Solution Manual

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27.8 Exercises

27.8.1

Find the gradient of $f(x,y) = xy$ and the gradient of $g(x,y) = x + y$:

$$\vec{\nabla} f(x,y) = \langle y, x \rangle$$
$$\vec{\nabla} g(x,y) = \langle 1, 1 \rangle$$

It is known that $f(x,y)$ attains extrema when $\vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y)$, that is at points (x,y) that satisfy the following system using Lagrange multipliers (recall one of the equations is the constraint itself):

$$y = \lambda$$
$$x = \lambda$$
$$x + y = 1$$

Evidently the only critical point is $(\frac{1}{2}, \frac{1}{2})$ - but because it is the only one we may choose to use the second derivative test or to use the properties of the function f under the constraint (for instance, parameterize (x,y) as $(t, 1-t)$ as given by the constraint to show that we have a downwards opening parabola) to determine that it is a maximum. Thus $f(x,y)$ constrained to $g(x,y) = 1$ has a maximum $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$ at that point.

27.8.3

Find the gradient of $f(x,y) = xy^2$ and the gradient of $g(x,y) = 2x^2 + y^2$:

$$\vec{\nabla} f(x,y) = \langle y^2, 2xy \rangle$$
$$\vec{\nabla} g(x,y) = \langle 4x, 2y \rangle$$

It is known that $f(x,y)$ attains extrema when $\vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y)$, that is at points (x,y) that satisfy the following system using Lagrange multipliers (recall one of the equations is the constraint itself):

$$y^2 = 4\lambda x$$

$$2xy = 2\lambda y$$

$$2x^2 + y^2 = 6$$

With some algebra (resolve the middle equation for x, then substitute x for lambda in the first equation, and then substitute $4x^2$ for y^2 and solve for x), it is apparent that $x = \pm 1$ $y = \pm 2$ (both still satisfy the constraint). Thus we have critical points at $(1, \pm 2)$, where $f(1, \pm 2) = 4$, and at $(-1, \pm 2)$, where $f(-1, \pm 2) = -4$. Hence we have $\max f = 4$ and $\min f = -4$

27.8.7

Find the gradient of $f(x, y) = Ax^2 + 2Bxy + Cy^2$ and the gradient of $g(x, y) = x^2 + y^2$:

$$\vec{\nabla} f(x, y) = \langle 2Ax + 2By, 2Bx + 2Cy \rangle$$

$$\vec{\nabla} g(x, y) = \langle 2x, 2y \rangle$$

It is known that $f(x, y)$ attains extrema when $\vec{\nabla} f(x, y) = \lambda \vec{\nabla} g(x, y)$, that is at points (x, y) that satisfy the following system using Lagrange multipliers (recall one of the equations is the constraint itself):

$$2Ax + 2By = 2\lambda x$$

$$2Bx + 2Cy = 2\lambda y$$

$$x^2 + y^2 = 1$$

There is a bit of algebra we have to do to find critical values. First, we can add the first two equations together and divide through by 2 to find

$$Ax + By + Bx + Cy = \lambda x + \lambda y \rightarrow (A + B)x + (B + C)y = \lambda x + \lambda y$$

This tells us that $A + B = \lambda$ and $B + C = \lambda$, furthermore that $A = C$ and A or C $= \lambda - B$. Then substitute that last equality in for A and C in the first equations like so:

$$(\lambda - B)x + By = \lambda x$$

$$Bx + (\lambda - B)y = \lambda y$$

Deduce then that $x = y$. Using the third equation it is apparent that critical points occur at $(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2})$, and what kind of extrema they form entirely depend on the choice of A, B, C.

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