1. Consider the integral discussed in the lecture

$$I = \int_C \vec{F}(\vec{r}) \cdot \mathrm{d}\vec{r}, \quad \vec{F} = \hat{\mathbf{\imath}} 2xy^2 + \hat{\mathbf{\jmath}} x^2.$$

   (a) Evaluate the integral along path $C = A$.

$$\begin{aligned} I = \int_A \vec{F}(\vec{r}) \cdot \mathrm{d}\vec{r} &= \int_{x_1}^{x_2} 2ty_1^2 \,\mathrm{d}t + \int_{y_1}^{y_2} x_2^2 \,\mathrm{d}t \\ &= \boxed{y_1^2(x_2^2 - x_1^2) + x_2^2(y_2 - y_1)}. \end{aligned}$$

   (b) Evaluate the integral along path $C = B$.

$$\begin{aligned} I = \int_B \vec{F}(\vec{r}) \cdot \mathrm{d}\vec{r} &= \int_{y_1}^{y_2} x_1^2 \,\mathrm{d}t + \int_{x_1}^{x_2} 2ty_2^2 \,\mathrm{d}t \\ &= \boxed{x_1^2(y_2 - y_1) + y_2^2(x_2^2 - x_1^2)}. \end{aligned}$$

   These integrals are **not** equivalent to each other.

2. In the presence of uniform charge density ρ and current density $\vec{j}$, Maxwell's equations are changed to

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \quad \vec{\nabla} \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j},$$

   with $\mu_0\epsilon_0 = 1/c^2$. The total energy stored in the medium in a volume V is $\mathcal{E} = \int_V \frac{1}{2}(\epsilon_0 E^2 + B^2/\mu_0)\,\mathrm{d}V$. Show that the rate of change $\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}$ is given by

$$-\int_V \vec{j} \cdot \vec{E} \,\mathrm{d}V - \frac{1}{\mu_0} \oint_S (\vec{E} \times \vec{B}) \cdot \mathrm{d}\vec{S},$$

   where S is the boundary of V.

   Proof. Because we assume both the electric field and magnetic field are sufficiently differentiable, it is safe to interchange the differentiation operator with taking the volume integral. Then

$$\begin{aligned} \frac{\partial}{\partial t}\mathcal{E} = \frac{\partial}{\partial t}\int_V \frac{1}{2}(\epsilon_0 E^2 + B^2/\mu_0)\,\mathrm{d}V &= \int_V \frac{1}{2}\frac{\partial}{\partial t}(\epsilon_0 E^2 + B^2/\mu_0)\,\mathrm{d}V \\ &= \frac{1}{2\mu_0}\int_V \frac{1}{c^2}\frac{\partial}{\partial t}\left(\vec{E}\cdot\vec{E}\right)\mathrm{d}V + \frac{1}{2\mu_0}\int_V \frac{\partial}{\partial t}\left(\vec{B}\cdot\vec{B}\right)\mathrm{d}V \\ &= \frac{1}{2\mu_0}\int_V \frac{1}{c^2}\left(2\vec{E}\cdot\frac{\partial \vec{E}}{\partial t}\right)\mathrm{d}V + \frac{1}{2\mu_0}\int_V \left(2\vec{B}\cdot\frac{\partial \vec{B}}{\partial t}\right)\mathrm{d}V \\ &= \frac{1}{\mu_0}\int_V \left(\vec{E}\cdot\left(\vec{\nabla}\times\vec{B} - \mu_0\vec{j}\right)\right)\mathrm{d}V - \frac{1}{\mu_0}\int_V \left(\vec{B}\cdot\left(\vec{\nabla}\times\vec{E}\right)\right)\mathrm{d}V \\ &= -\int_V \vec{j}\cdot\vec{E}\,\mathrm{d}V + \frac{1}{\mu_0}\int_V \left(\vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right) - \vec{B}\cdot\left(\vec{\nabla}\times\vec{E}\right)\right)\mathrm{d}V \\ &= -\int_V \vec{j}\cdot\vec{E}\,\mathrm{d}V - \frac{1}{\mu_0}\int_V \vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right)\mathrm{d}V \\ &= -\int_V \vec{j}\cdot\vec{E}\,\mathrm{d}V - \frac{1}{\mu_0}\oint_S (\vec{E}\times\vec{B})\cdot\mathrm{d}\vec{S}, \end{aligned}$$

   which is what we wanted to prove. □