For any group G the *Frattini subgroup* of G (denoted by $\Phi(G)$) is defined to be the intersection of all the maximal subgroups of G (if G has no maximal subgroups, set $\Phi(G) = G$).

1. (DF6.1.24) Say an element x of G is a *nongenerator* if for every proper subgroup H of G, $\langle x, H\rangle$ is also a proper subgroup of G. Prove that $\Phi(G)$ is the set of nongenerators of G (here $|G| > 1$).

   Proof. Let G be a finite group as given.

   Suppose that $x \in \Phi(G)$. By way of contradiction, suppose that x is *not* a nongenerator of G. This means that there exists a proper subgroup K of G such that $\langle x, K\rangle = G$. Observe that x cannot be an element of K, otherwise $\langle x, K\rangle = K < G$ which is not possible. By order considerations there must exist some maximal subgroup M which contains K. We have that $G = \langle x, K\rangle \subseteq \langle x, M\rangle$, and so $x \notin M$, otherwise $G \subseteq \langle x, M\rangle = M$ which is impossible because $M < G$. With $x \notin M$, we reach a contradiction since x must lie in every maximal subgroup by definition of $\Phi(G)$.

   Conversely, suppose that $x \in G$ is a nongenerator but by way of contradiction that $x \notin \Phi(G)$. Then x is not in some maximal subgroup M as a result, so that $\langle x, M\rangle$ is a subgroup of G properly containing M. Due to the maximality of M, the only subgroup $\langle x, M\rangle$ can take on is G. But this is in contradiction with the assumption that x was a nongenerator of G, since M is a proper subgroup of G but $\langle x, M\rangle$ is not a proper subgroup of G.

   Hence $\Phi(G)$ is the set of nongenerators of G. □

2. Auxiliary result. Show that automorphisms of finite groups send maximal subgroups to maximal subgroups, and as a result the Frattini subgroup of a finite group is a characteristic subgroup.

   Proof. Let G be a finite group and let α be any automorphism of G. Then let M be a maximal subgroup of G. We will show that $\alpha(M)$ is a maximal subgroup of G.

   Suppose by way of contradiction that $\alpha(M)$ was not a maximal subgroup. Then there exists a proper subgroup K of G which properly contains $\alpha(M)$. Then since α is an automorphism we may take the preimage of K, and observe that $\alpha^{-1}(K)$ is a proper subgroup of G which properly contains M. This is in contradiction with the assumption that M was a maximal subgroup of G, hence $\alpha(M)$ is a maximal subgroup of G.

   Hence maximal subgroups of G are sent to maximal subgroups of G by any automorphism.

   The Frattini subgroup $\Phi(G)$ is given by the intersection of all of the maximal subgroups of G, and since all of the maximal subgroups of G are sent to maximal subgroups of G by any automorphism, the intersection will remain the same. Hence $\Phi(G)$ is a characteristic subgroup of G. □

3. (DF6.1.25) Let G be a finite group. Prove that $\Phi(G)$ is nilpotent. [Use Frattini's Argument to prove that every Sylow subgroup of $\Phi(G)$ is normal in G.]

Proof. Let G be a finite group as given.

Because the Frattini subgroup $\Phi(G)$ is characteristic and hence normal in G, we can apply Frattini's Argument to $\Phi(G)$. Let P be any Sylow p-subgroup of $\Phi(G)$ for some prime p dividing the order of $\Phi(G)$. Then $G = \Phi(G)N_G(P)$.

We claim that $N_G(P) = G$. Suppose by way of contradiction that $N_G(P)$ is instead a proper subgroup of G, so that it is contained in a maximal normal subgroup M (due to order considerations). Then $G = \Phi(G)N_G(P) \leq \Phi(G)M = M$, where the last equality holds because all of the elements of $\Phi(G)$ are by definition found in M. This is in contradiction with the fact that M was a proper subgroup of G, so we must have that $N_G(P) = G$.

Since $\Phi(G) \leq G$, we have that $\Phi(G)$ normalizes P. Since P was an arbitrary Sylow p-subgroup, for any prime divisor p of $|\Phi(G)|$, every Sylow p-subgroup of $\Phi(G)$ is normal in $\Phi(G)$. By Theorem 3, it follows that $\Phi(G)$ is nilpotent. □

4. (DF6.1.31) For any group G a *minimal normal subgroup* is a normal subgroup M of G such tshat the only normal subgroups of G which are contained in M are 1 and M. Prove that every minimal normal subgroup of a finite solvable group is an elementary abelian p-group for some prime p. [If M is a minimal normal subgroup of G, consider its characteristic subgroups: M' and $\langle x^p \mid x \in M\rangle$.]

Proof. Let G be a finite solvable group as given.

Let H be a minimal (nontrivial) normal subgroup of G. Note that H must be solvable since G is solvable (subgroups of solvable groups are solvable). Then consider the commutator subgroup H' of H. If H' is not trivial, then either $H' = H$ or $H' < H$.

Neither can happen. If $H' = H$, then H is not solvable because the derived series of H is indefinite (each subgroup in the series will be H and the series cannot terminate at 1). This is in contradiction with the fact that H is solvable.

If $1 \neq H' < H$, because H' is a characteristic subgroup of H, it is normal in G as well. This is in contradiction to the minimality of H, as H' is a subgroup of H which is normal in G. So $H' = 1$, which implies that H is abelian.

Then we show that H is a p-group for some prime p dividing $|H|$. Let P be a Sylow p-subgroup of H. Then because H is abelian, P is unique and is hence normal in H. This forces $P = H$ since P is not trivial and also cannot be a proper characteristic subgroup of H since then it would be normal in G (contradicting the minimality of H). Thus H is a p-group.

Consider the characteristic subgroup $\langle x^p \mid x \in H\rangle$ (characteristic because any automorphism of H preserves exponentiation) of H. Then $\langle x^p \mid x \in H\rangle$ cannot be a proper nontrivial subgroup of H as again this would contradict the minimality of H.

Since H is a p-group there is a subgroup of order p (Cauchy's theorem), and as a result the subgroup $\langle x^p \mid x \in H\rangle$ is properly contained in H. (All of the elements in the subgroup of order p when raised to the

p-th power become the identity, so they do not contribute to $\langle x^p \mid x \in H\rangle$.) This forces $\langle x^p \mid x \in H\rangle = 1$, so that every element of H when raised to the p power must be 1.

Then H (with order p^n for some n) must take on the form $Z_p \times \cdots \times Z_p \cong \mathbb{F}_p^n$ (the order of elements in this group is at most $\operatorname{lcm}(p, \ldots, p) = p$). Hence H is an elementary abelian p-group for some prime p. $\square$

5. Auxiliary result. Show that the intersection of two normal subgroups is a normal subgroup.

Proof. Let G be a group with H_1, H_2 normal in G. Then for any $h \in H_1 \cap H_2$, observe that for any $g \in G$, we have that ghg^{-1} is an element of H_1 and is also an element of H_2, since h can be viewed as an element of each normal subgroup. This implies that $H_1 \cap H_2$ is normal in G. $\square$

6. Auxiliary result. Show that the center of a direct product is the direct product of the centers.

Proof. Let $G = G_1 \times G_2 \times \cdots \times G_n$. Then $Z(G)$ is the set of all n-tuples which commute with every element in G. Let $g = (g_1, \ldots, g_n) \in G$ and let $z = (z_1, \ldots, z_n) \in Z(G)$. Then $gz = (g_1z_1, \ldots, g_nz_n) = (z_1g_1, \ldots, z_ng_n) = zg$ if and only if $g_iz_i = z_ig_i$ for $1 \leq i \leq n$. We have that $z_i \in Z(G_i)$ so that $z \in Z(G_1) \times \cdots \times Z(G_n)$, and hence $Z(G) \subseteq Z(G_1) \times \cdots \times Z(G_n)$.

Similarly, let $z' = (z_1', \ldots, z_n') \in Z(G_1) \times \cdots \times Z(G_n)$. Then it follows that $gz' = (g_1z_1', \ldots, g_nz_n') = (z_1'g_1, \ldots, z_n'g_n) = zg$ since $g_iz_i = z_ig_i$ for $1 \leq i \leq n$. Thus $z' \in Z(G)$, and the reverse inclusion holds.

Hence the center of a direct product is the direct product of the centers. $\square$

7. Let $G = A \times A$ be the direct product of two simple groups. Prove that if A is nonabelian then the only normal subgroups of G other than G and the trivial subgroup are $A \times 1$ and $1 \times A$.

Show that this is false if A is abelian.

Proof. Let $G = A \times A$ be the direct product of two simple groups as given. We show that every nontrivial normal subgroup of G other than G is isomorphic to either $A \times 1$ or $1 \times A$. Write $G = A \times A$ as $(A \times 1)(1 \times A)$ (by order considerations this holds).

Let N be a nontrivial proper normal subgroup of G which is not equal to either $A \times 1$ or $1 \times A$. Observe that $A \times 1$ cannot be properly contained in N. If $A \times 1 < N$, then $N/(A \times 1)$ is a proper normal subgroup of $G/(A \times 1) \cong (1 \times A) \cong A$. By the simplicity of A, $N/(A \times 1) = 1$ which does not make sense since $N/(A \times 1)$ is not a trivial group (as $A \times 1 < N$). We reach a contradiction. By reversing the roles of $A \times 1$ and $1 \times A$ it follows that $1 \times A$ cannot be properly contained in N as well. Thus $A \times 1$ and $1 \times A$ are not contained in N.

Consider the commutator subgroup $[N, A \times 1] = \langle h^{-1}a^{-1}ha \mid h \in N, a \in A \times 1\rangle$. Because N and A are normal in G, N is normalized by $A \times 1$ and $A \times 1$ is normalized by N. For $h \in N$ and $a \in A$, the product $h^{-1}a^{-1}ha = h^{-1}(a^{-1}ha) = h^{-1}h' \in N$ and also $h^{-1}a^{-1}ha = (h^{-1}a^{-1}h)a = a'a \in A$. Thus any element of $[N, A \times 1]$ (a finite product of elements of the form $h^{-1}a^{-1}ha$) is in $N \cap (A \times 1)$.

Because the intersection of two normal subgroups is a normal subgroup, we have that $[N, A \times 1]$ is a proper normal subgroup of G and because N is not contained in $A \times 1$ we also have that $[N, A \times 1]$ is a proper normal subgroup of $A \times 1 \cong A$. By the simplicity of A, we have that $[N, A \times 1] = 1$ so that elements of N commute with elements of $A \times 1$.

Repeat the preceding argument with $1 \times A$ in place of $A \times 1$ to find that elements of N commute with elements of $1 \times A$ also. Since $G = (A \times 1)(1 \times A)$, it follows that $N \leq Z(G) = Z(A \times A) = Z(A) \times Z(A)$. And since A is nonabelian, $Z(A)$ is properly contained in A and by simplicity of A must be 1. Nence N is trivial, which is in contradiction to the assumption that N is nontrivial.

Nence the only normal subgroups of G other than G and the trivial subgroup are $A \times 1$ and $1 \times A$. $\square$

We can exhibit a nontrivial proper normal subgroup of $G = A \times A$ which is not $A \times 1$ or $1 \times A$ when A is abelian instead.

Consider the diagonal subgroup given by $\{(a, a) \mid a \in A\}$. Because A is abelian, $G = A \times A$ is also abelian and necessarily the diagonal subgroup is normal in G. It is clear that this nontrivial group is not $A \times 1$ or $1 \times A$, and is also properly contained in G since G contains elements of the form (a_1, a_2) with $a_1 \neq a_2$.

Thus the proposition of the previous problem is false when A is abelian.