The inverse Laplace transform

via the Bromwich integral

Sai Sivakumar

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outline

1. the Laplace transform
2. the inversion problem
3. a cute example

the Laplace transform

brief summary

Theorem (Laplace transform)

Let $F(t)$ be sectionally continuous on every finite interval $0 \leq t \leq T$ and also of exponential order, so that $|F(t)| \in O(e^{at})$ where $t \geq 0$.

Then for complex s the Laplace transform of $F(t)$ is

$$f(s) = \int_0^\infty e^{-st} F(t)\,\mathrm{d}t$$

and is analytic in the half plane where $\mathrm{Re}(s) > a$

properties of the transform

Linearity: $\mathcal{L}[f(t)+g(t)] = \mathcal{L}[f(t)] + \mathcal{L}[g(t)]$

Time shift: $\mathcal{L}[f(t-a)u(t-a)] = e^{-as}\mathcal{L}[f(t)]$, $a \geq 0$

Complex shift: $\mathcal{L}[e^{at}f(t)](s) = \mathcal{L}[f(t)](s-a)$

Transform of the n-th derivative:

$$\mathcal{L}[f^{(n)}(t)](s) = s^n\mathcal{L}[f(t)](s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

Convolution: $\mathcal{L}[(f*g)(t)] = \mathcal{L}[f(t)]\mathcal{L}[g(t)]$

non-unique inverses?

If two functions agree everywhere except at countably many points, their Laplace transforms will be the same.

Furthermore, if two continuous functions have the same Laplace transform, then they must be the same function.

See Lerch's theorem for more.

the inversion problem

using a table?

I guess so...

A TABLE OF LAPLACE TRANSFORMS

$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$	$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$
1. $f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$	**20.** $\frac{1}{\sqrt{t}}$	$\frac{\sqrt{\pi}}{\sqrt{s}}$
2. $e^{at}f(t)$	$F(s-a)$	**21.** $\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$
3. $f'(t)$	$sF(s)-f(0)$	**22.** $t^{n-(1/2)}, \quad n=1,2,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+(1/2)}}$
4. $f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)$	**23.** $t^r, \quad r>-1$	$\frac{\Gamma(r+1)}{s^{r+1}}$
	$-\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$	**24.** $\sin bt$	$\frac{b}{s^2+b^2}$
5. $t^nf(t)$	$(-1)^nF^{(n)}(s)$	**25.** $\cos bt$	$\frac{s}{s^2+b^2}$
6. $\frac{1}{t}f(t)$	$\int_s^\infty F(u)du$	**26.** $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
7. $\int_0^t f(v)dv$	$\frac{F(s)}{s}$	**27.** $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
8. $(f*g)(t)$	$F(s)G(s)$	**28.** $\sinh bt$	$\frac{b}{s^2-b^2}$
9. $f(t+T)=f(t)$	$\frac{\int_0^T e^{-st}f(t)dt}{1-e^{-sT}}$	**29.** $\cosh bt$	$\frac{s}{s^2-b^2}$
10. $f(t-a)u(t-a),\ a\geq 0$	$e^{-as}F(s)$	**30.** $\sin bt - bt\cos bt$	$\frac{2b^3}{(s^2+b^2)^2}$
11. $g(t)u(t-a), \quad a\geq 0$	$e^{-as}\mathscr{L}\{g(t+a)\}(s)$	**31.** $t\sin bt$	$\frac{2bs}{(s^2+b^2)^2}$
12. $u(t-a), \quad a\geq 0$	$\frac{e^{-as}}{s}$	**32.** $\sin bt + bt\cos bt$	$\frac{2bs^2}{(s^2+b^2)^2}$
13. $\prod_{a,b}(t), \quad 0<a<b$	$\frac{e^{-sa}-e^{-sb}}{s}$	**33.** $t\cos bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
14. $\delta(t-a), \quad a\geq 0$	e^{-as}	**34.** $\sin bt\cosh bt - \cos bt\sinh bt$	$\frac{4b^3}{s^4+4b^4}$
15. e^{at}	$\frac{1}{s-a}$	**35.** $\sin bt\sinh bt$	$\frac{2b^2s}{s^4+4b^4}$
16. $t^n, \quad n=1,2,\ldots$	$\frac{n!}{s^{n+1}}$	**36.** $\sinh bt - \sin bt$	$\frac{2b^3}{s^4-b^4}$
17. $e^{at}t^n, \quad n=1,2,\ldots$	$\frac{n!}{(s-a)^{n+1}}$	**37.** $\cosh bt - \cos bt$	$\frac{2b^2s}{s^4-b^4}$
18. $e^{at}-e^{bt}$	$\frac{(a-b)}{(s-a)(s-b)}$	**38.** $J_\nu(bt), \nu>-1$	$\frac{(\sqrt{s^2+b^2}-s)^\nu}{b^\nu\sqrt{s^2+b^2}}$
19. $ae^{at}-be^{bt}$	$\frac{(a-b)s}{(s-a)(s-b)}$		

handwavey motivation

We can start by extending Cauchy's integral formula for a specific class of functions.

Recall that Cauchy's integral formula gives you the value of an analytic function f at a point z_0 inside a simple closed contour C by the following:

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} \, \mathrm{d}z$$

extending Cauchy's integral formula

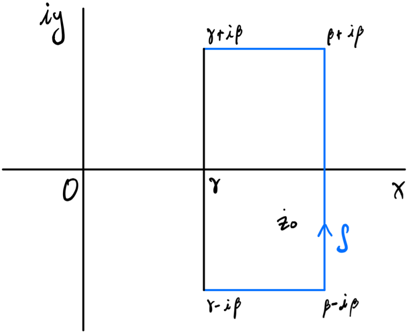
Theorem (Cauchy integral formula extension)

Let $f(z)$ *be analytic where* $\mathrm{Re}(z) \geq \gamma$ *and also be of order* $O(z^{-k})$*, where* γ, k *are real and* $k > 0$*.*

Then if z_0 *is a complex number where* $\mathrm{Re}(z_0) > \gamma$*,*

$$f(z_0) = -\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{f(z)}{z - z_0}\,\mathrm{d}z\,.$$

Construct the contour S:



Apply Cauchy's integral formula to find

$$f(z_0) = \frac{1}{2\pi i}\left[-\int_{\gamma-i\beta}^{\gamma+i\beta} \frac{f(z)}{z-z_0}\,\mathrm{d}z + \int_S \frac{f(z)}{z-z_0}\,\mathrm{d}z\right].$$

proof

To show that the second integral vanishes as β goes to infinity, use a bounding argument.

From the order condition imposed on $f(z)$, it follows that

$$\left|\frac{f(z)}{z-z_0}\right| \leq \frac{M}{|z|^k|z-z_0|} = \frac{M}{|z|^{k+1}|1-(z_0/z)|},$$

where M is the maximum value of $f(z)$ on S.

proof

Note that $|z| \geq \beta$, since $z \in S$.

If we take β to be large enough, say $\beta > 2|z_0|$, so that $|z_0| < |z|/2$ or otherwise $|z_0/z| < 1/2$, then by the reverse triangle inequality (or otherwise) see that

$$|1 - (z_0/z)| \geq ||1| - |z_0/z|| > 1 - |z_0/z| > 1/2.$$

proof

We can then bound the integrand above, so that

$$\left|\frac{f(z)}{z-z_0}\right| < \frac{2M}{|z|^{k+1}} \leq \frac{2M}{\left(\sqrt{2}\beta\right)^{k+1}} < \frac{2M}{\beta^{k+1}}.$$

See that the length of S is $\beta + \beta + (\beta - \gamma) + (\beta - \gamma) = 4\beta - 2\gamma$.

proof

We can bound the second integral, so that

$$\left|\int_S \frac{f(z)}{z - z_0}\,\mathrm{d}z\right| < \frac{2M}{\beta^{k+1}} \int_S |\mathrm{d}z| = \frac{2M}{\beta^k}\left(4 - \frac{2\gamma}{\beta}\right).$$

Then since $k > 0$, if we let β become arbitrarily large, the second integral vanishes.

proof

From Cauchy's integral formula we had that

$$f(z_0) = \frac{1}{2\pi i}\left[-\int_{\gamma-i\beta}^{\gamma+i\beta} \frac{f(z)}{z-z_0}\,\mathrm{d}z + \int_S \frac{f(z)}{z-z_0}\,\mathrm{d}z\right],$$

but because β was arbitrary, we may take $\beta \to \infty$, so that

$$f(z_0) = -\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{f(z)}{z-z_0}\,\mathrm{d}z\,.$$

This completes the proof.

handwaving begins here

Let $f(s)$ be analytic and of order $O(s^{-k}), k > 0$ for all points where $\mathrm{Re}(s) \geq \gamma$. Then from the extension to Cauchy's integral formula we have

$$f(s) = -\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{f(z)}{z - s} \, \mathrm{d}z \,,$$

which we can rewrite as

$$f(s) = \lim_{\beta \to \infty} \frac{1}{2\pi i} \int_{\gamma - i\beta}^{\gamma + i\beta} \left(\frac{1}{s - z} \right) f(z) \, \mathrm{d}z \,.$$

(holds for all s such that $\mathrm{Re}(s) > \gamma$)

handwaving ends after this

Then apply the inverse Laplace transform **formally** to $f(s)$, we have that

$$F(t) = \mathcal{L}^{-1}\left[f(s)\right] = \lim_{\beta\to\infty} \frac{1}{2\pi i} \int_{\gamma-i\beta}^{\gamma+i\beta} \mathcal{L}^{-1}\left[\frac{1}{s-z}\right] f(z)\,\mathrm{d}z$$

$$= \lim_{\beta\to\infty} \frac{1}{2\pi i} \int_{\gamma-i\beta}^{\gamma+i\beta} e^{zt} f(z)\,\mathrm{d}z\,.$$

This does not prove anything! it is just cute

the inversion integral

Theorem (Laplace inversion integral)

Let $f(s)$ *(*$s = x + iy$*) be analytic and of order* $O(s^{-k}), k > 1$ *where* $x \geq x_0$*, and also let* $f(x)$ *be real valued for* $x \geq x_0$*.*

Then for any $\gamma \geq x_0$*,*

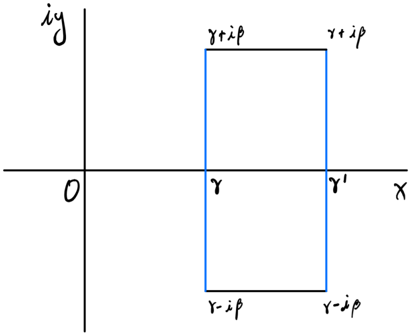
$$F(t) = \lim_{\beta \to \infty} \frac{1}{2\pi i} \int_{\gamma - i\beta}^{\gamma + i\beta} e^{zt} f(z) \, \mathrm{d}z \,,$$

where $F(t)$ *is a real valued function independent of* γ*.*

Furthermore, the Laplace transform of $F(t)$ *is indeed* $f(s)$*, and* $F(t)$ *is of order* $O(e^{x_0 t})$*.*

independence of choice of γ?

We can pick a second path where instead of γ we choose γ', where $\gamma' > \gamma$. Form a rectangular contour like so:



Observe that the integral around the rectangle is zero since $f(s)$ is analytic on the rectangle and in it.

inspect the horizontal bits

For points on BC, $z = x + i\beta$ (so $|z| > \beta$ if $\gamma > 0$), and from the order condition on $f(s)$ we have that

$$\left|e^{zt} f(z)\right| \leq e^{xt} |f(z)| \leq e^{xt} \frac{M}{|z|^k} < e^{xt} \frac{M}{\beta^k},$$

and so we can bound the integral over BC:

$$\left| \int_{\gamma + i\beta}^{\gamma' + i\beta} e^{zt} f(z) \, \mathrm{d}z \right| < \frac{M}{\beta^k} \int_{\gamma}^{\gamma'} e^{xt} \, \mathrm{d}x \, .$$

the horizontal parts vanish

Since β was arbitrary we may take $\beta \to \infty$, and see that the integrals over BC and similarly AD vanish.

Since the integral around the rectangle was zero, we have that

$$\lim_{\beta\to\infty}\left[\int_{\gamma-i\beta}^{\gamma+i\beta} e^{zt} f(z)\,\mathrm{d}z + \int_{\gamma'+i\beta}^{\gamma'-i\beta} e^{zt} f(z)\,\mathrm{d}z\right] = 0.$$

independence

So by rearranging and multiplying by $(2\pi i)^{-1}$ we have that

$$\lim_{\beta\to\infty} \frac{1}{2\pi i} \int_{\gamma-i\beta}^{\gamma+i\beta} e^{zt} f(z)\,\mathrm{d}z = \lim_{\beta\to\infty} \frac{1}{2\pi i} \int_{\gamma'-i\beta}^{\gamma'+i\beta} e^{zt} f(z)\,\mathrm{d}z\,,$$

which establishes the independence of the choice of γ in the inversion integral.

is this actually an inverse?

We wish to show that the Laplace transform of the inversion integral is equivalent to $f(s)$.

Proceed directly. Here, choose $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > \mathrm{Re}(z) = \gamma$.

$$\mathcal{L}[F(t)] = \lim_{T\to\infty} \int_0^T e^{-st} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt} f(z)\,\mathrm{d}z\,\mathrm{d}t$$

integral shenanigans, mostly handwaving

$$\mathcal{L}[F(t)] = \lim_{T\to\infty} \int_0^T e^{-st} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{zt} f(z) \, \mathrm{d}z \, \mathrm{d}t$$

$$\stackrel{?}{=} \text{(magic)}$$

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} f(z) \lim_{T\to\infty} \int_0^T e^{(z-s)t} \, \mathrm{d}t \, \mathrm{d}z$$

$$= -\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{f(z)}{z - s} \, \mathrm{d}z$$

something familiar

From the extension to Cauchy's integral formula we saw earlier, observe that

$$-\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{f(z)}{z-s}\,\mathrm{d}z = f(s).$$

using the residue theorem

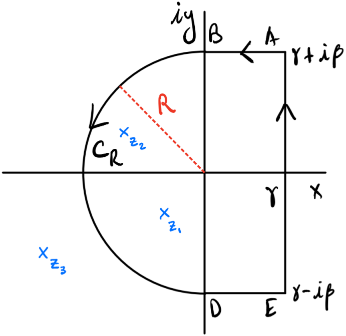
We can use the residue theorem to evaluate the inversion integral when there are isolated singularities where $\mathrm{Re}(z) < \gamma$.

Suppose $f(s)$ is a complex valued function which satisfies the conditions needed so that the inversion integral of $f(s)$ converges to $F(t)$ (whose Laplace transform is $f(s)$).

We can also take $k \geq 1$ in the order condition, but there are lots of details.

the Bromwich integral

Observe that the inversion integral is the integral over the straight part.



We take the integral over the whole contour and show that integrals over the other segments vanish as R or β become arbitrarily large.

contour surgery

So by the residue theorem

$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt} f(z)\,\mathrm{d}z + \int_{AB} e^{zt} f(z)\,\mathrm{d}z + \int_{C_R} e^{zt} f(z)\,\mathrm{d}z \\ + \int_{DE} e^{zt} f(z)\,\mathrm{d}z = 2\pi i \sum_k \mathrm{Res}\big(e^{zt} f(z), z_k\big).$$

the horizontal bits vanish

See that the integral over AB vanishes (the integral over DE vanishes similarly).

Here, $z = x + i\beta$, so $|z| \geq \beta$. Then

$$\left|e^{zt} f(z)\right| \leq e^{xt} |f(z)| \leq e^{xt} \frac{M}{|z|^k} \leq e^{xt} \frac{M}{\beta^k},$$

so

$$\left|\int_{AB} e^{zt} f(z) \, \mathrm{d}z\right| \leq \frac{M}{\beta^k} \int_{\gamma}^{0} e^{xt} \, \mathrm{d}x \,.$$

Hence as $\beta \to \infty$, the integral over AB (and DE) vanishes.

the integral over C_R vanishes

To show this, it is easier instead to invoke a variation of Jordan's lemma.

Lemma (variation of Jordan's lemma)

Let $g(z)$ be a continuous complex valued function defined on a semicircular contour $C_R = \left\{Re^{i\phi} \mid \phi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\right\}$, where $g(z)$ is of order $O(z^{-k})$, $k > 0$.

Then

$$\lim_{R\to\infty} \int_{C_R} e^{azt} g(z)\,\mathrm{d}z = 0$$

for real and positive a.

proof

We play the same bounding game for

$$\left|\int_{C_R} e^{azt} g(z)\,\mathrm{d}z\right|.$$

Parameterize $z = Re^{i\phi}$ for $\phi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

Then we wish to bound from above the following:

$$\left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{atRe^{i\phi}} g(Re^{i\phi}) iRe^{i\phi}\,\mathrm{d}\phi\right|$$

inequalities

$$\left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{atRe^{i\phi}} g(Re^{i\phi}) iRe^{i\phi}\,\mathrm{d}\phi\right| \leq \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{atR\cos(\phi)} \left(\frac{1}{R^k}\right)(R)\,\mathrm{d}\phi$$

Then let $\phi = \theta + \pi$ $(\mathrm{d}\phi = \mathrm{d}\theta)$ and observe some symmetry.

$$= \frac{1}{R^{k-1}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-atR\cos(\theta)}\,\mathrm{d}\theta = \frac{2}{R^{k-1}} \int_{0}^{\frac{\pi}{2}} e^{-atR\cos(\theta)}\,\mathrm{d}\theta$$

more bounding

We can continue by finding a function $h(\theta) \leq \cos(\theta)$, since if $a \leq b$, $e^{-b} \leq e^{-a}$. Let $h(\theta) = \left(1 - \frac{2}{\pi}\theta\right)$. Thus

$$e^{-atR\cos(\theta)} \leq e^{-atR\left(1-\frac{2}{\pi}\theta\right)} = e^{-atR}e^{atR\frac{2}{\pi}\theta},$$

and

$$\frac{2}{R^{k-1}}\int_0^{\frac{\pi}{2}} e^{-atR\cos(\theta)}\,\mathrm{d}\theta \leq \frac{2e^{-atR}}{R^{k-1}}\int_0^{\frac{\pi}{2}} e^{atR\frac{2}{\pi}\theta}\,\mathrm{d}\theta\,.$$

no more

$$\frac{2e^{-atR}}{R^{k-1}} \int_0^{\frac{\pi}{2}} e^{atR\frac{2}{\pi}\theta} \,\mathrm{d}\theta = \frac{2e^{-atR}}{R^{k-1}} \left(\frac{e^{atR} - 1}{atR\frac{2}{\pi}} \right)$$

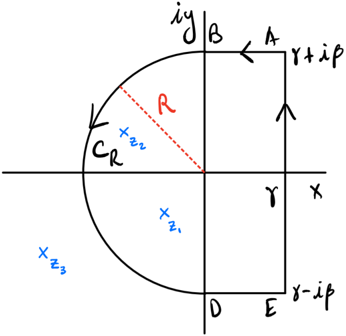
Hence

$$\left| \int_{C_R} e^{azt} g(z) \,\mathrm{d}z \right| \leq \frac{\pi}{R^k} \left(\frac{1 - e^{-atR}}{at} \right),$$

and as $R \to \infty$, we see that the integral vanishes.

back to the inversion integral

Notice above that in the case $a = 1$, we just showed that the inversion integral over C_R vanishes as R is taken arbitrarily large.



All three of the extraneous contours vanish as we take β (or R) to be arbitrarily large.

results of contour surgery

Hence the inversion integral can be computed by just summing up the residues of $e^{zt}f(z)$ at each isolated singularity to the left of the line on which the integral is taken.

$$F(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{zt} f(z)\,\mathrm{d}z = \sum_k \operatorname{Res}\big(e^{zt}f(z), z_k\big)$$

a cute example

compute the inverse transform of $\left(s^2+1\right)^{-1}$

(it is $\sin(t)$)

extra slide in case I had comments