Solution Manual

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35.6 Exercises

35.6.4

We do not need to change anything for the z bounds since they are given directly save for just applying $r = x^2 + y^2$. Then $0 \leq z \leq r^2$. Then since the region is bounded by a cylinder of unit radius it is apparent that $0 \leq r \leq 1$ and that $0 \leq \theta \leq 2\pi$.

35.6.7

Since E is the sphere of radius a in the first octant, it is easy to see that θ may only vary from 0 to $\frac{\pi}{2}$. Furthermore since the radius of the sphere is a, the quantity r may only vary from 0 to a. The surfaces for z that bound the surface is the plane $z = 0$ and $z = \sqrt{a^2 - x^2 - y^2} = \sqrt{a^2 - r^2}$. So $0 \leq z \leq \sqrt{a^2 - r^2}$

35.6.10

The region of integraion is a cylinder with a cylindrical cut out, where its base sits on the xy plane and is cut off above by the plane $z = x + y + 5$. This plane does not cut off the cylinder short on the xy plane, so the base is given by an annulus of inner radius 1 and outer radius 2. This is enough to deduce that $1 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$.

To find bounds in z apply the transformation $(x, y) \to (r\cos(\theta), r\sin(\theta))$ to the plane $z = x + y + 5$. The bottom bound is still 0. The upper bound becomes $r\cos(\theta) + r\sin(\theta) + 5$. Applying the same transformation to the integrand, the triple integral becomes:

$$\int_0^{2\pi} \int_1^2 \int_0^{r\cos(\theta)+r\sin(\theta)+5} r\sin(\theta)dz(r)drd\theta$$

$$\to \int_0^{2\pi} \int_1^2 \left(r^3 \sin(\theta)\cos(\theta) + r^3 \sin^2(\theta) + 5r^2 \sin(\theta)\right) drd\theta$$

$$\to \int_0^{2\pi} \int_1^2 \left(\frac{r^3}{2} + \frac{r^3}{2}\left(\sin(2\theta) - \cos(2\theta)\right) + 5r^2 \sin(\theta)\right) drd\theta$$

The last two terms will vanish due to the periodicity of the sine and cosine. Then the remaining integral is

$$\int_0^{2\pi}\int_1^2 \frac{r^3}{2} dr d\theta \to \pi \frac{r^4}{4}\Big|_1^2 = \frac{15}{4}\pi$$

35.6.13

The region is bounded above by the plane and below by the paraboloid. Rewrite the paraboloid equation as $z = \frac{1}{2}r^2$. The region of integration in the xy plane is the disk of radius 2 (find the boundary by solving $2 = \frac{1}{2}r^2$). Then it follows that $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$. The triple integral becomes

$$\int_0^{2\pi}\int_0^2\int_{\frac{1}{2}r^2}^2 (r^2)dz(r)drd\theta \to \int_0^{2\pi}\int_0^2 \left(2r^3 - \frac{1}{2}r^5\right) drd\theta \to \int_0^{2\pi} \frac{8}{3} d\theta = \frac{16}{3}\pi$$

35.6.20

The sphere indicates that ρ varies from 0 to a. Then the half planes (since $x \geq 0$) may be rewritten as

$$\frac{1}{2}y = \frac{\sqrt{3}}{2}x \implies \cos(\theta) = \frac{1}{2} \implies \theta = \frac{\pi}{3}$$
$$\frac{\sqrt{3}}{2}y = \frac{1}{2}x \implies \sin(\theta) = \frac{1}{2} \implies \theta = \frac{\pi}{6}$$

This means that $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$. These half planes make it so that it intersects a half arc of the greatest circle of the sphere of radius a. This means that ϕ varies from 0 to π, since only half of the full circumference is traced out by the intersection of the half planes and the sphere.

35.6.23

Give $x = \rho\cos(\phi)$, $y = \rho\sin(\phi)\cos(\theta)$, and $z = \rho\sin(\phi)\sin(\theta)$, where ϕ is the angle from the x axis outwards and θ is the angle swept from the positive y axis around towards the positive z axis. Also give $r = \sqrt{y^2+z^2}$ Then $x = \sqrt{1-r^2}$ and $x = \sqrt{4-r^2}$ are the hemispheres, of radius 1 and 2 respectively. So $1 \leq \rho \leq 2$. Then since these are positive hemispheres, they stop forming the rest of the sphere where x is negative. So ϕ varies from 0 to $\frac{\pi}{2}$. And conveniently since the hemispheres are fully formed about the x axis, θ takes on its natural range.

The triple integral becomes

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_1^2 (\rho\sin(\phi)\cos(\theta))^2 \left(\rho^2\sin(\phi)\right) d\rho d\phi d\theta$$
$$\to \left(\int_0^{2\pi}\cos^2(\theta)d\theta\right)\left(\int_0^{\frac{\pi}{2}}\sin^3(\phi)d\phi\right)\left(\int_1^2 \rho^4 d\rho\right)$$
$$\to (\pi)\left(\frac{2}{3}\right)\left(\frac{31}{5}\right) = \frac{62}{15}\pi$$

35.6.30

Form these three inequalities directly from the bounds of integration:

$$0 \leq \rho \leq \frac{2}{\cos(\phi)} \to 0 \leq z \leq 2$$

$$0 \leq \phi \leq \frac{\pi}{4} \to z = r \text{ is a conical boundary}$$

$$0 \leq \theta \leq \frac{\pi}{2}$$

From these we can deduce that the solid region is the part of the cone in the first octant that is bounded below by $z = r$ and above by $z = 2$ for $r = \sqrt{x^2 + y^2}$. The region of integration in the xy plane is quickly found to be the disk of radius 2 (since z=r=2). The triple integral for the volume in cylindrical coordinates is given below:

$$\int_0^{\frac{\pi}{2}} \int_0^2 \int_r^2 dz(r) dr d\theta \to \int_0^{\frac{\pi}{2}} \int_0^2 (2r - r^2) dr d\theta \to \int_0^{\frac{\pi}{2}} \frac{4}{3} d\theta = \frac{2}{3}\pi$$