Concepts in Calculus III

Multivariable Calculus

Solutions Manual

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CHAPTER 1

Vectors and the Space Geometry

1. Rectangular Coordinates in Space

1–**2**. Find the distance between the specified points.

1. $(1, 2, -3)$ and $(-1, 0, -2)$

SOLUTION: Let d denote the distance between the given points. Then, by the distance formula,

$$d = \sqrt{(-1-1)^2 + (0-2)^2 + (-2+3)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$$

2. $(-1, 3, -2)$ and $(-1, 2, -1)$

SOLUTION: Let d denote the distance between the given points. Then, by the distance formula,

$$d = \sqrt{(-1+1)^2 + (2-3)^2 + (-1+2)^2} = \sqrt{0+1+2} = \sqrt{2}$$

3–**4**. Determine whether the given points lie in a straight line.

3. $(8, 3, -3)$, $(-1, 6, 3)$, and $(2, 5, 1)$

SOLUTION: Given three points A, B, and C in $\mathbb{R}^n$, we can determine if they are colinear if

$$|AB| + |BC| = |AC|.$$

Therefore, we must compute the distances between each pair of points and see if one is the sum of the other two. Let $A = (8, 3, -3)$, $B = (-1, 6, 3)$, and $C = (2, 5, 1)$. By the distance formula,

$$|AB| = \sqrt{(8+1)^2 + (3-6)^2 + (-3-3)^2} = \sqrt{81+9+36} = \sqrt{126}$$
$$|BC| = \sqrt{(-1-2)^2 + (6-5)^2 + (3-1)^2} = \sqrt{9+1+4} = \sqrt{14}$$
$$|AC| = \sqrt{(8-2)^2 + (3-5)^2 + (-3-1)^2} = \sqrt{36+4+16} = \sqrt{56}$$

Note that $|AB| = |BC| + |AC|$, so they are colinear. To see this, consider squaring the right hand side,

$$\begin{aligned}(\sqrt{14}+\sqrt{56})^2 &= 14 + 2\sqrt{14}\sqrt{56} + 56\\ 126 &= 70 + 2\sqrt{7\cdot 2\cdot 7\cdot 8}\\ 126 &= 70 + 2(7)(4) = 70 + 56 = 126\end{aligned}$$

4. $(-1, 4, -2)$, $(1, 2, 2)$, and $(-1, 2, -1)$

Solution: Given three points A, B, and C in $\mathbb{R}^n$, we can determine if they are colinear if

$$|AB| + |BC| = |AC|.$$

Therefore, we must compute the distances between each pair of points and see if one is the sum of the other two. Let $A = (-1, 4, -2)$, $B = (1, 2, 2)$, and $C = (-1, 2, -1)$. By the distance formula,

$$\begin{aligned}|AB| &= \sqrt{(-1-1)^2 + (4-2)^2 + (-2-2)^2} = \sqrt{4+4+16} = \sqrt{24}\\ |BC| &= \sqrt{(1+1)^2 + (2-2)^2 + (2+1)^2} = \sqrt{4+0+9} = \sqrt{13}\\ |AC| &= \sqrt{(-1+1)^2 + (4-2)^2 + (-2+1)^2} = \sqrt{0+4+1} = \sqrt{5}\end{aligned}$$

Note that no one distance is the sum of the other two, hence the points are not colinear.

5. Determine whether the points $(1, 2, 3)$ and $(3, 2, 1)$ lie in the plane through $O = (1, 1, 1)$ and perpendicular to the line through O and $B = (2, 2, 2)$. If not, find a point in this plane.

Solution: Let $P_1 = (1, 2, 3)$ and $P_2 = (3, 2, 1)$. Let $\mathcal{P}$ be the plane that passes through a given point O and is perpendicular to the line through O and B. An arbitrary point $P \in \mathbb{R}^3$ will be in $\mathcal{P}$ iff OP, OB, and BP form a right triangle. Clearly, OP and OB must be perpendicular, hence we must have

$$|OB|^2 + |OP|^2 = |BP|^2$$

First checking P_1,

$$\begin{aligned}|OB| &= \sqrt{(2-1)^2 + (2-1)^2 + (2-1)^2} = \sqrt{3},\\ |BP_1| &= \sqrt{(2-1)^2 + (2-2)^2 + (2-3)^2} = \sqrt{2},\\ |OP_1| &= \sqrt{(1-1)^2 + (1-2)^2 + (1-3)^2} = \sqrt{5}.\end{aligned}$$

Since $3+5 \neq 2, P_1 \notin \mathcal{P}$. A similar argument shows that $P_2 \notin \mathcal{P}$. It turns out that, in this problem, $|OP_1| = |OP_2|$ and $|BP_1| = |BP_2|$.

To find a point in $\mathcal{P}$, let $P = (x, y, z)$. Then we must have that

$$3 + ((1-x)^2 + (1-y)^2 + (1-z)^2) = ((2-x)^2 + (2-y)^2 + (2-z)^2)$$
$$(2-x)^2 - (1-x)^2 + (2-y)^2 - (1-y)^2 + (2-z)^2 - (1-z)^2 - 3 = 0$$
$$(2-x-(1-x))(2-x+1-x) + (2-y-(1-y))(2-y+1-y)+$$
$$(2-z-(1-z))(2-z+1-z) - 3 = 0$$
$$3 - 2x + 3 - 2y + 3 - 2z - 3 = 0$$
$$2x + 2y + 2z = 6$$
$$x + y + z = 3$$

Thus any point $P = (x, y, z)$ satisfying $x + y + z = 3$ will be in $\mathcal{P}$. For example, $(3, 0, 0) \in \mathcal{P}$. In the future, you will learn an easier way to arrive at this formula.

6. Find the distance from the point $(1, 2, -3)$ to each of the coordinate planes and to each of the coordinate axes.

Solution: Let $P = (1, 2, -3)$. To find the distance between a point and a plane, we must find a point P^* on the plane such that PP^* is perpendicular to the plane. For the coordinate planes, this is easy – simply project the point onto the plane. Let d_{xz} represent the distance between P and the projection of P onto the xz plane, $P^* = (1, 0, -3)$. Thus, by the distance formula,

$$d_{xz} = \sqrt{(1-1)^2 + (2-0)^2 + (3-(-3))^2} = 2$$

Similarly, d_{yz} and d_{xy} are as follows

$$d_{yz} = \sqrt{(1-0)^2 + (2-2)^2 + (3-(-3))^2} = 1$$
$$d_{xy} = \sqrt{(1-1)^2 + (2-2)^2 + (3-0)^2} = 3$$

To find the distance between P and a coordinate axis, we must find our P^* and project it again onto the desired axis. For example, find the distance between P and the x-axis, we must project P onto the xz plane ($P^* = (1, 0, -3)$), and then project it again onto the x axis ($P^{**} = (1, 0, 0)$). Thus, d_x, the distance between P and the x-axis is

$$d_x = \sqrt{(1-1)^2 + (2-0)^2 + (3-0)^2} = \sqrt{13}$$

Similarly, d_y and d_z are as follows

$$d_y = \sqrt{(1-0)^2 + (2-2)^2 + (3-0)^2} = \sqrt{10}$$
$$d_z = \sqrt{(1-0)^2 + (2-0)^2 + (3-3)^2} = \sqrt{5}$$

7. Find the length of the medians of the triangle with vertices $A = (1, 2, 3)$, $B = (-3, 2, -1)$, and $C = (-1, -4, 1)$.

Solution: The coordinates of the midpoint of a straight line segment with given endpoints are the half-sums of the corresponding coordinates of the endpoints. Let P_a, P_b, and P_c be the midpoints of BC, AC, and AB, respectively. Then

$$P_a = (-2, -1, 0)\,, \quad P_b = (0, -1, 2)\,, \quad P_c = (-1, 2, 1)\,.$$

By the distance formula,

$$|AP_a| = \sqrt{(-2-1)^2 + (-1-2)^2 + (3-0)^2} = \sqrt{9+9+9} = 3\sqrt{3}\,,$$
$$|BP_b| = \sqrt{(0+3)^2 + (-1-2)^2 + (2+1)^2} = \sqrt{9+9+9} = 3\sqrt{3}\,.$$
$$|CP_c| = \sqrt{(-1+1)^2 + (2+4)^2 + (1-1)^2} = \sqrt{0+6^2+0} = 6\,.$$

8. Let the set $\mathcal{S}$ consist of points $(t, 2t, 3t)$ where $-\infty < t < \infty$. Find the point of $\mathcal{S}$ that is the closest to the point $(3, 2, 1)$. Sketch the set $\mathcal{S}$.

Solution: I will begin by answering the second part. Consider the following: $t(1, 2, 3)$. This is a restatement of the given set of points. Interpreting $(1, 2, 3)$ as a vector in space, we can see that t acts as a scalar. Thus the set of points is a straight line parallel to the vector $\langle 1, 2, 3\rangle$. To minimize the distance between our curve and the given point $P = (3, 2, 1)$, we must find a point Q on our curve such that the function $d = |PQ|$ is minimized. Let $Q = (x, 2x, 3x)$. So,

$$d(x) = \sqrt{(3-x)^2 + (2-2x)^2 + (1-3x)^2}$$
$$d(x)^2 = (3-x)^2 + (2-2x)^2 + (1-3x)^2$$
$$2d(x)d'(x) = -2(3-x) - 4(2-2x) - 6(1-3x)$$
$$d'(x) = \frac{-3 + x - 4 + 4x - 3 + 9x}{d(x)}$$
$$d'(x) = \frac{14x - 10}{d(x)}$$

Thus $d(x)$ is minimized when $x = \frac{10}{14} = \frac{5}{7}$, as $d(x)$ is never zero (the curve does not intersect P). So $Q = (\frac{5}{7}, \frac{10}{7}, \frac{15}{7})$. Our minimized distance then is

$$|PQ| = \sqrt{(3 - \frac{5}{7})^2 + (2 - \frac{10}{7})^2 + (1 - \frac{15}{7})^2)} = \frac{4}{7}\sqrt{21}$$

9–18. Give a geometrical description of the given set $\mathcal{S}$ of points defined algebraically and sketch the set:

9. $\mathcal{S} = \{(x, y, z) \,|\, x^2 + y^2 + z^2 - 2x + 4y - 6z = 0\}$

SOLUTION: By completing the square, we obtain

$$\begin{aligned} & (x-1)^2 - 1 + (y+2)^2 - 4 + (z-3)^2 - 9 = 0 \\ \Rightarrow \quad & (x-1)^2 + (y+2)^2 + (z-3)^2 = 14 \end{aligned}$$

Thus this is a sphere with radius $R = \sqrt{14}$ and center $(1, -2, 3)$.

10. $\mathcal{S} = \{(x, y, z) \,|\, x^2 + y^2 + z^2 \geq 4\}$

SOLUTION: Take all of space, then remove the ball $x^2 + y^2 + z^2 \leq 4$. Add back in the sphere $x^2 + y^2 + z^2 = 4$. This is the set (all of space with a spherical cavity).

11. $\mathcal{S} = \{(x, y, z) \,|\, x^2 + y^2 + z^2 \leq 4,\ z > 0\}$

SOLUTION: This is an intersection of two sets. The first is a ball of radius 2 centered at the origin. The other is all of space where $z > 0$. Thus the set is a hemisphere, minus a disk of radius 2 in the xy plane.

12. $\mathcal{S} = \{(x, y, z) \,|\, x^2 + y^2 - 4y < 0,\ z > 0\}$

SOLUTION: This is an intersection of two sets. Completing the square on the first condition yields

$$\begin{aligned} & x^2 + (y-2)^2 - 4 = 0 \\ \Rightarrow \quad & x^2 + (y-2)^2 = 4 \end{aligned}$$

This is a cylinder of radius 2 whose axis is parallel to the z axis and passes through $(0, 2, 0)$. The second set is all of space with $z > 0$. So the set is the portion of the cylinder above the xy plane.

13. $\mathcal{S} = \{(x, y, z) \,|\, 4 \le x^2 + y^2 + z^2 \le 9\}$

SOLUTION: This is a ball of radius 3 with a ball of radius 2 (except for its boundary) removed from it.

14. $\mathcal{S} = \{(x, y, z) \,|\, x^2 + y^2 \ge 1,\ x^2 + y^2 + z^2 \le 4\}$

SOLUTION: This is an intersection of two sets. The first the complement of a cylinder of radius 1 aligned along the z axis (i.e., all of space with such a cylinder removed from it). The second is a ball of radius 2. Thus this is a ball of radius 2 with a cylindrical cavity of radius 1 aligned along the z axis (think like an olive).

15. $\mathcal{S} = \{(x, y, z) \,|\, x^2 + y^2 + z^2 - 2z < 0,\ z > 1\}$

SOLUTION: This is an intersection of two sets. The first, after completing the square is

$$\begin{aligned} & x^2 + y^2 + (z-1)^2 - 1 < 0 \\ \Rightarrow\quad & x^2 + y^2 + (z-1)^2 < 1 \end{aligned}$$

This is a ball of radius 1 centered at $(0, 0, 1)$, without the boundary. The second set is all of space with $z > 1$. Thus this is a hemisphere, without the boundary.

16. $\mathcal{S} = \{(x, y, z) \,|\, x^2 + y^2 + z^2 - 2z = 0,\ z = 1\}$

SOLUTION: This is an intersection of two sets. The first, after completing the square is

$$\begin{aligned} & x^2 + y^2 + (z-1)^2 - 1 = 0 \\ \Rightarrow\quad & x^2 + y^2 + (z-1)^2 = 1 \end{aligned}$$

This is a sphere of radius 1 centered at $(0, 0, 1)$. The second set is the plane $z = 1$. Thus this is a disk of radius 1 in the plane $z = 1$, centered at $(0, 0, 1)$.

17. $\mathcal{S} = \{(x, y, z) \,|\, (x - a)(y - b)(z - c) = 0\}$

SOLUTION: Solutions to the above yield $x = a$, $y = a$, or $z = a$. These are three planes. Because any one can be true (notice the or), it is the union of these planes, not the intersection.

18. $\mathcal{S} = \{(x, y, z)\,|\,|x| \le 1,\ |y| \le 2,\ |z| \le 3\}$

SOLUTION: This is an intersection of three sets. The first, $|x| \le 1$, is all of space between the planes $x = -1$ and $x = 1$. The second is all of space between $y = -2$ and $y = 2$. The third is all of space between $z = -3$ and $z = 3$. The intersection of these gives a solid box centered at the origin with length 2 along the x-axis, 4 along the y-axis, and a height of 6 along the z-axis.

19–24. Sketch the given set of points and give its algebraic description.

19. A sphere whose diameter is the straight line segment AB, where $A = (1, 2, 3)$ and $B = (3, 2, 1)$.

SOLUTION: A sphere is determined by its radius and center. Since AB is a diameter, the radius is

$$R = \frac{1}{2}|AB| = \frac{1}{2}\sqrt{(3-1)^2 + (2-2)^2 + (1-3)^2} = \frac{1}{2}\sqrt{8} = \sqrt{2}$$

The center $P_0 = (x_0, y_0, z_0)$ is the midpoint of the segment AB:

$$P_0 = \left(\frac{1}{2}(1+3),\ \frac{1}{2}(2+2),\ \frac{1}{2}(3+1)\right) = (2,\ 2,\ 2)\,.$$

Therefore an equation for the sphere has the standard form:

$$\begin{aligned} &(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2 \\ \Rightarrow\quad &(x-2)^2 + (y-2)^2 + (z-2)^2 = 2\,. \end{aligned}$$

20. Three spheres centered at $(1, 2, 3)$ that just barely touch the xy, yz, and xz planes, respectively.

SOLUTION: Because a sphere in space is uniquely determined by its center and radius, we must find the radii of each sphere. Note that if a sphere "barely touches" a plane, then it is tangent to that plane. We can construct a line AP_0, where $P_0 = (x_0, y_0, z_0)$ is the center of the sphere and A is the point on the plane where the sphere touches it. This line is perpendicular to the plane; hence $|AP_0|$ is the shortest distance between P_0 and any point on the plane. Thus it is the radius of our sphere.

For the xy plane, we have that $A = (1, 2, 0)$. It is easy to verify that AP_0 is perpendicular to the plane. Hence our radius R is

$$R = |AP_0| = \sqrt{(1-1)^2 + (2-2)^2 + (0-3)^2} = 3$$

Recall that the standard form of a sphere is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$$

Thus our sphere is

$$(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 9$$

For the yz plane, we have that $B = (0, 2, 3)$. It is easy to verify that BP_0 is perpendicular to the plane. Hence our radius R is

$$R = |BP_0| = \sqrt{(0-1)^2 + (2-2)^2 + (3-3)^2} = 1$$

Recall that the standard form of a sphere is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$$

Thus our sphere is

$$(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 1$$

For the xz plane, we have that $C = (1, 0, 3)$. It is easy to verify that CP_0 is perpendicular to the plane. Hence our radius R is

$$R = |CP_0| = \sqrt{(1-1)^2 + (0-2)^2 + (3-3)^2} = 2$$

Recall that the standard form of a sphere is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$$

Thus our sphere is

$$(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 4$$

21. Three spheres centered at $(1, -2, 3)$ that just barely touch the x, y, and z coordinate axes, respectively.

Solution: Because a sphere in space is uniquely determined by its center and radius, we must find the radii of each sphere. Note that if a sphere "barely touches" a line, then it is tangent to that line. We can construct a line AP_0, where $P_0 = (x_0, y_0, z_0)$ is the center of the sphere and A is the point on the line where the sphere touches it. This line is perpendicular to the given line; hence $|AP_0|$ is the shortest distance between P_0 and any point on the given line. Thus it is the radius of our sphere.

For the x axis, we have that $A = (1, 0, 0)$. It is easy to verify that AP_0 is perpendicular to the line. Hence our radius R is

$$R = |AP_0| = \sqrt{(1-1)^2 + (0+2)^2 + (0-3)^2} = \sqrt{4+9} = \sqrt{13}$$

Recall that the standard form of a sphere is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$$

Thus our sphere is

$$(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = 13$$

For the y axis, we have that $B = (0, -2, 0)$. It is easy to verify that BP_0 is perpendicular to the line. Hence our radius R is

$$R = |BP_0| = \sqrt{(0-1)^2 + (-2+2)^2 + (0-3)^2} = \sqrt{1+9} = \sqrt{10}$$

Recall that the standard form of a sphere is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$$

Thus our sphere is

$$(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = 10$$

For the z axis, we have that $C = (0, 0, 3)$. It is easy to verify that CP_0 is perpendicular to the line. Hence our radius R is

$$R = |CP_0| = \sqrt{(0-1)^2 + (0+2)^2 + (3-3)^2} = \sqrt{1+4} = \sqrt{5}$$

Recall that the standard form of a sphere is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$$

Thus our sphere is

$$(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = 5$$

22. The largest solid cube that is contained in a ball of radius R centered at the origin. Solve the same problem if the ball is not centered at the origin. Compare the cases when the boundaries of the solid are included into the set or excluded from it.

Solution: For some problems, it helps to drop down into a lower dimension first; That is, if the question were asking to find the largest square contained in a circle of radius R. It is evident that the largest square will have its corners touching the circle and its center at the center of the circle. In other words, the distance from the center of the circle to any corner of the square is R, and so the diagonal is $2R$. Thus the length of the sides of the square is $d = \frac{1}{\sqrt{2}}R$.

We can now apply similar logic to the 3D case. Evidently, the largest solid cube will be that whose eight corners also touch points on the sphere and whose center is also the center of the sphere. Now we must find the length of each side of the cube. Suppose the side length is $2d$. Then we can find the distance between a corner and the center of a nearby face by $D^2 = d^2 + d^2$, so $D = \sqrt{2}d$. Next, the distance between this point and the center of the cube is d, so the distance between the center of the cube and a corner is $R^2 = D^2 + d^2 = 3d^2$, so $d = \frac{1}{\sqrt{3}}R$, and our side length is $\frac{2}{\sqrt{3}}R$.

Let the center of the sphere be at (x_0, y_0, z_0). Then our solid cube is given by

$$\mathcal{S} = \{(x,y,z)\,|\,|x - x_0| \leq \frac{1}{\sqrt{3}}R,\ |y - y_0| \leq \frac{1}{\sqrt{3}}R,\ |z - z_0| \leq \frac{1}{\sqrt{3}}R\}$$

Note that I did not include the 2 for each $\frac{1}{\sqrt{3}}R$. This is due to the absolute value.

23. The solid region that is a ball of radius R that has a cylindrical hole of radius $R/2$ whose axis is at a distance of $R/2$ from the center of the ball. Choose a convenient coordinate system. Compare the cases when the boundaries of the solid are included into the set or excluded from it.

Solution: Consider such a solid region in space. Then, up to a finite number of rigid transformations (distance and angle preserving), this region is identical to one centered at the origin of a standard xyz coordinate system, with the cylindrical hole aligned parallel to the z ("up") axis. The algebraic description for a ball (solid sphere) of radius R centered at the origin is simply

$$x^2 + y^2 + z^2 \leq R^2.$$

Next, we want the axis of the cylinder to be a distance $R/2$ from the origin. I will choose that the cylinder's axis goes through the point $(0, R/2, 0)$. Realistically, however, it can go through any point $(R/2\cos\theta, R/2\sin\theta, 0)$ for any real θ. The general form of a solid cylinder, whose axis passes through $(0, R/2, 0$, aligned with the z-axis, is

$$x^2 + (y - R/2)^2 < (R/2)^2$$

It is crucial that a $<$ is used rather than $\leq$. This will be explained later.
Since the cylinder is actually a hole, we must take the complement of

this,

$$x^2 + (y - R/2)^2 \geq R^2/4$$

If we had used $<$ earlier, the $\geq$ would now be a $>$. This means that the boundary between the solid and the hole would not be included. The region is simply the intersection of these two solids,

$$\mathcal{S} = \{(x, y, z) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R} \mid x^2+y^2+z^2 \leq R^2 \cap x^2+(y-R/2)^2 \ \geq R^2/4\}$$

And without boundaries,

$$\mathcal{S} = \{(x, y, z) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R} \mid x^2+y^2+z^2 < R^2 \cap x^2+(y-R/2)^2 > R^2/4\}$$

24. The part of a ball of radius R that lies between two parallel planes each of which is at a distance of $a < R$ from the center of the ball. Choose a convenient coordinate system. Compare the cases when the boundaries of the solid are included into the set or excluded from it.

Solution: Then, up to a finite number of rigid transformations (distance and angle preserving), this region is identical to one centered at the origin of a standard xyz coordinate system, with the parallel planes also parallel to the xy plane. The algebraic description for a ball (solid sphere) of radius R centered at the origin is simply

$$x^2 + y^2 + z^2 \leq R^2.$$

Let $a < R$ be a real number. The two planes in question can be seen as one above the center and another below the center. Recall that a plane parallel to the xy plane is simply the set of all points whose z coordinate is a constant value. Thus, the two planes in question are

$$z = a$$

$$z = -a$$

Now we want to classify all the points between these two planes. This will be all of the points below $z = a$ and above $z = -a$. So all the points between the two planes are all the points where $-a \leq z \leq a$, where we have leq instead of $<$ because we want the boundaries to be included. Then the set of all points in the ball of radius R and between the two planes will be the intersection of the two found sets,

$$\mathcal{S} = \{(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 + z^2 \leq R^2 \ \cap \ -a \leq z \leq a$$

And without boundaries,

$$\mathcal{S} = \{(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 + z^2 < R^2 \ \cap \ -a < z < a$$

It may help to consider the 2D analogue first to have a clear visual idea of the problem.

25. Consider the points P such that the distance from P to the point $(-3, 6, 9)$ is twice the distance from P to the origin. Show that the set of all such points is sphere, and find its center and radius.

SOLUTION: Let $P = (x, y, z)$ be an arbitrary point in space. The distance between P and the given point $P_0 = (-3, 6, 9)$ is

$$|PP_0| = \sqrt{(x+3)^2 + (y-6)^2 + (z-9)^2}$$

Next, the distance between P and the origin is

$$|PO| = \sqrt{x^2 + y^2 + z^2}$$

If the distance between P and P_0 is twice the distance from P to the origin, then we have

$$\begin{aligned}
& \sqrt{(x+3)^2 + (y-6)^2 + (z-9)^2} = 2\sqrt{x^2 + y^2 + z^2} \\
\Rightarrow \quad & (x+3)^2 + (y-6)^2 + (z-9)^2 = 4(x^2 + y^2 + z^2) \\
\Rightarrow \quad & 3x^2 - 6x - 9 + 3y^2 + 12y - 36 + 3z^2 + 18z - 81 = 0 \\
\Rightarrow \quad & 3(x^2 - 2x - 3) + 3(y^2 + 4y - 12) + 3(z^2 + 6z - 27) = 0 \\
\Rightarrow \quad & 3(x^2 - 2x + 1 - 4) + 3(y^2 + 4y + 4 - 16) + 3(z^2 + 6z + 9 - 36) = 0 \\
\Rightarrow \quad & 3(x^2 - 2x + 1) - 12 + 3(y^2 + 4y + 4) - 48 + 3(z^2 + 6z + 9) - 108 = 0 \\
\Rightarrow \quad & (x-1)^2 + (y+2)^2 + (z+3)^2 = \frac{12 + 48 + 108}{3} \\
\Rightarrow \quad & (x-1)^2 + (y+2)^2 + (z+3)^2 = 56\,.
\end{aligned}$$

Thus it is a sphere with center $(1, -2, -3)$ and radius $\sqrt{56}$.

26. Find the volume of the solid whose boundaries are the spheres $x^2 + y^2 + z^2 - 6z = 0$ and $x^2 + y^2 - 2y + z^2 - 6z = -9$.

SOLUTION: By completing the square, the equations of the spheres are written in the standard form:

$$x^2 + y^2 + (z-3)^2 = 9\,, \quad x^2 + (y-1)^2 + (z-3)^2 = 1\,.$$

It follows then that the first sphere has radius $R_1 = 3$ and is centered at $(0, 0, 3)$, while the second sphere lies inside the first one because it has radius $R_2 = 1$ and is centered at $(0, 1, 3)$. So the solid is the ball bounded by a sphere of radius R_1 with a spherical cavity of radius R_2. Therefore the volume of the solid is the difference of the volumes of the ball and the cavity:

$$V = \frac{4\pi}{3}R_1^3 - \frac{4\pi}{3}R_2^3 = \frac{4\pi}{3}(3^3 - 1^3) = \frac{104\pi}{3}\,.$$

27. Find the volume of the solid that is described by the inequalities $|x-1| \leq 2$, $|y-2| \leq 1$, and $|z+1| \leq 2$. Sketch the solid.

SOLUTION: The solid is a rectangular prism. Recall that if $|a| < b$, then $-b < a < b$, and the set of points satisfying this is bounded below by $-b$ and above by b. Thus the solid is bounded by the planes
$x-1=2 \Rightarrow x=3$ and $x-1=-2 \Rightarrow x=-1$,
$y-2=1 \Rightarrow y=3$ and $y-2=-1 \Rightarrow y=1$,
$z+1=2 \Rightarrow z=1$ and $z+1=-2 \Rightarrow z=-3$.
We can calculate the side lengths by finding the distance in between each set of parallel planes. Hence the side length parallel to x is $3-(-1)=4$, the side length parallel to y is $3-1=2$, and the side length parallel to z is $1-(-3)=4$. Notice that we had $|x-1| \leq 2$, and the side length along the x axis was $2*2$. This is no coincidence. If $|x-a| \leq b$, the side length along the x axis will be $2b$. Thus the volume is $V=4(2)(4)=32$.

28. The solid region is described by the inequalities $|x-a| \leq a$, $|y-b| \leq b$, $|z-c| \leq c$, and $(y-b)^2+(z-c)^2 \geq R^2$. If $R \leq \min(b,c)$, sketch the solid and find its volume.

SOLUTION: First note that the solid described by the inequalities $|x-a| \leq a$, $|y-b| \leq b$, and $|z-c| \leq c$ is a rectangular prism with side lengths $2a$, $2b$, and $2c$ (see problem 27 for a more detailed explanation). The rectangular prism is centered so that (a,b,c) is the center of it.

Next, we have a hole formed by $(y-b)^2+(z-c)^2 \geq R^2$. In 2D, this imagine having a square with a circle perfectly embedded inside it. Because we want the set of points greater than the radius, but bounded by the square, we will have a circular hole. In 3D, the inequality describes a cylinder, because of its independence of x. Furthermore, this tells us that the cylinder's axis is parallel to the x axis, and that its axis is along the line (t,b,c). That is, the axis of the cylinder cuts through the center of the prism. The height of the cylinder is $2a$, because it is parallel to the x axis.

Thus the volume of the cylinder is $V_c = \pi R^2(2a) = 2a\pi R^2$, where $R \leq \min(b,c)$ (Note: The min function is used to guarantee that the cylinder lies perfectly inside the prism). The volume of the prism is $V_s = 2a(2b)(2c) = 8abc$. Thus the volume of the solid is $V = 8abc - 2a\pi R^2$.

29. Sketch the set of all points in the xy plane that are equidistant from two given points A and B. Let A and B be $(1,2)$ and $(-2,-1)$, respectively. Give an algebraic description of the set.

SOLUTION: Imagine a square, with two opposite (diagonal) corners labeled A and B. It is obvious that the other two diagonal corners are equidistant to A and B. Furthermore, the midpoint of the diagonal is equidistant from A and B. This leads us to conclude that the set of all points equidistant from A and B is the set of points on the line through the other diagonal. Intuitively, this makes sense. This line is perpendicular to the diagonal containing A and B and goes through the midpoint. Call the midpoint O. Pick a point on the line, P. Then triangle OAP is congruent to triangle OBP by SAS, as they share the same side in common, and an angle in between (the right angle).

The slope of the line connecting A and B is $m = \frac{-1-2}{-2-1} = 1$. So a line perpendicular to this will have slope $M = \frac{-1}{m} = -1$. The midpoint of AB is $(\frac{-2+1}{2}, \frac{2-1}{2}) = (\frac{-1}{2}, \frac{1}{2})$. So the set of points equidistant from A and B make the line $y = -(x+\frac{1}{2}) + \frac{1}{2}) = -x$. So the set of points is $\mathcal{S} = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid y = -x\}$.

30. Sketch the set of all points in space that are equidistant from two given points A and B. Let A and B be $(1,2,3)$ and $(-3,-2,-1)$, respectively. Give an algebraic description of the set.

SOLUTION: Extending the idea of number 29 to 3D, the solution will be a plane through the midpoint of AB and perpendicular to it. Let $P = (x,y,z) \in \mathcal{P}$. Let O be the midpoint of AB, so $O = (\frac{1-3}{2}, \frac{2-2}{2}, \frac{3-1}{2}) = (-1,0,1)$. Because $\mathcal{P}$ is perpendicular to AB, triangle OAP is a right triangle. Then $|OA|^2 + |OP|^2 = |AP|^2$. $|OA|$=$((1+1)^2 + (2-0)^2 + (3-1)^2) = 12$. So,

$$\begin{aligned}
12 + ((x+1)^2 + (y-0)^2 + (z-1)^2) &= (x-1)^2 + (y-2)^2 + (z-3)^2 \\
(x-1)^2 - (x+1)^2 + (y-2)^2 - y^2 + (z-3)^2 - (z-1)^2 &= 12 \\
(-2)(2x) + (-2)(2y-2) + (-2)(2z-4) &= 12 \\
(x) + (y-1) + (z-2) &= -3 \\
x + y + z &= 0
\end{aligned}$$

Where I applied that $a^2 - b^2 = (a-b)(a+b)$ several times.

So $\mathcal{P} = \{(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \mid x + y + z = 0\}$

31. A point $P = (x, y)$ belongs to the set $\mathcal{S}$ in the xy plane if $|PA| + |PB| = c$ where $A = (a, 0)$, $B = (-a, 0)$, and $c > 2a$. Show that $\mathcal{S}$ is an ellipse.

Solution: One might remember that an ellipse can be generated by two foci A, B as above. If not, then we have

$$\begin{aligned}
\sqrt{(x-a)^2 + (y-0)^2} + \sqrt{(x-(-a))^2 + (y-0)^2} &= c \\
\sqrt{(x-a)^2 + y^2} + \sqrt{(x+a)^2 + y^2} &= c \\
(\sqrt{(x-a)^2 + y^2} + \sqrt{(x+a)^2 + y^2})^2 &= c^2 \\
(x-a)^2 + y^2 + 2\sqrt{(x-a)^2 + y^2}\sqrt{(x+a)^2 + y^2} + (x+a)^2 + y^2 &= c^2 \\
(x-a)^2 + (x+a)^2 + 2y^2 + 2\sqrt{(x-a)^2 + y^2}\sqrt{(x+a)^2 + y^2} &= c^2
\end{aligned}$$

We could continue by bringing all the radicals to one side and squaring again, but I find that tedious. Instead, return to the original equation and do the following:

$$\begin{aligned}
\sqrt{(x-a)^2 + y^2} + \sqrt{(x+a)^2 + y^2} &= c \\
\sqrt{(x-a)^2 + y^2} &= c - \sqrt{(x+a)^2 + y^2} \\
(x-a)^2 + y^2 &= (c - \sqrt{(x+a)^2 + y^2})^2 \\
(x-a)^2 + y^2 &= c^2 - 2c\sqrt{(x+a)^2 + y^2} + (x+a)^2 + y^2 \\
(x-a)^2 - (x+a)^2 &= c^2 - 2c\sqrt{(x+a)^2 + y^2} \\
(x-a+x+a)(x-a-(x+a)) &= c^2 - 2c\sqrt{(x+a)^2 + y^2} \\
-4ax &= c^2 - 2c\sqrt{(x+a)^2 + y^2} \\
2c\sqrt{(x+a)^2 + y^2} &= c^2 + 4ax \\
\sqrt{(x+a)^2 + y^2} &= \frac{c}{2} + \frac{2ax}{c}
\end{aligned}$$

Where the identity $a^2 - b^2 = (a+b)(a-b)$ has been used. We can obtain a similar formula by substituting a for $-a$,

$$\sqrt{(x-a)^2 + y^2} = \frac{c}{2} - \frac{2ax}{c}$$

Substituting these into the first equation we found yields

$$
\begin{aligned}
(x-a)^2+(x+a)^2+2y^2+2(\frac{c}{2}+\frac{2ax}{c})(\frac{c}{2}-\frac{2ax}{c}) &= c^2\\
(x-a)^2+(x+a)^2+2y^2+2((\frac{c}{2})^2-(\frac{2ax}{c})^2) &= c^2\\
x^2-2ax+a^2+x^2+2ax+a^2+2y^2+(\frac{c^2}{2}-\frac{8a^2x^2}{c^2}) &= c^2\\
2x^2-\frac{8a^2x^2}{c^2}+2a^2+2y^2 &= \frac{c^2}{2}\\
\frac{(2c^2-8a^2)x^2}{c^2}+2y^2 &= \frac{c^2}{2}-2a^2\\
\frac{4x^2}{c^2}+\frac{4y^2}{c^2-4a^2} &= 1
\end{aligned}
$$

This is an ellipse. Note that the coefficient of y^2 must be positive. hence $c^2-4a^2>0 \;\Leftrightarrow\; c^2>4a^2 \;\Leftrightarrow c>2a$ or $c<-2a$ Since $c>0$, it must be that $c>2a$, hence the condition given in the problem.

32. Determine whether the points $A=(1,0,-1)$, $B=(3,1,1)$, and $C=(2,2,-3)$ are vertices of a right-angled triangle.

Solution: Suppose that the given points A,B,C are points of a right-angled triangle. Then, by the pythagorean theorem, the square of the largest side length must be equal to the sum of the squares of the other perpendicular side lengths. So we must first find $|AB|$, $|BC|$, and $|AC|$ as follows

$$
\begin{aligned}
|AB| &= \sqrt{(1-3)^2+(0-1)^2+(-1-1)^2}=\sqrt{9}=3\\
|BC| &= \sqrt{(3-2)^2+(1-2)^2+(1+3)^2}=\sqrt{18}\\
|AC| &= \sqrt{(1-2)^2+(0-2)^2+(-1+3)^2}=\sqrt{9}=3
\end{aligned}
$$

It is evident that $|AB|^2+|AC|^2=|BC|$. Hence ABC is a right-angled triangle.

2. Vectors in Space

1–5. Find the components and norms of each of the following vectors.

1. $\overrightarrow{AB}$ where $A = (1, 2, 3)$ and $B = (-1, 5, 1)$.

SOLUTION: Recall that

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA},$$

where $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are the position vectors of A and B, respectively. So,

$$\overrightarrow{AB} = \langle -1, 5, 1\rangle - \langle 1, 2, 3\rangle = \langle -1-1, 5-2, 1-3\rangle = \langle -2, 3, -2\rangle$$

Furthermore,

$$\|\overrightarrow{AB}\| = \sqrt{(-2)^2 + 3^2 + (-2)^2} = \sqrt{17}$$

2. $\overrightarrow{BA}$ where $A = (1, 2, 3)$ and $B = (-1, 5, 1)$.

SOLUTION: Recall that

$$\overrightarrow{BA} = -\overrightarrow{AB}.$$

Then, by 1,

$$\overrightarrow{BA} = -\langle -2, 3, -2\rangle = \langle 2, -3, 2\rangle.$$

Since only the direction has changed,

$$\|\overrightarrow{BA}\| = \|\overrightarrow{AB}\| = \sqrt{17}$$

3. $\overrightarrow{AC}$ where C is the midpoint of the line segment AB with $A = (1, 2, 3)$ and $B = (-1, 5, 1)$.

SOLUTION: Since C is the midpoint of AB, we have that

$$\overrightarrow{AC} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\langle -2, 3, -2\rangle = \langle -1, \frac{3}{2}, -1\rangle.$$

Furthermore, because C is the midpoint of AB, we know that

$$|AC| = \frac{1}{2}|AB|$$

Thus

$$\|\overrightarrow{AC}\| = \frac{1}{2}\|\overrightarrow{AB}\| = \frac{1}{2}\sqrt{17}$$

4. The position vector of a point P obtained from the point $A = (-1, 2, -1)$ by moving the latter along a straight line by a distance of 3 units in the direction of the vector $\mathbf{u} = \langle 2, 2, 1\rangle$ then by a distance of

10 units in the direction of the vector $\mathbf{w} = \langle -3, 0, -4\rangle$.

SOLUTION: The position vector of the point A is $\mathbf{a} = \langle -1, 2, -1\rangle$. Let $\mathbf{p}$ be the position vector of the point P. Then by the rules of vector algebra

$$\mathbf{p} = \mathbf{a} + t\mathbf{u} + s\mathbf{w}$$

where positive numbers t and s must be chosen so that the first displacement vector $t\mathbf{u}$ has a length of 3 units and the second displacement vector $s\mathbf{w}$ has a length of 10 units:

$$3 = \|t\mathbf{u}\| = t\|\mathbf{u}\| = t\sqrt{2^2 + 2^2 + 1} = 3t \quad \Rightarrow \quad t = 1\,,$$
$$10 = \|s\mathbf{w}\| = s\|\mathbf{w}\| = s\sqrt{3^2 + 0 + 4^2} = 5s \quad \Rightarrow \quad s = 2\,.$$

Therefore

$$\mathbf{p} = \langle -1, 2, -1\rangle + \langle 2, 2, 1\rangle + 2\langle -3, 0, -4\rangle = \langle -5, 4, -8\rangle$$
$$\Rightarrow \quad P = (-5, 4, -8)\,.$$

5. The position vector of the vertex C of a triangle ABC in the first quadrant of the xy plane if A is at the origin, $B = (a, 0, 0)$, the angle at the vertex B is $2\pi/3$, and $|BC| = 2a$.

SOLUTION: First recognize that since the angle at vertex B is $2\pi/3$, the angle between $\overrightarrow{OB}$ and $\overrightarrow{OC}$ is actually $\pi/3$. This can be seen by first sketching triangle ABC and moving the tail of $\overrightarrow{OC}$ to the tail of $\overrightarrow{OB}$. Since $\overrightarrow{OB}$ is parallel to the x axis, we know that the angle between $\overrightarrow{OC}$ and the x axis is $\pi/3$. Next, recall that we can write the position vector of a point P in the xy plane as

$$\overrightarrow{OP} = \langle p\cos(\theta), p\sin(\theta), 0\rangle$$

where p is taken to be the distance between the origin and P and θ is the smallest angle between OP and the x axis. This is simply the polar representation of a point P in the xy plane. Thus,

$$\overrightarrow{BC} = \langle 2a\cos(\pi/3), 2a\sin(\pi/3), 0\rangle = \langle a, \sqrt{3}a, 0\rangle$$

Note that this is the vector for $\overrightarrow{BC}$! If this were $\overrightarrow{OC}$, then we would have that $|AC| = 2a$, but we are given $|BC| = 2a$. To find $\overrightarrow{OC}$, observe that

$$\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} = \langle a, 0, 0\rangle + \langle a, \sqrt{3}a, 0\rangle = \langle 2a, \sqrt{3}a, 0\rangle = a\langle 2, \sqrt{3}, 0\rangle$$

Furthermore,

$$\|\overrightarrow{OC}\| = a\sqrt{2^2 + \sqrt{3}^2} = \sqrt{7}a$$

6. Are the points $A = (-3, 1, 2)$, $B = (1, 5, -2)$, $C = (0, 3, -1)$, and $D = (-2, 3, 1)$ vertices of a parallelogram?

SOLUTION: If the given points are indeed the sides of a parallelogram, then we must have that two pairs of sides are equal in length and parallel. That is, the we need to find two pairs of vectors between two distinct points that are equal (up to a negative sign, which just indicates direction). So,

$$\begin{aligned}
\overrightarrow{AB} &= \langle 1, 5, -2\rangle - \langle -3, 1, 2\rangle = \langle 4, 4, -4\rangle \\
\overrightarrow{AC} &= \langle 0, 3, -1\rangle - \langle -3, 1, 2\rangle = \langle 3, 2, -3\rangle \\
\overrightarrow{AD} &= \langle -2, 3, 1\rangle - \langle -3, 1, 2\rangle = \langle 1, 2, -1\rangle \\
\overrightarrow{BC} &= \langle 0, 3, -1\rangle - \langle 1, 5, -2\rangle = \langle -1, -2, 1\rangle \\
\overrightarrow{BD} &= \langle -2, 3, 1\rangle - \langle 1, 5, -2\rangle = \langle -3, -2, 3\rangle \\
\overrightarrow{CD} &= \langle -2, 3, 1\rangle - \langle 0, 3, -1\rangle = \langle -2, 0, 2\rangle
\end{aligned}$$

Observe that

$$\begin{aligned}
\overrightarrow{AC} &= -\overrightarrow{BD} \\
\overrightarrow{AD} &= -\overrightarrow{BC}
\end{aligned}$$

So $ACBD$ is a parallelogram, with A and B opposite each other and C and D opposite each other.

7. If $A = (2, 0, 3)$, $B = (-1, 2, 0)$, and $C = (0, 3, 1)$, determine the point D such that A, B, C, and D are vertices of a parallelogram with sides AB, BC, CD, and DA.

SOLUTION: From the description of the parallelogram it follows that the side AB is parallel to CD and the side BC is parallel to DA. In particular, $\overrightarrow{AD} = \overrightarrow{BC}$. Let $\mathbf{a} = \langle 2, 0, 3\rangle$ be the position vector of the point A. Then, by the parallelogram rule, the position vector of the point D is

$$\mathbf{a} + \overrightarrow{AD} = \mathbf{a} + \overrightarrow{BC} = \langle 2, 0, 3\rangle + \langle 0 - (-1), 3 - 2, 1 - 0\rangle = \langle 3, 1, 4\rangle .$$

So, $D = (3, 1, 4)$.

8. A parallelogram has a vertex at $A = (1, 2, 3)$ and two sides $\mathbf{a} = \langle 1, 0, -2\rangle$ and $\mathbf{b} = \langle 3, -2, 6\rangle$ adjacent at A. Find the coordinates of the point of intersection of the diagonals of the parallelogram.

Solution: The point of intersection of the diagonals is the midpoint of either of them. By the parallelogram rule, the diagonal extended from A is $\mathbf{a}+\mathbf{b}$. Therefore the position vector of the midpoint relative to the point A is $\frac{1}{2}(\mathbf{a}+\mathbf{b})$ so that the position vector of the midpoint relative to the origin is

$$\begin{aligned}\overrightarrow{OA}+\frac{1}{2}(\mathbf{a}+\mathbf{b}) &= \langle 1,2,3\rangle + \frac{1}{2}\langle 1+3, 0-2, -2+6\rangle \\ &= \langle 1,2,3\rangle + \langle 2,-1,2\rangle = \langle 3,1,5\rangle\end{aligned}$$

Thus, the point in question is $(3,1,5)$.

9. Draw two vectors $\mathbf{a}$ and $\mathbf{b}$ that are neither parallel and nor perpendicular. Sketch each of the following vectors: $\mathbf{a}+2\mathbf{b}$, $\mathbf{b}-2\mathbf{a}$, $\mathbf{a}-\frac{1}{2}\mathbf{b}$, and $2\mathbf{a}+3\mathbf{b}$.

10. Draw three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in a plane, with none of them parallel to either of the others. Sketch each of the following vectors: $\mathbf{a}+(\mathbf{b}-\mathbf{c})$, $(\mathbf{a}+\mathbf{b})-\mathbf{c}$, $2\mathbf{a}-3(\mathbf{b}+\mathbf{c})$, and $(2\mathbf{a}-3\mathbf{b})-3\mathbf{c}$.

11. Let $\mathbf{a}=\langle 2,-1,-2\rangle$ and $\mathbf{b}=\langle -3,0,4\rangle$. Find unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$. Express $6\hat{\mathbf{a}}-15\hat{\mathbf{b}}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

Solution: We first have that

$$\begin{aligned}\|\mathbf{a}\| &= \sqrt{2^2+(-1)^2+(-2)^2} = 3 \\ \|\mathbf{b}\| &= \sqrt{(-3)^2+4^2} = 5\end{aligned}$$

So

$$\begin{aligned}\hat{\mathbf{a}} &= \frac{1}{\|\mathbf{a}\|}\mathbf{a} = \frac{1}{3}\mathbf{a} \\ \hat{\mathbf{b}} &= \frac{1}{\|\mathbf{b}\|}\mathbf{b} = \frac{1}{5}\mathbf{b}\end{aligned}$$

And thus

$$6\hat{\mathbf{a}}-15\hat{\mathbf{b}} = \frac{6}{3}\mathbf{a} - \frac{15}{5}\mathbf{b} = 2\mathbf{a}-3\mathbf{b}$$

12. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in the xy plane such that their sum $\mathbf{c}=\mathbf{a}+\mathbf{b}$ makes the angle $\pi/3$ with $\mathbf{a}$ and has the length twice the length of $\mathbf{a}$. Find $\mathbf{b}$ if $\mathbf{a}$ is based at the origin, has its terminal point in the first quadrant, makes an angle $\pi/3$ with the positive x-axis, and has length a. There are two vectors $\mathbf{b}$ with these properties. Find both of them.

SOLUTION: First we must find **a**. Since it has length a and makes an angle of $\pi/3$ with the positive x-axis, we can write **a** in polar coordinates as

$$\mathbf{a} = \langle a\cos(\pi/3), a\sin(\pi/3)\rangle = a\langle \frac{1}{2}, \frac{\sqrt{3}}{2}\rangle$$

Now, there are two options for **c** which will lead to our two answers for **b**, both of which are symmetric about **a**. The first is when **c** is "above **a**". That is, the angle it makes with the positive x-axis is greater than $\pi/3$. Since we know that the angle **c** makes with **a** is $\pi/3$, and the angle **a** makes with the positive x-axis is $\pi/3$, we know that the angle **c** makes with the positive x-axis in this case is $2\pi/3$. The other option is when **c** is "below **a**". In this case, the angle **c** will make with the positive x-axis is 0. Let us split these into two cases.

Case 1: **c** makes angle $2\pi/3$ with the positive x-axis.
In this case, we can write **c** in polar form as

$$\mathbf{c} = 2a\langle\cos(2\pi/3), \sin(2\pi/3)\rangle = a\langle -1, \sqrt{3}\rangle,$$

where the $2a$ came from the fact that $\|\mathbf{c}| = 2a$.
By vector algebra,

$$\mathbf{b} = \mathbf{c} - \mathbf{a} = a\langle -1, \sqrt{3}\rangle - a\langle \frac{1}{2}, \frac{\sqrt{3}}{2}\rangle = a\langle -\frac{3}{2}, \frac{\sqrt{3}}{2}\rangle$$

Case 2: **c** makes angle 0 with the positive x-axis.
In this case, we can write **c** in polar form as

$$\mathbf{c} = 2a\langle\cos(0), \sin(0)\rangle = 2a\langle 1, 0\rangle,$$

where the $2a$ came from the fact that $\|\mathbf{c}| = 2a$.
By vector algebra,

$$\mathbf{b} = \mathbf{c} - \mathbf{a} = a\langle 2, 0\rangle - a\langle \frac{1}{2}, \frac{\sqrt{3}}{2}\rangle = a\langle \frac{3}{2}, -\frac{\sqrt{3}}{2}\rangle$$

13. Consider a triangle ABC. Let **a** be a vector from the vertex A to the midpoint of the side BC, let **b** be a vector from B to the midpoint of AC, and let **c** be a vector from C to the midpoint of AB. Use vector algebra to find $\mathbf{a} + \mathbf{b} + \mathbf{c}$, that is, do not resort to writing vectors in component-form; just use properties of vector addition, subtraction, and multiplication by scalars.

SOLUTION: First observe the following: If P and Q are vertices of a triangle with midpoint M between them, then

$$\mathbf{m} = \frac{1}{2}(\mathbf{p} + \mathbf{q}).$$

Where $\mathbf{m}, \mathbf{p}, \mathbf{q}$ are taken to be the position vectors of M, P, Q respectively.
Let D be the midpoint of BC, E the midpoint of AC, and F the midpoint of AB. Then,

$$\begin{aligned}
\mathbf{a} + \mathbf{b} + \mathbf{c} &= \overrightarrow{DA} + \overrightarrow{EB} + \overrightarrow{FC} \\
&= \overrightarrow{OA} - \overrightarrow{OD} + \overrightarrow{OB} - \overrightarrow{OE} + \overrightarrow{OC} - \overrightarrow{OF} \\
&= \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} - \frac{1}{2}(\overrightarrow{OB} + \overrightarrow{OC}) - \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OC}) - \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB}) \\
&= \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} - \frac{1}{2}\overrightarrow{OA} - \frac{1}{2}\overrightarrow{OA} - \frac{1}{2}\overrightarrow{OB} - \frac{1}{2}\overrightarrow{OB} - \frac{1}{2}\overrightarrow{OC} - \frac{1}{2}\overrightarrow{OC} \\
&= \mathbf{0}
\end{aligned}$$

14. Let $\hat{\mathbf{u}}_k$, $k = 1, 2, ..., n$, be unit vectors in the plane such that the smallest angle between the two vectors $\hat{\mathbf{u}}_k$ and $\hat{\mathbf{u}}_{k+1}$ is $2\pi/n$. What is the sum $\mathbf{v}_n = \hat{\mathbf{u}}_1 + \hat{\mathbf{u}}_2 + \cdots + \hat{\mathbf{u}}_n$ for an even n? Sketch the sum for $n = 1$, $n = 3$, and $n = 5$. Compare the norms $\|\mathbf{v}_n\|$ for $n = 1, 3, 5$. Investigate the limit of $\mathbf{v}_n$ as $n \to \infty$ by studying the limit of $\|\mathbf{v}_n\|$ as $n \to \infty$.

SOLUTION: Consider a regular n-gon with its center located at the origin of the xy plane, where each vertex is one unit away from the origin. Then each $\hat{\mathbf{u}}_k$ is the position vector of a vertex of the n-gon. To see this, observe that we can split an entire angle (2π) into n equal sized pieces, each of which has magnitude $2\pi/n$ (like cutting a pizza). For the even n, it is obvious that one could rotate the n-gon by π and obtain the original image. This implies that for every vertex, there is another vertex exactly opposite. The sum of the position vectors of these vertices will cancel (equal magnitude but opposite direction). Thus $\mathbf{v}_n = \mathbf{0}$ for even n. The same will be true for odd n, with the exception of when $n = 1$. (Sketches to come).
Referring back to the fact that we can construct each $\hat{\mathbf{u}}_k$ as the vertex

of a regular n-gon as above, we have that:

$$
\begin{aligned}
\mathbf{v}_1 &= \langle\cos(0),\sin(0)\rangle = \langle 1,0\rangle\\
\mathbf{v}_3 &= \langle\cos(0),\sin(0)\rangle + \langle\cos(2\pi/3),\sin(2\pi/3)\rangle + \langle\cos(4\pi/3),\sin(4\pi/3)\rangle\\
&= \langle 1,0\rangle + \langle -\frac{1}{2},\frac{\sqrt{3}}{2}\rangle + \langle -\frac{1}{2},-\frac{\sqrt{3}}{2}\rangle\\
&= \mathbf{0}\\
\mathbf{v}_5 &= \langle\cos(0),\sin(0)\rangle + \langle\cos(2\pi/5),\sin(2\pi/5)\rangle + \langle\cos(4\pi/5),\sin(4\pi/5)\rangle\\
&\quad + \langle\cos(6\pi/5),\sin(6\pi/5) + \langle\cos(8\pi/5),\sin(8\pi/5)\rangle\\
&= \mathbf{0}
\end{aligned}
$$

Therefore $\mathbf{v}_n = \mathbf{0}$ as $n \to \infty$

15. Let $\hat{\mathbf{u}}_k$, $k = 1, 2, ..., n$, be unit vectors as defined in Exercise 14. Let $\mathbf{w}_k = \hat{\mathbf{u}}_{k+1} - \hat{\mathbf{u}}_k$ for $k = 1, 2, ..., n-1$ and $\mathbf{w}_n = \hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_n$. Find the limit of $\|\mathbf{w}_1\| + \|\mathbf{w}_2\| + \cdots + \|\mathbf{w}_n\|$ as $n \to \infty$. Hint: Use a geometrical interpretation of the sum.

Solution: As with Exercise 14, we can consider a regular n-gon centered at the origin of the xy plane with vertices distance 1 from the origin. Enumerate the vertices as $1, 2, ..., n$. Consider vertices k and $k + 1$. Recall that we discovered that the position vector for vertex k is $\hat{\mathbf{u}}_k$. Convince yourself that $\mathbf{w}_k = \hat{\mathbf{u}}_{k+1} - \hat{\mathbf{u}}_k$ is the vector from vertex k to vertex $k + 1$. Therefore the sum of all $\|\mathbf{w}_k\|$ from 0 to n is simply the perimeter of the n gon. As $n \to \infty$, the n-gon approaches a circle. Therefore $\|\mathbf{w}_1\| + \|\mathbf{w}_2\| + \cdots + \|\mathbf{w}_n\| = 2\pi$ as $n \to \infty$.
The fact that the sum of the $\mathbf{w}_k$ gives a length is important for the formulation of a line integral in later chapters.

16. Suppose a wind is blowing at a speed of u mi/h in the direction that is $0 < \alpha < 90°$ degrees west of the northerly direction. A pilot is steering a plane in the direction that is $0 < \beta < 90°$ degrees east of the northerly direction at an airspeed (speed in still air) of $v > u$ mi/h. The true course of the plane is the direction of the resultant of the velocity vectors of the plane and the wind. The ground speed is the magnitude of the resultant. Find the true course and the ground speed of the plane. If α, u, and v are fixed, what is the direction in which the pilot should steer the plane to make the true course north?

Solution: First we must define a coordinate system. Let the positive y direction be north. Then the vector representing the wind speed is given by

$$\begin{aligned}\mathbf{u} &= \langle u\cos(\alpha+\pi/2), u\sin(\alpha+\pi/2)\rangle \\ &= u\langle \cos(\alpha)\cos(\pi/2)-\sin(\alpha)\sin(\pi/2), \sin(\alpha)\cos(\pi/2)+\cos(\alpha)\sin(\pi/2)\rangle \\ &= u\langle -\sin(\alpha), \cos(\alpha)\rangle\end{aligned}$$

where we had to shift the arguments of the cos and sin by $\pi/2$ because north is aligned along the positive y-axis. To find $\mathbf{v}$, we will apply the same thing as follows:

$$\begin{aligned}\mathbf{v} &= \langle v\cos(-\beta+\pi/2), u\sin(-\beta+\pi/2)\rangle \\ &= v\langle \sin(\beta), \cos(\beta)\rangle\end{aligned}$$

where there is a negative sign in front of β in the first line because to move east, we must rotate clockwise, which is in the negative direction. The resultant vector, then, is

$$\mathbf{v}+\mathbf{u} = \langle v\sin(\beta)-u\sin(\alpha), v\cos(\beta)+u\cos(\alpha)\rangle$$

and the magnitude of the resultant is

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\| &= \sqrt{(v\sin(\beta)-u\sin(\alpha))^2+(v\cos(\beta)+u\cos(\alpha))^2} \\ &= \sqrt{v^2+u^2+2uv(\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta))} \\ &= \sqrt{v^2+u^2+2uv\cos(\alpha+\beta)}\end{aligned}$$

If the course is true north, then the resultant vector must be parallel to the y-axis. So the x-component of the resultant must be 0. Thus,

$$\begin{aligned}v\sin(\beta)-u\sin(\alpha) &= 0 \\ v\sin(\beta) &= u\sin(\alpha) \\ \sin(\beta) &= \frac{u}{v}\sin(\alpha) \\ \beta &= \arcsin(\frac{u}{v}\sin(\alpha))\end{aligned}$$

Note that if $u > v$, it may be that β cannot be computed. Hence the restriction that $v > u$ in the exercise.

17. Use vector algebra (do not resort to writing vectors in component-form) to show that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

SOLUTION: Let A, B, C be the vertices of a triangle. Let D, E be the midpoints of AC and BC, respectively. We must then prove that

$$\overrightarrow{DE} = \frac{1}{2}\overrightarrow{AB}.$$

First, by Exercise 13, we know that the position vector of the midpoint M of a line with endpoints P and Q is

$$\mathbf{m} = \frac{1}{2}(\mathbf{p} + \mathbf{q})$$

where $\mathbf{m}, \mathbf{p}, \mathbf{q}$ are taken to be the position vectors of M, P, Q respectively. So, we have that

$$\begin{aligned}\mathbf{d} &= \frac{1}{2}(\mathbf{a} + \mathbf{c})\\ \mathbf{e} &= \frac{1}{2}(\mathbf{a} + \mathbf{b})\end{aligned}$$

Then

$$\begin{aligned}\overrightarrow{DE} &= \overrightarrow{OE} - \overrightarrow{OD}\\ &= \frac{1}{2}(\mathbf{b} + \mathbf{c}) - \frac{1}{2}(\mathbf{a} + \mathbf{c})\\ &= \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c} - \frac{1}{2}\mathbf{a} - \frac{1}{2}\mathbf{c}\\ &= \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}\\ &= \frac{1}{2}(\overrightarrow{OB} - \overrightarrow{OA})\\ &= \frac{1}{2}\overrightarrow{AB}\end{aligned}$$

Note: Working from

$$\overrightarrow{DE} = \frac{1}{2}\overrightarrow{AB}$$

to

$$\overrightarrow{DE} = \overrightarrow{DE}$$

would be an incorrect proof; it is in the wrong direction.

18– 21. Describe geometrically the set of points whose position vectors $\mathbf{r}$ satisfy the given conditions.

18. $\|\mathbf{r} - \mathbf{a}\| = k$ and $\mathbf{r}$ lies in the xy plane, where $\mathbf{a}$ is a vector in the xy plane and $k > 0$.

Solution: First, we can recognize $\mathbf{a}$ as the position vector of some point A in the xy plane. Then, we can interpret $\|\mathbf{r} - \mathbf{a}\| = k$ as all the points a fixed distance (k) from A.

This is simply a circle with center A and radius k.

19. $\|\mathbf{r}-\mathbf{a}\|+\|\mathbf{r}-\mathbf{b}\|=k$ and $\mathbf{r}$ lies in the xy plane, where $\mathbf{a}$ and $\mathbf{b}$ are vectors in the xy plane and $k>\|\mathbf{a}-\mathbf{b}\|$.

SOLUTION: First, $\mathbf{a}$ and $\mathbf{b}$ represent the position vectors of two points in the xy plane, A and B. Recall that an ellipse can be generated by the set of all points such that the sum of the distances to two points is a fixed value (see Section 1.1, Exercise 31). This is exactly what is written above, so it is an ellipse with foci A and B.

20. $\|\mathbf{r}-\mathbf{a}\|=k$, where $\mathbf{a}$ is a vector in space and $k>0$.

SOLUTION: This is the 3D analogue to Exercise 18. Thus we have a sphere of raidus k whose center is at A, where A is the point with position vector $\mathbf{a}$.

21. $\|\mathbf{r}-\mathbf{a}\|+\|\mathbf{r}-\mathbf{b}\|=k$, where $\mathbf{a}$ and $\mathbf{b}$ are vectors in space and $k>\|\mathbf{a}-\mathbf{b}\|$.

SOLUTION: This is the 3D analogue to Exercise 19. Thus we have an ellipsoid with foci A and B, whose position vectors are $\mathbf{a}$ and $\mathbf{b}$, respectively.

22. Let point-like massive objects be positioned at P_i, $i=1,2,...,n$, and let m_i be the mass at P_i. The point P_0 is called the *center of mass* if

$$m_1\overrightarrow{P_0P_1}+m_2\overrightarrow{P_0P_2}+\cdots+m_n\overrightarrow{P_0P_n}=\mathbf{0}$$

Express the position vector $\mathbf{r}_0$ of P_0 in terms of the position vectors $\mathbf{r}_i$ of P_i. In particular, find the center of mass of three point masses, $m_1=m_2=m_3=m$, located at the vertices of a triangle ABC for $A=(1,2,3)$, $B=(-1,0,1)$, and $C=(1,1,-1)$.

SOLUTION: First observe that for any $1\leq i\leq n$,

$$\overrightarrow{P_0P_i}=\mathbf{r}_i-\mathbf{r}$$

So we have that

$$\begin{aligned}
m_1\overrightarrow{P_0P_1} + m_2\overrightarrow{P_0P_2} + \cdots + m_n\overrightarrow{P_0P_n} &= \mathbf{0} \\
\sum_{k=1}^{n} m_k\overrightarrow{P_0P_k} &= \mathbf{0} \\
\sum_{k=1}^{n} m_k(\mathbf{r}_k - \mathbf{r}) &= \mathbf{0} \\
\sum_{k=1}^{n} m_k\mathbf{r}_i - \sum_{k=1}^{n} m_k\mathbf{r} &= \mathbf{0} \\
\sum_{k=1}^{n} m_k\mathbf{r} &= \sum_{k=1}^{n} m_k\mathbf{r}_i \\
\mathbf{r} &= \frac{\sum_{k=1}^{n} m_k\mathbf{r}_i}{\sum_{k=1}^{n} m_k}
\end{aligned}$$

In the given problem, we are asked to find the center of mass of three point masses, each with mass m, located at $A = (1, 2, 3)$, $B = (-1, 0, 1)$, and $C = (1, 1, -1)$. So,

$$\begin{aligned}
\mathbf{r} &= \frac{\sum_{k=1}^{n} m_k\mathbf{r}_i}{\sum_{k=1}^{n} m_k} \\
&= \frac{m(\langle 1, 2, 3\rangle + \langle -1, 0, 1\rangle + \langle 1, 1 - 1\rangle)}{3m} \\
&= \frac{1}{3}\langle 1 - 1 + 1, 2 + 0 + 1, 3 + 1 - 1\rangle \\
&= \frac{1}{3}\langle 1, 3, 3\rangle
\end{aligned}$$

23. Consider the graph $y = f(x)$ of a differentiable function and the line tangent to it at a point $x = a$. Express components of a vector parallel to the line in terms of the derivative $f'(a)$ and find a vector perpendicular to the line. In particular, find such vectors for the graph $y = x^2$ at the point $x = 1$.

SOLUTION: Recall that we can write the equation for the tangent line to a differentiable function at $x = a$ as

$$y_t = f'(a)(x - a) + f(a)$$

and for the normal line as

$$y_n = -\frac{1}{f'(a)}(x-a) + f(a)$$

A vector parallel to this line can simply be the difference between the position vectors of two points on the line. I will use the points $A = (a, f(a))$ and $B_t = (0, y_t(0))$. So,

$$\begin{aligned} \mathbf{v}_t &= \overrightarrow{OA} - \overrightarrow{OB_t} \\ &= \langle a, f(a)\rangle - \langle 0, -af'(a) + f(a)\rangle \\ &= \langle a, af'(a)\rangle \quad || \quad \langle 1, f'(a)\rangle \end{aligned}$$

is a vector parallel to the tangent line, and using $B_n = (0, y_n(0))$,

$$\begin{aligned} \mathbf{v}_n &= \overrightarrow{OA} - \overrightarrow{OB_n} \\ &= \langle a, f(a)\rangle - \langle 0, \frac{a}{f'(a)} + f(a)\rangle \\ &= \langle a, -\frac{a}{f'(a)}\rangle \quad || \quad \langle f'(a), -1\rangle \end{aligned}$$

In particular when $f(x) = x^2$ and $x = 1$,

$$\mathbf{v}_t = \langle 1, f'(1))\rangle = \langle 1, 2\rangle$$

$$\mathbf{v}_n = \langle -f'(1), 1)\rangle = \langle -2, 1\rangle$$

24. Let the vectors **a**, **b**, and **c** have fixed lengths a, b, and c, respectively, while their direction may be changed. Is it always possible to achieve $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$? If not, formulate the most general condition under which it is possible.

Solution: No it is not always possible. Consider $\mathbf{a} = \langle 3, 0\rangle$, $\mathbf{b} = \mathbf{c} = \langle 1, 0\rangle$. Notice that the best way to get to the zero vector is to have $\mathbf{b}, \mathbf{c}$ be antiparallel to $\mathbf{a}$, as this will have the smallest magnitude of the resultant vector. In this case, $\mathbf{a} - \mathbf{b} - \mathbf{c} = \langle 3 - 1 - 1, 0\rangle = \langle 1, 0\rangle$, which is not the zero vector.
What we are trying to achieve is to form a triangle from the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. In this case, we were unable to form a triangle. So the question boils down to: Given three lines of lengths a, b, c, is there any way to make a triangle from them? Recall the cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

So we must have that

$$C = \arccos(\frac{c^2 - a^2 - b^2}{2ab})$$

For C to be defined,

$$\begin{aligned} |\frac{c^2 - a^2 - b^2}{2ab}| &\leq 1 \\ |c^2 - a^2 - b^2| &\leq |2ab| \end{aligned}$$

25. Let the vectors **a** and **b** have fixed lengths, while their directions may be changed. Put $c_\pm = \|\mathbf{a} \pm \mathbf{b}\|$. It is always possible to achieve that $c_- > c_+$, or $c_- = c_+$, or $c_- < c_+$? If so, give examples of the corresponding relative directions of **a** and **b**.

SOLUTION: Yes it is possible. First, note that c_- and c_+ are real numbers, hence there must be some way to relate them (via $<, =$, or $>$). Consider **a** and **b** to be vectors with length 1. If $\mathbf{a}, \mathbf{b}$ are parallel (in the same direction), then the lengths will simply add and we will have that $c_+ = 2$ and $c_- = 0$. If $\mathbf{a}, \mathbf{b}$ are parallel (in the opposite direction), then the lengths will subtract when the vectors are added, so $c_+ = 0$ and $c_- = 2$. If $\mathbf{a}, \mathbf{b}$ are perpendicular, then c_+ and c_- will simply be the length of the hypotenuse of the triangle formed by **a** and **b** and **a** and $-\mathbf{b}$, respectively. These two triangles share a common angle (perpendicular) between two sides of the same length, hence they are congruent (SAS), so their hypotenuses are the same.

26. A point object travels in the xy plane, starting from an initial point P_0. Its trajectory consists of straight line-segments, where the n^{th} segment starts at point P_{n-1} and ends at point P_n (for $n \geq 1$). When the object reaches P_n, it makes a 90–degree counterclockwise turn to proceed towards P_{n+1}. (Thus each segment, other than the first, is perpendicular to the preceding segment.) The length of the first segment is a. For $n \geq 2$, the n^{th} segment is s times as long as the $(n-1)^{\text{st}}$ segment, where s is a fixed number between 0 and 1 (strictly). If the object keeps moving forever,
(i) what is the farthest distance it ever gets from P_0, and
(ii) what is the distance between P_0 and the limiting position P_∞ that the object approaches?
Hint: Investigate the components of the position vector of the object in an appropriate coordinate system.

SOLUTION: First, suppose that the initial point P_0 is at the origin and its initial trajectory is along the positive x-axis. Notice that, since $0 < s < 1$, we have for all $n \geq 1$

$$|P_{n-1}P_n| > |P_nP_{n+1}|$$

(i) Consider the first 6 points $P_0, P_1, ...P_5$. If you were to plot these out, you would notice that they form a spiral. The point object, starting from P_0, moves to P_1, then turns and moves to P_2. Its distance from the origin is now greater than P_0P_1. Then it turns and moves to P_3 (in the opposite direction of the original direction), but since the lengths are decreasing, it will not reach the y-axis, but will get closer to it. Hence the distance to the origin has decreased. If you continue in this fashion, you will see that the maximum distance is at P_2. We know that P_2 is at (a, sa), so the maximum distance is $a\sqrt{1+s^2}$.

(ii) We must make two infinite series: one for the x-component and one for the y component. Notice that the object alternates traveling along the x and y directions. So one sum will be composed of odd powers of s and one will be composed of even powers of s. We can see that since $P_1 = (a, 0)$, the x-component sum will involve even powers of s. We also know that each sum will be an alternating series. We can continue finding the first couple coordinates as follows

$$\begin{aligned}
P_0 &= (0,0) \\
P_1 &= (a,0) \\
P_2 &= (a, sa) \\
P_3 &= (a - s^2a, sa) \\
P_4 &= (a - s^2a, sa - s^3a) \\
P_5 &= (a - s^2a + s^4a, sa - s^3a) \\
P_6 &= (a - s^2a + s^4a, sa - s^3a + s^5a)
\end{aligned}$$

I will only consider the points with even subscripts, to help simplify the formulation of the series. This will not matter in the limiting case. So we have that

$$P_{2n} = (a\sum_{k=0}^{n-1}(-1)^k s^{2k}, a\sum_{k=0}^{n-1}(-1)^k s^{(2k+1)})$$

$$P_{2n} = (a\sum_{k=0}^{n-1}(-s^2)^k, as\sum_{k=0}^{n-1}(-s^2)^k)$$

And the distance from the origin is simply

$$\begin{aligned} d(n) &= \sqrt{(a\sum_{k=0}^{n-1}(-s^2)^k)^2 + (sa\sum_{k=0}^{n-1}(-s^2)^k)^2} \\ &= a\sum_{k=0}^{n-1}(-s^2)^k\sqrt{1+s^2} \\ &= a\frac{1-(-s^2)^n}{1-(-s^2)}\sqrt{1+s^2} \\ &= a\frac{1-(-s^2)^n}{1+s^2}\sqrt{1+s^2} \end{aligned}$$

by recalling that

$$\sum_{k=0}^{n-1} ar^k = a\frac{1-r^n}{1-r}$$

an if $|r| < 1$, then $r^n \to 0$ as $n \to \infty$. So,

$$\lim_{n\to\infty} d(n) = \frac{a}{\sqrt{1+s^2}}$$

3. The Dot Product

1–5. Find the dot product $\mathbf{a} \cdot \mathbf{b}$ for the given vectors $\mathbf{a}$ and $\mathbf{b}$.

1. $\mathbf{a} = \langle 1, 2, 3\rangle$ and $\mathbf{b} = \langle -1, 2, 0\rangle$

SOLUTION: Recall that the dot product between two vectors $\mathbf{u} = \langle u_x, u_y, u_z\rangle$ and $\mathbf{v} = \langle v_x, v_y, v_z\rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_x v_x + u_y v_y + u_z v_z$$

So

$$\mathbf{a} \cdot \mathbf{b} = 1(-1) + 2(2) + 3(0) = 3$$

2. $\mathbf{a} = \overrightarrow{AB}$ and $\mathbf{b} = \overrightarrow{BC}$ where $A = (1, -2, 1)$, $B = (2, -1, 3)$, and $C = (1, 1, 1)$.

SOLUTION: First we have that

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} = \langle 2, -1, 3\rangle - \langle 1, -2, 1\rangle = \langle 1, 1, 2\rangle \\ \overrightarrow{BC} &= \overrightarrow{OC} - \overrightarrow{OB} = \langle 1, 1, 1\rangle - \langle 2, -1, 3\rangle = \langle -1, 2, -2\rangle \end{aligned}$$

So

$$\mathbf{a} \cdot \mathbf{b} = 1(-1) + 1(2) + 2(-2) = -3$$

3. $\mathbf{a} = \hat{\mathbf{e}}_1 + 3\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3$ and $\mathbf{b} = 3\hat{\mathbf{e}}_1 - 2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$

SOLUTION: Recall that $\hat{\mathbf{e_1}}, \hat{\mathbf{e_2}}, \hat{\mathbf{e_3}}$ are the unit direction vectors parallel to the x, y, z axes, respectively. So, $\mathbf{a} = \langle 1, 3, -1\rangle$ and $\mathbf{b} = \langle 3, -2, 1\rangle$, and

$$\mathbf{a} \cdot \mathbf{b} = 1(3) + 3(-2) - 1(1) = -4$$

4. $\mathbf{a} = \mathbf{u}_1 + 3\mathbf{u}_2 - \mathbf{u}_3$ and $\mathbf{b} = 3\mathbf{u}_1 - 2\mathbf{u}_2 + \mathbf{u}_3$ where $\mathbf{u}_n$, $n = 1, 2, 3$, are orthogonal vectors and $\|\mathbf{u}_n\| = n$.

SOLUTION: Recall that if two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal then $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$. So

$$\mathbf{a} \cdot \mathbf{b} = 1(3)\|\mathbf{u}_1\|^2 + 3(-2)\|\mathbf{u}_2\|^2 - 1(1)\|\mathbf{u}_3\|^2 = 3 - 6(4) - 9 = -30$$

5. $\mathbf{a} = 2\mathbf{c} - 3\mathbf{d}$ and $\mathbf{b} = \mathbf{c} + 2\mathbf{d}$ if $\mathbf{c}$ is a unit vector that makes the angle $\pi/3$ with the vector $\mathbf{d}$ and $\|\mathbf{d}\| = 2$

SOLUTION: Recall that, given two vectors $\mathbf{u}$ and $\mathbf{v}$, another way to write the dot product of them is

$$\mathbf{u} \cdot \mathbf{v} = \cos(\theta)|\mathbf{u}\|\|\mathbf{v}\|$$

where θ is the smallest positive angle between $\mathbf{u}$ and $\mathbf{v}$. So we have that

$$\mathbf{c} \cdot \mathbf{d} = \cos(\pi/3)(2) = 1$$

since the angle between $\mathbf{c}$ and $\mathbf{d}$ is $\pi/3$, c is a unit vector so $\|\mathbf{c}\| = 1$, and $\|\mathbf{d}\| = 2$ as given. Thus,

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= 2\mathbf{c} \cdot \mathbf{c} + 4\mathbf{c} \cdot \mathbf{d} - 3\mathbf{d} \cdot \mathbf{c} - 6\mathbf{d} \cdot \mathbf{d} \\ &= 2\|\mathbf{c}\|^2 + \mathbf{c} \cdot \mathbf{d} - 6\|\mathbf{d}\|^2 \\ &= 2 + 1 - 6(4) = -21 \end{aligned}$$

6. Let $\mathbf{a}$ be a nonzero vector. Show that $\mathbf{0}$ is the only vector that is both parallel and perpendicular to $\mathbf{a}$.

Solution: Let $\mathbf{a}$ be a nonzero vector. Suppose $\mathbf{u}$ is a vector that is both parallel and perpendicular to $\mathbf{a}$. Since $\mathbf{u}$ is parallel to $\mathbf{a}$, there exists a real number s such that $\mathbf{u} = s\mathbf{a}$. Since $\mathbf{u}$ is perpendicular to $\mathbf{a}$, $\cos(\theta) = 0 \Rightarrow \mathbf{a} \cdot \mathbf{u} = 0$. But $\mathbf{a} \cdot \mathbf{u} = \mathbf{a} \cdot s\mathbf{a} = s(\mathbf{a} \cdot \mathbf{a}) = s\|\mathbf{a}\|^2$. So $s\|\mathbf{a}\|^2 = 0$. Since $\mathbf{a}$ is a nonzero vector, $\|\mathbf{a}\| \neq 0$. Thus $s = 0$, and $\mathbf{u} = \mathbf{0}$

7–**9**. Are the given vectors $\mathbf{a}$ and $\mathbf{b}$ orthogonal, parallel, or neither?

7. $\mathbf{a} = \langle 5, 2\rangle$ and $\mathbf{b} = \langle -4, -10\rangle$

Solution: To determine the relationship between $\mathbf{a}$ and $\mathbf{b}$, we must investigate the angle θ between them. If two vectors are orthogonal, $\theta = \pi/2$. If two vectors are parallel, $\theta = 0$. If neither, then $\theta \neq 0, \pi/2$. We have that

$$\begin{aligned} \theta &= \arccos\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}\right) \\ \mathbf{a} \cdot \mathbf{b} &= 5(-4) + 2(-10) = -40 \\ \|\mathbf{a}\| &= \sqrt{5^2 + 2^2} = \sqrt{29} \\ \|\mathbf{b}\| &= \sqrt{(-4)^2 + (-10)^2} = 2\sqrt{29} \\ \theta &= \arccos\left(\frac{-40}{\sqrt{29}(2\sqrt{29})}\right) = \arccos\left(-\frac{20}{29}\right) \end{aligned}$$

Therefore $\mathbf{a}$ and $\mathbf{b}$ are neither parallel nor orthogonal.

8. $\mathbf{a} = \langle 1, -2, 1\rangle$ and $\mathbf{b} = \langle 0, 1, 2\rangle$

Solution: To determine the relationship between $\mathbf{a}$ and $\mathbf{b}$, we must investigate the angle θ between them. If two vectors are orthogonal, $\theta = \pi/2$. If two vectors are parallel, $\theta = 0$. If neither, then $\theta \neq 0, \pi/2$. We have that

$$\begin{aligned} \theta &= \arccos(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}) \\ \mathbf{a} \cdot \mathbf{b} &= 1(0) - 2(1) + 1(2) = 0 \\ \theta &= \arccos(\frac{0}{\|\mathbf{a}\|\|\mathbf{b}\|}) = \pi/2 \end{aligned}$$

Therefore $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

9. For what values of b are the vectors $\langle -6\,, b\,, 2\rangle$ and $\langle b\,, b^2\,, b\rangle$ orthogonal?

SOLUTION: The two given vectors will be orthogonal if their dot product vanishes. Hence,

$$\begin{aligned} \langle -6, b, 2\rangle \cdot \langle b, b^2, b\rangle &= 0 \\ -6(b) + b(b^2) + 2(b) &= 0 \\ b(b^2 - 4) &= 0 \\ b &= \pm 2, 0 \end{aligned}$$

10. Use the dot product to find all unit vectors that are perpendicular to the vectors $\langle 1, 1, -2\rangle$ and $\langle 1, -2, 4\rangle$.

SOLUTION: Suppose that $\mathbf{u} = \langle a, b, c\rangle$ is perpendicular to both of the given vectors. Then,

$$\begin{aligned} \mathbf{u} \cdot \langle 1, 1, -2\rangle = 0 \quad &\Longleftrightarrow \quad a + b - 2c = 0 \\ \mathbf{u} \cdot \langle 1, -2, 4\rangle = 0 \quad &\Longleftrightarrow \quad a - 2b + 4c = 0 \end{aligned}$$

which gives a system of two linear equations in three unknowns. We can solve for a, b as follows:

$$\begin{cases} a + b &= 2c \\ a - 2b &= -4c \end{cases}$$

By subtracting the first from the second,

$$\begin{cases} a + b &= 2c \\ -3b &= -6c \Longleftrightarrow b = 2c \end{cases}$$

And by substituting $b = 2c$ into the first,

$$\begin{cases} a = 0 \\ b = 2c \end{cases}$$

So $\mathbf{u} = \langle 0, 2c, c\rangle = c\langle 0, 2, 1\rangle$. Now we must determine what c is so that $\mathbf{u}$ is a unit vector. If $\mathbf{u}$ is a unit vector then $\|\mathbf{u}\| = 1$, so

$$\begin{aligned} |c|\sqrt{0^2 + 2^2 + 1^2} &= 1 \\ |c|\sqrt{5} &= 1 \\ c = \pm\frac{1}{\sqrt{5}} \end{aligned}$$

Therefore $\mathbf{u} = \pm\frac{1}{\sqrt{5}}\langle 0, 2, 1\rangle$.

11. Find the angle at the vertex A of a triangle ABC for $A = (1, 0, 1)$, $B = (1, 2, 3)$, and $= C(0, 1, 1)$. Express the answer in radians.

Solution: By the geometrical properties of the dot product, the angle θ between two vectors $\mathbf{a}$ and $\mathbf{b}$ is determined by the equation

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}$$

Setting $\mathbf{a} = \overrightarrow{AB} = \langle 0, 2, 2\rangle$ and $\mathbf{b} = \overrightarrow{AC} = \langle -1, 1, 0\rangle$, it follows that

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= 0\cdot(-1) + 2\cdot 1 + 2\cdot 0 = 2 \\ \|\mathbf{a}\| &= \sqrt{0^2 + 2^2 + 2^2} = 2\sqrt{2} \\ \|\mathbf{b}\| &= \sqrt{(-1)^2 + 1^2 + 0^2} = \sqrt{2} \end{aligned} \quad\Rightarrow\quad \cos\theta = \frac{2}{2(\sqrt{2})^2} = \frac{1}{2}$$

So, $\theta = \pi/3$.

12. Find the cosines of the angles of a triangle ABC for $A = (0, 1, 1)$, $B = (-2, 4, 3)$, and $C = (1, 2, -1)$.

Solution: We can easily find the cosines of the angles by representing finding vectors parallel to the sides of the triangle, and using the geometric interpretation of the dot product. So,

$$\begin{aligned} \overrightarrow{AB} &= \langle -2, 4, 3\rangle - \langle 0, 1, 1\rangle = \langle -2, 3, 2\rangle \\ \overrightarrow{BC} &= \langle 1, 2, -1\rangle - \langle -2, 4, 3\rangle = \langle 3, -2, -4\rangle \\ \overrightarrow{AC} &= \langle 1, 2, -1\rangle - \langle 0, 1, 1\rangle = \langle 1, 1, -2\rangle \end{aligned}$$

and their magnitudes are

$$\begin{aligned}
\|\overrightarrow{AB}\| &= \sqrt{(-2)^2+3^2+2^2} = \sqrt{17} \\
\|\overrightarrow{BC}\| &= \sqrt{3^2+(-2)^2+(-4)^2} = \sqrt{29} \\
\|\overrightarrow{AC}\| &= \sqrt{1^2+1^2+(-2)^2} = \sqrt{6}
\end{aligned}$$

So the cosines of the angles of triangle ABC are as follows:

$$\begin{aligned}
\cos\alpha &= \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\|\overrightarrow{AB}\|\|\overrightarrow{AC}\|} = \frac{-2(1)+3(1)+2(-2)}{\sqrt{6}\sqrt{17}} = -\frac{3}{\sqrt{6}\sqrt{17}} \\
\cos\beta &= \frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{\|\overrightarrow{BA}\|\|\overrightarrow{AB}\|} = \frac{2(3)-3(-2)-2(-4)}{\sqrt{17}\sqrt{29}} = \frac{20}{\sqrt{17}\sqrt{29}} \\
\cos\gamma &= \frac{\overrightarrow{CA}\cdot\overrightarrow{CB}}{\|\overrightarrow{CA}\|\|\overrightarrow{CB}\|} = \frac{-1(-3)-1(2)+2(4)}{\sqrt{6}\sqrt{29}} = \frac{9}{\sqrt{6}\sqrt{29}}
\end{aligned}$$

where α, β, γ are the angles at vertex A, B, and C respectively. Note that the each vector in a dot product start at the vertex where the angle is. This is to correctly find the angle they enclose. Otherwise you might get the obtuse angle when it is actually acute, and vice verse.

13. Consider a triangle whose one side is a diameter of a circle and the vertex opposite to this side is on the circle. Use vector algebra to prove that any such triangle is right-angled. *Hint*: Consider position vectors of the vertices relative to the center of the circle.

SOLUTION: Consider a circle centered at the origin with radius R. The triangle in question can be formed by three points as follows: $A = (R, 0)$, $B = (-R, 0)$, and $C = (R\cos\theta, R\sin\theta)$ for $0 \le \theta < 2\pi$. We wish to prove that $\overrightarrow{AC} \perp \overrightarrow{BC}$. To show this, we need only demonstrate that their dot product vanishes. So,

$$\begin{aligned}
\overrightarrow{AC} &= \overrightarrow{OC} - \overrightarrow{OA} = \langle R\cos\theta - R, R\sin\theta \\
\overrightarrow{BC} &= \overrightarrow{OC} - \overrightarrow{OA} = \langle R\cos\theta + R, R\sin\theta \\
\overrightarrow{AC}\cdot\overrightarrow{BC} &= (R\cos\theta - R)(R\cos\theta + R) + (R\sin\theta)^2 \\
&= (R\cos\theta)^2 - R^2 + (R\sin\theta)^2 \\
&= R^2 - R^2 = 0
\end{aligned}$$

14. Let $\mathbf{a} = s\hat{\mathbf{u}} + \hat{\mathbf{v}}$ and $\mathbf{b} = \hat{\mathbf{u}} + s\hat{\mathbf{v}}$ where the angle between unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ is $\pi/3$. Find the values of s for which the dot product $\mathbf{a} \cdot \mathbf{b}$ is maximal, minimal, or zero if such values exist.

SOLUTION: First let us compute $\hat{\mathbf{u}} \cdot \hat{\mathbf{v}}$. By the geometric properties of the dot product,

$$\begin{aligned} \hat{\mathbf{u}} \cdot \hat{\mathbf{v}} &= \frac{\cos\theta}{\|\hat{\mathbf{u}}\|\|\hat{\mathbf{v}}\|} \\ &= \frac{\cos\pi/3}{(1)(1)} = \frac{1}{2} \end{aligned}$$

Note that since the dot product is commutative $\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = \hat{\mathbf{v}} \cdot \hat{\mathbf{u}}$, and that $\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} = \|\hat{\mathbf{u}}\|$. Next we will compute $\mathbf{a} \cdot \mathbf{b}$ as follows

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (s\hat{\mathbf{u}} + \hat{\mathbf{v}}) \cdot (\hat{\mathbf{u}} + s\hat{\mathbf{v}}) \\ &= s\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} + s\hat{\mathbf{u}} \cdot s\hat{\mathbf{v}} + \hat{\mathbf{v}} \cdot \hat{\mathbf{u}} + \hat{\mathbf{v}} \cdot s\hat{\mathbf{v}} \\ &= s\|\hat{\mathbf{u}}\| + (s^2+1)\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} + s\|\hat{\mathbf{v}}\| \\ &= s + (s^2+1)(\frac{1}{2}) + s \\ &= \frac{1}{2}(s^2+4s+1) = \frac{1}{2}(s^2+4s+4-3) = \frac{1}{2}(s+2)^2 - \frac{3}{2} \end{aligned}$$

The function $f(s) = \frac{1}{2}(s+2)^2 - \frac{3}{2}$ is a parabola opening upwards. Hence there is no maximum value. There is a single minimum occurring at the vertex, when $s = -2$. The zeroes occur when $f(s) = 0$,

$$\begin{aligned} \frac{1}{2}(s+2)^2 - \frac{3}{2} &= 0 \\ (s+2)^2 &= 3 \\ s+2 &= \pm\sqrt{3} \\ s &= -2 \pm \sqrt{3} \end{aligned}$$

15. Consider a cube whose edges have length a. Find the angle between its largest diagonal and any edge adjacent to the diagonal.

SOLUTION: Consider a cube of side length a situated in the first octant, with one corner at the origin and side lengths parallel to the x, y, z axes. The largest diagonal (of four, actually, but since this is a cube they are all equivalent) is the one connecting the corner at $(0,0,0)$ with the corner at (a,a,a). We can find the angle between any edge adjacent to the diagonal by considering the dot product between the vectors parallel to these. Clearly a vector parallel to the largest diagonal is $\langle a,a,a\rangle$. Any adjacent edge is simply an edge of the cube that meets the diagonal at a corner. Notice that the angles must be the same, first because the problem asks for "the" angle, but also because you can rotate the cube around its diagonal and still have a cube. So I will consider the angle between the largest diagonal and the edge along the x-axis. A vector parallel to this edge is $\langle a, 0, 0\rangle$. So we have that

$$\begin{aligned}\theta &= \arccos\left(\frac{\langle a,0,0\rangle \cdot \langle a,a,a\rangle}{\|\langle a,0,0\rangle\|\|\langle a,a,a\rangle\|}\right) \\ &= \arccos\left(\frac{a^2}{(a)(a\sqrt{3})}\right) \\ &= \arccos\left(\frac{1}{\sqrt{3}}\right)\end{aligned}$$

16. Consider a parallelepiped with adjacent sides $\mathbf{a} = \langle 1, -2, 2\rangle$, $\mathbf{b} = \langle -2, -2, 1\rangle$, and $\mathbf{c} = \langle -1, -1, -1\rangle$ (see the definition of a parallelepiped in Study Problem **??**). It has four vertex-to-opposite-vertex diagonals. Express them in terms of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ and find the largest one. Find the angle between the largest diagonal and the adjacent sides of the parallelepiped.

Solution: There are four diagonals we must find. Recall that a parallelogram formed from two non-parallel vectors $\mathbf{u}$ and $\mathbf{v}$ has diagonal $\mathbf{u} + \mathbf{v}$, and recall that the faces of a parallelpiped are parallelograms. Notice that each diagonal can be formed as the sum/difference of a diagonal of a face and the other vector which does not make up the face. It may be help to form a sketch. For example, consider a face formed from the vectors $\mathbf{a}$ and $\mathbf{b}$. The diagonal of this face is $\mathbf{a} + \mathbf{b}$, and the other vector that does not make up this face is $\mathbf{c}$. One of the diagonals in question then is $\mathbf{d}_1 = \mathbf{a}+\mathbf{b}+\mathbf{c}$, and another is $\mathbf{d}_2 = \mathbf{a}+\mathbf{b}-\mathbf{c}$. Next consider the face formed by the vectors $\mathbf{a}$ and $\mathbf{c}$. Then the diagonal of this face is $\mathbf{a} + \mathbf{c}$. So the third diagonal in question is $\mathbf{d}_3 = \mathbf{a} + \mathbf{c} - \mathbf{b}$. Finally, consider the face formed by the vectors $\mathbf{b}$ and $\mathbf{c}$. Then the diagonal of this face is $\mathbf{b} + \mathbf{c}$. So the fourth diagonal in question is $\mathbf{d}_4 = \mathbf{b} + \mathbf{c} - \mathbf{a}$. Clearly, the largest diagonal is $\mathbf{d}_1$ (subtracting any vector will decrease the magnitude of the resultant).

Next we must find the angle between the largest diagonal, $\mathbf{d}_1 = \langle -2, -5, 2\rangle$, and the adjacent sides $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. First let us find the magnitudes of each of these:

$$\begin{aligned}\|\mathbf{d}_1\| &= \sqrt{(-2)^2+(-5)^2+2^2} = \sqrt{33}\\ \|\mathbf{a}\| &= \sqrt{1^2+(-2)^2+2^2} = 3\\ \|\mathbf{b}\| &= \sqrt{(-2)^2+(-2)^2+1^2} = 3\\ \|\mathbf{c}\| &= \sqrt{(-1)^2+(-1)^2+(-1)^2} = \sqrt{3}\end{aligned}$$

So the angles between them are:

$$\begin{aligned}\alpha &= \arccos(\frac{\mathbf{d}_1\cdot\mathbf{a}}{\|\mathbf{d}_1\|\|\mathbf{a}\|}) = \arccos(\frac{12}{3\sqrt{33}}) = \arccos(\frac{4}{\sqrt{33}})\\ \beta &= \arccos(\frac{\mathbf{d}_1\cdot\mathbf{b}}{\|\mathbf{d}_1\|\|\mathbf{b}\|}) = \arccos(\frac{16}{3\sqrt{33}})\\ \gamma &= \arccos(\frac{\mathbf{d}_1\cdot\mathbf{c}}{\|\mathbf{d}_1\|\|\mathbf{c}\|}) = \arccos(\frac{5}{\sqrt{3}\sqrt{33}}) = \arccos(\frac{5}{3\sqrt{11}})\end{aligned}$$

Where α, β, γ represent the angles between $\mathbf{d}_1$ and $\mathbf{a}$, $\mathbf{d}_1$ and $\mathbf{b}$, and $\mathbf{d}_1$ and $\mathbf{c}$, respectively.

17. Let $\mathbf{a} = \langle 1, 2, 2\rangle$. For the vector $\mathbf{b} = \langle -2, 3, 1\rangle$, find the scalar and vector projections of $\mathbf{b}$ onto $\mathbf{a}$ and construct the orthogonal decomposition $\mathbf{b} = \mathbf{b}_\perp + \mathbf{b}_{||}$ relative to $\mathbf{a}$.

Solution: Recall that the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is given by

$$b_{||} = \frac{\mathbf{b}\cdot\mathbf{a}}{\|\mathbf{a}\|}.$$

And the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is given by

$$\mathbf{b}_{||} = b_{||}\hat{\mathbf{a}} = \frac{b_{||}}{\|\mathbf{a}\|}\mathbf{a}$$

And so,

$$\begin{aligned}b_{||} &= \frac{-2(1)+3(2)+1(2)}{\sqrt{1^2+2^2+2^2}} = \frac{6}{\sqrt{9}} = 2\\ \mathbf{b}_{||} &= \frac{2}{3}\mathbf{a}\\ \mathbf{b}_\perp &= \mathbf{b} - \mathbf{b}_{||} = \mathbf{b} - \frac{2}{3}\mathbf{a}\end{aligned}$$

18. Find the scalar and vector projections of $\overrightarrow{AB}$ onto $\overrightarrow{BC}$ if $A = (0, 0, 4)$, $B = (0, 3, -2)$, and $C = (3, 6, 2)$.

SOLUTION: First we must compute $\overrightarrow{AB}$ and $\overrightarrow{BC}$ as follows:

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} = \langle 0, 3, -2\rangle - \langle 0, 0, 4\rangle = \langle 0, 3, -6\rangle \\ \overrightarrow{BC} &= \overrightarrow{OC} - \overrightarrow{OB} = \langle 3, 6, 2\rangle - \langle 0, 3, -2\rangle\rangle = \langle 3, 3, 4\rangle\end{aligned}$$

And so

$$\begin{aligned}AB_{||} &= \frac{\overrightarrow{AB} \cdot \overrightarrow{BC}}{\|\overrightarrow{BC}\|} = \frac{0(3) + 3(3) - 6(4)}{\sqrt{3^2 + 3^2 + 4^2}} = -\frac{15}{\sqrt{34}} \\ \overrightarrow{AB}_{||} &= \frac{AB_{||}}{\|\overrightarrow{BC}\|}\overrightarrow{BC} = -\frac{15}{34}\langle 3, 3, 4\rangle\end{aligned}$$

are the scalar and vector projections of $\overrightarrow{AB}$ onto $\overrightarrow{BC}$, respectively.

19. Find all vectors that have a given length a and make an angle $\pi/3$ with the positive x axis and the angle $\pi/4$ with the positive z axis.

SOLUTION: Recall that the direction of a vector in space is uniquely determined by three angles, the so-called direction angles α, β, and γ. These angles satisfy the following equation:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

We are given two direction angles already. Set $\alpha = \pi/3$ and $\gamma = \pi/4$, then

$$\begin{aligned}\cos^2\pi/3 + \cos^2\beta + \cos^2\pi/4 &= 1 \\ \frac{1}{4} + \cos^2\beta + \frac{1}{2} &= 1 \\ \cos^2\beta &= \frac{1}{4} \\ \cos\beta &= \pm\frac{1}{2}\end{aligned}$$

Further, recall that a unit vector with these three direction angles is of the form

$$\hat{\mathbf{u}} = \langle\cos\alpha, \cos\beta, \cos\gamma\rangle.$$

Since we require that the vector must be of length a, we simply have that

$$\mathbf{u} = a\hat{\mathbf{u}} = a\langle\frac{1}{2}, \pm\frac{1}{2}, \frac{1}{\sqrt{2}}\rangle$$

20. Find the components of all unit vectors $\hat{\mathbf{u}}$ that make an angle ϕ with the positive z axis. *Hint:* Put $\hat{\mathbf{u}} = a\hat{\mathbf{v}} + b\hat{\mathbf{e}}_3$, where $\hat{\mathbf{v}}$ is a unit vector in the xy plane. Find a, b, and all $\hat{\mathbf{v}}$ using the polar angle in the xy plane.

Solution: First recognize that $\hat{\mathbf{u}}$, $a\hat{\mathbf{v}}$, and $b\hat{\mathbf{e}}_3$ form a triangle with hypotenuse 1 and side lengths a, b where the side with length a is opposite of ϕ. This means that $\tan\phi = a/b$. Thus $a = b\tan\phi$. Next we have that $\hat{\mathbf{u}}$ is a unit vector, so

$$\begin{aligned} \sqrt{a^2+b^2} &= 1 \\ a^2+b^2 &= 1 \\ b^2\tan^2\phi + b^2 &= 1 \\ b^2(\tan^2\phi + 1) &= 1 \\ b^2(\sec^2\phi) &= 1 \\ b^2 &= \cos^2\phi \\ b &= \pm\cos\phi \end{aligned}$$

As it turns out, later we will be able to discard the negative solution. In any case, we can now express a as

$$a = \pm\cos\phi\tan\phi = \pm\sin\phi$$

Next we must find $\hat{\mathbf{v}}$. We can write it in polar form as

$$\hat{\mathbf{v}} = \langle\cos\theta, \sin\theta, 0\rangle$$

where θ is taken to be an angle measured counterclockwise from the positive x axis in the xy plane (the standard polar angle). So we have that

$$\begin{aligned} \hat{\mathbf{u}} &= \langle\pm\cos\theta\sin\phi, \pm\sin\theta\sin\phi, 0\rangle + \langle 0, 0, \cos\phi\rangle \\ &= \langle\pm\cos\theta\sin\phi, \pm\sin\theta\sin\phi, \cos\phi\rangle \end{aligned}$$

Consider the negative solution for b,

$$\hat{\mathbf{u}}' = \langle -\cos\theta'\sin\phi, -\sin\theta'\sin\phi, \cos\phi\rangle$$

Next, consider the transformation $\theta' = \theta + \pi$. Then we will have

$$\hat{\mathbf{u}}' = \langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$$

So we may discard the negative solution, and simply have

$$\hat{\mathbf{u}} = \langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$$

You will learn later that this is the conversion from spherical coordinates to Cartesian coordinates.

21. If $\mathbf{c} = \|\mathbf{a}\|\mathbf{b} + \|\mathbf{b}\|\mathbf{a}$, where $\mathbf{a}$ and $\mathbf{b}$ are non zero vectors, show that $\mathbf{c}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$. *Hint*: compare the angle between $\mathbf{c}$ and $\mathbf{a}$ to the angle between $\mathbf{c}$ and $\mathbf{b}$.

Solution: First we will compute the angle α between $\mathbf{a}$ and $\mathbf{c}$. By the geometric properties of the dot product, we have that

$$\begin{aligned}\alpha &= \arccos \frac{\mathbf{a}\cdot\mathbf{c}}{\|\mathbf{a}\|\|\mathbf{c}\|}\\ &= \arccos \frac{\|\mathbf{a}\|(\mathbf{a}\cdot\mathbf{b}) + \|\mathbf{b}\|\|\mathbf{a}\|^2}{\|\mathbf{a}\|\|\mathbf{c}\|}\\ &= \arccos \frac{\mathbf{a}\cdot\mathbf{b} + \|\mathbf{b}\|\|\mathbf{a}\|}{\|\mathbf{c}\|}\end{aligned}$$

Similarly, we must compute the angle β between $\mathbf{b}$ and $\mathbf{c}$

$$\begin{aligned}\beta &= \arccos \frac{\mathbf{b}\cdot\mathbf{c}}{\|\mathbf{b}\|\|\mathbf{c}\|}\\ &= \arccos \frac{\|\mathbf{b}\|^2\|\mathbf{a}\| + \|\mathbf{b}\|(\mathbf{b}\cdot\mathbf{a})}{\|\mathbf{b}\|\|\mathbf{c}\|}\\ &= \arccos \frac{\|\mathbf{b}\|\|\mathbf{a}\| + \mathbf{b}\cdot\mathbf{a}}{\|\mathbf{c}\|}\end{aligned}$$

Evidently $\alpha = \beta$, so $\mathbf{c}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

22. A rhombus is a parallelogram with sides of equal length. Prove that the diagonals of a rhombus meet at right angles.

Solution: The adjacent sides of a rhombus can be formed from two vectors $\mathbf{a}$ and $\mathbf{b}$ where $\|\mathbf{a}\| = \|\mathbf{b}\|$. The two diagonals of the rhombus are $\mathbf{d}_1 = \mathbf{a} + \mathbf{b}$ and $\mathbf{d}_2 = \mathbf{b} - \mathbf{a}$. Convince yourself of this by sketching a picture, if necessary. Next we have that

$$\mathbf{d}_1 \cdot \mathbf{d}_2 = \mathbf{a}\cdot\mathbf{b} - \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \mathbf{b}\cdot\mathbf{a} = 0 \;\Rightarrow\; \mathbf{d}_1 \perp \mathbf{d}_2$$

since $\|\mathbf{a}\| = \|\mathbf{b}\|$.

23. Consider a parallelogram with adjacent sides of length a and b. If d_1 and d_2 are the lengths of the diagonals, prove the parallelogram law: $d_1^2 + d_2^2 = 2(a^2 + b^2)$. *Hint:* Consider the vectors $\mathbf{a}$ and $\mathbf{b}$ that are adjacent sides of the parallelogram and express the diagonals via $\mathbf{a}$

and $\mathbf{b}$. Use the dot product to evaluate $d_1^2 + d_2^2$.

SOLUTION: The adjacent sides of a parallelogram can be formed from two vectors $\mathbf{a}$ and $\mathbf{b}$, as stated in the hint. The two diagonals of the parallelogram are $\mathbf{d}_1 = \mathbf{a} + \mathbf{b}$ and $\mathbf{d}_2 = \mathbf{b} - \mathbf{a}$. Convince yourself of this by sketching a picture, if necessary. Notice that

$$\mathbf{d}_1 \cdot \mathbf{d}_1 = d_1^2$$

On the other hand, we also have that

$$\mathbf{d}_1 \cdot \mathbf{d}_1 = a^2 + 2\mathbf{a} \cdot \mathbf{b} + b^2$$

So $d_1^2 = a^2 + 2\mathbf{a} \cdot \mathbf{b} + b^2$.
Similarly for $\mathbf{d}_2$ we have that

$$\mathbf{d}_2 \cdot \mathbf{d}_2 = d_2^2$$

On the other hand, we also have that

$$\mathbf{d}_2 \cdot \mathbf{d}_2 = a^2 - 2\mathbf{a} \cdot \mathbf{b} + b^2$$

So $d_2^2 = a^2 - 2\mathbf{a} \cdot \mathbf{b} + b^2$. Thus

$$d_1^2 + d_2^2 = a^2 + a^2 + 2\mathbf{a} \cdot \mathbf{b} - 2\mathbf{a} \cdot \mathbf{b} + b^2 + b^2 = 2(a^2 + b^2)$$

24. Consider a right-angled triangle whose adjacent sides at the right angle have lengths a and b. Let P be a point in space at a distance c from all three vertices of the triangle ($c \geq a/2$ and $c \geq b/2$). Find the angles between the line segments connecting P with the vertices of the triangle. *Hint*: Consider vectors with the initial point P and terminal points at the vertices of the triangle.

SOLUTION: The solution I have is much easier than doing anything with dot products. Notice that the solid formed by the vertices of the triangle and P is a triangular pyramid. So each face is a triangle. Consider one such triangle (that is not the base). Two sides are of length c and one is of length l, where l is either a, b, or $\sqrt{a^2 + b^2}$. Call the angle between the sides of length c θ. Draw a line from P to the side of length l, which will be a perpendicular bisector. We now have two right triangles with side lengths $l/2, c$ and another irrelevant one, and an angle $\theta/2$. Consider $\sin\theta/2$. The opposite side has length $l/2$ and the hypoteneuse has length c. So $\sin\theta/2 = l/2c$. Therefore $\theta/2 = \arcsin l/2c$, and $\theta = 2\arcsin l/2c$. So the angles in question are $\alpha = 2\arcsin a/2c$, $\beta = 2\arcsin b/2c$, $\gamma = 2\arcsin\sqrt{a^2 + b^2}/2c$

25. Show that the vectors $\mathbf{u}_1 = \langle 1, 1, 2\rangle$, $\mathbf{u}_2 = \langle 1, -1, 0\rangle$, and $\mathbf{u}_3 = \langle 2, 2, -2\rangle$ are mutually orthogonal. For a vector $\mathbf{a} = \langle 4, 3, 4\rangle$ find the scalar orthogonal projections of $\mathbf{a}$ onto $\mathbf{u}_i$, $i = 1, 2, 3$, and the numbers s_i such that $\mathbf{a} = s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + s_3\mathbf{u}_3$.

SOLUTION: To show that $\mathbf{u}_j$ are mutually orthogonal for $j = 1, 2, 3$, we must show that their dot products vanish as follows:

$$\begin{aligned}
\mathbf{u}_1 \cdot \mathbf{u}_2 &= 1(1) + 1(-1) + 2(0) = 1 - 1 = 0 \\
\mathbf{u}_2 \cdot \mathbf{u}_3 &= 1(2 + -1(2) + 0(-2) = 2 - 2 = 0 \\
\mathbf{u}_1 \cdot \mathbf{u}_3 &= 1(2) + 1(2) + 2(-2) = 2 + 2 - 4 = 0
\end{aligned}$$

So they are mutually orthogonal. Next consider $\mathbf{a} \cdot \mathbf{u_1}$. This will be $s_1\|\mathbf{u}_1\|^2 + 0 + 0$. In general,

$$s_j = \frac{\mathbf{a} \cdot \mathbf{u_j}}{\|\mathbf{u}_j\|^2}$$

for $j = 1, 2, 3$. So,

$$\begin{aligned}
s_1 &= \frac{4(1) + 3(1) + 4(2)}{\sqrt{1^2 + 1^2 + 2^2}^2} = \frac{4 + 3 + 8}{6} = \frac{5}{2} \\
s_2 &= \frac{4(1) + 3(-1) + 4(0)}{\sqrt{1^2 + (-1)^2 + 0^2}^2} = \frac{4 - 3}{2} = \frac{1}{2} \\
s_3 &= \frac{4(2) + 3(2) + 4(-2)}{\sqrt{2^2 + 2^2 + (-2)^2}^2} = \frac{8 + 6 - 8}{12} = \frac{1}{2}
\end{aligned}$$

Recall that the the scalar projection of $\mathbf{y}$ onto $\mathbf{x}$ is

$$y_{||} = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|},$$

which we can rewrite as

$$y_{||} = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|^2}\|\mathbf{x}\|$$

Let $\mathbf{y} = \mathbf{a}$ and $\mathbf{x} = \mathbf{u}_j$. Then the scalar projection of $\mathbf{a}$ onto $\mathbf{u}_j$ is

$$a_{||} = \frac{\mathbf{u}_j \cdot \mathbf{a}}{\|\mathbf{u}_j\|^2}\|\mathbf{u}_j\| = s_j\|\mathbf{u}_j\|$$

So the necessary scalar projections are

$$\begin{aligned} s_1\|\mathbf{u}_1\| &= \frac{5}{2}\sqrt{6} = \frac{5\sqrt{6}}{2} \\ s_2\|\mathbf{u}_2\| &= \frac{1}{2}\sqrt{2} = \frac{\sqrt{2}}{2} \\ s_2\|\mathbf{u}_3\| &= \frac{1}{2}\sqrt{12} = \sqrt{3} \end{aligned}$$

26. For two nonzero vectors **a** and **b** find all vectors coplanar with **a** and **b** that have the same vector projection onto **a** as the vector **b**. Express these vectors in terms of **a** and **b**.

SOLUTION: First recall that the vector projection of **b** onto **a** is uniquely determined by

$$b_{||} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}$$

If **a** is a given vector, then all we can change in this equation is $\mathbf{a}\cdot\mathbf{b}$. So, suppose **v** is a vector with the same scalar projection onto **a** as **b**. Then,

$$\frac{\mathbf{a}\cdot\mathbf{v}}{\|\mathbf{a}\|} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|} \;\Rightarrow\; \mathbf{a}\cdot\mathbf{v} = \mathbf{a}\cdot\mathbf{b} \;\Rightarrow\; \mathbf{a}\cdot(\mathbf{v}-\mathbf{b}) = 0$$

So $\mathbf{v}-\mathbf{b}$ is orthogonal to **a**. This means that $\mathbf{v}-\mathbf{b}$ is parallel to $\mathbf{b}_\perp$, so

$$\mathbf{v}-\mathbf{b} = t\mathbf{b}_\perp \;\Rightarrow\; \mathbf{v} = \mathbf{b} + t\mathbf{b}_\perp$$

for all $t \in \mathbb{R}$.

27. A point object traveled 3 meters from a point A in a particular direction, then it changed the direction by 60° and traveled 4 meters, and then it changed the direction again so that it was traveling at 60° with each of the previous two directions. If the last stretch was 2 meters long, how far from A is the object?

SOLUTION: Let A, B, C, D be the four points the object stops or changes direction at. Then we have that

$$\|\overrightarrow{AB}\| = 3,\;\; \|\overrightarrow{BC}\| = 4,\;\; \|\overrightarrow{CD}\| = 2$$

and we are being asked to find $\|\overrightarrow{AD}\|$. We are also given that the object travels in one direction from A, then changes direction at B and travels

60° from the original direction. So

$$\overrightarrow{AB}\cdot\overrightarrow{BC} = \|\overrightarrow{AB}\|\|\overrightarrow{BC}\|\cos\theta = 3(4)(\cos 60^\circ) = 6.$$

The object then changes direction again at C and travels 60° from both of the previous directions (i.e., $\overrightarrow{AB}$ and $\overrightarrow{BC}$). So

$$\begin{aligned}\overrightarrow{AB}\cdot\overrightarrow{CD} &= \|\overrightarrow{AB}\|\|\overrightarrow{CD}\|\cos\theta = 3(2)(\cos 60^\circ) &&= 3\\ \overrightarrow{BC}\cdot\overrightarrow{CD} &= \|\overrightarrow{BC}\|\|\overrightarrow{CD}\|\cos\theta = 4(2)(\cos 60^\circ) &&= 4\end{aligned}$$

Recall that $\|\mathbf{u}\|^2 = \mathbf{u}\cdot\mathbf{u}$ for a vector $\mathbf{u}$. So,

$$\begin{aligned}\|\overrightarrow{AD}\|^2 &= \overrightarrow{AD}\cdot\overrightarrow{AD}\\ &= (\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD})(\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD})\\ &= \|\overrightarrow{AB}\|^2+\|\overrightarrow{BC}\|^2+\|\overrightarrow{CD}\|^2+2\overrightarrow{AB}\cdot\overrightarrow{BC}+2\overrightarrow{AB}\cdot\overrightarrow{CD}+2\overrightarrow{BC}\cdot\overrightarrow{CD}\\ &= 3^2+4^2+2^2+2(6+3+4)\\ &= 55\\ \|\overrightarrow{AD}\| &= \sqrt{55}\end{aligned}$$

28. Two balls of the same mass m are connected by a piece of rope of length h. Then the balls are attached to different points on a horizontal ceiling by a piece of rope with the same length h so that the distance L between the points is greater than h but less than $3h$. Find the equilibrium positions of the balls and the magnitude of tension forces in the ropes.

Solution: The setup involves two balls and three ropes, where the ropes and the ceiling make a regular trapezoid with the largest base facing up. The free body diagram for the left ball involves three forces: Gravity, the tension $\mathbf{T}_1$ from the rope connecting the ball to the ceiling, and the tension $\mathbf{T}_2$ connecting the ball to the other ball. Note that $\mathbf{F}_g$ points down, $\mathbf{T}_1$ to the up and left, and $\mathbf{T}_2$ horizontally to the right. If the system is in equilibrium, then the vector sum of the forces must vanish. Let θ denote the angle between the ceiling and $\mathbf{T}_1$. Then we must have

$$\begin{aligned}-T_1\sin\theta + T_2 &= 0\\ T_1\cos\theta - F_g &= 0 \;\Rightarrow T_1\cos\theta = mg\end{aligned}$$

where the first equation is the force balance of the x-components, the second is the force balance of the y-components, and right and up are

taken to be the positive x and y directions, respectively.
Next, refer back to the trapezoidal setup. Imagine bringing the horizontal rope of length h up onto the ceiling (keeping it horizontal and in line with where it previously was), then connect the end points with the balls. This forms a rectangle within the trapezoid and two right triangles, with bases $\frac{1}{2}(L-h)$. Look at the left triangle. The angle between the base and the hypotenuse is θ, as defined before. The hypotenuse is of length h, as stated in the problem (the rope that connects the balls to the ceiling is length h), so the vertical side is of length $\sqrt{h^2+(\frac{1}{2}(L-h))^2}$. Then $\cos\theta = \frac{L-h}{2h}$. Referring back to our free body diagram equations, we have that

$$\begin{aligned} T_1 &= \frac{mg}{\cos\theta} = \frac{2mgh}{L-h} \\ T_2 &= T_1\sin\theta = mg\tan\theta = \frac{2mg\sqrt{h^2+(\frac{1}{2}(L-h)^2}}{L-h} \end{aligned}$$

are the tensions in the ropes connecting the balls each other and to the ceiling, respectively (the tensions in the right ball are the same as those in the left due to symmetry).

29. A cart is pulled up a 20° slope a distance of 10 meters by a horizontal force of 30 newtons. Determine the work.

Solution: Recall that the work done by a force $\mathbf{F}$ over a distance d on an object is given by

$$W = \mathbf{F}\cdot\overrightarrow{P_1P_2}$$

where P_1 is the initial point of the object and P_2 is the final point, and $|P_1P_2| = d$. So

$$W = \mathbf{F}\cdot\overrightarrow{P_1P_2} = Fd\cos\theta = 30(10)\cos 20^\circ = 300\cos 20^\circ \text{ J}$$

30. Two tug boats are pulling a barge against the river stream. One tug is pulling with the force of magnitude 20 (in some units) and at the angle 45° to the stream and the second with the force of magnitude 15 at the angle 30° so that the angle between the pulling ropes is $45^\circ + 30^\circ = 75^\circ$. If the barge does not move in the direction of the stream, what is the drag force exerted by the stream on the barge? Does the barge move in the direction perpendicular to the stream?

Solution: First we must set up our problem. Suppose the stream flows in the negative x direction, where positive x is defined to be right and positive y is defined to be up. Then we can write the force of the two ropes and the drag force as follows:

$$\begin{aligned}\mathbf{T}_1 &= \langle T_1\cos\theta_1, T_1\sin\theta_1\rangle = \langle 20\cos 45^\circ, 20\sin 45^\circ\rangle\\ \mathbf{T}_2 &= \langle T_2\cos\theta_2, T_2\sin\theta_2\rangle = \langle 15\cos(-30^\circ), 15\sin(-30^\circ)\rangle\\ \mathbf{D} &= \langle -D, 0\rangle\end{aligned}$$

where $\theta_2 = -30^\circ$ and not 30° because we must have that $\theta_1 - \theta_2 = 75^\circ$. Notice that $\mathbf{D}$ points in the negative x direction because that is how the stream is trying to pull the barge. Next we must have that the x component of the vector sum vanishes (because the barge does not move in the direction of the stream), so

$$20\cos 45^\circ + 15\cos(-30^\circ) - D = 0 \;\Rightarrow\; D = 10\sqrt{2} + \frac{15}{2}\sqrt{3}$$

Furthermore since only the x-component vanishes, the barge moves in the y direction, which is perpendicular to the direction of the stream (negative x).

31. A ball of mass m is attached by three ropes of the same length a to a horizontal ceiling so that the attachment points on the ceiling form a triangle with sides of length a. Find the magnitude of the tension force in the ropes.

Solution: Notice that the three ropes and the ceiling form a triangular pyramid. Furthermore, the face is an equilateral triangle with side lengths a. Each of $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{T}_3$ have the same magnitude as the problem is symmetric (rotating the setup by 120° does not change anything). Consider one rope and its tension, say $\mathbf{T}_1$. Construct a line segment from the ball to the center of the triangular base and another line from the center of the base to the attachment point of the rope. Call the angle between the rope and the ceiling θ. Then $T_1\sin\theta$ describes the vertical component of $\mathbf{T}_1$. We must find the length of the base of the triangle, b. We know its hypotenuse is length a and the base connects the center of an equilateral triangle with sides of length a to a vertex. Evidently, the height of the entire triangle is $\frac{\sqrt{3}}{2}a$. Draw three lines from each vertex to the opposite side. These lines should be perpendicular to the opposite side and have length equal to the height. Observe that we can split one height into two pieces, one of length b and another of length c. Choose another line of length b, and this will form a right-angled triangle with interior angles 60° and 30°. We have that $b = \sec(30^\circ)\frac{1}{2}a = \frac{a}{\sqrt{3}}$. So the height of the first triangle we were looking at (with base b and hypotenuse a) is $h = \sqrt{a^2 - b^2} = \sqrt{a^2 - \frac{a^2}{3}} = a\frac{\sqrt{2}}{\sqrt{3}}$. Then $\sin\theta = \frac{h}{a} = \frac{\sqrt{2}}{\sqrt{3}}$. Since there are three tension vectors, we have that

$$\begin{aligned} 3T_1 \sin\theta &= F_g \\ T_1 &= \frac{mg}{3}\frac{\sqrt{3}}{\sqrt{2}} \\ &= \frac{mg}{\sqrt{6}} \end{aligned}$$

And so $T = T_1 = T_2 = T_3 = \frac{mg}{\sqrt{6}}$.

32. Four dogs are at the vertices of a square. Each dog starts running toward its neighbor on the right. The dogs run with the same speed v. At every moment of time each dog keeps running in the direction of its right neighbor (its velocity vector always points to the neighbor). Eventually, the dogs meet in the center of the square. When will this happen if the sides of the square have length a? What is the distance traveled by each dog? *Hint*: Is there a particular direction from the center of the square relative to which the velocity vector of a dog has the same component at each moment of time?

Solution: First, recognize that the problem is symmetric. Rotating the setup by 90° at any time will result in essentially the same thing prior to rotation. It is easy, then, to see that at any time, the coordinates of the dogs form a square with the same center as the original (at $t = 0$). This can be seen by doing the following: take a line segment of length d and rotate it around a point by 90° so that one endpoint of the new segment lands on an endpoint of the original segment, and keep doing this, you will get a square.
With this in mind, the velocity vector of any dog is at an angle of $3\pi/4$ with its position vector. This is because the velocity vector of a dog will point towards the dog it is chasing (i.e., along the edge of the square). Choose a dog. Let the position vector of the dog be $\overrightarrow{OD}$. Then the scalar projection of the dog's velocity vector onto its position vector is given by

$$v_{||} = \frac{\mathbf{v} \cdot \overrightarrow{OD}}{\|\overrightarrow{OD}\|} = \frac{v\|\overrightarrow{OD}\| \cos 3\pi/4}{\|\overrightarrow{OD}\|} = v\sqrt{2}.$$

where v is the speed of the dog, a constant value. This decomposes $\mathbf{v}$ into two vectors. One is along $\overrightarrow{OD}$ with magnitude $v\sqrt{2}$. The other is perpendicular to this, which gives rise to an angular velocity, motion in a circle. This is seen by considering that the tangent line to a point P on a circle is orthogonal to $\overrightarrow{CP}$, from the center C of the circle. Recall that the velocity vector is parallel to the tangent line at P of a particle's motion $(dy/dx\ s(x) = v(x))$. So, the true motion of the dog is the superposition of a circular motion around the center of the square and a motion along a straight line towards the center. Since the circular motion does not affect the distance of the dog to the center , the time it takes for the dog to reach the center is the same as if the dog moved only in a straight line ("radial"). The radial distance traveled is $a\sqrt{2}$, and so the time taken to reach the center is

$$T = \frac{a\sqrt{2}}{v\sqrt{2}} = \frac{a}{v}$$

So the distance traveled by the dog, given by its velocity multiplied by the time taken to reach the center, is

$$D = v\frac{a}{v} = a$$

Notice that it is as if the dog is chasing a stationary object.

4. The Cross Product

1–7. Find the cross product $\mathbf{a} \times \mathbf{b}$ for the given vectors $\mathbf{a}$ and $\mathbf{b}$.

1. $\mathbf{a} = \langle 1, 2, 3\rangle$ and $\mathbf{b} = \langle -1, 0, 1\rangle$

SOLUTION: Recall that the definition of the cross product $\mathbf{u} \times \mathbf{v}$ for two vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ is:

$$\mathbf{u}\times\mathbf{v} = \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} = \langle \det\begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix}, -\det\begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix}, \det\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}\rangle$$

Thus, by the definition of the cross product,

$$\mathbf{a}\times\mathbf{b} = \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 1 & 2 & 3 \\ -1 & 0 & 1 \end{pmatrix} = \langle \det\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}, -\det\begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}, \det\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}\rangle = \langle 2, -4, 2\rangle$$

2. $\mathbf{a} = \langle 1, -1, 2\rangle$ and $\mathbf{b} = \langle 3, -2, 1\rangle$

SOLUTION: Given the definition above, we can find the cross product as

$$\mathbf{a}\times\mathbf{b} = \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 1 & -1 & 2 \\ 3 & -2 & 1 \end{pmatrix} = \langle \det\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}, -\det\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, \det\begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix}\rangle = \langle 3, 5, 1\rangle$$

3. $\mathbf{a} = \hat{\mathbf{e}}_1 + 3\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3$ and $\mathbf{b} = 3\hat{\mathbf{e}}_1 - 2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$

SOLUTION: Given the definition above, we can find the cross product as

$$\mathbf{a}\times\mathbf{b} = \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 1 & 3 & -1 \\ 3 & -2 & 1 \end{pmatrix} = \langle \det\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}, -\det\begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}, \det\begin{pmatrix} 1 & 3 \\ 3 & -2 \end{pmatrix}\rangle = \langle 1, -4, -11\rangle$$

4. $\mathbf{a} = 2\mathbf{c} - \mathbf{d}$, $\mathbf{b} = 3\mathbf{c} + 4\mathbf{d}$ where $\mathbf{c} \times \mathbf{d} = \langle 1, 2, 3\rangle$.

SOLUTION: Recall the following properties of the cross product:
(1) For any vector $\mathbf{u}$, we have that $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
(2) For any vectors $\mathbf{u}$ and $\mathbf{v}$, we have that $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
(3) It is left and right distributive
(4) For any vectors $s\mathbf{u}$ and $\mathbf{v}$, where $s \in \mathbb{R}$ is a scalar, we have

$(2\mathbf{u}) \times \mathbf{v} = 2(\mathbf{u} \times \mathbf{v})$. Thus,

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= (2\mathbf{c} - \mathbf{d}) \times (3\mathbf{c} + 4\mathbf{d}) \\
&= 2\mathbf{c} \times (3\mathbf{c} + 4\mathbf{d}) - \mathbf{d} \times (3\mathbf{c} + 4\mathbf{d}) \\
&= 2\mathbf{c} \times 3\mathbf{c} + 2\mathbf{c} \times 4\mathbf{d} - \mathbf{d} \times 3\mathbf{c} - \mathbf{d} \times 4\mathbf{d} \\
&= 8\mathbf{c} \times \mathbf{d} + 3\mathbf{c} \times \mathbf{d} \\
&= 11\langle 1, 2, 3\rangle
\end{aligned}$$

5. $\mathbf{a} = \mathbf{u}_1 + 2\mathbf{u}_2 + 3\mathbf{u}_3$ and $\mathbf{b} = \mathbf{u}_1 - \mathbf{u}_2 + \mathbf{u}_3$ where the vectors $\mathbf{u}_i$, $i = 1, 2, 3$, are mutually orthogonal, have the same length 3, and $\mathbf{u}_1 \times \mathbf{u}_2 = 3\mathbf{u}_3$. Express the answer as a linear combination of $\mathbf{u}_i$ (not their cross products).

SOLUTION: Recall that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$. Therefore we have

$$\begin{aligned}
\mathbf{u}_1 \times 3\mathbf{u}_3 &= \mathbf{u}_1 \times (\mathbf{u}_1 \times \mathbf{u}_2) = \mathbf{u}_1(\mathbf{u}_1 \cdot \mathbf{u}_2) - \mathbf{u}_2(\mathbf{u}_1 \cdot \mathbf{u}_1) \\
&= -\mathbf{u}_2\|\mathbf{u}_1\|^2 = -9\mathbf{u}_2 \\
&\Rightarrow\ \mathbf{u}_1 \times \mathbf{u}_3 = -\frac{9}{3}\mathbf{u}_2 = -3\mathbf{u}_2
\end{aligned}$$

Similarly, $\mathbf{u}_2 \times \mathbf{u}_3 = 3\mathbf{u}_1$

Using the properties of the cross product,

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \mathbf{u}_1 \times \mathbf{u}_1 + \mathbf{u}_1 \times (-\mathbf{u}_2) + \mathbf{u}_1 \times \mathbf{u}_3 + 2\mathbf{u}_2 \times \mathbf{u}_1 + 2\mathbf{u}_2 \times (-\mathbf{u}_2) \\
&\quad + 2\mathbf{u}_2 \times \mathbf{u}_3 + 3\mathbf{u}_3 \times \mathbf{u}_1 + 3\mathbf{u}_3 \times (-\mathbf{u}_2) + 3\mathbf{u}_3 \times \mathbf{u}_3 \\
&= -\mathbf{u}_1 \times \mathbf{u}_2 + \mathbf{u}_1 \times \mathbf{u}_3 + 2\mathbf{u}_2 \times \mathbf{u}_1 + 2\mathbf{u}_2 \times \mathbf{u}_3 + 3\mathbf{u}_3 \times \mathbf{u}_1 - 3\mathbf{u}_3 \times \mathbf{u}_2 \\
&= -\mathbf{u}_1 \times \mathbf{u}_2 - 2\mathbf{u}_1 \times \mathbf{u}_2 + \mathbf{u}_1 \times \mathbf{u}_3 - 3\mathbf{u}_1 \times \mathbf{u}_3 + 2\mathbf{u}_2 \times \mathbf{u}_3 + 3\mathbf{u}_2 \times \mathbf{u}_3 \\
&= -3\mathbf{u}_1 \times \mathbf{u}_2 - 2\mathbf{u}_1 \times \mathbf{u}_3 + 5\mathbf{u}_2 \times \mathbf{u}_3 \\
&= -3(3\mathbf{u}_3) - 2(-3\mathbf{u}_2) + 5(3\mathbf{u}_1) \\
&= 3(5\mathbf{u}_1 + 2\mathbf{u}_2 - 3\mathbf{u}_3)
\end{aligned}$$

It turns out that this is the same as $\langle 1, 2, 3\rangle \times \langle 1, -1, 1\rangle$ multiplied by 3. This is no coincidence.

6. $\mathbf{a}$ has length 3 units, lies in the xy plane, and points from the origin to the first quadrant at the angle $\pi/3$ to the x axis and $\mathbf{b}$ has length 2 units and points from the origin in the direction of the z axis.

SOLUTION: From the given information, we can write $\mathbf{a}$ in polar form as $\mathbf{a} = 3\langle \cos\pi/3, \sin\pi/3, 0\rangle = \frac{3}{2}\langle 1, \sqrt{3}, 0\rangle$ and we can write $\mathbf{b}$ as $\mathbf{b} = 2\langle 0, 0, 1\rangle$. So,

$$\mathbf{a}\times\mathbf{b} = \frac{3}{2}*2\det\begin{pmatrix}\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ 1 & \sqrt{3} & 0\\ 0 & 0 & 1\end{pmatrix} = 3\langle\det\begin{pmatrix}\sqrt{3} & 0\\ 0 & 1\end{pmatrix}, -\det\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \det\begin{pmatrix}1 & \sqrt{3}\\ 0 & 0\end{pmatrix}\rangle = 3\langle\sqrt{3}, -1, 0\rangle$$

7. $\mathbf{a}$ and $\mathbf{b}$ point from the origin to the second and first quadrant of the xy plane, respectively, so that $\mathbf{a}$ makes the angle $15°$ with the y axis and $\mathbf{b}$ makes the angle $75°$ with the x axis, and $\|\mathbf{a}\| = 2$, $\|\mathbf{b}\| = 3$.

Solution: If $\mathbf{a}$ makes an angle of $15°$ with the y axis and is in the second quadrant, then it makes an angle of $105°$ with the positive x axis. So the angle between $\mathbf{a}$ and $\mathbf{b}$ is $105° - 75° = 30°$. Recall that for vectors $\mathbf{u}$ and $\mathbf{v}$ we have that

$$\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$$

But $\mathbf{a}$ and $\mathbf{b}$ both lie in the xy plane. So their cross product must be parallel to the z axis, i.e.

$$\mathbf{a}\times\mathbf{b} = \langle 0, 0, k\rangle = k\langle 0, 0, 1\rangle$$

for some scalar k. Clearly, $\|\mathbf{a}\times\mathbf{b}\| = k$. On the other hand, $\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta = 2*3\sin 30° = 3$. So $k = 3$, and $\mathbf{a}\times\mathbf{b} = 3\langle 0, 0, 1\rangle$.

8. Let $\mathbf{a} = \langle 3, 2, 1\rangle$, $\mathbf{b} = \langle -2, 1, -1\rangle$, and $\mathbf{c} = \langle 1, 0, -1\rangle$. Find $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$, $\mathbf{b}\times(\mathbf{c}\times\mathbf{a})$, and $\mathbf{c}\times(\mathbf{a}\times\mathbf{b})$. Verify the Jacobi identity.

Solution: Recall the "bac-cab" rule, as follows:

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$$

The other cross products can be calculated similarly, by permuting the vectors in the formula above. If $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \to \mathbf{b}\times(\mathbf{c}\times\mathbf{a})$, then we have moved each vector to the left once, and moved the front one to the back. The effect of this is to permute the vectors in the first term of the right-hand side to the right once and to permute the vectors in the second term to the left once. I.e, $\mathbf{b}(\mathbf{a}\cdot\mathbf{c}) \to \mathbf{c}(\mathbf{b}\cdot\mathbf{a})$ and $\mathbf{c}(\mathbf{a}\cdot\mathbf{b}) \to \mathbf{a}(\mathbf{b}\cdot\mathbf{c})$. If you have trouble remembering this, look at what happens to the first vector in the LHS when you permute the LHS. In this case $\mathbf{a}$ turns into $\mathbf{b}$. That means in the RHS, wherever $\mathbf{a}$ appears, $\mathbf{b}$ must now appear. Shift the vectors accordingly. Try to practice by determining what $\mathbf{c} \times (\mathbf{a} \times \mathbf{b})$ should be before continuing.

We therefore have the following identities:

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \\ \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) &= \mathbf{c}(\mathbf{b} \cdot \mathbf{a}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) \\ \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) &= \mathbf{a}(\mathbf{c} \cdot \mathbf{b}) - \mathbf{b}(\mathbf{c} \cdot \mathbf{a}) \end{aligned}$$

And so,

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \mathbf{b}(3(1) + 2(0) + 1(-1)) - \mathbf{c}(3(-2) + 2(1) + 1(-1)) = 2\mathbf{b} + 5\mathbf{c} = \langle 1, 2, -7 \rangle \\ \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) &= \mathbf{c}(-2(3) + 1(2) - 1(1)) - \mathbf{a}(-2(1) + 1(0) - 1(-1)) = -5\mathbf{c} + \mathbf{a} = \langle -2, 2, 6 \rangle \\ \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) &= \mathbf{a}(1(-2) + 0(1) - 1(-1)) - \mathbf{b}(1(3) + 0(2) - 1(1)) = -\mathbf{a} - 2\mathbf{b} = \langle 1, -4, 1 \rangle \end{aligned}$$

Thus,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$$

9. Let $\mathbf{a}$ be a unit vector orthogonal to $\mathbf{b}$ and $\mathbf{c}$. If $\mathbf{c} = \langle 1, 2, 2 \rangle$, find the length of the vector $\mathbf{a} \times [(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{b} + \mathbf{c})]$

SOLUTION: Recall that if $\mathbf{u}$ is a unit vector, then $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2 = 1^2 = 1$. Furthermore, it $\mathbf{u}$ and $\mathbf{v}$ are two perpendicular vectors, then $\mathbf{u} \cdot \mathbf{v} = 0$. Thus, using the "bac-cab" rule on $\mathbf{a} \times [(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{b} + \mathbf{c})]$ yields

$$\begin{aligned} \mathbf{a} \times [(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{b} + \mathbf{c})] &= (\mathbf{a} + \mathbf{b})(\mathbf{a} \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c})) - (\mathbf{a} + \mathbf{b} + \mathbf{c})(\mathbf{a} \cdot (\mathbf{a} + \mathbf{b}) \\ &= (\mathbf{a} + \mathbf{b})(\mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}) - (\mathbf{a} + \mathbf{b} + \mathbf{c})(\mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b}) \\ &= (\mathbf{a} + \mathbf{b})(1) - (\mathbf{a} + \mathbf{b} + \mathbf{c})(1) \\ &= -\mathbf{c} \end{aligned}$$

So

$$\|\mathbf{a} \times [(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{b} + \mathbf{c})]\| = \sqrt{(1)^2 + (2)^2 + (2)^2} = 3$$

10. Given two nonparallel vectors $\mathbf{a}$ and $\mathbf{b}$, show that the vectors $\mathbf{a}$, $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a} \times (\mathbf{a} \times \mathbf{b})$ are mutually orthogonal, and, hence, form an orthogonal basis in space.

SOLUTION: By the definition of the cross product, $\mathbf{a}$ is orthogonal to $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a} \times (\mathbf{a} \times \mathbf{b})$. Thus we must only verify that $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a} \times (\mathbf{a} \times \mathbf{b})$ are orthogonal. Certinaly this is also true, since for any two nonparallel vectors $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u} \times \mathbf{v}$ will be orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. Since $\mathbf{a}$ and $\mathbf{b}$ are nonparallel, their cross product is nonzero and orthogonal to both vectors. So $\mathbf{a} \times \mathbf{b}$ is not parallel to $\mathbf{a}$ ($\mathbf{0}$ is the only vector simultaneously parallel and orthogonal to any vector). So $\mathbf{a} \times (\mathbf{a} \times \mathbf{b})$ is orthogonal to $\mathbf{a} \times \mathbf{b}$

11. Suppose $\mathbf{a}$ lies in the xy plane, its initial point is at the origin, and its terminal point is in first quadrant of the xy plane. Let $\mathbf{b}$ be parallel to $\hat{\mathbf{e}}_3$. Use the right-hand rule to determine whether the angle between $\mathbf{a} \times \mathbf{b}$ and the unit vectors parallel to the coordinate axes lies in the interval $(0, \pi/2)$ or $(\pi/2, \pi)$ or equals $\pi/2$.

Solution: By the right hand rule, $\mathbf{a} \times \mathbf{b}$ will be in the xy plane in the fourth quadrant. Let α, β, and γ be the angles between $\mathbf{a} \times \mathbf{b}$ and $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$. Clearly, $\alpha \in (0, \pi/2)$. Since $\mathbf{a} \times \mathbf{b}$ is in the fourth quadrant, it is closer to the negative y-axis than the positive y-axis, thus $\beta \in (\pi/2, \pi)$. Since $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{b}$, $\gamma = \pi/2$.

12. If vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ have the initial point at the origin and lie, respectively, in the positive quadrants of the xy, yz, and xz planes, determine the octants in which the pairwise cross products of these vectors lie by specifying the signs of the components of the cross products.

Solution: To denote the sign of a component, I will simply write the sign in the component. Thus, a vector with positive x and z components, but a negative y component will be written as $\langle +, -, + \rangle$.

First examine $\mathbf{a} \times \mathbf{b}$. Note that $\mathbf{a}$ is $\langle +, +, 0 \rangle$ and $\mathbf{b}$ is $\langle 0, +, + \rangle$By the right hand rule $\mathbf{a} \times \mathbf{b}$ will be $\langle +, -, + \rangle$. This can be seen by considering the "dot product" as follows:

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = \langle +, +, 0 \rangle \cdot \langle +, -, + \rangle = +(+) \ + \ +(-) \ + \ 0(+)$$

It is feasible to make this equal 0, with appropriate choices for the components. But couldn't we switch the + and - in the cross product and also achieve this? To refute this, investigate the "dot product" with $\mathbf{b}$ as follows:

$$\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = \langle 0, +, + \rangle \cdot \langle +, -, + \rangle = 0(+) \ + \ +(-) \ + \ +(+)$$

If the x and y component signs of the cross product are reversed, then the "dot product" with $\mathbf{b}$ will always be positive. It follows, in general, that the component two of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ share must be positive/negative in the cross product, with the other two components the opposite sign. Thus $\mathbf{a} \times \mathbf{c}$ will be $\langle +, -, - \rangle$ and $\mathbf{b} \times \mathbf{c}$ will be $\langle +, +, - \rangle$, by the right-hand rule and the above discussion.

13. If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are coplanar vectors, find $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{b} \times \mathbf{c})$.

Solution: If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are coplanar, then each of $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{c}$

will be perpendicular to the plane $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ span. Therefore $\mathbf{a} \times \mathbf{b}$ is parallel to $\mathbf{b} \times \mathbf{c}$, and their cross product will be $\mathbf{0}$.

14. Find a unit vector perpendicular to the vectors $\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 - 2\hat{\mathbf{e}}_3$ and $\hat{\mathbf{e}}_1 - 2\hat{\mathbf{e}}_2 + 4\hat{\mathbf{e}}_3$.

SOLUTION: A unit vector perpendicular to the given vectors will simply be a unit vector parallel to their cross product. Let $\mathbf{a} = \langle 1, 1, -2\rangle$ and $\mathbf{b} = \langle 1, -2, 4\rangle$. Then

$$\mathbf{a}\times\mathbf{b} = \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 1 & 1 & -2 \\ 1 & -2 & 4 \end{pmatrix} = \langle \det\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}, -\det\begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}, \det\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\rangle = \langle 0, -6, -3\rangle$$

Let $\mathbf{u} = \mathbf{a} \times \mathbf{b}$. Then

$$\|\mathbf{u}\| = \sqrt{0^2 + (-6)^2 + (-3)^2} = 3\sqrt{5}$$

Thus a unit vector perpendicular to the given vectors is

$$\hat{\mathbf{u}} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{3\sqrt{5}}\langle 0, -6, -3\rangle = -\frac{1}{\sqrt{5}}\langle 0, 2, 1\rangle$$

15. Find the area of a triangle whose vertices lie on the different coordinate axes at distances a, b, and c from the origin.

SOLUTION: The vertices of the triangle are $A = (a, 0, 0)$, $B = (0, b, 0)$, and $C = (0, 0, c)$. Given the coordinates of the vertices, the area is

$$A_\Delta = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\|\,.$$

Therefore

$$\begin{aligned} \overrightarrow{AB} &= \langle -a, b, 0\rangle \\ \overrightarrow{AC} &= \langle -a, 0, c\rangle \\ \overrightarrow{AB} \times \overrightarrow{AC} &= \langle bc - 0,\ -(-ac - 0),\ 0 - (-ab)\rangle = \langle bc, ac, ab\rangle \\ A_\Delta &= \frac{1}{2}\sqrt{b^2c^2 + a^2c^2 + a^2b^2}\,. \end{aligned}$$

16. Find the area of a triangle ABC for $A(1, 0, 1)$, $B(1, 2, 3)$, and $C(0, 1, 1)$ and a nonzero vector perpendicular to the plane containing the triangle.

SOLUTION: Let $\mathbf{a} = \overrightarrow{AB} = \langle 0, 2, 2\rangle$ and $\mathbf{b} = \overrightarrow{AC} = \langle -1, 1, 0\rangle$. By the geometrical properties of the cross product, the vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ and, hence, to the plane containing the triangle. Furthermore, $\|\mathbf{a}\times\mathbf{b}\|$ is the area of the parallelogram with adjacent sides $\mathbf{a}$ and $\mathbf{b}$ and, hence, the area of the triangle is $\frac{1}{2}\|\mathbf{a}\times\mathbf{b}\|$. One has

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= \det\begin{pmatrix}\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ 0 & 2 & 2\\ -1 & 1 & 0\end{pmatrix} = \hat{\mathbf{e}}_1(0-2)-\hat{\mathbf{e}}_2(0+2)+\hat{\mathbf{e}}_3(0+2)\\ &= \langle -2,-2,2\rangle = 2\langle -1,-1,1\rangle\\ \|\mathbf{a}\times\mathbf{b}\| &= 2\|\langle -1,-1,1\rangle\| = 2\sqrt{3}\end{aligned}$$

Therefore the area of the triangle is $\frac{1}{2}\cdot 2\sqrt{3} = \sqrt{3}$ and the vector $\langle -1,-1,1\rangle$ is orthogonal to the plane in which the triangle lies.

17. Use the cross product to show that the area of the triangle whose vertices are midpoints of the sides of a triangle with area A is $A/4$. *Hint*: Define sides of the triangle of area A as vectors and express the sides of the other triangle in question in terms of these vectors.

Solution: Without loss of generality, suppose one vertex of the triangle is at the origin. Let the two other vertices be B and C. The area of triangle OBC is given by

$$A = \frac{1}{2}\|\overrightarrow{OB}\times\overrightarrow{OC}\|$$

because $|\overrightarrow{OB}\times\overrightarrow{OC}\|$ gives the area of the parallelogram spanned by $\overrightarrow{OB}$ and $\overrightarrow{OC}$, and the parallelogram is formed from two copies of OBC. Let the midpoint of OB, OC, and BC be D, E, and F, respectively. Then $\overrightarrow{OD} = \frac{1}{2}\overrightarrow{OB}$, $\overrightarrow{OE} = \frac{1}{2}\overrightarrow{OC}$, and $\overrightarrow{OF} = \frac{1}{2}(\overrightarrow{OB}+\overrightarrow{OC}) = \overrightarrow{OD}+\overrightarrow{OE}$. Then, two sides of the triangle are given by $\overrightarrow{FD} = -\frac{1}{2}\overrightarrow{OC}$ and $\overrightarrow{FE} = -\frac{1}{2}\overrightarrow{OB}$. Thus the area of the triangle is

$$A' = \frac{1}{2}\|\overrightarrow{FD}\times\overrightarrow{FE}\| = \frac{1}{2}\|-\frac{1}{2}\overrightarrow{OC}\times(-\frac{1}{2}\overrightarrow{OB})\| = \frac{1}{4}(\frac{1}{2}\|\overrightarrow{OB}\times\overrightarrow{OC}\|) = \frac{1}{4}A$$

18. Consider a triangle whose vertices are midpoints of any three sides of a parallelogram. If the area of the parallelogram is A, find the area of the triangle. *Hint*: Define adjacent sides of the parallelogram as vectors and express the sides of the triangle in terms of these vectors.

Solution: This problem can actually be easily done without cross products. I will give both methods.

First, consider the parallelogram spanned by two vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$, where A is on to the x-axis (all parallelograms are isomorphic to such a parallelogram, up to rigid motions). Then the area of this parallelogram is

$$A = \|\overrightarrow{OA} \times \overrightarrow{OB}\|$$

Next, consider three midpoints D_1, D_2, D_3, where D_1 is the midpoint of OA, D_2 the midpoint of OB, and D_3 is the midpoint of the line segment formed by connecting A to the last vertex.
Cross product method: Then $\overrightarrow{OD_1} = \frac{1}{2}\overrightarrow{OA}$, $\overrightarrow{OD_2} = \frac{1}{2}\overrightarrow{OB}$, and $\overrightarrow{OD_3} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB}$. Two sides of the triangle formed from D_1,D_2, and D_3 are given by $\overrightarrow{D_3D_1} = -\frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$ and $\overrightarrow{D_3D_1} = -\overrightarrow{OA}$. So the area is given by

$$A' = \frac{1}{2}\|\overrightarrow{D_3D_1}\overrightarrow{D_3D_2}\| = \frac{1}{2}\|-\frac{1}{2}(\overrightarrow{OA}+\overrightarrow{OB})\times(-\overrightarrow{OA}) = \frac{1}{4}\|\overrightarrow{OA}\times\overrightarrow{OB}\| = \frac{1}{4}A$$

Because $\overrightarrow{OA} \times \overrightarrow{OA} = \mathbf{0}$.

Other method: Recognize that the line D_2D_3 splits the parallelogram in two. Thus there are two regions, one of area $A/2$ with the triangle, and one of the same area without the triangle. Look at the triangles OD_1D_2 and AD_1D_3. Let the height of the parallelogram be h and the base be b. Then the triangles share the same base length and height, namely $h/2$ and $b/2$. Thus their area is $A_\Delta = 1/2(h/2b/2) = 1/2(hb/4)$. So their combined area is $hb/4$, which is $A/4$. The only remaining area is that of the triangle in question. Thus its area is $A-A/2-A/4 = A/4$.

19. Let $A = (1, 2, 1)$ and $B = (-1, 0, 2)$ be vertices of a parallelogram. If the other two vertices are obtained by moving A and B along straight lines by a distance of 3 units in the direction of the vector $\mathbf{a} = \langle 2, 1, -2\rangle$, find the area of the parallelogram.

Solution: First, $\overrightarrow{AB} = \langle -2, -2, 1\rangle$ Translating a point in the direction of $\mathbf{a}$ by 3 units is done by adding $3\hat{\mathbf{a}}$ to the position vector of the point. So first we must find $3\hat{\mathbf{a}}$.

$$\|\mathbf{a}\| = 3 \Rightarrow \frac{3}{\|\mathbf{a}\|}\mathbf{a} = \mathbf{a}$$

Let A' be the translated point of A. Then $\overrightarrow{AA'} = \mathbf{a}$. So the area of the parallelogram, S, is

$$S = \|\overrightarrow{AA'} \times \overrightarrow{AB}\| = \|\mathbf{a} \times \overrightarrow{AB}\|$$

The cross product is computed as follows:

$$\mathbf{a}\times\overrightarrow{AB} = \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix} = \langle \det\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}, -\det\begin{pmatrix} 2 & -2 \\ -2 & 1 \end{pmatrix}, \det\begin{pmatrix} 2 & 1 \\ -2 & -2 \end{pmatrix}\rangle = \langle -3, 2, -2\rangle$$

Thus

$$S = \|\mathbf{a}\times\overrightarrow{AB}\| = \sqrt{(-3)^2 + 2^2 + (-2)^2} = \sqrt{17}$$

20. Consider four points in space. Suppose that the coordinates of the points are known. Describe a procedure based on the properties of the cross product to determine whether the points are in one plane. In particular, are the points $(1,2,3)$, $(-1,0,1)$, $(1,3,-1)$, and $(0,1,2)$ in one plane?

21. Let the sides of a triangle have lengths a, b, and c and let the angles at the vertices opposite to the sides a, b, and c be, respectively, α, β, and γ. Prove that

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}.$$

Hint: Define the sides as vectors and express the area of the triangle in terms of the vectors adjacent at each vertex of the triangle.

SOLUTION: Let A, B, C be the vertices of a triangle. Let α be the angle at vertex A, β at B, and γ at C. Let $|BC| = a$, $|AC| = b$, and $|AB| = c$. Then a is opposite α, b is opposite β, and c is opposite γ. The area of the triangle can be computed in three ways as follows:

$$\begin{aligned} A_\Delta &= \frac{1}{2}\|\overrightarrow{AB}\times\overrightarrow{AC}\| = \frac{1}{2}\|\overrightarrow{AB}\|\|\overrightarrow{AC}\|\sin\alpha = \frac{1}{2}(bc)\sin\alpha \\ &= \frac{1}{2}\|\overrightarrow{BA}\times\overrightarrow{BC}\| = \frac{1}{2}\|\overrightarrow{BA}\|\|\overrightarrow{BC}\|\sin\beta = \frac{1}{2}(ac)\sin\beta \\ &= \frac{1}{2}\|\overrightarrow{CA}\times\overrightarrow{CB}\| = \frac{1}{2}\|\overrightarrow{CA}\|\|\overrightarrow{CB}\|\sin\gamma = \frac{1}{2}(ab)\sin\gamma \end{aligned}$$

We can equate these as follows

$$\begin{aligned} \frac{1}{2}(bc)\sin\alpha &= \frac{1}{2}(ac)\sin\beta \;\Rightarrow\; \frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \\ \frac{1}{2}(ac)\sin\beta &= \frac{1}{2}(ab)\sin\gamma \;\Rightarrow\; \frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \end{aligned}$$

22. A polygon $ABCD$ in a plane is a part of the plane bounded by four straight line segments AB, BC, CD, and DA. Suppose that the polygon is *convex*, that is, a straight line segment connecting any two points in the polygon lies in the polygon. If the coordinates of the vertices are specified, describe the procedure based on vector algebra to calculate the area of the polygon. In particular, let a convex polygon be in the xy plane and put $A = (0, 0)$, $B = (x_1, y_1)$, $C = (x_2, y_2)$, and $D = (x_3, y_3)$. Express the area in terms of x_i and y_i, $i = 1, 2, 3$.

Solution: The polygon ABCD can be divided into two parts along the line AC. This forms two triangles, ABC and ACD. The area of the polygon will be the sum of the area of each triangle. Since the polygon is convex, each interior angle will be less than 180°, so ABC and ACD are indeed triangles. Let the area of triangle ABC be A_1 and that of triangle ACD be A_2. The total area, A_t, then is $A_1 + A_2$. So,

$$\begin{aligned} A_1 &= \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| \\ A_2 &= \frac{1}{2}\|\overrightarrow{AC} \times \overrightarrow{AD}\| \\ A_t &= A_1 + A_2 \end{aligned}$$

In particular, when $A = (0, 0)$, $\overrightarrow{AB}, \overrightarrow{AC}$, and $\overrightarrow{OD}$ are the position vectors of B, C, and D. Since each point lies in the xy plane, the cross products above will only have a z component. Furthermore, their magnitudes will be that z component. So,

$$\begin{aligned} \overrightarrow{AB} \times \overrightarrow{AC} &= \langle 0, 0, \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \rangle = \langle 0, 0, x_1 y_2 - x_2 y_1 \rangle \\ A_1 = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| &= \frac{1}{2}|x_1 y_2 - x_2 y_1| \\ \overrightarrow{AC} \times \overrightarrow{AD} &= \langle 0, 0, \det \begin{pmatrix} x_2 & y_2 \\ x_3 & y_3 \end{pmatrix} \rangle = \langle 0, 0, x_2 y_3 - x_3 y_2 \rangle \\ A_2 = \frac{1}{2}\|\overrightarrow{AC} \times \overrightarrow{AD}\| &= \frac{1}{2}|x_2 y_3 - x_3 y_2| \end{aligned}$$

Thus the area of the polygon is

$$A_t = \frac{1}{2}|x_1 y_2 + x_2 y_3 - x_2 y_1 - x_3 y_2|$$

This is known as the shoelace method.

23. Consider a parallelogram. Construct another parallelogram whose adjacent sides are diagonals of the first parallelogram. Find the relation between the areas of the parallelograms.

Solution: Let $\mathbf{a}$ and $\mathbf{b}$ be vectors that form a parallelogram. The diagonals of this parallelogram are given by $\mathbf{d}_1 = \mathbf{a} + \mathbf{b}$ and $\mathbf{d}_2 = \mathbf{a} - \mathbf{b}$.

Then the area of the parallelogram spanned by these vectors is

$$A' = \|\mathbf{d}_1\times\mathbf{d}_2\| = \|(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})\| = \|\mathbf{a}\times\mathbf{a}-\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{a}-\mathbf{b}\times\mathbf{b}\| = \|2(\mathbf{b}\times\mathbf{a})\| = 2A$$

where A is the area of the original parallelogram.

24. Given two nonparallel vectors $\mathbf{a}$ and $\mathbf{b}$, show that any vector $\mathbf{r}$ in space can be written as a linear combination $\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{a}\times\mathbf{b}$ and that the numbers x, y, and z are unique for every $\mathbf{r}$. Express z in terms of $\mathbf{r}$, $\mathbf{a}$ and $\mathbf{b}$. In particular, put $\mathbf{a} = \langle 1, 1, 1\rangle$ and $\mathbf{b} = \langle 1, 1, 0\rangle$. Find the coefficients x, y, and z for $\mathbf{r} = \langle 1, 2, 3\rangle$. *Hint:* See Study Problems **??** and **??**.

Solution: This is actually a fundamental fact of linear algebra (see direct sums of subspaces for more information). First, let $\mathcal{P}$ represent the plane spanned by $\mathbf{a}$ and $\mathbf{b}$, that is $\mathcal{P} = \{s\mathbf{a} + t\mathbf{b}|s, t \in \mathbb{R}\}$. Then $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathcal{P}$. Let $\mathbf{r}$ be a vector in space. Then $\mathbf{r}$ can be uniquely decomposed into two vectors $\mathbf{r}_{||}$ and $\mathbf{r}_{\perp}$ (Corollary 3.1). We know that

$$\mathbf{r}_{||} = \frac{r_{||}}{\|\mathbf{a}\times\mathbf{b}\|}\mathbf{a}\times\mathbf{b} = \frac{(\mathbf{a}\times\mathbf{b})\cdot\mathbf{r}}{\|\mathbf{a}\times\mathbf{b}\|^2}\,\mathbf{a}\times\mathbf{b}$$

Next, we have that $\mathbf{r}_{\perp} = \mathbf{r} - \mathbf{r}_{||}$, where $\mathbf{r}_{\perp} \in \mathcal{P}$ since $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathcal{P}$. So $\mathbf{r}_{\perp} = x\mathbf{a} + y\mathbf{b}$ for some unique $x, y \in \mathbb{R}$. This can easily be shown as follows: Let $\mathbf{r}_{\perp} = \langle r_1, r_2, r_3\rangle$, $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$. So if $\mathbf{r}_{\perp} = x\mathbf{a} + y\mathbf{b}$, then we must have that

$$\begin{cases} xa_1 + yb_1 & = r_1 \\ xa_2 + yb_2 & = r_2 \\ xa_3 + yb_3 & = r_3 \end{cases}$$

Where each equation was found by setting the components equal to each other. Now, since $\mathbf{a}$ and $\mathbf{b}$ are not parallel, there exist some i, j between 1 and 3 where $a_i \neq sa_j$ and $b_i \neq sb_j$ for all $s \in \mathbb{R}$. What this means is that at least two of the equations above are linearly independent. So we reduce our system of equations to the following

$$\begin{cases} xa_i + yb_i & = r_i \\ xa_j + yb_j & = r_j \end{cases}$$

for some i, j between 1 and 3. Now we can rewrite this as

$$\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix} = \begin{pmatrix} r_i \\ r_j \end{pmatrix}$$

Now recall that the determinant of $A = \begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix}$ is $a_i b_j - b_i a_j$. It ends up being that $A^{-1} = \frac{1}{A}\begin{pmatrix} b_j & -b_i \\ -a_j & a_i \end{pmatrix}$, so its existence is dependent on the existence of $\det(A)$. Were it true that $a_i = sa_j$ and $b_i = sb_j$, then $\det(A) = (sa_j)b_j - (sb_j)a_j = 0$. So it would be that A^{-1} wouldn't exist, but this is not true, so we are okay. What this means is that A^{-1} exists, and it is unique (evident since each of its entries are unique and the determinant is too). Thus,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r_i \\ r_j \end{pmatrix} A^{-1}$$

and x, y are unique.

Now suppose two vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are such that their x, y are equal, but $\mathbf{r}_1 \neq \mathbf{r}_2$. It remains to be shown that their z are unequal. Consider the vector $x\mathbf{a} + y\mathbf{b}$. Draw a line perpendicular to $\mathcal{P}$ through the point whose position vector is $x\mathbf{a} + y\mathbf{b}$. By construction, both $\mathbf{r}_1$ and $\mathbf{r}_2$ will be position vectors for points on this line. Consider the quantity

$$z = \frac{(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{r}}{\|\mathbf{a} \times \mathbf{b}\|} = \frac{\|\mathbf{r}\| \cos\theta}{\|\mathbf{a} \times \mathbf{b}\|}$$

Thus, given $\mathbf{a}$ and $\mathbf{b}$, z is determined by the magnitude of $\mathbf{r}$ and its angle to $\mathbf{a} \times \mathbf{b}$, parallel to the constructed line. Suppose $\|\mathbf{r}_1\| = \|\mathbf{r}_2\|$. There are exactly two points on the line a distance $\|\mathbf{r}_1\|$ (so long as $\mathbf{r}_1$ and $\mathbf{r}_2$ do not lie in the plane, but if they do, there is only one point, and so $\mathbf{r}_1 = \mathbf{r}_2$, a contradiction). Thus, either $\mathbf{r}_1 = \mathbf{r}_2$ or $\mathbf{r}_1$ corresponds to one of these points and $\mathbf{r}_2$ corresponds to the other. The former is a contradiction so assume the latter. This, however, implies that θ_1, the angle between $\mathbf{r}_1$ and $\mathbf{a} \times \mathbf{b}$ is unequal to θ_2 (indeed, it should be that $\cos\theta_1 = -\cos\theta_2$). Thus their z are of opposite sign and hence unequal. Now suppose that $\theta_1 = \theta_2$. Then it must be, for both vectors to lie on the line, that they are equivalent, a contradiction. So z is unique.

Thus the first part of the question is proved, since the triplet (x, y, z) is unique.

For the second part, we first calculate $\mathbf{a} \times \mathbf{b}$ where $\mathbf{a} = \langle 1, 1, 1 \rangle$ and $\mathbf{b} = \langle 1, 1, 0 \rangle$. So,

$$\mathbf{a} \times \mathbf{b} = \det \begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \langle \det \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, -\det \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \det \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rangle = \langle -1, 1, 0 \rangle$$

By the above, we have that

$$z = \frac{(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{r}}{\|\mathbf{a} \times \mathbf{b}\|^2} = \frac{\langle -1, 1, 0\rangle \cdot \langle 1, 2, 3\rangle}{\sqrt{(-1)^2 + 1^2 + 0^2}^2} = \frac{1}{2}$$

And also,

$$\begin{aligned} \mathbf{r} - \mathbf{r}_{||} &= x\mathbf{a} + y\mathbf{y} \\ \langle 1, 2, 3\rangle - \frac{1}{2}\langle -1, 1, 0\rangle &= x\langle 1, 1, 1\rangle + y\langle 1, 1, 0\rangle \\ \langle \frac{3}{2}, \frac{3}{2}, 3\rangle &= \langle x + y, x + y, x\rangle \end{aligned}$$

It is clear then that $x = 3$ and $x + y = \frac{3}{2}$, so $y = -\frac{3}{2}$.

25. A tetrahedron is a solid with four vertices and four triangular faces. Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ be vectors with lengths equal to the areas of the faces and directions perpendicular to the faces and pointing outward. Show that $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$. *Hint*: Set up vectors being the egdes of a tetrahedron. There are six edges. So all these vectors can be expressed as linear combinations of particular three non-coplanar vectors. Use the cross product to find the vectors $\mathbf{v}_j$, $j = 1, 2, 3, 4$, in terms of the three non-coplanar vectors.

Solution: Let A, B, C and D be the coordinates of the vertices of the tetrahedron, in the following way: With A oriented at the top, looking down, vertices B, C, and D are coplanar and go counterclockwise (so B can be at the bottom right of the base, C at the top, and D at the bottom left, with A still overhead). Then $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{BC}$, $\overrightarrow{BD}$, $\overrightarrow{CD}$ are the six edges of the tetrahedron. We have then that four vectors perpendicular to the faces of the tetrahedron are $\overrightarrow{AB} \times \overrightarrow{AC}$, $\overrightarrow{AC} \times \overrightarrow{AD}$, $\overrightarrow{AD} \times \overrightarrow{AB}$, and $\overrightarrow{BD} \times \overrightarrow{BC}$. The first three can be found by orienting the tetrahedron such that vertex A points at the top, and then looking from the top down. There will be three faces visible, whose normal vectors correspond to the first three found. The order in which the vectors are in the cross product can be found using the right-hand rule, and that the normal vectors must point out. Because the bottom face does not include vertex A, the last vector can be found by choosing any two nonparallel vectors not including A, for example $\overrightarrow{BC}$ and $\overrightarrow{BD}$. The order in which the cross product is taken is based on the right-hand rule.

Next, we set $\mathbf{v}_1 = \frac{1}{2}\overrightarrow{AB} \times \overrightarrow{AC}$, $\mathbf{v}_2 = \frac{1}{2}\overrightarrow{AC} \times \overrightarrow{AD}$, $\mathbf{v}_3 = \frac{1}{2}\overrightarrow{AD} \times \overrightarrow{AB}$, and

$\mathbf{v}_4 = \frac{1}{2}\overrightarrow{BD} \times \overrightarrow{BC}$. Recall that the area of a triangle spanned by vertices O, P, and Q is $A_\Delta = \frac{1}{2}\|\overrightarrow{OP} \times \overrightarrow{OQ}\|$. This explains the $\frac{1}{2}$ in each of $\mathbf{v}_j$ for $j = 1, 2, 3, 4$, because $\|\mathbf{v}_j\|$ needs to be equal to the area of each triangle face. Next we must expand each of $\mathbf{v}_j$ as follows, because in their current form they are not useful.

$$\begin{aligned}
\mathbf{v}_1 &= \frac{1}{2}\overrightarrow{AB} \times \overrightarrow{AC} \\
&= \frac{1}{2}(\overrightarrow{OB} - \overrightarrow{OA}) \times (\overrightarrow{OC} - \overrightarrow{OA}) \\
&= \frac{1}{2}(\overrightarrow{OB} \times (\overrightarrow{OC} - \overrightarrow{OA}) - \overrightarrow{OA} \times (\overrightarrow{OC} - \overrightarrow{OA})) \\
&= \frac{1}{2}(\overrightarrow{OB} \times \overrightarrow{OC} - \overrightarrow{OB} \times \overrightarrow{OA} - \overrightarrow{OA} \times \overrightarrow{OC})
\end{aligned}$$

Similarly, we conclude that

$$\begin{aligned}
\mathbf{v}_2 &= \frac{1}{2}(\overrightarrow{OC} \times \overrightarrow{OD} - \overrightarrow{OC} \times \overrightarrow{OA} - \overrightarrow{OA} \times \overrightarrow{OD}) \\
\mathbf{v}_3 &= \frac{1}{2}(\overrightarrow{OD} \times \overrightarrow{OB} - \overrightarrow{OD} \times \overrightarrow{OA} - \overrightarrow{OA} \times \overrightarrow{OB}) \\
\mathbf{v}_4 &= \frac{1}{2}(\overrightarrow{OD} \times \overrightarrow{OC} - \overrightarrow{OD} \times \overrightarrow{OB} - \overrightarrow{OB} \times \overrightarrow{OC})
\end{aligned}$$

Computing $\mathbf{v}_1 + \mathbf{v}_2$ yields

$$\begin{aligned}
\mathbf{v}_1 + \mathbf{v}_2 &= \frac{1}{2}(\overrightarrow{OB} \times \overrightarrow{OC} - \overrightarrow{OB} \times \overrightarrow{OA} - \overrightarrow{OA} \times \overrightarrow{OC}) + \frac{1}{2}(\overrightarrow{OC} \times \overrightarrow{OD} - \overrightarrow{OC} \times \overrightarrow{OA} - \overrightarrow{OA} \times \overrightarrow{OD}) \\
&= \frac{1}{2}(\overrightarrow{OB} \times \overrightarrow{OC} - \overrightarrow{OB} \times \overrightarrow{OA} - \overrightarrow{OA} \times \overrightarrow{OC} + \overrightarrow{OC} \times \overrightarrow{OD} - \overrightarrow{OC} \times \overrightarrow{OA} - \overrightarrow{OA} \times \overrightarrow{OD}) \\
&= \frac{1}{2}(\overrightarrow{OB} \times \overrightarrow{OC} - \overrightarrow{OB} \times \overrightarrow{OA} - \overrightarrow{OA} \times \overrightarrow{OC} - \overrightarrow{OC} \times \overrightarrow{OA} + \overrightarrow{OC} \times \overrightarrow{OD} - \overrightarrow{OA} \times \overrightarrow{OD}) \\
&= \frac{1}{2}(\overrightarrow{OB} \times \overrightarrow{OC} - \overrightarrow{OB} \times \overrightarrow{OA} - \overrightarrow{OA} \times \overrightarrow{OC} + \overrightarrow{OA} \times \overrightarrow{OC} + \overrightarrow{OC} \times \overrightarrow{OD} - \overrightarrow{OA} \times \overrightarrow{OD}) \\
&= \frac{1}{2}(\overrightarrow{OB} \times \overrightarrow{OC} - \overrightarrow{OB} \times \overrightarrow{OA} + \overrightarrow{OC} \times \overrightarrow{OD} - \overrightarrow{OA} \times \overrightarrow{OD})
\end{aligned}$$

Computing $\mathbf{v}_3 + \mathbf{v}_4$ yields

$$\begin{aligned}
\mathbf{v}_3 + \mathbf{v}_4 &= \frac{1}{2}(\overrightarrow{OD} \times \overrightarrow{OB} - \overrightarrow{OD} \times \overrightarrow{OA} - \overrightarrow{OA} \times \overrightarrow{OB}) + \frac{1}{2}(\overrightarrow{OD} \times \overrightarrow{OC} - \overrightarrow{OD} \times \overrightarrow{OB} - \overrightarrow{OB} \times \overrightarrow{OC}) \\
&= \frac{1}{2}(\overrightarrow{OD} \times \overrightarrow{OB} - \overrightarrow{OD} \times \overrightarrow{OA} - \overrightarrow{OA} \times \overrightarrow{OB} + \overrightarrow{OD} \times \overrightarrow{OC} - \overrightarrow{OD} \times \overrightarrow{OB} - \overrightarrow{OB} \times \overrightarrow{OC}) \\
&= \frac{1}{2}(\overrightarrow{OD} \times \overrightarrow{OB} - \overrightarrow{OD} \times \overrightarrow{OB} - \overrightarrow{OD} \times \overrightarrow{OA} - \overrightarrow{OA} \times \overrightarrow{OB} + \overrightarrow{OD} \times \overrightarrow{OC} - \overrightarrow{OB} \times \overrightarrow{OC}) \\
&= \frac{1}{2}(-\overrightarrow{OD} \times \overrightarrow{OA} - \overrightarrow{OA} \times \overrightarrow{OB} + \overrightarrow{OD} \times \overrightarrow{OC} - \overrightarrow{OB} \times \overrightarrow{OC}) \\
&= \frac{1}{2}(-\overrightarrow{OB} \times \overrightarrow{OC} + \overrightarrow{OB} \times \overrightarrow{OA} - \overrightarrow{OC} \times \overrightarrow{OD} + \overrightarrow{OA} \times \overrightarrow{OD})
\end{aligned}$$

where in the last step, terms have been rearranged and the equality $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$ has been used multiple times. Comparing $\mathbf{v}_1 + \mathbf{v}_2$ to $\mathbf{v}_3 + \mathbf{v}_4$, it is now easy to see the cancellation. Hence $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$

26. If $\mathbf{a}$ is a non-zero vector, $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, and $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?

SOLUTION: By basic algebraic properties of the cross and dot products, the stated equations can be cast in the following form which is then analyzed using the geometrical properties of the products:

$$\begin{aligned}
\mathbf{a} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} \quad &\Rightarrow \quad \mathbf{a} \times (\mathbf{c} - \mathbf{b}) = \mathbf{0} \quad \Rightarrow \quad \mathbf{a} \parallel \mathbf{c} - \mathbf{b} \\
\mathbf{a} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b} \quad &\Rightarrow \quad \mathbf{a} \cdot (\mathbf{c} - \mathbf{b}) = 0 \quad \Rightarrow \quad \mathbf{a} \perp \mathbf{c} - \mathbf{b}
\end{aligned}$$

But there exists only one vector that is perpendicular *and* parallel to a non-zero vector $\mathbf{a}$; it is the zero vector. Thus, $\mathbf{c} - \mathbf{b} = \mathbf{0}$ or $\mathbf{c} = \mathbf{b}$.

27. Given two non-parallel vectors $\mathbf{a}$ and $\mathbf{b}$, construct three mutually orthogonal unit vectors $\hat{\mathbf{u}}_i$, $i = 1, 2, 3$, one of which is parallel to $\mathbf{a}$. Are such unit vectors unique? In particular, put $\mathbf{a} = \langle 1, 2, 2 \rangle$ and $\mathbf{b} = \langle 1, 0, 2 \rangle$ and find $\hat{\mathbf{u}}_i$.

SOLUTION: It is easy to see that $\mathbf{a}$, $\mathbf{a} \times \mathbf{b}$, and $\mathbf{a} \times (\mathbf{a} \times \mathbf{b})$ are mutually orthogonal. In fact, this was shown in problem 10. Therefore we set $\hat{\mathbf{u}}_1 = \frac{1}{\|\mathbf{a}\|}\, \mathbf{a}$, $\hat{\mathbf{u}}_2 = \frac{1}{\|\mathbf{a} \times \mathbf{b}\|}\, \mathbf{a} \times \mathbf{b}$, and $\hat{\mathbf{u}}_3 = \frac{1}{\|\mathbf{a} \times (\mathbf{a} \times \mathbf{b})\|}\, \mathbf{a} \times (\mathbf{a} \times \mathbf{b})$. These vectors are not unique, as we could have chosen the negatives of $\hat{\mathbf{u}}_i$ $i = 1, 2, 3$.

In particular, when $\mathbf{a} = \langle 1, 2, 2\rangle$ and $\mathbf{b} = \langle 1, 0, 2\rangle$, we have

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 1 & 2 & 2 \\ 1 & 0 & 2 \end{pmatrix} = \langle \det\begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}, -\det\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \det\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\rangle = \langle 4, 0, -2\rangle \\
\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) &= \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 1 & 2 & 2 \\ 4 & 0 & -2 \end{pmatrix} = \langle \det\begin{pmatrix} 2 & 2 \\ 0 & -2 \end{pmatrix}, -\det\begin{pmatrix} 1 & 2 \\ 4 & -2 \end{pmatrix}, \det\begin{pmatrix} 1 & 2 \\ 4 & 0 \end{pmatrix}\rangle = \langle -4, 10, -8\rangle
\end{aligned}$$

It follows that

$$\begin{aligned}
\hat{\mathbf{u}}_1 &= \frac{1}{\sqrt{1^2 + 2^2 + 2^2}}\langle 1, 2, 2\rangle = \frac{1}{3}\langle 1, 2, 2\rangle \\
\hat{\mathbf{u}}_2 &= \frac{1}{\sqrt{4^2 + 0^2 + (-2)^2}}\langle 4, 0, -2\rangle = \frac{1}{\sqrt{5}}\langle 2, 0, -1\rangle \\
\hat{\mathbf{u}}_3 &= \frac{1}{\sqrt{(-4)^2 + 10^2 + (-8)^2}}\langle -4, 10, -8\rangle = -\frac{1}{3\sqrt{5}}\langle 2, -5, 4\rangle
\end{aligned}$$

28. Find the area of a quadrilateral $ACDB$ if $A = (1, 0, -1)$, $B = (2, 1, 2)$, $C = (0, 1, 2)$, and $\overrightarrow{AD} = 2\overrightarrow{AC} - \overrightarrow{AB}$.

29. Find the area of a quadrilateral $ABCD$ whose vertices are obtained as follows. The vertex B is the result of moving A by a distance of 6 units along the vector $\mathbf{u} = \langle 2, 1, -2\rangle$, C is obtained from B by moving the latter by a distance of 5 units along the vector $\mathbf{v} = \langle -3, 0, 4\rangle$, and $\overrightarrow{CD} = \mathbf{v} - \mathbf{u}$.

30. Let $\hat{\mathbf{u}}_i$, $i = 1, 2, 3$, be an orthonormal basis in space with the property that $\hat{\mathbf{u}}_3 = \hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_2$. If a_1, a_2, and a_3 are the components of vector $\mathbf{a}$ relative to this basis and b_1, b_2, and b_3 are the components of $\mathbf{b}$, show that the components of the cross product $\mathbf{a} \times \mathbf{b}$ can also be computed by the determinant rule given in Definition **??** where $\hat{\mathbf{e}}_i$ are replaced by $\hat{\mathbf{u}}_i$. *Hint*: Use the "bac-cab" rule to find all pairwise cross products of the basis vectors $\hat{\mathbf{u}}_i$.

SOLUTION: We have that $\mathbf{a} = a_1\hat{\mathbf{u}}_1 + a_2\hat{\mathbf{u}}_2 + a_3\hat{\mathbf{u}}_3$ and $\mathbf{b} = b_1\hat{\mathbf{u}}_1 + b_2\hat{\mathbf{u}}_2 + b_3\hat{\mathbf{u}}_3$. Using the "bac-cab" rule and that $\hat{\mathbf{u}}_3 = \hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_2$, we have

$$\begin{aligned}
\hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_3 &= \hat{\mathbf{u}}_1 \times (\hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_2) \\
&= \hat{\mathbf{u}}_1(\hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_2) - \hat{\mathbf{u}}_2(\hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_1) \\
&= -\hat{\mathbf{u}}_2\|\hat{\mathbf{u}}_1\|^2 \\
&= -\hat{\mathbf{u}}_2
\end{aligned}$$

where I used the fact that $\hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_2 = 0$ since they are mutually orthogonal, and that $\hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_1 = \|\hat{\mathbf{u}}_1\|^2 = 1$, since it is a unit vector.

Similarly, we have that $\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_3 = \hat{\mathbf{u}}_1$. Thus,

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= (a_1\hat{\mathbf{u}}_1 + a_2\hat{\mathbf{u}}_2 + a_3\hat{\mathbf{u}}_3) \times (b_1\hat{\mathbf{u}}_1 + b_2\hat{\mathbf{u}}_2 + b_3\hat{\mathbf{u}}_3) \\
&= a_1\hat{\mathbf{u}}_1 \times (b_1\hat{\mathbf{u}}_1 + b_2\hat{\mathbf{u}}_2 + b_3\hat{\mathbf{u}}_3) + a_2\hat{\mathbf{u}}_2 \times (b_1\hat{\mathbf{u}}_1 + b_2\hat{\mathbf{u}}_2 + b_3\hat{\mathbf{u}}_3) \\
&\quad + a_3\hat{\mathbf{u}}_3 \times (b_1\hat{\mathbf{u}}_1 + b_2\hat{\mathbf{u}}_2 + b_3\hat{\mathbf{u}}_3) \\
&= a_1\hat{\mathbf{u}}_1 \times (b_1\hat{\mathbf{u}}_1) + a_1\hat{\mathbf{u}}_1 \times (b_2\hat{\mathbf{u}}_2) + a_1\hat{\mathbf{u}}_1 \times (b_3\hat{\mathbf{u}}_3) + a_2\hat{\mathbf{u}}_2 \times (b_1\hat{\mathbf{u}}_1) + a_2\hat{\mathbf{u}}_2 \times (b_2\hat{\mathbf{u}}_2) \\
&\quad + a_2\hat{\mathbf{u}}_2 \times (b_3\hat{\mathbf{u}}_3) + a_3\hat{\mathbf{u}}_3 \times (b_1\hat{\mathbf{u}}_1) + a_3\hat{\mathbf{u}}_3 \times (b_2\hat{\mathbf{u}}_2) + a_3\hat{\mathbf{u}}_3 \times (b_3\hat{\mathbf{u}}_3) \\
&= a_1b_1\hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_1 + a_1b_2\hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_2 + a_1b_3\hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_3 + a_2b_1\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_1 + a_2b_2\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_2 \\
&\quad + a_2b_3\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_3 + a_3b_1\hat{\mathbf{u}}_3 \times \hat{\mathbf{u}}_1 + a_3b_2\hat{\mathbf{u}}_3 \times \hat{\mathbf{u}}_2 + a_3b_3\hat{\mathbf{u}}_3 \times \hat{\mathbf{u}}_3 \\
&= a_1b_2\hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_2 + a_1b_3\hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_3 + a_2b_1\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_1 + a_2b_3\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_3 + a_3b_1\hat{\mathbf{u}}_3 \times \hat{\mathbf{u}}_1 + a_3b_2\hat{\mathbf{u}}_3 \times \hat{\mathbf{u}}_2 \\
&= a_1b_2\hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_2 + a_1b_3\hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_3 - a_2b_1\hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_2 + a_2b_3\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_3 - a_3b_1\hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_3 - a_3b_2\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_3 \\
&= a_1b_2\hat{\mathbf{u}}_3 + a_1b_3(-\hat{\mathbf{u}}_2) - a_2b_1\hat{\mathbf{u}}_3 + a_2b_3\hat{\mathbf{u}}_1 - a_3b_1(-\hat{\mathbf{u}}_2) - a_3b_2\hat{\mathbf{u}}_1 \\
&= a_1b_2\hat{\mathbf{u}}_3 - a_1b_3\hat{\mathbf{u}}_2 - a_2b_1\hat{\mathbf{u}}_3 + a_2b_3\hat{\mathbf{u}}_1 + a_3b_1\hat{\mathbf{u}}_2 - a_3b_2\hat{\mathbf{u}}_1 \\
&= (a_2b_3 - a_3b_2)\hat{\mathbf{u}}_1 - (a_1b_3 - a_3b_1)\hat{\mathbf{u}}_2 + (a_1b_2 - a_2b_1)\hat{\mathbf{u}}_3 \\
&= \det\begin{pmatrix} a_2 & a_3 \\ b_2 & b_3 \end{pmatrix}\mathbf{u}_1 - \det\begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix}\hat{\mathbf{u}}_2 + \det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\hat{\mathbf{u}}_3 \\
&= \det\begin{pmatrix} \mathbf{u}_1 & \hat{\mathbf{u}}_2 & \hat{\mathbf{u}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}
\end{aligned}$$

31. Let the angle between the rigid rods in Example **??** be $0 < \varphi < \pi$. Find the equilibrium position of the system.

Solution: Recall that $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$. It may be helpful to sketch the problem, where the rods are pointing down and fixed at some common point. Note that the position vector of the ball with mass m_1, $\mathbf{r}_1$, is coplanar with the gravitational force on that ball, $\mathbf{F}_1 = m_1\mathbf{g}$ (they both lie in the plane that the rods lie in). This means that the torque produced, $\boldsymbol{\tau}_1$ is orthogonal to this plane. The same goes for $\boldsymbol{\tau}_2$, so they are both orthogonal to the same plane. Then $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are coplanar. For the rod system to be in equilibrium, it must be that the sum of its torques must be $\mathbf{0}$. Since the two vectors are coplanar, their magnitudes simply add. By the right hand rule, the two torques are in opposite directions. So, we have that $\tau_1 - \tau_2 = 0 \Rightarrow \tau_1 = \tau_2$.
Recall that $\tau_1 = L_1F_1 \sin\phi_1$ where ϕ_1 is the smallest angle between $\mathbf{r}_1$ and $\mathbf{F}_1$. Sketch these two vectors, noting that the gravitational force points down. Create a horizontal line and define the angle θ_1 to be the smallest angle between $\mathbf{r}_1$ and the horizontal. It should be seen that $\theta_1 + \phi_1 = \pi/2$. Extend $\mathbf{r}_1$ enough to connect $\mathbf{r}_2$ to it,

and extend the horizontal line, so that the horizontal line, $\mathbf{r}_1$, and $\mathbf{r}_2$ form a triangle. It should be that the lower two angles of the triangle are exactly θ_1 and θ_2. As per the problem, we have $\varphi + \theta_1 + \theta_2 = \pi$, so $\theta_1 + \theta_2 = \pi - \varphi$. Finally, using the trigonometric identity $\sin(\pi/2 - \theta) = \cos(\theta)$, we have that $\tau_1 = L_1 F_1 \sin(\phi_1) = L_1 F_1 \cos(\theta_1)$. Similarly, $\tau_2 = L_2 F_2 \cos(\theta_2) = L_2 F_2 \cos(\pi - \varphi - \theta_1)$. Recall that $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$. So $\cos(\pi - \varphi - \theta_1) = \cos(\pi - \varphi)\cos(\theta_1) + \sin(\pi - \varphi)\sin(\theta_1) = -\cos(\varphi)\cos(\theta_1) + \sin(\varphi)\sin(\theta_1)$

Returning to our equilibrium condition, $\tau_1 = \tau_2$ it follows that

$$\begin{aligned}
\tau_1 &= \tau_2 \\
L_1 m_1 g \cos(\theta_1) &= L_2 m_2 g(-\cos(\varphi)\cos(\theta_1) + \sin(\varphi)\sin(\theta_1)) \\
\frac{L_1 m_1}{L_2 m_2} &= -\cos(\varphi) + \sin(\varphi)\tan(\theta_1)
\end{aligned}$$

32. Two rigid rods of the same length are rigidly attached to a ball of mass m so that the angle between the rods is $\pi/2$. A ball of mass $2m$ is attached to one of the free ends of the system. The remaining free end is used to hang the system. Find the angle between the rod connecting the pivot point and the ball of mass m and the vertical axis along which the gravitational force is acting. Assume that the masses of the rods can be neglected as compared to m.

Solution: Construct a triangle OAB in the xy plane such that A is in the third quadrant and B in the fourth, where $|OA| = |AB| = L$, the angle OAB is $\pi/2$, and the smallest angle between OA and the vertical is ϕ. This effectively models the situation, with the ball at mass m at point A and the ball of mass $2m$ at point B. The rigid rods are represented by the line segments OA and AB.
Recall the definition of torque as follows:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

Where $\mathbf{r}$ is the position vector to where the force $\mathbf{F}$ is being applied. There are two torques in the problem, each produced by the gravitational force acting on a ball, $\mathbf{F}_1 = m\mathbf{g}$ and $\mathbf{F_2} = 2m\mathbf{g}$. So $\mathbf{r}$ for each torque will simply be $\overrightarrow{OA}$ or $\overrightarrow{OB}$. Thus we have

$$\|\boldsymbol{\tau}_1\| = \tau_1 = \|\overrightarrow{OA}\|\mathbf{F}_1\| \sin\theta_1 = L(mg)\sin\theta_1$$
$$\|\boldsymbol{\tau}_2\| = \tau_2 = \|\overrightarrow{OB}\|\|\mathbf{F}_2\| \sin\theta_2 = L(2mg)\sin\theta_2$$

where θ_1 and θ_2 are defined to be the smallest angles between $\overrightarrow{OA}$ and $\mathbf{F}_1$ and $\overrightarrow{OB}$ and $\mathbf{F}_2$, respectively. Moreover, the directions of $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are opposite, by the right hand rule. Thus in equilibrium, it follows that

$$\tau_1 - \tau_2 = 0 \ \Leftrightarrow \tau_1 = \tau_2$$

By definition, we have that $\theta_1 = \phi$. This can be seen by moving the end point of $\mathbf{F}_1$ to the origin. Since $\mathbf{F}_1$ is vertical and pointing downwards, the smallest angle is ϕ, which we had defined earlier to be exactly this smallest angle. Next, observe that OAB is a right triangle, where the other two angles are both $\pi/4$. Translate $\mathbf{F}_2$ so its endpoint lands at the origin. It follows that the smallest angle between $\overrightarrow{OB}$ and $\mathbf{F}_2$ is $\theta_2 = \pi/4 - \phi$. Recall that $\sin(A - B) = \sin(A)\cos(B) - \sin(B)\cos(A)$. Therefore,

$$\begin{aligned}
L(mg)\sin\phi &= \sqrt{2}L(2mg)\sin(\pi/4 - \phi) \\
\sin\phi &= 2\sqrt{2}\sin(\pi/4 - \phi) \\
&= 2\sqrt{2}\left(\frac{\sqrt{2}}{2}\cos\phi - \frac{\sqrt{2}}{2}\sin\phi\right) \\
&= 2(\cos\phi - \sin\phi) \\
\frac{1}{2} &= \cot\phi - 1 \\
\frac{3}{2} &= \cot\phi \\
\tan\phi &= \frac{2}{3} \\
\phi &= \arctan\frac{2}{3}
\end{aligned}$$

33. Three rigid rods of the same length are rigidly joined by one end so that the rods lie in a plane and the other end of each rod is free. Let three balls of masses m_1, m_2, and m_3 are attached to the free ends of the rods. The system is hanged by the joining point and can rotate freely about it. Assume that the masses of the rods can be neglected as compared to the masses of the balls. Find the angles between the rods at which the system remains in a horizontal plane under gravitational forces acting vertically.

Solution: Without loss of generality, we can assume that the pivot point is at the origin and one rigid rod is parallel to the negative y-axis. Let this be the rod with mass m_3. This is convenient as it eliminates the torque that would be produced by the ball attached to this rod.

The torque vanishes since the position vector and the gravitational force are both parallel to the negative y-axis. Notice that the solution will be determined by only two angles, as the third can be found from subtracting the sum of the two from 2π.

5. The Triple Product

1–5. Find the triple products $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, $\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})$, and $\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$ for given vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

1. $\mathbf{a}=\langle 1,-1,2\rangle$, $\mathbf{b}=\langle 2,1,2\rangle$, and $\mathbf{c}=\langle 2,1,3\rangle$.

SOLUTION: Recall that the definition of the triple product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ for three vectors $\mathbf{a}=\langle a_1,a_2,a_3\rangle$, $\mathbf{b}=\langle b_1,b_2,b_3\rangle$ and $\mathbf{c}=\langle c_1,c_2,c_3\rangle$ is:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\det\begin{pmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{pmatrix}=a_1\det\begin{pmatrix}b_2 & b_3\\ c_2 & c_3\end{pmatrix}-a_2\det\begin{pmatrix}b_1 & b_3\\ c_1 & c_3\end{pmatrix}+a_3\det\begin{pmatrix}b_1 & b_2\\ c_1 & c_2\end{pmatrix}$$

A fact of linear algebra is that swapping any two rows in a matrix inverts the sign of its determinant. Therefore in the triple product, switching any two vectors inverts the result. Thus $\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})=-\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$. Moreover, switching any two vectors twice will not change the sign. Switching any two vectors twice can be viewed as permuting the vectors in the triple product. Thus $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$. So,

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\det\begin{pmatrix}1 & -1 & 2\\ 2 & 1 & 2\\ 2 & 1 & 3\end{pmatrix}=1\det\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}+1\det\begin{pmatrix}2 & 2\\ 2 & 3\end{pmatrix}+2\det\begin{pmatrix}2 & 1\\ 2 & 1\end{pmatrix}=1(1)+1(2)+2(0)=3$$

Therefore the answers are $3,-3,3$ respectively.

2. $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are, respectively, position vectors of the points $A=(1,2,3)$, $B=(1,-1,1)$, and $C=(2,0,-1)$ relative to the point $O=(1,1,1)$.

SOLUTION: First we must compute $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ as follows:

$$\begin{aligned}\mathbf{a}&=\overrightarrow{OA}=\langle 1,2,3\rangle-\langle 1,1,1\rangle=\langle 0,1,2\rangle\\ \mathbf{b}&=\overrightarrow{OB}=\langle 1,-1,1\rangle-\langle 1,1,1\rangle=\langle 0,-2,0\rangle\\ \mathbf{c}&=\overrightarrow{OC}=\langle 2,0,-1\rangle-\langle 1,1,1\rangle=\langle 1,-1,-2\rangle\end{aligned}$$

Thus we have that

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\det\begin{pmatrix}0 & 1 & 2\\ 0 & -2 & 0\\ 1 & -1 & -2\end{pmatrix}=0\det\begin{pmatrix}-2 & 0\\ -1 & -2\end{pmatrix}-1\det\begin{pmatrix}0 & 0\\ 1 & -2\end{pmatrix}+2\det\begin{pmatrix}0 & -2\\ 1 & -1\end{pmatrix}=4$$

Therefore the answers are $4,-4,4$, respectively.

3. $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are coplanar so that $\mathbf{c} = 2\mathbf{a} - 3\mathbf{b}$.

SOLUTION: Note that if $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are coplanar then $\mathbf{b}\times\mathbf{c}$ is not only orthogonal to $\mathbf{b}$ and $\mathbf{c}$, but also $\mathbf{a}$, as they all lie in the same plane and $\mathbf{b}\times\mathbf{c}$ is orthogonal to this plane. Therefore the dot product between them vanishes, and the triple product is 0.

4. $\mathbf{a} = \mathbf{u}_1 + 2\mathbf{u}_2$, $\mathbf{b} = \mathbf{u}_1 - \mathbf{u}_2 + 2\mathbf{u}_3$, and $\mathbf{c} = \mathbf{u}_2 - 3\mathbf{u}_3$ if $\mathbf{u}_1 \cdot (\mathbf{u}_2 \times \mathbf{u}_3) = 2$

SOLUTION: We have that

$$\begin{aligned}
\mathbf{b}\times\mathbf{c} &= (\mathbf{u}_1 - \mathbf{u}_2 + 2\mathbf{u}_3) \times (\mathbf{u}_2 - 3\mathbf{u}_3)\\
&= \mathbf{u}_1 \times (\mathbf{u}_2 - 3\mathbf{u}_3) - \mathbf{u}_2 \times (\mathbf{u}_2 - 3\mathbf{u}_3) + 2\mathbf{u}_3 \times (\mathbf{u}_2 - 3\mathbf{u}_3)\\
&= \mathbf{u}_1 \times \mathbf{u}_2 - 3\mathbf{u}_1 \times \mathbf{u}_3 - \mathbf{u}_2 \times \mathbf{u}_2 + 3\mathbf{u}_2 \times \mathbf{u}_3 + 2\mathbf{u}_3 \times \mathbf{u}_2 - 6\mathbf{u}_3 \times \mathbf{u}_3\\
&= \mathbf{u}_1 \times \mathbf{u}_2 - 3\mathbf{u}_1 \times \mathbf{u}_3 + 3\mathbf{u}_2 \times \mathbf{u}_3 + 2\mathbf{u}_3 \times \mathbf{u}_2\\
&= \mathbf{u}_1 \times \mathbf{u}_2 - 3\mathbf{u}_1 \times \mathbf{u}_3 + \mathbf{u}_2 \times \mathbf{u}_3\\
&= \mathbf{u}_1 \times (\mathbf{u}_2 - 3\mathbf{u}_3) + \mathbf{u}_2 \times \mathbf{u}_3
\end{aligned}$$

And so,

$$\begin{aligned}
\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= (\mathbf{u}_1 + 2\mathbf{u}_2) \cdot (\mathbf{u}_1 \times (\mathbf{u}_2 - 3\mathbf{u}_3) + \mathbf{u}_2 \times \mathbf{u}_3)\\
&= \mathbf{u}_1 \cdot (\mathbf{u}_1 \times (\mathbf{u}_2 - 3\mathbf{u}_3) + \mathbf{u}_2 \times \mathbf{u}_3) + 2\mathbf{u}_2 \cdot (\mathbf{u}_1 \times (\mathbf{u}_2 - 3\mathbf{u}_3) + \mathbf{u}_2 \times \mathbf{u}_3)\\
&= \mathbf{u}_1 \cdot (\mathbf{u}_1 \times (\mathbf{u}_2 - 3\mathbf{u}_3)) + \mathbf{u}_1 \cdot (\mathbf{u}_2 \times \mathbf{u}_3) + 2\mathbf{u}_2 \cdot (\mathbf{u}_1 \times (\mathbf{u}_2 - 3\mathbf{u}_3)) + 2\mathbf{u}_2 \cdot (\mathbf{u}_2 \times \mathbf{u}_3)\\
&= \mathbf{u}_1 \cdot (\mathbf{u}_2 \times \mathbf{u}_3) + 2\mathbf{u}_2 \cdot (\mathbf{u}_1 \times (\mathbf{u}_2 - 3\mathbf{u}_3))\\
&= \mathbf{u}_1 \cdot (\mathbf{u}_2 \times \mathbf{u}_3) + 2\mathbf{u}_2 \cdot (\mathbf{u}_1 \times \mathbf{u}_2) - 6\mathbf{u}_2 \cdot (\mathbf{u}_1 \times \mathbf{u}_3)\\
&= \mathbf{u}_1 \cdot (\mathbf{u}_2 \times \mathbf{u}_3) - 6\mathbf{u}_2 \cdot (\mathbf{u}_1 \times \mathbf{u}_3)\\
&= \mathbf{u}_1 \cdot (\mathbf{u}_2 \times \mathbf{u}_3) + 6\mathbf{u}_1 \cdot (\mathbf{u}_2 \times \mathbf{u}_3)\\
&= 7\mathbf{u}_1 \cdot (\mathbf{u}_2 \times \mathbf{u}_3)
\end{aligned}$$

Therefore the answers are $14, -14, 14$, respectively.

5. $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are pairwise perpendicular and $\|\mathbf{a}\| = 1$, $\|\mathbf{b}\| = 2$, and $\|\mathbf{c}\| = 3$. Is the answer unique under the specified conditions?

SOLUTION: Recall that the absolute value of the triple product gives the volume of the parallelpiped generated by the three vectors. In this case, since they are all pairwise perpendicular, the volume is simply $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| = \|\mathbf{a}\|\|\mathbf{b}\|\|\mathbf{c}\| = 1(2)(3) = 6$.

Therefore $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \pm 6$. The answer is not unique; it is either $6, -6, 6$ or $-6, 6, -6$, respectively.

6. Verify whether the vectors $\mathbf{a} = \hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3$, $\mathbf{b} = 2\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$, and $\mathbf{c} = 3\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 - 2\hat{\mathbf{e}}_3$ are coplanar.

SOLUTION: By the geometric properties of the triple product, three vectors are coplanar iff their triple product vanishes. Thus we compute the triple product as follows:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \det\begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & 1 & -2 \end{pmatrix} = \det\begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix} - 2\det\begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix} - \det\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} = 1+14-5 = 10$$

So they are not coplanar.

7. Consider the vectors $\mathbf{a} = \langle 1,2,3\rangle$, $\mathbf{b} = \langle -1,0,1\rangle$ and $\mathbf{c} = \langle s,1,2s\rangle$ where s is a number.
(i) Find all values of s, if any, for which these vectors are coplanar.
(ii) If such s exists, find the area of the quadrilateral whose three vertices have position vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ relative to the fourth vertex.
Hint: Determine which of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is a diagonal of the quadrilateral.

SOLUTION: (i) By the geometric properties of the triple product, three vectors are coplanar iff their triple product vanishes. Thus we must compute the triple product as follows:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \det\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ s & 1 & 2s \end{pmatrix} = \det\begin{pmatrix} 0 & 1 \\ 1 & 2s \end{pmatrix} - 2\det\begin{pmatrix} -1 & 1 \\ s & 2s \end{pmatrix} + 3\det\begin{pmatrix} -1 & 0 \\ s & 1 \end{pmatrix} = -1+6s-3 = 6s-4$$

For the triple product to vanish then, it follows that $s = \frac{2}{3}$.
(ii) Evidently the diagonal will be the vector whose magnitude is the largest. It is easy to see that this is $\mathbf{a}$ by comparing its components to those of $\mathbf{b}$ and $\mathbf{c}$. So the area of the quadrilateral will be

$$\begin{aligned} A &= \|\mathbf{b}\times\mathbf{c}\| \\ &= \|\det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ -1 & 0 & 1 \\ \frac{2}{3} & 1 & \frac{4}{3} \end{pmatrix}\| \\ &= \|\langle \det\begin{pmatrix} 0 & 1 \\ 1 & \frac{4}{3} \end{pmatrix}, -\det\begin{pmatrix} -1 & 1 \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix}, \det\begin{pmatrix} -1 & 0 \\ \frac{2}{3} & 1 \end{pmatrix}\rangle\| \\ &= \|\langle -1, 2, -1\rangle\| = \sqrt{6} \end{aligned}$$

8. Determine whether the points $A = (1, 2, 3)$, $B = (1, 0, 1)$, $C = (-1, 1, 2)$, and $D = (-2, 1, 0)$ are in one plane and, if not, find the volume of the parallelepiped with adjacent edges AB, AC, and AD.

Solution: First compute $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$.

$$\begin{aligned}
\overrightarrow{AB} &= \langle 1, 0, 1\rangle - \langle 1, 2, 3\rangle = \langle 0, -2, -2\rangle \\
\overrightarrow{AC} &= \langle -1, 1, 2\rangle - \langle 1, 2, 3\rangle = \langle -2, -1, -1\rangle \\
\overrightarrow{AD} &= \langle -2, 1, 0\rangle - \langle 1, 2, 3\rangle = \langle -3, -1, -3\rangle
\end{aligned}$$

By the geometric properties of the triple product, the triple product gives the volume of the parallelpiped spanned by three vectors. Thus we need only compute the triple product to answer the question.

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \det \begin{pmatrix} 0 & -2 & -2 \\ -2 & -1 & -1 \\ -3 & -1 & -3 \end{pmatrix} = 0 \det \begin{pmatrix} -1 & -1 \\ -1 & -3 \end{pmatrix} + 2 \det \begin{pmatrix} -2 & -1 \\ -3 & -3 \end{pmatrix} - 2 \det \begin{pmatrix} -2 & -1 \\ -3 & -1 \end{pmatrix} \\
&= 2(3) - 2(-1) = 8
\end{aligned}$$

Thus the points are not in one plane, and the volume of the parallelpiped is $V = 8$.

9. Find:

(i) all values of s at which the points $A = (s, 0, s)$, $B = (1, 0, 1)$, $C = (s, s, 1)$, and $D = (0, 1, 0)$ are in the same plane;

(ii) all values of s at which the volume of the parallelepiped with adjacent edges AB, AC, and AD is 9 units.

Solution: Let $\mathbf{a} = \overrightarrow{AB} = \langle 1 - s, 0, 1 - s\rangle$, $\mathbf{b} = \overrightarrow{AC} = \langle 0, s, 1 - s\rangle$, and $\mathbf{c} = \overrightarrow{AD} = \langle -s, 1, -s\rangle$. Then the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are coplanar (and the points in question are in one plane) if and only if their triple product vanishes. So the points in question are in one plane at the values of s at which the triple vanishes (Part (i)). Furthermore, the absolute value of the triple product is the volume of the parallelepiped in question. The latter condition give an equation for s to be solved (Part (ii)). One has

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \det \begin{pmatrix} 1 - s & 0 & 1 - s \\ 0 & s & 1 - s \\ -s & 1 & -s \end{pmatrix} \\
&= (1 - s)[-s^2 - (1 - s)] + (1 - s)[0 + s^2] \\
&= (1 - s)[-s^2 - (1 - s) + s^2] = -(1 - s)^2
\end{aligned}$$

The triple product vanishes at $s = 1$ and, hence, A, B, C, and D are in one plane when $s = 1$. The volume is

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = |-(1-s)^2| = (s-1)^2 .$$

The condition $V = 9$ yields

$$(s-1)^2 = 9 \quad \Rightarrow \quad s - 1 = \pm 3 \quad \Rightarrow \quad s = 1 \pm 3 .$$

The volume of the parallelepiped is 9 units when $s = 4$ or $s = -2$.

10. Let $\mathbf{a} = \langle 1, 2, 3\rangle$, $\mathbf{b} = \langle 2, 1, 0\rangle$, and $\mathbf{c} = \langle 3, 0, 1\rangle$. Find the volume of the parallelepiped with adjacent sides $s\mathbf{a}+\mathbf{b}$, $\mathbf{c}-t\mathbf{b}$, and $\mathbf{a}-p\mathbf{c}$ if the numbers s, t, and p satisfy the condition $stp = 1$.

Solution: By the geometric properties of the triple product, we know its absolute value gives the volume of the parallelpiped spanned by the three vectors. We first use the properties of the dot and cross product to help simplify the problem:

$$\begin{aligned}
V &= |(s\mathbf{a}+\mathbf{b}) \cdot ((\mathbf{c}-t\mathbf{b}) \times (\mathbf{a}-p\mathbf{c}))| = |(s\mathbf{a}+\mathbf{b}) \cdot (\mathbf{c} \times (\mathbf{a}-p\mathbf{c}) - t\mathbf{b} \times (\mathbf{a}-p\mathbf{c}))| \\
&= |(s\mathbf{a}+\mathbf{b}) \cdot (\mathbf{c} \times \mathbf{a} - p\mathbf{c} \times \mathbf{c} - t\mathbf{b} \times (\mathbf{a}-p\mathbf{c}))| = |(s\mathbf{a}+\mathbf{b}) \cdot (\mathbf{c} \times \mathbf{a} - t\mathbf{b} \times (\mathbf{a}-p\mathbf{c}))| \\
&= |(s\mathbf{a} \cdot (\mathbf{c} \times \mathbf{a} - t\mathbf{b} \times (\mathbf{a}-p\mathbf{c})) + \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a} - t\mathbf{b} \times (\mathbf{a}-p\mathbf{c}))| \\
&= |(s\mathbf{a} \cdot (\mathbf{c} \times \mathbf{a} - t(\mathbf{b} \times \mathbf{a}) + tp(\mathbf{b} \times \mathbf{c})) + \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) - t\mathbf{b} \cdot (\mathbf{b} \times (\mathbf{a}-p\mathbf{c}))| \\
&= |s\mathbf{a} \cdot (\mathbf{c} \times \mathbf{a}) - st\mathbf{a} \cdot (\mathbf{b} \times \mathbf{a}) + stp\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})) + \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| \\
&= |(stp+1)||\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|
\end{aligned}$$

So we must calculate $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ now:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \det\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 2\det\begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} + 3\det\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} = 1 - 2(2) + 3(-3) = -12$$

Thus $V = |stp+1||-12| = 24$.

11. Let the numbers u, v, and w be such that $uvw = 1$ and $u^3 + v^3 + w^3 = 1$. Are the vectors $\mathbf{a} = u\hat{\mathbf{e}}_1 + v\hat{\mathbf{e}}_2 + w\hat{\mathbf{e}}_3$, $\mathbf{b} = v\hat{\mathbf{e}}_1 + w\hat{\mathbf{e}}_2 + u\hat{\mathbf{e}}_3$, and $\mathbf{c} = w\hat{\mathbf{e}}_1 + u\hat{\mathbf{e}}_2 + v\hat{\mathbf{e}}_3$ coplanar? If not, what is the volume of the parallelepiped with adjacent edges $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$?

SOLUTION: We have that

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \det\begin{pmatrix} u & v & w\\ v & w & u\\ w & u & v\end{pmatrix} = u\det\begin{pmatrix} w & u\\ u & v\end{pmatrix} - v\det\begin{pmatrix} v & u\\ w & v\end{pmatrix} + w\det\begin{pmatrix} v & w\\ w & u\end{pmatrix}$$
$$= u(vw-u^2) - v(v^2-uw) + w(uv-w^2) = uvw - u^3 - v^3 + uvw + uvw - w^3$$
$$= 3uvw - (u^3+v^3+w^3) = 3-1 = 2$$

Thus the vectors are not coplanar

12. Prove that

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \det\begin{pmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c}\\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d}\end{pmatrix}.$$

Hint: Put $\mathbf{n} = \mathbf{a}\times\mathbf{b}$. Use the invariance of the triple product under cyclic permutations of vectors in it and the "bac-cab" rule (**??**).

SOLUTION: Let $\mathbf{n} = \mathbf{a}\times\mathbf{b}$. Then, by the invariance of the triple product under cyclic permutations of its vectors, we have

$$\mathbf{n}\cdot(\mathbf{c}\times\mathbf{d}) = \mathbf{c}\cdot(\mathbf{d}\times\mathbf{n}) = \mathbf{c}\cdot(\mathbf{d}\times(\mathbf{a}\times\mathbf{b}))$$

Using the "bac-cab" rule to evaluate $\mathbf{d}\times(\mathbf{a}\times\mathbf{b})$ yields

$$\mathbf{c}\cdot(\mathbf{d}\times(\mathbf{a}\times\mathbf{b})) = \mathbf{c}\cdot(\mathbf{a}(\mathbf{d}\cdot\mathbf{b})-\mathbf{b}(\mathbf{d}\cdot\mathbf{a})) = (\mathbf{c}\cdot\mathbf{a})(\mathbf{d}\cdot\mathbf{b})-(\mathbf{c}\cdot\mathbf{b})(\mathbf{d}\cdot\mathbf{a}) = \det\begin{pmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c}\\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d}\end{pmatrix}$$

This completes the proof.

13. Let P be a parallelepiped of volume V. Find the volumes of all parallelepipeds whose adjacent edges are diagonals of the adjacent faces of P.

SOLUTION: Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be the adjacent sides of P. Then $V = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$. The adjacent faces of P are parallelograms with adjacent sides being the pairs of these vectors. Therefore the diagonals of the faces are $\mathbf{d}_1 = \mathbf{a}+\mathbf{b}$, $\mathbf{d}_2 = \mathbf{a}+\mathbf{c}$, and $\mathbf{d}_3 = \mathbf{b}+\mathbf{c}$. Using the basic algebraic properties of the dot, cross, and triple products, the volume in question is

$$\begin{aligned}
V' &= |\mathbf{d}_1 \cdot (\mathbf{d}_2 \times \mathbf{d}_3)| \\
&= |\mathbf{d}_1 \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} + \mathbf{c} \times \mathbf{b} + \mathbf{c} \times \mathbf{c})| \\
&= |(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} + \mathbf{c} \times \mathbf{b} + \mathbf{0})| \\
&= |\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) + \mathbf{a} \cdot (\mathbf{a} \times \mathbf{c}) + \mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) + \mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) + \mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) + \mathbf{b} \cdot (\mathbf{c} \times \mathbf{b})| \\
&= |0 + 0 + \mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) + 0 + \mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) + 0| \\
&= |-\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) - \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| \\
&= 2|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = 2V\,.
\end{aligned}$$

14. Let P be a parallelepiped of volume V. Find the volumes of all parallelepipeds whose two adjacent edges are diagonals of two non-parallel faces of P, while the third adjacent edge is a diagonal of P (the segment connecting two vertices of P that does not lie in a face of P).

Solution: Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be the adjacent edges of P. First, the diagonals of all the faces are:
Face 1,2: $\mathbf{d}_1 = \mathbf{a} + \mathbf{b}$ Face 3,4: $\mathbf{d}_2 = \mathbf{b} + \mathbf{c}$ Face 5,6: $\mathbf{d}_3 = \mathbf{a} + \mathbf{c}$.

Next, the diagonals of P are $\mathbf{D}_1 = \mathbf{a} + \mathbf{b} + \mathbf{c}$, $\mathbf{D}_2 = \mathbf{a} - \mathbf{b} + \mathbf{c}$, $\mathbf{D}_3 = \mathbf{a} + \mathbf{b} - \mathbf{c}$, and $\mathbf{D}_4 = -\mathbf{a} + \mathbf{b} + \mathbf{c}$.

Let V_{ijk} be the volume of the parallelpiped formed from diagonals $\mathbf{d}_i, \mathbf{d}_j$, and $\mathbf{D}_k$. Notice that the only difference between each $\mathbf{D}_k$ is the inclusion of a single minus sign and its location. To save time with computation, I will generally call $\mathbf{D}_k = s_{ka}\mathbf{a} + s_{kb}\mathbf{b} + s_{kc}\mathbf{c}$, where each s is either a $+$ or a $-$, with a maximum of one $-$. Furthermore, notice that each $\mathbf{d}_i$ can be written as $s_{ia}\mathbf{a} + s_{ib}\mathbf{b} + s_{ic}\mathbf{c}$, where each s is either a $+$ or a 0, with a maximum of one 0. Thus, for the parallelpipeds formed from $\mathbf{d}_i$, $\mathbf{d}_j$, and $\mathbf{D}_k$ we have

$$\begin{aligned}
V_{ijk} &= \mathbf{d}_i \cdot ((s_{ja}\mathbf{a} + s_{jb}\mathbf{b} + s_{jc}\mathbf{c}) \times (s_{ka}\mathbf{a} + s_{kb}\mathbf{b} + s_{kc}\mathbf{c})) \\
&= \mathbf{d}_i \cdot (s_{ja}\mathbf{a} \times (s_{ka}\mathbf{a} + s_{kb}\mathbf{b} + s_{kc}\mathbf{c}) + s_{jb}\mathbf{b} \times (s_{ka}\mathbf{a} + s_{kb}\mathbf{b} + s_{kc}\mathbf{c}) \\
&\qquad + s_{jc}\mathbf{c} \times (s_{ka}\mathbf{a} + s_{kb}\mathbf{b} + s_{kc}\mathbf{c})) \\
&= \mathbf{d}_i \cdot (s_{ja}s_{ka}\mathbf{a} \times \mathbf{a} + s_{ja}s_{kb}\mathbf{a} \times \mathbf{b} + s_{ja}s_{kc}\mathbf{a} \times \mathbf{c} + s_{jb}s_{ka}\mathbf{b} \times \mathbf{a} + s_{jb}s_{kb}\mathbf{b} \times \mathbf{b} + s_{jb}s_{kc}\mathbf{b} \times \mathbf{c} \\
&\qquad + s_{jc}s_{ka}\mathbf{c} \times \mathbf{a} + s_{jc}s_{kb}\mathbf{c} \times \mathbf{b} + s_{jc}s_{kc}\mathbf{c} \times \mathbf{c}) \\
&= \mathbf{d}_i \cdot (s_{ja}s_{kb}\mathbf{a} \times \mathbf{b} + s_{ja}s_{kc}\mathbf{a} \times \mathbf{c} + s_{jb}s_{ka}\mathbf{b} \times \mathbf{a} + s_{jb}s_{kc}\mathbf{b} \times \mathbf{c} + s_{jc}s_{ka}\mathbf{c} \times \mathbf{a} + s_{jc}s_{kb}\mathbf{c} \times \mathbf{b})
\end{aligned}$$

$$\begin{aligned}
V_{ijk} &= (s_{ia}\mathbf{a} + s_{ib}\mathbf{b} + s_{ic}\mathbf{c}) \cdot ((s_{ja}s_{kb} - s_{jb}s_{ka})\mathbf{a} \times \mathbf{b} + (s_{ja}s_{kc} - s_{jc}s_{ka})\mathbf{a} \times \mathbf{c} \\
&\quad + (s_{jb}s_{kc} - s_{jc}s_{kb})\mathbf{b} \times \mathbf{c}) \\
&= s_{ia}\mathbf{a} \cdot ((s_{ja}s_{kb} - s_{jb}s_{ka})\mathbf{a} \times \mathbf{b} + (s_{ja}s_{kc} - s_{jc}s_{ka})\mathbf{a} \times \mathbf{c} + (s_{jb}s_{kc} - s_{jc}s_{kb})\mathbf{b} \times \mathbf{c}) \\
&\quad + s_{ib}\mathbf{b} \cdot ((s_{ja}s_{kb} - s_{jb}s_{ka})\mathbf{a} \times \mathbf{b} + (s_{ja}s_{kc} - s_{jc}s_{ka})\mathbf{a} \times \mathbf{c} + (s_{jb}s_{kc} - s_{jc}s_{kb})\mathbf{b} \times \mathbf{c}) \\
&\quad + s_{ic}\mathbf{c} \cdot ((s_{ja}s_{kb} - s_{jb}s_{ka})\mathbf{a} \times \mathbf{b} + (s_{ja}s_{kc} - s_{jc}s_{ka})\mathbf{a} \times \mathbf{c} + (s_{jb}s_{kc} - s_{jc}s_{kb})\mathbf{b} \times \mathbf{c}) \\
&= s_{ia}(s_{jb}s_{kc} - s_{jc}s_{kb})\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + s_{ib}(s_{ja}s_{kc} - s_{jc}s_{ka})\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) \\
&\quad + s_{ic}((s_{ja}s_{kb} - s_{jb}s_{ka})\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \\
&= s_{ia}(s_{jb}s_{kc} - s_{jc}s_{kb})\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) - s_{ib}(s_{ja}s_{kc} - s_{jc}s_{ka})\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \\
&\quad + s_{ic}((s_{ja}s_{kb} - s_{jb}s_{ka})\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \\
&= (s_{ia}(s_{jb}s_{kc} - s_{jc}s_{kb}) - s_{ib}(s_{ja}s_{kc} - s_{jc}s_{ka}) + s_{ic}((s_{ja}s_{kb} - s_{jb}s_{ka}))\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \\
&= \det \begin{pmatrix} s_{ia} & s_{ib} & s_{ic} \\ s_{ja} & s_{jb} & s_{jc} \\ s_{ka} & s_{kb} & s_{kc} \end{pmatrix} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det \begin{pmatrix} s_{ia} & s_{ib} & s_{ic} \\ s_{ja} & s_{jb} & s_{jc} \\ s_{ka} & s_{kb} & s_{kc} \end{pmatrix} V
\end{aligned}$$

Now it is simply a matter of doing all the computations, substituting all the proper entries into the matrix. As an example, here is the computation of V_{121}

$$V_{121} = \det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} V = (\det \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \det \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + 0 \det \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix})V = (0 - (-1) + 0)V = V$$

All the possible volumes are as follows:

$$\begin{array}{llll}
V_{121} = V & V_{122} = 3V & V_{123} = V & V_{124} = V \\
V_{131} = V & V_{132} = V & V_{133} = V & V_{134} = 3V \\
V_{231} = V & V_{232} = V & V_{233} = 3V & V_{234} = V
\end{array}$$

15. Given two non-parallel vectors $\mathbf{a}$ and $\mathbf{b}$, find the most general vector $\mathbf{r}$ that satisfies the conditions $\mathbf{a} \cdot (\mathbf{r} \times \mathbf{b}) = 0$ and $\mathbf{b} \cdot \mathbf{r} = 0$.

SOLUTION: Using the invariance of the triple product under cyclic permutations of its vectors, we have that $\mathbf{a} \cdot (\mathbf{r} \times \mathbf{b}) = \mathbf{r} \cdot (\mathbf{b} \times \mathbf{a}) = 0$. It follows that r is perpendicular to both $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b}$. The first condition implies that $\mathbf{r}$ lies in the plane spanned by $\mathbf{a}$ and $\mathbf{b}$, and the latter condition requires that r is orthogonal to $\mathbf{b}$. By the first condition, $\mathbf{r} = s\mathbf{a} + t\mathbf{b}$ for any real numbers s, t. Adding the second condition requires that $s(\mathbf{a}\cdot\mathbf{b}) + t\|\mathbf{b}\|^2 = 0 \ \Leftrightarrow\ t = -\frac{s(\mathbf{a}\cdot\mathbf{b})}{\|\mathbf{b}\|^2}$. Thus

$$\mathbf{r} = s\mathbf{a} - \frac{s(\mathbf{a}\cdot\mathbf{b})}{\|\mathbf{b}\|^2}\mathbf{b}$$

for any real number s is the most general vector $\mathbf{r}$ satisfying the conditions.

16. Let a set $\mathcal{S}_1$ be the circle $x^2 + y^2 = 1$ and let a set $\mathcal{S}_2$ be the line through the points $(0, 2)$ and $(2, 0)$. Find the distance between the sets $\mathcal{S}_1$ and $\mathcal{S}_2$.

SOLUTION: Recall that the distance D between two sets $\mathcal{S}_1$ and $\mathcal{S}_2$ is $\inf |A_1A_2|$, where $A_1 \in \mathcal{S}_1$ and $A_2 \in \mathcal{S}_2$. In this scenario, we must find two points A_1 and A_2 such that the line segment A_1A_2 is perpendicular to both sets. Since $\mathcal{S}_1$ is a circle, any straight line through its center (here, the origin) that intersects it will be perpendicular to it. The slope of the line passing through $(0, 2)$ and $(2, 0)$ is $m = \frac{2-0}{0-2} = -1$. Thus a line perpendicular to it will have slope $M = -\frac{1}{m} = 1$. So a straight line perpendicular to both sets must pass through the origin and have slope $M = 1$. There is only one such line, $y = x$, that passes through $\mathcal{S}_1$ at $(\sqrt{2}/2, \sqrt{2}/2)$ and $\mathcal{S}_2$ at $(1, 1)$. Thus the distance is

$$D = \sqrt{(1-\sqrt{2}/2)^2 + (1-\sqrt{2}/2)^2} = \sqrt{2(1-\sqrt{2}/2)^2} = \sqrt{(\sqrt{2}-1)^2} = \sqrt{2}-1$$

17. Consider a plane through three points $A = (1, 2, 3)$, $B = (2, 3, 1)$, and $C = (3, 1, 2)$. Find the distance between the plane and a point P obtained from A by moving the latter 3 units along a straight line segment parallel to the vector $\mathbf{a} = \langle -1, 2, 2\rangle$.

SOLUTION: The distance can be interpreted as the distance between two parallel planes. One plane goes through A, B, and C. The other plane is parallel to this and passes through P. Recall that the distance between two planes is

$$D = \frac{|\overrightarrow{AP}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})|}{\|\overrightarrow{AB}\times\overrightarrow{AC}\|}$$

First we compute $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AP}$ as follows:

$$\begin{aligned}
\overrightarrow{AB} &= \langle 2,3,1\rangle - \langle 1,2,3\rangle = \langle 1,1,-2\rangle \\
\overrightarrow{AC} &= \langle 3,1,2\rangle - \langle 1,2,3\rangle = \langle 2,-1,-1\rangle \\
\overrightarrow{AP} &= \frac{3}{\|\mathbf{a}\|}\,\mathbf{a} = \frac{3}{\sqrt{1+4+4}}\,\mathbf{a} = \mathbf{a}
\end{aligned}$$

It follows that $\overrightarrow{AB} \times \overrightarrow{AC}$ is

$$\overrightarrow{AB}\times\overrightarrow{AC} = \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 1 & 1 & -2 \\ 2 & -1 & -1 \end{pmatrix} = \langle \det\begin{pmatrix} 1 & -2 \\ -1 & -1 \end{pmatrix}, -\det\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}, \det\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\rangle = \langle -3,-3,-3\rangle$$

Thus $\|\overrightarrow{AB} \times \overrightarrow{AC}\|$ is

$$\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \|\langle -3,-3,-3\rangle\| = \|-3\langle 1,1,1\rangle\| = 3\|\langle 1,1,1\rangle\| = 3\sqrt{3}$$

Furthermore we have that $|\overrightarrow{AP} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})|$ is

$$|\overrightarrow{AP}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})| = |\langle -1,2,2\rangle\cdot\langle -3,-3,-3\rangle| = |-1(-3)+2(-3)+2(-3)| = 9$$

Thus the distance between the plane and the point P is

$$D = \frac{9}{3\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

18. Consider two lines. The first line passes through the points $(1,2,3)$ and $(2,-1,1)$, while the other passes through the points $(-1,3,1)$ and $(1,1,3)$. Find the distance between the lines.

Solution: First we must determine if the two lines are skew, parallel, coincident, or incident. Set $A = (1,2,3)$, $B = (2,-1,1)$, $C = (-1,3,1)$, and $P = (1,1,3)$. Recall that two lines are skew iff

$$\overrightarrow{AC} \cdot (\overrightarrow{AB} \times \overrightarrow{CP}) \neq 0$$

where A and B are distinct points on line $\mathcal{L}_1$ and C and D are distinct points on line $\mathcal{L}_2$. We compute each of $\overrightarrow{AC}$, $\overrightarrow{AB}$, and $\overrightarrow{CP}$ as follows:

$$\begin{aligned}
\overrightarrow{AC} &= \langle -1,3,1\rangle - \langle 1,2,3\rangle = \langle -2,1,-2\rangle \\
\overrightarrow{AB} &= \langle 2,-1,1\rangle - \langle 1,2,3\rangle = \langle 1,-3,-2\rangle \\
\overrightarrow{CP} &= \langle -1,3,1\rangle - \langle 1,1,3\rangle = \langle -2,2,-2\rangle
\end{aligned}$$

It follows that $\overrightarrow{AB} \times \overrightarrow{CP}$ is

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{CP} &= \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 1 & -3 & -2 \\ -2 & 2 & -2 \end{pmatrix} = \langle \det\begin{pmatrix} -3 & -2 \\ 2 & -2 \end{pmatrix}, -\det\begin{pmatrix} 1 & -2 \\ -2 & -2 \end{pmatrix}, \det\begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix} \rangle \\ &= \langle 10, 6, -4 \rangle\end{aligned}$$

Thus $\overrightarrow{AB} \cdot (\overrightarrow{AB} \times \overrightarrow{CP})$ is

$$\overrightarrow{AB} \cdot (\overrightarrow{AB} \times \overrightarrow{CP}) = \langle -2, 1, -2 \rangle \cdot \langle 10, 6, -4 \rangle = -2(10) + 1(6) - 2(-4) = -20 + 6 + 8 = -20 + 14 = -6$$

So the lines are skew. We therefore use the formula for the distance between skew lines, which is as follows:

$$D = \frac{|\overrightarrow{AB} \cdot (\overrightarrow{AB} \times \overrightarrow{CP})|}{\|\overrightarrow{AB} \times \overrightarrow{CP}\|}$$

We must calculate $\|\overrightarrow{AB} \times \overrightarrow{CP}\|$, which is

$$\|\overrightarrow{AB} \times \overrightarrow{CP}\| = \sqrt{10^2 + 6^2 + (-4)^2} = \sqrt{100 + 36 + 16} = \sqrt{152} = 2\sqrt{38}$$

It follows that the distance is

$$D = \frac{|-6|}{2\sqrt{38}} = \frac{3}{\sqrt{38}}$$

19. Find the distance between the line through the points $(1, 2, 3)$ and $(2, 1, 4)$ and the plane through the points $(1, 1, 1)$, $(3, 1, 2)$, and $(1, 2, -1)$. *Hint:* If the line is not parallel to the plane, then they intersect and the distance is 0. So check first whether the line is parallel to the plane. How can this be done?

SOLUTION: Let $\mathbf{r}$ be a vector parallel to a line $\mathcal{L}$. Observe that if $\mathcal{L}$ is parallel to the plane $\mathcal{P}$, for every $O \in \mathcal{P}$ there exists a $P \in \mathcal{P}$ such that $\overrightarrow{OP} = \mathbf{r}$. It is sufficient to check this for only one such O. This can be interpreted in the following way: Take a point in the plane, and try to make $\mathbf{r}$ from it. If the vector is not in the plane, then clearly it simply passes through, and it is not parallel.

First we compute $\mathbf{r}$ using the points given in the line

$$\mathbf{r} = \langle 2, 1, 4 \rangle - \langle 1, 2, 3 \rangle = \langle 1, -1, 1 \rangle$$

Let $O = (1, 1, 1)$, $A = (3, 1, 2)$, and $B = (1, 2, -1)$. Then

$$\begin{aligned}\overrightarrow{OA} &= \langle 3, 1, 2 \rangle - \langle 1, 1, 1 \rangle = \langle 2, 0, 1 \rangle \\ \overrightarrow{OB} &= \langle 1, 2, -1 \rangle - \langle 1, 1, 1 \rangle = \langle 0, 1, -2 \rangle\end{aligned}$$

Suppose there exists a $P \in \mathcal{P}$ such that $\overrightarrow{OP} = \mathbf{r}$. Then P is in the plane iff $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OP}$ are coplanar. Thus we compute the triple product

$$\overrightarrow{OP} \cdot (\overrightarrow{OA} \times \overrightarrow{OB}) = \det \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix} = (1)\det \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} - (-1)\det \begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix} + (1)\det \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= -1 + (-4) + 2 = -3$$

So the triple product does not vanish. It follows that the vectors are not coplanar, hence no P exists such that $\overrightarrow{OP} = \mathbf{r}$. Thus $\mathcal{L}$ is not parallel to $\mathcal{P}$. So the distance is 0.

20. Consider the line through the points $(1, 2, 3)$ and $(2, 1, 2)$. If a second line passes through the points $(1, 1, s)$ and $(2, -1, 0)$, find all values of s, if any, at which the distance between the lines is $3/2$ units.

Solution: Let $A = (1, 2, 3)$, $B = (2, 1, 2)$, $C = (1, 1, s)$, and $P = (2, -1, 0)$. Then

$$\overrightarrow{AB} = \langle 2, 1, 2\rangle - \langle 1, 2, 3\rangle = \langle 1, -1, -1\rangle$$
$$\overrightarrow{CP} = \langle 2, -1, 0\rangle - \langle 1, 1, s\rangle = \langle 1, -2, -s\rangle$$

It is clear that there does not exist a real number s such that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CP}$. Thus the lines will always be skew. It follows that we can use the formula for the distance between skew lines, which reads

$$D = \frac{|\overrightarrow{AC} \cdot (\overrightarrow{AB} \times \overrightarrow{CP})|}{\|\overrightarrow{AB} \times \overrightarrow{CP}\|}$$

for any distinct points A, B on one line and C, P on the other line. We first calculate $\overrightarrow{AC}$, which is

$$\overrightarrow{AC} = \langle 1, 1, s\rangle - \langle 1, 2, 3\rangle = \langle 0, -1, s - 3\rangle$$

Then $\overrightarrow{AB} \times \overrightarrow{CP}$ is

$$\overrightarrow{AB} \times \overrightarrow{CP} = \det \begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 1 & -1 & -1 \\ 1 & -2 & -s \end{pmatrix} = \langle \det \begin{pmatrix} -1 & -1 \\ -2 & -s \end{pmatrix}, -\det \begin{pmatrix} 1 & -1 \\ 1 & -s \end{pmatrix}, \det \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix} \rangle$$
$$= \langle s - 2, s - 1, -1\rangle$$

It follows that

$$\begin{aligned}\|\overrightarrow{AB} \times \overrightarrow{CP}\| &= \sqrt{(s-2)^2 + (s-1)^2 + (-1)^2} = \sqrt{s^2 - 4s + 4 + s^2 - 2s + 1 + 1} \\ &= \sqrt{2s^2 - 6s + 6}\end{aligned}$$

Moreover, $|\overrightarrow{AC} \cdot (\overrightarrow{AB} \times \overrightarrow{CP})|$ is

$$\begin{aligned}|\overrightarrow{AC} \cdot (\overrightarrow{AB} \times \overrightarrow{CP})| &= |\langle 0, -1, s-3\rangle \cdot \langle s-2, s-1, -1\rangle| = |0(s-2) - (s-1) + (s-3)(-1)| \\ &= |2s - 4| = 2|s-2|\end{aligned}$$

So the distance between the lines is

$$D = \frac{2|s-2|}{\sqrt{2s^2 - 6s + 6}}$$

The problem asks for the values of s such that $D = 3/2$, so

$$\begin{aligned}\frac{2|s-2|}{\sqrt{2s^2 - 6s + 6}} &= \frac{3}{2} \\ 4|s-2| &= 3\sqrt{2s^2 - 6s + 6} \\ 16(s-2)^2 &= 9(2s^2 - 6s + 6) \\ 16s^2 - 64s + 64 &= 18s^2 - 54s + 54 \\ 2s^2 + 10s - 10 &= = 0 \\ s^2 + 5s - 5 &= = 0\end{aligned}$$

To which applying the quadratic formula yields

$$s = \frac{-5 \pm 3\sqrt{5}}{2}$$

as the solutions.

21. Consider two parallel straight line segments in space. Formulate an algorithm to compute the distance between them if the coordinates of their end points are given. In particular, find the distance between AB and CD if:

(i) $A = (1, 1, 1)$, $B = (4, 1, 5)$, $C = (2, 3, 3)$, $D = (5, 3, 7)$;
(ii) $A = (1, 1, 1)$, $B = (4, 1, 5)$, $C = (3, 5, 5)$, $D = (6, 5, 9)$

Note that this distance does not generally coincide with the distance between the parallel lines containing AB and CD. The segments may even be in the same line at a nonzero distance.

Solution: The issue with simply applying the distance between parallel lines formula is that it assumes the lines extend infinitely, or at least that there exist points L_1 in $\mathcal{L}_1$ and L_2 in $\mathcal{L}_2$ where L_1L_2

is perpendicular to both lines. However, consider the line segment $\mathcal{L}_1 = \{(x, 2)|x \in [-2, -1]\}$ (line segment of length 1 centered at $-3/2$) and the line segment $[0, 1]$. Clearly no L_1, L_2 exist to satisfy the required criteria. The distance then is simply the shortest distance between the endpoint of one line and the endpoint of the other line (in this case, the distance between $(-1, 2)$ and $(0, 0)$). It can be seen that L_1 and L_2 exist when one line is "above" another, with respect to the plane the two line segments are in. First, everything will be conducted where $|AB| \geq |CD|$. Thus step 1 of the algorithm is to compute the length of each line segment. Whichever is longer, call the endpoints A and B, and whichever is shorter, call the endpoints C and D (note: these may not match the names given in the problem. Ignore those and rename them accordingly). Step 2 is to calculate $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{BC}$, and $\overrightarrow{BD}$, where the first vector is the vector parallel to the line segments and the other vectors are the vectors parallel to the lines passing through all the endpoint pairs, where one endpoint is from one line and the other is from the other. We have the following cases for what can occur:

Case 1:
$$\overrightarrow{AB} \mid\mid \overrightarrow{AC} \mid\mid \overrightarrow{AD} \mid\mid \overrightarrow{BC} \mid\mid \overrightarrow{BD}$$
In this case, the two line segments lie on the same line. The distance is $\inf(S)$ where $S = \{\|\overrightarrow{AC}\|, \|\overrightarrow{AD}\|, \|\overrightarrow{BC}\|, \|\overrightarrow{BD}\|\}$.

Case 2:
There will be many sub-cases here. First, it is important to compute all the following dot products:
$$d_1 = \overrightarrow{AB} \cdot \overrightarrow{AC},\ d_2 = \overrightarrow{AB} \cdot \overrightarrow{AD},\ d_3 = \overrightarrow{BA} \cdot \overrightarrow{BC},\ d_4 = \overrightarrow{BA} \cdot \overrightarrow{BD}$$
The sign of each of these dot products will determine which sub-case we are in, as the sign determines whether the angle between two vectors is acute (positive), right (zero), or obtuse (negative).

a).
Two dot products are negative.
In this case, segment CD is not "above" AB. Thus, we cannot use the distance formula, and the distance is $\inf(S)$, where S is defined as above.
This case is the one used in the example above.

b).

One dot product is negative.
In this case, one end point of segment CD is "above" AB. Thus we can form a perpendicular segment from that endpoint to a point on AB. Thus the distance formula can be used.

c).
No dot products are negative.
In this case, both endpoints of segment CD are above AB. We can do something similar to case 2b, and hence we can use the distance formula.

Overall, we can use the distance formula when one or none of the dot products are negative. Otherwise, compute $\inf(S)$.

(i): The length of the line segments are

$$\begin{aligned} l_1 &= \sqrt{(4-1)^2 + (1-1)^2 + (5-1)^2} = \sqrt{9+16} = 5 \\ l_2 &= \sqrt{(5-2)^2 + (3-3)^2 + (7-3)^2} = \sqrt{9+16} = 5 \end{aligned}$$

Since the lengths are equal, it does not matter which segment we call AB. Thus use the designations in the problem. Next, we compute the required vectors

$$\begin{aligned} \overrightarrow{AB} &= \langle 4,1,5\rangle - \langle 1,1,1\rangle = \langle 3,0,4\rangle \\ \overrightarrow{AC} &= \langle 2,3,3\rangle - \langle 1,1,1\rangle = \langle 1,2,2\rangle \\ \overrightarrow{AD} &= \langle 5,3,7\rangle - \langle 1,1,1\rangle = \langle 4,2,6\rangle \\ \overrightarrow{BC} &= \langle 2,3,3\rangle - \langle 4,1,5\rangle = \langle -2,2,-2\rangle \\ \overrightarrow{BD} &= \langle 5,3,7\rangle - \langle 4,1,5\rangle = \langle 1,2,2\rangle \end{aligned}$$

And finally compute the required dot products

$$\begin{aligned} d_1 &= \overrightarrow{AB} \cdot \overrightarrow{AC} = \langle 3,0,4\rangle \cdot \langle 1,2,2\rangle = 3(1) + 0(2) + 4(2) = 11 \\ d_2 &= \overrightarrow{AB} \cdot \overrightarrow{AD} = \langle 3,0,4\rangle \cdot \langle 4,2,6\rangle = 3(4) + 0(2) + 4(6) = 36 \\ d_3 &= \overrightarrow{BA} \cdot \overrightarrow{BC} = -\langle 3,0,4\rangle \cdot \langle -2,2,-2\rangle = -(3(-2) + 0(2) + 4(-2)) = 14 \\ d_2 &= \overrightarrow{BA} \cdot \overrightarrow{BD} = -\langle 3,0,4\rangle \cdot \langle 1,2,2\rangle = -(3(1) + 0(2) + 4(2)) = -11 \end{aligned}$$

Since only one dot product is negative, we can use the distance formula as follows:

$$D = \frac{\|\overrightarrow{AB} \times \overrightarrow{AC}\|}{\|\overrightarrow{AB}\|}$$

We have already calculated the denominator; it is $|AB|$. Thus we must calculate $\|\overrightarrow{AB} \times \overrightarrow{AC}\|$.

$$\overrightarrow{AB}\times\overrightarrow{AC} = \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 3 & 0 & 4 \\ 1 & 2 & 2 \end{pmatrix} = \langle \det\begin{pmatrix} 0 & 4 \\ 2 & 2 \end{pmatrix}, -\det\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}, \det\begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} \rangle = \langle -8, -2, 6 \rangle$$

It follows that

$$\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \sqrt{(-8)^2 + (-2)^2 + 6^2} = \sqrt{104} = 2\sqrt{26}$$

Therefore the distance is

$$D = \frac{2\sqrt{26}}{5}$$

(ii): The length of the line segments are

$$\begin{aligned} l_1 &= \sqrt{(4-1)^2 + (1-1)^2 + (5-1)^2} = \sqrt{9+16} = 5 \\ l_2 &= \sqrt{(6-3)^2 + (5-5)^2 + (9-5)^2} = \sqrt{9+16} = 5 \end{aligned}$$

Since the lengths are equal, it does not matter which segment we call AB. Thus use the designations in the problem. Next, we compute the required vectors

$$\begin{aligned} \overrightarrow{AB} &= \langle 4,1,5\rangle - \langle 1,1,1\rangle = \langle 3,0,4\rangle \\ \overrightarrow{AC} &= \langle 3,5,5\rangle - \langle 1,1,1\rangle = \langle 2,4,4\rangle \\ \overrightarrow{AD} &= \langle 6,5,9\rangle - \langle 1,1,1\rangle = \langle 5,4,8\rangle \\ \overrightarrow{BC} &= \langle 3,5,5\rangle - \langle 4,1,5\rangle = \langle -1,4,0\rangle \\ \overrightarrow{BD} &= \langle 5,3,7\rangle - \langle 4,1,5\rangle = \langle 2,4,4\rangle \end{aligned}$$

And finally compute the required dot products

$$\begin{aligned} d_1 &= \overrightarrow{AB}\cdot\overrightarrow{AC} = \langle 3,0,4\rangle\cdot\langle 2,4,4\rangle = 3(2)+0(4)+4(4) = 22 \\ d_2 &= \overrightarrow{AB}\cdot\overrightarrow{AD} = \langle 3,0,4\rangle\cdot\langle 5,4,8\rangle = 3(5)+0(4)+4(8) = 47 \\ d_3 &= \overrightarrow{BA}\cdot\overrightarrow{BC} = -\langle 3,0,4\rangle\cdot\langle -1,4,0\rangle = -(3(-1)+0(4)+4(0)) = 3 \\ d_2 &= \overrightarrow{BA}\cdot\overrightarrow{BD} = -\langle 3,0,4\rangle\cdot\langle 2,4,4\rangle = -(3(2)+0(4)+4(4)) = -22 \end{aligned}$$

Since only one dot product is negative, we can use the distance formula as follows:

$$D = \frac{\|\overrightarrow{AB} \times \overrightarrow{AC}\|}{\|\overrightarrow{AB}\|}$$

We have already calculated the denominator; it is $|AB|$. Thus we must calculate $\|\overrightarrow{AB} \times \overrightarrow{AC}\|$.

$$\overrightarrow{AB}\times\overrightarrow{AC} = \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 3 & 0 & 4 \\ 2 & 4 & 4 \end{pmatrix} = \langle \det\begin{pmatrix} 0 & 4 \\ 4 & 4 \end{pmatrix}, -\det\begin{pmatrix} 3 & 4 \\ 2 & 4 \end{pmatrix}, \det\begin{pmatrix} 3 & 0 \\ 2 & 4 \end{pmatrix} \rangle = \langle -16, -4, 12 \rangle$$

It follows that

$$\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \sqrt{(-16)^2 + (-4)^2 + 12^2} = \sqrt{416} = 4\sqrt{26}$$

Therefore the distance is

$$D = \frac{4\sqrt{26}}{5}$$

22-25. Consider the parallelepiped with adjacent edges AB, AC, and AD where $A = (3, 0, 1)$, $B = (-1, 2, 5)$, $C = (5, 1, -1)$, $D = (0, 4, 2)$. Find the specified distances.

22. The distances between the edge AB and all other edges parallel to it.

23. The distances between the edge AC and all other edges parallel to it;

24. The distances between the edge AD and all other edges parallel to it;

25. The distances between all parallel planes containing the faces of the parallelepiped.

26. The distances between all skew lines containing the edges of the parallelepiped.

6. Planes in Space

1. Find an equation of the plane through the origin and parallel to the plane $2x - 2y + z = 4$. What is the distance between the two planes?

SOLUTION: Recall that the standard form of a plane is

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$$

where $\mathbf{n}$ is a vector normal to the plane and $\mathbf{r}_0$ is the position vector of a point in the plane. Thus for a plane to be parallel to the given one, it must have the same normal vector. For the plane to pass through the origin, we can choose $\mathbf{r}_0$ to be the position vector of the origin. So $\mathbf{n} \cdot \mathbf{r}_0 = 0$, and

$$2x - 2y + z = 0$$

is a plane parallel to the given one passing through the origin.
Given two parallel planes

$$\mathbf{n} \cdot \mathbf{r} = d_1, \quad \mathbf{n} \cdot \mathbf{r} = d_2$$

The distance between the planes is defined to be

$$D = \frac{|d_2 - d_1|}{\|\mathbf{n}\|}$$

Applying this formula to the problem yields

$$D = \frac{|4 - 0|}{\sqrt{2^2 + (-2)^2 + 1^2}} = \frac{4}{\sqrt{9}} = \frac{4}{3}$$

2. Do the planes $2x + y - z = 1$ and $4x + 2y - 2z = 10$ intersect?

SOLUTION: An equivalent question is if the planes are parallel or coincide. The planes are parallel or coincide iff their normal vectors are parallel. The normal vector of plane $\mathcal{P}_1$ defined by $2x+y-z = 1$ is $\mathbf{n}_1 = \langle 2, 1, -1\rangle$. The normal vector of plane $\mathcal{P}_2$ defined by $4x + 2y - 2z = 10$ is $\mathbf{n}_2 = \langle 4, 2, -2\rangle$. Suppose there is a real number s where $\mathbf{n}_1 = s\mathbf{n}_2$. Then $2 = s(4)$, $1 = s(2)$, and $-1 = s(-2)$ by equating the components. Solving the first yields $s = 1/2$. Substituting into the other two equations yields true statements, so the normal vectors are parallel. By definition, intersecting planes have nonparallel normal vectors. So the planes do not intersect.

3. Determine whether the planes $2x + y - z = 3$ and $x + y + z = 1$ are intersecting. If they are, find the angle between them.

Solution: An equivalent question is if the planes are parallel or coincide. The planes are parallel or coincide iff their normal vectors are parallel. The normal vector of plane $\mathcal{P}_1$ defined by $2x + y - z = 3$ is $\mathbf{n}_1 = \langle 2, 1, -1\rangle$. The normal vector of plane $\mathcal{P}_2$ defined by $x+y+z = 1$ is $\mathbf{n}_2 = \langle 1, 1, 1\rangle$. Suppose there is a real number s where $\mathbf{n}_1 = s\mathbf{n}_2$. Then $2 = s(1)$, $1 = s(1)$, and $-1 = s(1)$ by equating the components.Solving the first yields $s = 2$, but substituting into the second yields $1 = 2$, a contradiction. Thus the vectors are not parallel. So the planes are not parallel nor coincidental, and must intersect.

The angle between two planes is defined as

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|}$$

Thus,

$$\theta = \arccos \frac{\langle 2, 1, -1\rangle \cdot \langle 1, 1, 1\rangle}{\sqrt{2^2 + 1^2 + (-1)^2}\sqrt{1^2 + 1^2 + 1^2}} = \arccos \frac{2 + 1 - 1}{\sqrt{6}\sqrt{3}} = \arccos \frac{\sqrt{2}}{3}$$

4–**6**. Consider a parallelepiped with one vertex at the origin O at which the adjacent sides are the vectors $\mathbf{a} = \langle 1, 2, 3\rangle$, $\mathbf{b} = \langle 2, 1, 1\rangle$, and $\mathbf{c} = \langle -1, 0, 1\rangle$. Let OP be its diagonal extended from the vertex O. Find equations of the following planes.

4. The planes that contain the faces of the parallelepiped.

Solution: There are 6 faces to find, 3 through the origin and 3 through the point P. First computing $\overrightarrow{OP}$ gives

$$\overrightarrow{OP} = \mathbf{a} + \mathbf{b} + \mathbf{c} = \langle 1, 2, 3\rangle + \langle 2, 1, 1\rangle + \langle -1, 0, 1\rangle = \langle 2, 3, 5\rangle$$

Next we must find the normal vector of each plane. These normal vectors will be $\mathbf{a} \times \mathbf{b}$, $\mathbf{b} \times \mathbf{c}$, and $\mathbf{a} \times \mathbf{c}$.

$$\begin{aligned} n_1 = \mathbf{a} \times \mathbf{b} &= \det \begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix} = \langle \det \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}, -\det \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}, \det \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \rangle \\ &= \langle (2 - 3), -(1 - (6)), (1 - (4)\rangle = \langle -1, 5, -3\rangle \\ n_2 = \mathbf{b} \times \mathbf{c} &= \det \begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 2 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} = \langle \det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, -\det \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}, \det \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \rangle \\ &= \langle 1, -(2 - (-1)), (0 - (-1)\rangle = \langle 1, -3, 1\rangle \end{aligned}$$

$$n_3 = \mathbf{a} \times \mathbf{c} = \det \begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 1 & 2 & 3 \\ -1 & 0 & 1 \end{pmatrix} = \langle \det \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}, -\det \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}, \det \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \rangle$$
$$= \langle 2, -(1-(-3)), (0-(-2)) \rangle = \langle 2, -4, 2 \rangle$$

Recall that the standard equation of a plane is

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$$

where $\mathbf{n}$ is a normal vector to the plane, $\mathbf{r}$ is a general position vector, and $\mathbf{r}_0$ is the position vector of a point in the plane. The left hand side for each plane can now be computed using the dot product, with $\mathbf{r} = \langle x, y, z \rangle$

$$\mathbf{n}_1 \cdot \mathbf{r} = -x + 5y - 3z$$
$$\mathbf{n}_2 \cdot \mathbf{r} = x - 3y + z$$
$$\mathbf{n}_3 \cdot \mathbf{r} = 2x - 4y + 2z$$

The right hand side can be calculated using the dot product, with $\mathbf{r}_0$ as either $\langle 0, 0, 0 \rangle$ or $\overrightarrow{OP}$. The former dot product is trivially 0 for each plane. The latter is

$$\mathbf{n}_1 \cdot \overrightarrow{OP} = -1(2) + 5(3) - 3(5) = -2$$
$$\mathbf{n}_2 \cdot \overrightarrow{OP} = 1(2) - 3(3) + (5) = -2$$
$$\mathbf{n}_3 \cdot \overrightarrow{OP} = 2(2) - 4(3) + 2(5) = 2$$

Thus the three planes passing through the origin are

$$x - 5y + 3z = 0 \quad x - 3y + z = 0 \quad x - 2y + z = 0$$

and the three planes passing through $\overrightarrow{OP}$ are

$$x - 5y + 3z = 2 \quad x - 3y + z = -2 \quad x - 2y + z = 1$$

5. The planes that contain the diagonal OP and the diagonal of each of three its faces adjacent at P.

SOLUTION: The diagonals at the faces adjacent at P are $\mathbf{d}_1 = -(\mathbf{a} + \mathbf{b})$, $\mathbf{d}_2 = -(\mathbf{b} + \mathbf{c})$, and $\mathbf{d}_3 = -(\mathbf{a} + \mathbf{c})$. If a plane contains two vectors, then the normal vector must be parallel to the cross product of those vectors. Thus we can let $\mathbf{n}_1 = \overrightarrow{OP} \times \mathbf{d}_1$, $\mathbf{n}_2 = \overrightarrow{OP} \times \mathbf{d}_2$, and $\mathbf{n}_3 = \overrightarrow{OP} \times \mathbf{d}_3$. We calculate $\overrightarrow{OP} \times \mathbf{a}$, $\overrightarrow{OP} \times \mathbf{b}$, and $\overrightarrow{OP} \times \mathbf{c}$ separately below

$$\overrightarrow{OP} \times \mathbf{a} = \det \begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 2 & 3 & 5 \\ 1 & 2 & 3 \end{pmatrix} = \langle \det \begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix}, -\det \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}, \det \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \rangle = \langle -1, -1, 1 \rangle$$

$$\overrightarrow{OP} \times \mathbf{b} = \det \begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 2 & 3 & 5 \\ 2 & 1 & 1 \end{pmatrix} = \langle \det \begin{pmatrix} 3 & 5 \\ 1 & 1 \end{pmatrix}, -\det \begin{pmatrix} 2 & 5 \\ 2 & 1 \end{pmatrix}, \det \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \rangle = \langle -2, 8, -4 \rangle$$

$$\overrightarrow{OP} \times \mathbf{c} = \det \begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 2 & 3 & 5 \\ -1 & 0 & 1 \end{pmatrix} = \langle \det \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix}, -\det \begin{pmatrix} 2 & 5 \\ -1 & 1 \end{pmatrix}, \det \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix} \rangle = \langle 3, -7, 3 \rangle$$

Then, we have that

$$\mathbf{n}_1 = \overrightarrow{OP} \times \mathbf{d}_1 = -(\langle -1, -1, 1 \rangle + \langle -2, 8, -4 \rangle) = -\langle -3, 7, -3 \rangle = \langle 3, -7, 3 \rangle$$
$$\mathbf{n}_2 = \overrightarrow{OP} \times \mathbf{d}_2 = -(\langle -2, 8, -4 \rangle + \langle 3, -7, 3 \rangle) = -\langle 1, 1, -1 \rangle = \langle -1, -1, 1 \rangle$$
$$\mathbf{n}_3 = \overrightarrow{OP} \times \mathbf{d}_3 = -(\langle -1, -1, 1 \rangle + \langle 3, -7, 3 \rangle) = -\langle 2, -8, 4 \rangle = \langle = 2, 8, -4 \rangle$$

These normal vectors should seem very familiar. This is no coincidence, because $\overrightarrow{OP} = \mathbf{a} + \mathbf{b} + \mathbf{c}$ So, for example, $-\overrightarrow{OP} \times (\mathbf{a}+\mathbf{b}) = -\overrightarrow{OP} \times (\mathbf{a}+\mathbf{b}+\mathbf{c}-\mathbf{c}) = -\overrightarrow{OP} \times (\overrightarrow{OP} - \mathbf{c}) = \overrightarrow{OP} \times \mathbf{c}$. Notice that each $\mathbf{n}_i$ for $1 \leq i \leq 3$ is the cross product of some nonzero vector and $\overrightarrow{OP}$. Thus the dot product of each normal vector with $\overrightarrow{OP}$ will be 0. It follows that the equation for each plane is

$$3x - 7y + 3z = 0 \quad x + y - z = 0 \quad x - 4y + 2z = 0$$

6. The planes that contain parallel diagonals in the opposite faces of the parallelepiped.

Solution: The following argument may be hard to visualize. There is a set of symmetry operations that will map parallel diagonals on opposite faces to each other, while keeping the diagonal fixed. Specifically, take any two parallel faces. Constructing a line segment from the center of these faces gives an axis. Rotate about this axis. Construct a plane where the rotation axis is orthogonal to it. Reflecting the parallelpiped over this plane provides the above set of operations. This implies that the planes that go through the origin are exactly those in problem 5. However, there are still three more to compute. The normal vectors will be $\mathbf{n}_1 = \overrightarrow{OP} \times (\mathbf{a} - \mathbf{b})$,

$\mathbf{n}_2 = \overrightarrow{OP} \times (\mathbf{b} - \mathbf{c})$, and $\mathbf{n}_3 = \overrightarrow{OP} \times (\mathbf{a} - \mathbf{c})$. So,

$$\begin{aligned}
\mathbf{n}_1 &= \overrightarrow{OP} \times \mathbf{d}_1 = \langle -1, -1, 1\rangle - \langle -2, 8, -4\rangle = \langle 1, -9, 5\rangle \\
\mathbf{n}_2 &= \overrightarrow{OP} \times \mathbf{d}_2 = \langle -2, 8, -4\rangle - \langle 3, -7, 3\rangle = \langle -5, 15, -7\rangle \\
\mathbf{n}_3 &= \overrightarrow{OP} \times \mathbf{d}_3 = \langle -1, -1, 1\rangle - \langle 3, -7, 3\rangle = \langle -4, 6, -2\rangle
\end{aligned}$$

NEEDS TO BE COMPLETED

7. Find an equation of the plane with x intercept $a \neq 0$, y intercept $b \neq 0$, and z intercept $c \neq 0$. What is the distance between the origin and the plane? Find the angles between the plane and the coordinate planes.

Solution: If the plane $\mathcal{P}$ has intercepts $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$ (a, b, c nonzero) then the plane must pass through these points. The general form of a plane is

$$n_x x + n_y y + n_z z = d$$

for real numbers n_x, n_y, n_z, d. Thus, $n_x a = d$, $n_y b = d$, $n_z c = d$ by substituting each intercept into the equation for the plane. So, $n_x = d/a$, $n_y = d/b$, and $n_z = d/c$, which are valid since a, b, c are nonzero. It follows that

$$(d/a)x + (d/b)y + (d/c)z = d \;\Leftrightarrow\; \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

by substituting n_x, n_y, n_z into the equation for the plane.

The distance between the plane and the origin can be answered by finding the distance between the plane and a parallel plane passing through the origin, $x/a + y/b + z/c = 0$. Applying the distance formula yields

$$D = \frac{|1 - 0|}{\sqrt{(\frac{1}{a})^2 + (\frac{1}{b})^2 + (\frac{1}{c})^2}} = \frac{abc}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}$$

Recall that the angle between two planes whose normals are $\mathbf{n}_1$ and $\mathbf{n}_2$, respectively, is

$$\theta = \arccos |\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2|$$

We first find $\hat{\mathbf{n}}_1$ to be

$$\hat{\mathbf{n}}_1 = \frac{1}{\sqrt{(\frac{1}{a})^2 + (\frac{1}{b})^2 + (\frac{1}{c})^2}}\langle \frac{1}{a}, \frac{1}{b}, \frac{1}{c}\rangle = \frac{1}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}\langle bc, ac, ab\rangle$$

Next, $\hat{\mathbf{n}}_2$ will be one of $\langle 1,0,0\rangle$, $\langle 0,1,0\rangle$, or $\langle 0,0,1\rangle$, for the angle between the plane and the yz, xz, or xy plane, respectively. Thus we have

$$\theta_{yz} = \arccos|\frac{1}{\sqrt{b^2c^2+a^2c^2+a^2b^2}}\langle bc,ac,ab\rangle\cdot\langle 1,0,0\rangle| = |\frac{bc}{\sqrt{b^2c^2+a^2c^2+a^2b^2}}| = \frac{D}{|a|}$$
$$\theta_{xz} = \arccos|\frac{1}{\sqrt{b^2c^2+a^2c^2+a^2b^2}}\langle bc,ac,ab\rangle\cdot\langle 0,1,0\rangle| = |\frac{ac}{\sqrt{b^2c^2+a^2c^2+a^2b^2}}| = \frac{D}{|b|}$$
$$\theta_{xy} = \arccos|\frac{1}{\sqrt{b^2c^2+a^2c^2+a^2b^2}}\langle bc,ac,ab\rangle\cdot\langle 0,0,1\rangle| = |\frac{ab}{\sqrt{b^2c^2+a^2c^2+a^2b^2}}| = \frac{D}{|c|}$$

8. Show that the points $A=(1,1,1)$, $B=(1,2,3)$, $C=(2,0,-1)$ and $D=(3,1,0)$ are not in a plane and therefore vertices of a tetrahedron. Any two of the four faces of the tetrahedron are intersecting along one of its six edges. Find the angles of intersection of the face BCD with the other three faces.

Solution: To answer the first part, we can find an equation for the plane containing face BCD (which we will need anyway later), and show that point A does not satisfy the equation. The normal to the plane can be $\mathbf{n}_1 = \overrightarrow{BC}\times\overrightarrow{CD}$. Thus we must find $\overrightarrow{BC}$ and $\overrightarrow{BD}$ as follows:

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \langle 2,0,-1\rangle - \langle 1,2,3\rangle = \langle 1,-2,-4\rangle$$
$$\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB} = \langle 3,1,0\rangle - \langle 1,2,3\rangle = \langle 2,-1,-3\rangle$$

It follows that

$$\mathbf{n}_1 = \det\begin{pmatrix}\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ 1 & -2 & -4\\ 2 & -1 & -3\end{pmatrix} = \langle \det\begin{pmatrix}-2 & -4\\ -1 & -3\end{pmatrix}, -\det\begin{pmatrix}1 & -4\\ 2 & -3\end{pmatrix}, \det\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}\rangle = \langle 2,-5,3\rangle$$

Next, point B must be a part of the plane. So we can let $d_1 = \mathbf{n}_1\cdot\overrightarrow{OB}$. Thus,

$$d_1 = \langle 2,-5,3\rangle\cdot\langle 1,2,3\rangle = 2(1)-5(2)+3(3) = 1$$

Thus an equation for the plane in question is

$$2x-5y+3z=1$$

The reader can verify that points C and D indeed satisfy this equation. Substitution of point A into the left hand side gives

$$2(1)-5(1)+3(1) = 5-5 = 0 \neq 1$$

Thus A is not a point in the plane, and the four points are not coplanar.

The other three faces can be found with the following normal vectors $\mathbf{n}_2 = \overrightarrow{BA} \times \overrightarrow{BC}$, $\mathbf{n}_3 = \overrightarrow{BA} \times \overrightarrow{BD}$, $\mathbf{n}_4 = \overrightarrow{CA} \times \overrightarrow{CD}$. These vectors were chosen simply to reduce the amount of computations. The necessary vectors are

$$\begin{aligned}
\overrightarrow{BA} &= \overrightarrow{OA} - \overrightarrow{OB} = \langle 1,1,1\rangle - \langle 1,2,3\rangle = \langle 0,-1,-2\rangle \\
\overrightarrow{CA} &= \overrightarrow{OA} - \overrightarrow{OC} = \langle 1,1,1\rangle - \langle 2,0,-1\rangle = \langle -1,1,2\rangle \\
\overrightarrow{CD} &= \overrightarrow{OD} - \overrightarrow{OC} = \langle 3,1,0\rangle - \langle 2,0,-1\rangle = \langle 1,1,1\rangle
\end{aligned}$$

The necessary normal vectors are

$$\mathbf{n}_2 = \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 0 & -1 & -2 \\ 1 & -2 & -4 \end{pmatrix} = \langle \det\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix}, -\det\begin{pmatrix} 0 & -2 \\ 1 & -4 \end{pmatrix}, \det\begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}\rangle = \langle 0,-2,1\rangle$$

$$\mathbf{n}_3 = \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 0 & -1 & -2 \\ 2 & -1 & -3 \end{pmatrix} = \langle \det\begin{pmatrix} -1 & -2 \\ -1 & -3 \end{pmatrix}, -\det\begin{pmatrix} 0 & -2 \\ 2 & -3 \end{pmatrix}, \det\begin{pmatrix} 0 & -1 \\ 2 & -1 \end{pmatrix}\rangle = \langle 1,-4,2\rangle$$

$$\mathbf{n}_2 = \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ -1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \langle \det\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, -\det\begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}, \det\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\rangle = \langle -1,3,-2\rangle$$

The right hand side for each plane can be calculated as $d_2 = \mathbf{n}_2 \cdot \overrightarrow{OB}$, $d_3 = \mathbf{n}_3 \cdot \overrightarrow{OB}$, and $d_4 = \mathbf{n}_4 \cdot \overrightarrow{OC}$. The points were chosen to be the ones that "repeated" in the cross product vectors. The constants are, then,

$$\begin{aligned}
d_2 &= \langle 0,-2,1\rangle \cdot \langle 1,2,3\rangle = 0(1) - 2(2) + 1(3) = -1 \\
d_3 &= \langle 1,-4,2\rangle \cdot \langle 1,2,3\rangle = 1(1) - 4(2) + 2(3) = -1 \\
d_4 &= \langle -1,3,-2\rangle \cdot \langle 2,0,-1\rangle = -1(2) + 3(0) - 2(-1) = 0
\end{aligned}$$

Thus, an equation for each plane (containing face XYZ) is:

$$\begin{aligned}
BCD:&\ \ 2x - 5y + 3z = 1 \\
ABC:&\ \ -2y + z = -1 \\
ABD:&\ \ x - 4y + 2z = -1 \\
ACD:&\ \ -x + 3y - 2z = 0
\end{aligned}$$

Let α, β, and γ denote the angle between plane BCD and plane ABC, ABD, or ACD, respectively. The angle between two planes, whose normal vectors are $\mathbf{n}_1$ and $\mathbf{n}_2$ is:

$$\theta = \arccos\frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|}$$

Then,

$$\alpha = \arccos \frac{|\langle 2,-5,3\rangle \cdot \langle 0,-2,1\rangle|}{\sqrt{2^2+(-5)^2+3^2}\sqrt{0^2+(-2)^2+1^2}} = \frac{|2(0)-5(-2)+3(1)|}{\sqrt{38}\sqrt{5}} = \frac{13}{\sqrt{38}\sqrt{5}}$$

$$\beta = \arccos \frac{|\langle 2,-5,3\rangle \cdot \langle 1,-4,2\rangle|}{\sqrt{2^2+(-5)^2+3^2}\sqrt{1^2+(-4)^2+2^2}} = \frac{|2(1)-5(-4)+3(2)|}{\sqrt{38}\sqrt{21}} = \frac{28}{\sqrt{38}\sqrt{21}}$$

$$\gamma = \arccos \frac{|\langle 2,-5,3\rangle \cdot \langle -1,3,-2\rangle|}{\sqrt{2^2+(-5)^2+3^2}\sqrt{(-1)^2+3^2+(-2)^2}} = \frac{|2(-1)-5(3)+3(-2)|}{\sqrt{38}\sqrt{21}} = \frac{23}{\sqrt{38}\sqrt{14}}$$

9. Find equations of all planes that are perpendicular to the line through $(1,-1,1)$ and $(3,0,-1)$ and that are at the distance 2 from the point $(1,2,3)$.

Solution: Since the planes in question are perpendicular to the line through two given points, a normal vector of all such planes may be chosen as the vector whose initial and terminal points are the given points, $\mathbf{n} = \langle 3-1, 0-(-1), -1-1\rangle = \langle 2,1,-2\rangle$, so that $\|\mathbf{n}\| = 3$. All planes perpendicular to the line are described by equations

$$2x+y-2z=d$$

where d is real. So the question is reduced to determining the values of d at which the distance D from the point with the position vector $\mathbf{r}_1 = \langle 1,2,3\rangle$ to the plane is $D=2$. If $\mathbf{r}_0$ is the position vector of a particular point in the plane, then $d = \mathbf{n}\cdot\mathbf{r}_0$. The position vector of the point $(1,2,3)$ relative to a particular point in the plane is $\mathbf{r}_1 - \mathbf{r}_0$. The distance in question is the absolute value of the scalar projection of this vector onto $\mathbf{n}$:

$$D = \frac{|\mathbf{n}\cdot(\mathbf{r}_1-\mathbf{r}_0)|}{\|\mathbf{n}\|} = \frac{|\mathbf{n}\cdot\mathbf{r}_1 - d|}{\|\mathbf{n}\|} = \frac{|2+2-6-d|}{3} = \frac{|d+2|}{3},$$

$$D=2 \quad\Leftrightarrow\quad |d+2|=6 \quad\Leftrightarrow\quad d+2=\pm 6 \quad\Leftrightarrow\quad d=-2\pm 6$$

So equations of the planes in question are $2x+y-2z=-8$ and $2x+y-2z=4$.

10. Find an equation for the set of points that are equidistant from the points $(1,2,3)$ and $(-1,2,1)$. Give a geometrical description of the set.

Solution: Because we're in the section on planes, this is obviously a plane. However, an intuition for why this is a plane can be made by dropping down into the second dimension. Consider two distinct points. Connect the two with a line segment, then draw a normal line through the midpoint of this segment. This line is the set of all points equidistant to the two given points. This can be seen by choosing a point, and making line segments from it to the two given ones. Two triangles will be made, which are clearly congruent. Extending this idea into the third dimension transforms the line into a plane. The algebra for it is as follows:

Let $A = (1, 2, 3)$, $B = (-1, 2, 1)$, and $\mathbf{n} = \overrightarrow{AB}$. Then the plane through A whose normal is $\mathbf{n}$, $\mathcal{P}_A$, is given by $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \overrightarrow{OA}$. Similarly, the plane through B whose normal is $\mathbf{n}$, $\mathcal{P}_B$, is given by $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \overrightarrow{OB}$. Consider the plane $\mathcal{P}$ given by $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot (\overrightarrow{OA} + \overrightarrow{OB})/2$. Let P be a point in this plane. Then the distance between A and P is given by $D = \|\overrightarrow{AP}\|$. Consider the point M in $\mathcal{P}$ such that $\overrightarrow{AM}$ is normal to $\mathcal{P}$. Then $\|\overrightarrow{AM}\|$ is the distance between $\mathcal{P}_A$ and $\mathcal{P}$, which is

$$\|\overrightarrow{AM}\| = \frac{|d_2 - d_1|}{\|\mathbf{n}\|} = \frac{|\mathbf{n} \cdot (\overrightarrow{OA} + \overrightarrow{OB})/2 - \mathbf{n} \cdot \overrightarrow{OA}|}{\|\overrightarrow{AB}\|} = \frac{|\mathbf{n} \cdot (-\overrightarrow{OA} + \overrightarrow{OB})|}{2\|\overrightarrow{AB}\|} = \frac{|\overrightarrow{AB} \cdot \overrightarrow{AB}|}{2\|\overrightarrow{AB}\|} = \frac{\|\overrightarrow{AB}\|}{2}$$

A nearly identical argument shows that $\|\overrightarrow{BM}\| = \|\overrightarrow{AB}\|/2 = \|\overrightarrow{AM}\|$. Thus M is equidistant to both A and B, and AM, BM are normal to the plane. This implies that AM and BM are orthogonal to MP. Consider the distance $D' = \|\overrightarrow{BP}\|$. Triangles AMP and BMP are congruent by SAS ($|AM| = |BM|$, $|MP| = |MP|$, and a $\pi/2$ angle enclosed between these sides), thus $|AP| = |BP|$, and $D' = D$. So $\mathcal{P}$ is the set of points equidistant to A and B.

In this problem, we have that $\mathbf{n} = \overrightarrow{OB} - \overrightarrow{OA} = \langle -1, 2, 1\rangle - \langle 1, 2, 3\rangle = \langle -2, 0, -2\rangle$ and $1/2(\overrightarrow{OA} + \overrightarrow{OB}) = 1/2(\langle 1, 2, 3\rangle + \langle -1, 2, 1\rangle) = 1/2\langle 0, 4, 4\rangle = \langle 0, 2, 2\rangle$. The equation for the set of points that are equidistant from the given ones is the plane

$$\langle -2, 0, -2\rangle \cdot \mathbf{r} = \langle -2, 0, -2\rangle \cdot \langle 0, 2, 2\rangle \;\Leftrightarrow\; -2x - 2z = -2(0) + 0(2) - 2(2) \;\Leftrightarrow\; x + z = 2$$

11. Find an equation of the plane that is perpendicular to the plane $x + y + z = 1$ and contains the line through the points $(1, 2, 3)$ and $(-1, 1, 0)$.

Solution: A plane is determined by a particular point P_0 in the plane and a normal $\mathbf{n}$ (a non-zero vector perpendicular to the plane).

Since the plane contains the segment AB, one can take

$$P_0 = A = (1, 2, 3)\,.$$

The vector $\mathbf{n}_1 = \langle 1, 1, 1\rangle$ is a normal of the given plane. Since the plane in question is perpendicular to the given plane, their normals must be perpendicular. Furthermore, the normal $\mathbf{n}$ is perpendicular to the vector $\overrightarrow{AB} = \langle -2, -1, -3\rangle$ because the segment AB lies in the plane in question. By the geometrical properties of the cross product, the vector $\mathbf{n}_1 \times \overrightarrow{AB}$ is perpendicular to both $\mathbf{n}_1$ and $\overrightarrow{AB}$ and can be taken as a normal to the plane in question:

$$\left.\begin{array}{l}\mathbf{n} \perp \mathbf{n}_1\\ \mathbf{n} \perp \overrightarrow{AB}\end{array}\right\} \quad\Rightarrow\quad \mathbf{n} = \mathbf{n}_1 \times \overrightarrow{AB} = \det\begin{pmatrix}\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ 1 & 1 & 1\\ -2 & -1 & -3\end{pmatrix} = \langle -2\,, 1\,, 1\rangle$$

Therefore an equation of the plane in question reads:

$$-2(x-1) + (y-2) + (z-3) = 0 \quad\Rightarrow\quad -2x + y + z = 3\,.$$

12. To which of the planes $x + y + z = 1$ and $x + 2y - z = 2$ is the point $(1, 2, 3)$ the closest?

Solution: The point $A = (1, 2, 3)$ is not in the plane $x + y + z = 1$ because its coordinates do not satisfy the equation of the plane, but A lies in the second plane because $1 + 2\cdot 2 - 3 = 2$. Thus, A is the closest to the second plane.

Alternative solution: The plane $x + y + z = 1$ has a normal $\mathbf{n}_1 = \langle 1, 1, 1\rangle$ and passes through the point $P_1 = (1, 0, 0)$. Therefore the distance from the point $A = (1, 2, 3)$ to this plane is

$$D_1 = \frac{|\mathbf{n}_1 \cdot \overrightarrow{P_1A}|}{\|\mathbf{n}_1\|} = \frac{|\langle 1, 1, 1\rangle \cdot \langle 0, 2, 3\rangle|}{\|\langle 1, 1, 1\rangle\|} = \frac{|2+3|}{\sqrt{3}} = \frac{5}{\sqrt{3}}\,.$$

Similarly, the plane $x + 2y - z = 2$ has a normal $\mathbf{n}_2 = \langle 1, 2, -1\rangle$ and passes through the point $P_2 = (2, 0, 0)$. Therefore the distance from the point $A = (1, 2, 3)$ to this plane is

$$D_2 = \frac{|\mathbf{n}_2 \cdot \overrightarrow{P_2A}|}{\|\mathbf{n}_2\|} = \frac{|\langle 1, 2, -1\rangle \cdot \langle -1, 2, 3\rangle|}{\|\langle 1, 2, -1\rangle\|} = \frac{|-1+4-3|}{\sqrt{6}} = 0 < D_1\,.$$

Thus, the point A is closer to the second plane.

13–15. Give a geometrical description of each of the following families of planes where c is a numerical parameter.

13. $x + y + z = c$.

Solution: This is the collection of all planes whose normal vector is $\langle 1, 1, 1\rangle$.

14. $x + y + cz = 1$.

Solution: This is the collection of all planes whose normal vector is $\langle 1, 1, c\rangle$. Each plane intersects the coordinate axes at $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1/c)$. Furthermore, each plane contains the line $x + y = 1$, seen by substituting $x + y = 1$ and $z = 0$ into the equation.

15. $x \sin c + y \cos c + z = 1$.

Solution: This is the collection of all planes whose normal vector is $\langle \sin c, \cos c, 1\rangle$, passing through the z axis at $(0, 0, 1)$. This normal vector can be seen as a sum of a unit vector in the xy plane and $\hat{\mathbf{e}}_3$. Each plane is tangent to the unit circle in the xy plane. Therefore the planes can be seen as taking one plane (e,g, $y + z = 1$) and rotating it about the z axis. The space enclosed by the union of all these planes and the disk $\{(x, y, z)|x^2+y^2 = 1,\ z = 0\}$ is in the shape of a cone.

16. Find values of c for which the plane $x + y + cz = 1$ is closest to the point $P = (1, 2, 1)$ and farthest from P.

Solution: An equivalent problem is to find the constants c that give a maximum or minimum distance between the plane $x + y + cz = 1$ and a parallel plane through the point P. The distance between two parallel planes $\mathbf{n} \cdot \mathbf{r} = d_1$ and $\mathbf{n} \cdot \mathbf{r} = d_2$ is

$$D = \frac{|d_2 - d_1|}{\|\mathbf{n}\|}$$

We have that $\mathbf{n} = \langle 1, 1, c\rangle$. For one plane, set $d_1 = 1$. For the plane passing through P, let $d_2 = \mathbf{n} \cdot \overrightarrow{OP} = \langle 1, 1, c\rangle \cdot \langle 1, 2, 1\rangle = 1(1) + 1(2) + c(1) = 3 + c$. Thus the distance between them is

$$D = \frac{|3 + c - 1|}{\sqrt{1^2 + 1^2 + c^2}} = \frac{|c + 2|}{\sqrt{2 + c^2}}$$

Clearly, D is minimal when $c = -2$. This is the case when the plane $x + y + cz = 1$ is coincidental with the plane passing through P (they are the same plane). Finding when D is maximal could be done by differentiating D with respect to c, finding the zeroes, and so on. But that is tedious, so instead let's do a geometric investigation. Notice that in all cases, when $z = 0$, we have that $x + y = 1$. Thus, no matter what c is, the plane will always intersect the line $x + y = 1$ in the xy plane (see problem 14). This implies that varying c rotates the plane along an axis produced by the line $x + y = 1$ in the xy plane. Start with the plane passing through P. The plane can be divided into two "parts" by the line $x + y = 1$; one part with $z > 0$ and one with $z \leq 0$. In this state, one part goes through P and the other one is maximally far away. Rotating the plane causes the part that had intersected with P to move away from P, and the part that was farthest from P to move towards it. Eventually, you will rotate the plane until both parts are equidistant; this is when there exists a point Q on the line $x+y = 1$ such that $\overrightarrow{PQ}$ is normal to the plane. Continuing to rotate the plane will bring one of the parts closer to P again, thus the previous state yields a maximum distance between the plane and P. Thus, let Q be a point on the line $x+y = 1$ in the xy plane. So Q satisfies $(a, 1-a, 0)$ for some real a. Next, we require $\overrightarrow{PQ}$ to be parallel to the normal of the plane. Thus, $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = s\langle 1, 1, c\rangle \;\Leftrightarrow\; \langle a - 1, -1 - a, -1\rangle = s\langle 1, 1, c\rangle$ for some real s. Setting the first two components equal to each other yields $a - 1 = s$ and $-1 - a = s$. So $a - 1 = -1 - a \;\Leftrightarrow\; a = 0$. Thus $s = -1$. Equating the third components gives $-1 = sc$, so $c = 1$.

17. Consider three planes with normals $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ such that each pair of the planes is intersecting. Under what condition on the normals are the three lines of intersection parallel or even coincide?

Solution: Consider planes $\mathcal{P}_1$ and $\mathcal{P}_2$, whose normals are $\mathbf{n}_1$ and $\mathbf{n}_2$, respectively. Then line $\mathcal{L}_{12}$, formed from the intersection of planes $\mathcal{P}_1$ and $\mathcal{P}_2$, must be parallel to a vector that is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$. Let P and Q be two distinct points on the line. Then P and Q also lie in $\mathcal{P}_1$, so $\overrightarrow{PQ}$ is parallel to the plane. Thus $\overrightarrow{PQ}$ is perpendicular to $\mathbf{n}_1$. The same argument applies for plane 2. It follows that $\overrightarrow{PQ}$ is parallel to $\mathbf{n}_1 \times \mathbf{n}_2$. Suppose that the three lines of intersection formed from the three planes pairwise intersecting are parallel. Then $\mathbf{n}_1 \times \mathbf{n}_2 \;||\; \mathbf{n}_1 \times \mathbf{n}_3 \;||\; \mathbf{n}_2 \times \mathbf{n}_3$. Note that since $\mathbf{n}_1 \times \mathbf{n}_2$ and $\mathbf{n}_2 \times \mathbf{n}_3$ are parallel, their cross product is 0. Hence,

$$(\mathbf{n}_1 \times \mathbf{n}_2) \times (\mathbf{n}_2 \times \mathbf{n}_3) = 0$$

By the BAC-CAB rule,

$$(\mathbf{n}_1\times\mathbf{n}_2)\times(\mathbf{n}_2\times\mathbf{n}_3) = \mathbf{n}_2((\mathbf{n}_1\times\mathbf{n}_2)\cdot(\mathbf{n}_3))-\mathbf{n}_3((\mathbf{n}_1\times\mathbf{n}_2)\cdot(\mathbf{n}_2)) = \mathbf{n}_2((\mathbf{n}_1\times\mathbf{n}_2)\cdot(\mathbf{n}_3))$$

It easily follows that

$$\mathbf{n}_2((\mathbf{n}_1 \times \mathbf{n}_2) \cdot (\mathbf{n}_3)) = 0$$

Because $\mathbf{n}_2$ is not $\mathbf{0}$ (as required in the definition of a plane), it must be that $(\mathbf{n}_1 \times \mathbf{n}_2) \cdot (\mathbf{n}_3) = 0$. Thus the normals must be coplanar.

18. Find equations of all the planes that are perpendicular to the plane $x + y + z = 1$, have the angle $\pi/3$ with the plane $x + y = 1$, and pass through the point $(1, 1, 1)$.

Solution: Let $\mathbf{n} = \langle n_x, n_y, n_z\rangle$ be the normal vector of any such plane. By the first condition, the normal vectors of the two planes must be orthogonal. So, $n_x + n_y + n_z = 0$. By the second condition, the normal vectors must have an angle of $\pi/3$. So,

$$\cos\pi/3 = \frac{|\langle n_x, n_y, n_z\rangle \cdot \langle 1, 1, 0\rangle|}{\sqrt{n_x^2 + n_y^2 + n_z^2}\sqrt{1^2 + 1^2}} \;\Leftrightarrow\; |n_x+n_y| = \frac{\sqrt{2}}{2}\sqrt{n_x^2 + n_y^2 + n_z^2}$$

However, we can do the following

$$\begin{aligned}
|n_x + n_y| &= \frac{\sqrt{2}}{2}\sqrt{n_x^2 + n_y^2 + n_z^2}\\
\sqrt{2}|n_x + n_y| &= \sqrt{n_x^2 + n_y^2 + n_z^2}\\
2(n_x + n_y)^2 &= n_x^2 + n_y^2 + n_z^2\\
2(n_x + n_y)^2 &= n_x^2 + 2n_xn_y - 2n_xn_y + n_y^2 + n_z^2\\
2(n_x + n_y)^2 &= (n_x + n_y)^2 + n_z^2 - 2n_xn_y\\
(n_x + n_y)^2 - n_z^2 &= -2n_xn_y\\
(n_x + n_y - n_z)(n_x + n_y + n_z) &= -2n_xn_y\\
0 = -2n_xn_y &
\end{aligned}$$

where the result from the first condition has been used in the second to last line. It follows that at least one of n_x or n_y (or both) must be zero. If both n_x and n_y are zero, then by the original deduction from the second condition, it must be that $n_z = 0$. This, however, would result in the normal vector being the zero vector. Thus only one of n_x or n_y must be zero.

Any such plane will be described by an equation of the form

$$n_x x + n_y y + n_z z = d$$

for some real number d. By the third condition $(1, 1, 1)$ is a point on the plane and therefore must satisfy this equation. So $n_x + n_y + n_z = d \iff d = 0$, by the first condition.

Suppose first that $n_x = 0$. Returning to the first condition, we know that $n_x + n_y + n_z = 0$. Since $n_x = 0$, we have $n_y = -n_z$. In this case, the normal vector is $\langle 0, -n_z, n_z\rangle = n_z\langle 0, -1, 1\rangle$ for some real n_z. Thus an equation for the plane would be

$$n_z(-y + z) = 0 \iff y = z$$

Suppose now that $n_y = 0$. A nearly exact argument shows that $n_x = -n_z$. In this case, the normal vector is $\langle -n_z, 0, n_z\rangle = n_z\langle -1, 0, 1\rangle$ for some real n_z. Thus an equation for the plane would be

$$n_z(-x + z) = 0 \iff x = z$$

19. Let $\mathbf{a} = \langle 1, 2, 3\rangle$ and $\mathbf{b} = \langle 1, 0, -1\rangle$. Find an equation of the plane that contains the point $(1, 2, -1)$, the vector $\mathbf{a}$ and a vector orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Solution: A vector orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ is $\mathbf{a} \times \mathbf{b}$. We can therefore let $\mathbf{n} = \mathbf{a} \times (\mathbf{a} \times \mathbf{b})$. We can compute $\mathbf{n}$ using the BAC-CAB rule as follows:

$$\mathbf{n} = \mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\mathbf{a} \cdot \mathbf{b}) - \mathbf{b}(\mathbf{a} \cdot \mathbf{a}) = \mathbf{a}(\mathbf{a} \cdot \mathbf{b}) - \mathbf{b}\|\mathbf{a}\|^2$$

In this problem, with $\mathbf{a} = \langle 1, 2, 3\rangle$ and $\mathbf{b} = \langle 1, 0, -1\rangle$, we have that

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \langle 1, 2, 3\rangle \cdot \langle 1, 0, -1\rangle = 1(1) + 2(0) + 3(-1) = -2 \\ \|\mathbf{a}\| &= \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14} \\ \mathbf{n} &= -2\langle 1, 2, 3\rangle - 14\langle 1, 0, -1\rangle = \langle -16, -4, 8\rangle = -4\langle 4, 1, -2\rangle \end{aligned}$$

The general form of a plane is

$$\mathbf{n} \cdot \mathbf{r} = d$$

for a general position vector r. Thus if $P = (1, 2, -1)$ is a point on the plane, its position vector must satisfy the above equation. It follows that

$$d = \mathbf{n} \cdot \overrightarrow{OP} = -4\langle 4, 1, -2\rangle \cdot \langle 1, 2, -1\rangle = -4(4(1) + 1(2) - 2(-1)) = -4(8)$$

Thus an equation for the plane is

$$-4(4x + y - 2z) = -4(8) \;\Leftrightarrow\; 4x + y - 2z = 8$$

20. Consider the plane $\mathcal{P}$ through three points $A = (1, 1, 1)$, $B = (2, 0, 1)$ and $C = (-1, 3, 2)$. Find all the planes that contain the segment AB and have the angle $\pi/3$ with the plane $\mathcal{P}$. Hint: see Study Problem **??**.

Solution: To find the angle between two planes, we must first find their normal vectors. The normal of $\mathcal{P}$ is parallel to $\overrightarrow{AB} \times \overrightarrow{AC}$, where $\overrightarrow{AB} = \langle 2, 0, 1\rangle - \langle 1, 1, 1\rangle = \langle 1, -1, 0\rangle$ and $\overrightarrow{AC} = \langle -1, 3, 2\rangle - \langle 1, 1, 1\rangle = \langle -2, 2, 1\rangle = -2\overrightarrow{AB} + \langle 0, 0, 1\rangle$. Thus the cross product reduces to $\overrightarrow{AB} \times (-2\overrightarrow{AB} + \langle 0, 0, 1\rangle) = \overrightarrow{AB} \times \langle 0, 0, 1\rangle$. We have

$$\overrightarrow{AB} \times \langle 0, 0, 1\rangle = \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \langle \det\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, -\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \det\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\rangle = \langle -1, -1, 0\rangle$$

So we may set the normal of $\mathcal{P}$ to be $\mathbf{n}_1 = \langle 1, 1, 0\rangle$.
Next, let the normal of the plane in question be given by $\mathbf{n} = \langle n_x, n_y, n_z\rangle$ where $\mathbf{n}$ is a normal vector. Then the angle between the two planes is

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}|}{\|\mathbf{n}_1\|\|\mathbf{n}\|} = \frac{|n_x + n_y|}{\sqrt{2}}$$

It follows that since $\theta = \pi/3$ that

$$|n_x + n_y| = \frac{\sqrt{2}}{2}$$

Next, we require AB to be in the plane. Therefore $\overrightarrow{AB}$ must be orthogonal to its normal. So,

$$\overrightarrow{AB} \cdot \mathbf{n} = 0 \;\Leftrightarrow\; n_x - n_y = 0$$

Squaring each obtained equation gives

$$n_x^2 + n_y^2 + 2n_x n_y = \frac{1}{2}$$

$$n_x^2 + n_y^2 - 2n_x n_y = 0$$

Adding the two, then multiplying by $1/2$ gives

$$n_x^2 + n_y^2 = \frac{1}{4}$$

Since $\mathbf{n}$ is a unit vector, it must be that

$$\sqrt{n_x^2 + n_y^2 + n_z^2} = 1 \;\Leftrightarrow\; (\frac{1}{4}) + n_z^2 = 1 \;\Leftrightarrow\; n_z = \pm\frac{\sqrt{3}}{2}$$

To find n_x and n_y, we must have two different cases.

Case 1: $n_x + n_y = 1$
We also have that $n_x - n_y = 0$. Adding the two then multiplying by $1/2$ gives $n_x = \frac{1}{2}$. So $n_y = \frac{1}{2}$. The possible normal vectors then are

$$\mathbf{n}_{11} = \langle\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{3}}{2}\rangle$$

$$\mathbf{n}_{12} = \langle\frac{1}{2}, \frac{1}{2}, -\frac{\sqrt{3}}{2}\rangle$$

Case 2: $n_x + n_y = -1$
We also have that $n_x - n_y = 0$. Adding the two then multiplying by $1/2$ gives $n_x = -\frac{1}{2}$. So $n_y = -\frac{1}{2}$. The possible normal vectors then are

$$\mathbf{n}_{21} = \langle-\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{3}}{2}\rangle$$

$$\mathbf{n}_{22} = \langle-\frac{1}{2}, -\frac{1}{2}, -\frac{\sqrt{3}}{2}\rangle$$

However, notice that $\mathbf{n}_{21} = -\mathbf{n}_{12}$ and $\mathbf{n}_{22} = -\mathbf{n}_{11}$. Therefore the planes generated in Case 1 are identical to those in Case 2.

The planes then are

$$\mathbf{n}_{11} \cdot \mathbf{r} = \mathbf{n}_{11} \cdot \overrightarrow{OA}, \quad \mathbf{n}_{12} \cdot \mathbf{r} = \mathbf{n}_{12} \cdot \overrightarrow{OA}$$

21. Find an equation of the plane that contains the line through $(1, 2, 3)$ and $(2, 1, 1)$ and cuts the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z = 0$ into two hemispheres.

Solution: In order for a plane to cut a sphere into two hemisphere, the plane must contain the center of the sphere. By completing the squares, the given equation is reduced to the standard form

$$\begin{aligned} & (x-1)^2 - 1 + (y+2)^2 - 4 + (z-3)^2 - 9 = 0 \\ \Rightarrow\quad & (x-1)^2 + (y+2)^2 + (z-3)^2 = 14 \end{aligned}$$

So, the center of the sphere is $C = (1, -2, 3)$. Since the plane in question contains there known points, a normal $\mathbf{n}$ of the plane is proportional to the cross product

$$\begin{aligned} & \overrightarrow{AB} \times \overrightarrow{AC} = \langle 1, -1, -2\rangle \times \langle 0, -4, 0\rangle = \langle -8, 0, -4\rangle = -4\langle 2, 0, 1\rangle \\ \Rightarrow \quad & \mathbf{n} = \langle 2, 0, 1\rangle \end{aligned}$$

Therefore, taking A as a particular point of the plane, the equation reads

$$2(x-1) + (z-3) = 0 \quad \text{or} \quad 2x + z = 5\,.$$

22. Find all planes perpendicular to $\mathbf{n} = \langle 1, 1, 1\rangle$ whose intersection with the ball $x^2 + y^2 + z^2 \le R^2$ is a disk of area $\pi R^2/4$.

Solution: For the intersection to be a disk of area $\pi R^2/4$, it must be that the radius of the disk is $r = R/2$. Construct a line $\mathcal{L}$ parallel to $\mathbf{n} = \langle 1, 1, 1\rangle$ that goes through the center of the sphere. There are two points on the line that, when a plane perpendicular to the line passes through the point, a disk of radius $r = R/2$ is formed. Next, consider the triangle connecting one such point, the center of the sphere, and a point where the disk of radius $r = R/2$ is tangent to the sphere. It will have one leg of length $R/2$ with a hypotenuse of length R. By the pythagoren theorem, the length of the other leg must be $\sqrt{R^2 - (R/2)^2} = \sqrt{3R^2/4} = \sqrt{3}R/2$. This is the distance between the center and one of the two points. The length of $\mathbf{n}$ is $\|\mathbf{n}\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$. Therefore we can find the two points by traveling parallel and antiparallel to $R/2\mathbf{n}$ from the origin. This gives $P = (R/2, R/2, R/2)$ and $Q = (-R/2, -R/2, -R/2)$ as the two points. An equation for each plane is as follows

$$x + y + z = \langle 1, 1, 1\rangle \cdot \langle \pm R/2, \pm R/2, \pm R/2\rangle = \pm 3R/2$$

23. Find an equation of the plane that is tangent to the sphere $x^2 + y^2 + z^2 - 2x - 4y - 6z + 11 = 0$ at the point $(2, 1, 2)$. *Hint*: What is the angle between a line tangent to a circle at a point P and the segment OP where O is the center of the circle? Extend this observation to a plane tangent to a sphere to determine a normal of the tangent plane.

Solution: The hint shows that $\overrightarrow{OP}$ is perpendicular to a line tangent to the circle at P. Therefore given a point P on a sphere, a plane tangent to the sphere at P will have a normal vector of $\overrightarrow{OP}$.

By completing the square, the given equation for the sphere becomes

$$\begin{aligned} &(x-1)^2 - 1 + (y-2)^2 - 4 + (z-3)^2 - 9 + 11 = 0 \\ \Rightarrow\quad &(x-1)^2 + (y-2)^2 + (z-3)^2 = 3 \end{aligned}$$

The center of the circle then is $O = (1, 2, 3)$. Therefore,

$$\overrightarrow{OP} = \langle 2, 1, 2\rangle - \langle 1, 2, 3\rangle = \langle 1, -1, -1\rangle$$

Thus an equation for a plane tangent to the given sphere at $(2, 1, 2)$ is

$$x - y - z = \langle 1, -1, -1\rangle \cdot \langle 2, 1, 2\rangle = 1(2) - 1(1) - 1(2) = -1$$

24. Find the family of planes through the point $(0, 0, a)$, $a > R$, that are tangent to the sphere $x^2+y^2+z^2 = R^2$. *Hint*: Compare this family with the family of planes in Exercise **15**.

SOLUTION: First notice that the family of planes can be generated by taking one plane that passes through $(0, 0, a)$ (for $a > R$) that is tangent to the sphere and rotating it about the z axis. Choose a plane. Suppose it is tangent to the point P on the sphere. Rotating this plane around the z axis corresponds to translating P along a circular path. This circular path lies in another plane containing P, parallel to the xy plane.

We are interested in finding the radius of this circular path, r. Because the path is parallel to the xy plane, it must be that there exists a point on the sphere whose y coordinate is 0. Call this point $S = (r, 0, s_z)$. Construct a line from $(0, 0, a)$ to S. By construction, this line lies in one of the planes in the family, so it must be that this line is tangent to the circular path, and thus the sphere, at S. Let the line be given by $z = -cx + a$ in the xz plane. The line intersects the sphere at the points whose x coordinate satisfy

$$x^2 + 0^2 + (-cx + a)^2 = R^2 \;\Leftrightarrow (1 + c^2)x^2 + (-2ca)x + (a^2 - R^2) = 0$$

For the line to be tangent, we only want one point. Thus there must be only one x coordinate, and only one solution to the above quadratic. This is achieved when the discriminant vanishes,

$$-(-2ca)^2 - 4(1 + c^2)(a^2 - R^2) = 0 \;\Leftrightarrow c = \sqrt{a^2 - R^2}/R$$

It follows that

$$r = \frac{-(-2ca)}{2(1 + c^2)} = \frac{\frac{a\sqrt{a^2-R^2}}{R}}{1 + \frac{a^2-R^2}{R^2}} = \frac{aR\sqrt{a^2 - R^2}}{R^2 + a^2 - R^2} = \frac{R\sqrt{a^2 - R^2}}{a}$$

We can also find s_z as follows:

$$r^2 + 0^2 + s_z^2 = R^2 \;\Leftrightarrow\; s_z^2 = R^2 - \left(\frac{R\sqrt{a^2 - R^2}}{a}\right)^2 = \frac{R^4}{a^2}$$

So $s_z = R^2/a$, where the positive root has been chosen since $a > R > 0$. The constant s_z can be interpreted as the height of the circular path above the xy plane.

We know that the normal of the plane must be parallel to $\overrightarrow{OP}$. We can easily find $\overrightarrow{OP}$ using our construction. First, recognize that it is the sum of two vectors, $s_z\hat{\mathbf{e}}_3$ and the vector connecting the center of the circular path to P (this latter vector is also horizontal). Notice that a vector from the center of the circular path to a point on the path will be of the form $r\langle \cos b, \sin b, 0\rangle$ for some b. The quantity b revolves the vector around the circular path, achieving the revolution necessary. This vector is very similar in form to that of exercise 15. Thus,

$$\overrightarrow{OP} = r\langle \cos b, \sin b, 0\rangle + s_z\hat{\mathbf{e}}_3 = \left\langle \frac{R\sqrt{a^2 - R^2}}{a}\cos b, \frac{R\sqrt{a^2 - R^2}}{a}\sin b, \frac{R^2}{a}\right\rangle$$

Finally, the plane must pass through the point $(0, 0, a)$. Therefore $d = \overrightarrow{OP} \cdot \langle 0, 0, a\rangle = R^2$. It follows that an equation for a distinct plane in the family is

$$\left(\frac{R\sqrt{a^2 - R^2}}{a}\cos b\right)x + \left(\frac{R\sqrt{a^2 - R^2}}{a}\sin b\right)y + \frac{R^2}{a}z = R^2$$

which can be reduced to

$$(\sqrt{a^2 - R^2}\cos b)x + (\sqrt{a^2 - R^2}\sin b)y + Rz = aR$$

25. Consider a sphere of radius R centered at the origin and two points P_1 and P_2 whose position vectors are $\mathbf{r}_1$ and $\mathbf{r}_2$. Suppose that $\|\mathbf{r}_1\| > R$ and $\|\mathbf{r}_2\| > R$ (the points are outside the sphere). Find the equation $\mathbf{n}\cdot\mathbf{r} = d$ of the plane through P_1 and P_2 whose distance from the sphere is maximal. What is the distance? Hint: Show first that a normal of the plane can always be written in the form $\mathbf{n} = \mathbf{r}_1 + c(\mathbf{r}_2 - \mathbf{r}_1)$. Then find a condition to determine the constant c.

7. Lines in Space

1–7. Find vector, parametric, and symmetric equations of the specified line.

1. The line containing the segment AB where $A = (1, 2, 3)$ and $B = (-1, 2, 4)$.

Solution: The vector, parametric, and symmetric equations of a line parallel to a vector $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ through a point $P = (x_0, y_0, z_0)$ are

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} \quad -\infty < t < \infty$$

$$x = x_0 + tv_1,\ y = y_0 + tv_2,\ z = z_0 + tv_3,\ -\infty < t < \infty$$

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}$$

Note that the last one, the symmetric equation of a line, only applies if $\mathbf{v} \neq \mathbf{0}$. If one component is zero, say $v_1 = 0$, then the symmetric equation becomes:

$$x = x_0,\ \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}$$

Therefore we need only find $\mathbf{v}$ and P. $\mathbf{v}$ is a vector parallel to the line, so we may simply choose $\mathbf{v} = \overrightarrow{AB} = \langle -1, 2, 4\rangle - \langle 1, 2, 3\rangle = \langle -2, 0, 1\rangle$. P is any point on the line, so we may choose $P = A = (1, 2, 3)$. Thus we have

$$\mathbf{r} = \langle 1, 2, 3\rangle + t\langle -2, 0, 1\rangle,\ -\infty < t < \infty$$

$$x = 1 - 2t,\ y = 2,\ z = 3 + t,\ -\infty < t < \infty$$

$$\frac{x - 1}{-2} = z - 3,\ y = 2$$

2. The lines containing the diagonals of the parallelogram whose adjacent sides at the vertex $(1, 0, -1)$ are $\mathbf{a} = \langle 1, 2, 3\rangle$ and $\mathbf{b} = \langle -1, 2, 1\rangle$.

Solution: Let the vertices of the parallelogram be $OABC$, where $O = (1, 0, -1)$, $\mathbf{a} = \overrightarrow{OA}$, and $\mathbf{b} = \overrightarrow{OB}$. Then the final vertex C is given by $\mathbf{a} + \mathbf{b}$. It follows that a line parallel to diagonal OC is parallel to $\mathbf{a} + \mathbf{b} = \langle 1, 2, 3\rangle + \langle -1, 2, 1\rangle = \langle 0, 4, 4\rangle$. Thus for the first line we have that a vector parallel to it is $\mathbf{v}_1 = \langle 0, 1, 1\rangle$. We must now choose any point on the line, say $P_1 = O = (1, 0, -1)$. So the vector, parametric, and symmetric equations of the line through diagonal OC are given by

$$\mathbf{r} = \langle 1, 0, -1\rangle + t\langle 0, 1, 1\rangle,\ -\infty < t < \infty$$

$$x = 1,\ y = t,\ z = -1 + t,\ -\infty < t < \infty$$

$$x = 1, \ y = z + 1$$

A vector parallel to diagonal AB is given by $\mathbf{a} - \mathbf{b} = \langle 1, 2, 3\rangle - \langle -1, 2, 1\rangle = \langle 2, 0, 2\rangle$. Thus a vector parallel to this line is $\mathbf{v}_2 = \langle 1, 0, 1\rangle$. We must now find a point on the line. We can choose $P_2 = A = (1, 0, -1) + (1, 2, 3) = (2, 2, 2)$. So the vector, parametric, and symmetric equations of the line through diagonal AB are given by

$$\mathbf{r} = \langle 2, 2, 2\rangle + t\langle 1, 0, 1\rangle, \ -\infty < t < \infty$$

$$x = 2 - t, \ y = 2, \ z = 2 - t, \ -\infty < t < \infty$$

$$\frac{x-2}{-1} = \frac{z-2}{-1}, \ y = 2 \ \Leftrightarrow \ x = z \ y = 2$$

3. The line through the vertex A of a triangle ABC and perpendicular to the sides AB and AC if $A = (1, 0, -1)$, $B = (-1, 1, 2)$, and $C = (2, -1, -2)$

Solution: A line is determined by a particular point P_0 in it and a non-zero vector $\mathbf{v}$ parallel to the line. Since the line passes through A,

$$P_0 = A = (1, 0, -1) = (x_0, y_0, z_0)\,.$$

The vector $\mathbf{v}$ must be perpendicular to the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and hence, by the geometrical properties of the cross product, be parallel to their cross product:

$$\left.\begin{array}{l} \mathbf{v} \perp \ \overrightarrow{AB} = \langle -2, 1, 3\rangle \\ \mathbf{v} \perp \overrightarrow{AC} = \langle 1, -1, -1\rangle \end{array}\right\} \quad \Rightarrow \quad \mathbf{v} = \overrightarrow{AB}\times\overrightarrow{AC} = \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ -2 & 1 & 3 \\ 1 & -1 & -1 \end{pmatrix} = \langle 2, 1, 1\rangle\,.$$

The symmetric and parametric equations of the line read, respectively,

$$\frac{x-x_0}{v_1} = \frac{y-y_0}{v_2} = \frac{z-z_0}{v_3} \quad \Rightarrow \quad \frac{x-1}{2} = y = z + 1$$

$$\left\{\begin{array}{l} x = x_0 + v_1 t \\ y = y_0 + v_2 t \\ z = z_0 + v_3 t \end{array}\right. \quad \Rightarrow \quad \left\{\begin{array}{l} x = 1 + 2t \\ y = t \\ z = -1 + t \end{array}\right.$$

4. The line through the vertex C of a triangle ABC and parallel to the edge AB if $A = (1, 0, -1)$, $B = (-1, 1, 2)$, and $C = (2, -1, -2)$

Solution: The line in question must be parallel to $\overrightarrow{AB}$. We have that $\overrightarrow{AB} = \langle -1, 1, 2\rangle - \langle 1, 0, -1\rangle = \langle = \langle -2, 1, 3\rangle$. Thus we can set $\mathbf{v} = \langle -2, 1, 3\rangle$. Since the line must pass through C, we can let

$P = C = (2, -1, -2)$. Thus the vector, parametric, and symmetric equations of a line through C parallel to side AB are given by

$$\mathbf{r} = \langle 2, -1, -2\rangle + t\langle -2, 1, 3\rangle, \ -\infty < t < \infty$$

$$x = 2 - 2t, \ y = -1 + t, \ z = -2 + 3t, \ -\infty < t < \infty$$

$$\frac{x-2}{-2} = y + 1 = \frac{z+2}{3}$$

5. A line through the origin that makes an angle of $60°$ with the x and y axes. Is such a line unique? Explain.

Solution: First, since such a line must pass through the origin, we can have $P = (0, 0, 0)$ for all such lines. Next, such a line is not unique. Let $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ be a unit vector parallel to the line. The angle between two lines is defined to be

$$\cos(\theta) = |\hat{\mathbf{v}}_1 \cdot \hat{\mathbf{v}}_2|$$

Note that the x and y axes can be represented as lines parallel to $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$, respectively, through the origin. So we must have that

$$|\mathbf{v} \cdot \hat{\mathbf{e}}_1| = |v_1| = \cos 60° = \frac{1}{2}$$

$$|\mathbf{v} \cdot \hat{\mathbf{e}}_2| = |v_2| = \cos 60° = \frac{1}{2}$$

So we have that $\mathbf{v} = \langle \pm 1/2, \pm 1/2, v_3\rangle$. Since we required $\mathbf{v}$ to be a unit vector, it must be that

$$\begin{aligned} \sqrt{v_1^2 + v_2^2 + v_3^2} &= 1 \\ 1/4 + 1/4 + v_3^2 &= 1 \\ v_3^2 &= 1/2 \\ v_3 &= \pm\sqrt{2}/2 \end{aligned}$$

Therefore we have that $\mathbf{u} = \langle \pm 1, \pm 1, \pm\sqrt{2}\rangle$ is a vector parallel to such a line (note that $\mathbf{u}$ is simply $2\mathbf{v}$). Thus the vector, parametric, and symmetric equations for such a line are given by

$$\mathbf{r} = \langle 0, 0, 0\rangle + t\langle \pm 1, \pm 1, \pm\sqrt{2}\rangle, \ -\infty < t < \infty$$

$$x = \pm t, \ y = \pm t, \ z = \pm\sqrt{2}t, \ -\infty < t < \infty$$

$$\pm x = \pm y = \pm\frac{z}{\sqrt{2}}$$

Note: There are $2^3 = 8$ combinations for the plus and minus, but since $-\mathbf{v}$ is parallel to $\mathbf{v}$, there are only four distinct lines.

6. The line through the vertex $A = (1, 2, 3)$ of a parallelogram with adjacent sides at A being $\mathbf{a} = \langle 1, 2, 2\rangle$ and $\mathbf{b} = \langle -2, 1, -2\rangle$ that bisects the angle of the parallelogram at A. *Hint*: See Exercise **21** in Section **??**.

Solution: By Exercise **21** in Section **??**, we know that $\mathbf{c} = \|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b}$ bisects the angle between vectors $\mathbf{a}$ and $\mathbf{b}$. Thus we have

$$\mathbf{c} = \sqrt{(-2)^2 + 1^2 + (-2)^2}\langle 1, 2, 2\rangle + \sqrt{1^2 + 2^2 + 2^2}\langle -2, 1, -2\rangle = 3\langle -1, 3, 0\rangle$$

The line in question must be parallel to this vector. Thus we may choose $\mathbf{v} = \langle -1, 3, 0\rangle$. Since the line must pass through $A = (1, 2, 3)$, we may choose $P = A$. Thus the line through vertex A that bisects the angle of the parallelogram at A is given by

$$\mathbf{r} = \langle 1, 2, 3\rangle + t\langle -1, 3, 0\rangle, \ -\infty < t < \infty$$
$$x = 1 - t, \ y = 2 + 3t, \ z = 3, \ -\infty < t < \infty$$
$$1 - x = \frac{y - 2}{3}, \ z = 3$$

7. The line parallel to the vector $\langle 1, -2, 0\rangle$ that contains a diameter of the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z = 0$.

Solution: By completing the square, the given equation for the sphere becomes

$$\begin{aligned} & (x-1)^2 - 1 + (y+2)^2 - 4 + (z-3)^2 - 9 = 0 \\ \Rightarrow \quad & (x-1)^2 + (y+2)^2 + (z-3)^2 = 14 \end{aligned}$$

Any line that contains a diameter of the sphere must include its center. Thus any such line passes through $(1, 2, 3)$. We may choose this point as P. We are given the vector $\mathbf{v}$ that the line is parallel to. Thus we have that the vector, parametric, and symmetric equations for such a line are

$$\mathbf{r} = \langle 1, 2, 3\rangle + t\langle 1, -2, 0\rangle, \ -\infty < t < \infty$$
$$x = 1 + t, \ y = 2 - 2t, \ z = 3, \ -\infty < t < \infty$$
$$x - 1 = \frac{2 - y}{2}, \ z = 3$$

8. Show that the line through $P_1 = (1, 2, -1)$ and parallel to $\mathbf{v}_1 = \langle 1, -1, 3\rangle$ coincides with the line through $P_2 = (0, 3, -4)$ and parallel to $\mathbf{v}_2 = \langle -2, 2, -6\rangle$ as points sets in space.

Solution: Since $\mathbf{v}_1 = -2\mathbf{v}_2$, the two lines are at least parallel. They coincide now iff they share the given points P_1 and P_2. If they share

P_1 and P_2, then $\overrightarrow{P_1P_2}$ must be parallel to $\mathbf{v}_1$ or $\mathbf{v}_2$ (it doesn't matter which since they are parallel). Thus we have

$$\overrightarrow{P_1P_2} = \langle 0, 3, -4\rangle - \langle 1, 2, -1\rangle = \langle -1, 1, -3\rangle = -\mathbf{v}_1$$

So $\overrightarrow{P_1P_2}$ is parallel to $\mathbf{v}_1$. Thus the lines coincide.

9. Do the lines $x - 1 = 2y = 3z$ and $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, where $\mathbf{r}_0 = \langle 7, 3, 2\rangle$ and $\mathbf{v} = \langle 6, 3, 2\rangle$, coincide as point sets in space?

Solution: Converting the second line into its symmetric equation yields

$$\frac{x-7}{6} = \frac{y-3}{3} = \frac{z-2}{2} \Leftrightarrow x-7 = 2y-6 = 3z-6 \Leftrightarrow x-1 = 2y = 3z$$

Since the lines share the same symmetric equation, they must be the same line. Hence they coincide.

10. Find parametric equations of the line through the point $(1, 2, 3)$ and perpendicular to the plane $x + y + 2z = 1$. Find the point of intersection of the line and the plane.

Solution: If the line is perpendicular to the given plane, then it must be parallel to the plane's normal vector, $\mathbf{n} = \langle 1, 1, 2\rangle$. We may choose $P = (1, 2, 3)$. Thus the parametric equations of the line are given by

$$x = 1 + t,\ y = 2 + t,\ z = 3 + 2t,\ -\infty < t < \infty$$

We can find the value of t when the line intersects with the plane as follows

$$(1 + t) + (2 + t) + 2(3 + 2t) = 1 \Leftrightarrow 6t + 9 = 1 \Leftrightarrow t = -\frac{4}{3}$$

So the point of intersection is

$$(1 - \frac{4}{3}, 2 - \frac{4}{3}, 3 - \frac{8}{3}) = (-\frac{1}{3}, \frac{2}{3}, \frac{1}{3})$$

11. Find parametric and symmetric equations of the line of intersection of the planes $x + y + z = 1$ and $2x - 2y + z = 1$.

Solution: Since the line of intersection lies in both planes, a vector parallel to the line must be orthogonal to both $\mathbf{n}_1 = \langle 1, 1, 1\rangle$ and

$\mathbf{n}_2 = \langle 2, -2, 1\rangle$. Thus we must compute $\mathbf{n}_1 \times \mathbf{n}_2$.

$$\mathbf{n}_1 \times \mathbf{n}_2 = \det \begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 1 & 1 & 1 \\ 2 & -2 & 1 \end{pmatrix} = \langle \det \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}, -\det \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \det \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \rangle = \langle 3, 1, -4\rangle$$

Thus we may set $\mathbf{v} = \langle 3, 1, -4\rangle$. Now we must find a point of intersection. We are free to choose one coordinate at will, so long as the corresponding component of $\mathbf{v}$ does not vanish (otherwise the parametric equation for this component is simply a constant). For example, we may choose $x_0 = 0$. By inspection, we can let $z_0 = 1$ and it easily follows that $y_0 = 0$. So we may let $P = (0, 0, 1)$. The parametric and symmetric equations of the line of intersection then are

$$x = 3t, \; y = t, \; z = 1 - 4t, \; -\infty < t < \infty$$

$$\frac{x}{3} = y = \frac{1 - z}{4}$$

12–13. Determine whether the given lines are are parallel, skew, or intersecting. If they intersect, find the point of intersection and the angle between the lines.

12. The line through the points $(1, 2, 3)$ and $(2, -1, 1)$ and the line through the points $(0, 1, 3)$ and $(1, 0, 2)$

SOLUTION: Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two lines, parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively, and passing through P_1 and P_2, respectively. Let $\mathbf{r}_{12} = \overrightarrow{P_1P_2}$, where P_1 is a point on $\mathcal{L}_1$ and P_2 is a point on $\mathcal{L}_2$. Two lines can be skew, intersecting, parallel, or coincidental. To determine which, it is useful to first compute $\mathbf{r}_{12} \cdot (\mathbf{v}_1 \times \mathbf{v}_2)$. Let $P_1 = (1, 2, 3)$ and $P_2 = (0, 1, 3)$. First,

$$\mathbf{v}_1 = \langle 1, 2, 3\rangle - \langle 2, -1, 1\rangle = \langle -1, 3, 2\rangle$$

$$\mathbf{v}_2 = \langle 1, 0, 2\rangle - \langle 0, 1, 3\rangle = \langle 1, -1, -1\rangle$$

$$\mathbf{r}_{12} = \langle 0, 1, 3\rangle - \langle 1, 2, 3\rangle = \langle -1, -1, 0\rangle$$

The triple product then is

$$\mathbf{r}_{12} \cdot (\mathbf{v}_1 \times \mathbf{v}_2) = \det \begin{pmatrix} -1 & -1 & 0 \\ -1 & 3 & 2 \\ 1 & -1 & -1 \end{pmatrix} = -\det \begin{pmatrix} 3 & 2 \\ -1 & -1 \end{pmatrix} + \det \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} + 0 \det \begin{pmatrix} -1 & 3 \\ 1 & -1 \end{pmatrix} = 0$$

So the three vectors are coplanar, and therefore the lines are not skew. Notice further that $\mathbf{v}_1$ and $\mathbf{v}_2$ are not parallel, so the lines are neither parallel nor coincidental. Thus they are intersecting. The parametric equations for each line are as follows

$$x = 1 - t,\ y = 2 + 3t,\ z = 3 + 2t,\ -\infty < t < \infty$$

$$x = s,\ y = 1 - s,\ z = 3 - s,\ -\infty < t < \infty$$

Notice that two different parameters have been used. This is because the lines may intersect at different "times" (the parameters may attain different values at the point of intersection). Equating each coordinate yields

$$\begin{cases} 1 - t = s \\ 2 + 3t = 1 - s \\ 3 + 2t = 3 - s \end{cases} \Rightarrow \begin{cases} t - 1 = -s \\ 2 + 3t = 1 - s \\ 3 + 2t = 3 - s \end{cases}$$

We can then add 1 to each side of the first equation and substitute it into the second, which gives

$$2 + 3t = t \iff t = -1$$

Therefore the point of intersection is at

$$(1 - (-1), 2 + 3(-1), 3 + 2(-1)) \iff (2, -1, 1)$$

We can verify this is correct by solving for s. Using the first equation, $s = 1 - (-1) = 2$. Substituting this into the parametric equations for y and z yield $y = 1 - (2) = -1$ and $z = 3 - (2) = 1$, which are correct.

Next we must find the angle between the two lines. The angle is defined to be

$$\cos\theta = \frac{|\mathbf{v}_1 \cdot \mathbf{v}_2|}{\|\mathbf{v}_1\|\|\mathbf{v}_2\|}$$

Thus the angle between the two lines is given by

$$\cos\theta = \frac{|\langle -1, 3, 2\rangle \cdot \langle 1, -1, -1\rangle|}{\sqrt{(-1)^2 + 3^2 + 2^2}\sqrt{1^2 + (-1)^2 + (-1)^2}} = \frac{|-1(1) + 3(-1) + 2(-1)|}{\sqrt{14}\sqrt{3}} = \frac{6}{\sqrt{14}\sqrt{3}}$$

So $\theta = \arccos \frac{6}{\sqrt{14}\sqrt{3}}$.

13. The lines $x = 1 + 2t$, $y = 3t$, $z = 2 - t$ and $x + 1 = y - 4 = (z - 1)/3$.

Solution: Using the intuition developed in Exercise 12, we must find $\mathbf{r}_{12} \cdot (\mathbf{v_1} \times \mathbf{v}_2)$. Let $P_1 = (1, 3, 2)$, a point on the first line (this is found by looking at the constant terms in the parametric equations). Let $P_2 = (-1, 4, 1)$, a point on the second line (this is found by recalling the definition of the symmetric equation of a line). We find $\mathbf{v}_1 = \langle 2, 3, -1\rangle$ and $\mathbf{v}_2 = \langle 1, 1, 3\rangle$. We calculate $\mathbf{r}_{12}$ as $\mathbf{r}_{12} = \langle -1, 4, 1\rangle - \langle 1, 3, 2\rangle = \langle -2, 1, -1\rangle$ Then the triple product is

$$\mathbf{r}_{12}\cdot(\mathbf{v}_1\times\mathbf{v}_2) = \det\begin{pmatrix} -2 & 1 & -1 \\ 2 & 3 & -1 \\ 1 & 1 & 3 \end{pmatrix} = -2\det\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} - \det\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} - \det\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} = -26$$

So the vectors are not coplanar. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are not parallel, it must be that the vectors are skew, and thus do not intersect.

14. Let $\overrightarrow{AB} = \langle 1, 2, 2\rangle$, $\overrightarrow{AC} = \langle 2, -1, -2\rangle$, and $\overrightarrow{AD} = \langle 0, 3, 4\rangle$ be the adjacent sides of a parallelepiped. Show that the diagonal of the parallelepiped extended from the vertex A intersects the diagonal extended from the vertex D and find the angle between the diagonals.

Solution: First, the diagonal extending from vertex A is parallel to $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD}$. So we may let $\mathbf{v}_1 = \overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD}$. Second, the diagonal extending from vertex D is parallel to $\overrightarrow{AB} + \overrightarrow{AC} - \overrightarrow{AD}$. So we may let $\mathbf{v}_2 = \overrightarrow{AB} + \overrightarrow{AC} - \overrightarrow{AD}$. We may let $P_1 = A$ as a point on the first line (the diagonal through A) and $P_2 = D$ as a point on the second line (the diagonal through D). Let $\mathbf{r}_{12} = \overrightarrow{P_1P_2} = \overrightarrow{AD}$. For the diagonals to be intersecting, the triple product $\mathbf{r}_{12}\cdot(\mathbf{v}_1\times\mathbf{v}_2)$ must vanish. Computing the triple product yields:

$$\begin{aligned}
\mathbf{r}_{12}\cdot(\mathbf{v}_1\times\mathbf{v}_2) &= \mathbf{r}_{12}\cdot((\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD})\times(\overrightarrow{AB} + \overrightarrow{AC} - \overrightarrow{AD})) \\
&= \mathbf{r}_{12}\cdot((\overrightarrow{AB} + \overrightarrow{AC})\times(\overrightarrow{AB} + \overrightarrow{AC}) - (\overrightarrow{AB} + \overrightarrow{AC})\times\overrightarrow{AD} + (\overrightarrow{AB} + \overrightarrow{AC})\times\overrightarrow{AD} \\
&\qquad + \overrightarrow{AD}\times\overrightarrow{AD}) \\
&= \mathbf{r}_{12}\cdot\mathbf{0} = 0
\end{aligned}$$

Next we must check if $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel. Their dot product is

$$\begin{aligned}
\mathbf{v}_1\cdot\mathbf{v}_2 &= (\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD})\cdot(\overrightarrow{AB} + \overrightarrow{AC} - \overrightarrow{AD}) \\
&= (\overrightarrow{AB} + \overrightarrow{AC})\cdot(\overrightarrow{AB} + \overrightarrow{AC}) - (\overrightarrow{AB} + \overrightarrow{AC})\cdot\overrightarrow{AD} + (\overrightarrow{AB} + \overrightarrow{AC})\cdot\overrightarrow{AD} \\
&\qquad + \overrightarrow{AD}\cdot\overrightarrow{AD} \\
&= \|\overrightarrow{AB} + \overrightarrow{AC}\|^2 + \|\overrightarrow{AD}\|^2
\end{aligned}$$

Suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel. Then

$$\begin{aligned}|\mathbf{v}_1 \cdot \mathbf{v}_2| &= \|\mathbf{v}_1\|\|\mathbf{v}_2\| \\ &= \|\overrightarrow{AB} + \overrightarrow{AB} + \overrightarrow{AD}\|\|\overrightarrow{AB} + \overrightarrow{AB} - \overrightarrow{AD}\| \\ &\leq (\|\overrightarrow{AB} + \overrightarrow{AC}\| + \|\overrightarrow{AD}\|)(\|\overrightarrow{AB} + \overrightarrow{AC}\| - \|\overrightarrow{AD}\|) \\ &\leq \|\overrightarrow{AB} + \overrightarrow{AC}\|^2 + \|\overrightarrow{AD}\|\|\overrightarrow{AB} + \overrightarrow{AC}\| - \|\overrightarrow{AD}\|\|\overrightarrow{AB} + \overrightarrow{AC}\| - \|\overrightarrow{AD}\|^2 \\ &\leq \|\overrightarrow{AB} + \overrightarrow{AC}\|^2 - \|\overrightarrow{AD}\|^2\end{aligned}$$

where the Cauchy-Schwarz inequality has been used. Comparing this to what we previously obtained yields

$$|\|\overrightarrow{AB} + \overrightarrow{AC}\|^2 + \|\overrightarrow{AD}\|^2| \leq \|\overrightarrow{AB} + \overrightarrow{AC}\|^2 - \|\overrightarrow{AD}\|^2 \;\Leftrightarrow\; 2\|\overrightarrow{AD}\|^2 \leq 0$$

Since $\|\overrightarrow{AD}\| \geq 0$ by definition, it must be that $\|\overrightarrow{AD}\| = 0$. But this is not true. Hence $\mathbf{v}_1$ and $\mathbf{v}_2$ are not parallel. Therefore they must intersect.

To find the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$, we must first compute them as follows

$$\mathbf{v}_1 = \overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} = \langle 1, 2, 2\rangle + \langle 2, -1, -2\rangle + \langle 0, 3, 4\rangle = \langle 3, 4, 4\rangle$$
$$\mathbf{v}_2 = \overrightarrow{AB} + \overrightarrow{AC} - \overrightarrow{AD} = \langle 1, 2, 2\rangle + \langle 2, -1, -2\rangle - \langle 0, 3, 4\rangle = \langle 3, -2, -4\rangle$$

The angle them is

$$\theta = \arccos\left(\frac{|\langle 3, 4, 4\rangle \cdot \langle 3, -2, -4\rangle|}{\sqrt{3^2 + 4^2 + 4^2}\sqrt{3^2 + (-2)^2 + (-4)^2}}\right) = \arccos \frac{15}{\sqrt{41}\sqrt{29}}$$

15. Are the four lines containing the diagonals of a parallelepiped intersecting at a point? Prove your answer. If they are intersecting, find the position vector of the point of intersection relative to a vertex at which the adjacent sides of the parallelepiped are $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

Solution: Let A, B, C, D be points such that $\overrightarrow{AB} = \mathbf{b}$, $\overrightarrow{AC} = \mathbf{c}$, and $\overrightarrow{AD} = \mathbf{c}$. Notice that in Exercise 14, the choice of using the diagonal extended from point D was more or less arbitrary. We could have chosen point B, and the vector would have been $\mathbf{v}_3 = -\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD}$, or we could have chosen point C, and the vector would have been $\mathbf{v}_3 = \overrightarrow{AB} - \overrightarrow{AC} + \overrightarrow{AD}$. Note that all that changes is where the minus sign is. If we chose point B, we could rename it to something like D' and rename D to B'. The argument then looks nearly identical to the one presented in Exercise 15, with the exception of changing B to B'

and D to D'. We can do the same thing if we had chosen point C. Thus all the diagonal extending from each of B, C, and D intersect the one extending from A.

Let us continue working with the diagonal extending from point D. The line that contains this diagonal is given by the vector equation

$$\mathbf{r} = \overrightarrow{OD} + t(\overrightarrow{AB} + \overrightarrow{AC} - \overrightarrow{AD}) = \overrightarrow{OD} + t(\mathbf{a} + \mathbf{b} - \mathbf{c}), \ -\infty < t < \infty$$

and the vector equation for the line containing the diagonal extended from point A is given by

$$\mathbf{r} = \overrightarrow{OA} + s(\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD}) = \overrightarrow{OA} + s(\mathbf{a} + \mathbf{b} + \mathbf{c}), \ -\infty < s < \infty$$

note that a different parameter has to be used.

We know these lines intersect, so we can equate them

$$\begin{aligned} \overrightarrow{OD} + t(\mathbf{a} + \mathbf{b} - \mathbf{c}) &= \overrightarrow{OA} + s(\mathbf{a} + \mathbf{b} + \mathbf{c}) \\ \overrightarrow{AD} + t(\mathbf{a} + \mathbf{b} - \mathbf{c}) - s(\mathbf{a} + \mathbf{b} + \mathbf{c}) &= 0 \\ \mathbf{c} + (t - s)\mathbf{a} + (t - s)\mathbf{b} + (-t - s)\mathbf{c} &= 0 \\ (t - s)\mathbf{a} + (t - s)\mathbf{b} + (1 - t - s)\mathbf{c} &= 0 \end{aligned}$$

The only way this can be true for any arbitrary noncoplanar, nonparallel, nonzero vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ must be if $t - s = 0$ and $1 - t - s = 0$. So $t = s$, and substituting this into the second equation yields

$$1 - s - s = 0 \ \Leftrightarrow \ 1 = 2s \ \Leftrightarrow \ s = \frac{1}{2}$$

We can substitute this into the second vector equation to give

$$\mathbf{r} = \overrightarrow{OA} + \frac{1}{2}(\mathbf{a} + \mathbf{b} + \mathbf{c})$$

Relative to A, this point of intersection is at $\frac{1}{2}(\mathbf{a} + \mathbf{b} + \mathbf{c})$.
The same answer is produced by using the first equation, but some tedious vector manipulations are involved.

The same argument as in the beginning paragraph can be used to show this is where all the diagonals intersect.

16. Find vector and parametric equations of the straight line segment from the point $(1, 2, 3)$ to the point $(-1, 1, 2)$.

Solution: Let $A = (1, 2, 3)$ and $B = (-1, 1, 2)$. The line segment from A to B must be parallel to $\overrightarrow{AB} = \langle -1, 1, 2\rangle - \langle 1, 2, 3\rangle = \langle -2, -1, -1\rangle$.

Thus we may choose it to be $\mathbf{v} = \langle -2, -1, -1\rangle$. Since we want the line segment to emanate from A, we should choose it as our representative point. Thus the vector and parametric equations are:

$$\mathbf{r} = \langle 1, 2, 3\rangle + t\langle -2, -1, -1\rangle$$

$$x = 1 - 2t,\ y = 2 - t,\ z = 3 - t$$

You may have noticed that I did not include an interval for the parameter t. This is because we must find it ourselves. We want t to range so that $\mathbf{r}$ ranges from $\overrightarrow{OA}$ to $\overrightarrow{OB}$. Thus we have:

$$\langle 1, 2, 3\rangle = \langle 1, 2, 3\rangle + t\langle -2, -1, -1\rangle \iff t = 0$$

$$\langle -1, 1, 2\rangle = \langle 1, 2, 3\rangle + t\langle -2, -1, -2\rangle \iff \langle -2, -1, -1\rangle = t\langle -2, -1, -2\rangle \iff t = 1$$

The vector and parametric equations then are

$$\mathbf{r} = \langle 1, 2, 3\rangle + t\langle -2, -1, -1\rangle,\ 0 \le t \le 1$$

$$x = 1 - 2t,\ y = 2 - t,\ z = 3 - t,\ 0 \le t \le 1$$

17. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be position vectors of two points in space. Find a vector equation of the straight line segment from $\mathbf{r}_1$ to $\mathbf{r}_2$.

Solution: Clearly the line must be parallel to the vector from $\mathbf{r}_1$ to $\mathbf{r}_2$. This vector is easily found to be $\mathbf{r}_2 - \mathbf{r}_1$ (use the parallelogram law to visualize if necessary). We should choose the point whose position vector is $\mathbf{r}_1$ as our point, because this is where the line starts from. Thus we have

$$\mathbf{r} = \mathbf{r}_1 + t(\mathbf{r}_2 - \mathbf{r}_1)$$

You may have noticed I did not include an interval for t. Every parametric equation must include an interval that defines the possible values of t. Here, we must find it ourselves. We need $\mathbf{r}$ to range from $\mathbf{r}_1$ to $\mathbf{r}_2$, so we have

$$\mathbf{r}_1 = \mathbf{r}_1 + t(\mathbf{r}_2 - \mathbf{r}_1) \iff t = 0$$

$$\mathbf{r}_2 = \mathbf{r}_1 + t(\mathbf{r}_2 - \mathbf{r}_1) \iff (1-t)\mathbf{r}_2 = (1-t)\mathbf{r}_1 \iff t = 1$$

Thus we have $0 \le t \le 1$. So the vector equation is

$$\mathbf{r} = \mathbf{r}_1 + t(\mathbf{r}_2 - \mathbf{r}_1),\ 0 \le t \le 1$$

18. Find the distance from the point $(1, 2, 3)$ to the line $2x = y + 1$, $z = 3$.

Solution: For an arbitrary line, with $\mathbf{v}$ a vector parallel to it, P_0

a point on the line, and P_1 an arbitrary point, the distance between the line and P_1 is given by

$$D = \frac{\|\mathbf{v} \times \overrightarrow{P_0P_1}\|}{\|\mathbf{v}\|}$$

We find from the symmetric equation of the line that it is parallel to $\mathbf{v} = \langle 1, 2, 0\rangle$ and passes through $(0, -1, 3)$. Let $P_0 = (0, -1, 3)$ and $P_1 = (1, 2, 3)$. Then $\overrightarrow{P_0P_1} = \langle 1, 2, 3\rangle - \langle 0, -1, 3\rangle = \langle 1, 3, 0\rangle$. So,

$$\mathbf{v}\times\overrightarrow{P_0P_1} = \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{pmatrix} = \langle \det\begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix}, -\det\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \det\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\rangle = \langle 0, 0, 1\rangle$$

Then $\|\mathbf{v} \times \overrightarrow{P_0P_1}\| = 1$. Furthermore,

$$\|\mathbf{v}\| = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}$$

It follows that the distance is

$$D = \frac{1}{\sqrt{5}}$$

19. Consider the plane $x + y - z = 0$ and a point $P = (1, 1, 2)$ in it. Find parametric equations of the lines through the origin that lie in the plane and are at a distance of 1 unit from P. *Hint*: A vector parallel to these lines can be taken in the form $\mathbf{v} = \langle 1, c, 1 + c\rangle$ where c is to be determined. Explain why!

Solution: Let $\mathbf{v}$ be a vector parallel to the line. Then $\mathbf{v}$ is orthogonal to the normal of the plane, $\mathbf{n} = \langle 1, 1, -1\rangle$. Recall that the distance between a line and a point P_1 is given by

$$D = \frac{\|\mathbf{v} \times \overrightarrow{P_0P_1}\|}{\|\mathbf{v}\|}$$

where $\mathbf{v}$ is parallel to the line and P_0 is a point on the line. Since the line in question passes through the origin, we may choose $P_0 = O = (0, 0, 0)$. Note that $P_1 = (1, 1, 2)$ is a point on the plane, and the plane passes through the origin. Then $\overrightarrow{OP_1}$ is in the plane. So $\overrightarrow{OP_1}$ is orthogonal to the normal as well. Then $\mathbf{v} \times \overrightarrow{P_0P_1}$ is parallel to the normal of the plane. Let $\mathbf{v} \times \overrightarrow{P_0P_1} = s\mathbf{n}$, where $\mathbf{v} = \langle v_x, v_y, v_z\rangle$. Then,

$$\begin{aligned}\mathbf{v} \times \overrightarrow{P_0P_1} &= \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ v_x & v_y & v_z \\ 1 & 1 & 2 \end{pmatrix} = \langle \det\begin{pmatrix} v_y & v_z \\ 1 & 2 \end{pmatrix}, -\det\begin{pmatrix} v_x & v_z \\ 1 & 2 \end{pmatrix}, \det\begin{pmatrix} v_x & v_y \\ 1 & 1 \end{pmatrix}\rangle \\ &= \langle 2v_y - v_z, v_z - 2v_x, v_x - v_y\rangle\end{aligned}$$

Notice that adding the first two equations gives a multiple of the third equation. This means there are too many degrees of freedom. We are free to choose one of the coordinates, say $v_x = 1$. Then the second and third equations become $v_z = s+2$ and $v_y = 1+s$. The transformation $c = s+1$ gives the vector in the hint. So we have $\mathbf{v} = \langle 1, c, 1+c\rangle$ for some constant c. Note that $\mathbf{v} \times \overrightarrow{P_0P_1} = (c-1)\mathbf{n}$. So

$$\|\mathbf{v} \times \overrightarrow{P_0P_1}\| = |(c-1)|\|\mathbf{n}\| = |c-1|\sqrt{1^2+1^2+(-1)^2} = |c-1|\sqrt{3}$$

Next, we have

$$\|\mathbf{v}\| = \sqrt{1^2+c^2+(1+c)^2}\sqrt{2c^2+2c+2}$$

By the distance formula, we have

$$1 = \frac{\|\mathbf{v} \times \overrightarrow{P_0P_1}\|}{\|\mathbf{v}\|} \;\Leftrightarrow\; \sqrt{2c^2+2c+2} = |c-1|\sqrt{3}$$

Then,

$$\begin{aligned} \sqrt{2c^2+2c+2} &= |c-1|\sqrt{3} \\ 2c^2+2c+2 &= 3(c-1)^2 \\ 2c^2+2c+2 &= 3c^2-6c+3 \\ c^2-8c+1 &= 0 \end{aligned}$$

Applying the quadratic formula to this gives

$$c = \frac{-(-8) \pm \sqrt{(-8)^2-4(1)(1)}}{2(1)} = \frac{8 \pm \sqrt{64-4}}{2} = \frac{8 \pm 2\sqrt{15}}{2} = 4 \pm \sqrt{15}$$

So $\mathbf{v} = \langle 1, 4 \pm \sqrt{15}, 5 \pm \sqrt{15}\rangle$. Then the parametric equations of the lines are

$$x = t,\; y = (4 \pm \sqrt{15})t,\; z = (5 \pm \sqrt{15})t,\; -\infty < t < \infty$$

20. Find the parallel lines intersecting the line $x = 2+t$, $y = 1+t$, $z = 2+2t$ at a right angle and parallel to the plane $x+2y-2z = 1$ that are at a distance of 1 unit from the plane. *Hint*: Find values of t at which the distance from a point in the given line to the plane is 1. This determines the points of intersection of the lines in question with the given line.

Solution: First using the hint, we should find the two points on the line that are a distance of 1 unit from the plane. The distance between a point on the line and the plane is the length of the perpendicular segment from the point to the plane. Note that this segment is perpendicular to the plane, and is thus parallel to its normal, $\mathbf{n} = \langle 1, 2, -2\rangle$. We find when the line intersects the plane as follows:

$$(2+t) + 2(1+t) - 2(2+2t) = 1 \iff t = -1$$

Let $P = (1, 0, 0)$, the point of intersection. Let Q be a point on the line a distance of 1 unit from the plane. Then $\overrightarrow{PQ} = \langle 2+t, 1+t, 2+2t\rangle - \langle 1, 0, 0\rangle = \langle 1+t, 1+t, 2+2t\rangle = (1+t)\langle 1, 1, 2\rangle$. The absolute value of the scalar projection of $\overrightarrow{PQ}$ onto $\mathbf{n}$ gives the distance of Q to the plane. Recall the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$b_{||} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}$$

So we have that the scalar projection s is

$$s = \frac{(1+t)\langle 1,1,2\rangle\cdot\langle 1,2,-2\rangle}{\sqrt{1^2+2^2+(-2)^2}} = \frac{(1+t)(1(1)+1(2)+2(-2)}{\sqrt{9}} = \frac{(1+t)(-1)}{3}$$

By the above analysis, we require $|s| = 1$, so

$$\frac{|1+t|}{3} = 1 \iff 1+t = \pm 3 \iff t = -4, 2$$

We can find the coordinates of Q as follows:

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} \iff \overrightarrow{OQ} = \overrightarrow{PQ} + \overrightarrow{OP}$$

So the possible coordinates are:

$$\begin{aligned}\overrightarrow{OQ_1} &= \langle 1-4, 1-4, 2+2(-4)\rangle + \langle 1,0,0\rangle = \langle -2,-3,-6\rangle\\ \overrightarrow{OQ_2} &= \langle 1+2, 1+2, 2+2(2)\rangle + \langle 1,0,0\rangle = \langle 4,3,6\rangle\end{aligned}$$

Any such line in question is parallel to the plane. Thus they are orthogonal to the plane's normal vector. The problem requires the lines to also be perpendicular to the given line, so they must be orthogonal to $\mathbf{v} = \langle 1, 1, 2\rangle$. This means the vector parallel to any line in question must be orthogonal to both $\mathbf{n}$ and $\mathbf{v}$. In other words it is parallel to $\mathbf{n}\times\mathbf{v}$. The cross product is

$$\mathbf{n}\times\mathbf{v} = \det\begin{pmatrix}\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ 1 & 2 & -2\\ 1 & 1 & 2\end{pmatrix} = \langle \det\begin{pmatrix}2 & -2\\ 1 & 2\end{pmatrix}, -\det\begin{pmatrix}1 & -2\\ 1 & 2\end{pmatrix}, \det\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}\rangle = \langle 6,-4,-1\rangle$$

Let $\mathbf{v}_1 = \langle 6, -4, -1\rangle$. Then the two lines in question are given by

$$\mathbf{r} = \overrightarrow{OQ_1} + t\mathbf{v}_1,\ -\infty < t < \infty$$

$$\mathbf{r} = \overrightarrow{OQ_2} + t\mathbf{v}_1,\ -\infty < t < \infty$$

21. Find parametric equations of the line that is parallel to $\mathbf{v} = \langle 2, -1, 2\rangle$ and goes through the center of the sphere $x^2 + y^2 + z^2 = 2x + 6z - 6$. Restrict the range of the parameter to describe the part of the line that is inside the sphere.

SOLUTION: To determine the center of the sphere (and, hence, a particular point of the line in question), the equation of the sphere should be transformed to the standard form by completing the squares:

$$\begin{aligned} & x^2 + y^2 + z^2 - 2x - 6z = -6 \\ \Rightarrow\quad & (x-1)^2 - 1 + y^2 + (z-3)^2 - 9 = -6 \\ \Rightarrow\quad & (x-1)^2 + y^2 + (z-3)^2 = 4 \end{aligned}$$

So the sphere has radius 2 and its center is at the point $(1, 0, 3)$. Hence the parametric equations of the line are

$$x = 1 + 2t\,, \quad y = -t\,, \quad z = 3 + 2t$$

The points of intersection of the line and the sphere are determined by the values of t at which the point $(1 + 2t, -t, 3 + 2t)$ satisfies the equation of the sphere:

$$(2t)^2 + (-t)^2 + (2t)^2 = 4 \quad \Leftrightarrow \quad 9t^2 = 4 \quad \Leftrightarrow \quad t = \pm\frac{2}{3}$$

Since the center of the sphere corresponds to $t = 0$, the line segment inside the sphere is described by restricting the range of t to the interval: $-\frac{2}{3} \le t \le \frac{2}{3}$.

22. Let the line $\mathcal{L}_1$ pass through the point $A(1, 1, 0)$ parallel to the vector $\mathbf{v} = \langle 1, -1, 2\rangle$ and let the line $\mathcal{L}_2$ pass through the point $B(2, 0, 2)$ parallel to the vector $\mathbf{w} = \langle -1, 1, 2\rangle$. Show that the lines are intersecting. Find the point C of intersection and parametric equations of the line $\mathcal{L}_3$ through C that is perpendicular to $\mathcal{L}_1$ and $\mathcal{L}_2$.

SOLUTION: The parametric equations for lines $\mathcal{L}_1$ and $\mathcal{L}_2$ are

$$x = 1 + t,\ y = 1 - t,\ z = 2t,\ -\infty < t < \infty$$

$$x = 2 - s,\ y = s,\ z = 2 + 2s,\ -\infty < s < \infty$$

If the lines are intersecting, there will be a unique pair of real numbers (t_0, s_0) such that substituting the respective parameter into the parametric equations yield identical points. Thus we attempt to find a solution by setting the parametric equation for each coordinate equal to each other. The following three equations are obtained: $1+t = 2-s$,

$1-t=s$, $2t=2+2s \;\Leftrightarrow\; t=1+s$. Substituting the third equation into the first yields

$$1+(1+s)=2-s \;\Leftrightarrow\; 2+s=2-s \;\Leftrightarrow\; s=0$$

Thus $t_0=1$ and $s_0=0$. Using the parametric equations for $\mathcal{L}_1$, the point of intersection is

$$(1+1,1-1,2(1))=(2,0,2)$$

This is actually point B, so line $\mathcal{L}_2$ trivially passes through it.

A line $\mathcal{L}_3$ that passes through B and is perpendicular to $\mathcal{L}_1$ and $\mathcal{L}_2$ must be parallel to a vector that is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$. Thus it is parallel to their cross product

$$\mathbf{v}\times\mathbf{w}=\det\begin{pmatrix}\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ 1 & -1 & 2\\ -1 & 1 & 2\end{pmatrix}=\langle\det\begin{pmatrix}-1 & 2\\ 1 & 2\end{pmatrix},-\det\begin{pmatrix}1 & 2\\ -1 & 2\end{pmatrix},\det\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\rangle=\langle -4,-4,0\rangle$$

Thus we may choose $\mathbf{u}=\langle 1,1,0\rangle=-\frac{1}{4}(\mathbf{v}\times\mathbf{w})$. The parametric equations of $\mathcal{L}_3$ then are

$$x=2+t,\; y=t,\; z=2,\; -\infty<t<\infty$$

23. Find parametric equations of the line through $(1,2,5)$ that is perpendicular to the line $x-1=1-y=z$ and intersects this line.

Solution: Parametric equations of the given line are $x=1+t$, $y=1-t$, $z=t$. This line is parallel to $\mathbf{u}=\langle 1,-1,1\rangle$. The point $P_t=(1+t,1-t,t)$ is a generic point of the given line. Any line through A and P_t intersects the given line, but not at the right angle. However, there should exists a particular value of t at which the vector $\overrightarrow{AP_t}=\langle t,-1-t,t-5\rangle$ is perpendicular to the given line or $\mathbf{u}$ and, hence, the line parallel to such $\overrightarrow{AP_t}$ is the line in question:

$$\overrightarrow{AP_t}\perp\mathbf{u}\quad\Rightarrow\quad\overrightarrow{AP_t}\cdot\mathbf{u}=0\quad\Rightarrow\quad t+(1+t)+t-5=0\quad\Rightarrow\quad t=\frac{4}{3}$$

Thus, parametric equations of the line in question $\mathcal{L}$ are:

$$\mathcal{L}\parallel\mathbf{v}=\overrightarrow{AP_t}\Big|_{t=\frac{4}{3}}=\frac{1}{3}\langle 4,\,-7,\,-11\rangle$$
$$\Rightarrow\quad x=1+4t\,,\; y=2-7t\,,\; z=5-11t\,.$$

Note that a non-zero scaling factor in $\mathbf{v}$ may always be changed.
Alternative solutions: A vector parallel to the line in question

may also be found in other ways. Let $\mathbf{w} = \overrightarrow{AP_0} = \langle 0, -1, -5\rangle$ (the vector from A to a particular point of the given line). Then

$$\mathbf{v} = \mathbf{u} \times (\mathbf{u} \times \mathbf{w})$$

Indeed, by the geometrical properties of the cross product, $\mathbf{u} \times \mathbf{w}$ is perpendicular to the plane containing A and the given line. Therefore, $\mathbf{v}$ is perpendicular to both $\mathbf{u} \times \mathbf{w}$ and $\mathbf{u}$ and, hence, parallel to the line in question. One can also make an orthogonal decomposition of $\mathbf{w}$ relative to $\mathbf{u}$: $\mathbf{w} = \mathbf{w}_\parallel + \mathbf{w}_\perp$ where $\mathbf{w}_\parallel$ is the vector projection of $\mathbf{w}$ onto $\mathbf{u}$. Then $\mathbf{w}_\perp$ is parallel to the line in question:

$$\mathbf{v} = \mathbf{w}_\perp = \mathbf{w} - \mathbf{w}_\parallel = \mathbf{w} - \frac{\mathbf{w} \cdot \mathbf{u}}{\|\mathbf{u}\|^2}\,\mathbf{u}$$

The two above solutions are related to one another through the "bac-cab" rule applied to the double cross cross product in the former solution.

24. Find the distance between the lines $x = y = z$ and $x + 1 = y/2 = z/3$.

Solution: Let $\mathcal{L}_1$ be defined by the symmetric equation $x = y = z$ and $\mathcal{L}_2$ be defined by the symmetric equation $x + 1 = y/2 = z/3$. Then $\mathcal{L}_1$ is parallel to $\mathbf{v}_1 = \langle 1, 1, 1\rangle$ and $\mathcal{L}_2$ is parallel to $\mathbf{v}_2 = \langle 1, 2, 3\rangle$. Clearly, $\mathbf{v}_1$ and $\mathbf{v}_2$ are not parallel. It is also fairly obvious that the lines do not intersect, for if they did, then we would have $x+1 = x/2 = x/3$. Solving $x/2 = x/3$ gives $x = 0$. But $0 + 1 \neq 0$, so there is no solution. Thus the lines are skew. Recall the distance between skew lines is given by

$$D = \frac{|\overrightarrow{AC} \cdot (\overrightarrow{AB} \times \overrightarrow{CP})|}{\|\overrightarrow{AB} \times \overrightarrow{CP}\|}$$

where A, B are distinct points on $\mathcal{L}_1$ and C, P are distinct points on $\mathcal{L}_2$. We may choose A, B and C, P such that $\overrightarrow{AB} = \mathbf{v}_1$ and $\overrightarrow{CP} = \mathbf{v}_2$. Furthermore, we may also choose $A = (0, 0, 0)$ and $C = (-1, 0, 0)$. Thus $\overrightarrow{AC} = \overrightarrow{OC} = \langle -1, 0, 0\rangle$. We must then compute $\mathbf{v}_1 \times \mathbf{v}_2$ as follows:

$$\mathbf{v}_1 \times \mathbf{v}_2 = \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \langle \det\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, -\det\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}, \det\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\rangle = \langle 1, -2, 1\rangle$$

Then we have

$$\|\mathbf{v}_1 \times \mathbf{v}_2\| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$$

The triple product then is

$$\overrightarrow{OC} \cdot (\mathbf{v}_1 \times \mathbf{v}_2) = \langle -1, 0, 0 \rangle \cdot \langle 1, -2, 1 \rangle = -1$$

It follows that the distance is

$$D = \frac{|-1|}{\sqrt{6}} = \frac{1}{\sqrt{6}}$$

25. A small meteor moves with the speed v along a straight line parallel to a unit vector $\hat{\mathbf{u}}$. If the meteor passed the point $\mathbf{r}_0$, find the condition on $\hat{\mathbf{u}}$ so that the meteor hits an asteroid of the shape of a sphere of radius R centered at the point $\mathbf{r}_1$. Determine the position vector of the impact point.

Solution: Let $\mathcal{L}$ be the line $\mathbf{r} = \mathbf{r}_0 + tv\hat{\mathbf{u}},\ 0 \le t < \infty$. Notice the inclusion of v, otherwise the units do not make sense (t is time). Let $\mathcal{P}$ be the plane containing $\mathcal{L}$ and the center of the sphere. We need only consider what is happening within this plane. Since the plane passes through the center of the sphere, the portion of the sphere within the plane is a circle of radius R. Suppose $\mathcal{L}$ is just tangent with the sphere. Then $\mathcal{L}$ is also just tangent with this circle. Let P be the point of tangency. Then the triangle with vertices P, where the meteor started from, and the center of the circle is right. Note that the hypotenuse of this triangle is parallel to $\mathbf{r}_1 - \mathbf{r}_0$, and one leg is parallel to $\hat{\mathbf{u}}$. Let θ be the angle between $\mathbf{r}_1 - \mathbf{r}_0$ and $\hat{\mathbf{u}}$. Then $\sin\theta = R/\|\mathbf{r}_1 - \mathbf{r}_0\| \Leftrightarrow \|\mathbf{r}_1 - \mathbf{r}_0\| \sin\theta = R$. This is the "maximal" case. If the left hand side exceeds R, then the line is no longer intersecting.
Suppose now that $\mathcal{L}$ passes through the center of the sphere. Clearly $\mathcal{L}$ intersects the sphere. Then $\hat{\mathbf{u}}$ is parallel to $\mathbf{r}_1 - \mathbf{r}_0$. So $\sin\theta = 0$.

Combining these two facts, we know it must be that

$$0 \le \|\mathbf{r}_1 - \mathbf{r}_0\| \sin\theta \le R \Leftrightarrow 0 \le \|\mathbf{r}_1 - \mathbf{r}_0\| \|\hat{\mathbf{u}}\| \sin\theta \le R \Leftrightarrow \|(\mathbf{r}_1 - \mathbf{r}_0) \times \hat{\mathbf{u}}\| \le R$$

Where the fact that $\|\hat{\mathbf{u}}\| = 1$ and $\|(\mathbf{r}_1 - \mathbf{r}_0) \times \hat{\mathbf{u}}\| = \|\mathbf{r}_1 - \mathbf{r}_0\| \|\hat{\mathbf{u}}\| \sin\theta$ have been used.

To find the position vector of the point of intersection, we must first find the smallest positive value of t such that $\mathcal{L}|_t$ is a point on the sphere. Recall that $\mathcal{L}$ is given by

$$\mathbf{r} = \mathbf{r}_0 + tv\hat{\mathbf{u}},\ 0 \le t < \infty$$

and the sphere is given by

$$\|\mathbf{r} - \mathbf{r}_1\| = R$$

Substituting the equation for the line into that of the sphere yields

$$\begin{aligned}
\|\mathbf{r}_0 + tv\hat{\mathbf{u}} - \mathbf{r}_1\| &= R \\
\|tv\hat{\mathbf{u}} + \mathbf{r}_0 - \mathbf{r}_1\|^2 &= R^2 \\
t^2v^2\|\hat{\mathbf{u}}\|^2 + 2tv\hat{\mathbf{u}}\cdot(\mathbf{r}_0 - \mathbf{r}_1) + \|\mathbf{r}_0 - \mathbf{r}_1\|^2 &= R^2 \\
v^2t^2 + 2tv\hat{\mathbf{u}}\cdot(\mathbf{r}_0 - \mathbf{r}_1) + \|\mathbf{r}_0 - \mathbf{r}_1\|^2 - R^2 &= 0
\end{aligned}$$

This is a second degree polynomial in t, so we can solve for t using the quadratic formula.

$$\begin{aligned}
t &= \frac{-(2v\hat{\mathbf{u}}\cdot(\mathbf{r}_0 - \mathbf{r}_1)) \pm \sqrt{(2v\hat{\mathbf{u}}\cdot(\mathbf{r}_0 - \mathbf{r}_1))^2 - 4(v^2)(\|\mathbf{r}_0 - \mathbf{r}_1\|^2 - R^2)}}{2(v^2)} \\
&= \frac{-2v\hat{\mathbf{u}}\cdot(\mathbf{r}_0 - \mathbf{r}_1) \pm 2v\sqrt{(\|\hat{\mathbf{u}}\|\|\mathbf{r}_0 - \mathbf{r}_1\|\cos\theta)^2 - (\|\mathbf{r}_0 - \mathbf{r}_1\|^2 - R^2)}}{2v^2} \\
&= \frac{-\hat{\mathbf{u}}\cdot(\mathbf{r}_0 - \mathbf{r}_1) \pm \sqrt{\|\mathbf{r}_0 - \mathbf{r}_1\|^2(\cos\theta)^2 - \|\mathbf{r}_0 - \mathbf{r}_1\|^2 + R^2}}{v} \\
&= \frac{-\hat{\mathbf{u}}\cdot(\mathbf{r}_0 - \mathbf{r}_1) \pm \sqrt{R^2 - \|\mathbf{r}_0 - \mathbf{r}_1\|^2(1 - (\cos\theta)^2)}}{v} \\
&= \frac{-\hat{\mathbf{u}}\cdot(\mathbf{r}_0 - \mathbf{r}_1) \pm \sqrt{R^2 - \|\mathbf{r}_0 - \mathbf{r}_1\|^2(\sin\theta)^2}}{v} \\
&= \frac{-\hat{\mathbf{u}}\cdot(\mathbf{r}_0 - \mathbf{r}_1) \pm \sqrt{R^2 - \|\hat{\mathbf{u}}\times(\mathbf{r}_0 - \mathbf{r}_1)\|^2}}{v}
\end{aligned}$$

Since we require $t \geq 0$, it must be that

$$t = \frac{\sqrt{R^2 - \|\hat{\mathbf{u}}\times(\mathbf{r}_0 - \mathbf{r}_1)\|^2} - \hat{\mathbf{u}}\cdot(\mathbf{r}_0 - \mathbf{r}_1)}{v}$$

Substitution of this into the equation for the line gives the desired position vector.

26. A projectile is fired in the direction $\mathbf{v} = \langle 1, 2, 3\rangle$ from the point $(1, 1, 1)$. Let the target be a disk of radius R centered at $(2, 3, 6)$ in the plane $2x - 3y + 4z = 19$. If the trajectory of the projectile is a straight line, determine whether it hits a target in two cases $R = 2$ and $R = 3$.

27. Consider a triangle ABC where $A = (1, 1, 1)$, $B = (3, 1, -1)$, $C = (1, 3, 1)$. Find the area of a polygon $DPQB$ where the vertices D and Q are the midpoints of CB and AB, respectively, and the vertex P is the intersection of the segments CQ and AD.

8. Euclidean Spaces.

9. Quadric Surfaces

1–10. Use traces to sketch and identify each of the following surfaces:

1. $y^2 = x^2 + 9z^2$;

SOLUTION: First letting $z = 0$, we have that $y^2 = x^2 \Leftrightarrow y = \pm x$. These are two orthogonal, straight lines through the origin.
Next, letting $y = 0$ we have that $x^2 + 9z^2 = 0$. This is similar to a circle or ellipse of radius 0, and is simply a point at the origin.
Finally, letting $x = 0$ we have that $y^2 = 9z^2 \Leftrightarrow y = \pm 3z$. These are also two straight lines through the origin.

This is an elliptic double cone aligned along the y axis.

2. $y = x^2 - z^2$;

SOLUTION: First letting $z = 0$, we have that $y = x^2$. This is a parabola, pointing up, whose vertex is the origin.
Next, letting $y = 0$ we have that $x^2 - z^2 = 0 \Leftrightarrow x = \pm z$. These are two orthogonal, straight lines through the origin.
Finally, letting $x = 0$ we have that $y = -z^2$. This is a parabola, pointing down, whose vertex is the origin.

This is a saddle (hyperbolic paraboloid) aligned along the y axis.

3. $4x^2 + 2y^2 + z^2 = 4$;

SOLUTION: First letting $z = 0$, we have that $4x^2 + 2y^2 = 4 \Leftrightarrow x^2 + y^2/2 = 1$. This is an ellipse, centered at the origin, with a major axis along the y axis of length $\sqrt{2}$ and a minor axis along the x axis of length 1.
Next, letting $y = 0$ we have that $4x^2+z^2 = 4 \Leftrightarrow x^2+z^2/4 = 1$. This is an ellipse, centered at the origin, with a major axis along the z axis of length 2 and a minor axis along the x axis of length 1.
Finally, letting $x = 0$ we have that $2y^2 + z^2 = 4 \Leftrightarrow y^2/2 + z^2/4 = 1$. This is an ellipse, centered at the origin, with a major axis along the z axis of length 2 and a minor axis along

the y axis of length $\sqrt{2}$.

This is an ellipsoid with $a = 1$, $b = \sqrt{2}$, and $c = 2$.

4. $x^2 - y^2 + z^2 = -1$;

SOLUTION: First letting $z = 0$, we have that $x^2 - y^2 = -1 \Leftrightarrow y = \pm\sqrt{1 + x^2}$. This is a hyperbola opening up and down along the y axis. For sufficiently large x, the hyperbola begins to look like $y = \pm x$
Next, letting $y = 0$ we have that $x^2 + z^2 = -1$. There are no solutions to this.
Finally, letting $x = 0$ we have that $-y^2 + z^2 = -1 \Leftrightarrow y = \pm\sqrt{1 + z^2}$. This is a hyperbola opening up and down along the y axis. For sufficiently large z, the hyperbola begins to look like $y = \pm z$

This is a hyperboloid of two sheets, aligned along the y axis.

5. $y^2 + 4z^2 = 16$;

SOLUTION: First letting $z = 0$, we have that $y^2 = 16 \Leftrightarrow y = \pm 4$. These are lines parallel to the x axis in the xy plane.
Next, letting $y = 0$ we have that $4z^2 = 16 \Leftrightarrow z = \pm 4$. These are lines parallel to to the x axis in the xz plane.
Finally, letting it does not matter what we let x equal, since the equation is independent of it. Therefore the surface will be aligned along the x axis and will be identical for any cross section $x = k$. Every cross section will look like $y^2 + 4z^2 = 16 \Leftrightarrow y^2/16 + z^2/4 = 1$, an ellipse centered at the origin with major axis along the y axis of length 4 and minor axis along the z axis of length 2.

This is an elliptic cylinder, aligned along the x axis.

6. $x^2 - y^2 + z^2 = 1$;

SOLUTION: First letting $z = 0$, we have that $x^2 - y^2 = 1 \Leftrightarrow x = \pm\sqrt{1 + y^2}$. This is a hyperbola opening up and down along the x axis. For sufficiently large y, the hyperbola begins to look like $x = \pm y$

Next, letting $y = 0$ we have that $x^2 + z^2 = 1$. This is a circle of radius 1 centered at the origin.
Finally, letting $x = 0$ we have that $-y^2 + z^2 = 1 \;\Leftrightarrow\; z = \pm\sqrt{1 + y^2}$. This is a hyperbola opening up and down along the z axis. For sufficiently large y, the hyperbola begins to look like $z = \pm y$

This is a hyperboloid of one sheet, aligned along the y axis. (One way to try and visualize this is to connect the traces. There should be a circle and two hyperboloids tangent to the circle, where you can rotate one by $\pi/2$ to get the other. Filling in the rotations gives the full surface).

7. $x^2 + 4y^2 - 9z^2 + 1 = 0$;

Solution: First letting $z = 0$, we have that $x^2 + 4y^2 + 1 = 0 \;\Leftrightarrow\; x^2 + 4y^2 = -1$. This has no solutions.
Next, letting $y = 0$ we have that $x^2 - 9z^2 + 1 = 0 \;\Leftrightarrow\; z = \pm 1/3\sqrt{1 + x^2}$. This is a hyperbola opening up and down along the z axis. For sufficiently large x, the hyperbola begins to look like $z = \pm 1/3x$
Finally, letting $x = 0$ we have that $4y^2 - 9z^2 + 1 = 0 \;\Leftrightarrow\; z = \pm 2/3\sqrt{1 + y^2}$. This is a hyperbola opening up and down along the z axis. For sufficiently large y, the hyperbola begins to look like $z = \pm 2/3y$

This is a hyperboloid of two sheets, aligned along the z axis.

8. $x^2 + z = 0$;

Solution: First letting $z = 0$, we have that $x^2 = 0 \;\Leftrightarrow x = 0$. This is the y axis.
Next, it does not matter what y we choose, for the surface is independent of y. Therefore at every cross section $y = k$ for real k, the surface will be identical. At every such cross section, the surface looks like $x^2 + z = 0 \;\Leftrightarrow\; z = -x^2$, a parabola opening down with vertex at the origin.
Finally, letting $x = 0$ we have that $z = 0$. This the y axis once again.

This is parabolic cylinder, parallel to the y axis.

9. $x^2 + 9y^2 + z = 0$;

Solution: First letting $z = 0$, we have that $x^2 + 9y^2 = 0 \Leftrightarrow x = 0$. This is simply the origin.
Next, letting $y = 0$ we have that $x^2 + z = 0 \Leftrightarrow z = -x^2$, a parabola opening down with vertex at the origin.
Finally, letting $x = 0$ we have that $9y^2 + z = 0 \Leftrightarrow z = -9y^2$, a parabola opening down with vertex at the origin.

This is an elliptic paraboloid opening downwards parallel to the z axis.

10. $y^2 - 4z^2 = 16$.

Solution: First letting $z = 0$, we have that $y^2 = 16 \Leftrightarrow y = \pm 4$. These are lines parallel to the x axis in the xy plane.
Next, letting $y = 0$ we have that $-4z^2 = 16$. There are no solutions to this.
Finally, letting it does not matter what we let x equal, since the equation is independent of it. Therefore the surface will be aligned along the x axis and will be identical for any cross section $x = k$ for real k. Every cross section will look like $y^2 - 4z^2 = 16 \Leftrightarrow y = \pm 4\sqrt{1 + z^2/4}$, a hyperbola opening up and down along y. For sufficiently large z, the hyperbola begins to look like $y = \pm 2x$

This is a hyperbolic cylinder, aligned along the x axis.

11–15. Reduce each of the following equations to one of the standard forms, classify the surface, and sketch it:

11. $x^2 + y^2 + 4z^2 - 2x + 4y = 0$;

Solution: By completing the square, the given equation becomes

$$\begin{aligned}
& (x-1)^2 - 1 + (y+2)^2 - 4 + 4z^2 = 0 \\
\Rightarrow \quad & (x-1)^2 + (y+2)^2 + 4z^2 = 5 \\
\Rightarrow \quad & \frac{(x-1)^2}{\sqrt{5}^2} + \frac{(y+2)^2}{\sqrt{5}^2} + \frac{z^2}{(\frac{\sqrt{5}}{2})^2} = 1
\end{aligned}$$

This is an ellipsoid centered at $(1,-2,0)$ with $a=b=\sqrt{5}$ and $c=\sqrt{5}/2$.

12. $x^2-y^2+z^2+2x-2y+4z+2=0$;

SOLUTION: By completing the square, the given equation becomes

$$\begin{aligned} & (x+1)^2-1-(y+1)^2+1+(z+2)^2-4=-2 \\ \Rightarrow\quad & (x+1)^2-(y+1)^2+(z+2)^2=2 \end{aligned}$$

This is a hyperboloid of one sheet aligned along the y axis centered at $(-1,-1,-2)$.

13. $x^2+4y^2-6x+z=0$;

SOLUTION: By completing the square, the given equation becomes

$$\begin{aligned} & (x-3)^2-9+4y^2+z=0 \\ \Rightarrow\quad & (z-9)=-((x-3)^2+4y^2) \end{aligned}$$

This is an elliptic paraboloid concave down with vertex at $(3,0,9)$.

14. $y^2-4z^2+2y-16z=0$;

SOLUTION: By completing the square, the given equation becomes

$$\begin{aligned} & (y+1)^2-1-4(z+2)^2+16=0 \\ \Rightarrow\quad & -(y+1)^2+4(z+2)^2=15 \end{aligned}$$

This is a hyperbolic cylinder parallel to the x axis, whose axis passes through $(0,-1,-2)$.

15. $x^2-y^2+z^2-2x+2y=0\,.$

SOLUTION: By completing the square, the given equation becomes

$$\begin{aligned} & (x-1)^2-1-(y-1)^2+1+z^2=0 \\ \Rightarrow\quad & (y-1)^2=(x-1)^2+z^2 \end{aligned}$$

This is an elliptic double cone whose axis is parallel to the y axis and with vertex at $(1, 1, 0)$.

16–20. Use rotations in the appropriate coordinate plane to reduce each of the following equations to one of the standard forms and classify the surface:

16. $6xy + x^2 + y^2 = 1$;

SOLUTION: Let $x = x' \cos\phi - y' \sin\phi$ and $y = y' \cos\phi + x' \sin\phi$. Then

$$\begin{aligned} x^2 &= (x' \cos\phi - y' \sin\phi)^2 \\ &= x'^2 \cos\phi^2 - 2x'y' \cos\phi \sin\phi + y'^2 \sin\phi^2 \\ &= \frac{1}{2}(1 + \cos(2\phi))x'^2 + \frac{1}{2}(1 - \cos(2\phi))y'^2 - \sin(2\phi)x'y' \end{aligned}$$

$$\begin{aligned} y^2 &= (y' \cos\phi + y' \sin\phi)^2 \\ &= y'^2 \cos\phi^2 + 2x'y' \cos\phi \sin\phi + x'^2 \sin\phi^2 \\ &= \frac{1}{2}(1 - \cos(2\phi))x'^2 + \frac{1}{2}(1 + \cos(2\phi))y'^2 + \sin(2\phi)x'y' \end{aligned}$$

$$\begin{aligned} xy &= (x' \cos\phi - y' \sin\phi)(y' \cos\phi + x' \sin\phi) \\ &= x'y'(\cos\phi^2 - \sin\phi^2) + (x'^2 - y'^2) \cos\phi \sin\phi \\ &= \frac{1}{2} \sin(2\phi)(x'^2 - y'^2) + \cos(2\phi)x'y' \end{aligned}$$

We are interested in finding an angle ϕ such that the mixed term $x'y'$ is "annihilated". Thus we need only consider the contributions of the mixed terms for now. We have then that

$$6(\cos(2\phi)) + (-\sin(2\phi)) + (\sin(2\phi)) = 0 \;\Leftrightarrow\; \cos(2\phi) = 0$$

We may then choose $\phi = \pi/4$. So,

$$\begin{aligned} x^2 &= \frac{1}{2}(1 + \cos(\frac{2\pi}{4}))x'^2 + \frac{1}{2}(1 - \cos(\frac{2\pi}{4}))y'^2 - \sin(\frac{2\pi}{4})x'y' \\ &= \frac{1}{2}x'^2 + \frac{1}{2}y'^2 - x'y' \end{aligned}$$

$$\begin{aligned} y^2 &= \frac{1}{2}(1 - \cos(\frac{2\pi}{4}))x'^2 + \frac{1}{2}(1 + \cos(\frac{2\pi}{4}))y'^2 + \sin(\frac{2\pi}{4})x'y' \\ &= \frac{1}{2}x'^2 + \frac{1}{2}y'^2 + x'y' \end{aligned}$$

$$\begin{aligned} xy &= \frac{1}{2}\sin(\frac{2\pi}{4})(x'^2 - y'^2) + \cos(\frac{2\pi}{4})x'y' \\ &= \frac{1}{2}(x'^2 - y'^2) \end{aligned}$$

Substitution of these into the given equation yields

$$\begin{aligned} 6(\frac{1}{2}(x'^2 - y'^2)) + (\frac{1}{2}x'^2 + \frac{1}{2}y'^2 - x'y') + (\frac{1}{2}x'^2 + \frac{1}{2}y'^2 + x'y') &= 1 \\ 3x'^2 - 3y'^2 + x'^2 + y'^2 &= 1 \\ 4x'^2 - 2y'^2 &= 1 \end{aligned}$$

So this is a hyperbolic cylinder.

17. $3y^2 + 3z^2 - 2yz = 1$;

Solution: We can play the same game as in Exercise 16, except in the yz plane. Thus we let $y = y'\cos\phi - z'\sin\phi$ and $z = z'\cos\phi + y'\sin\phi$. Then

$$y^2 = \frac{1}{2}(1 + \cos(2\phi))y'^2 + \frac{1}{2}(1 - \cos(2\phi))z'^2 - \sin(2\phi)y'z'$$

$$z^2 = \frac{1}{2}(1 - \cos(2\phi))y'^2 + \frac{1}{2}(1 + \cos(2\phi))z'^2 + \sin(2\phi)z'z'$$

$$yz = \frac{1}{2}\sin(2\phi)(y'^2 - z'^2) + \cos(2\phi)y'z'$$

We are interested in finding an angle ϕ such that the mixed term is annihilated. So we need only consider the mixed term contribution of each of the above equations. Thus,

$$3(-\sin(2\phi)) + 3(\sin(2\phi)) - 2(\cos(2\phi)) = 0 \;\Leftrightarrow\; \cos(2\phi) = 0$$

So we may let $\phi = \pi/4$. Thus we have

$$\begin{aligned} y^2 &= \frac{1}{2}(1 + \cos(\frac{2\pi}{4}))y'^2 + \frac{1}{2}(1 - \cos(\frac{2\pi}{4}))z'^2 - \sin(\frac{2\pi}{4})y'z' \\ &= \frac{1}{2}y'^2 + \frac{1}{2}y'^2 - y'z' \end{aligned}$$

$$\begin{aligned} z^2 &= \frac{1}{2}(1 - \cos(\frac{2\pi}{4}))y'^2 + \frac{1}{2}(1 + \cos(\frac{2\pi}{4}))z'^2 + \sin(\frac{2\pi}{4})y'z' \\ &= \frac{1}{2}y'^2 + \frac{1}{2}y'^2 + y'z' \end{aligned}$$

$$\begin{aligned} yz &= \frac{1}{2}\sin(\frac{2\pi}{4})(y'^2 - z'^2) + \cos(\frac{2\pi}{4})y'z' \\ &= \frac{1}{2}(y'^2 - z'^2) \end{aligned}$$

Substitution of these into the given equation yields

$$\begin{aligned} 3(\frac{1}{2}y'^2 + \frac{1}{2}y'^2 - y'z') + 3(\frac{1}{2}y'^2 + \frac{1}{2}y'^2 + y'z') - 2(\frac{1}{2}(y'^2 - z'^2)) &= 1 \\ \frac{3}{2}y'^2 + \frac{3}{2}y'^2 + \frac{3}{2}y'^2 + \frac{3}{2}y'^2 - y'z' + y'z' - y'^2 + z'^2 &= 1 \\ 2y'^2 + 4z'^2 &= 1 \end{aligned}$$

This is an elliptical cylinder centered at the origin with axis parallel to the x axis.

18. $x - yz = 0$;

Solution: As with Exercise 17, we will be rotating in the yz plane to attempt to annihilate the mixed yz term. Thus we let $y = y'\cos\phi - z'\sin\phi$ and $z = z'\cos\phi + y'\sin\phi$. Then

$$yz = \frac{1}{2}\sin(2\phi)(y'^2 - z'^2) + \cos(2\phi)y'z'$$

We are only interested in the mixed contributions of this equation. So, $-(\cos(2\phi)) = 0$ and we may let $\phi = \pi/4$. Then we have

$$\begin{aligned} yz &= \frac{1}{2}\sin(\frac{2\pi}{4})(y'^2 - z'^2) + \cos(\frac{2\pi}{4})y'z' \\ &= \frac{1}{2}(y'^2 - z'^2) \end{aligned}$$

Substitution of this into the given equation yields

$$x - \frac{1}{2}y'^2 + \frac{1}{2}z'^2 = 0 \;\Leftrightarrow\; 2x = y'^2 - z'^2$$

This is a hyperbolic paraboloid.

19. $xy - z^2 = 0$;

Solution: As in Exercise 16, we will be rotating in the xy plane, attempting to annihilate the mixed xy term. Let

$x = x' \cos\phi - y' \sin\phi$ and $y = y' \cos\phi + x' \sin\phi$. Then

$$xy\frac{1}{2}\sin(2\phi)(x'^2 - y'^2) + \cos(2\phi)x'y'$$

Since we are interested in annihilating the mixed term, we need only consider the mixed contributions from the above equation. Thus, $\cos(2\phi) = 0$, and we may let $\phi = \pi/4$. So,

$$xy = \frac{1}{2}(x'^2 - y'^2)$$

Substituting this into the given equation yields

$$\frac{1}{2}(x'^2 - y'^2) - z^2 = 0 \iff x'^2 - y'^2 - 2z^2 = 0 \iff x'^2 = y'^2 + 2z^2$$

This is an elliptic double cone parallel to the x axis. However, prior to rotation it would have been parallel to the line $y = x$ in the xy plane.

20. $2xz + 2x^2 - y^2 = 0.$

Solution: We will need to rotate in the xz plane in order to annihilate the mixed xz term. Thus, let $x = x' \cos\phi - z' \sin\phi$ and $z = z' \cos\phi + x' \sin\phi$. Then

$$x^2 = \frac{1}{2}(1 + \cos(2\phi))x'^2 + \frac{1}{2}(1 - \cos(2\phi))z'^2 - \sin(2\phi)x'z'$$

$$xz = \frac{1}{2}\sin(2\phi)(x'^2 - z'^2) + \cos(2\phi)x'z'$$

We need only use the mixed contribution of these to find a value of ϕ to kill the mixed term. Thus we have

$$2\cos(2\phi) + 2(-\sin(2\phi)) = 0 \iff \tan(2\phi) = 1$$

and we may let $2\phi = \pi/4 \iff \phi = \pi/8$. So

$$\begin{aligned} x^2 &= \frac{1}{2}(1 + \cos(\frac{2\pi}{8}))x'^2 + \frac{1}{2}(1 - \cos(\frac{2\pi}{8}))z'^2 - \sin(\frac{2\pi}{8})x'z' \\ &= \frac{2+\sqrt{2}}{4}x'^2 + \frac{2-\sqrt{2}}{4}z'^2 - \frac{\sqrt{2}}{2}x'z' \end{aligned}$$

$$\begin{aligned} xy &= \frac{1}{2}\sin(\frac{2\pi}{8})(x'^2 - z'^2) + \cos(\frac{2\pi}{8})x'z' \\ &= \frac{\sqrt{2}}{4}(x'^2 - z'^2) + \frac{\sqrt{2}}{2}x'z' \end{aligned}$$

Substituting these into the given equation yields

$$2(\frac{\sqrt{2}}{4}(x'^2 - z'^2) + \frac{\sqrt{2}}{2}x'z') + 2(\frac{2+\sqrt{2}}{4}x'^2 + \frac{2-\sqrt{2}}{4}z'^2 - \frac{\sqrt{2}}{2}x'z') - y^2 = 0$$

$$\frac{\sqrt{2}}{2}(x'^2 - z'^2) + \sqrt{2}x'z' + \frac{2+\sqrt{2}}{2}x'^2 + \frac{2-\sqrt{2}}{2}z'^2 - \sqrt{2}x'z' - y^2 = 0$$

$$\frac{2+2\sqrt{2}}{2}x'^2 + \frac{2-2\sqrt{2}}{2}z'^2 - y^2 = 0$$

$$(1+\sqrt{2})x'^2 + (1-\sqrt{2})z'^2 = y^2$$

This is an elliptic double cone.

21. Find an equation for the surface obtained by rotating the line $y = 2x$ about the y axis. Classify the surface.

Solution: First, this is an elliptic double cone. This can be seen by looking at cross sections through $y = k$, which are circles, and through the xy and yz planes, which are intersecting lines. When $z = 0$, it follows that we should have $y = \pm 2x \Leftrightarrow y^2 = 4x^2$. Furthermore, when $x = 0$ we should have $y = \pm 2z \Leftrightarrow y^2 = 4z^2$ (this can be seen by rotating $y = 2x$ until it lies in the yz plane). Combining these with the general formula for an elliptic double cone yields

$$y^2 = 4x^2 + 4z^2 \ \Leftrightarrow\ \frac{1}{4}y^2 = x^2 + z^2$$

22. Find an equation for the surface obtained by rotating the curve $y = 1 + z^2$ about the y axis. Classify the surface.

Solution: This is an elliptic paraboloid. The trace when $x = 0$ should be $y = 1 + z^2$. The trace when $z = 0$ should look identical, so we should have $y = 1 + x^2$. Combining these with the general form of an elliptic paraboloid yields

$$y = 1 + x^2 + z^2 \ \Leftrightarrow\ (y-1) = x^2 + z^2$$

23. Find equations for the family of surfaces obtained by rotating the curves $x^2 - 4y^2 = k$ about the y axis where k is real. Classify the surfaces.

Solution: We must separate this into three cases: $k = 0$, $k < 0$, and $k > 0$.
Case 1: $k = 0$

The traces in the xy plane look like $x^2 - 4y^2 = 0$, which are intersecting lines. This is indicative of an elliptic double cone. In the yz plane, we obtain the same traces, just replacing x with z. So we have $z^2 - 4y^2 = 0$. Combining these two with the general form of an elliptic double cone yields

$$x^2 + z^2 - 4y^2 = 0 \Leftrightarrow 4y^2 = x^2 + z^2$$

Case 2: $k < 0$
The traces in the xy plane look like $x^2 - 4y^2 = k$. These are hyperboloids opening up towards y. So, rotation of these around the y axis produces the same traces in the yz plane, with x replaced by z. So we have $z^2 - 4y^2 = k$.Combining the two equations, we have

$$x^2 + z^2 - 4y^2 = k \Leftrightarrow -x^2 + 4y^2 - z^2 = -k$$

In the form on the right, it is clear we have a hyperboloid of two sheets (note that $-k > 0$).

Case 3: $k > 0$
The traces in the xy plane look like $x^2 - 4y^2 = k$. These are hyperboloids opening up towards x. So, rotation of these around the y axis produces the same traces in the yz plane, with x replaced by z. So we have $z^2 - 4y^2 = k$.Combining the two equations, we have

$$x^2 + z^2 - 4y^2 = k$$

In the form on the right, it is clear we have a hyperboloid of one sheet.

24. Find an equation for the surface consisting of all points that are equidistant from the point $(1, 1, 1)$ and the plane $z = 2$.

SOLUTION: Some may be familiar that a parabola is generated by a point (a focus) and a line (a directrix), where the parabola is the set of points equidistant from the focus and the directrix. The 3D analogue of this is described in the problem. Thus we will have an elliptic paraboloid.
First, notice that the distance between a point $P = (x_0, y_0, z_0)$ and a plane $z = k$ is simply $|z_0 - k|$. This is because $Q = (x_0, y_0, k)$ is a point on the plane, and PQ is orthogonal to the plane. So the distance between P and the plane is simply $|PQ|$, which is trivially $|z_0 - k|$.

Thus we have

$$\begin{aligned}
\sqrt{(x-1)^2+(y-1)^2+(z-1)^2} &= |z-2| \\
(x-1)^2+(y-1)^2+(z-1)^2 &= (z-2)^2 \\
(x-1)^2+(y-1)^2+(z-1)^2-(z-2)^2 &= 0 \\
(x-1)^2+(y-1)^2+(z-1-(z-2))(z-1+z-2) &= 0 \\
(x-1)^2+(y-1)^2+(2z-3) &= 0 \\
(2z-3) &= -((x-1)^2+(y-1)^2)
\end{aligned}$$

which is an elliptic paraboloid, as hypothesized, that is concave down with vertex at $(1, 1, \frac{3}{2})$

25. Sketch the solid region bounded by the surface $z = \sqrt{x^2+y^2}$ from below and by $x^2+y^2+z^2-2z=0$ from above.

Solution: The surface $z = \sqrt{x^2+y^2}$ is the top part of an elliptic double cone. The surface $x^2+y^2+z^2-2z=0 \Leftrightarrow x^2+y^2+(z-1)^2=1$ is a sphere of radius 1 centered at $(0,0,1)$. Incidentally, when $z=1$, the surface $z=\sqrt{x^2+y^2}$ is a circle of radius 1. Thus the double cone cuts through the sphere at all points along this circle. Since the circle is parallel to the xy plane and contains a diameter of the sphere, you can imagine the top portion as just the upper hemisphere. The solid now looks like an ice-cream cone.

26. Sketch the solid region bounded by the surfaces $y = 2-x^2-z^2$, $y = x^2+z^2-2$, and $x^2+z^2=1$.

Solution: The surface $y = 2-x^2-z^2 \Leftrightarrow (y-2) = -(x^2+z^2)$ is an elliptic paraboloid aligned along the y axis, concave down, with vertex at $(0,2,0)$. The surface $y = x^2+z^2-2 \Leftrightarrow (y+2) = x^2+z^2$ is an elliptic paraboloid aligned along the y axis, concave up, with vertex at $(0,-2,0)$. The surface $x^2+z^2=1$ is a cylinder or radius 1 parallel to the y axis, whose axis is the y axis. The solid region can first be viewed as an "egg" shape inside the two paraboloids. The upper part of the egg looks like $y = 2-x^2$ above the y axis, and the lower looks like $y = x^2-2$ below the y axis. Then, shave away everything lying outside the cylinder $x^2+z^2=1$. The solid looks like that bounded by $y=2-x^2$, $y=x^2-2$, $x=\pm 1$ revolved about the y axis.

27. Sketch the solid region bounded by the surfaces $x^2+y^2=R^2$ and $x^2+z^2=R^2$

Solution: The surface $x^2 + y^2 = R^2$ is a cylinder of radius R aligned along the z axis. The surface $x^2 + z^2 = R^2$ is a cylinder of radius R aligned along the y axis. The solid generated is fairly hard to visualize. First, we can set the two equations equal to each other and obtain $y = \pm z$. This means that the intersection of the two solids lies in the planes $y = \pm z$. Construct these planes, with only one cylinder in mind. Focus only on one of the four regions of this cross such that the axis of the cylinder does not go through it (i.e., there is no hole), then rotate it by $\pi/2, \pi$, and $3\pi/2$ around the axis of the other cylinder. For more information see Steinmetz Solid.

28. Find an equation for the surface consisting of all points P for which the distance from P to the y axis is twice the distance from P to the zx plane. Identify the surface.

Solution: First note that the xz plane is given by $y = 0$. We have established in Exercise 24 that the distance between a point $P = (x_0, y_0, z_0)$ and a plane $y = k$ is simply $|y_0 - k|$. The projection of P onto the y axis is given by $P' = (0, y_0, 0)$. So the distance between P and the y axis is simply $|PP'|$, since PP' is orthogonal to the y axis. Combining these we have

$$\begin{aligned} \sqrt{(x-0)^2 + (y-y)^2 + (z-0)^2} &= 2|y-0| \\ \sqrt{x^2 + z^2}^2 &= 2^2|y|^2 \\ 4y^2 &= x^2 + z^2 \end{aligned}$$

This is an elliptic double cone.

29. Show that if the point (a, b, c) lies on the hyperbolic paraboloid $z = y^2 - x^2$, then the lines through (a, b, c) and parallel to $\mathbf{v} = \langle 1, 1, 2(b-a)\rangle$ and $\mathbf{u} = \langle 1, -1, -2(a+b)\rangle$ both lie entirely on this paraboloid. Deduce from this result that the hyperbolic paraboloid can be generated by the motion of a straight line. Show that hyperboloids of one sheet, cones, and cylinders can also be obtained by the motion of a straight line.

Remark. The fact that hyperboloids of one sheet are generated by the motion of a straight line is used to produce gear transmissions. The cogs of the gears are the generating lines of the hyperboloids.

30. Find an equation for the cylinder of radius R whose axis goes through the origin and is parallel to a vector $\mathbf{v}$.

SOLUTION: Let $\mathbf{r}$ be a point on the cylinder. Connect this point to the axis of the cylinder. A right triangle with the point, the origin, and the point on the axis is now formed. The leg along the axis of the cylinder is given by $s\mathbf{v}$ for some real s. Then the are of the triangle is given by

$$\frac{1}{2}\|\mathbf{r} \times s\mathbf{v}\| = \frac{s}{2}\|\mathbf{r} \times \mathbf{v}\|$$

On the other hand, the area of the triangle is also given by

$$\frac{1}{2}R\|s\mathbf{v}\| = \frac{s}{2}R\|\mathbf{v}\|$$

Combining these we have

$$\frac{s}{2}\|\mathbf{r} \times \mathbf{v}\| = \frac{s}{2}R\|\mathbf{v}\| \;\Leftrightarrow\; s\|\mathbf{r} \times \mathbf{v}\| = sR\|\mathbf{v}\|$$

Suppose $s \neq 0$, then we have

$$\|\mathbf{r} \times \mathbf{v}\| = R\|\mathbf{v}\|$$

If $s = 0$, then $\mathbf{r}$ is orthogonal to $\mathbf{v}$ and $\|\mathbf{r}\| = R$, so the above equation still holds.

31. Show that the curve of intersection of the surfaces $x^2 - 2y^2 + 3z^2 - 2x + y - z = 1$ and $2x^2 - 4y^2 + 6z^2 + x - y + 2z = 4$ lies in a plane.

SOLUTION: Subtracting twice the first equation from the second yields

$$2x^2 - 2x^2 - 4y^2 + 4y^2 + 6z^2 - 6z^2 + x + 4x - y - 2y + 2z + 2z = 4 - 2 \;\Leftrightarrow\; 5x - 3y + 4z = 2$$

This is clearly a plane.

32. The projection of a point set $\mathcal{S}$ onto the xy plane is obtained by setting the z coordinates of all points of $\mathcal{S}$ to zero. The projections of $\mathcal{S}$ onto the other two coordinate axes are defined similarly. What are the curves that bound the projections of the ellipsoid $x^2 + y^2 + z^2 - xy = 1$ onto the coordinate planes?

SOLUTION: RETURN TO THIS, IT'S WRONG

The surface given is an ellipsoid (imagine just removing the mixed xy term for some insight). An important distinction is that we cannot immediately just substitute $x = 0$, $y = 0$, or $z = 0$ to get the necessary projections. The curves must bind the projection of any trace onto the respective coordinate plane. Imagine a sphere or radius 1 centered at $(0, 0, 1)$. The sphere is just tangent to the point $(0, 0, 0)$, which is the only point on the surface where $z = 0$. But the projection of the sphere onto the xy plane should be a circle of radius 1, the largest circle produced by any trace $z = k$ for real k. So we must ensure that the largest curve is already in the appropriate coordinate plane before setting $x = 0$, $y = 0$, or $z = 0$. For ellipsoids, this will be the case when the center is at the origin. The ellipsoid given is indeed one centered at the origin, albeit rotated. Thus we are free to continue simply by substituting $x = 0$, $y = 0$, or $z = 0$.

Case 1: Projection onto the xy plane
Here we set $z = 0$. So we have

$$x^2 + y^2 - xy = 1$$

Let $x = x' \cos\phi - y' \sin\phi$ and $y = y' \cos\phi + x' \sin\phi$. Then

$$x^2 = \frac{1}{2}(1 + \cos(2\phi))x'^2 + \frac{1}{2}(1 - \cos(2\phi))y'^2 - \sin(2\phi)x'y'$$

$$y^2 = \frac{1}{2}(1 - \cos(2\phi))x'^2 + \frac{1}{2}(1 + \cos(2\phi))y'^2 + \sin(2\phi)x'y'$$

$$xy = \frac{1}{2}\sin(2\phi)(x'^2 - y'^2) + \cos(2\phi)x'y'$$

We are interested in finding an angle ϕ such that the mixed term $x'y'$ is "annihilated". Thus we need only consider the contributions of the mixed terms for now. We have then that

$$-\sin(2\phi) + \sin(2\phi) - \cos(2\phi) = 0 \;\Leftrightarrow\; \cos(2\phi) = 0$$

We may then choose $\phi = \pi/4$. So,

$$\begin{aligned} x^2 &= \frac{1}{2}(1 + \cos(\frac{2\pi}{4}))x'^2 + \frac{1}{2}(1 - \cos(\frac{2\pi}{4}))y'^2 - \sin(\frac{2\pi}{4})x'y' \\ &= \frac{1}{2}x'^2 + \frac{1}{2}y'^2 - x'y' \end{aligned}$$

$$\begin{aligned} y^2 &= \frac{1}{2}(1 - \cos(\frac{2\pi}{4}))x'^2 + \frac{1}{2}(1 + \cos(\frac{2\pi}{4}))y'^2 + \sin(\frac{2\pi}{4})x'y' \\ &= \frac{1}{2}x'^2 + \frac{1}{2}y'^2 + x'y' \end{aligned}$$

$$\begin{aligned} xy &= \frac{1}{2}\sin(\frac{2\pi}{4})(x'^2 - y'^2) + \cos(\frac{2\pi}{4})x'y' \\ &= \frac{1}{2}(x'^2 - y'^2) \end{aligned}$$

Substitution of these into the above equation yields

$$\begin{aligned}\frac{1}{2}x'^2+\frac{1}{2}y'^2-x'y'+\frac{1}{2}x'^2+\frac{1}{2}y'^2+x'y'+\frac{1}{2}(x'^2-y'^2)&=1\\ \frac{1}{2}x'^2+\frac{1}{2}x'^2+\frac{1}{2}x'^2+\frac{1}{2}y'^2+\frac{1}{2}y'^2-\frac{1}{2}y'^2-x'y'+x'y'&=1\\ \frac{3}{2}x'^2+\frac{1}{2}y'^2&=1\end{aligned}$$

This is an ellipse with $a=\sqrt{2/3}$ and $b=\sqrt{2}$. However, to obtain the curve pre-rotation, we must rotate this by $\pi/4$ clockwise. The ellipse then is aligned with major axis along $y=x$.

Case 2: Projection onto the xz plane
Here we set $y=0$. So we have

$$x^2+z^2=1$$

This is a circle of radius 1.

Case 2: Projection onto the yz plane
Here we set $x=0$. So we have

$$y^2+z^2=1$$

This is a circle of radius 1.

CHAPTER 2

Vector Functions

1. Curves in Space and Vector Functions

1–5. Find the domain of each of the following vector functions:

1. $\mathbf{r}(t) = \langle t,\ t^2,\ e^t\rangle$;

SOLUTION: Recall that a vector function $\mathbf{r}(t)$ may be written as $\mathbf{r}(t) = \langle x(t),\ y(t),\ z(t)\rangle$ for functions $x(t), y(t), z(t)$. The domain of $\mathbf{r}$ is the intersection of the domains of each component. Here, $x(t) = t,\ y(t) = t^2,\ z(t) = e^t$. The domains are as follows:

$$\begin{aligned} x(t) = t &: -\infty < t < \infty \\ y(t) = t^2 &: -\infty < t < \infty \\ z(t) = e^t &: -\infty < t < \infty \end{aligned}$$

since polynomials and exponential functions are defined on the entire real number line. Thus, the domain of $\mathbf{r}(t)$ is the intersection of the three domains, and is $-\infty < t < \infty$.

2. $\mathbf{r}(t) = \langle \sqrt{t},\ t^2,\ e^t\rangle$;

SOLUTION: We find the domain in a similar manner as above. Here, $x(t) = \sqrt{t},\ y(t) = t^2,\ z(t) = e^t$. The latter two are the same as in Exercise 1, so we need only find the domain of $x(t)$, which is wherever $\sqrt{t}$ is defined. Clearly, this is when $0 \le t$. Thus the domain of $\mathbf{r}(t)$ is $0 \le t$.

3. $\mathbf{r}(t) = \langle \sqrt{9 - t^2},\ \ln t,\ \cos t\rangle$;

SOLUTION: We find the domain in a similar manner as above. Here, $x(t) = \sqrt{9 - t^2},\ y(t) = \ln t,\ z(t) = \cos t$. The domains are

$$\begin{aligned} x(t) = \sqrt{9 - t^2} &: 0 \le 9 - t^2 \ \Leftrightarrow\ -3 \le t \le 3 \\ y(t) = \ln t &: 0 < t \\ z(t) = \cos t &: -\infty < t < \infty \end{aligned}$$

Thus the domain of $\mathbf{r}(t)$ is $0 < t \leq 3$.

4. $\mathbf{r}(t) = \langle \ln(9-t^2),\ \ln|t|,\ (1+t)/(2+t)\rangle$;

SOLUTION: We find the domain in a similar manner as above. Here, $x(t) = \ln(9-t^2),\ y(t) = \ln|t|,\ z(t) = (1+t)/(2+t)$. The domains are

$$\begin{aligned} x(t) = \ln(9-t^2) &: 0 < 9 - t^2 \ \Leftrightarrow\ -3 < t < 3 \\ y(t) = \ln|t| &: t \neq 0 \\ z(t) = (1+t)/(2+t) &: t \neq -2 \end{aligned}$$

Thus the domain of $\mathbf{r}(t)$ is $(-3,-2) \cup (-2,0) \cup (0,3)$

5. $\mathbf{r}(t) = \langle \sqrt{t-1},\ \ln t,\ \sqrt{1-t}\rangle$.

SOLUTION: We find the domain in a similar manner as above. Here, $x(t) = \sqrt{t-1},\ y(t) = \ln t,\ z(t) = \sqrt{1-t}$. The domains are

$$\begin{aligned} x(t) = \sqrt{t-1} &: 0 \leq t-1 \ \Leftrightarrow\ 1 \leq t \\ y(t) = \ln t &: 0 < t \\ z(t) = \sqrt{1-t} &: 0 \leq 1-t \ \Leftrightarrow\ t \leq 1 \end{aligned}$$

Thus the domain of $\mathbf{r}(t)$ is $t = 1$.

6–15. Find each of the following limits or show that it does not exist:

6. $\lim_{t\to 1}\langle \sqrt{t},\ 2-t-t^2,\ 1/(t^2-2)\rangle$;

SOLUTION: Recall that a vector function $\mathbf{r}(t)$ may be written as $\mathbf{r}(t) = \langle x(t),\ y(t),\ z(t)\rangle$ for functions $x(t), y(t), z(t)$. The limit of $\mathbf{r}$ as t approaches some value can be found by finding the limits of the each component. If any one of the limits does not exist, the limit of $\mathbf{r}(t)$ does not either. Note that the components are simply functions of one variable, hence normal limit laws can be used. Here, $x(t) = \sqrt{t},\ y(t) = 2-t-t^2,\ z(t) = 1/(t^2-2)$. The limits are as follows:

$$\begin{aligned} \lim_{t\to 1} x(t) &= \sqrt{1} = 1 \\ \lim_{t\to 1} y(t) &= 2 - 1 - 1^2 = 0 \\ \lim_{t\to 1} z(t) &= \frac{1}{1^2-2} = -1 \end{aligned}$$

If a function is continuous at the limit point, then you may simply substitute. Polynomials are continuous everywhere, and rational functions are continuous everywhere such that the denominator is nonzero. Thus,

$$\lim_{t\to 1}\langle\sqrt{t},\ 2-t-t^2,\ 1/(t^2-2)\rangle = \langle 1,0,-1\rangle$$

7. $\lim_{t\to 1}\langle\sqrt{t},\ 2-t-t^2,\ 1/(t^2-1)\rangle$;

SOLUTION: We find the limit in a similar manner as above. Note that $z(t) = 1/(t^2-1)$ is not defined at $t = 1$, and the limit tends to infinity. Thus the limit does not exist.

8. $\lim_{t\to 0}\langle e^t,\ \sin t,\ t/(1-t)\rangle$;

SOLUTION: We find the limit in a similar manner as above. Here, $x(t) = e^t,\ y(t) = \sin t,\ z(t) = t/(1-t)$. The limits are

$$\begin{aligned}\lim_{t\to 0} x(t) &= e^0 = 1\\ \lim_{t\to 0} y(t) &= \sin 0 = 0\\ \lim_{t\to 0} z(t) &= \frac{0}{1-0} = 0\end{aligned}$$

Thus,

$$\lim_{t\to 0}\langle e^t,\ \sin t,\ t/(1-t)\rangle = \langle 1,0,0\rangle$$

9. $\lim_{t\to\infty}\langle e^{-t},\ 1/t^2,\ 4\rangle$;

SOLUTION: We find the limit in a similar manner as above. Here, $x(t) = e^{-t},\ y(t) = 1/t^2,\ z(t) = 4$. The limits are

$$\begin{aligned}\lim_{t\to\infty} x(t) &= 0\\ \lim_{t\to\infty} y(t) &= 0\\ \lim_{t\to\infty} z(t) &= 4\end{aligned}$$

Thus,

$$\lim_{t\to\infty}\langle e^{-t},\ 1/t^2,\ 4\rangle = \langle 0,0,4\rangle$$

10. $\lim_{t\to\infty}\langle e^{-t},\ (1-t^2)/t^2,\ \sqrt[3]{t}/(\sqrt{t}+t)\rangle$;

SOLUTION: We find the limit in a similar manner as above.

Here, $x(t) = e^{-t},\ y(t) = (1-t^2)/t^2,\ z(t) = \sqrt[3]{t}/(\sqrt{t}+t)$. The limits are

$$\begin{aligned}
\lim_{t\to\infty} x(t) &= 0\\
\lim_{t\to\infty} y(t) &= \lim_{t\to\infty}(1/t^2 - 1) = -1\\
\lim_{t\to\infty} z(t) &= \lim_{t\to\infty}\left(\frac{1}{\sqrt[6]{t}+\sqrt[3]{t^2}}\right) = 0
\end{aligned}$$

Thus,

$$\lim_{t\to\infty}\langle e^{-t},\ (1-t^2)/t^2,\ \sqrt[3]{t}/(\sqrt{t}+t)\rangle = \langle 0,-1,0\rangle$$

11. $\lim_{t\to-\infty}\langle 2,\ t^2,\ 1/\sqrt[3]{t}\rangle$;

SOLUTION: We find the limit in a similar manner as above. Note that $y(t) = t^2$, and the limit tends to infinity. Thus the limit does not exist.

12. $\lim_{t\to0+}\langle (e^{2t}-1)/t,\ (\sqrt{1+t}-1)/t,\ t\ln t\rangle$;

SOLUTION: We find the limit in a similar manner as above. Here, $x(t) = (e^{2t}-1)/t,\ y(t) = (\sqrt{1+t}-1)/t,\ z(t) = t\ln t$. The limits are

$$\begin{aligned}
\lim_{t\to0+} x(t) &= \lim_{t\to0+}\frac{\sum_{n=0}^{\infty}(2t)^n/(n)! - 1}{t} = \lim_{t\to0+}\frac{\sum_{n=1}^{\infty}(2t)^n/(n)!}{t} = \lim_{t\to0+}\sum_{n=1}^{\infty}2^n t^{n-1}/(n)! = 2\\
\lim_{t\to0+} y(t) &= \lim_{t\to0+}\frac{1/2(1+t)^{-1/2}}{1} = \frac{1}{2\sqrt{1+0}} = \frac{1}{2}\\
\lim_{t\to0+} z(t) &= \lim_{t\to0+}\frac{\ln t}{1/t} = \lim_{t\to0+}\frac{1/t}{-1/t^2} = -(0) = 0
\end{aligned}$$

In calculating the limit of $x(t)$, you need only find the constant term in the Taylor expansion. You could also use L'Hopitals rule. I did this for $y(t),\ z(t)$. Thus,

$$\lim_{t\to0+}\langle (e^{2t}-1)/t,\ (\sqrt{1+t}-1)/t,\ t\ln t\rangle = \langle 2, 1/2, 0\rangle$$

13. $\lim_{t\to0}\langle \sin^2(2t)/t^2,\ t^2+2,\ (\cos t - 1)/t^2\rangle$;

SOLUTION: We find the limit in a similar manner as above. Here, $x(t) = \sin^2(2t)/t^2,\ y(t) = t^2+2,\ z(t) = (\cos t - 1)/t^2$.

The limits are

$$\lim_{t\to 0} x(t) = \lim_{t\to 0} \frac{4\cos(2t)\sin(2t)}{2t} = \lim_{t\to 0} \frac{\sin(4t)}{t} = \lim_{t\to 0} \frac{4\cos(4t)}{1} = 4$$
$$\lim_{t\to 0} y(t) = 0^2 + 2 = 2$$
$$\lim_{t\to 0} z(t) = \lim_{t\to 0} \frac{\sum_{n=0}^{\infty}(-1)^n t^{2n}/(2n)! - 1}{t^2} = \lim_{t\to 0} \frac{\sum_{n=1}^{\infty}(-1)^n t^{2n}/(2n)!}{t^2} = \lim_{t\to 0} \sum_{n=1}^{\infty}(-1)^n t^{2n-2}/(2n)!$$
$$= (-1)^1/(2)! = -1/2$$

where L'Hopitals rule has been used in the first two limits, and a Taylor series in the third. Thus,

$$\lim_{t\to 0}\langle \sin^2(2t)/t^2,\ t^2+2,\ (\cos t - 1)/t^2\rangle = \langle 4, 2, -1/2\rangle$$

14. $\lim_{t\to 0}\langle (e^{2t}-1)/t,\ t\cos t,\ \sqrt{1+t}\rangle$;

Solution: We find the limit in a similar manner as above. Here, $x(t) = (e^{2t}-1)/t$, $y(t) = t\cos t$, $z(t) = \sqrt{1+t}$. The limits are

$$\lim_{t\to 0} x(t) = 2$$
$$\lim_{t\to 0} y(t) = 0\cos 0 = 0$$
$$\lim_{t\to 0} z(t) = \sqrt{1+0} = 1$$

Thus,

$$\lim_{t\to 0}\langle (e^{2t}-1)/t,\ t\cos t,\ \sqrt{1+t}\rangle = \langle 2, 0, 1\rangle$$

15. $\lim_{t\to\infty}\langle e^{2t}/\cosh^2 t,\ t^{2012}e^{-t},\ e^{-2t}\sinh^2 t\rangle$.

Solution: We find the limit in a similar manner as above. Here, $x(t) = e^{2t}/\cosh^2 t$, $y(t) = t^{2012}e^{-t}$, $z(t) = e^{-2t}\sinh^2 t$. First, $\cosh(t) = (e^t + e^{-t})/2$ and $\sinh(t) = (e^t - e^{-t})/2$. From these it can be shown that $d/dt\cosh(t) = \sinh(t)$ and $d/dt\sinh(t) = \cosh(t)$. The limits are

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{4e^{2t}}{(e^t+e^{-t})^2} = \lim_{t\to\infty} \frac{4e^{2t}}{e^{2t}+2+e^{-2t}} = \lim_{t\to\infty} \frac{4}{1+2e^{-2t}+e^{-4t}} = 4$$
$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{t^{2012}}{e^t} = \lim_{t\to\infty} \frac{2012t^{2011}}{e^t} = \lim_{t\to\infty} \frac{2012 * 2011t^{2010}}{e^t} = \ldots = \lim_{t\to\infty} \frac{2012!}{e^t} = 0$$
$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} \frac{(e^t-e^{-t})^2}{4e^{2t}} = \lim_{t\to\infty} \frac{e^{2t}-2+e^{-2t}}{4e^{2t}} = \lim_{t\to\infty}(1/4 - 1/2e^{-2t} + 1/4e^{-4t}) = 1/4$$

where repeated L'Hopitals rule has been used in finding the limit of $y(t)$. Thus,

$$\lim_{t\to\infty} \langle e^{2t}/\cosh^2 t,\ t^{2012}e^{-t},\ e^{-2t}\sinh^2 t\rangle = \langle 4, 0, 1/4\rangle$$

16–22. Sketch each of the following curves and identify the direction in which the curve is traced out as the parameter t increases:

16. $\mathbf{r}(t) = \langle t,\ \cos(3t),\ \sin(3t)\rangle$;

SOLUTION: For these problems, the general strategy will be to split $\mathbf{r}(t)$ into a sum of two vectors, $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$, one with zero as one component and the other with zero in the other two components. For example, here it would be wise to let $\mathbf{r}_1(t) = \langle t, 0, 0\rangle$ and $\mathbf{r}_2(t) = \langle 0, \cos(3t), \sin(3t)\rangle$. It is clear then that projected in the yz plane, $\mathbf{r}(t)$ traces a circle. The action of $\mathbf{r}_1(t)$ is to stretch out this circle along the x axis – to make a helix of radius 1. To determine the direction, simply choose some sample points. In the yz plane, it should be evident that the curve is traced counterclockwise. As t increases, the curve extends in the positive x direction.

17. $\mathbf{r}(t) = \langle 2\sin(5t),\ 4,\ 3\cos(5t)\rangle$;

SOLUTION: Let $\mathbf{r}_1(t) = \langle 2\sin(5t), 0, 3\cos(5t)\rangle$. This traces an ellipse with major axis of length 6 along the z axis and minor axis of length 4 along the x axis. Let $bfr_2(t) = \langle 0, 4, 0\rangle$. This has the action of shifting $\mathbf{r}_1(t)$ to the plane $y = 4$ rather than $y = 0$. In the xz plane, it traverses clockwise.

18. $\mathbf{r}(t) = \langle 2t\sin t,\ 3t\cos t,\ t\rangle$;

SOLUTION: Let $\mathbf{r}_1(t) = \langle 2t\sin t,\ 3t\cos t, 0\rangle$. This curve resembles that of the above, notably it seems to be t times the previous $\mathbf{r}_1(t)$. The multiplication of t has the effect of causing the ellipse to spiral out, as $|\mathbf{r}_1(t)| = t$. Letting $\mathbf{r}_2(t) = \langle 0, 0, t\rangle$ reveals that the curve traced is helical in nature. In fact, this curve lies on the surface of the double cone $x^2/4 + y^2/9 = z^2$. It has a clockwise orientation viewed from above.

19. $\mathbf{r}(t) = \langle \sin t,\ \cos t,\ \ln t\rangle$;

SOLUTION: Let $\mathbf{r}_1(t) = \langle \sin t,\ \cos t, 0\rangle$. This is clearly a circle oriented clockwise. Let $\mathbf{r}_2(t) = \langle 0, 0, \ln t\rangle$. Since $\ln t$ is always increasing and nonnegative, this is a helical shape above the x axis of radius 1. Notably, it is simply $\ln t$ wrapped around the cylinder $x^2 + y^2 = 1$.

20. $\mathbf{r}(t) = \langle t,\ 1-t,\ (t-1)^2\rangle$;

SOLUTION: Let $\mathbf{r}_1(t) = \langle t,\ 1-t, 0\rangle = \langle 0,1,0\rangle + t\langle 1,-1,0$. This is a line through $(0,1,0)$ parallel to $(1,-1,0)$, lying in the xy plane. Let $\mathbf{r}_2(t) = \langle 0,0,(t-1)^2\rangle$. The effect of this is to raise up the line by $(t-1)^2$. The curve is a parabola with vertex at $(1,0,0)$ rotated so that it is in the plane orthogonal to $z = 0$ containing $\mathbf{r}_1(t)$.

21. $\mathbf{r}(t) = \langle t^2,\ t,\ \sin^2(\pi t)\rangle$;

SOLUTION: Let $\mathbf{r}_1(t) = \langle t^2, t, 0\rangle$. This is a parabola with vertex at $(0,0)$ opening towards the positive x axis, lying in the xy plane. Let $\mathbf{r}_2(t) = \langle 0,0,\sin^2(\pi t)\rangle$. The effect of this is to wrap $\sin^2(\pi t)$ so that it sits on top of the parabola.

22. $\mathbf{r}(t) = \langle \sin t,\ \sin t,\ \sqrt{2}\cos t\rangle$.

SOLUTION: Let $\mathbf{r}_1(t) = \langle 0, \sin t, \sqrt{2}\cos t\rangle$. Then $\mathbf{r}_1(t)$ traces an ellipse in the yz plane with major axis of length $\sqrt{2}$ along the z axis, oriented clockwise. Let $\mathbf{r}_2(t) = \langle \sin t, 0, 0\rangle$. What is the effect of adding $\mathbf{r}_2(t)$ to $\mathbf{r}_1(t)$? Notice that both have the same period, 2π. Therefore the shape of $\mathbf{r}_1(t)$ will be more or less maintained, but maybe skewed – it will not form a helix. The effect of $\mathbf{r}_2(t)$ is to move the points of the ellipse along the x axis by some amount (according to $\sin t$). Starting with $t = 0$ and traversing to $t = \pi$, the ellipse is pushed in the positive x direction. It reaches a maximum at $t = \pi/2$, where the ellipse is pushed by a full one unit. Starting now with $t = \pi$ and traversing to $t = 2\pi$, the ellipse is pushed in the negative x direction, eventually reaching the start again. Overall, this effect just skews the ellipse. You can imagine that the ellipse, first in the yz plane, is rotated by 45° from the y axis toward the x axis. Alternatively, because the x and y coordinates are equal, the curve must lie in the plane $y = x$. In fact, it can be shown that this is the intersection of $x^2 + y^2 + z^2 = 2$ with

$y = x$ as follows:
Let $x(t) = y(t) = \sin t$, then

$$\sin^2 t + \sin^2 t + z^2 = 2 \Leftrightarrow z^2 = 2(1 - \sin^2 t) = 2\cos^2 t \Leftrightarrow z(t) = \pm\sqrt{2}\cos t$$

where the sign signifies the starting point and the orientation.

23. Two objects are said to collide if they are at the same position *at the same time*. Two trajectories are said to intersect if they have common points. Let t be the physical time. Let two objects travel along the space curves $\mathbf{r}_1(t) = \langle t,\ t^2,\ t^3\rangle$ and $\mathbf{r}_2(t) = \langle 1+2t,\ 1+6t,\ 1+14t\rangle$. Do the objects collide? Do their trajectories intersect? If so, find the collision and intersection points.

Solution: We first answer if their trajectories intersect. To do this, use different parameters t, s. If it happens that $t = s$, then the objects collide. Thus we are looking for a (t_0, s_0) such that $\mathbf{r}_1(t_0) = \mathbf{r}_2(s_0)$ and determining if $t_0 = s_0$ or not. So we have

$$t = 1 + 2s,\ t^2 = 1 + 6s,\ t^3 = 1 + 14s$$

Squaring both sides of the first gives an equation that we may substitute into the second, as follows:

$$(1+2s)^2 = 1+6s \Leftrightarrow 1+4s+4s^2 = 1+6s \Leftrightarrow 4s^2 - 2s = 0 \Leftrightarrow s = 0, 1/2$$

Substituting these into the first equation gives $(1, 0)$ and $(2, 1/2)$ as possible solutions. Both of these are consistent in the third equation. Therefore there are two points of intersection but the objects do not collide. The two points of intersection are:

$$\mathbf{r}_1(0) = \langle 1, 1, 1\rangle, \quad \mathbf{r}_2(2) = \langle 2, 4, 8\rangle$$

24. Find a simple parameterization of the curve of intersection of the surfaces $x^2 + y^2/4 + z^2/9 = 1, y \geq 0$ and $z = x^2$. Sketch the curve. Let $x(t) = t$, a standard choice because it is easy to work with. Then $z(t) = t^2$. Substitution of these into the first equation yields

$$t^2 + y(t)^2/4 + t^4/9 = 1 \Leftrightarrow 36t^2 + 9y(t)^2 + 4t^4 = 36 \Leftrightarrow y(t) = \pm 2/3\sqrt{9 - 9t^2 - t^4}$$

This parameterization is simple because $x(t)$ and $z(t)$ are monotonically increasing.

25–32. Find two vector functions that traverse a given curve C in the opposite directions if C is the curve of intersection of two surfaces:

25. $y = x^2$ and $z = 1$;

Solution: Simply let $x(t) = t$. Then $y(t) = x(t)^2 = t^2$.

So $\mathbf{r}(t) = \langle t, t^2, 1\rangle$ describes the intersection. To traverse C in the opposite direction, simply replace t with $-t$. This yields $\mathbf{r}(t) = \langle -t, t^2, 1\rangle$.

26. $x = \sin y$ and $z = x$;

SOLUTION: Let $y(t) = t$. Then $z(t) = x(t) = \sin y(t) = \sin t$. So the intersection is given by $\mathbf{r}(t) = \langle \sin t, t, \sin t\rangle$, and in the opposite direction is given by $\mathbf{r}(t) = \langle -\sin t, -t, -\sin t\rangle$.

27. $x^2 + y^2 = 9$ and $z = xy$;

SOLUTION: Let $x(t) = 3\cos t$, since when an x^2 and y^2 appears it is natural to use sine and cosine. Then $y(t) = \pm 3\sqrt{1-\cos^2 t} = \pm 3\sin t$. Choose $y(t) = 3\sin t$ to avoid having a negative sign. Then $z(t) = x(t)y(t) = 9\cos t\sin t = 9/2\sin(2t)$. So the intersection is given by $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 9/2\sin(2t)\rangle$ in one direction and $\mathbf{r}(t) = \langle 3\cos t, -3\sin t, -9/2\sin(2t)\rangle$ in the opposite direction.

28. $x^2 + y^2 = z^2$ and $x + y + z = 1$;

SOLUTION: I am going to employ a little trick first. I will define $x(t) = z(t)\cos t$ and $y(t) = z(t)\sin t$, despite the fact that we do not know what $z(t)$ is. These satisfy the first equation naturally. To satisfy the second equation, it must be that

$$z(t)(\cos t + \sin t) + z(t) = 1 \;\Leftrightarrow\; z(t) = \frac{1}{1+\cos t+\sin t}$$

So $\mathbf{r}(t) = \langle \cos t/(1+\cos t+\sin t), \sin t/(1+\cos t+\sin t), 1/(1+\cos t+\sin t)\rangle$ gives the curve of intersection in one direction and $\mathbf{r}(t) = \langle \cos t/(1+\cos t-\sin t), -\sin t/(1+\cos t-\sin t), 1/(1+\cos t-\sin t)\rangle$ in the opposite. I believe there are actually two curves of intersection, but I am not sure how to get the other.

29. $z = x^2 + y^2$ and $y = x^2$;

SOLUTION: Let $x(t) = t$. Then $y(t) = t^2$ and $z(t) = x(t)^2 + y(t)^2 = t^2 + t^4$. So $\mathbf{r}(t) = \langle t, t, t^2 + t^4\rangle$ gives the curve of intersection in one direction and $\mathbf{r}(t) = \langle -t, t^2, t^2+t^4\rangle$ in the other.

30. $x^2/4 + y^2/9 = 1$ and $x + y + z = 1$;

SOLUTION: Let $x(t) = 2\cos t$ and $y(t) = 3\sin t$. These satisfy the first equation. To satisfy the second equation it must be that $z(t) = 1 - 2\cos t - 3\sin t$. So $\mathbf{r}(t) = \langle 2\cos t, 3\sin t, 1 - 2\cos t - 3\sin t\rangle$ gives the curve of intersection in one direction and $\mathbf{r}(t) = \langle 2\cos t, -3\sin t, 1 - 2\cos t + 3\sin t\rangle$ in the other.

31. $x^2/2 + y^2/2 + z^2/9 = 1$ and $x - y = 0$;

SOLUTION: By the second equation we have $x(t) = y(t)$. Substitution of this into the first yields $x(t)^2 + z(t)^2/9 = 1$. So let $x(t) = y(t) = \cos t$. Then $z(t) = 3\sin t$. So $\mathbf{r}(t) = \langle \cos t, \cos t, 3\sin t\rangle$ gives the curve of intersection in one direction and $\mathbf{r}(t) = \langle \cos t, \cos t, -3\sin t\rangle$ in the opposite.

32. $x^2 + y^2 - 2x = 0$ and $z = x^2 + y^2$.

SOLUTION: Note that substitution of the second equation into the first gives that $z(t) = 2x(t)$. Completing the square in the first yields

$$(x-1)^2 + y^2 = 1$$

So let $x(t) = 1+\cos t$ and $y(t) = \sin t$. Then $z(t) = 2(1+\cos t)$. Thus $\mathbf{r}(t) = \langle 1 + \cos t, \sin t, 2(1 + \cos t)\rangle$ gives the curve of intersection in one direction and$\mathbf{r}(t) = \langle 1 + \cos t, -\sin t, 2(1 + \cos t)\rangle$ in the opposite.

33. Specify the parts of the curve $\mathbf{r}(t) = \langle \sin t, \cos t, 4\sin^2 t\rangle$ that lie above the plane $z = 1$ by restricting the range of the parameter t.

SOLUTION: To solve this we need only find the t such that $4\sin^2 t > 1$. So

$$\sin^2 t > 1/4 \;\Leftrightarrow\; \sin t < -1/2, \sin t > 1/2$$

So we are looking for the t where $-1 \le \sin t < -1/2$ and $1/2 < \sin t \le 1$. Solving these gives $7\pi/6 < t < 11\pi/6$ and $\pi/6 < t < 5\pi/6$. Inclusion of the endpoints allows for $\mathbf{r}(t)$ to intersect $z = 1$.

34. Find the values of the parameters a and b at which the curve $\mathbf{r}(t) = \langle 1 - at^2, b - t, t^3\rangle$ passes through the point $(1, 2, 8)$.

SOLUTION: Equating the third coordinates yields $t = 2$. Substituting this gives $\mathbf{r}(2) = \langle 1 - 4a, b - 2, 8\rangle = \langle 1, 2, 8\rangle$. Equating the first coordinates yields $1 - 4a = 1 \Leftrightarrow a = 0$ and equating the second coordinates yields $b - 2 = 2 \Leftrightarrow b = 4$.

35–39. Find the values of a, b, and c, if any, for which each of the following vector functions is continuous: $\mathbf{r}(0) = \langle a, b, c\rangle$ and, for $t \neq 0$,

35. $\mathbf{r}(t) = \langle t,\ \cos^2 t,\ 1 + t + t^2\rangle$;

SOLUTION: Recall that a vector function $\mathbf{r}(t)$ is continuous at $t = t_0$ iff $\lim_{t\to t_0} = \mathbf{r}(t_0)$. So, to make each of these function continuous, we must define $\mathbf{r}(0) = \lim_{t\to 0} \mathbf{r}(t)$, if it exists. In this problem, the limit exists since polynomials are continuous and sine and cosine are continuous. So $a = \lim_{t\to 0} x(t) = 0$, $b = \lim_{t\to 0} y(t) = 1$, and $c = \lim_{t\to 0} z(t) = 1$ and $\mathbf{r}(0) = \langle 0, 1, 1\rangle$.

36. $\mathbf{r}(t) = \langle t,\ \cos^2 t,\ \sqrt{1 + t^2}\rangle$;

SOLUTION: As with above, we must define $\mathbf{r}(0) = \lim_{t\to 0} \mathbf{r}(t)$, if it exists. In this problem, the limit exists since polynomials are continuous, sine and cosine are continuous, and $\sqrt{1 + t^2}$ is continuous in its domain. So $a = \lim_{t\to 0} x(t) = 0$, $b = \lim_{t\to 0} y(t) = 1$, and $c = \lim_{t\to 0} z(t) = 1$ and $\mathbf{r}(0) = \langle 0, 1, 1\rangle$.

37. $\mathbf{r}(t) = \langle t,\ \cos^2 t,\ \ln|t|\rangle$;

SOLUTION: As with above, we must define $\mathbf{r}(0) = \lim_{t\to 0} \mathbf{r}(t)$, if it exists. In this problem, the limit does not exist since $\ln|t|$ is not continuous at $t = 0$.

38. $\mathbf{r}(t) = \langle \sin(2t)/t,\ \sinh(3t)/t,\ t\ln|t|\rangle$;

SOLUTION: As with above, we must define $\mathbf{r}(0) = \lim_{t\to 0} \mathbf{r}(t)$, if it exists. To determine if the limit exist, we examine the limit of each component individually

$$a = \lim_{t\to 0} x(t) = \lim t_{\to} 0 \frac{\sum_{n=0}^{\infty}(-1)^n 2^{2n+1} t^{2n+1}/(2n+1)!}{t} = \lim_{t\to 0} \sum_{n=0}^{\infty}(-1)^n 2^{2n+1} t^{2n}/(2n+1)! = 2$$

$$b = \lim_{t\to 0} y(t) = \lim_{t\to 0} \frac{e^{3t} - e^{-3t}}{2t} = \lim_{t\to 0} \frac{3e^{3t} + 3e^{-3t}}{2} = 3$$

$$c = \lim_{t\to 0} z(t) = \lim_{t\to 0} \frac{\ln|t|}{1/t} = \lim_{t\to 0} \frac{1/t}{-1/t^2} = \lim_{t\to 0}(-t) = 0$$

Thus $\mathbf{r}(0) = \langle 2, 3, 0\rangle$.

39. $\mathbf{r}(t) = \langle t\cot(2t),\ t^{1/3}\ln|t|,\ t^2+2\rangle$;

SOLUTION: As with above, we must define $\mathbf{r}(0) = \lim_{t\to 0}\mathbf{r}(t)$, if it exists. To determine if the limit exist, we examine the limit of each component individually

$$a = \lim_{t\to 0} x(t) = \lim_{t\to 0}\frac{t\cos(2t)}{\sin(2t)} = \lim_{t\to 0}\frac{\cos(2t) - 2t\sin(2t)}{2\cos(2t)} = \frac{1}{2}$$
$$b = \lim_{t\to 0} y(t) = \lim_{t\to 0}\frac{\ln|t|}{t^{-1/3}} = \lim_{t\to 0}\frac{1/t}{-1/3t^{-4/3}} = \lim_{t\to 0}(-3t^{1/3}) = 0$$

where L'Hopitals rule has been used. Polynomials are continuous so we need not find the limit. Thus $\mathbf{r}(0) = \langle 1/2, 0, 2\rangle$.

40. Suppose that the limits $\lim_{t\to a}\mathbf{v}(t)$ and $\lim_{t\to a}\mathbf{u}(t)$ exist. Prove the basic laws of limits for the following vector functions:

$$\lim_{t\to a}(\mathbf{v}(t) + \mathbf{u}(t)) = \lim_{t\to a}\mathbf{v}(t) + \lim_{t\to a}\mathbf{u}(t), \tag{1.1}$$
$$\lim_{t\to a}(s\mathbf{v}(t)) = s\lim_{t\to a}\mathbf{v}(t), \tag{1.2}$$
$$\lim_{t\to a}\mathbf{v}(t)\cdot\mathbf{u}(t) = \lim_{t\to a}\mathbf{v}(t)\cdot\lim_{t\to a}\mathbf{u}(t), \tag{1.3}$$
$$\lim_{t\to a}\mathbf{v}(t)\times\mathbf{u}(t) = \lim_{t\to a}\mathbf{v}(t)\times\lim_{t\to a}\mathbf{u}(t). \tag{1.4}$$

SOLUTION:
Throughout the proofs, let $v_i(t)$, $v_j(t)$, and $v_k(t)$ be the components of $\mathbf{v}(t)$. Moreover, let $\lim_{t\to a} v_i(t) = v_i(a)$ and $\lim_{t\to a} v_j(t) = v_j(a)$. Similarly, let $u_i(t)$, $u_j(t)$, and $u_k(t)$ be the components of $\mathbf{u}(t)$. Moreover, let $\lim_{t\to a} u_i(t) = u_i(a)$ and $\lim_{t\to a} u_j(t) = u_j(a)$. Note that the components of these vector functions are functions of a single variable, so basic limit laws apply.

Proof of (1.1):
Look at one component of each, say $v_i(t)$ and $u_i(t)$. Then

$$\lim_{t\to a}(v_i(t) + u_i(t)) = \lim_{t\to a} v_i(t) + \lim_{t\to a} u_i(t) = v_i(a) + u_i(a)$$

Hence the limit exists for each component. By Theorem **10.1** the proof is complete.

Proof of (1.2):

Look at one component of $\mathbf{v}(t)$, say $v_i(t)$. Then

$$\lim_{t\to a}(sv_i(t)) = s\lim_{t\to a} v_i(t) = sv_i(a)$$

Hence the limit exists for each component. By Theorem **10.1** the proof is complete.

Proof of (1.3):
Look at one component of each, say $v_i(t)$ and $u_i(t)$. Then

$$\lim_{t\to a}(v_i(t)u_i(t)) = (\lim_{t\to a} v_i(t))(\lim_{t\to a} u_i(t)) = v_i(a)u_i(a)$$

Hence the limit exists for each component. By Theorem **10.1** the proof is complete.

Proof of (1.4):
Look at one component of the cross product, say the ith component. Then

$$\begin{aligned}\lim_{t\to a}(v_j(t)u_k(t) - v_k(t)u_i(t)) &= \lim_{t\to a}(v_j(t)u_k(t)) - \lim_{t\to a}(v_k(t)u_j(t)) \\ &= (\lim_{t\to a} v_j(t))(\lim_{t\to a} u_k(t)) - (\lim_{t\to a} v_k(t))(\lim_{t\to a} u_j(t)) \\ &= v_j(a)u_k(a) - v_k(a)u_j(a)\end{aligned}$$

Hence the limit exists for each component. By Theorem **10.1** the proof is complete.

41. Prove the last limit law in Exercise **40** directly from Definition **10.2**, i.e., without using Theorem **10.1**. Hint: see Study Problem **10.7**.

SOLUTION: The proof is more or less identical to the one given in Study Problem **10.7**, just replace all dots with crosses. The only noticeable is to prove a Cauchy-Schwarz inequality for the cross product, the proof of which is below:

$$|\mathbf{a}\times\mathbf{b}| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta \le \|\mathbf{a}\|\|\mathbf{b}\|$$

owing to the fact that $-1 \le \sin\theta \le 1$.

42–**47**. Let

$$\begin{aligned}\mathbf{v}(t) &= \langle (e^{2t}-1)/t,\ (\sqrt{1+t}-1)/t,\ t\ln|t|\rangle, \\ \mathbf{u}(t) &= \langle \sin^2(2t)/t^2,\ t^2+2,\ (\cos t - 1)/t^2\rangle, \\ \mathbf{w}(t) &= \langle t^{2/3},\ 2/(1-t),\ 1+t-t^2+t^3\rangle.\end{aligned}$$

Use the basic laws of limits established in Exercise **40** to find:

42. $\lim_{t\to 0}(2\mathbf{v}(t) - \mathbf{u}(t) + \mathbf{w}(t))$;

SOLUTION: It would first be wise to find the individual limits $\lim_{t\to 0}\mathbf{v}(t)$, $\lim_{t\to 0}\mathbf{u}(t)$, and $\lim_{t\to 0}\mathbf{w}(t)$. We first find $\lim_{t\to 0}\mathbf{v}(t)$:

$$\begin{aligned}
\lim_{t\to 0} x(t) &= \lim_{t\to 0}\frac{e^{2t}-1}{t} = \lim_{t\to 0} 2e^{2t} = 2\\
\lim_{t\to 0} y(t) &= \lim_{t\to 0}\frac{\sqrt{1+t}-1}{t} = \lim_{t\to 0}\frac{1}{2\sqrt{1+t}} = \frac{1}{2}\\
\lim_{t\to 0} z(t) &= \lim_{t\to 0}\frac{\ln|t|}{1/t} = \lim_{t\to 0}\frac{1/t}{-1/t^2} = \lim_{t\to 0}(-t) = 0
\end{aligned}$$

Thus $\lim_{t\to 0}\mathbf{v}(t) = \langle 2, 1/2, 0\rangle$.

We next find $\lim_{t\to 0}\mathbf{u}(t)$:

$$\begin{aligned}
\lim_{t\to 0} x(t) &= \lim_{t\to 0}\frac{\sin^2(2t)}{t^2} = \lim_{t\to 0}\frac{4\sin(2t)\cos(2t)}{2t} = \lim_{t\to 0}\frac{2\sin(4t)}{2t} = \lim_{t\to 0} 4\cos(4t) = 4\\
\lim_{t\to 0} y(t) &= \lim_{t\to 0}(t^2+2) = 2\\
\lim_{t\to 0} z(t) &= \lim_{t\to 0}\frac{\cos t - 1}{t^2} = \lim_{t\to 0}\frac{-\sin t}{2t} = \lim_{t\to 0}\frac{-\cos t}{2} = -\frac{1}{2}
\end{aligned}$$

Thus $\lim_{t\to 0}\mathbf{u}(t) = \langle 4, 2, -1/2\rangle$.

Finally we find $\lim_{t\to 0}\mathbf{w}(t)$:

$$\begin{aligned}
\lim_{t\to 0} x(t) &= t^{2/3} = 0\\
\lim_{t\to 0} y(t) &= \lim_{t\to 0}\frac{2}{1-t} = 2\\
\lim_{t\to 0} z(t) &= \lim_{t\to 0}(1+t-t^2+t^3) = 1
\end{aligned}$$

Thus $\lim_{t\to 0}\mathbf{w}(t) = \langle 0, 2, 1\rangle$.

With these in hand we may now proceed. The limit in question reduces to

$$\begin{aligned}
\lim_{t\to 0}(2\mathbf{v}(t) - \mathbf{u}(t) + \mathbf{w}(t)) &= 2\lim_{t\to 0}\mathbf{v}(t) - \lim_{t\to 0}\mathbf{u}(t) + \lim_{t\to 0}\mathbf{w}(t)\\
&= 2\langle 2, 1/2, 0\rangle - \langle 4, 2, -1/2\rangle + \langle 0, 2, 1\rangle\\
&= \langle 0, 1, 3/2\rangle
\end{aligned}$$

43. $\lim_{t\to 0}(\mathbf{v}(t)\cdot\mathbf{u}(t))$;

SOLUTION: From Exercise **40** the limit can be evaluated as $\lim_{t\to 0}\mathbf{v}(t)\cdot\lim_{t\to 0}\mathbf{u}(t)$. Thus,

$$\lim_{t\to 0}(\mathbf{v}(t)\cdot\mathbf{u}(t)) = \langle 2, 1/2, 0\rangle\cdot\langle 4, 2, -1/2\rangle = 8+1=9$$

44. $\lim_{t\to 0}(\mathbf{v}(t)\times\mathbf{u}(t))$;

SOLUTION: From Exercise **40** the limit can be evaluated as $\lim_{t\to 0}\mathbf{v}(t)\times\lim_{t\to 0}\mathbf{u}(t)$. Thus,

$$\lim_{t\to 0}(\mathbf{v}(t)\times\mathbf{u}(t)) = \langle 2, 1/2, 0\rangle\cdot\langle 4, 2, -1/2\rangle = \langle -1/4, 1, 2\rangle$$

45. $\lim_{t\to 0}[\mathbf{w}(t)\cdot(\mathbf{v}(t)\times\mathbf{u}(t))]$;

SOLUTION: From Exercise **40** the limit can be evaluated as

$$\lim_{t\to 0}[\mathbf{w}(t)\cdot(\mathbf{v}(t)\times\mathbf{u}(t))] = \lim_{t\to 0}\mathbf{w}(t)\cdot\lim_{t\to 0}(\mathbf{v}(t)\times\mathbf{u}(t)).$$

Thus,

$$\lim_{t\to 0}[\mathbf{w}(t)\cdot(\mathbf{v}(t)\times\mathbf{u}(t))] = \langle 0, 2, 1\rangle\cdot\langle -1/4, 1, 2\rangle = 2+2=4$$

46. $\lim_{t\to 0}[\mathbf{w}(t)\times(\mathbf{v}(t)\times\mathbf{u}(t))]$;

SOLUTION: From Exercise **40** the limit can be evaluated as

$$\lim_{t\to 0}[\mathbf{w}(t)\times(\mathbf{v}(t)\times\mathbf{u}(t))] = \lim_{t\to 0}\mathbf{w}(t)\times\lim_{t\to 0}(\mathbf{v}(t)\times\mathbf{u}(t)).$$

Thus,

$$\lim_{t\to 0}[\mathbf{w}(t)\cdot(\mathbf{v}(t)\times\mathbf{u}(t))] = \langle 0, 2, 1\rangle\times\langle -1/4, 1, 2\rangle = \langle 4-1, -(-1/4), 0-2(-1/4)\rangle = \langle 3, 1/4, 1/2\rangle$$

47. $\lim_{t\to 0}[\mathbf{w}(t)\times(\mathbf{v}(t)\times\mathbf{u}(t)) + \mathbf{v}(t)\times(\mathbf{u}(t)\times\mathbf{w}(t)) + \mathbf{u}(t)\times(\mathbf{w}(t)\times\mathbf{v}(t))]$.

SOLUTION: By the Jacobi identity, $[\mathbf{w}(t)\times(\mathbf{v}(t)\times\mathbf{u}(t)) + \mathbf{v}(t)\times(\mathbf{u}(t)\times\mathbf{w}(t)) + \mathbf{u}(t)\times(\mathbf{w}(t)\times\mathbf{v}(t))] = \mathbf{0}$. So

$$\lim_{t\to 0}[\mathbf{w}(t)\times(\mathbf{v}(t)\times\mathbf{u}(t)) + \mathbf{v}(t)\times(\mathbf{u}(t)\times\mathbf{w}(t)) + \mathbf{u}(t)\times(\mathbf{w}(t)\times\mathbf{v}(t))] = \mathbf{0}$$

.

47. Suppose that the vector functions $\mathbf{v}(t)\times\mathbf{u}(t)$ where $\mathbf{v}(t)$ and $\mathbf{u}(t)$ are non-vanishing. Does this imply that the both vector functions $\mathbf{v}(t)$ and $\mathbf{u}(t)$ are continuous? Support your arguments by examples.

SOLUTION: Consider $\mathbf{v}(t) = \langle t, 1, 0\rangle$ and $\mathbf{u}(t) = \langle -1, \ln|t|, 0\rangle$, which are both non-vanishing. Then $\mathbf{v}(t)\times\mathbf{u}(t) = \langle 0, 0, t\ln|t|+1\rangle$. It has already been shown that this vector function is continuous. Yet $\mathbf{u}(t)$ is not continuous at $t = 0$.

48. Suppose that the vector functions $\mathbf{v}(t) \times \mathbf{u}(t)$ and $\mathbf{v}(t) \neq 0$ are continuous. Does this imply that the vector function $\mathbf{u}(t)$ is continuous? Support your arguments by examples.

SOLUTION: See the above example.