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7. A basis for $\mathbb{R}^3$ will have exactly 3 vectors. To construct this basis we may pick one of the vectors, say $u_1 = (2, -3, 1)$. Then we simply need to pick another vector not in the spanning set of u_1, that is a vector not in $\{cu_1 : c \in \mathbb{R}\}$. By inspection we can find that $u_2 = (1, 4, -2)$ is a vector that satisfies this criteria, since $\frac{1}{2}u_1 \neq u_2$. Then all that remains is to find one more vector that is not in $\operatorname{span}(\{u_1, u_2\})$.

The spanning set for $\operatorname{span}(\{u_1, u_2\})$ is $\{(2a + b, -3a + 4b, a - 2b) : a, b \in \mathbb{R}\}$. We can exhaust through the other vectors and find that u_5 is not in the spanning set. The vector u_5 cannot be expressed as a linear combination of u_1, u_2, as demonstrated by the result of row reducing the following linear system:

$$\begin{aligned} 2a + b &= -3 \\ -3a + 4b &= -5 \\ a - 2b &= 8 \end{aligned} \rightarrow \begin{aligned} a + 0b &= 0 \\ 0a + b &= 0 \\ 0a + 0b &= 1 \end{aligned}$$

The system is unsolvable, so we may take u_5 and include it within the set that should now be our basis: $\boxed{\{u_1, u_2, u_5\}}$.

9. Because $\{u_1, u_2, u_3, u_4\}$ form a basis for $\mathbb{F}^4$, then if we can find a representation of (a_1, a_2, a_3, a_4) as a linear combination of u_1, u_2, u_3, u_4, then it will be unique due to the linear independence of the basis.

All that remains is to find a solution to the linear system of equations $c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4 = (a_1, a_2, a_3, a_4)$, for $c_i \in \mathbb{F}$, represented like so:

$$\begin{aligned} c_1 + 0c_2 + 0c_3 + 0c_4 &= a_1 \\ c_1 + c_2 + 0c_3 + 0c_4 &= a_2 \\ c_1 + c_2 + c_3 + 0c_4 &= a_3 \\ c_1 + c_2 + c_3 + c_4 &= a_4 \end{aligned}$$

Since the system is already expressed in some kind of lower triangular reduced form we can immediately deduce the solutions are:

$$\begin{aligned} c_1 &= a_1 \\ c_2 &= a_2 - a_1 \\ c_3 &= a_3 - a_2 \\ c_4 &= a_4 - a_3 \end{aligned}$$

Thus $(a_1)\,u_1 + (a_2 - a_1)\,u_2 + (a_3 - a_2)\,u_3 + (a_4 - a_3)\,u_4$ is the unique linear combination representing (a_1, a_2, a_3, a_4).

12. Prove that if $\{u, v, w\}$ is a basis for V, then $\{u + v + w, v + w, w\}$ is also a basis for V.

Proof. Suppose $\{u, v, w\}$ is a basis for V. Then $\{u, v, w\}$ is a linearly independent generator of V. So then the goal is to show that $\{u+v+w, v+w, w\}$ is also a linearly independent generator of V.

Linear independence of $\{u+v+w, v+w, w\}$**:** Suppose by way of contradiction that $\{u+v+w, v+w, w\}$ is not linearly independent, that is, that there exist scalars $a, b, c \in \mathbb{F}$, not all zero, such that $a(u+v+w)+b(v+w)+c(w)=0$. With some simplification we may rewrite this combination as $(a)u+(a+b)v+(a+b+c)w=0$. Then since not all of a, b, c were zero, this implies that at least one of $a, a+b, a+b+c$ is not zero, which means that there is some nontrivial linear combination of u, v, w that produced the zero vector. This is in contradiction with the assumption that $\{u, v, w\}$ is linearly independent, so we must have that $\{u+v+w, v+w, w\}$ is linearly independent.

Showing $\operatorname{span}(\{u+v+w, v+w, w\})=\mathsf{V}$**:** Take some arbitrary vector $v \in \mathsf{V}$. We know that since $\{u, v, w\}$ is a basis for V, there exist scalars $a, b, c \in \mathbb{F}$ such that $v = au+bv+cw$.

Notice that $u=(u+v+w)-(v+w)$ and $v=(v+w)-(w)$. Then we may rewrite the linear combination as follows: $v = a((u+v+w)-(v+w))+b((v+w)-(w))+c(w) = a(u+v+w)+(b-a)(v+w)+(c-b)(w)$. We can deduce then that this vector is really an element of the spanning set for $\{u+v+w, v+w, w\}$. So $\mathsf{V} \subseteq \operatorname{span}(\{u+v+w, v+w, w\})$.

Similarly, take any vector in $\operatorname{span}(\{u+v+w, v+w, w\})$, say $v = a(u+v+w)+b(v+w)+c(w)$ for scalars $a, b, c \in \mathbb{F}$. Then we may rewrite the combination as $(a)u+(a+b)v+(a+b+c)w$. Since $\{u, v, w\}$ is a basis for V, then $\operatorname{span}(\{u, v, w\}) = \mathsf{V}$. Then $(a)u+(a+b)v+(a+b+c)w \in \operatorname{span}(\{u, v, w\})$, so really $\operatorname{span}(\{u+v+w, v+w, w\}) \subseteq \operatorname{span}(\{u, v, w\}) \implies \operatorname{span}(\{u+v+w, v+w, w\}) \subseteq \mathsf{V}$.

Hence $\operatorname{span}(\{u+v+w, v+w, w\}) = \mathsf{V}$.

Therefore $\{u+v+w, v+w, w\}$ is a basis for V. $\square$

13. We are tasked with finding the set of solutions expressed in the form (x_1, x_2, x_3), and to find a basis for the subspace of $\mathbb{R}^3$ that the set of solutions forms.

First solve the system of linear equations like so:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 - 3x_2 + x_3 &= 0 \end{aligned} \rightarrow \begin{aligned} 1x_1 + 0x_2 - 1x_3 &= 0 \\ 0x_1 + 1x_2 - 1x_3 &= 0 \end{aligned}$$

Deduce that $x_1 = x_3$ and $x_2 = x_3$. Then all solutions of the form (x_1, x_2, x_3) are really just (x_3, x_3, x_3). This solution set is actually just the span of $(1,1,1)$, since $(x_3, x_3, x_3) = x_3(1,1,1)$, and x_3 can take on any real value. So a basis for the subspace of $\mathbb{R}^3$ that is the solution set for this system of linear equations is $\{(1,1,1)\}$.

20. Let V be a vector space having dimension n, and let S be a subset of V that generates V.

(a). Prove that there is a subset of S that is a basis for V.

Proof. We can construct a subset of S by the following algorithm: Take one nonzero vector from S, and put it in a set, call it B. Then B contains one nonzero vector, and as deduced from earlier, B will be linearly independent.

Then check to see if $\operatorname{span}(B) = \mathsf{V}$. If this is true, then we can halt and nothing more needs to be done, as this subset of S generates V and is linearly independent. If this is not true, we find another vector in S that is not contained within $\operatorname{span}(B)$ (to ensure linear independence of B), and add it to B. Then again check to see if B generates V, and again stop if it does or add another vector from S not in $\operatorname{span}(B)$ to B. Keep checking and repeating (if needed) to see if this linearly independent set generates V. This process terminates, because if there are n linearly independent vectors in B, a subset of S and therefore a subset of V, then B is a basis for V. Thus we have constructed a subset of S that is a basis for V. □

(b). Prove that S contains at least n vectors.

Proof. From the proof of (a), we know that the set B, which is a basis for V, is a subset of S and contains exactly n vectors by construction. Therefore S contains at least n vectors. □