1. Evaluate $\displaystyle\int_0^\infty \frac{\arctan(\pi x)-\arctan(x)}{x}\,\mathrm{d}x$.

$\displaystyle\int_0^\infty \frac{\arctan(\pi x)-\arctan(x)}{x}\,\mathrm{d}x = \frac{1}{2}\pi\log(\pi)$.

Continuously extend the integrand by defining it to be $\pi-1$ when $x=0$ so that we do not need to worry about taking an improper integral as x tends to 0. We still have to take the principal value of an improper integral as x tends to ∞. Simplifying, we have

$$\int_0^\infty \frac{\arctan(\pi x)-\arctan(x)}{x}\,\mathrm{d}x = \lim_{R\to\infty}\left[\int_0^R \frac{\arctan(\pi x')}{x'}\,\mathrm{d}x' - \int_0^R \frac{\arctan(x)}{x}\,\mathrm{d}x\right]$$

Using the substitution $x=\pi x'$, $\mathrm{d}x = \pi\,\mathrm{d}x'$ in the first integral, continue:

$$\begin{aligned} &= \lim_{R\to\infty}\left[\int_0^{\pi R} \frac{\arctan(x)}{x}\,\mathrm{d}x - \int_0^R \frac{\arctan(x)}{x}\,\mathrm{d}x\right] \\ &= \lim_{R\to\infty}\int_R^{\pi R} \frac{\arctan(x)}{x}\,\mathrm{d}x\,. \end{aligned}$$

Then we argue that the limit is $\pi\log(\pi)/2$. Given any $\varepsilon>0$, we can find $R'>0$ such that for $x>R'$, we have that $|\pi/2-\arctan(x)|<\varepsilon$.

Observe that $\int_R^{\pi R}\pi/(2x)\,\mathrm{d}x = \pi(\log(\pi R)-\log(R))/2 = \pi\log(\pi)/2$. Then by choosing R greater than R',

$$\begin{aligned} \left|\frac{1}{2}\pi\log(\pi) - \int_R^{\pi R}\frac{\arctan(x)}{x}\,\mathrm{d}x\right| = \left|\int_R^{\pi R}\frac{\pi/2-\arctan(x)}{x}\,\mathrm{d}x\right| &\le \int_R^{\pi R}\frac{|\pi/2-\arctan(x)|}{x}\,\mathrm{d}x \\ &< \int_R^{\pi R}\frac{\varepsilon}{x}\,\mathrm{d}x \\ &= \varepsilon\log(\pi). \end{aligned}$$

Since ε was arbitrary, as $\varepsilon\to 0$ and as $R\to\infty$, it follows that

$$\lim_{R\to\infty}\int_R^{\pi R}\frac{\arctan(x)}{x}\,\mathrm{d}x = \frac{1}{2}\pi\log(\pi)$$

as desired.

8. Evaluate $\displaystyle\int_1^\infty \frac{\mathrm{d}x}{e^{x+1}+e^{3-x}}$.

$\displaystyle\int_1^\infty \frac{\mathrm{d}x}{e^{x+1}+e^{3-x}} = \frac{\pi}{4e^2}$.

Start by making the change of variables $x'=x+1, \mathrm{d}x'=\mathrm{d}x$ so that

$$\int_1^\infty \frac{\mathrm{d}x'}{e^{x'+1}+e^{3-x'}} = \int_0^\infty \frac{\mathrm{d}x}{e^{x+2}+e^{-(x-2)}}.$$

Then with some algebra and another change of variables $u = e^x, \mathrm{d}u = e^x \,\mathrm{d}x$, we have

$$\begin{aligned}
\int_0^\infty \frac{\mathrm{d}x}{e^{x+2} + e^{-(x-2)}} &= \frac{1}{e^2} \int_0^\infty \frac{\mathrm{d}x}{e^x + e^{-x}} \\
&= \frac{1}{e^2} \int_0^\infty \frac{e^x \,\mathrm{d}x}{1 + e^{2x}} \\
&= \frac{1}{e^2} \int_1^\infty \frac{\mathrm{d}u}{1 + u^2} \\
&= \frac{1}{e^2} \left(\arctan(u) \Big|_1^\infty \right) \\
&= \frac{1}{e^2} \left(\frac{\pi}{2} - \frac{\pi}{4} \right) = \frac{\pi}{4e^2}
\end{aligned}$$

as desired.