20.1 Consider an analytic function $W(z) = u(x, y) + iv(x, y)$.

(a) Assuming all required derivatives exist, show that $\nabla^2 u = \nabla^2 v = 0$.

Proof. Let $W(z) = u(x, y) + iv(x, y)$ be sufficiently differentiable as given. Then observe that

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y}$$

and

$$\frac{\partial}{\partial y}\frac{\partial v}{\partial x} = \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2} = -\frac{\partial}{\partial y}\frac{\partial u}{\partial y}.$$

Then

$$\boxed{\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial x} - \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} - \frac{\partial^2 v}{\partial y \partial x} = 0},$$

where the equality of mixed partial derivatives was used. Similarly, we have that

$$\frac{\partial}{\partial x}\frac{\partial v}{\partial x} = \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial}{\partial x}\frac{\partial u}{\partial y}$$

and

$$\frac{\partial}{\partial y}\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial v}{\partial y}$$

imply for the same reasons that

$$\boxed{\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 u}{\partial y \partial x} = 0}.$$

□

(b) Show that

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0.$$

Proof. By the C-R equations,

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = \left(\frac{\partial v}{\partial y}\right)\left(-\frac{\partial v}{\partial x}\right) + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0.$$

□

20.2 Find if $f(z) = \mathrm{Re}(z) = x$ is analytic.

The function f is not analytic.

Proof. Write $f(z) = u(x, y) + iv(x, y)$ with $u(x, y) = x$ and $v(x, y) = 0$. The partial derivatives

$$\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 0$$

exist and are continuous for all $z \in \mathbb{C}$. But $\frac{\partial u}{\partial x} = 1 \neq 0 = \frac{\partial v}{\partial y}$, so the Cauchy-Riemann equations do not hold. Hence $f(z) = \mathrm{Re}(z) = x$ is not differentiable, and hence is not analytic. □

20.3 Evaluate $\oint_C \frac{\mathrm{d}z}{z^2-1}$ where C is the circle $|z|=2$ (traversed only once around).

Let C be the circle $|z|=2$ traversed only once around, and let C_1 be the circle $|z-1|=\frac{1}{2}$ and C_{-1} be the circle $|z+1|=\frac{1}{2}$ both traversed only once around. Then due to contour surgery and Cauchy's integral theorem we have

$$\begin{aligned}\oint_C \frac{\mathrm{d}z}{z^2-1} &= \oint_{C_1} \frac{1/2}{z-1}\,\mathrm{d}z - \oint_{C_{-1}} \frac{1/2}{z+1}\,\mathrm{d}z \\ &= 2\pi i(1/2) - 2\pi i(1/2) \\ &= 0.\end{aligned}$$