Another proof for the infinitude of primes

using elementary methods

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outline

1. setting up
2. getting leverage

statement

Theorem

There are infinitely many prime numbers.

Proof. (variation of Euler's?)

$$\vdots$$

setting up

the number of primes less than x

Let $\pi(x) \coloneqq |\mathbb{P}_{\leq x}|$, where $\mathbb{P}_{\leq x} = \{p \leq x \mid p \in \mathbb{P}\}$, be the number of primes which are less than or equal to the real number x.

Number the primes in ascending order so that we can write

$$\mathbb{P} = \{p_1, p_2, p_3, \dots\} \text{ and } \mathbb{P}_{\leq x} = \left\{p_1, p_2, \dots, p_{\pi(x)}\right\},$$

where $p_1 < p_2 < \cdots < p_{\pi(x)} < \cdots$.

We wish to show that $\pi(x)$ is unbounded.

comparing areas

Let $n = \lfloor x \rfloor \leq x < n + 1$ and consider the two integrals

$$\int_1^x \frac{1}{t}\,\mathrm{d}t \text{ and } \int_1^{n+1} \frac{1}{\lfloor t \rfloor}\,\mathrm{d}t\,.$$

The first integral is equal to $\log(x)$ (by definition) and the second one is equal to

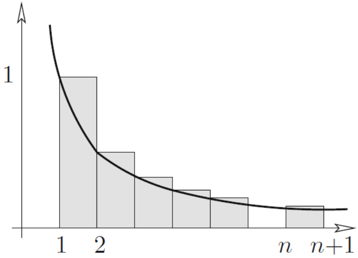
$$\sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} + \frac{1}{n}.$$

comparing areas

Furthermore, note that

$$\log(x) \leq \sum_{i=1}^{n} \frac{1}{i}.$$

This is most apparent with the image from the textbook.



getting leverage

new summation

Now consider the infinite sum

$$\sum \frac{1}{m},$$

where we sum over all $m \in \mathbb{N}$ which may have only prime divisors which are less than or equal to x. In other words, only primes $p \in \mathbb{P}_{\leq x}$ could divide each m.

this sum is larger?

We will show that this new sum, $\sum m^{-1}$, is larger than $\sum_{i=1}^{n} i^{-1}$ and hence larger than $\log(x)$.

Observe that each of the terms in the sum $\sum_{i=1}^{n} i^{-1}$ are terms in this new sum, as every i is less than x and as a result all the prime divisors of each i will be less than x. Then there will be infinitely many more extra nonzero terms in the sum, so we must have that this new sum exceeds $\log(x)$.

rewriting the new sum

We want to rewrite the sum

$$\sum \frac{1}{m}$$

in a more convenient manner.

Recall the fundamental theorem of arithmetic says that every positive integer has a unique prime factorization (up to permutation of the prime factors).

So each m has a unique prime factorization given by some $\displaystyle\prod_{p \in \mathbb{P}_{\leq x}} p^{k_p}$.

cooler way to write the sum

We write

$$\sum \frac{1}{m} = \prod_{p \in \mathbb{P}_{\leq x}} \left(\sum_{k=0}^{\infty} \frac{1}{p^k} \right),$$

and to see how these are equal, observe how the full expansion can produce every term in the original sum:

$$\left(\frac{1}{p_1} + \frac{1}{p_1^2} + \cdots \right) \left(\frac{1}{p_2} + \frac{1}{p_2^2} + \cdots \right) \cdots \left(\frac{1}{p_{\pi(x)}} + \frac{1}{p_{\pi(x)}^2} + \cdots \right)$$

algebra

Each sum in the product is a geometric series, so

$$\log(x) \leq \prod_{p\in\mathbb{P}_{\leq x}} \left(\sum_{k=0}^{\infty} \frac{1}{p^k} \right) = \prod_{p\in\mathbb{P}_{\leq x}} \frac{1}{1-\frac{1}{p}} = \prod_{p\in\mathbb{P}_{\leq x}} \frac{p}{p-1} = \prod_{j=1}^{\pi(x)} \frac{p_j}{p_j-1}.$$

Furthermore, $p_j \geq j+1$, so we bound above each factor in the product:

$$\frac{p_j}{p_j-1} = \frac{(p_j-1)+1}{p_j-1} = 1 + \frac{1}{p_j-1} \leq 1 + \frac{1}{j} = \frac{j+1}{j}$$

satisfying part

Finally see that

$$\log(x) \leq \prod_{j=1}^{\pi(x)} \frac{p_j}{p_j - 1} \leq \prod_{j=1}^{\pi(x)} \frac{j+1}{j} = \pi(x) + 1.$$

So $\log(x) - 1 \leq \pi(x)$, meaning we found a lower bound on the number of primes less than x. The lower bound, $\log(x) - 1$, is unbounded, which implies $\pi(x)$ is unbounded as well and so we find there are infinitely many primes. $\square$