Homework 21 Sai Sivakumar

21.1 Consider the complex function given by

$$f(z) = \frac{z\exp(iz)}{z^2 + a^2} = \frac{z\exp(iz)}{(z+ia)(z-ia)}.$$

(a) Find all of the poles of f.

By inspection, the poles are $z = \pm ia$ (1st order pole)

(b) Find the corresponding residues of f at the poles.

$z = ia$: $\lim_{z\to ia} (z-ia)\left[\frac{z\exp(iz)}{(z+ia)(z-ia)}\right] = \lim_{z\to ia} \frac{z\exp(iz)}{z+ia} = \boxed{\frac{\exp(-a)}{2}}$

$z = -ia$: $\lim_{z\to -ia} (z+ia)\left[\frac{z\exp(iz)}{(z+ia)(z-ia)}\right] = \lim_{z\to -ia} \frac{z\exp(iz)}{z-ia} = \boxed{\frac{\exp(a)}{2}}$

21.2 Consider the complex function given by

$$f(z) = \frac{a}{4z^4+5z^2+1} = \frac{a}{(4z^2+1)(z^2+1)} = \frac{a}{(2z+i)(2z-i)(z+i)(z-i)}.$$

(a) Find all of the poles of f.

By inspection, the poles are $z = \pm i/2, \pm i$ (first order poles)

(b) Find the corresponding residues of f at the poles.

$z = -i/2$: $\lim_{z\to -i/2} (z+i/2)\frac{a}{(2z+i)(2z-i)(z+i)(z-i)} = \lim_{z\to -i/2} \frac{a}{2(2z-i)(z^2+1)} = \boxed{\frac{a}{-3i}}$

$z = i/2$: $\lim_{z\to i/2} (z-i/2)\frac{a}{(2z+i)(2z-i)(z+i)(z-i)} = \lim_{z\to i/2} \frac{a}{2(2z+i)(z^2+1)} = \boxed{\frac{a}{3i}}$

$z = +i$: $\lim_{z\to i} (z-i)\frac{a}{(2z+i)(2z-i)(z+i)(z-i)} = \lim_{z\to i} \frac{a}{(4z^2+1)(z+i)} = \boxed{\frac{a}{-6i}}$

$z = -i$: $\lim_{z\to -i} (z+i)\frac{a}{(2z+i)(2z-i)(z+i)(z-i)} = \lim_{z\to -i} \frac{a}{(4z^2+1)(z-i)} = \boxed{\frac{a}{6i}}$

21.3 Consider the complex function given by

$$f(z) = \frac{a^2}{(z^2+a^2)^2} = \frac{a^2}{(z+ia)^2(z-ia)^2} .$$

(a) Find all of the poles of f.

By inspection, the poles are $z = \pm ia$ (second order poles)

(b) Find the corresponding residues of f at the poles.

$z = ia$: $$\lim_{z\to ia}\left(\frac{d}{dz}\left[(z-ia)^2\frac{a^2}{(z+ia)^2(z-ia)^2}\right]\right) = \lim_{z\to ia}\left(\frac{d}{dz}\frac{a^2}{(z+ia)^2}\right)$$

$$= \lim_{z\to ia}\frac{-2a^2}{(z+ia)^3} = \frac{-2a^2}{(2ia)^3} = \boxed{\frac{1}{4ia}}$$

$z = -ia$: $$\lim_{z\to -ia}\left(\frac{d}{dz}\left[(z+ia)^2\frac{a^2}{(z+ia)^2(z-ia)^2}\right]\right) = \lim_{z\to -ia}\left(\frac{d}{dz}\frac{a^2}{(z-ia)^2}\right)$$

$$= \lim_{z\to -ia}\frac{-2a^2}{(z-ia)^3} = \frac{-2a^2}{(-2ia)^3} = \boxed{\frac{1}{-4ia}}$$