1 previous chapters TODO

2 (ch2)

2.1 Analytic Functions

A function is analytic

3 Elementary Functions

...

3.1 Branches of the Logarithm

Let $\alpha \in \mathbb{R}$. Let

$$D_\alpha = \left\{ z = re^{i\theta} : r > 0, \alpha < \theta < \alpha + 2\pi \right\}.$$

For $z \in D_\alpha$ define

$$\log_\alpha(z) := \log(|z|) + i \arg(z)$$

where $\alpha < \arg(z) < \alpha + 2\pi$

Note. In the notation for $\log_\alpha(z)$, α does <u>NOT</u> denote the base of the logarithm. This function is called a branch of the logarithm function.

Theorem 3.1. *Let $\alpha \in \mathbb{R}$. The function*

$$\log_\alpha : D_\alpha \to \mathbb{C}$$

is analytic and

$$\frac{\mathrm{d}}{\mathrm{d}z} \log_\alpha(z) = \frac{1}{z}.$$

Example. Let $\alpha = \frac{\pi}{2}$. Find
(i) $\log_{\frac{\pi}{2}}(-i)$ and (ii) $\log_{\frac{\pi}{2}}(1)$.

(i) $\arg(-i) = \frac{-\pi}{2} + 2\pi n$ ($n \in \mathbb{Z}$) Choose $n = 1$. Then $\arg(-i) = \frac{-\pi}{2} + 2\pi = \frac{3\pi}{2}$. Then $\log_{\frac{\pi}{2}}(-i) = \log(|-i|) + i\left(\frac{3\pi}{2}\right) = \log(1) + \frac{3\pi i}{2} = \frac{3\pi i}{2}$.

(2) $\log_{\frac{\pi}{2}}(1) = \log(|1|) + i\,(0 + 2\pi) = 2\pi i$.

Note that $\log_{\frac{\pi}{2}}(z) = \log(|z|) + i\arg(z)$, where $\frac{\pi}{2} < \arg(z) < \frac{5\pi}{2} = \frac{\pi}{2} + 2\pi$.

Note. The principal branch of the logarithm, $\mathrm{Log}(z)$, is equal to $\log_{-\pi}(z)$. (Choose $\alpha = -\pi$)

Properties

Let $z, z_1, z_2 \in \mathbb{C}$.

1. $\exp(\log(z)) = z$ if $z \neq 0$.
2. $\log(\exp(z)) = z + 2\pi n i$ where $n \in \mathbb{Z}$.
3. $\operatorname{Log}(\exp(z)) = z$ if $-\pi < \operatorname{Im}(z) \leq \pi$.
4. $\log(z_1 z_2) = \log(z_1) + \log(z_2)$ if $z_1, z_2 \neq 0$.

Proof. (3.) Let $z = x + iy$, where $x, y \in \mathbb{R}$. Then $\operatorname{Log}(\exp(z)) = \operatorname{Log}(e^x e^{iy}) = \log\big(|e^x e^{iy}|\big) + i\operatorname{Arg}(e^x e^{iy}) = \log(e^x) + iy = x + iy = z$ when $-\pi < y = \operatorname{Im}(z) \leq \pi$. □

Note. In general it is <u>NOT TRUE</u> that

$$\operatorname{Log} z_1 z_2 = \operatorname{Log}(z_1) + \operatorname{Log}(z_2).$$

But $\log(z_1 z_2) = \log(z_1) + \log(z_2)$ if $z_1, z_2 \neq 0$.

Example. Let $z_1 = z_2 = -1 + i$.

Then $|z_1| = |z_2| = \sqrt{(2)}$ and $\operatorname{Arg}(z_1) = \operatorname{Arg}(z_2) = \frac{3\pi}{4} \in (-\pi, \pi]$.

Also $z_1 z_2 = (-1+i)^2 = 1 - 2i + i^2 = -2i$, so $|z_1 z_2| = |-2i| = 2$, and $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(-2i) = \frac{-\pi}{2}$.

Thus $\operatorname{Log}(z_1 z_2) = \log(|z_1 z_2|) + i\operatorname{Arg}(z_1 z_2) = \log(2) + i\left(\frac{-\pi}{2}\right)$.

However, $\operatorname{Log}(z_1) + \operatorname{Log}(z_2) = 2\operatorname{Log}(z_1) = 2\left(\log(|z_1|) + i\operatorname{Arg}(z_1)\right) = 2\left(\log\left(\sqrt{(2)}\right) + i\left(\frac{3\pi}{4}\right)\right) = \log(2) + i\left(\frac{3\pi}{2}\right)$.

So for $z_1 = z_2 = -1 + i$, $\operatorname{Log}(z_1 z_2) \neq \operatorname{Log}(z_1) + \operatorname{Log}(z_2)$.

3.2 Trigonometric Functions

Let $\theta \in \mathbb{R}$. Then

$$\begin{aligned}
e^{i\theta} &= \cos(\theta) + i\sin(\theta) \\
e^{-i\theta} &= \cos(-\theta) + i\sin(-\theta) = \cos(\theta) - i\sin(\theta) \\
2\cos(\theta) &= e^{i\theta} + e^{-i\theta} \\
2i\sin(\theta) &= e^{i\theta} - e^{-i\theta}
\end{aligned}$$

We find that

$$\cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

and

$$\sin(\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

Definition 3.1. Let $z \in \mathbb{C}$. We define

$$\cos(\theta) := \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

and

$$\sin(\theta) := \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right).$$

Properties

1. $\sin(z)$ and $\cos(z)$ are entire functions.
2. $\frac{\mathrm{d}}{\mathrm{d}z}\sin(z) = \cos(z)$ and $\frac{\mathrm{d}}{\mathrm{d}z}\cos(z) = -\sin(z)$.
3. $\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2)$.
4. $\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2)$.
5. $\sin^2(z) + \cos^2(z) = 1$.
6. $\sin\left(z + \frac{\pi}{2}\right) = \cos(z)$.
7. $\sin(z + \pi) = -\sin(z)$.
8. $\cos(z + \pi) = -\cos(z)$.
9. $\sin(z + 2\pi) = \sin(z)$ and $\cos(z + 2\pi) = \cos(z)$
10. $|\sin(z)|^2 =$
11. $|\cos(z)|^2 =$

Proof. (1.) Since $\exp(z)$ is entire, so is $\exp(iz)$ and $\exp(-iz)$. Hence

$$\begin{aligned}\sin(z) &= \frac{1}{2i}\left(\exp(iz) - \exp(-iz)\right)\\ \cos(z) &= \frac{1}{2}\left(\exp(iz) + \exp(-iz)\right)\end{aligned}$$

are entire.

(2.)

$$\begin{aligned}\sin(z) &= \frac{1}{2i}\left(\exp(iz) - \exp(-iz)\right)\\ \frac{\mathrm{d}}{\mathrm{d}z}\sin(z) &= \frac{1}{2i}\left(i\exp(iz) - (-i\exp(-iz))\right)\\ &= \frac{i}{2i}\left(\exp(iz) + \exp(-iz)\right)\\ &= \frac{1}{2}\left(\exp(iz) + \exp(-iz)\right)\\ &= \cos(z)\end{aligned}$$

Exercise. Show $\frac{\mathrm{d}}{\mathrm{d}z}\cos(z) = -\sin(z)$.

(3.) $\sin(z_1 + z_2) = \frac{1}{2i}\left(\exp(i(z_1 + z_2)) - \exp(-i(z_1 + z_2))\right)$ But $\sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2) =$ $\frac{1}{4i}\left(\exp(iz_1) - \exp(-iz_2)\right)\left(\exp(iz_2) + \exp(-iz_1)\right)$... fix and simplify

(5.) $\square$

3.3 Review of Hyperbolic Functions

Let $x \in \mathbb{R}$. Define

$$\sinh(x) \coloneqq \frac{1}{2}\left(e^x - e^{-x}\right)$$

and

$$\cosh(x) \coloneqq \frac{1}{2}\left(e^x + e^{-x}\right).$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh(x) = \frac{1}{2}\left(e^x + e^{-x}\right) = \cosh(x)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh(x) = \frac{1}{2}\left(e^x - e^{-x}\right) = \sinh(x)$$

See that $\sinh^2(x) = \left(\frac{1}{2}\left(e^x - e^{-x}\right)\right)^2 = \frac{1}{4}\left(e^{2x} - 2 + e^{2x}\right)$ and $\cosh^2(x) = \left(\frac{1}{2}\left(e^x + e^{-x}\right)\right)^2 =$. Then $\cosh^2(x) - \sinh^2(x) = stuff = 1$.

some more stuff...

$$\overline{\exp(z)} = \exp(\overline{z})$$