4. Prove that any set of n integers contains a nonempty subset the sum of whose elements is divisible by n.

Every subset of the empty set is empty, so I assume we take $n > 0$.

Proof. Let $\{s_1, s_2, \ldots, s_n\}$ be a set of $n > 0$ integers. Then there are $2^n - 1$ nonempty subsets of this set (the powerset without the empty set). We do not need to investigate all of these subsets. Consider the n nested nonempty subsets $\{s_1\} \subset \{s_1, s_2\} \subset \cdots \subset \{s_1, s_2, \ldots, s_n\}$. Then each of the subsets have the corresponding sums:

$$\begin{aligned} S_1 &= s_1 \\ S_2 &= s_1 + s_2 \\ &\vdots \\ S_n &= s_1 + \cdots + s_n \end{aligned}$$

Divide each sum S_i by n using the division algorithm to obtain nonnegative remainders r_i strictly less than n.

If some r_i is equal to zero, then the subset corresponding to the sum S_i is a nonempty subset the sum of whose elements is divisible by n.

If none of the r_i are equal to zero, we invoke the pigeonhole principle to place each r_i in $n-1$ boxes; these boxes being the $n-1$ nonzero residue classes in $\mathbb{Z}/n\mathbb{Z}$. There are n remainders, and $n-1$ nonzero residue classes, so it follows that at least two remainders r_k and r_j are equal to each other. Without loss of generality, take $k > j$ so that the subset corresponding to S_j is contained in the subset corresponding to S_k.

If $r_k = r_j$, then $n \mid S_k - S_j$. But $S_k - S_j = s_{j+1} + \cdots + s_k$. But $\{s_{j+1}, \ldots, s_k\} \subseteq \{s_1, s_2, \ldots, s_n\}$, and we have exhibited a nonempty subset the sum of whose elements is divisible by n. $\square$

7. Prove that the set $\{11, 111, 1111, \ldots\}$ contains no squares.

Proof. Rewrite the set $\{11, 111, 1111, \ldots\}$ as $\{8+3, 108+3, 1108+3, \ldots\}$. Each of these elements are of the form $4k+3$ for some $k \in \mathbb{Z}$, since both 10^n for $n \geq 2$ and 8 are divisible by 4.

But there are no square integers of the form $4k+3$. Consider the even integers which take on either the form $4n+0$ or $4n+2$ for integers n. Then $(4n)^2 = 16n^2 \equiv 0 \pmod 4$ and $(4n+2)^2 = 4(4n^2+4n+1) \equiv 0 \pmod 4$.

The odd integers take on either the form $4n+1$ or $4n+3$ for integers n. Then again $(4n+1)^2 = 4(4n^2+2n)+1 \equiv 1 \pmod 4$ and $(4n+3)^2 = 4(4n^2+6n+2)+1 \equiv 1 \pmod 4$.

So for all integers after squaring them and reducing modulo 4, none take on the form $4k+3$ for some integer k. But every element in the set takes on this form, so it is impossible for there to be a squared integer within the set $\{11, 111, 1111, \ldots\}$. $\square$