d}x &= \frac{e^{i\pi\alpha}}{2\sin(\pi\alpha)}\int_0^{2\pi} e^{-i(n+\alpha)x}\,\mathrm{d}x \\ &= \frac{e^{i\pi\alpha}}{2\sin(\pi\alpha)}\left(\frac{e^{-i\alpha x}}{-i(n+\alpha)}\bigg|_0^{2\pi}\right) \\ &= \frac{e^{-i\pi\alpha}-e^{i\pi\alpha}}{-2i\sin(\pi\alpha)(n+\alpha)} \\ &= \frac{1}{n+\alpha}. \end{aligned}$$

Hence the Fourier series of $\frac{\pi}{\sin(\pi\alpha)}e^{i(\pi-x)\alpha}$ is given by

$$\frac{\pi}{\sin(\pi\alpha)}e^{i(\pi-x)\alpha} \sim \sum_{n=-\infty}^{\infty}\frac{e^{inx}}{n+\alpha}.$$

Hence by applying Parseval's formula we have

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\pi e^{i(\pi-x)\alpha}}{\sin(\pi\alpha)}\right|^2\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\pi}{\sin(\pi\alpha)}\right)^2\mathrm{d}x = \frac{\pi^2}{\sin^2(\pi\alpha)} &= \sum_{n=-\infty}^{\infty}\left|\frac{e^{inx}}{n+\alpha}\right|^2 \\ &= \sum_{n=-\infty}^{\infty}\frac{1}{(n+\alpha)^2}. \end{aligned}$$

10. Show that the total energy $E(t) = \frac{1}{2}\rho\int_0^L\left(\frac{\partial u}{\partial t}\right)^2\mathrm{d}x + \frac{1}{2}\tau\int_0^L\left(\frac{\partial u}{\partial x}\right)^2\mathrm{d}x$ of a vibrating string whose displacement $u(x,t)$ satisfies the wave equation $\rho u''_{tt} = \tau u''_{xx}$ with initial conditions $u(x,0) = f(x)$ and $u'_t(x,0) = g(x)$ is constant.

Proof. The total energy is constant if $E'(t)$ vanishes. With $u(x,t)$ smooth enough and satisfying the wave equation, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\rho\int_0^L\left(\frac{\partial u}{\partial t}\right)^2\mathrm{d}x + \frac{1}{2}\tau\int_0^L\left(\frac{\partial u}{\partial x}\right)^2\mathrm{d}x\right) \\ &= \frac{1}{2}\rho\int_0^L\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial u}{\partial t}\right)^2\mathrm{d}x + \frac{1}{2}\tau\int_0^L\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial u}{\partial x}\right)^2\mathrm{d}x \\ &= \rho\int_0^L\frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t}\,\mathrm{d}x + \tau\int_0^L\frac{\partial^2 u}{\partial x\partial t}\frac{\partial u}{\partial x}\,\mathrm{d}x \\ &= \rho\int_0^L\frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t}\,\mathrm{d}x + \tau\int_0^L\frac{\partial^2 u}{\partial t\partial x}\frac{\partial u}{\partial x}\,\mathrm{d}x \\ &= \rho\int_0^L\frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t}\,\mathrm{d}x + \tau\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\bigg|_0^L\right) - \tau\int_0^L\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial t}\,\mathrm{d}x \\ &= \rho\int_0^L\frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t}\,\mathrm{d}x + \tau\left(\frac{\partial u}{\partial x}(L,t)\frac{\partial u}{\partial t}(L,t) - \frac{\partial u}{\partial x}(0,t)\frac{\partial u}{\partial t}(0,t)\right) - \tau\int_0^L\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial t}\,\mathrm{d}x \\ &= \rho\int_0^L\frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t}\,\mathrm{d}x - \tau\int_0^L\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial t}\,\mathrm{d}x = \int_0^L\frac{\partial u}{\partial t}\left(\rho\frac{\partial^2 u}{\partial t^2} - \tau\frac{\partial^2 u}{\partial x^2}\right)\mathrm{d}x = \int_0^L 0\,\mathrm{d}x = 0, \end{aligned}$$

where we used integration by parts and the periodicity of $u'(x,t)$ in x to arrive at $E'(t)=0$, which means that $E(t)$ is constant for all time, and in particular, is equal to $E(0)$. $\square$

11.

(a) *Proof.* Let f be T-periodic, continuous, and piecewise C^1 with $\int_0^T f(t)\,\mathrm{d}t=0$ as given. Then observe that the condition $\int_0^T f(t)\,\mathrm{d}t=0$ yields $\hat{f}(0)=0$. Since f is C^1, we can use integration by parts to find that for $n\neq 0$,

$$\begin{aligned} a_n=\hat{f}(n)&=\frac{1}{T}\int_0^T f(t)e^{int(2\pi/T)}\,\mathrm{d}t\\ &=f(t)\frac{Te^{int(2\pi/T)}}{-2\pi in}\Bigg|_0^T+\left(\frac{T}{2\pi in}\right)\frac{1}{T}\int_0^T f'(t)e^{int(2\pi/T)}\,\mathrm{d}t\\ &=\frac{T}{2\pi in}\widehat{f'}(n)=\frac{T}{2\pi in}b_n. \end{aligned}$$

Then we can apply Parseval's identity with $\hat{f}(0)=0$ to $\int_0^T |f(t)|^2\,\mathrm{d}t$ to find

$$\begin{aligned} \int_0^T |f(t)|^2\,\mathrm{d}t&=T\sum_{n\neq 0}|a_n|^2\\ &=T\sum_{n\neq 0}\left|\frac{T}{2\pi in}b_n\right|^2\\ &=\frac{T^3}{4\pi^2}\sum_{n\neq 0}\frac{|b_n|^2}{n^2}\\ &\le\frac{T^3}{4\pi^2}\sum_{n=-\infty}^{\infty}|b_n|^2\\ &=\frac{T^2}{4\pi^2}\int_0^T\left|f'(t)\right|^2\mathrm{d}t\,. \end{aligned}$$

We have equality if for $|n|\geq 2$, $a_n=0$ (we also need $b_0=0$). This is because we must have $n^2=1$ for $n=\pm 1$, and this requirement also forces $b_0=0$ since $f(t)=a_1e^{int(2\pi/T)}+a_{-1}e^{-int(2\pi/T)}=A\sin(2\pi t/T)+B\cos(2\pi t/T)$, whose derivative is clearly periodic over T as well. $\square$

(b) *Proof.* Let g be C^1 and T-periodic as given, with $a_n=\hat{f}(n)$, $b_n=\hat{g}(n)$, and $c_n=\widehat{g'}(n)$. Note that for $n\neq 0$,

$b_n = \frac{T}{2\pi i n} c_n$. Then with $a_0 = 0$ and using the Cauchy-Schwarz inequality,

$$\begin{aligned}
\left| \int_0^T \overline{f(t)} g(t) \, \mathrm{d}t \right|^2 &= T^2 \left| \sum_{n=-\infty}^{\infty} \overline{a_n} b_n \right|^2 = T^2 \left| \sum_{n \neq 0} \overline{a_n} b_n \right|^2 \\
&\leq \left(T \sum_{n \neq 0} |a_n|^2 \right) \left(T \sum_{n \neq 0} |b_n|^2 \right) \\
&\leq \left(T \sum_{n=-\infty}^{\infty} |a_n|^2 \right) \left(\frac{T^3}{4\pi^2} \sum_{n \neq 0} |c_n|^2 + T c_0 \right) \\
&\leq \int_0^T |f(t)|^2 \, \mathrm{d}t \left(\frac{T^3}{4\pi^2} \sum_{n=-\infty}^{\infty} |c_n|^2 \right) \\
&= \frac{T^2}{4\pi^2} \int_0^T |f(t)|^2 \, \mathrm{d}t \int_0^T \left| g'(t) \right|^2 \mathrm{d}t \,,
\end{aligned}$$

so $\left| \int_0^T \overline{f(t)} g(t) \, \mathrm{d}t \right|^2 \leq \frac{T^2}{4\pi^2} \int_0^T |f(t)|^2 \, \mathrm{d}t \int_0^T |g'(t)|^2 \, \mathrm{d}t$ as desired. □

(c) *Proof.* Let $[a, b]$ be any compact interval, and let f be any continuously differentiable function such that $f(a) = f(b) = 0$ as given. Then extend f to be an odd function centered at a (that is, $f(t + a)$ is an odd function about the origin) and periodic with period $T = 2(b - a)$, which means any integral over an interval of length T, like $\int_{2a-b}^{b} f(t) \, \mathrm{d}t$, vanishes. Then this extended f satisfies the hypotheses of part (a) up to a change of variables (translation), so

$$\int_a^b |f(t)|^2 \, \mathrm{d}t = \frac{1}{2} \int_{2a-b}^{b} |f(t)|^2 \, \mathrm{d}t \leq \frac{(b-a)^2}{2\pi^2} \int_{2a-b}^{b} \left| f'(t) \right|^2 \mathrm{d}t = \frac{(b-a)^2}{\pi^2} \int_a^b \left| f'(t) \right|^2 \mathrm{d}t \,,$$

where by construction, $|f'(t)|^2$ is symmetric (even) about a in an interval of length T. In the case of equality, from part (a) we have $f(t) = A \sin(2\pi(t - a)/T) + B \cos(2\pi(t - a)/T)$ which is centered at a, but it should not have the cosine term since f is odd about a. Then $f(t) = A \sin(2\pi(t - a)/T) = A \sin\left(\pi \frac{t-a}{b-a}\right)$. □

14. Prove that the Fourier series of a continuously differentiable function f on the circle is absolutely convergent.

Proof. With $a_n = \hat{f}(n)$ and $b_n = \widehat{f'}(n)$, the Fourier series for f is given by $\sum_{n=-\infty}^{\infty} a_n e^{int}$. Note that $a_n = (in)^{-1} b_n$ for $n \neq 0$, and observe that the Fourier series converges if the series $\sum_{n \neq 0} a_n e^{int}$ converges (removing a finite number of terms does not affect convergence). Then

$$\begin{aligned}\left|\sum_{n\neq 0} a_n e^{int}\right| \leq \sum_{n\neq 0} |a_n| &= \sum_{n\neq 0} \left|(in)^{-1} b_n\right| \\ &= \sum_{n\neq 0} |n|^{-1} |b_n| \\ &\leq \left(2\sum_{n\geq 1} |n|^{-2}\right)^{\frac{1}{2}} \left(\sum_{n\neq 0} |b_n|^2\right)^{\frac{1}{2}} \\ &\leq \sqrt{\frac{\pi^2}{3} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|f'(t)\right|^2 \mathrm{d}t} < \infty.\end{aligned}$$

Hence the Fourier series of f converges absolutely. $\square$