Section 2.2 Problems 7, 11, 13, 14, 30, 31 and Exercise from Section 1 of the Week 2 Supplement

7. $x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y^3}$ Solve.

No equilibria.

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y^3} \to y^3\,\mathrm{d}y = \frac{1}{x}\,\mathrm{d}x \to \int y^3\,\mathrm{d}y = \int \frac{1}{x}\,\mathrm{d}x \to \frac{y^4}{4} = \ln|x| + C$$

Hence the integral curve is of the form

$$y = (4\ln|x| + C)^{\frac{1}{4}}$$

for $x \in \mathbb{R} \setminus \{0\}$.

11. $x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1-4v^2}{3v}$ Solve.

There are equilibrium solutions that form where $v = \pm\frac{1}{2}$.

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1-4v^2}{3v} \to \frac{3v}{1-4v^2}\,\mathrm{d}v = \frac{1}{x}\,\mathrm{d}x \to \int \frac{3v}{1-4v^2}\,\mathrm{d}v = \int \frac{1}{x}\,\mathrm{d}x$$

Let $u = 1 - 4v^2$ and $\mathrm{d}u = -8v\,\mathrm{d}v$.

$$-\frac{3}{8}\int \frac{1}{u}\,\mathrm{d}u = \int \frac{1}{x}\,\mathrm{d}x \to \ln|u| = -\frac{8}{3}\ln|x| + C \to \left|1-4v^2\right| = C|x|^{-\frac{8}{3}} \to 1 - 4v^2 = \pm C|x|^{-\frac{8}{3}}$$

Hence the integral curve is of the form

$$v = \pm\sqrt{\frac{1 \mp C|x|^{-\frac{8}{3}}}{4}}$$

for $x \in \mathbb{R} \setminus \{0\}$.

13. $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2\left(1+y^2\right)^{\frac{3}{2}}$ Solve.

No equilibria.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2\left(1+y^2\right)^{\frac{3}{2}} \to \frac{1}{(1+y^2)^{\frac{3}{2}}}\,\mathrm{d}y = 3x^2\,\mathrm{d}x \to \int \frac{1}{(1+y^2)^{\frac{3}{2}}}\,\mathrm{d}y = \int 3x^2\,\mathrm{d}x$$

Let $y = \tan t$ and $\mathrm{d}y = \sec^2 t\,\mathrm{d}t$.

$$\int \frac{\sec^2 t}{(1+\tan^2 t)^{\frac{3}{2}}}\,\mathrm{d}t = x^3 + C \to \int \cos t\,\mathrm{d}t = x^3 + C \to \sin t = x^3 + C \to \sin\arctan y = x^3 + C$$

Using trigonometry, it is apparent that $\sin\arctan y = \frac{y}{\sqrt{1+y^2}}$.

Then:

$$\frac{y}{\sqrt{1+y^2}} = x^3 + C \to \frac{y^2}{1+y^2} = \left(x^3+C\right)^2 \to y^2 = \left(x^3+C\right)^2 + y^2\left(x^3+C\right)^2 \to y^2\left(1 - \left(x^3+C\right)^2\right) = \left(x^3+C\right)^2$$

Hence the integral curve is of the form

$$y = \pm\sqrt{\frac{\left(x^3+C\right)^2}{\left(1-\left(x^3+C\right)^2\right)}}$$

for $x \in \mathbb{R}$.

14. $\frac{\mathrm{d}x}{\mathrm{d}t} - x^3 = x$ Solve.

There is an equilibrium solution that forms where $x = 0$.

$$\frac{\mathrm{d}x}{\mathrm{d}t} - x^3 = x \to \frac{1}{x+x^3}\,\mathrm{d}x = \mathrm{d}t \to \int \frac{1}{x\left(1+x^2\right)}\,\mathrm{d}x = \int \mathrm{d}t$$

Let $x = \tan u$ and $\mathrm{d}x = \sec^2 u\,\mathrm{d}u$. Then with a trivial substitution afterwards:

$$\int \frac{\sec^2 u}{\tan u\left(1+\tan^2 u\right)}\,\mathrm{d}u = t + C \to \int \frac{\cos u}{\sin u}\,\mathrm{d}u = t + C \to \ln|\sin u| = t + C \to \sin\arctan x = \pm C\exp(t)$$

Using trigonometry, it is apparent that $\sin\arctan x = \frac{x}{\sqrt{1+x^2}}$.

Then:

$$\frac{x}{\sqrt{1+x^2}} = \pm C\exp(t) \to \frac{x^2}{1+x^2} = C\exp(2t) \to x^2 = C\exp(2t) + x^2 C\exp(2t) \to x^2\left(1 - C\exp(2t)\right) = C\exp(2t)$$

Hence the integral curve is of the form

$$x = \pm\sqrt{\frac{C\exp(2t)}{\left(1-C\exp(2t)\right)}}$$

for $t \in \mathbb{R}$.

30.

(a) $\frac{\mathrm{d}y}{\mathrm{d}x} = (x-3)(y+1)^{\frac{2}{3}}$ Solve.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x-3)(y+1)^{\frac{2}{3}} \to (y+1)^{-\frac{2}{3}}\,\mathrm{d}y = (x-3)\,\mathrm{d}x \to \int (y+1)^{-\frac{2}{3}}\,\mathrm{d}y = \int (x-3)\,\mathrm{d}x \to 3(y+1)^{\frac{1}{3}} = \frac{x^2}{2} - 3x + C$$

Hence the integral curve is of the form

$$y = -1 + \left(\frac{x^2}{6} - x + C\right)^3$$

for $x \in \mathbb{R}$.

(b) Observe that $y \equiv -1 \implies \frac{\mathrm{d}(-1)}{\mathrm{d}x} = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x}|_{y=-1} = (x-3)\left((-1)+1\right)^{\frac{2}{3}} = 0$

(c) Note that in order to find that $y = -1$, we require $\frac{x^2}{6} - x + C = 0$ and there is no **constant** value of C that can be chosen. Hence we have lost the $y = -1$ solution.

31.

(a) $\frac{\mathrm{d}y}{\mathrm{d}x} = xy^3$ Solve.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy^3 \to y^{-3}\,\mathrm{d}y = x\,\mathrm{d}x \to \int y^{-3}\,\mathrm{d}y = \int x\,\mathrm{d}x \to \frac{1}{-2y^2} = \frac{x^2}{2} + C$$

Hence the integral curve is of the form

$$y = \pm\sqrt{\frac{1}{-x^2 + C}}$$

for $x \in \left(-\sqrt{C}, \sqrt{C}\right)$.

(b) and (c) :

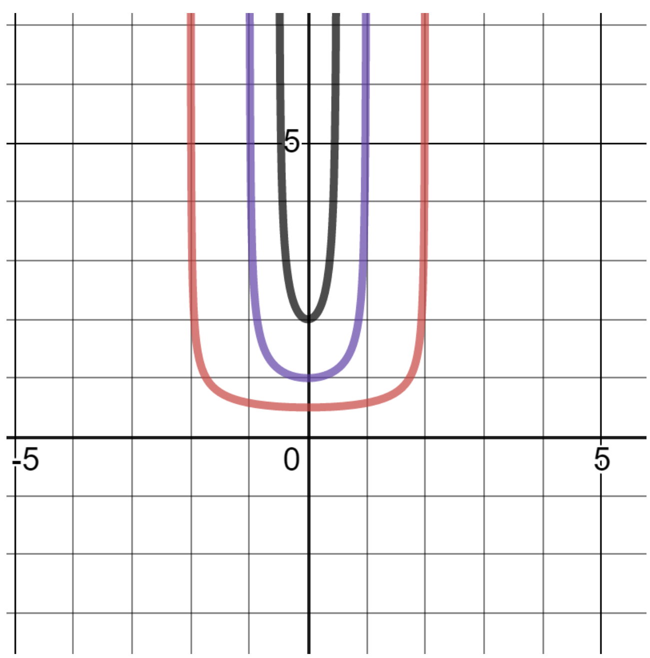
$(1) = \pm\sqrt{\frac{1}{-(0)^2+C}} \implies C = 1 \implies y = \pm\sqrt{\frac{1}{-x^2+1}}$ for $x \in (-1, 1)$.
$\left(\frac{1}{2}\right) = \pm\sqrt{\frac{1}{-(0)^2+C}} \implies C = 4 \implies y = \pm\sqrt{\frac{1}{-x^2+4}}$ for $x \in (-2, 2)$.
$(2) = \pm\sqrt{\frac{1}{-(0)^2+C}} \implies C = \frac{1}{4} \implies y = \pm\sqrt{\frac{1}{-x^2+\frac{1}{4}}}$ for $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$.

(d) :

$(a) = \pm\sqrt{\frac{1}{-(0)^2+C}} \implies C = \frac{1}{a^2} \implies y = \pm\sqrt{\frac{1}{-x^2+\frac{1}{a^2}}}$ for $x \in \left(-\frac{1}{a}, \frac{1}{a}\right)$.

If we take $\lim_{a\to 0^+} \frac{1}{a}$, then we find that it tends to ∞, and thus the domain of the integral curve is $x \in (-\infty, \infty)$. Likewise, if we take $\lim_{a\to\infty} \frac{1}{a}$ we find that it tends to 0 and so the function will only be defined on $x = 0$ since the adjacent real numbers cause the function to explode.

(e) Graphing the positive curves (red: $a = \pm\frac{1}{2}$, purple: $a = \pm 1$, black: $a = \pm 2$):



Exercise from Section 1 of the Week 2 Supplement:

At every point (t, x) on the integral curve $x = \varphi(t)$ all tangent vectors come in the form $\vec{w} = k(1, \dot{x})$. Then the dot product of all tangent vectors $\vec{w}$ with the vector field $\vec{u}(t, x)$ is 0:

$$\langle \vec{u}(t,x), \vec{w} \rangle \to \left\langle \left(g(t), -\frac{1}{h(x)} \right), k\left(1, g(t)h(x)\right) \right\rangle \to k\left(g(t) + (-g(t))\right) = 0$$