

p.58: 1, 2, 4, 5, 6, 7, 11

1.

*Proof.* (1) For  $x \in \mathbb{R}$ , observe that  $f(x) = f(x + 2\pi) = f(x - 2\pi)$  because  $f$  is  $2\pi$ -periodic. Then in the integral

$$\int_{a+2\pi}^{b+2\pi} f(x) \, dx,$$

make the change of variables  $u = x - 2\pi$ ,  $du = dx$ , so that

$$\int_{a+2\pi}^{b+2\pi} f(x) \, dx = \int_a^b f(u + 2\pi) \, du = \int_a^b f(u) \, du = \int_a^b f(x) \, dx.$$

Similarly apply the change of variables  $u = x + 2\pi$ ,  $du = dx$  to

$$\int_{a-2\pi}^{b-2\pi} f(x) \, dx$$

to find

$$\int_{a-2\pi}^{b-2\pi} f(x) \, dx = \int_a^b f(u - 2\pi) \, du = \int_a^b f(u) \, du = \int_a^b f(x) \, dx.$$

□

*Proof.* (2) Using the change of variables  $u = x + a$ ,  $du = dx$ , we have that

$$\int_{-\pi}^{\pi} f(x + a) \, dx = \int_{-\pi+a}^{\pi+a} f(u) \, du = \int_{-\pi+a}^{\pi+a} f(x) \, dx.$$

Then for any  $a$  the following is true:

$$\int_{-\pi+a}^{\pi+a} f(x) \, dx = \int_{-\pi+a}^{-\pi} f(x) \, dx + \int_{-\pi}^{\pi} f(x) \, dx + \int_{\pi}^{\pi+a} f(x) \, dx.$$

But from (1), we can adjust the bounds of the first integral in the sum, so that

$$\int_{-\pi+a}^{-\pi} f(x) \, dx = \int_{-\pi+a+2\pi}^{-\pi+2\pi} f(x) \, dx = \int_{\pi+a}^{\pi} f(x) \, dx = - \int_{\pi}^{\pi+a} f(x) \, dx.$$

So the summation becomes

$$\int_{-\pi+a}^{\pi+a} f(x) \, dx = - \int_{\pi}^{\pi+a} f(x) \, dx + \int_{-\pi}^{\pi} f(x) \, dx + \int_{\pi}^{\pi+a} f(x) \, dx = \int_{-\pi}^{\pi} f(x) \, dx.$$

□

2.

(a) *Proof.* The Fourier series for the  $2\pi$ -periodic function  $f$  is written as

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta},$$

where

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)e^{-in\theta} d\theta.$$

For  $n \geq 1$ ,

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2} \left[ \hat{f}(n) + \hat{f}(-n) + \hat{f}(n) - \hat{f}(-n) \right] \\ \hat{f}(-n) &= \frac{1}{2} \left[ \hat{f}(-n) + \hat{f}(n) + \hat{f}(-n) - \hat{f}(n) \right],\end{aligned}$$

so that

$$\begin{aligned}\hat{f}(n)e^{in\theta} &= \frac{1}{2} \left[ \hat{f}(n) + \hat{f}(-n) \right] e^{in\theta} + \frac{1}{2} \left[ \hat{f}(n) - \hat{f}(-n) \right] e^{in\theta} \\ \hat{f}(-n)e^{-in\theta} &= \frac{1}{2} \left[ \hat{f}(n) + \hat{f}(-n) \right] e^{-in\theta} - \frac{1}{2} \left[ \hat{f}(n) - \hat{f}(-n) \right] e^{-in\theta} \\ \hat{f}(n)e^{in\theta} + \hat{f}(-n)e^{-in\theta} &= \left[ \hat{f}(n) + \hat{f}(-n) \right] \cos(n\theta) + i \left[ \hat{f}(n) - \hat{f}(-n) \right] \sin(n\theta).\end{aligned}$$

Then by pairing up the  $n$  and  $-n$  terms in the Fourier series for  $f(\theta)$ ,

$$\begin{aligned}f(\theta) &\sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta} = \hat{f}(0) + \sum_{n \geq 1} \left( \hat{f}(n)e^{in\theta} + \hat{f}(-n)e^{-in\theta} \right) \\ &= \hat{f}(0) + \sum_{n \geq 1} \left( \left[ \hat{f}(n) + \hat{f}(-n) \right] \cos(n\theta) + i \left[ \hat{f}(n) - \hat{f}(-n) \right] \sin(n\theta) \right).\end{aligned}$$

□

(b) *Proof.* If  $f$  is even, then

$$\hat{f}(n) - \hat{f}(-n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} -f(\theta) \left( e^{in\theta} - e^{-in\theta} \right) d\theta = i \frac{1}{\pi} \int_{-\pi}^{\pi} -f(\theta) \sin(\theta) d\theta = 0,$$

because the product of an even function and an odd function is still odd and so the integral vanishes over the symmetric bounds, so  $\hat{f}(n) = \hat{f}(-n)$ , and the sine component of the sum above vanishes. Similarly the integrand in  $\hat{f}(n) + \hat{f}(-n)$  is an even function, so the integral will not vanish and so the cosine terms will not vanish unless  $f$  is the zero function. Thus the series above becomes a cosine series. □

(c) *Proof.* If  $f$  is odd, then

$$\hat{f}(n) + \hat{f}(-n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \left( e^{in\theta} + e^{-in\theta} \right) d\theta = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(\theta) d\theta = 0,$$

because the product of an even function and an odd function is still odd and so the integral vanishes over the symmetric bounds, so  $\hat{f}(n) = -\hat{f}(-n)$ , and the cosine component of the sum above vanishes. Similarly the integrand in  $\hat{f}(n) - \hat{f}(-n)$  is an even function, so the integral will not vanish and so the sine terms will not vanish unless  $f$  is the zero function. Thus the series above becomes a sine series. □

(d) *Proof.* For odd  $n$ , write  $n = 2k + 1$  where  $k \in \mathbb{Z}$ . Then

$$\hat{f}(n) = \hat{f}(2k + 1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-i(2k+1)\theta} d\theta,$$

and by the change of variables  $\phi = \theta - \pi$  and  $d\phi = d\theta$ ,

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-i(2k+1)\theta} d\theta = \frac{1}{2\pi} \int_{-2\pi}^0 -f(\phi + \pi) e^{-i(2k+1)\phi} d\phi = \frac{1}{2\pi} \int_{-2\pi}^0 -f(\theta) e^{-i(2k+1)\theta} d\theta.$$

Note that because  $f$  is  $\pi$ -periodic, then it is also  $2\pi$  periodic, and we can use earlier results to find that

$$\begin{aligned} \frac{1}{2\pi} \int_{-2\pi}^0 -f(\theta) e^{-i(2k+1)\theta} d\theta &= \frac{1}{2\pi} \int_{-2\pi}^{-\pi} -f(\theta) e^{-i(2k+1)\theta} d\theta + \frac{1}{2\pi} \int_{-\pi}^0 -f(\theta) e^{-i(2k+1)\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{\pi} -f(\theta) e^{-i(2k+1)\theta} d\theta + \frac{1}{2\pi} \int_{-\pi}^0 -f(\theta) e^{-i(2k+1)\theta} d\theta \\ &= -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-i(2k+1)\theta} d\theta = -\hat{f}(n). \end{aligned}$$

So for all odd  $n$ ,  $\hat{f}(n) = -\hat{f}(n)$ , which implies that  $\hat{f}(n) = 0$ . □

(e) *Proof.* Forwards direction. Suppose  $\overline{\hat{f}(n)} = \hat{f}(-n)$  for all  $n$ . Then

$$\overline{\hat{f}(n)} = \overline{\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(\theta)} e^{in\theta} d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{in\theta} d\theta = \hat{f}(-n),$$

and we can rearrange terms to find that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (f(\theta) - \overline{f(\theta)}) e^{in\theta} d\theta = \frac{i}{\pi} \int_{-\pi}^{\pi} \text{Im}(f(\theta)) e^{in\theta} d\theta = 0.$$

This integral can only vanish when  $\text{Im}(f(\theta))$  is simultaneously even and odd, which forces  $\text{Im}(f(\theta))$  to be zero. This means  $f(\theta)$  is real valued. □

4.

(a)

(b) The Fourier coefficients of  $f$  are

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^0 \theta(\pi + \theta) e^{-in\theta} d\theta + \frac{1}{2\pi} \int_0^{\pi} \theta(\pi - \theta) e^{-in\theta} d\theta,$$

which with a substitution in the first integral (shift theta forwards by  $\pi$ ), see that

$$\hat{f}(n) = \frac{1 + (-1)^{n+1}}{2\pi} \int_0^\pi \theta(\pi - \theta)e^{-in\theta} d\theta.$$

Clearly for all even  $n$  the integral vanishes, so we take  $n$  to be an odd integer. Then since  $\theta(\pi - \theta)$  has roots at 0 and  $\pi$ , it is easy to compute the Fourier coefficients:

$$\begin{aligned} \hat{f}(n) &= \frac{1 + (-1)^{n+1}}{2\pi} \int_0^\pi \theta(\pi - \theta)e^{-in\theta} d\theta = \frac{1}{2\pi} \left( \frac{\theta(\pi - \theta)e^{-in\theta}}{-in} \Big|_0^\pi + \frac{-(\pi - 2\theta)e^{-in\theta}}{(-in)^2} \Big|_0^\pi + \frac{-2e^{-in\theta}}{(-in)^3} \Big|_0^\pi \right) \\ &= -\frac{4i}{\pi n^3}. \end{aligned}$$

Then the Fourier series for  $f$  is

$$\sum_{n \text{ odd}} -\frac{4i}{\pi n^3} e^{in\theta} = \sum_{n \text{ odd}} \left( \frac{4 \sin(n\theta)}{\pi n^3} - \frac{4i \cos(n\theta)}{\pi n^3} \right) = \frac{8}{\pi} \sum_{n \text{ odd} \geq 1} \frac{\sin(n\theta)}{n^3},$$

where in the last equality because both sine and the cubing function are odd we may join the negative  $n$  sine terms with the positive  $n$  sine terms, and similarly, the imaginary part vanishes because the cosine function is even but the cubing function is odd. Observe that the series absolutely converges due to the  $n^3$  term in the denominator, so we may say that

$$f(\theta) = \frac{8}{\pi} \sum_{n \text{ odd} \geq 1} \frac{\sin(n\theta)}{n^3}.$$

5. Compute the Fourier coefficients of  $f$ ,

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\delta}^0 \left(1 + \frac{\theta}{\delta}\right) e^{-in\theta} d\theta + \frac{1}{2\pi} \int_0^\delta \left(1 - \frac{\theta}{\delta}\right) e^{-in\theta} d\theta.$$

In the first integral observe that changing variables from  $\theta$  to  $-\theta$  makes it so that

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^\delta \left(1 - \frac{\theta}{\delta}\right) (e^{in\theta} + e^{-in\theta}) d\theta = \frac{1}{\pi} \int_0^\delta \left(1 - \frac{\theta}{\delta}\right) \cos(n\theta) d\theta,$$

and observe that  $\hat{f}(0) = \delta/2\pi$ . Then

$$\hat{f}(n) = \frac{1}{\pi} \int_0^\delta \left(1 - \frac{\theta}{\delta}\right) \cos(n\theta) d\theta = \frac{1}{\pi} \left( \frac{(1 - \frac{\theta}{\delta}) \sin(n\theta)}{n} \Big|_0^\delta + \frac{-\cos(n\theta)}{n^2 \delta} \Big|_0^\delta \right) = \frac{1 - \cos(n\delta)}{n^2 \pi \delta}.$$

The Fourier series for  $f$  is given by

$$\frac{\delta}{2\pi} + \sum_{n=-\infty}^{-1} \left( \frac{1 - \cos(n\delta)}{n^2 \pi \delta} \right) e^{in\theta} + \sum_{n=1}^{\infty} \left( \frac{1 - \cos(n\delta)}{n^2 \pi \delta} \right) e^{in\theta},$$

where because squaring and the cosine function are even, we rewrite the first sum instead to find the following more convenient form:

$$\frac{\delta}{2\pi} + \sum_{n=1}^{\infty} \left( \frac{1 - \cos(n\delta)}{n^2 \pi \delta} \right) e^{-in\theta} + \sum_{n=1}^{\infty} \left( \frac{1 - \cos(n\delta)}{n^2 \pi \delta} \right) e^{in\theta} = \frac{\delta}{2\pi} + 2 \sum_{n=1}^{\infty} \left( \frac{1 - \cos(n\delta)}{n^2 \pi \delta} \right) \cos(n\theta).$$

The sum converges absolutely because of the  $n^2$  term in the denominator, so we may write

$$f(\theta) = \frac{\delta}{2\pi} + 2 \sum_{n=1}^{\infty} \left( \frac{1 - \cos(n\delta)}{n^2\pi\delta} \right) \cos(n\theta).$$

6.

(a)

(b) The Fourier coefficients of  $f$  are

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^0 (-\theta) e^{-in\theta} d\theta + \frac{1}{2\pi} \int_0^{\pi} \theta e^{-in\theta} d\theta.$$

In the first integral change variables from  $\theta$  to  $-\theta$  to find that

$$\begin{aligned} \hat{f}(n) &= \frac{1}{2\pi} \int_0^{\pi} \theta e^{in\theta} d\theta + \frac{1}{2\pi} \int_0^{\pi} \theta e^{-in\theta} d\theta = \frac{1}{\pi} \int_0^{\pi} \theta \cos(n\theta) d\theta \\ &= \frac{-1 + (-1)^n}{\pi n^2} \text{ for } n \neq 0. \end{aligned}$$

Observe that  $\hat{f}(0) = \pi/2$  as a result.

(c) Then the Fourier series of  $f$  in terms of sines and cosines is

$$f(\theta) \sim \frac{\pi}{2} + \sum_{n=-\infty}^{-1} \left( \frac{-1 + (-1)^n}{\pi n^2} \right) e^{in\theta} + \sum_{n=1}^{\infty} \left( \frac{-1 + (-1)^n}{\pi n^2} \right) e^{in\theta},$$

and we again flip the first summation to the positive integers by replacing  $n$  with  $-n$  to see that

$$f(\theta) \sim \frac{\pi}{2} + \sum_{n=1}^{\infty} \left( \frac{-1 + (-1)^n}{\pi n^2} \right) e^{-in\theta} + \sum_{n=1}^{\infty} \left( \frac{-1 + (-1)^n}{\pi n^2} \right) e^{in\theta} = \frac{\pi}{2} + 2 \sum_{n=1}^{\infty} \left( \frac{-1 + (-1)^n}{\pi n^2} \right) \cos(n\theta),$$

but the inner fraction vanishes for all even  $n$ , so we can sum over the odd  $n$  (when  $n$  is odd the fraction is  $-2/\pi n^2$ ). Furthermore this sum converges absolutely due to the  $n^2$  term in the denominator, so we may write

$$f(\theta) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd } \geq 1} \frac{\cos(n\theta)}{n^2}.$$

(d) Let  $\theta = 0$ . Then

$$f(0) = 0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd} \geq 1} \frac{1}{n^2} \implies \frac{\pi^2}{8} = \sum_{n \text{ odd} \geq 1} \frac{1}{n^2}.$$

Furthermore,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n \text{ odd} \geq 1} \frac{1}{n^2} + \sum_{n \text{ even} \geq 1} \frac{1}{n^2},$$

but in the third sum since  $n$  is even ( $2 \mid n$  yields  $4 \mid n^2$ ), we may factor out  $1/4$  from it to find the first sum. So

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n \text{ odd} \geq 1} \frac{1}{n^2} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} \implies \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3} \cdot \frac{\pi^2}{8} = \frac{\pi^2}{6}.$$

7.

(a) *Proof.* Let  $\{a_n\}_{n=1}^N, \{b_n\}_{n=1}^N$  be finite sequences of complex numbers as given. Then

$$\begin{aligned} \sum_{n=M}^N a_n b_n &= \sum_{n=M}^N a_n (B_n - B_{n-1}) \\ &= \sum_{n=M}^N a_n B_n - \sum_{n=M}^N a_n B_{n-1} \\ &= a_N B_N + \sum_{n=M}^{N-1} a_n B_n - \sum_{n=M-1}^{N-1} a_{n+1} B_n \\ &= a_N B_N - a_M B_{M-1} + \sum_{n=M}^{N-1} a_n B_n - \sum_{n=M}^{N-1} a_{n+1} B_n \\ &= a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n. \end{aligned}$$

Thus the summation by parts formula holds. □

(b) *Proof.* Let the partial sums  $B_n$  be bounded above by  $B$ , and let  $\{a_n\}_{n=1}^N$  be a sequence of (positive) real numbers decreasing monotonically to 0. Also require that  $N \geq M$ . Then

$$\begin{aligned} \left| \sum_{n=M}^N a_n b_n \right| &= \left| a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n \right| \\ &\leq B \left( a_N + a_M + \sum_{n=M}^{N-1} (a_{n+1} - a_n) \right) \\ &= B \left( a_N + a_M + \sum_{n=M}^{N-1} a_n - \sum_{n=M+1}^N a_n \right) \\ &= 2Ba_M, \end{aligned}$$

but because we can choose  $M$ , we can make  $2Ba_M$  arbitrarily small and so the sequence of partial sums of  $\sum a_n b_n$  is a Cauchy sequence as a result, which converges. □

11.

*Proof.* Let  $\{f_k\}_{k=1}^{\infty}$  be a sequence of Riemann integrable functions on the interval  $[0, 1]$  such that

$$\int_0^1 |f_k(x) - f(x)| \, dx \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Then for every  $\varepsilon > 0$  there exists  $K \in \mathbb{N}$  such that for  $k > K$ ,  $\int_0^1 |f_k(x) - f(x)| \, dx < \varepsilon$ . So for any  $\varepsilon$ , take  $k > K$  and see that

$$\left| \hat{f}_k(n) - \hat{f}(n) \right| = \left| \int_0^1 (f_k(x) - f(x)) e^{-2\pi i n x} \, dx \right| \leq \int_0^1 |f_k(x) - f(x)| \, dx < \varepsilon,$$

which means that  $\hat{f}_k(n) \rightarrow \hat{f}(n)$  uniformly in  $n$  as  $k \rightarrow \infty$ . □