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Let $f(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order. Find

$$\lim_{s\to\infty} \mathcal{L}\{f\}$$

where $\mathcal{L}\{f\}$ is the Laplace transform of $f(t)$. *Let s be real.*

$\mathcal{L}\{f\} = 0.$

Proof. Since $f(t)$ is of exponential order, there exist $\alpha, M > 0, T > 0 \in \mathbb{R}$ such that $|f(t)| \leq Me^{\alpha t}$ for all $t > T$.

With

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t\,,$$

observe that since $f(t)$ is of exponential order, $\mathcal{L}\{f\}$ exists for all $s > \alpha$ (the integral converges). We have that

$$|\mathcal{L}\{f\}| = \left|\int_0^\infty e^{-st} f(t)\,\mathrm{d}t\right| \leq \int_0^\infty \left|e^{-st} f(t)\right|\mathrm{d}t = \int_0^\infty \left|e^{-st}\right||f(t)|\,\mathrm{d}t$$
$$\leq \int_0^T e^{-st}|f(t)|\,\mathrm{d}t + \int_T^\infty e^{-st} Me^{\alpha t}\,\mathrm{d}t\,.$$

Then since $f(t)$ is piecewise continuous on $[0, \infty)$, we have that there exists a real value $C \geq 0$ such that for $t \in [0, T]$, $|f(t)| \leq C$. Then

$$\int_0^T e^{-st}|f(t)|\,\mathrm{d}t + \int_T^\infty e^{-st} Me^{\alpha t}\,\mathrm{d}t \leq \int_0^T e^{-st} C\,\mathrm{d}t + \int_T^\infty e^{-st} Me^{\alpha t}\,\mathrm{d}t\,,$$

and so what remains is to take the limit as s tends to infinity.

Since e^{-st} and $e^{(\alpha-s)t}$ converge uniformly to the identically zero function as $s \to \infty$, we can compute the limit

$$\lim_{s\to\infty}\left(\int_0^T e^{-st} C\,\mathrm{d}t + \int_T^\infty e^{-st} Me^{\alpha t}\,\mathrm{d}t\right) = \lim_{s\to\infty}\int_0^T e^{-st} C\,\mathrm{d}t + \lim_{s\to\infty}\int_T^\infty e^{-st} Me^{\alpha t}\,\mathrm{d}t$$

by passing the limit into the integral. Then

$$\lim_{s\to\infty}\int_0^T e^{-st} C\,\mathrm{d}t + \lim_{s\to\infty}\int_T^\infty e^{-st} Me^{\alpha t}\,\mathrm{d}t = C\int_0^T \lim_{s\to\infty}\left(e^{-st}\right)\mathrm{d}t + M\int_T^\infty \lim_{s\to\infty}\left(e^{(\alpha-s)t}\right)\mathrm{d}t$$
$$= C\int_0^T (0)\,\mathrm{d}t + M\int_T^\infty (0)\,\mathrm{d}t = 0.$$

Hence

$$\lim_{s\to\infty} |\mathcal{L}\{f\}| \leq 0,$$

and so by the squeeze theorem (and continuity of absolute value), $\mathcal{L}\{f\} = 0.$ $\square$