26.1 Consider an operator A in the Shrödinger picture which has no explicit time dependence and which does not evolve in time. The vector $|\psi\rangle$ evolves in time according to (26.4). Consider the expectation value $\langle A\rangle = \langle\psi| A |\psi\rangle$.

(a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle A\rangle = \frac{i}{\hbar}\langle [H, A]\rangle.$$

Proof. Let A be an operator which has no explicit time dependence and which does not evolve in time, and let $|\psi\rangle$ evolve in time according to (26.4). Using the product rule, we have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle A\rangle = \frac{\mathrm{d}}{\mathrm{d}t}(\langle\psi| A |\psi\rangle) &= \langle\psi(0)| \frac{\mathrm{d}}{\mathrm{d}t}\Big(U^\dagger(t) A U(t)\Big) |\psi(0)\rangle \\
&= \langle\psi(0)| \left(\frac{\partial U^\dagger}{\partial t} A U + U^\dagger A \frac{\partial U}{\partial t} + \frac{\partial A}{\partial t}\right) |\psi(0)\rangle \\
&= \langle\psi(0)| \left(\frac{i}{\hbar} H U^\dagger A U - \frac{i}{\hbar} U^\dagger A H U\right) |\psi(0)\rangle \\
&= \frac{i}{\hbar} \langle\psi| (HA - AH) |\psi\rangle \\
&= \frac{i}{\hbar} \langle\psi| [H, A] |\psi\rangle = \frac{i}{\hbar}\langle [H, A]\rangle,
\end{aligned}$$

as desired. □

(b) Use

$$H = \frac{p_x^2}{2m} + V(x) \quad \text{with } p_x = -i\hbar\frac{\partial}{\partial x}$$

and the result of (a) to show that

$$(i) \quad \frac{\mathrm{d}\langle p_x\rangle}{\mathrm{d}t} = -\left\langle \frac{\partial V}{\partial x}\right\rangle \quad \text{and} \quad (ii) \quad \frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} = \frac{\langle p_x\rangle}{m}.$$

Proof. Directly, we have

$$\begin{aligned}
\frac{\mathrm{d}\langle p_x\rangle}{\mathrm{d}t} &= \frac{i}{\hbar}\left\langle \left[\frac{p_x^2}{2m} + V(x), p_x\right]\right\rangle \\
&= \frac{i}{\hbar}\left\langle \left(\left(\frac{p_x^3}{2m} + V(x)p_x\right) - \left(\frac{p_x^3}{2m} + p_x V(x) + V(x)p_x\right)\right)\right\rangle \\
&= \left\langle \left(-\frac{i}{\hbar} p_x V(x)\right)\right\rangle = -\left\langle \frac{\partial V}{\partial x}\right\rangle
\end{aligned}$$

and

$$\begin{aligned}
\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} &= \frac{i}{\hbar}\left\langle\left[\frac{p_x^2}{2m}+V(x),x\right]\right\rangle \\
&= \frac{i}{\hbar}\left\langle\left(\left(\frac{1}{2m}(p_x(xp_x-i\hbar))+V(x)x\right)-\left(x\frac{p_x^2}{2m}+V(x)x\right)\right)\right\rangle \\
&= \frac{i}{\hbar}\left\langle\left(\left(\frac{1}{2m}(-2i\hbar p_x+xp_x^2)+V(x)x\right)-\left(x\frac{p_x^2}{2m}+V(x)x\right)\right)\right\rangle \\
&= \frac{\langle p_x\rangle}{m},
\end{aligned}$$

as desired. □