HOMEWORK 13

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Recall the function $t : [0,1] \to \mathbb{R}$ defined by

$$t(x) = \begin{cases} \frac{1}{q} & \text{if } x \in \mathbb{Q} \cap (0,1], \quad x = \frac{p}{q}, \quad p, q \in \mathbb{N}, \text{ and } \gcd(p,q) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

from the course notes. Abbott refers to it as Thomae's function.

For $k \in \mathbb{N}$, define $f_k : [0,1] \to \mathbb{R}$ by

$$f_k(x) = \begin{cases} \frac{1}{q} & \text{if } x \in \mathbb{Q} \cap (0,1], \quad x = \frac{p}{q}, \quad p, q \in \mathbb{N}, \quad \gcd(p,q) = 1 \text{ and } q \leq k \\ 0 & \text{otherwise.} \end{cases}$$

(a) Prove, if $g : [a,b] \to \mathbb{R}$ is 0 except on a finite set $G \subseteq [a,b]$, then g is Riemann integrable and

$$\int_a^b g \, dx = 0.$$

(b) Show (f_k) converges uniformly to t.

(c) Conclude t is Riemann integrable and

$$\int_0^1 t \, dx = 0.$$

Proof. Let $t\colon [0,1] \to \mathbb{R}$ and $f_k\colon [0,1] \to \mathbb{R}$ for $k \in \mathbb{N}$ be as given.

(a) With $g\colon [a,b] \to \mathbb{R}$ zero everywhere except on a finite set $G \subseteq [a,b]$, we show by induction on $n = |G|$ that g is Riemann integrable.

Consider the case with one discontinuity ($|G| = 1$) at the point $a < c_1 < b$ (if g is not continuous at a or b, g is still integrable). Then the restrictions of g given by $g|_{[a,c_1]}\colon [a,c_1] \to \mathbb{R}$ and $g|_{[c_1,b]}\colon [c_1,b] \to \mathbb{R}$ are both integrable on their domains. This follows since $g|_{[a,c_1]}$ and $g|_{[c_1,b]}$ are bounded; furthermore, for every $a < x < c_1$ the function $g|_{[a,c_1]}$ is integrable on $[a,x]$ and for every $c_1 < y < b$ the function $g|_{[c_1,b]}$ is integrable on $[y,b]$.

It follows that g is integrable. Thus suppose that g is integrable when $|G| = n$ and add one more point c_{n+1} to G. Then repeat the same argument as above in the case for one discontinuity, with c_{n+1} in place of c_1. Thus g is still integrable after including c_{n+1} in G. Hence by induction g is integrable if the set of points on which g is discontinuous is finite.

We estimate the integral $\int_a^b g \, \mathrm{d}x$. Let $\varepsilon > 0$ be given. With $|G| = n > 0$ (if $|G| = 0$ then the integral $\int_a^b g \, \mathrm{d}x$ is automatically zero), write

$$G = \{c_i \colon g \text{ is discontinuous at } c_i \text{ for } 1 \leq i \leq n\}.$$

Then let m and M be the infimum and supremum of the set $\{g(x) \colon x \in [a, b]\}$, and form the partition $P = \{a, c_1 - \varepsilon/n, c_1 + \varepsilon/n, \ldots, c_n - \varepsilon/n, c_1 + \varepsilon/n, b\}$. Then

$$2m\varepsilon = n \cdot 2m\varepsilon/n \leq L(g, P) \leq \int_a^b g \, \mathrm{d}x \leq U(g, P) \leq n \cdot M\varepsilon/n = 2M\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\int_a^b g \, \mathrm{d}x = 0$.

(b) Let $\varepsilon > 0$ be given. Let K be a natural number strictly larger than $1/\varepsilon$, and for $k > K$, it follows that

$$\begin{aligned} |t(x) - f_k(x)| &\leq |t(x) - f_K(x)| + |f_K(x) - f_k(x)| \\ &\leq \frac{1}{K+1} + \frac{1}{k+1} \\ &\leq \frac{2}{K} < 2\varepsilon. \end{aligned}$$

Thus (f_k) converges uniformly to t.

(c) Every function f_k for $k \in \mathbb{N}$ has finitely many discontinuities, so each f_k is integrable on $[0, 1]$. Then because (f_k) converges uniformly to t, it follows that t is also integrable. Furthermore, the sequence $(\int_0^1 f_k \, \mathrm{d}x)$ converges to $\int_0^1 t \, \mathrm{d}x$. But because f_k for every $k \in \mathbb{N}$ is zero except at its (finitely many) points of discontinuity, $(\int_0^1 f_k \, \mathrm{d}x)$ is the zero sequence, which converges to zero. Hence $\int_0^1 t \, \mathrm{d}x = 0$.

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