HOMEWORK 9

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Given $L \in \mathbb{R}$ and $f : [0, \infty) \to \mathbb{R}$, the function f *has limit L at infinity,* written,

$$\lim_{x \to \infty} f(x) = L,$$

if for every $\epsilon > 0$ there is a $C > 0$ such that if $x > C$, then $|f(x) - L| < \epsilon$.

Prove if $f : [0, \infty) \to \mathbb{R}$ is continuous, $L \in \mathbb{R}$ and has limit L at infinity, then f is uniformly continuous.

I worked with Jude Flynn, Nicholas Kapsos, Elaine Danielson, and Silas Rickards to come up with the idea for the proof.

Proof. Let $f \colon [0, \infty) \to \mathbb{R}$ be continuous with limit $L \in \mathbb{R}$ at infinity as given.

Given $\varepsilon > 0$, because f has limit L at infinity, there exists $C > 0$ such that if $z > C$, then $|f(z) - L| < \varepsilon/2$.

The function f is continuous on $[0, \infty)$, and so f is continuous on the nonempty compact interval $[0, C+1]$. Hence f is uniformly continuous on $[0, C+1]$. Therefore when $x, y \in [0, C+1]$, we can choose $0 < \delta < 1$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

When x, y are not both in $[0, C + 1]$ (with $x, y \in [0, \infty)$) and $|x - y| < \delta$, it follows that both $x, y > C$ since $\delta < 1$ (without loss of generality, if $x \notin [0, C+1]$, then $C+1 < x$, which implies $C + 1 - \delta < y$). Thus the only other scenario which remains is to consider when $x, y > C$. When $x, y > C$,

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - L + L - f(y)| \\ &\leq |f(x) - L| + |L - f(y)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Note that in this case we did not use the hypothesis that $|x - y| < \delta$. Hence f is uniformly continuous on $[C, \infty)$. Thus f is uniformly continuous on $[0, C + 1]$ and $[C, \infty)$.

It follows that given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in [0, \infty)$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$ (in particular the choice of $\delta > 0$ is the one made in the first case). Hence f is uniformly continuous on $[0, \infty)$. $\square$