2.1: 6, 9(b,c), 10, 13, 14(a,b)

6. Prove that T is a linear transformation, and find bases for both $\mathsf{N}(\mathsf{T})$ and $\mathsf{R}(\mathsf{T})$. Then compute the nullity and rank of T, and verify the dimension theorem. Finally, determine if T is one-to-one or onto.

The transformation T is linear iff $\mathsf{T}(cA+B) = c\mathsf{T}(A) + \mathsf{T}(B)$, for $c \in \mathbb{F}$ and $A, B \in \mathsf{M}_{n\times n}(\mathbb{F})$.

$$\begin{aligned}\mathsf{T}(cA+B) = \operatorname{tr}(cA+B) &= \sum_{i=1}^{n} (cA+B)_{ii} = \sum_{i=1}^{n} \left[(cA)_{ii} + B_{ii}\right] = \sum_{i=1}^{n} c\left(A_{ii}\right) + \sum_{i=1}^{n} B_{ii} \\ &= c\sum_{i=1}^{n} \left(A_{ii}\right) + \sum_{i=1}^{n} B_{ii} = c\operatorname{tr}(A) + \operatorname{tr}(B) = c\mathsf{T}(A) + \mathsf{T}(B)\end{aligned}$$

Hence T is linear.

The zero vector in $\mathbb{F}$ is the scalar 0 itself, so we must find all those matrices in $\mathsf{M}_{n\times n}(\mathbb{F})$ whose trace is zero.

So let e_{ij} be the matrix where in position ij there is a 1 and in any other position the matrix contains zeros.

So matrices that easily has a trace of zero are those in the set $\{e_{ij} : i \neq j\}$, since all the diagonal entries will be zero. This set is easily seen to be linearly independent since each matrix here will have 1 in *different* positions (but not on the diagonal!) and 0 everywhere else. Then to form the other ones, take the set of matrices in the form $\{e_{ii} - e_{(i+1)(i+1)} : 1 \leq i \leq n-1\}$. We must show that this set is linearly independent, by showing the only linear combination of these matrices that produce the zero matrix is the trivial combination: Take scalars $a_i \in \mathbb{F}$ and see that

$$\begin{aligned}&a_1\left(e_{11} - e_{22}\right) + a_2\left(e_{22} - e_{33}\right) + \cdots + a_{n-1}\left(e_{(n-1)(n-1)} - e_{nn}\right) \\
&= a_1 \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} + \cdots + a_{n-1} \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & -a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & -a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & a_{n-1} & 0 \\ 0 & \cdots & 0 & -a_{n-1} \end{pmatrix} \\
&= \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_2 - a_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_3 - a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} - a_{n-2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & -a_{n-1} \end{pmatrix} = \mathbf{0}_{n\times n}\end{aligned}$$

This forms the following systems of equations:

$$\begin{aligned} a_1 &= 0 \\ a_2 - a_1 &= 0 \\ &\vdots \\ a_{n-1} - a_{n-2} &= 0 \\ -a_{n-1} &= 0 \end{aligned}$$

It is easily seen that this forces all the a_i to be zero and so $\{e_{ii} - e_{(i+1)(i+1)} : 1 \leq i \leq n-1\}$ is linearly independent. Furthermore $\{e_{ii} - e_{(i+1)(i+1)} : 1 \leq i \leq n-1\}$ is not in the span of $\{e_{ij} : i \neq j\}$ and vice versa, so the union of the two is linearly independent. Therefore a basis for $\mathsf{N}(\mathsf{T})$ is $\{e_{ij} : i \neq j\} \cup \{e_{ii} - e_{(i+1)(i+1)} : 1 \leq i \leq n-1\}$. The nullity then will be $\left(n^2 - n\right) + (n-1) = n^2 - 1$.

The basis for $\mathsf{R}(\mathsf{T})$ will just be the basis for $\mathbb{F}$, since $\mathbb{F}$ is the codomain for the transformation T. A basis for $\mathbb{F}$ will simply be $\{1\}$, since $\mathbb{F}$ is just a field. Then the rank of T is 1 since there is one item in the basis for $\mathsf{R}(\mathsf{T})$.

To confirm the dimension theorem, we must check to see that the sum of the nullity and the rank of T is equivalent to the dimension of $\mathsf{M}_{n\times n}(\mathbb{F})$ (which is equal to n^2). Indeed, $\left(n^2 - 1\right) + 1 = n^2$ as expected.

Linear transformations are only one-to-one iff the nullity is 0. Here the nullity is $1 \neq 0$, so T is not one-to-one. To see if T is onto, for every element c in $\mathbb{F}$, we must be able to produce an element (preimage) A in $\mathsf{M}_{n\times n}(\mathbb{F})$ such that $\mathsf{T}(A) = c$. So take some c in $\mathbb{F}$ and observe that we can choose $A = ce_{11}$ so that $\mathsf{T}(A) = \mathsf{T}(ce_{11}) = c \cdot \operatorname{tr}(e_{11}) = c \cdot 1 = c$. Therefore T is onto.

9.

(b) If the transformation T is linear, then $\mathsf{T}(sx+y) = s\mathsf{T}(x) + \mathsf{T}(y)$, for vectors $x, y \in \mathbb{R}^2$, $s \in \mathbb{R}$. We will show this is not the case.

Let $x = (a,b)$, and $y = (c,d)$ for $a,b,c,d \in \mathbb{R}$. Then $\mathsf{T}(sx+y) = \mathsf{T}(s(a,b) + (c,d)) = \mathsf{T}((sa+c, sb+d)) = (sa+c, (sa+c)^2) = (sa+c, (sa)^2 + 2sac + c^2)$. This is not equal to $(sa+c, sa^2 + c^2) = s\mathsf{T}(x) + \mathsf{T}(y)$, so we fail linearity - T is not linear.

(c) If the transformation T is linear, then $\mathsf{T}(sx+y) = s\mathsf{T}(x) + \mathsf{T}(y)$, for vectors $x, y \in \mathbb{R}^2$, $s \in \mathbb{R}$. We will show this is not the case.

Let $x = (a,b)$, and $y = (c,d)$ for $a,b,c,d \in \mathbb{R}$. Then $\mathsf{T}(sx+y) = \mathsf{T}(s(a,b) + (c,d)) = \mathsf{T}((sa+c, sb+d)) = (\sin(sa+c), 0)$. This is not equal to $(s\sin(a) + \sin(c), 0) = \mathsf{T}(x) + \mathsf{T}(y)$ (it is known that $\sin(sa+c) \neq s\sin(a) + \sin(c)$), so we fail linearity - T is not linear.

10. We may decompose $(1,1)$ into $(1,0)+(0,1)$, so that by linearity of T we have $\mathsf{T}((1,1)) = \mathsf{T}((1,0)+(0,1)) = \mathsf{T}((1,0)) + \mathsf{T}((0,1)) = (1,4) + \mathsf{T}((0,1)) = (2,5)$. Then since $\mathbb{R}^2$ forms a vector space we may add to both sides the correct inverse to find $\mathsf{T}((0,1)) = (1,1)$. Since $\{(1,0),(0,1)\}$ form a basis for $\mathbb{R}^2$, we know all of the information about T. Similarly decompose $(2,3)$ into $2(1,0)+3(0,1)$. Then $\mathsf{T}((2,3)) = \mathsf{T}(2(1,0)+3(0,1)) = 2\mathsf{T}((1,0)) + 3\mathsf{T}((0,1)) = 2(1,4) + 3(1,1) = (5,11)$.

To show that T is one-to-one, we can show that only the zero vector belongs to $\mathsf{N}(\mathsf{T})$, so that the nullity becomes zero. This amounts to showing that for any (x, y) (where $x, y \in \mathbb{R}$), the only (x, y) that satisfies $\mathsf{T}((x, y)) = (0, 0)$ is $(0, 0)$. To show this consider the system of equations that form out of the linearity of T:

$$\begin{aligned} x + y &= 0 \\ 4x + y &= 0 \end{aligned}$$

The solutions to this system (by elimination) are that $x = 0$ and $y = 0$. Therefore the only vector that gets mapped into the zero vector is the zero vector itself and so the nullity is zero, which implies that T is one-to-one.

13. Let V and W be vector spaces, let $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ be linear, and let $\{w_1, w_2, \ldots, w_k\}$ be a linearly independent subset of $\mathsf{R}(\mathsf{T})$. Prove that if $S = \{v_1, v_2, \ldots, v_k\}$ is chosen so that $\mathsf{T}(v_i) = w_i$ for $i = 1, 2, \ldots, k$, then S is linearly independent.

Proof. Suppose by way of contradiction that S was instead linearly dependent, so that without loss of generality, $v_k = c_1 v_1 + c_2 v_2 + \cdots + c_{k-1} v_{k-1}$. Then notice that $w_k = \mathsf{T}(v_k) = \mathsf{T}(c_1 v_1 + c_2 v_2 + \cdots + c_{k-1} v_{k-1}) = c_1 \mathsf{T}(v_1) + c_2 \mathsf{T}(v_2) + \cdots + c_{k-1} \mathsf{T}(v_{k-1}) = c_1 w_1 + c_2 w_2 + \cdots + c_{k-1} w_{k-1}$. Thus w_k can be expressed as a linear combination of other vectors in $\{w_1, w_2, \ldots, w_k\}$, which is in contradiction to the assumption that $\{w_1, w_2, \ldots, w_k\}$ was linearly independent.

Hence if $S = \{v_1, v_2, \ldots, v_k\}$ is chosen so that $\mathsf{T}(v_i) = w_i$ for $i = 1, 2, \ldots, k$, then S is linearly independent. □

14. Let V and W be vector spaces and $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ be linear.

(a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.

Proof. Forwards direction. Suppose T carries linearly independent subsets of V onto linearly independent subsets of W, that is if we take a linearly independent subset $\{v_1, v_2, \ldots, v_n\}$ of V, then
$\mathsf{T}(\{v_1, v_2, \ldots, v_n\}) = \{\mathsf{T}(v_1), \mathsf{T}(v_2), \ldots, \mathsf{T}(v_n)\}$ is a linearly independent subset of W.

Suppose by way of contradiction that T is not one-to-one, that is, the nullity of T is nonzero and so there exists a nonzero vector v in V that maps to the zero vector in W. However, $\{v\}$ is a linearly independent subset of V because v is not the zero vector. But $\mathsf{T}(\{v\}) = \{\vec{0}_{\mathsf{W}}\}$, which is not a linearly independent subset of W since a nontrivial combination of the zero vector is still the zero vector. This is in contradiction with the assumption that Suppose T carries linearly independent subsets of V onto linearly independent subsets of W, and so we must have that T is one-to-one.

Converse. Suppose T is one-to-one.

Then suppose by way of contradiction that T does not always carry linearly independent subsets of V onto linearly independent subsets of W, that is, there is a subset $\{v_1, v_2, \ldots, v_n\}$ of V that is linearly independent whose image formed a linearly dependent subset $\{w_1, w_2, \ldots, w_n\}$ of W. Without loss of generality, let $w_n =$

$c_1w_1 + c_2w_2 + \cdots + c_{n-1}w_{n-1}$. Then since each $w_i = \mathsf{T}(v_i)$, we may substitute and use the linearity of T to simplify: $w_n = \mathsf{T}(v_n) = c_1\mathsf{T}(v_1) + c_2\mathsf{T}(v_2) + \cdots + c_{n-1}\mathsf{T}(v_{n-1}) = \mathsf{T}(c_1v_1 + c_2v_2 + \cdots + c_{n-1}v_{n-1})$. Since T is one to one, we can deduce that $v_n = c_1v_1 + c_2v_2 + \cdots + c_{n-1}v_{n-1}$, which is in contradiction to the assumption that $\{v_1, v_2, \ldots, v_n\}$ is linearly independent. Thus we must have that T carries linearly independent subsets of V onto linearly independent subsets of W.

Therefore T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W. □

(b) Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if $\mathsf{T}(S)$ is linearly independent. For convenience let $S = \{v_1, v_2, \ldots, v_n\}$ and $\mathsf{T}(S) = \{w_1, w_2, \ldots, w_n\}$, and $w_i = \mathsf{T}(v_i)$.

Proof. Forwards direction. Suppose $\mathsf{T}(S)$ is linearly independent.

By way of contradiction, suppose S is linearly dependent, so that without loss of generality we can say $v_n = c_1v_1+c_2v_2+\cdots+c_{n-1}v_{n-1}$. Then by construction and linearity of T, $w_n = \mathsf{T}(v_n) = \mathsf{T}(c_1v_1+c_2v_2+\cdots+c_{n-1}v_{n-1}) = c_1\mathsf{T}(v_1)+c_2\mathsf{T}(v_2)+\cdots+c_{n-1}\mathsf{T}(v_{n-1}) = \mathsf{T}(c_1v_1+c_2v_2+\cdots+c_{n-1}v_{n-1}) = c_1w_1+c_2w_2+\cdots+c_{n-1}w_{n-1}$. This means that w_n can be expressed as a linear combination of vectors in $\{w_1, w_2, \ldots, w_n\}$, which means $\{w_1, w_2, \ldots, w_n\}$ is linearly dependent. This is a contradiction so we must have that S is linearly independent.

Converse. Suppose S is linearly independent, and again by way of contradiction suppose $\mathsf{T}(S)$ is linearly dependent. Then without loss of generality $w_n = \mathsf{T}(v_n) = c_1w_1+c_2w_2+\cdots+c_{n-1}w_{n-1} = c_1\mathsf{T}(v_1)+c_2\mathsf{T}(v_2)+\cdots+c_{n-1}\mathsf{T}(v_{n-1}) = \mathsf{T}(c_1v_1 + c_2v_2 + \cdots + c_{n-1}v_{n-1})$. Since T is one-to-one we may deduce that $v_n = c_1v_1 + c_2v_2 + \cdots + c_{n-1}v_{n-1}$, which means that v_n can be expressed as a linear combination of the other vectors in S and so S is not linearly independent This is in contradiction to the assumption that S is linearly independent. Therefore $\mathsf{T}(S)$ must be linearly independent.

Hence S is linearly independent if and only if $\mathsf{T}(S)$ is linearly independent. □