HOMEWORK 2

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For $n \in \mathbb{N}$, let $a_n = \sum_{j=1}^{n} j^{-3}$. Prove by induction that $a_n \leq 2 - \frac{1}{n^2}$ for all $n \in \mathbb{N}$.

Proof. We use mathematical induction. Let $a_n = \sum_{j=1}^{n} j^{-3}$ for $n \in \mathbb{N}$ be as given. Observe for $n = 1$,

$$1 = a_1 \leq 1 = 2 - \frac{1}{1^2}.$$

Then suppose $n \in \mathbb{N}$ and $a_n \leq 2 - \frac{1}{n^2}$, that is, $2 - a_n \geq \frac{1}{n^2}$. It follows that

$$\begin{aligned} 2 - \frac{1}{(n+1)^2} - a_{n+1} &= (2 - a_n) - \frac{1}{(n+1)^2} - \frac{1}{(n+1)^3} \\ &\geq \frac{1}{n^2} - \frac{1}{(n+1)^2} - \frac{1}{(n+1)^3} \\ &= \frac{1}{n^2} - \frac{n+2}{(n+1)^3} \\ &= \frac{(n+1)^3 - n^2(n+2)}{n^2(n+1)^3} \\ &= \frac{n^2 + 3n + 1}{n^2(n+1)^3}, \end{aligned}$$

where we used the inductive hypothesis in writing the inequality. Since $n \geq 1$ ($n \in \mathbb{N}$),

$$\frac{n^2 + 3n + 1}{n^2(n+1)^3} \geq 0,$$

which implies that

$$2 - \frac{1}{(n+1)^2} - a_{n+1} \geq 0.$$

This is equivalent to writing

$$a_{n+1} \leq 2 - \frac{1}{(n+1)^2},$$

and so by induction we have that $a_n \leq 2 - \frac{1}{n^2}$ for all $n \in \mathbb{N}$. □