19.1 Consider an LCR circuit with resistance R, capacitance C, and inductance L and driven by a time-dependent voltage $V(t) = V_0 \cos(\omega t)$. The resulting time-dependent charge $Q(t)$ in the circuit obeys the differential equation

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V_0 \cos(\omega t).$$

(a) With ansatz $Q(t) = Q_0 e^{i\omega t}$ and 'complex voltage' $V(t) = V_0 e^{i\omega t}$, we have

$$\begin{aligned} L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V_0 \cos(\omega t) &\iff -LQ_0\omega^2 e^{i\omega t} + RiQ_0\omega e^{i\omega t} + \frac{1}{C}e^{i\omega t} = V_0 e^{i\omega t} \\ &\iff Q_0(-L\omega^2 + Ri + \frac{1}{C}) = V_0, \end{aligned}$$

so that

$$Q_0 = \frac{V_0}{(-L\omega^2 + Ri\omega + \frac{1}{C})}.$$

(b) Computing $I(t) = \dot{Q}(t)$, we have

$$\begin{aligned} I(t) = \dot{Q}(t) &= i\omega Q_0 e^{i\omega t} \\ &= \frac{i\omega V_0 e^{i\omega t}}{(-L\omega^2 + Ri\omega + \frac{1}{C})} \\ &= \frac{V(t)}{\frac{-1}{\omega}(-Li\omega^2 - R\omega + \frac{i}{C})} \\ &= \frac{V(t)}{Z} \end{aligned}$$

with $Z = \frac{-1}{\omega}(-Li\omega^2 - R\omega + \frac{i}{C})$.

Then by inspection $\text{Re}(Z) = R$ and $\text{Im}(Z) = L\omega - \frac{1}{C\omega}$ Furthermore,

$$|Z| = \sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}.$$

(c) It is easier to express the denominator of the quantity defining Q_0 in exponential form before taking the reciprocal. Note that V_0 is a real value, and assume it is nonnegative.

So

$$\frac{1}{\rho} = \left|-L\omega^2 + Ri\omega + \frac{1}{C}\right| = \sqrt{\left(-L\omega^2 + \frac{1}{C}\right)^2 + (R\omega)^2},$$

and

$$-\phi = \arctan\left(\frac{R\omega}{-L\omega^2 + \frac{1}{C}}\right),$$

so that

$$Q_0 = \rho e^{i\phi} = \frac{V_0}{\sqrt{\left(-L\omega^2 + \frac{1}{C}\right)^2 + (R\omega)^2}} e^{-i\arctan\left(\frac{R\omega}{-L\omega^2 + \frac{1}{C}}\right)}.$$

(d) When $\phi = -\pi/2$, because $\arctan(\pi/2)$ is undefined (due to division by zero as $\cos(\pi/2) = 0$), we demand

$$\frac{R\omega}{-L\omega^2 + \frac{1}{C}}$$

to also be undefined by division by zero; that is, $-L\omega^2 + \frac{1}{C}$ must be zero. Thus $\omega = 1/\sqrt{LC}$.

Sketching $\rho^2(\omega)$, we have something which looks like:

The width of the resonance is determined by the circuit parameter R (comparing with the solution found in the notes and adapting $\rho^2(\omega)$ to fit that form up to a factor).