p.23: 3, 4, 5, 6, 8, 10, 11

3.

(a) *Proof.* Suppose via contradiction that for a converging sequence of complex numbers $\{w_n\}_{n=1}^{\infty}$, there is more than one limit of the sequence. So we can choose any two of these limits, $w_1, w_2 \in \mathbb{C}$ such that

$$\lim_{n\to\infty} |w_n - w_1| = 0 \text{ and } \lim_{n\to\infty} |w_n - w_2| = 0.$$

The goal is to find a contradiction. First, let $\varepsilon = |w_2 - w_1|/2$. Then by definition there exists $N_1, N_2 \in \mathbb{N}$ such that $|w_n - w_1| < \varepsilon$ whenever $n > N_1$ and $|w_n - w_2| < \varepsilon$ whenever $n > N_2$. Then let $N = \max\{N_1, N_2\}$, and so whenever $n > N$, we have that $|w_n - w_1| < \varepsilon$ and $|w_n - w_2| < \varepsilon$ hold.

Then consider the quantity $|w_2 - w_1|$. Note that by the triangle inequality

$$|w_2 - w_1| = |w_2 - w_n - (w_1 - w_n)| \leq |w_n - w_1| + |w_n - w_2| < 2 \cdot |w_2 - w_1|/2,$$

where the last inequality holds as long as $n > N$. A real number cannot be less than itself, so we have our contradiction. Hence converging sequences in $\mathbb{C}$ have unique limits. □

(b) *Proof.* Forwards direction. Suppose that a sequence $\{w_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Then this means that there exists a positive integer N such that

$$|w_n - w_m| < \varepsilon$$

whenever $n, m > N$. Write $w_n = x_n + iy_n$ and $w_m = x_m - iy_m$, and see that

$$|w_n - w_m| = |x_n - x_m + i(y_n - y_m)| \leq |x_n - x_m| + |y_n - y_m| \leq \varepsilon$$

whenever n, m exceed N. Hence $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are real Cauchy sequences, so they converge to limits x and y respectively. Hence the Cauchy sequence $\{w_n\}_{n=1}^{\infty}$ converges to $w = x + iy$.

Reverse direction. Conversely, suppose a complex valued sequence $\{w_n\}_{n=1}^{\infty}$ converges to a limit w. Then for every $\varepsilon/2$ there exist N_n, N_m such that $|w_n - w| < \varepsilon/2$ whenever $n > N_n$ and $|w_m - w| < \varepsilon/2$ whenever $m > N_m$. Then see that

$$|w_n - w_m| = |w_n - w + w - w_m| \leq |w_n - w| + |w_m - w| < \varepsilon$$

whenever $n > \max\{N_n, N_m\}$. Hence the sequence $\{w_n\}_{n=1}^{\infty}$ is a Cauchy sequence. □

(c) *Proof.* Let $\{a_n\}_{n=1}^{\infty}$ be a non-negative real-valued sequence as given, so that the series $\sum_n a_n$ converges (Note that the limit of this sequence is 0). Then let $\{z_n\}_{n=1}^{\infty}$ be a complex valued sequence where $|z_n| \leq a_n$ for all n.

Then the sequence

$$S_N = \sum_{n=1}^{N} a_n$$

converges. For any $\varepsilon > 0$, there exists $m, n > N$, where without loss of generality let $m \le n$, such that

$$\left|\sum_{i=1}^{n} z_i - \sum_{i=1}^{m} z_i\right| = \left|\sum_{i=m+1}^{n} z_i\right| \le \sum_{i=m+1}^{n} |z_i| \le \sum_{i=m+1}^{n} a_i = |S_n - S_m| \le \varepsilon,$$

where we use the convergence of Cauchy sequences to see that $\sum_n z_n$ converges. □

4.

(a) *Proof.* Use the ratio test directly:

$$\lim_{n\to\infty}\left|\frac{z^{n+1}/(n+1)!}{z^n/n!}\right| = \left|\lim_{n\to\infty}\frac{z}{n+1}\right| = 0,$$

which means that for any choice of $z \in \mathbb{C}$, the complex exponential converges. To show uniform convergence on every bounded subset S of $\mathbb{C}$, consider any $\varepsilon > 0$, and let $f_n(z) = \sum_{k=1}^{n} z^k/k!$. Then let $M = \max\{|z| \colon z \in S\}$. Then

$$|f_n(z) - e^z| = \left|\sum_{k=0}^{n} z^k/k! - \sum_{k=0}^{\infty} z^k/k!\right| \le \left|\sum_{k=n+1}^{\infty}\frac{z^k}{k!}\right| \le \sum_{k=n+1}^{\infty}\frac{M^k}{k!},$$

where the last sum tends to 0 as n becomes arbitrarily large. Hence we can pick a large enough n to ensure that the error is smaller than ε, thus the exponential function converges uniformly on every bounded subset of $\mathbb{C}$. □

(b) *Proof.* Let $z_1, z_2 \in \mathbb{C}$ be as given. From the definition of the exponential function, see that

$$e^{z_1+z_2} = \sum_{k=0}^{\infty}\frac{(z_1+z_2)^k}{k!} = \sum_{k=0}^{\infty}\frac{\sum_{i=0}^{k}\binom{k}{i} z_1^i z_2^{k-i}}{k!} = \sum_{k=0}^{\infty}\sum_{i=0}^{k}\frac{z_1^i}{i!}\cdot\frac{z_2^{k-i}}{(k-i)!},$$

which we can compare with

$$e^{z_1}e^{z_2} = \left(1 + \frac{z_1^2}{2!} + \frac{z_1^3}{3!} + \cdots\right)\left(1 + \frac{z_2^2}{2!} + \frac{z_2^3}{3!} + \cdots\right)$$

to see that they are the same (counting argument). □

(c) *Proof.* Let $z = iy$ with $z \in \mathbb{C}$ and $y \in \mathbb{R}$ as given. Then

$$e^{iy} = \sum_{k=0}^{\infty}\frac{(iy)^k}{k!} = 1 + iy - \frac{y^2}{2} - \frac{iy^3}{6} + \frac{y^4}{24} + \frac{iy^5}{120} + \cdots = \sum_{k=0}^{\infty}\frac{(-1)^k y^{2k}}{(2k)!} + i\sum_{k=0}^{\infty}\frac{(-1)^k y^{2k+1}}{(2k+1)!} = \cos(y) + i\sin(y)$$

□

(d) *Proof.* Let $x, y \in \mathbb{R}$ as given. Then

$$\left|e^{x+iy}\right| = |e^x(\cos(y) + i\sin(y))| = |e^x||\cos(y) + i\sin(y)| = e^x\sqrt{\sin^2(y) + \cos^2(y)} = e^x.$$

□

(e) *Proof.* Forwards direction. Suppose $z = 2\pi ki$ for some integer k. Then

$$e^z = e^{2\pi ki} = \cos(2\pi k) + i\sin(2\pi k) = 1 + i(0) = 1.$$

Conversely (reverse direction), suppose $e^z = 1$. Then see that $|1| = 1$, so that $|e^z|$ must also be 1. Thus $e^{\mathrm{Re}(z)} = 1$, and since the real-valued logarithm is injective, $\mathrm{Re}(z) = \log(1) = 0$. Then we still need $e^{i\,\mathrm{Im}(z)} = 1$, which is only possible when $\mathrm{Im}(z) = 2\pi k$ (k is any integer), since $e^{i\,\mathrm{Im}(z)} = \cos(\mathrm{Im}(z)) + i\sin(\mathrm{Im}(z))$, and we demand that the sine part vanishes (and the cosine part goes to 1). Hence $z = 2\pi ki$ for any integer k. □

(f) *Proof.* Let $z = x + iy$ for $x, y \in \mathbb{R}$. Then let $r = \sqrt{x^2 + y^2}$ and $\tan(\theta) = y/x$. Observe that by trigonometry that $\sin(\theta) = y/\sqrt{x^2 + y^2}$ and $\cos(\theta) = x/\sqrt{x^2 + y^2}$. Then

$$z = x + iy = \sqrt{x^2 + y^2}\left(\frac{x}{\sqrt{x^2 + y^2}} + i\frac{y}{\sqrt{x^2 + y^2}}\right) = r\left(\cos(\theta) + i\sin(\theta)\right),$$

where because the square root is surjective every complex number has a r that corresponds with its real and imaginary parts, and that solutions to $\tan(\theta) = y/x$ differ by multiples of 2π we may unambiguously take the least nonnegative value for θ to establish uniqueness classes (so we compute the principal value for $\arctan(y/x) = \theta$). Of course for $z = 0$ we do not compute θ as $r = 0$ and because $\arctan(0)$ is not defined. □

(g) The geometric meaning of multiplying a complex number by i is to rotate that complex number by $\pi/2$ radians about the origin, counterclockwise. If instead we multiply by $e^{i\theta}$, where $\theta \in \mathbb{R}$, the complex number rotates through θ radians about the origin, counterclockwise.

(h) See that (keep in mind the odd/even properties of the trigonometric functions)

$$\frac{1}{2}\left(e^{i\theta} + e^{i(-\theta)}\right) = \frac{1}{2}\left(\cos(\theta) + i\sin(\theta) + \cos(-\theta) + i\sin(-\theta)\right) = \frac{1}{2} \cdot 2\cos(\theta) = \cos(\theta)$$

and

$$\frac{1}{2i}\left(e^{i\theta} - e^{i(-\theta)}\right) = \frac{1}{2i}\left(\cos(\theta) + i\sin(\theta) - \cos(-\theta) - i\sin(-\theta)\right) = \frac{1}{2i} \cdot 2i\sin(\theta) = \sin(\theta).$$

(i) First give $\cos(\theta + \vartheta)$ as $\mathrm{Re}(e^{i(\theta+\vartheta)})$. Then

$$e^{i(\theta+\vartheta)} = e^{i\theta}e^{i\vartheta} = \left(\cos(\theta) + i\sin(\theta)\right)\left(\cos(\vartheta) + i\sin(\vartheta)\right),$$

and we only need to compute the real part of this product which is $\cos(\theta)\cos(\vartheta) - \sin(\theta)\sin(\vartheta)$, so $\cos(\theta + \vartheta) = \cos(\theta)\cos(\vartheta) - \sin(\theta)\sin(\vartheta)$. The imaginary part, $\sin(\theta + \vartheta)$, is thus equal to the imaginary part of the product, $\sin(\theta)\cos(\vartheta) + \cos(\theta)\sin(\vartheta)$.

Then

$$\cos(\theta - \varphi) - \cos(\theta + \varphi) = \cos(\theta)\cos(-\varphi) - \sin(\theta)\sin(-\varphi) - \cos(\theta)\cos(\varphi) + \sin(\theta)\sin(\varphi) = 2\sin(\theta)\sin(\varphi)$$

and

$$\sin(\theta + \varphi) + \sin(\theta - \varphi) = \sin(\theta)\cos(\varphi) + \cos(\theta)\sin(\varphi) + \sin(\theta)\cos(-\varphi) + \cos(\theta)\sin(-\varphi) = 2\sin(\theta)\cos(\varphi).$$

5. We can compare e^{inx} with $e^{in(x+2\pi)}$ $(n \in \mathbb{Z})$:

$$\begin{aligned} e^{in(x+2\pi)} &= \cos(n(x+2\pi)) + i\sin(n(x+2\pi)) \\ &= \cos(nx)\cos(2\pi n) - \sin(nx)\sin(2\pi n) + i\sin(nx)\cos(2\pi n) + i\cos(nx)\sin(2\pi n) \\ &= \cos(nx) + i\sin(nx) = e^{inx} \end{aligned}$$

and see that they are equal, so e^{inx} is periodic with period 2π. Then see that for $n \neq 0$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\,\mathrm{d}x = \frac{e^{inx}}{2\pi in}\bigg|_{-\pi}^{\pi} = \frac{\cos(n\pi) + i\sin(n\pi) - \cos(-n\pi) - i\sin(-n\pi)}{2\pi in}$$

$$= 0$$

And then naturally when $n = 0$, the integral becomes

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} 1\,\mathrm{d}x = \frac{2\pi}{2\pi} = 1.$$

Then from the hint we compute

$$e^{inx}e^{-imx} + e^{inx}e^{imx} = 2\cos(mx)\cos(nx) + 2i\cos(mx)\sin(nx)$$

and

$$e^{inx}e^{-imx} - e^{inx}e^{imx} = -2i\cos(nx)\sin(mx) + 2\sin(mx)\sin(nx).$$

So evidently,

$$I_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\cos(nx)\cos(mx)\,\mathrm{d}x = \frac{1}{2\pi}\operatorname{Re}\left(\int_{-\pi}^{\pi}\left(e^{inx}e^{-imx} + e^{inx}e^{imx}\right)\mathrm{d}x\right),$$

and when $n - m = 0 \iff n = m$,

$$I_1 = \frac{1}{2\pi}\operatorname{Re}\left(\int_{-\pi}^{\pi}\left(1 + e^{i(n+m)x}\right)\mathrm{d}x\right) = \frac{1}{2\pi}\operatorname{Re}\left(2\pi + \frac{e^{i(n+m)x}}{i(n+m)}\bigg|_{-\pi}^{\pi}\right) = 1,$$

and when $n - m \neq 0 \iff n \neq m$, so that $n - m$ and $n + m$ are both nonzero use the earlier result to see that

$$I_1 = \frac{1}{2\pi}\operatorname{Re}\left(\int_{-\pi}^{\pi}\left(e^{i(n-m)x} + e^{i(n+m)x}\right)\mathrm{d}x\right) = \frac{1}{2\pi}\operatorname{Re}\left(\frac{e^{i(n-m)x}}{i(n-m)}\bigg|_{-\pi}^{\pi} + \frac{e^{i(n+m)x}}{i(n+m)}\bigg|_{-\pi}^{\pi}\right) = 0.$$

Similarly we can reduce the next integral like so:

$$I_2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\sin(nx)\sin(mx)\,\mathrm{d}x = \frac{1}{2\pi}\operatorname{Re}\left(\int_{-\pi}^{\pi}\left(e^{inx}e^{-imx} - e^{inx}e^{imx}\right)\mathrm{d}x\right),$$

so that similarly if $n - m = 0$,

$$I_2 = \frac{1}{2\pi}\operatorname{Re}\left(\int_{-\pi}^{\pi}\left(1 - e^{i(n+m)x}\right)\mathrm{d}x\right) = \frac{1}{2\pi}\operatorname{Re}\left(2\pi - \frac{e^{i(n+m)x}}{i(n+m)}\bigg|_{-\pi}^{\pi}\right) = 1,$$

and likewise when $n - m \neq 0$,

$$I_2 = \frac{1}{2\pi}\operatorname{Re}\left(\int_{-\pi}^{\pi}\left(e^{i(n-m)x} - e^{i(n+m)x}\right)\mathrm{d}x\right) = \frac{1}{2\pi}\operatorname{Re}\left(\left.\frac{e^{i(n-m)x}}{i(n-m)}\right|_{-\pi}^{\pi} - \left.\frac{e^{i(n+m)x}}{i(n+m)}\right|_{-\pi}^{\pi}\right) = 0.$$

Then

$$I_3 = \int_{-\pi}^{\pi} 2\sin(nx)\cos(mx)\,\mathrm{d}x = \operatorname{Im}\left(\int_{-\pi}^{\pi}\left(e^{inx}e^{-imx} + e^{inx}e^{imx}\right)\mathrm{d}x\right) = \operatorname{Im}\left(\left.\frac{e^{i(n-m)x}}{i(n-m)}\right|_{-\pi}^{\pi} + \left.\frac{e^{i(n+m)x}}{i(n+m)}\right|_{-\pi}^{\pi}\right)$$
$$= 0$$

since in every case the imaginary part is zero.

6.

Proof. Let f be a twice continuously differentiable function on $\mathbb{R}$ which is a solution to the equation $f''(t)+c^2 f(t) = 0$.

Then let $g(t) = f(t)\cos(ct) - c^{-1}f'(t)\sin(ct)$ and $h(t) = f(t)\sin(ct) + c^{-1}f'(t)\cos(ct)$, and see that

$$-cg'(t) = c^2 f(t)\sin(ct) + f''\sin(t) = 0 \implies g'(t) = 0 \implies g(t) = a,$$

and

$$ch'(t) = c^2 f(t)\cos(ct) + f''\cos(t) = 0 \implies h'(t) = 0 \implies h(t) = b,$$

for some $a, b \in \mathbb{R}$. Then

$$\begin{aligned} a\cos(t) + b\sin(t) &= g(t)\cos(ct) + h(t)\sin(ct) \\ &= f(t)\cos^2(ct) - c^{-1}f'(t)\sin(ct)\cos(ct) + f(t)\sin^2(ct) + c^{-1}f'(t)\sin(ct)\cos(ct) \\ &= f(t) \end{aligned}$$

Hence there exist constants $a, b \in \mathbb{R}$ such that $f(t) = a\cos(ct) + b\sin(ct)$. □

8. Use Taylor expansion centered at a:

$$F(x) = \sum_{k=0}^{\infty}\frac{F^{(n)}(a)}{k!}(x-a)^k,$$

and center the series at x so that

$$F(t) = \sum_{k=0}^{\infty}\frac{F^{(n)}(x)}{k!}(t-x)^k.$$

Then

$$F(x+h) = \sum_{k=0}^{\infty}\frac{h^k}{k!}F^{(n)}(x) = F(x) + hF'(x) + \frac{h^2}{2}F''(x) + h^2\varphi(h),$$

where $h^2\varphi(h) \in O(h^3)$ (it is the remaining terms in the infinite sum) so that $\varphi(h)$ is at least linear in degree so that $\varphi(h) \to 0$ as $h \to 0$.

Then using this series expansion, see that

$$\begin{aligned}
&\frac{F(x+h)+F(x-h)-2F(x)}{h^2} \\
&= \frac{F(x)+hF'(x)+\frac{h^2}{2}F''(x)+h^2\varphi_+(h)+F(x)-hF'(x)+\frac{h^2}{2}F''(x)-h^2\varphi_-(h)-2F(x)}{h^2} \\
&= F''(x)+(\varphi_+(h)-\varphi_-(h)),
\end{aligned}$$

where this quantity tends to $F''(x)$ as $h \to 0$ since $(\varphi_+(h)-\varphi_-(h))$ tends to zero (each φ is at least linear and each tends to zero individually).

10. Let $u(x,y)$ be a function of two variables as given, which is sufficiently differentiable. For $x = r\cos(\theta)$ and $y = r\sin(\theta)$, see that

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}x} + \frac{\partial u}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}x} \\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}y} + \frac{\partial u}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}y},
\end{aligned}$$

then that

$$\begin{aligned}
\frac{\partial r}{\partial r} &= 1 = \frac{\partial r}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial r}{\partial y}\frac{\partial y}{\partial r} \\
\frac{\partial r}{\partial \theta} &= 0 = \frac{\partial r}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial r}{\partial y}\frac{\partial y}{\partial \theta} \\
\frac{\partial \theta}{\partial r} &= 0 = \frac{\partial \theta}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial y}{\partial r} \\
\frac{\partial \theta}{\partial \theta} &= 1 = \frac{\partial \theta}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial \theta}{\partial y}\frac{\partial y}{\partial \theta}
\end{aligned}$$

.

We can then form the following matrix product:

$$\begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix}\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix}\begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so that

$$\begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix}^{-1} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -r^{-1}\sin(\theta) & r^{-1}\cos(\theta) \end{pmatrix}.$$

Hence

$$\frac{\partial r}{\partial x} = \cos(\theta), \frac{\partial r}{\partial y} = \sin(\theta), \frac{\partial \theta}{\partial x} = -r^{-1}\sin(\theta), \frac{\partial \theta}{\partial y} = r^{-1}\cos(\theta),$$

so that

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\cos(\theta) - \frac{\partial u}{\partial \theta}r^{-1}\sin(\theta) \\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\sin(\theta) + \frac{\partial u}{\partial \theta}r^{-1}\cos(\theta).
\end{aligned}$$

Then take derivatives again:

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\cos(\theta) - \frac{\partial u}{\partial \theta} r^{-1}\sin(\theta)\right)\frac{\mathrm{d}r}{\mathrm{d}x} + \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\cos(\theta) - \frac{\partial u}{\partial \theta} r^{-1}\sin(\theta)\right)\frac{\mathrm{d}\theta}{\mathrm{d}x} \\
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\sin(\theta) + \frac{\partial u}{\partial \theta} r^{-1}\cos(\theta)\right)\frac{\mathrm{d}r}{\mathrm{d}y} + \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\sin(\theta) + \frac{\partial u}{\partial \theta} r^{-1}\cos(\theta)\right)\frac{\mathrm{d}\theta}{\mathrm{d}y}.
\end{aligned}$$

$$\begin{aligned}
&\implies \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial r^2}\cos^2(\theta) - \frac{\partial^2 u}{\partial \theta \partial r} r^{-1}\sin(\theta)\cos(\theta) + \frac{\partial u}{\partial \theta} r^{-2}\sin\theta\cos(\theta) \\
&- \frac{\partial^2 u}{\partial r \partial \theta} r^{-1}\sin(\theta)\cos(\theta) + \frac{\partial u}{\partial r} r^{-1}\sin^2(\theta) + \frac{\partial^2 u}{\partial \theta^2} r^{-2}\sin^2(\theta) + \frac{\partial u}{\partial \theta} r^{-2}\sin(\theta)\cos(\theta) \\
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 u}{\partial r^2}\sin^2(\theta) + \frac{\partial^2 u}{\partial \theta \partial r} r^{-1}\sin(\theta)\cos(\theta) - \frac{\partial u}{\partial \theta} r^{-2}\sin(\theta)\cos(\theta) \\
&+ \frac{\partial^2 u}{\partial r \partial \theta} r^{-1}\sin(\theta)\cos(\theta) + \frac{\partial u}{\partial r} r^{-1}\cos^2(\theta) + \frac{\partial^2 u}{\partial \theta^2} r^{-2}\cos^2(\theta) - \frac{\partial u}{\partial \theta} r^{-2}\sin(\theta)\cos(\theta)
\end{aligned}$$

And so finally $\Delta u(x, y)$ given in polar coordinates is

$$\Delta U(r,\theta) = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}.$$

Then use results from earlier to see that

$$\begin{aligned}
\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 = \left|\frac{\partial u}{\partial r}\right|^2 \cos(\theta) - 2\left|\frac{\partial u}{\partial r}\right|\left|\frac{\partial u}{\partial \theta}\right|\sin(\theta)\cos(\theta) + \left|\frac{\partial u}{\partial \theta}\right|^2 r^{-2}\sin^2(\theta) \\
+ \left|\frac{\partial u}{\partial r}\right|^2 \sin(\theta) + 2\left|\frac{\partial u}{\partial r}\right|\left|\frac{\partial u}{\partial \theta}\right|\sin(\theta)\cos(\theta) + \left|\frac{\partial u}{\partial \theta}\right|^2 r^{-2}\cos^2(\theta) \\
= \left|\frac{\partial u}{\partial r}\right|^2 + \frac{1}{r^2}\left|\frac{\partial u}{\partial \theta}\right|^2.
\end{aligned}$$

11. Examining solutions to $r^2F''(r) + rF'(r) - n^2F(r) = 0$, where $r > 0$.

Suppose for a twice differentiable solution $F(r)$ we can write $F(r) = g(r)r^n$, and substitute to find after dividing through by r^n, and doing some algebra, that $g''(t)r + g'(t) + 2g(t)n = (g'(t)r)' + 2ng' = 0$, which implies that for some c, $2ng(t) + g'(t)r = c$.

When $n = 0$, we solve $g'(t)r = c$ and find that $g(r)$ is a linear combination be one of $\log(r)$ or 1 (from the previous equation), so $F(r)$ in this case takes a linear combination of 1 or $\log(r)$. If n is nonzero, then we can find that $g(r)$ is a linear combination of 1 and r^{-2n}, so that $F(r)$ is a linear combination of r^n and r^{-n}.