p.120: 1, 2; p.161 2, 3, 5, 7

p.120

1. Let $\gamma\colon [a,b] \to \mathbb{R}^2$ be a parameterization for the closed curve Γ.

(a) Prove that γ is a parameterization by arc-length if and only if the length of the curve from $\gamma(a)$ to $\gamma(s)$ is precisely $s-a$, that is,

$$\int_a^s \left|\gamma'(t)\right| \mathrm{d}t = s - a.$$

Proof. First suppose that γ is a parameterization by arc-length. Then $|\gamma'(s)| = 1$ for all s, so that

$$\int_a^s \left|\gamma'(t)\right| \mathrm{d}t = \int_a^s 1 \, \mathrm{d}t = s - a.$$

Conversely, suppose that $\int_a^s |\gamma'(t)| \, \mathrm{d}t = s - a$. We know that γ is of class C^1 and $\gamma'(t) \neq 0$ for all t, so that $|\gamma'|$ is continuous and positive. Furthermore, $s - a = \int_a^s 1 \, \mathrm{d}t$, so we must have that $0 < |\gamma'(t)| \leq 1$. If $|\gamma'|$ is less than 1 at any point $p \in [a,s]$, because $|\gamma'|$ is continuous we may find a $\delta > 0$ small enough so that the integration outside of the δ-neighborhood of p

$$\int_{[a,p-\delta)\cup(p+\delta,s)} \left|\gamma'(t)\right| \mathrm{d}t$$

is bounded above by $s - a - 2\delta$. Then the integration $\int_{[p-\delta,p+\delta]} |\gamma'(t)| \, \mathrm{d}t$ is strictly less than 2δ, so the total integration over $[a,s]$ is less than $s-a$. This is a contradiction, so we must have that $|\gamma'(t)| = 1$; that is, γ is a parameterization by arc-length. □

(b) Prove that any curve Γ admits a parameterization by arc-length.

Proof. Let Γ be any curve and let η be any C^1 parameterization of Γ where $\eta'(t) \neq 0$. Then let $h(s) = \int_a^s |\eta'(t)| \, \mathrm{d}t$, so that the composition $\gamma = \eta \circ h^{-1}$ is differentiable. Then by directly computing, we have

$$\left|\gamma'(t)\right| = \left|(h^{-1})'(t) \cdot \frac{\mathrm{d}\eta(h^{-1}(t))}{\mathrm{d}h^{-1}(t)}\right| = \left|\left(\frac{\mathrm{d}h(h^{-1}(t))}{\mathrm{d}h^{-1}(t)}\right)^{-1} \cdot \frac{\mathrm{d}\eta(h^{-1}(t))}{\mathrm{d}h^{-1}(t)}\right| = \left|\frac{\mathrm{d}\eta(h^{-1}(t))}{\mathrm{d}h^{-1}(t)}\right|^{-1} \cdot \left|\frac{\mathrm{d}\eta(h^{-1}(t))}{\mathrm{d}h^{-1}(t)}\right|,$$

which means that $|\gamma'(t)| = 1$ for all t. This means that γ is a parameterization by arc-length, which means Γ admits a parameterization by arc-length. □

2. Suppose $\gamma\colon [a,b] \to \mathbb{R}^2$ is a parameterization for a closed curve Γ, with $\gamma(t) = (x(t), y(t))$.

(a) Show that

$$\frac{1}{2}\int_a^b (x(s)y'(s) - y(s)x'(s)) \, \mathrm{d}s = \int_a^b x(s)y'(s) \, \mathrm{d}s = -\int_a^b y(s)x'(s) \, \mathrm{d}s$$

Proof. Using integration by parts, we have

$$\begin{aligned}\int_a^b x(s)y'(s)\,\mathrm{d}s &= y(t)x(t)\Big|_a^b - \int_a^b y(s)x'(s)\,\mathrm{d}s \\ &= -\int_a^b y(s)x'(s)\,\mathrm{d}s\,,\end{aligned}$$

where we used the fact that Γ was a closed curve.

Thus

$$\frac{1}{2}\int_a^b (x(s)y'(s) - y(s)x'(s))\,\mathrm{d}s = 2\cdot\frac{1}{2}\int_a^b x(s)y'(s)\,\mathrm{d}s = 2\cdot\frac{-1}{2}\int_a^b y(s)x'(s)\,\mathrm{d}s\,.$$

$\square$

(b) Define the **reverse parameterization** of γ by $\gamma^- \colon [a,b] \to \mathbb{R}^2$ with $\gamma^-(t) = \gamma(b+a-t)$. The image of γ^- is precisely Γ, except that the points $\gamma^-(t)$ and $\gamma(t)$ travel in opposite directions. Thus γ^- "reverses" the orientation of the curve. Prove that

$$\int_\gamma (x\,\mathrm{d}y - y\,\mathrm{d}x) = -\int_{\gamma^-} (x\,\mathrm{d}y - y\,\mathrm{d}x).$$

In particular, we may assume (after a possible change in orientation) that

$$\mathcal{A} = \frac{1}{2}\int_a^b (x(s)y'(s) - y(s)x'(s))\,\mathrm{d}s = \int_a^b x(s)y'(s)\,\mathrm{d}s\,.$$

Proof. Changing variables from $b+a-t \mapsto t$ and $-\,\mathrm{d}t \mapsto \mathrm{d}t$, we have

$$\begin{aligned}\int_\gamma (x\,\mathrm{d}y - y\,\mathrm{d}x) &= \int_a^b (x(t)y'(t)\,\mathrm{d}t - y(t)x'(t)\,\mathrm{d}t) \\ &= -\int_{b+a-a}^{b+a-b} (x(b+a-t)y'(b+a-t)\,\mathrm{d}t - y(b+a-t)x'(b+a-t)\,\mathrm{d}t) \\ &= -\int_{\gamma^-} (x\,\mathrm{d}y - y\,\mathrm{d}x)\end{aligned}$$

where we used the fact that $\gamma^-(t) = \gamma(b+a-t)$. $\square$

p.161

2. Let f and g be the functions defined by

$$f(x) = \chi_{[-1,1]}(x) = \begin{cases} 1 & \text{if } |x| \le 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1-|x| & \text{if } |x| \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

Although f is not continuous, the integral defining its Fourier series still makes sense. Show that

$$\hat{f}(\xi) = \frac{\sin(2\pi\xi)}{\pi\xi} \quad \text{and} \quad \hat{g}(\xi) = \left(\frac{\sin(\pi\xi)}{\pi\xi}\right)^2,$$

with the understanding that $\hat{f}(0) = 2$ and $\hat{g}(0) = 1$.

Proof. Direct computation of the Fourier transforms yield

$$\begin{aligned}\hat{f}(\xi) = \int_{-\infty}^{\infty} \chi_{[-1,1]}(x) e^{-2\pi i x\xi}\,\mathrm{d}x &= \int_{-1}^{1} e^{-2\pi i x\xi}\,\mathrm{d}x \\ &= \frac{e^{-2\pi i x\xi}}{-2\pi i\xi}\bigg|_{-1}^{1} \\ &= \frac{\sin(2\pi\xi)}{\pi\xi}\end{aligned}$$

and

$$\begin{aligned}\hat{g}(\xi) = \int_{-\infty}^{\infty} g(x) e^{-2\pi i x\xi}\,\mathrm{d}x &= \int_{-1}^{1} (1-|x|) e^{-2\pi i x\xi}\,\mathrm{d}x \\ &= 2\int_{0}^{1} (1-x)\cos(2\pi x\xi)\,\mathrm{d}x \\ &= \frac{1-\cos(2\pi\xi)}{2\pi^2\xi^2} = \left(\frac{\sin(\pi\xi)}{\pi\xi}\right)^2\end{aligned}$$

with $\hat{f}(0) = \lim_{\xi\to 0} \frac{\sin(2\pi\xi)}{\pi\xi} = 2$, and $\hat{g}(0) = \lim_{\xi\to 0}\left(\frac{\sin(\pi\xi)}{\pi\xi}\right)^2 = 1$. $\square$

3. The following exercise illustrates the principle that the decay of $\hat{f}$ is related to the continuity properties of f.

(a) Suppose that f is a function of moderate decrease on $\mathbb{R}$ whose Fourier transform $\hat{f}$ is continuous and satisfies

$$\hat{f}(\xi) = O\left(\frac{1}{|\xi|^{1+\alpha}}\right) \quad \text{as } |\xi| \to \infty$$

for some $0 < \alpha < 1$. Prove that f satisfies a Hölder condition of order α, that is, that

$$|f(x+h) - f(x)| \le M|h|^{\alpha} \quad \text{for some } M > 0 \text{ and all } x, h \in \mathbb{R}.$$

Proof. Use the Fourier inversion formula to write

$$f(x+h) - f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x\xi}\left(e^{2\pi i h\xi} - 1\right)\mathrm{d}\xi\,,$$

so that

$$\begin{aligned}
|f(x+h)-f(x)||h|^{-\alpha} &\leq |h|^{-\alpha}\left|\int_{-\infty}^{\infty}\hat{f}(\xi)e^{2\pi i x\xi}\left(e^{2\pi i h\xi}-1\right)\mathrm{d}\xi\right| \\
&\leq \int_{-\infty}^{\infty}\frac{C|h|^{-\alpha}}{1+|\xi|^{1+\alpha}}\left|2ie^{\pi i h\xi}\right|\left|\frac{e^{\pi i h\xi}-e^{-\pi i h\xi}}{2i}\right|\mathrm{d}\xi \\
&\leq \int_{-\infty}^{\infty}\frac{2C|\sin(\pi h\xi)|}{|h|^{\alpha}+|h|^{-1}|h\xi|^{1+\alpha}}\mathrm{d}\xi \\
&\leq \frac{2C}{\pi^{1+\alpha}}\int_{-\infty}^{\infty}\frac{|\sin(t)|}{|h|^{1+\alpha}+|t|^{1+\alpha}}\mathrm{d}t \\
&\leq \frac{4C}{\pi}\int_{0}^{\infty}\frac{|\sin(t)|}{t^{1+\alpha}}\mathrm{d}t \\
&\leq \frac{4C}{\pi}\int_{0}^{\infty}\frac{1}{t^{1+\alpha}}\mathrm{d}t \\
&\leq M,
\end{aligned}$$

so that $|f(x+h)-f(x)| \leq M|h|^{\alpha}$. □

(b) Let f be a continuous function on $\mathbb{R}$ which vanishes for $|x| \geq 1$, with $f(0) = 0$ and which is equal to $1/\log(1/|x|)$ for all x in a neighborhood of the origin. Prove that $\hat{f}$ is not of moderate decrease. In fact there is no $\varepsilon > 0$ so that $\hat{f}(\xi) = O(1/|\xi|^{1+\varepsilon})$ as $|\xi| \to \infty$.

Proof. Investigating $f(0) = 0$ and $f(h) = 1/\log\left(h^{-1}\right)$, we have that $|f(h)-f(0)|/|h|^{\alpha} = 1/(|h|^{\alpha}\log(h))$, but for any fixed α we can choose h as small as we like so that this quantity becomes unbounded. So by the contrapositive to part (a), we should not have that f is of moderate decrease. □

5. Suppose f is continuous and of moderate decrease.

(a) Prove that $\hat{f}$ is continuous and $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

Proof. We have

$$\begin{aligned}
\left|\hat{f}(\xi+h)-\hat{f}(\xi)\right| &= \left|\int_{-\infty}^{\infty}f(x)e^{-2\pi i x\xi}(e^{-2\pi i x h}-1)\,\mathrm{d}x\right| \\
&\leq \int_{-\infty}^{\infty}|f(x)|\left|-2ie^{-\pi i x h}\right|\left|\frac{e^{\pi i x h}-e^{-\pi i x h}}{2i}\right|\mathrm{d}x \\
&\leq \int_{-\infty}^{\infty}\frac{2C|\sin(\pi x h)|}{1+x^2}\,\mathrm{d}x\,.
\end{aligned}$$

Then for any $\varepsilon > 0$, we may choose K large enough so that $\int_{|x|>K}\frac{2C|\sin(\pi x h)|}{1+x^2}\,\mathrm{d}x < \varepsilon/2$ and choose δ small enough so that for $|h| < \delta$ we have $\int_{|x|\leq K}\frac{2C|\sin(\pi x h)|}{1+x^2}\,\mathrm{d}x < \varepsilon/2$ since $\sin(\pi x h)$ can be made as small as we like if we take h to be small. With this choice of δ (which depended on K) we have $\left|\hat{f}(\xi+h)-\hat{f}(\xi)\right| < \varepsilon$, so that $\hat{f}$ is continuous.

Then observe that

$$\begin{aligned}\frac{1}{2}\int_{-\infty}^{\infty}[f(x)-f(x-1/(2\xi))]e^{-2\pi ix\xi}\,\mathrm{d}x &= \frac{1}{2}\hat{f}(\xi)+\frac{1}{2}\int_{-\infty}^{\infty}-f(x-1/(2\xi))e^{-2\pi ix\xi}\,\mathrm{d}x\\ &= \frac{1}{2}\hat{f}(\xi)+\int_{-\infty}^{\infty}-f(x)e^{-2\pi ix\xi}e^{-\pi i}\,\mathrm{d}x\\ &= \frac{1}{2}\hat{f}(\xi)+\frac{1}{2}\hat{f}(\xi)=\hat{f}(\xi).\end{aligned}$$

Then as $|\xi|\to\infty$, we have $x-1/(2\xi)\to x$. So using the Lebesgue dominated convergence theorem (as $f(x-1/(2\xi))$ tends to $f(x)$ and $f(x-1/(2\xi))$ is dominated above by $f(x)+C$ for large enough C),

$$\begin{aligned}\left|\hat{f}(\xi)\right| &\leq \int_{-\infty}^{\infty}|f(x)-f(x-1/(2\xi))|\,\mathrm{d}x\\ &\leq 0 \quad \text{as } |\xi|\to\infty.\end{aligned}$$

Hence $\hat{f}$ is continuous and $\hat{f}(\xi)\to 0$ as $|\xi|\to\infty$. □

(b) Show that if $\hat{f}(\xi)=0$ for all ξ, then f is identically zero.

Proof. Let $\hat{f}(\xi)=0$ for all ξ.

In general, by interchanging the order of integration whenever $g\in\mathcal{S}(\mathbb{R})$, we have

$$\int_{-\infty}^{\infty}f(x)\hat{g}(x)\,\mathrm{d}x=\int_{-\infty}^{\infty}f(x)\int_{-\infty}^{\infty}g(y)e^{-2\pi iyx}\,\mathrm{d}y\,\mathrm{d}x=\int_{-\infty}^{\infty}g(y)\int_{-\infty}^{\infty}f(x)e^{-2\pi iyx}\,\mathrm{d}x\,\mathrm{d}y=\int_{-\infty}^{\infty}\hat{f}(y)g(y)\,\mathrm{d}y\,.$$

The Gauss kernel $K_\delta(t-x)$ viewed as a function of x is in the Schwartz space, so it has a preimage $g(x)\in\mathcal{S}(\mathbb{R})$ under the Fourier transformation. We have for all $\delta>0$ and any t that

$$0=\int_{-\infty}^{\infty}\hat{f}(x)g(x)\,\mathrm{d}x=\int_{-\infty}^{\infty}f(x)K_\delta(t-x)\,\mathrm{d}x\,,$$

and because the Gauss kernel is a good kernel, as $\delta\to 0$, we have that the integral on the right converges uniformly to $f(t)$. So for any t, $f(t)\equiv 0$. □

7. Prove that the convolution of two functions of moderate decrease is a function of moderate decrease.

Proof. Let f,g be functions of moderate decrease. Then

$$\begin{aligned}|f*g|=\left|\int_{-\infty}^{\infty}f(x-y)g(y)\,\mathrm{d}y\right| &\leq \int_{|y|\leq|x|/2}|f(x-y)||g(y)|\,\mathrm{d}y+\int_{|y|\geq|x|/2}|f(x-y)||g(y)|\,\mathrm{d}y\\ &\leq \int_{|y|\leq|x|/2}\frac{C_1|g(y)|}{1+(x-y)^2}\,\mathrm{d}y+\int_{|y|\geq|x|/2}\frac{C_2|f(x-y)|}{1+y^2}\,\mathrm{d}y\\ &\leq \int_{|y|\leq|x|/2}\frac{C_1|g(y)|}{1+(x/2)^2}\,\mathrm{d}y+\int_{|y|\geq|x|/2}\frac{C_2|f(x-y)|}{1+(x/2)^2}\,\mathrm{d}y\\ &\leq \frac{4C_1}{4+x^2}A+\frac{4C_2}{4+x^2}B\\ &\leq \frac{C_3}{1+x^2}.\end{aligned}$$

The convolution is an integral so it is continuous. Thus $f*g$ is of moderate decrease. □

5. Suppose f is continuous and of moderate decrease.

(a) Prove that $\hat{f}$ is continuous and $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

Proof. We have

$$\begin{aligned}\left|\hat{f}(\xi+h) - \hat{f}(\xi)\right| &= \left|\int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi}(e^{-2\pi i x h} - 1)\, \mathrm{d}x\right| \\ &\leq \int_{-\infty}^{\infty} |f(x)| \left|-2ie^{-\pi i x h}\right| \left|\frac{e^{\pi i x h} - e^{-\pi i x h}}{2i}\right| \mathrm{d}x \\ &\leq \int_{-\infty}^{\infty} \frac{2C|\sin(\pi x h)|}{1+x^2}\, \mathrm{d}x\,.\end{aligned}$$

Then for any $\varepsilon > 0$, we may choose K large enough so that $\int_{|x|>K} \frac{2C|\sin(\pi x h)|}{1+x^2}\, \mathrm{d}x < \varepsilon/2$ and choose δ small enough so that for $|h| < \delta$ we have $\int_{|x|\leq K} \frac{2C|\sin(\pi x h)|}{1+x^2}\, \mathrm{d}x < \varepsilon/2$ since $\sin(\pi x h)$ can be made as small as we like if we take h to be small. With this choice of δ (which depended on K) we have $\left|\hat{f}(\xi+h) - \hat{f}(\xi)\right| < \varepsilon$, so that $\hat{f}$ is continuous.

Then observe that

$$\begin{aligned}\frac{1}{2}\int_{-\infty}^{\infty} [f(x) - f(x - 1/(2\xi))] e^{-2\pi i x \xi}\, \mathrm{d}x &= \frac{1}{2}\hat{f}(\xi) + \frac{1}{2}\int_{-\infty}^{\infty} -f(x - 1/(2\xi)) e^{-2\pi i x\xi}\, \mathrm{d}x \\ &= \frac{1}{2}\hat{f}(\xi) + \int_{-\infty}^{\infty} -f(x) e^{-2\pi i x \xi} e^{-\pi i}\, \mathrm{d}x \\ &= \frac{1}{2}\hat{f}(\xi) + \frac{1}{2}\hat{f}(\xi) = \hat{f}(\xi).\end{aligned}$$

Then as $|\xi| \to \infty$, we have $x - 1/(2\xi) \to x$. So using the Lebesgue dominated convergence theorem (as $f(x - 1/(2\xi))$ tends to $f(x)$ and $f(x - 1/(2\xi))$ is dominated above by $f(x) + C$ for large enough C),

$$\begin{aligned}\left|\hat{f}(\xi)\right| &\leq \int_{-\infty}^{\infty} |f(x) - f(x - 1/(2\xi))|\, \mathrm{d}x \\ &\leq 0 \quad \text{as } |\xi| \to \infty.\end{aligned}$$

Hence $\hat{f}$ is continuous and $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$. $\square$