Let G be a group and let A be a nonempty set.

1. (DF4.1.1) Let G act on the set A. Prove that if $a, b \in A$ and $b = g \cdot a$ for some $g \in G$, then $G_b = gG_ag^{-1}$. Deduce that if G acts transitively on A then the kernel of the action is $\cap_{g \in G}\, gG_ag^{-1}$.

   Proof. Let G act on A with $b = g \cdot a$ for $a, b \in A$ for some $g \in G$. We also have that $g^{-1} \cdot b = g^{-1} \cdot (g \cdot a) = (g^{-1}g) \cdot a = 1 \cdot a = a$.

   With $gG_ag^{-1} = \left\{gxg^{-1} \mid x \in G_a\right\}$, observe that any element $gxg^{-1} \in gG_ag^{-1}$ satisfies

   $$(gxg^{-1}) \cdot b = (gx) \cdot (g^{-1} \cdot b) = g \cdot (x \cdot a) = g \cdot a = b,$$

   so that $gxg^{-1} \in G_b$. Hence $gG_ag^{-1} \subseteq G_b$.

   Similarly, observe that for any $y \in G_b$, we may find $ghg^{-1} \in gG_ag^{-1}$ such that $y = ghg^{-1}$. Choose $h = g^{-1}yg$, where indeed $h = g^{-1}yg \in G_a$ because

   $$h \cdot a = (g^{-1}yg) \cdot a = (g^{-1}y) \cdot b = g^{-1} \cdot b = a.$$

   Then $y \in gG_ag^{-1}$, and hence $G_b \subseteq gG_ag^{-1}$.

   If G acts transitively on A; that is, there is only one orbit and so for any $a, c \in A$, there is some $g \in G$ such that $a = g \cdot c$. We may obtain the kernel of this action by finding $\cap_{c \in A}\, G_c$, but because this action is transitive on A, we may use the above result to rewrite this set intersection.

   For $a, c \in A$, there exists $g \in G$ such that $G_c = gG_ag^{-1}$; as a result, if we fix a and let g take on every element in G, then the sets gG_ag^{-1} take on every G_c for $c \in A$. Hence $\cap_{c \in A}\, G_c = \cap_{g \in G}\, gG_ag^{-1}$, which is the kernel of the transitive action of G on A. □

2. (DF4.1.4) Let S_3 act on the set Ω of ordered pairs: $\{(i, j) \mid 1 \le i, j \le 3\}$ by $\sigma((i, j)) = (\sigma(i), \sigma(j))$. Find the orbits of S_3 on Ω. For each $\sigma \in S_3$ find the cycle decomposition of σ under this action (i.e., find its cycle decomposition when σ is considered as an element of S_9 — first fix a labelling of these nine ordered pairs). For each orbit $\mathcal{O}$ of S_3 acting on these nine points pick some $a \in \mathcal{O}$ and find the stabilizer of a in S_3.

   The orbit of S_3 containing $a \in \Omega$ takes on the form $\{\sigma(a) \mid \sigma \in S_3\}$, and we know that the group action will partition A into disjoint orbits of this form. We find the orbits by taking the six permutations of S_3 and applying them to $(1, 1)$ and $(1, 2)$; we need not try any others since after this point we find all of the elements in Ω. The following table exhibits this method:

σ	$\sigma((1,1))$	$\sigma((1,2))$
1	$(1,1)$	$(1,2)$
$(1\,2)$	$(2,2)$	$(2,1)$
$(2\,3)$	$(1,1)$	$(1,3)$
$(1\,3)$	$(3,3)$	$(3,2)$
$(1\,2\,3)$	$(2,2)$	$(2,3)$
$(1\,3\,2)$	$(3,3)$	$(3,1)$

So the two orbits that form are $\{(c, c) \mid 1 \leq c \leq 3\}$ (the first column) and $\{(a, b), (b, a) \mid a \neq b, 1 \leq a, b \leq 3\}$ (the second column). Notice they are disjoint and their union forms Ω as expected.

We use a suggestive notation to simplify forming the cycle decomposition of σ under this group action. Using the matrices below we can establish a labelling of the elements of Ω:

$$\begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \\ \mathbf{7} & \mathbf{8} & \mathbf{9} \end{pmatrix} = \begin{pmatrix} (1,1) & (1,2) & (1,3) \\ (2,1) & (2,2) & (2,3) \\ (3,1) & (3,2) & (3,3) \end{pmatrix}$$

Then by tracking how each element in S_3 permutes $\{\mathbf{1}, \dots, \mathbf{9}\}$, we can find the following cycle decompositions (viewing them as elements of S_9):

σ	cycle decomposition for σ
1	1
$(1\,2)$	$(\mathbf{1\,5})(\mathbf{2\,4})(\mathbf{3\,6})(\mathbf{7\,8})(\mathbf{9})$
$(2\,3)$	$(\mathbf{2\,3})(\mathbf{4\,7})(\mathbf{5\,9})(\mathbf{6\,8})(\mathbf{1})$
$(1\,3)$	$(\mathbf{1\,9})(\mathbf{2\,8})(\mathbf{3\,7})(\mathbf{4\,6})(\mathbf{5})$
$(1\,2\,3)$	$(\mathbf{1\,5\,9})(\mathbf{2\,6\,7})(\mathbf{3\,4\,8})$
$(1\,3\,2)$	$(\mathbf{1\,9\,5})(\mathbf{2\,7\,6})(\mathbf{3\,8\,4})$

It is clear from these cycle decompositions that for $a \in \{(c, c) \mid 1 \leq c \leq 3\}$ (the first orbit), the stabilizer of a in S_3 is $S_{3a} = \{1, (x\,y) \mid x, y \neq a, 1 \leq x, y \leq 3\}$; for example, $(1\,2)((3, 3)) = (3, 3)$ since 3 is not found in the cycle $(1\,2)$. Then for $b \in \{(a, b), (b, a) \mid a \neq b, 1 \leq a, b \leq 3\}$ (the second orbit), the stabilizer of b in S_3 is $S_{3b} = \{1\}$, since the only 1-cycles present in any of the cycle decompositions above are those that fix elements from the first orbit.

3. (DF4.1.10) Let H and K be subgroups of the group G. For each $x \in G$ define the HK *double coset* of x in G to be the set

$$HxK = \{hxk \mid h \in H, k \in K\}.$$

(a) Prove that HxK is the union of the left cosets $x_1K, \dots, x_nK$ where $\{x_1K, \dots, x_nK\}$ is the orbit containing xK of H acting by left multiplication on the set of left cosets of K.

Proof. Let $\{x_1K, \dots, x_nK\}$ be the orbit containing xK of H acting by left multiplication on the set of left cosets of K (without loss of generality each x_iK here is distinct). This means that this set is equal to $\{h \cdot xK \mid h \in H\}$ (there is a bijection between the sets), so that hxK for each $h \in H$ is equal to some x_iK for $1 \leq i \leq n$. Similarly, each x_iK for $1 \leq i \leq n$ is equal to hxK for some $h \in H$.

Consider any element $g \in HxK$, so that we may write $g = hxk$ for some $h \in H$ and $k \in K$. Then $g = hxk \in hxK$, but there exists an x_jK in the orbit containing xK of H such that $g \in x_jK$ with $hxK = x_jK$. Then it is clear that $g \in \cup_{i=1}^n x_iK$, and because g was an arbitrary element of HxK, $HxK \subseteq \cup_{i=1}^n x_iK$.

Then take any element $g \in \cup_{i=1}^{n} x_iK$, so that g lies in some coset x_jK. There exists hxK for some $h \in H$ such that $x_jK = hxK$, so $g \in hxK$. Then because $hxK \subseteq HxK$, it follows that $g \in HxK$. Hence $\cup_{i=1}^{n} x_iK \subseteq HxK$, which means that $\cup_{i=1}^{n} x_iK = HxK$. □

(b) Prove that HxK is a union of right cosets of H.

Proof. Similarly, let $\{Hx_1, \ldots, Hx_n\}$ be the orbit containing Hx of K acting by right multiplication on the set of right cosets of H (without loss of generality each Hx_i here is distinct). Then $\{Hx_1, \ldots, Hx_n\} = \{Hx \cdot k \mid k \in K\}$, so that Hxk for each $k \in K$ is equal to some Hx_i for $1 \le i \le n$ and each Hx_i for $1 \le i \le n$ is equal to Hxk for some $k \in K$.

Then for any $g \in HxK$, we have that $g = hxk$ for some $k \in K$ and $h \in H$. Then $g = hxk \in Hxk$, but there exists some Hx_j in the orbit containing Hx of K such that $g \in Hx_j = Hxk$. Thus $g \in \cup_{i=1}^{n} Hx_i$, so that $HxK \subseteq \cup_{i=1}^{n} Hx_i$.

For any $g \in \cup_{i=1}^{n} Hx_i$, we have that $g \in Hx_j$ for some $1 \le j \le n$. Then since Hx_j is in the orbit containing Hx of K, there exists Hxk for some $k \in K$ such that $g \in Hxk = Hx_j$, which means that $g \in HxK$. Hence $\cup_{i=1}^{n} Hx_i \subseteq HxK$, so $HxK = \cup_{i=1}^{n} Hx_i$. □

(c) Show that HxK and HyK are either the same set or are disjoint for all $x, y \in G$. Show that the set of HK double cosets partitions G.

Proof. For any $g \in G$, we have that $g = 1g1 \in HgK$, since H, K are subgroups of G. Then each HgK for $g \in G$ contains g, so that $G = \cup_{g \in G} HgK$. What remains is to show that for two double cosets HuK, HvK for $u, v \in G$, either $HuK \cap HvK = \varnothing$ or $HuK = HvK$. If $u = v$ then $HuK = HvK$, so suppose $u \neq v$.

Then suppose that $HuK \cap HvK \neq \varnothing$, so that there exists $h_1uk_1 \in HuK$ and $h_2vk_2 \in HvK$ such that $h_1uk_1 = h_2vk_2$. Then it follows that $u = h_1^{-1}h_2vk_2k_1^{-1} = h_3vk_3$ for some $h_3 \in H, k_3 \in K$ (or $v = h_3uk_3$).

So then for any element $huk \in HuK$, we have that $huk = hh_3vk_3k = h_4vk_4 \in HvK$, so that $HuK \subseteq HvK$. By interchanging the roles of u and v it follows that $HvK \subseteq HuK$, so that $HuK = HvK$ whenever $HuK \cap HvK \neq \varnothing$.

Thus the set of HK double cosets partitions G, since each double coset is disjoint from each other and the union of the HK double cosets was shown to be G. □

(d) Prove that $|HxK| = |K| \cdot \left|H\colon H \cap xKx^{-1}\right|$.

Proof. Consider the action of H acting by left multiplication on the set of left cosets of K. Observe that the stabilizer H_{xK} equals $H \cap xKx^{-1}$ because for $h \in H_{xK}$,

$$h \cdot xK = hxK = xK$$

implies that $x^{-1}hx \in K$, so that $h \in xKx^{-1}$. So combined with $h \in H$, we have that $H_{xK} = H \cap xKx^{-1}$.

We saw from part **(a)** that the orbit containing xK of H was $\mathrm{Orb}(xK) = \{x_1K, \dots, x_nK\}$ (disjoint cosets), so $|\mathrm{Orb}(xK)| = n$. Furthermore, we saw that $HxK = \cup_{i=1}^n x_iK$, which because each x_iK is disjoint and has the same cardinality as K, $|HxK| = |\cup_{i=1}^n x_iK| = n|K|$.

Hence $|HxK|/|K| = n = |\mathrm{Orb}(xK)|$, so by the Orbit-Stabilizer Theorem, we have that

$$|HxK|/|K| = |H\colon H_{xK}| = \left|H\colon H \cap xKx^{-1}\right|,$$

which implies that $|HxK| = |K| \cdot \left|H\colon H \cap xKx^{-1}\right|$. □

(e) Prove that $|HxK| = |H| \cdot \left|K\colon K \cap x^{-1}Hx\right|$.

Proof. Similarly, consider the action of K acting by right multiplication on the set of right cosets of H. Observe that the stabilizer K_{Hx} equals $K \cap x^{-1}Hx$ because for $k \in K_{Hx}$,

$$Hx \cdot k = Hxk = Hx$$

implies that $xkx^{-1} = H$, so that $k \in x^{-1}Hx$. With $k \in K$, we have that $K_{Hx} = k \cap x^{-1}Hx$.

From part **(b)**, the orbit containing Hx of K was $\mathrm{Orb}(Hx) = \{Hx_1, \dots, Hx_n\}$ (disjoint cosets), so $|\mathrm{Orb}(Hx)| = n$. We also had that $HxK = \cup_{i=1}^n Hx_i$, and because each Hx_i is disjoint and has the same cardinality as H, $|HxK| = |\cup_{i=1}^n Hx_i| = n|H|$.

Hence $|HxK|/|H| = n = |\mathrm{Orb}(Hx)|$, so by the Orbit-Stabilizer Theorem, we have that

$$|HxK|/|H| = |K\colon K_{Hx}| = \left|K\colon k \cap x^{-1}Hx\right|,$$

which implies that $|HxK| = |H| \cdot \left|K\colon K \cap x^{-1}Hx\right|$. □

4. Q4. Let G be a finite group and H a subgroup. Consider the partition of G into double cosets HgH as in problem 10.

(a) Prove that every left coset contained in a given double coset has nonempty intersection with every right coset contained in the same double coset.

Proof. Let HgH be some given double coset, and let xH be any left coset contained in HgH, and let Hy be any right coset contained in HgH. Then due to containment, we may write $xH = h_1gH$ and $Hy = Hgh_2$, for $h_1, h_2 \in H$. Because $h_1, h_2 \in H$, we have that $h_1gh_2 \in h_1gH$ and $h_1gh_2 \in Hgh_2$, which means that $h_1gh_2 \in h_1gH \cap Hgh_2 = xH \cap Hy$, so that the intersection is nonempty.

Because xH and Hy were arbitrary cosets contained in HgH, it follows that every left coset contained in a given double coset has nonempty intersection with every right coset contained in the same double coset. □

(b) Deduce that if $n = |G : H|$ then there exist elements $g_1, \dots, g_n$ in G that belong to distinct left cosets and to distinct right cosets.

This means that G is the disjoint union of the Hg_i and also the disjoint union of the g_iH.

Proof. Let $n = |G : H|$, so that there are exactly n disjoint left cosets of H in G, and exactly n disjoint right cosets of H in G.

Considering the partition of G into distinct double cosets HgH, observe that for each HgH, there are left cosets $h_1gH \subseteq HgH$ and right cosets $Hgh_2 \subseteq HgH$ such that for any $h_1, h_2 \in H$, we have $h_1gH \cap Hgh_2 \neq \varnothing$ from part **(a)**. We saw earlier that one such element in the intersection was h_1gh_2, so that this element is a representative for the left and right coset we identified. So we can write $h_1gH = h_1gh_2H$ and $Hgh_2 = Hh_1gh_2$. What remains is to find n many left (right) cosets in this form which are all disjoint from each other, because there are exactly n disjoint left (right) cosets of H in G.

In the double coset HgH there may be $n' \leq n$ many left (right) cosets that are contained in HgH. For some left coset $a_iH \subseteq HgH$, it will have a nonempty intersection with some right coset $Hb_i \subseteq HgH$, with g_i being an element of this intersection. Hence it is a representative for both, so that $a_iH = g_iH$ and $Hb_i = Hg_i$. We can pick some other left coset $a_jH \subseteq HgH$ and a right coset $Hb_j \subseteq HgH$ where a_j is disjoint from a_i and b_j is disjoint from b_i, and again find a single representative g_j to form the disjoint left (right) cosets g_jH and Hg_j.

We may do this n' many times to find $n' \leq n$ many disjoint left and right cosets g_iH and Hg_i of G. After exhausting HgH of all of these cosets, if we do not have n many disjoint cosets, we consider another distinct double coset $Hg'H$, and find more (say n'' many more) distinct cosets g_kH and Hg_k which because HgH and $Hg'H$ are disjoint, it forces these new cosets g_kH and Hg_k to be disjoint to the old ones g_iH and Hg_i (left and right cosets respectively).

We may do this for every double coset which is part of the partition of G because they are disjoint from each other. Because the union of each of these double cosets is G, we must necessarily be able to find n many left and right cosets g_iH and Hg_i in total, after finding a smaller number (or equal number if there is only one double coset) of left and right cosets in each of the HgH that partition G.

Therefore there exist $g_1, \ldots, g_n$ in G that belong to distinct left cosets and to distinct right cosets. Thus the disjoint unions $\cup_{i=1}^n g_iH$ and $\cup_{i=1}^n Hg_i$ are equal to G. $\square$