HOMEWORK 3

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As a suggested problem, prove the following proposition.

Proposition 0.1. *Suppose $X \subseteq Y \subseteq \mathbb{R}$. If X is not empty and Y is bounded above, then X and Y both have suprema and*

$$\sup(X) \leq \sup(Y).$$

Let $A_1, A_2, \ldots$ be a collection of subsets of $\mathbb{R}$ such that A_n is not empty for each n. Suppose $A = \cup_{n=1}^{\infty} A_n$ is bounded above. Since, $A_n \subseteq A$, from the Proposition above, A, and each A_n, has a supremum and moreover $\sup(A_n) \leq \sup(A)$ for all $n \in \mathbb{N}$.

Let $\alpha_n = \sup(A_n)$ (thus $\alpha_n \leq \sup(A)$). Let $B = \{\alpha_n : n \in \mathbb{N}\}$ and note $\sup(A)$ is an upper bound for B. Since B is not empty, it follows that B has a supremum and moreover $\sup(B) \leq \sup(A)$.

For homework 3, complete the steps below (or otherwise) to show

$$\sup(A) = \sup(B).$$

(i) Show, if $y < \sup(A)$, then there is a $k \in \mathbb{N}$ that $y < \alpha_k$.

Since $y < \sup(A)$, it follows that there exists $x \in A$ such that $y < x < \sup(A)$. Then since $x \in A$, there exists $k \in \mathbb{N}$ such that $x \in A_k$. So $y < x \leq \alpha_k$.

(ii) Show, if $y < \sup(A)$, then $y < \sup(B)$.

If $y < \sup(A)$, we saw that there exists $k \in \mathbb{N}$ such that $y < \alpha_k$. Since $\alpha_k \leq \sup(B)$, it follows that $y < \sup(B)$.

(iii) Show $\sup(A) \leq \sup(B)$.

If $y < \sup(A)$ implies $y < \sup(B)$, then it follows that $\sup(A) \leq \sup(B)$.

Hence $\sup(A) = \sup(B)$. □