28.1 (a) Find the Fourier series of the function $f(x) = x$ in the range $-\pi < x \leq \pi$.

The Fourier coefficients for $f(x) = x$ are computed as

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,\mathrm{d}x = 0$$

and

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}\,\mathrm{d}x = \frac{-i\sin(\pi n)}{\pi n^2} + \frac{i\cos(\pi n)}{n} = \frac{i(-1)^n}{n}$$

so that the Fourier series for $f(x) = x$ on $(-\pi, \pi]$ is given by

$$\begin{aligned} f(x) = x \sim \sum_{n\neq 0} \frac{i(-1)^n}{n} e^{inx} &= \sum_{n=1}^{\infty} \frac{i(-1)^n}{n}\left(e^{inx} - e^{-inx}\right) \\ &= \sum_{n=1}^{\infty} \frac{-2\,(-1)^n}{n}\sin(nx). \end{aligned}$$

(b) Use (a) to show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

With $x = \pi/2$, we have

$$\frac{\pi}{2} = \sum_{n=1}^{\infty} \frac{-2\,(-1)^n}{n}\sin\!\left(\frac{n\pi}{2}\right)$$

so that

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

as desired.

28.2 For the function $f(x) = 1 - x$ for $0 \leq x \leq 1$, find

(a) the Fourier sine series:

The function f is extended to be an odd function on $[-1, 1]$ so that the Fourier coefficients are given by

$$a_n = 2\int_0^1 (1-x)\sin(\pi n x)\,\mathrm{d}x = \frac{2\,(n\pi - \sin(n\pi))}{n^2\pi^2}$$

so that the Fourier sine series for the odd extension of $f(x) = 1 - x$ is given by

$$\sum_{n=1}^{\infty} \frac{2\,(n\pi - \sin(n\pi))}{n^2\pi^2}\sin(\pi n x)$$

(b) the Fourier cosine series:

The function f is extended to be an even function on $[-1, 1]$ so that the Fourier coefficients are given by

$$a_0 = \int_0^1 (1-x)\,\mathrm{d}x = \frac{1}{2}$$

and

$$a_n = 2\int_0^1 (1-x)\cos(\pi n x)\,\mathrm{d}x = \frac{2(1-\cos(n\pi))}{n^2\pi^2}$$

so that the Fourier cosine series for the even extension of $f(x) = 1-x$ is given by

$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{2(1-\cos(n\pi))}{n^2\pi^2}\cos(\pi n x)$$

(c) and which of the two series is better for a numerical evaluation:

The cosine series is better since in the odd extension of $f(x) = 1-x$, there is a jump discontinuity at $x = 0$, so there would be unusual behaviour of the Fourier series. The Fourier series for a given function tends to converge better for continuous functions, and for this example in particular after plotting the series for some large finite number of terms the behavior around 0 for the sine series was shady at best due to the series being continuous, despite the odd extension of $f(x) = 1-x$ not being continuous.

28.3 Verify that

$$f(\phi_1 - \phi_2) \equiv \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi_1-\phi_2)}$$

a Dirac delta function.

Proof. Directly, by convoluting f with a smooth enough function g (which without loss of generality is defined on the circle $(-\pi,\pi]$) at x_0 we have

$$\begin{aligned}
f * g(x_0) &= \int_{-\pi}^{\pi} g(x) f(x_0 - x)\,\mathrm{d}x \\
&= \int_{-\pi}^{\pi} g(x)\left(\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(x_0-x)}\right)\mathrm{d}x \\
&= \sum_{m=-\infty}^{\infty} e^{imx_0}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-imx}\,\mathrm{d}x\right) \\
&= \sum_{m=-\infty}^{\infty} a_m e^{imx_0},
\end{aligned}$$

where a_m is the m-th Fourier coefficient of g. Because g was sufficiently smooth, the Fourier series of g converges to g and so the last sum can be observed to be the Fourier series of g evaluated at x_0, which is exactly $g(x_0)$ as desired. Hence f is a Dirac delta function. $\square$