HOMEWORK 12

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Suppose $f : [a, b] \to \mathbb{R}$ is bounded. Let

$$C = \sup\{|f(x)| : x \in [a, b]\}.$$

(i) Show, for any $a \le x, y \le b$, that

$$|f(x)^2 - f(y)^2| \le 2C|f(x) - f(y)|.$$

(ii) Fix $a \le c < d \le b$ and let m, M denote the infimum and supremum of the set $\{f(x) : x \in [c, d]\}$ respectively. Show, for any $c \le x, y \le d$, that

$$|f(x)^2 - f(y)^2| \le 2C(M - m).$$

(iii) Let m^*, M^* denote the infimum and supremum of the set $\{f(x)^2 : x \in [c, d]\}$. Prove

$$M^* - m^* \le 2C(M - m).$$

(iv) Prove if f is integrable, then so is f^2.

I spoke with Nicholas Kapsos about details of the proof.

Proof. Let $f\colon [a, b] \to \mathbb{R}$ be bounded with $C = \sup\{|f(x)| : x \in [a, b]\}$ as given.

(i) For $x, y \in [a, b]$,

$$\begin{aligned} \left|f(x)^2 - f(y)^2\right| &= |f(x) + f(y)||f(x) - f(y)| \\ &\le (|f(x)| + |f(y)|)|f(x) - f(y)| \\ &\le 2C|f(x) - f(y)|. \end{aligned}$$

(ii) Fix $a \le c < d \le b$ and let m and M denote the infimum and supremum of the set $\{f(x) : x \in [c, d]\}$ respectively. We have $m \le f(x), f(y) \le M$, for $x, y \in [c, d]$. It follows that $f(x)-f(y) \le M-m$, and by taking the absolute value $|f(x) - f(y)| \le M - m$. Then

$$\left|f(x)^2 - f(y)^2\right| \le 2C|f(x) - f(y)| \le 2C(M - m).$$

(iii) Let m^* and M^* denote the infimum and supremum of the set $\{f(x)^2 : x \in [c, d]\}$ respectively. Then given any $\eta > 0$, there exists $u, v \in [c, d]$ such that

$$\begin{aligned} M^* - \eta &\le f(u)^2 \\ m^* + \eta &\ge f(v)^2, \end{aligned}$$

so that $M^* - m^* - 2\eta \leq f(u)^2 - f(v)^2$. Since $m^* \leq M^*$, it follows that

$$\begin{aligned} M^* - m^* &\leq f(u)^2 - f(v)^2 + 2\eta \\ &\leq \left| f(u)^2 - f(v)^2 \right| + 2\eta \\ &\leq 2C(M - m) + 2\eta. \end{aligned}$$

Since $\eta > 0$ was arbitrary, $M^* - m^* \leq 2C(M - m)$.

(iv) Suppose f is integrable. Since the space of integrable functions is a vector space, it follows that $2Cf$ is integrable. Then given $\varepsilon > 0$, there is a partition $P = \{a = x_0, x_1, \ldots, x_{n-1}, x_n = b\}$ such that $U(2Cf, P) - L(2Cf, P) \leq \varepsilon$.

Let m_i^* and M_i^* denote the infimum and supremum of the set $\{f(x)^2 : x \in [x_i, x_{i+1}]\}$ respectively and let m_i and M_i denote the infimum and supremum of the set $\{f(x) : x \in [x_i, x_{i+1}]\}$ respectively. Then by the result of (iii), we have

$$\begin{aligned} U(f^2, P) - L(f^2, P) &= \sum_i (M_i^* - m_i^*)(x_{i+1} - x_i) \\ &\leq \sum_i 2C(M_i - m_i)(x_{i+1} - x_i) \\ &= U(2Cf, P) - L(2Cf, P) \leq \varepsilon. \end{aligned}$$

Hence f^2 is integrable.

□