HOMEWORK 5

SAI SIVAKUMAR

Define a sequence from $\mathbb{R}$ as follows. Fix $r > 1$. Let $a_1 = 1$ and define recursively,

$$a_{n+1} = \frac{1}{r}(a_n + r + 1).$$

Show (a_n) converges and find its limit. [Suggestion: Show (a_n) is bounded above by $\frac{r+1}{r-1}$.] Note: I worked with Jude Flynn, Nicholas Kapsos, and Silas Rickards to work out the main ideas of this proof.

Proof. Let (a_n) be a recursively defined sequence as given. To show that (a_n) converges, it suffices to show that the sequence is increasing and that it is bounded as well.

To see that the sequence is increasing, we show by induction that for all $n \in \mathbb{N}$, we have $a_n \geq a_{n-1}$. Observe for $n = 1$, we have $a_2 = \frac{1}{r}(a_1+r+1) = \frac{1}{r}(r+2) = 1+\frac{2}{r} \geq 1 = a_1$, since $a_1 = 1$ and we fixed $r > 1$. Then suppose that $a_n \geq a_{n-1}$, and see that

$$\begin{aligned} a_{n+1} - a_n &= \frac{1}{r}(a_n + r + 1) - \frac{1}{r}(a_{n-1} + r + 1) \\ &= \frac{1}{r}(a_n - a_{n-1}) \\ &\geq 0, \end{aligned}$$

because $r > 0$ and $a_n \geq a_{n-1}$. With $a_{n+1} - a_n \geq 0$, we have by induction that for $n \in \mathbb{N}$, $a_n \geq a_{n-1}$. Hence (a_n) is an increasing sequence.

The sequence (a_n) is bounded above by $\frac{r+1}{r-1}$, which we show by induction. For $n = 1$, $a_n = 1 \leq \frac{r+1}{r-1}$, because $r > 1$. Then suppose that $a_n \leq \frac{r+1}{r-1}$, and we have that

$$\begin{aligned} a_{n+1} &= \frac{1}{r}(a_n + r + 1) \\ &\leq \frac{1}{r}\left(\frac{r+1}{r-1} + r + 1\right) \\ &= \frac{r+1}{r} \cdot \frac{r}{r-1} = \frac{r+1}{r-1}. \end{aligned}$$

Hence all terms a_n are bounded above by $\frac{r+1}{r-1}$, which means the sequence (a_n) is bounded above by $\frac{r+1}{r-1}$.

Since (a_n) is an increasing and bounded sequence, (a_n) converges to a unique real number. Because (a_n) converges, it is a Cauchy sequence. This means that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n > N$, $|a_{n+1} - a_n| < \left(\frac{r-1}{r}\right)\varepsilon$. Since $a_{n+1} \geq a_n$ and for all $n \in \mathbb{N}$, $a_n \geq a_1 = 1$, we may omit the absolute value signs. Then

$$\frac{1}{r}(a_n + r + 1) = a_{n+1} \leq \left(\frac{r-1}{r}\right)\varepsilon + a_n,$$

which with $r > 1$ and some algebra, we have that

$$r + 1 \leq (r-1)\varepsilon + (r-1)a_n,$$

which implies that

$$\frac{r+1}{r-1} - \varepsilon \leq a_n.$$

But we showed earlier that an upper bound for the sequence (a_n) was $\frac{r+1}{r-1}$, which we may increment by ε since $\varepsilon > 0$. Hence

$$\frac{r+1}{r-1} - \varepsilon \leq a_n \leq \frac{r+1}{r-1} + \varepsilon,$$

which means that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$\left| a_n - \frac{r+1}{r-1} \right| \leq \varepsilon.$$

It follows that $\frac{r+1}{r-1}$ is the limit of the sequence (a_n). $\square$