Solution Manual

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33.7 Exercises

33.7.1

33.7.4

33.7.7

We are asked to transform from the (u, v) coordinates into the (x, y) coordinates. Since the region D' is a square, we may represent the boundary of the square with four line segments lying on these four lines: $u = 0$, $u = 1$, $v = 0$, and $v = 1$.

Using the equations given in the problem we may solve for the boundary of D in the (x, y) coordinate system. It is immediate that $x = 0$ and $x = 1$ are lines that form part of the new boundary.

Since y depends on both u and v, we may want to hold one of those variables constant and observe what happens if we vary the other (while still maintaining the bounds given by D'). Trying with $v = 0$, it is apparent that $y = 0$. That is fine. Then if we try with $v = 1$, we can still vary u. According to the definition of D', u varies from 0 to 1, exactly as x does (as per $x = u$). Therefore we may give $y = 1 - x^2$.

Hence D is the area bounded by the line $x = 0$, $y = 0$, and $y = 1 - x^2$. (the area under the parabola in the first quadrant)

33.7.11

We first find equations of lines passing through those four coordinate points. They are given by $y = x + 4$, $y = x - 4$, $y = -\frac{1}{3}x + \frac{8}{3}$, and $y = -\frac{1}{3}x$. We may convert each of these lines to their corresponding lines in the (u, v) coordinate system using the transformation given in the problem. Simply substitute the given equations for y and x into each of the lines and resolve into coordinate curves.

We find the following curves: $u = \pm 4$, $v = 0$, and $v = 8$. This is a rectangular region so it is convenient in the integral. The Jacobian of transformation for this change of variables is given by the determinant:

$$\left| \det \begin{pmatrix} -\frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} \right| = \frac{1}{4}$$

The integral becomes

$$\int_0^8 \int_{-4}^4 \left(8\left(\frac{v-3u}{4}\right)+4\left(\frac{u+v}{4}\right)\right)\left(\frac{1}{4}\right) dudv \to 6\int_0^8 vdv = 192$$

33.7.13

We first want to find the new bounds in the (u, v) coordinate system. Observe that

$$1 = x^2 - y^2 = (u\cosh(v))^2 - (u\sinh(v))^2 = u^2$$
$$4 = x^2 - y^2 = (u\cosh(v))^2 - (u\sinh(v))^2 = u^2$$

and since we are in the first quadrant we will also take u to be positive, so $1 \le u \le 2$. Then for v we use the other two equations given earlier:

$$x = 2y \to u\cosh(v) = 2u\sinh(v) \to \frac{1}{2} = \tanh(v) \to v = \tanh^{-1}\left(\frac{1}{2}\right)$$

$$x = 4y \to u\cosh(v) = 4u\sinh(v) \to \frac{1}{4} = \tanh(v) \to v = \tanh^{-1}\left(\frac{1}{4}\right)$$

Using the known equation for the inverse hyperbolic tangent

$$\tanh^{-1}(z) = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

deduce that $\frac{1}{2}\ln\left(\frac{5}{3}\right) \le v \le \frac{1}{2}\ln(3)$. This is a square region in the new coordinate system, which is convenient. Compute the Jacobian matrix for the coordinate transformation given in the problem:

$$J = \left|\det\begin{pmatrix} x'_u & y'_u \\ x'_v & y'_v \end{pmatrix}\right| = \left|\det\begin{pmatrix} \cosh(v) & \sinh(v) \\ u\sinh(v) & u\cosh(v) \end{pmatrix}\right|$$

Find that $J = |u(\cosh^2(v) - \sinh^2(v))| = u$. Now we may set up the new integral and solve:

$$\int_{\frac{1}{2}\ln\left(\frac{5}{3}\right)}^{\frac{1}{2}\ln(3)} \int_1^2 (u^2\cosh^2(v) - u^2\cosh^2(v))^{-\frac{1}{2}} ududv = \int_{\frac{1}{2}\ln\left(\frac{5}{3}\right)}^{\frac{1}{2}\ln(3)} \int_1^2 (1)dudv$$

$$\int_{\frac{1}{2}\ln\left(\frac{5}{3}\right)}^{\frac{1}{2}\ln(3)} (1)dv = \ln(3) - \frac{1}{2}\ln(5)$$

33.7.14

From the original bounds we can rewrite them as $\frac{x}{y} = 1$, $\frac{x}{y} = \frac{1}{2}$, $x + y = 1$, and $x + y = 2$. Then knowing that $u = \frac{x}{y}$ and $v = x + y$, it is apparent that

$\frac{1}{2} \leq u \leq 1$ and $1 \leq v \leq 2$. The Jacobian of transformation is found by observing the following:

$$dudv = Jdxdy \to \frac{1}{J}dudv = dxdy$$

$$J = \left| \det \begin{pmatrix} u'_x & v'_x \\ u'_y & v'_y \end{pmatrix} \right| = \left| \det \begin{pmatrix} \frac{1}{y} & 1 \\ \frac{-x}{y^2} & 1 \end{pmatrix} \right| = \frac{x+y}{y^2}$$

Remembering to take the reciprocal of J, the integral becomes

$$\iint e^{\frac{x}{y}} \frac{(x+y)^3}{y^2} \left(\frac{y^2}{x+y} \right) dudv \to \int_1^2 \int_{\frac{1}{2}}^1 e^u v^2 dudv \to \left(e^u|_1^2 \right) \left(\frac{v^3}{3}\bigg|_1^2 \right)$$

$$= \frac{7}{3}(e - \sqrt{e})$$

33.7.19

The lazy way out is sometimes the easiest way out. Give $u = xy$ and $v = xy^2$. Then it is evident that $1 \leq u \leq 2$ and $1 \leq v \leq 2$ from the definitions of the bounding curves in the (x, y) coordinate system. Then to compute the Jacobian we may want to compute it for the reverse transformation, that is to compute J for

$$dudv = Jdxdy$$

So compute partial derivatives of u and v and compute the following determinant:

$$J = \left| \det \begin{pmatrix} y & 2xy \\ x & x^2 \end{pmatrix} \right| = yx^2$$

So $dudv = (yx^2)dxdy$, which is already in the integral. Conveniently the integral becomes

$$\int_1^2 \int_1^2 dudv = 1$$

33.7.21

Give $u = x + y$ and $v = y - x^3$, so that $1 \leq u \leq 2$ and $0 \leq v \leq 1$. Computing the Jacobian J that satisfies $\frac{1}{J}dudv = dydx$ by taking the partial derivatives of u and v, we find

$$J = \left| \det \begin{pmatrix} 1 & -3x^2 \\ 1 & 1 \end{pmatrix} \right| = 1 + 3x^2$$

This makes the integral convenient since it makes the new integrand 1. The double integral becomes (by geometry)

$$\int_0^1 \int_1^2 dudv = 1$$

33.7.22

With some rearranging of the parabola equations it becomes apparent that we may give $u = xy$ and $v = y - x^2$ to find that $-1 \leq u \leq 1$ and $1 \leq v \leq 2$. Then we may compute the Jacobian where

$$dudv = Jdxdy$$

This is given by the following determinant:

$$J = \left| \det \begin{pmatrix} u'_x & v'_x \\ u'_y & v'_y \end{pmatrix} \right| = \left| \det \begin{pmatrix} y & -2x \\ x & 1 \end{pmatrix} \right| = y + 2x^2$$

So $dudv = (y + 2x^2)dxdy$, which is conveniently already in the integral. The integral then becomes

$$\int_1^2 \int_{-1}^1 dudv = 2$$

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