p.163: 9, 10, 11, 13, 14

9. If f is of moderate decrease, then

$$\int_{-R}^{R}\left(1-\frac{|\xi|}{R}\right)\hat{f}(\xi)e^{2\pi i x\xi}\,\mathrm{d}\xi=(f*\mathcal{F}_R)(x), \tag{1}$$

where the Fejér kernel on the real line is defined by

$$\mathcal{F}_R(t)=\begin{cases}R\left(\frac{\sin(\pi tR)}{\pi tR}\right)^2 & \text{if } t\neq 0,\\ R & \text{if t = 0.}\end{cases}$$

Show that $\{\mathcal{F}_R\}$ is a family of good kernels as $R\to\infty$, and therefore (1) tends uniformly to $f(x)$ as $R\to\infty$. This is the analogue of Fejér's theorem for Fourier series in the context of the Fourier transform.

Proof. Let $\{\mathcal{F}_R\}$ be as given. We show the three conditions which define good kernels hold.

For every R, $\mathcal{F}_R(t)$ is well behaved on $\mathbb{R}$, so that

$$\begin{aligned}\int_{\mathbb{R}}\mathcal{F}_R(t)\,\mathrm{d}t&=\int_{\mathbb{R}}R\left(\frac{\sin(\pi tR)}{\pi tR}\right)^2\mathrm{d}t\\&=\int_{\mathbb{R}}\left(\frac{\sin(\pi u)}{\pi u}\right)^2\mathrm{d}u\,.\end{aligned}$$

We evaluate this integral using Plancherel's identity; to that end, observe that the inverse Fourier transformation of $\sin(\pi\xi)/\pi\xi$ is the characteristic function on $[-1/2,1/2]$ (the Fourier transformation of $\chi_{[-1/2,1/2]}(x)$ is $\sin(\pi\xi)/\pi\xi$). It follows that

$$1=\int\limits_{[-1/2,1/2]}1^2\,\mathrm{d}x=\int_{\mathbb{R}}\left(\chi_{[-1/2,1/2]}(x)\right)^2\mathrm{d}x=\int_{\mathbb{R}}\left(\frac{\sin(\pi u)}{\pi u}\right)^2\mathrm{d}u=\int_{\mathbb{R}}\mathcal{F}_R(t)\,\mathrm{d}t\,.$$

Because $|\mathcal{F}_R(t)|=\mathcal{F}_R(t)$, both the first and second properties of good kernels are satisfied.

We show that for any $\eta>0$,

$$\int\limits_{|t|>\eta}\mathcal{F}_R(t)\,\mathrm{d}t\to 0\quad\text{as } R\to\infty.$$

Observe that $|\mathcal{F}_R(t)|\leq 1/\pi^2Rt^2$, so that

$$\int\limits_{|t|>\eta}\mathcal{F}_R(t)\,\mathrm{d}t\leq\frac{2}{\pi^2R}\int\limits_{[\eta,\infty)}t^{-2}\,\mathrm{d}t=\frac{2}{\pi^2R\eta}.$$

As $R\to\infty$, it follows that the integration must tend to 0.

Hence $\{\mathcal{F}_R\}$ is a family of good kernels as $R\to\infty$, and therefore (1) tends uniformly to $f(x)$ as $R\to\infty$. □

10. Below is an outline of a different proof of the Weierstrass approximation theorem.

Define the **Landau** kernels by

$$L_n(x) = \begin{cases} \frac{(1-x^2)^n}{c_n} & \text{if } -1 \leq x \leq 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

where c_n is chosen so that $\int_{-\infty}^{\infty} L_n(x)\,\mathrm{d}x = 1$. Prove that $\{L_n\}_{n\geq 0}$ is a family of good kernels as $n \to \infty$. As a result, show that if f is a continuous function supported in $[-1/2, 1/2]$, then $(f * L_n)(x)$ is a sequence of polynomials on $[-1/2, 1/2]$ which converges uniformly to f.

Proof. Let $\{L_n\}_{n\geq 0}$ be as given. For every $n \geq 0$, it follows that $\int_{-\infty}^{\infty} L_n(x)\,\mathrm{d}x = 1$ due to the choice of c_n. Furthermore, since each $c_n > 0$ (as $1 - x^2 \geq 0$ on $[-1, 1]$), it follows that $|L_n(x)| = L_n(x)$ for every $n \geq 0$, the second property of good kernels is also satisfied.

Then let $\eta > 0$ be given. If $\eta \geq 1$, then the following integral vanishes trivially, so let $0 < \eta < 1$. Note that since $1 - x^2 \geq 1 - x \geq 0$ on $[-1, 1]$, we have that $1 = \int_{-\infty}^{\infty} (1-x^2)^n/c_n\,\mathrm{d}x = 2\int_0^1 (1-x^2)^n/c_n\,\mathrm{d}x \geq 2\int_0^1 (1-x)^n/c_n\,\mathrm{d}x = 2/[(n+1)c_n]$. Hence $1/c_n \leq (n+1)/2$. It follows that

$$\begin{aligned} \int\limits_{|x|>\eta} L_n(x)\,\mathrm{d}x &= \int\limits_{1\geq|x|>\eta} \frac{(1-x^2)^n}{c_n}\,\mathrm{d}x \\ &= 2\int_\eta^1 \frac{(1-x^2)^n}{c_n}\,\mathrm{d}x \\ &\leq \int_\eta^1 (n+1)(1-\eta^2)^n\,\mathrm{d}x\,. \end{aligned}$$

But we can choose n large enough so that $(n+1)(1-\eta^2)^n \leq \varepsilon$ for any $\varepsilon > 0$ (since $(1-\eta^2)^n < 1$ tends to zero exponentially), so that the integration over $[\eta, 1]$ is less than $(1-\eta)\varepsilon$. This we can make as small as we like, so as $n \to \infty$, we have that $\int\limits_{|x|>\eta} L_n(x)\,\mathrm{d}x \to 0$.

Thus $\{L_n\}_{n\geq 0}$ is a family of good kernels as $n \to \infty$.

Let f be a continuous function supported in $[-1/2, 1/2]$. As each $L_n(x)$ is a polynomial in x on $[-1/2, 1/2]$, it follows that $(f * L_n)(x) = \int_{-1/2}^{1/2} f(y)L_n(x-y)\,\mathrm{d}y$ is a sequence of polynomials on $[-1/2, 1/2]$ in x which converges uniformly to f. $\square$

11. Suppose that u is the solution to the heat equation given by $u = f * \mathcal{H}_t$ where $f \in \mathcal{S}(\mathbb{R})$. If we also set $u(x, 0) = f(x)$, prove that u is continuous on the closure of the upper half-plane, and vanishes at infinity, that is,

$$u(x,t) \to 0 \quad \text{as } |x| + t \to \infty.$$

Proof. Since the heat kernel $\mathcal{H}_t$ is a good kernel, it follows that as t tends to zero, $u = f * \mathcal{H}_t$ must uniformly converge to f. Hence $\lim_{t\to 0} u(x,t) = f(x)$ and $u(x, 0) = f(x)$ so that u is continuous on the upper half plane and now also its closure.

We have that $u(x,t) = \int_{-\infty}^{\infty} f(x-y)(4\pi t)^{-1/2}\exp\bigl(-y^2/4t\bigr)\,\mathrm{d}y$. Since $-y^2 \in [0, -\infty)$, it follows that $|u(x,t)| \leq (4\pi t)^{-1/2}\int_{-\infty}^{\infty} |f(x-y)|\,\mathrm{d}y < C/\sqrt{t}$.

Since f is in $\mathcal{S}(\mathbb{R})$, we have that $|f(x-y)| \leq C/(1+|x-y|)^N$ for any $N \geq 0$. So then if $|y| \leq |x|/2$, we have that $|f(x-y)| \leq D/(1+|x|)^N$. Then from the same definition of $u(x,t)$ used earlier we can bound above in another way:

$$\begin{aligned}|u(x,y)| &\leq \frac{D}{(1+|x|)^N} \int_{|y| \leq \frac{|x|}{2}} \mathcal{H}_t(y)\,\mathrm{d}y + At^{-1/2}\exp\big(-x^2/16t\big) \int_{|y| \geq \frac{|x|}{2}} |f(x-y)|\,\mathrm{d}y \\ &\leq \frac{D}{(1+|x|)^N} + Ct^{-1/2}\exp\big(-x^2/16t\big).\end{aligned}$$

In this way if $|x|+t \to \infty$, if $|x| \geq t$ so that $|x| \to \infty$, from the second bounding argument we can see that $u(x,t)$ must tend to zero. If $t \geq |x|$ so that $t \to \infty$, then from the first bounding argument it is also clear that $u(x,t)$ must tend to zero. □

13. Prove the following uniqueness theorem for harmonic functions in the strip $\{(x,y)\colon 0<y<1, -\infty<x<\infty\}$: if u is harmonic in the strip, continuous on its closure with $u(x,0)=u(x,1)=0$ for all $x \in \mathbb{R}$, and u vanishes at infinity, then $u=0$.

Proof. We will use the mean-value property of harmonic functions. Without loss of generality, let u be real valued (if u is not; then repeat the proof for both the real and imaginary components of u).

Suppose by way of contradiction that $u(x,y)$ is positive valued somewhere in the strip, say at (a,b). Then by taking a sufficiently sized open disc of radius ρ centered at (a,b) (which is completely contained in the strip, so $0<\rho<\min\{b,1-b\}$) we may find a point (x_0,y_0) in the disc which is where u attains a local maximum $M>0$ due to u being continuous. Without loss of generality let the local maximum occur at $(x_0,y_0)=(a,b)$ so that we may use the same disc we chose earlier.

Then by the mean-value property, we have that

$$u(a,b) = \frac{1}{2\pi}\int_0^{2\pi} u(a+\rho\cos(\theta), b+\rho\sin(\theta))\,\mathrm{d}\theta\,.$$

But all values $u(a+\rho\cos(\theta), b+\rho\sin(\theta)) \leq M$. In order to maintain the maximality of $u(a,b)$ on the disc, $u(a+\rho\cos(\theta), b+\rho\sin(\theta)) = M$ for all θ. If we extend the radius of the disc chosen earlier enough so that $\rho \to \min\{b,1-b\}$, then it implies that $u(a,0)$ or $u(a,1)$ (depending on the value of $\min\{b,1-b\}$) equals M, which is in contradiction to the assumption that u vanished on the boundary of the strip.

Since (a,b) was arbitrary, it follows that u is identically zero on the strip. □

14. Prove that the periodization of the Fejér kernel $\mathcal{F}_N$ on the real line is equal to the Fejér kernel for periodic functions of period 1. In other words,

$$\sum_{n=-\infty}^{\infty} \mathcal{F}_N(x+n) = F_N(x),$$

when $N \geq 1$ is an integer, and where

$$F_N(x) = \sum_{n=-N}^{N}\left(1-\frac{|n|}{N}\right)e^{2\pi i n x} = \frac{1}{N}\frac{\sin^2(N\pi x)}{\sin^2(\pi x)}.$$

Proof. We wish to use the Poisson summation formula to evaluate $F_N(x)$, but in order to do so we need to find the Fourier transformation of the Fejér kernel. We have this immediately by considering the function given by

$$\widehat{\mathcal{F}}(\xi) = \begin{cases} 1 - \frac{|\xi|}{N} & \text{if } |\xi| \leq N, \\ 0 & \text{otherwise.} \end{cases}$$

Then by taking the inversion integral $f(x)$ when $x \neq 0$ (when $x = 0$, $f(x)$ is clearly equal to N), we have

$$\begin{aligned} f(x) = \int_{\mathbb{R}} \widehat{\mathcal{F}}(\xi) \exp(2\pi i x\xi) \, \mathrm{d}\xi &= \int_{-N}^{N} \left(1 - \frac{|\xi|}{N}\right) \exp(2\pi i x\xi) \, \mathrm{d}\xi \\ &= \int_{0}^{N} 2 \left(1 - \frac{\xi}{N}\right) (\exp(2\pi i x\xi) + \exp(-2\pi i x\xi)) \, \mathrm{d}\xi \\ &= \int_{0}^{N} 2 \left(1 - \frac{\xi}{N}\right) \cos(2\pi x\xi) \, \mathrm{d}\xi \\ &= \frac{2}{N} \left((N - \xi) \frac{\sin(2\pi x\xi)}{2\pi x}\right) \Big|_{0}^{N} + \frac{2}{N} \left(\frac{-\cos(2\pi x\xi)}{(2\pi x)^2}\right) \Big|_{0}^{N} \\ &= N \left(\frac{\sin(\pi x N)}{(\pi x N)}\right)^2, \end{aligned}$$

and so comparing with the definition given in Exercise 9, we have that $f(x)$ is equivalent to the Fejér kernel as desired.

Then apply the Poisson summation formula to see that

$$\begin{aligned} F_N(x) &= \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x} \\ &= \frac{1}{N} \sum_{n=-(N-1)}^{N-1} (N - |n|) e^{2\pi i n x} \\ &= \frac{D_0(2\pi x) + \cdots + D_{N-1}(2\pi x)}{N} \\ &= \frac{1}{N} \frac{\sin^2(N\pi x)}{\sin^2(\pi x)}, \end{aligned}$$

which is indeed the Fejér kernel for functions of period 1. $\square$