Section 2.4 Problems 9, 13, 32, Section 2.6 Problems 9, 11, 23 and Exercise from Section 2 of the Week 4 Supplement

Section 2.4

9. $(2xy+3)\,\mathrm{d}x+\left(x^2-1\right)\mathrm{d}y=0$ Solve.

Confirm if the mixed partial derivatives are equal to see if the differential equation is exact:

$$\frac{\partial}{\partial y}(2xy+3)=2x=\frac{\partial}{\partial x}\left(x^2-1\right)=2x$$

Hence it is exact. Then integrate:

$$\int 2xy+3\,\mathrm{d}x=x^2+3x+c(y)=F(x,y)$$

Differentiate and compare with the other partial derivative of $F(x,y)$:

$$\frac{\partial}{\partial y}\left(x^2y+3x+c(y)\right)=x^2+c'(y)=x^2-1\to c'(y)=-1\to c(y)=-y+C$$

Hence the solution curve is of the form:

$$F(x,y)=x^2y+3x-y=C$$

13. $e^t(y-t)\,\mathrm{d}t+\left(1+e^t\right)\mathrm{d}y=0$ Solve.

Confirm if the mixed partial derivatives are equal to see if the differential equation is exact:

$$\frac{\partial}{\partial y}e^t(y-t)=e^t=\frac{\partial}{\partial t}\left(1+e^t\right)=e^t$$

Hence it is exact. Then integrate:

$$\int 1+e^t\,\mathrm{d}y=y+ye^t+c(t)=F(t,y)$$

Differentiate and compare with the other partial derivative of $F(t,y)$:

$$\frac{\partial}{\partial t}\left(y+ye^t+c(t)\right)=ye^t+c'(t)=e^t(y-t)\to c'(t)=-te^t$$

$$c(t)=\int -te^t\,\mathrm{d}t=-te^t+e^t+C$$

Hence the solution curve is of the form:

$$F(t,y)=y+ye^t-te^t+e^t=C$$

32.

(a) Take the negative reciprocal of $\frac{\mathrm{d}y}{\mathrm{d}x}$ and resolve terms:

$$\text{new } \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial x}} \to \frac{\partial F}{\partial x}\,\mathrm{d}y = \frac{\partial F}{\partial y}\,\mathrm{d}x$$

Immediately the above equality shows that

$$\frac{\partial F}{\partial x}\,\mathrm{d}y - \frac{\partial F}{\partial y}\,\mathrm{d}x = 0$$

since both terms are just equal to each other.

(b) Substitute the partial derivatives and solve for explicit solutions $y = \phi(x)$.

$$2y\,\mathrm{d}x - 2x\,\mathrm{d}y = 0 \to \int \frac{1}{x}\,\mathrm{d}x = \int \frac{1}{y}\,\mathrm{d}y \to \ln|x| + C = \ln|y|$$

Hence the solution curve is of the form

$$y = C|x|$$

which are straight lines through the origin (It might be okay to omit the absolute value signs).

(c) Substitute the partial derivatives and solve for implicit solutions $F(x, y) = C$.

$$x\,\mathrm{d}x - y\,\mathrm{d}y = 0 \to \int x\,\mathrm{d}x = \int y\,\mathrm{d}y = x^2 = y^2 + C$$

Hence the solution curve is of the form

$$x^2 - y^2 = C$$

which are hyperbolas.

Section 2.6

9. $\left(xy + y^2\right)\mathrm{d}x - x^2\,\mathrm{d}y = 0$ Solve.

This is homogeneous, so substitute $v = \frac{y}{x}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ and solve.

$$v + v^2 = v + x\frac{\mathrm{d}v}{\mathrm{d}x} \to \int \frac{1}{x}\,\mathrm{d}x = \int \frac{1}{v^2}\,\mathrm{d}v \to \ln|x| + C = -\frac{x}{y}$$

Hence the solution curve is of the form:

$$y = -\frac{x}{\ln|x| + C}$$

11. $\left(y^2 - xy\right) \mathrm{d}x + x^2 \mathrm{d}y = 0$ Solve.

This is homogeneous, so substitute $v = \frac{y}{x}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ and solve.

$$v - v^2 = v + x\frac{\mathrm{d}v}{\mathrm{d}x} \to \int \frac{1}{x}\,\mathrm{d}x = \int -\frac{1}{v^2}\,\mathrm{d}v \to \ln|x| + C = \frac{x}{y}$$

Hence the solution curve is of the form:

$$y = \frac{x}{\ln|x| + C}$$

23. $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2y}{x} = -x^2y^2$ Solve.

Multiply through by x^2y^{-2} and apply the Bernoulli substitution $v = y^{-1}$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = -y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$-x^2y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2x}{y} = x^4 \to x^2\frac{\mathrm{d}v}{\mathrm{d}x} + 2xv = x^4 \to \left(x^2v\right)' = x^4$$

$$\int 1\,\mathrm{d}\left(x^2v\right) = \int x^4\,\mathrm{d}x \to y^{-1} = \frac{x^3}{5} + Cx^{-2}$$

Hence the solution curve is of the form:

$$y = \left(\frac{x^3}{5} + Cx^{-2}\right)^{-1}$$

Exercise from Section 2 of the Week 4 Supplement:

1) Since $F(t, x)$ is a polynomial in t and x, it is smooth and so we have:

$$\varphi'(t) = \frac{-\left(4t\left(t^2 + x^2\right) - 8t\right)}{4x\left(t^2 + x^2\right) + 8x}$$

We can express the level curve as an explicit function $x = \varphi(t)$ so long as the denominator in the derivative above does not vanish. So for a neighborhood around $\left(\sqrt{\frac{3}{2}}, \sqrt{\frac{1}{2}}\right)$, the denominator does not vanish and so we can, but around both $(2, 0)$ and $(0, 0)$ the denominator vanishes and so we cannot conclude anything for sure.

2) Likewise we can do the same for trying to express the level curve as $t = \psi(x)$, by checking the denominator of its derivative:

$$\psi'(x) = \frac{-\left(4x\left(t^2 + x^2\right) + 8x\right)}{4t\left(t^2 + x^2\right) - 8t}$$

Around $\left(\sqrt{\frac{3}{2}}, \sqrt{\frac{1}{2}}\right)$ the denominator vanishes so we cannot conclude anything for sure, likewise around $(0, 0)$. We can, however, around $(2, 0)$ since the denominator does not vanish.