HOMEWORK 1

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Suppose $\|\cdot\|$ is a norm on $\mathbb{R}^2$ such that, if $0 \le a_1 \le a_2$ and $0 \le b_1 \le b_2$, then

$$\|(a_1, b_1)\| \le \|(a_2, b_2)\|$$

and (X, d_X) and (Y, d_Y) are metric spaces. Let $Z = X \times Y$ and define $d : Z \times Z \to [0, \infty)$ by

$$d(z_1, z_2) = \|(d_X(x_1, x_2), d_Y(y_1, y_2))\|,$$

for $z_j = (x_j, y_j) \in Z$. Show d is a metric on Z. Explain the relation between this problem and Exercise 1.1.12.

Proof. Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$ with the property as given above, and let $d\colon Z \times Z \to [0, \infty)$ for $Z = X \times Y$ be as given, with X, Y being metric spaces with metrics d_X, d_Y, respectively. Let $z_1, z_2, z_3 \in Z$ with $z_k = (x_k, y_k)$ for $1 \le k \le 3$.

It is clear that d is positive definite because $\|\cdot\|$ maps into $[0, \infty)$, so that d maps into $[0, \infty)$. We check that $d(z_1, z_2) = 0$ if and only if $z_1 = z_2$. Suppose $z_1 = z_2$. Then

$$d(z_1, z_2) = \|(d_X(x_1, x_2), d_Y(y_1, y_2))\| = \|((x_1, x_1), (y_1, y_1))\| = \|(0, 0)\| = 0.$$

Conversely, suppose that $d(z_1, z_2) = 0$. Since $\|\cdot\|$ is a norm on $\mathbb{R}^2$, we must have that $(d_X(x_1, x_2), d_Y(y_1, y_2)) = (0, 0)$. Then because d_X, d_Y are metrics on X, Y respectively, we must have that $x_1 = x_2$ and $y_1 = y_2$. Hence $z_1 = z_2$.

Symmetry of d follows from symmetry of d_X and d_Y:

$$d(z_1, z_2) = \|(d_X(x_1, x_2), d_Y(y_1, y_2))\| = \|(d_X(x_2, x_1), d_Y(y_2, y_1))\| = d(z_2, z_1).$$

The function d satisfies the triangle inequality due to d_X, d_Y, and $\|\cdot\|$ satisfying the triangle inequality as well as the norm $\|\cdot\|$ having the property given above. With $0 \le d_X(x_1, x_3) \le d_X(x_1, x_2) + d_X(x_2, x_3)$ and $0 \le d_Y(y_1, y_3) \le d_Y(y_1, y_2) + d_Y(y_2, y_3)$, we have

$$\begin{aligned} d(z_1, z_3) &= \|(d_X(x_1, x_3), d_Y(y_1, y_3))\| \\ &\le \|(d_X(x_1, x_2) + d_X(x_2, x_3), d_Y(y_1, y_2) + d_Y(y_2, y_3))\| \\ &\le \|(d_X(x_1, x_2), d_Y(y_1, y_2))\| + \|(d_X(x_2, x_3), d_Y(y_2, y_3))\| \\ &= d(z_1, z_2) + d(z_2, z_3). \end{aligned}$$

Since z_1, z_2, z_3 were arbitrary, it follows that d is a metric on Z. $\square$

Exercise 1.1.12 is a special case of this exercise in that Exercise 1.1.12 is this exercise when we take $\|\cdot\|$ to be the ℓ^1 norm on $\mathbb{R}^2$ given by $\|(a, b)\| = |a| + |b|$ for $a, b \in \mathbb{R}$. In Exercise 1.1.12 the absolute value bars are omitted because d_X, d_Y map into the

nonnegative real numbers. Observe that $\|\cdot\|$ is indeed a norm on $\mathbb{R}^2$ and that it satisfies the property that, for $a_1, a_2, b_1, b_2 \in \mathbb{R}$ with $0 \leq a_1 \leq a_2$ and $0 \leq b_1 \leq b_2$, we have $\|(a_1, b_1)\| \leq \|(a_2, b_2)\|$:

$$\|(a_1, b_1)\| = a_1 + b_1 \leq a_2 + b_1 \leq a_2 + b_2 = \|(a_2, b_2)\|$$