Homework 31. Sai Sivakumar

31.1

$u(x,t)$ satisfying $v^2 u_{xx} = u_{tt}$

and (i) $u(0,t) = u(L,t)$

$u_t|_{t=0} = 0$ and (ii) $u(x,0) = \begin{cases} ax & 0 < x < L/2 \\ a(L-x) & L/2 < x < L \end{cases}$.

$a \in \mathbb{R}$.

a) Separation of variables.

Assert $u = X(x)T(t)$. Then

$$v^2 u_{xx} = v^2 X''(x)T(t) = X(x)T''(t) = u_{tt}.$$

Then $$\frac{X''(x)}{X(x)} = \frac{T''(t)}{v^2 T(t)} = -k^2. \qquad k \in \mathbb{Z}.$$

ODEs:
$$\boxed{\begin{aligned} X''(x) + k^2 X(x) &= 0 \\ T''(t) + v^2k^2 T(t) &= 0 \end{aligned}}$$

b) $u(0,t) = u(L,t) = 0$ implies solution to $X''(x) + k^2 X(x) = 0$ is a sine function.

Hence $k = \frac{n\pi}{L}, n \in \mathbb{Z}$, so that soln is of form $u = \begin{cases} \sin(n\pi x/L)\sin(n\pi vt/L) \\ \sin(n\pi x/L)\cos(n\pi vt/L) \end{cases}$.

Then $u_t|_{t=0} = 0$: $u_t = \begin{cases} \sin(n\pi x/L)\cos(n\pi vt/L)\, n\pi v/L \\ -\sin(n\pi x/L)\sin(n\pi vt/L)\, n\pi v/L \end{cases}$,

at $t=0$ $u_t|_{t=0} = \begin{cases} \sin(n\pi x/L)\cdot 1 \cdot n\pi v/L \\ -\sin(n\pi x/L) \cdot 0 \cdot n\pi v/L \end{cases} = 0$,

implies we should discard the second solution.

Hence $\boxed{u = \sum_n b_n \sin(n\pi x/L)\cos(n\pi vt/L)}$ is the most general form satisfying (i).

c) BC at $t=0$.

Fourier (sine series) coefficients of $u(x,0) = \begin{cases} ax & 0 < x < L/2 \\ a(L-x) & L/2 < x < L \end{cases}$:

$$b_n = \frac{2}{L}\int_0^L u(x,0)\sin(n\pi x/L)\,dx = \frac{2}{L}\int_0^{L/2} ax\sin(n\pi x/L)\,dx$$

$n \neq 0$

(since sine series, $\underline{n \text{ odd also.}}$

$$+ \frac{2}{L}\int_{L/2}^{L} a(L-x)\sin(n\pi x/L)\,dx$$

$$= 2La\sin(\pi n/2)/\pi^2 n^2$$
$$+ 2La\sin(\pi n/2)/\pi^2 n^2$$

$$= 4La\sin(\pi n/2)/\pi^2 n^2$$

(odd n means $\sin(\pi n/2)$ is $(-1)^{n+1}$)

$$\boxed{u = \sum_{\text{odd } n} \frac{4La(-1)^{n+1}}{(\pi n)^2}\sin\left(\frac{\pi n x}{L}\right)\cos\left(\frac{\pi n v t}{L}\right)} \longleftarrow \boxed{\underset{(\text{odd } n)}{b_n} = \frac{4La(-1)^{n+1}}{(\pi n)^2}}$$

31.2. Particle in a box:

$$-\frac{\hbar^2}{2M}\left(\psi_{xx} + \psi_{yy}\right) = E\psi$$

a) Assert $\psi = X(x)Y(y)$ and then

$$-\frac{\hbar^2}{2M}\left(X''(x)Y(y) + X(x)Y''(y)\right) = EX(x)Y(y)$$

$$\rightarrow -\frac{\hbar^2}{2M}\left(\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}\right) = E = E_x + E_y$$

$$\rightarrow \begin{cases} -\frac{\hbar^2}{2ME_x}\frac{X''(x)}{X(x)} = -k_x^2 \\ -\frac{\hbar^2}{2ME_y}\frac{Y''(y)}{Y(y)} = -k_y^2 \end{cases} \quad \text{odes.}$$

$E_x + E_y = E$ components of energy come out since x, y independent?

since ψ vanishes, the solutions to odes are sine curves:

$$X = \sin\left(\sqrt{\tfrac{2ME_x}{\hbar^2}}\,x\right) \quad \text{so that}$$
$$Y = \sin\left(\sqrt{\tfrac{2ME_y}{\hbar^2}}\,y\right)$$

$$\boxed{\psi = \sum_{n,m} b_{n,m}\sin\left(\sqrt{\tfrac{2ME_x}{\hbar^2}}\,x\right)\sin\left(\sqrt{\tfrac{2ME_y}{\hbar^2}}\,y\right)}$$

* (E_x and E_y depend on n, m)

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b) Considering the boundary condition where ψ vanishes at the boundary of the box, either when $\sin\left(\sqrt{\frac{2mE_x}{\hbar^2}}L_x\right) = 0$ or $\sin\left(\sqrt{\frac{2mE_y}{\hbar^2}}L_y\right) = 0$ independently. So $\sqrt{\frac{2mE_x}{\hbar^2}}L_x = n\pi$ and $\sqrt{\frac{2mE_y}{\hbar^2}}L_y = m\pi$ $(n, m \in \mathbb{Z})$

$$\Rightarrow \quad E_x = \frac{\hbar^2}{2m}\left(\frac{n^2\pi^2}{L_x^2}\right)$$

$$E_y = \frac{\hbar^2}{2m}\left(\frac{m^2\pi^2}{L_y^2}\right)$$

so that $\boxed{E = E_x + E_y = \frac{\hbar^2\pi^2}{2m}\left(\frac{n^2}{L_x^2} + \frac{m^2}{L_y^2}\right)}$.