1. For the double pendulum system, the given Lagrangian was

$$\mathcal{L} = \frac{1}{2}(m_1+m_2)\ell_1^2\dot{\theta_1}^2 + \frac{1}{2}m_2\ell_2^2\dot{\theta_2}^2 + m_2\ell_1\ell_2\cos(\theta_1-\theta_2)\dot{\theta_1}\dot{\theta_2} + (m_1+m_2)g\ell_1\cos(\theta_1) + m_2 g\ell_2\cos(\theta_2).$$

Then

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\theta_1} &= -m_2\ell_1\ell_2\sin(\theta_1-\theta_2)\dot{\theta_1}\dot{\theta_2} - (m_1+m_2)g\ell_1\sin(\theta_1)\\
\frac{\partial\mathcal{L}}{\partial\dot{\theta_1}} &= (m_1+m_2)\ell_1^2\dot{\theta_1} + m_2\ell_1\ell_2\cos(\theta_1-\theta_2)\dot{\theta_2}\\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\theta_1}} &= (m_1+m_2)\ell_1^2\ddot{\theta_1} - m_2\ell_1\ell_2\sin(\theta_1-\theta_2)(\dot{\theta_1}-\dot{\theta_2})\dot{\theta_2} + m_2\ell_1\ell_2\cos(\theta_1-\theta_2)\ddot{\theta_2}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\theta_2} &= m_2\ell_1\ell_2\sin(\theta_1-\theta_2)\dot{\theta_1}\dot{\theta_2} - m_2 g\ell_2\sin(\theta_2)\\
\frac{\partial\mathcal{L}}{\partial\dot{\theta_2}} &= m_2\ell_2^2\dot{\theta_2} + m_2\ell_1\ell_2\cos(\theta_1-\theta_2)\dot{\theta_1}\\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\theta_2}} &= m_2\ell_2^2\ddot{\theta_2} - m_2\ell_1\ell_2\sin(\theta_1-\theta_2)(\dot{\theta_1}-\dot{\theta_1})\dot{\theta_1} + m_2\ell_1\ell_2\cos(\theta_1-\theta_2)\ddot{\theta_1}.
\end{aligned}$$

Hence the Euler-Lagrange system of equations for the double pendulum can be expressed as

$$\begin{aligned}
0 = (m_1+m_2)\ell_1^2\ddot{\theta_1} - m_2\ell_1\ell_2\sin(\theta_1-\theta_2)(\dot{\theta_1}-\dot{\theta_2})\dot{\theta_2} + m_2\ell_1\ell_2\cos(\theta_1-\theta_2)\ddot{\theta_2}&\\
+ m_2\ell_1\ell_2\sin(\theta_1-\theta_2)\dot{\theta_1}\dot{\theta_2} + (m_1+m_2)g\ell_1\sin(\theta_1)&
\end{aligned}$$

$$\begin{aligned}
0 = m_2\ell_2^2\ddot{\theta_2} - m_2\ell_1\ell_2\sin(\theta_1-\theta_2)(\dot{\theta_1}-\dot{\theta_2})\dot{\theta_1} + m_2\ell_1\ell_2\cos(\theta_1-\theta_2)\ddot{\theta_1}&\\
- m_2\ell_1\ell_2\sin(\theta_1-\theta_2)\dot{\theta_1}\dot{\theta_2} + m_2 g\ell_2\sin(\theta_2)&
\end{aligned}$$

2. Define a Lagrangian density

$$\mathcal{L} = \mathcal{L}(\phi, \dot{\phi}, \phi'; t),$$

such that the action S is given by

$$S[\phi(x,t)] = \int_{-\infty}^{\infty}\int_{t_i}^{t_f} \mathcal{L}(\phi,\dot{\phi},\phi';t)\,\mathrm{d}t\,\mathrm{d}x\,.$$

(a) In minimizing the action, $\delta S = 0$. Thus

$$\begin{aligned}
0 = \delta S &= \int_{-\infty}^{\infty}\int_{t_i}^{t_f}\Big(\mathcal{L}(\phi+\mathrm{d}\phi\,,\dot{\phi}+\mathrm{d}\dot{\phi},\phi'+\mathrm{d}\phi'\,;t) - \mathcal{L}(\phi,\dot{\phi},\phi';t)\Big)\,\mathrm{d}t\,\mathrm{d}x\\
&= \int_{-\infty}^{\infty}\int_{t_i}^{t_f}\left(\frac{\partial\mathcal{L}}{\partial\phi}\,\mathrm{d}\phi + \frac{\partial\mathcal{L}}{\partial\dot{\phi}}\mathrm{d}\dot{\phi} + \frac{\partial\mathcal{L}}{\partial\phi'}\,\mathrm{d}\phi'\right)\mathrm{d}t\,\mathrm{d}x\\
&= \int_{-\infty}^{\infty}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\,\mathrm{d}\phi\bigg|_{t_i}^{t_f} + \frac{\partial\mathcal{L}}{\partial\phi'}\,\mathrm{d}\phi\bigg|_{-\infty}^{\infty} + \int_{t_i}^{t_f}\left(\frac{\partial\mathcal{L}}{\partial\phi}\,\mathrm{d}\phi - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\,\mathrm{d}\phi - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\mathcal{L}}{\partial\phi'}\,\mathrm{d}\phi\right)\mathrm{d}t\right)\mathrm{d}x\\
&= \int_{-\infty}^{\infty}\int_{t_i}^{t_f}\left(\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\mathcal{L}}{\partial\phi'}\right)\mathrm{d}\phi\,\mathrm{d}t\,\mathrm{d}x\,,
\end{aligned}$$

where $\mathrm{d}\phi\,(t_i) = \mathrm{d}\phi\,(t_f) = 0$ (fixed ends) and $\lim_{x\to-\infty} \mathrm{d}\phi = \lim_{x\to\infty} \mathrm{d}\phi = 0$ (it would be sad if this did not vanish). Then the last integral can only vanish when

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial \mathcal{L}}{\partial \phi'} = 0,$$

which is the modified Euler-Lagrange equation.

(b) Let

$$\mathcal{L}(\phi, \dot{\phi}, \phi'; t) = \frac{1}{2}\rho\dot{\phi}^2 - \frac{1}{2}\tilde{\mathcal{K}}\phi'^2.$$

Taking derivatives,

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \phi} &= 0 \\
\frac{\partial \mathcal{L}}{\partial \dot{\phi}} &= \rho\dot{\phi} \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} &= \rho\ddot{\phi} \\
\frac{\partial \mathcal{L}}{\partial \phi'} &= -\tilde{\mathcal{K}}\phi' \\
\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial \mathcal{L}}{\partial \phi'} &= -\tilde{\mathcal{K}}\phi'',
\end{aligned}$$

which by the modified Euler-Lagrange equation, yields the famous partial differential equation, the wave equation,

$$\tilde{\mathcal{K}}\phi'' = \rho\ddot{\phi}.$$