The Laplace transform (starred lines are important)

$$\star\ \mathcal{L}[f(t)](s) = \lim_{b\to\infty}\int_0^b e^{-st} f(t)\,\mathrm{d}t$$

Derivation of Laplace transform for e^{at}, $\cos(at)$, $\sin(at)$:

$$\star\ \mathcal{L}[e^{at}](s) = \lim_{b\to\infty}\int_0^b e^{-st}e^{at}\,\mathrm{d}t = \frac{1}{s-a}$$

$$\star\ \mathcal{L}[\cos(at)](s) = \frac{s}{s^2-a^2}$$

$$\star\ \mathcal{L}[\sin(at)](s) = \frac{a}{s^2-a^2}$$

For the exponential:

$$\mathcal{L}[e^{at}](s) = \lim_{b\to\infty}\int_0^b e^{-st}e^{at}\,\mathrm{d}t = \lim_{b\to\infty}\left[\frac{e^{(a-s)b}}{a-s} - \frac{1}{a-s}\right] = \frac{1}{s-a}$$

This only happens when $a < s$. A remark to make is that the case when $a = s$ will also fail to converge. As for the sine and cosine, to find their Laplace transforms we want to observe the following:

$$e^{iat} = \cos(at) + i\sin(at) \implies \mathrm{Re}\left(e^{iat}\right) = \cos(at),\ \ \mathrm{Im}\left(e^{iat}\right) = \sin(at)$$

Also know that since integration is a linear operation (it acts termwise on a sum), it is true that

$$\int \mathrm{Re}\left(f(z)\right)\mathrm{d}z = \mathrm{Re}\left(\int f(z)\,\mathrm{d}z\right)$$

$$\mathcal{L}[\mathrm{Re}\left(f(t)\right)](s) = \lim_{b\to\infty}\int_0^b e^{-st}\,\mathrm{Re}\{f(t)\}\,\mathrm{d}t = \mathrm{Re}\left(\lim_{b\to\infty}\int_0^b e^{-st} f(t)\,\mathrm{d}t\right) = \mathrm{Re}\left(\mathcal{L}[f(t)](s)\right)$$

We can use this fact to find the following Laplace transforms:

$$\mathcal{L}[\cos(at)](s) = \mathcal{L}[\mathrm{Re}\left(e^{iat}\right)](s) = \mathrm{Re}\left(\mathcal{L}[e^{iat}](s)\right)$$

$$\mathcal{L}[\sin(at)](s) = \mathcal{L}[\mathrm{Im}\left(e^{iat}\right)](s) = \mathrm{Im}\left(\mathcal{L}[e^{iat}](s)\right)$$

We must find the Laplace transform of e^{iat} first (not rigorously but it will do):

$$\mathcal{L}[e^{iat}](s) = \lim_{b\to\infty}\int_0^b e^{-st}\left(e^{iat}\right)\mathrm{d}t = \frac{1}{s-ia} = \frac{s+ia}{s^2-a^2}$$

$$\mathcal{L}[\cos(at)](s) = \mathrm{Re}\left(\frac{s+ia}{s^2-a^2}\right) = \frac{s}{s^2-a^2}$$

$$\mathcal{L}[\sin(at)](s) = \mathrm{Im}\left(\frac{s+ia}{s^2-a^2}\right) = \frac{a}{s^2-a^2}$$