p.58: 1, 2, 4, 5, 6, 7, 11

1.

Proof. (1) For $x \in \mathbb{R}$, observe that $f(x) = f(x+2\pi) = f(x-2\pi)$ because f is 2π-periodic. Then in the integral

$$\int_{a+2\pi}^{b+2\pi} f(x)\,\mathrm{d}x\,,$$

make the change of variables $u = x - 2\pi$, $\mathrm{d}u = \mathrm{d}x$, so that

$$\int_{a+2\pi}^{b+2\pi} f(x)\,\mathrm{d}x = \int_a^b f(u+2\pi)\,\mathrm{d}u = \int_a^b f(u)\,\mathrm{d}u = \int_a^b f(x)\,\mathrm{d}x\,.$$

Similarly apply the change of variables $u = x + 2\pi$, $\mathrm{d}u = \mathrm{d}x$ to

$$\int_{a-2\pi}^{b-2\pi} f(x)\,\mathrm{d}x$$

to find

$$\int_{a-2\pi}^{b-2\pi} f(x)\,\mathrm{d}x = \int_a^b f(u-2\pi)\,\mathrm{d}u = \int_a^b f(u)\,\mathrm{d}u = \int_a^b f(x)\,\mathrm{d}x\,.$$

□

Proof. (2) Using the change of variables $u = x + a$, $\mathrm{d}u = \mathrm{d}x$, we have that

$$\int_{-\pi}^{\pi} f(x+a)\,\mathrm{d}x = \int_{-\pi+a}^{\pi+a} f(u)\,\mathrm{d}u = \int_{-\pi+a}^{\pi+a} f(x)\,\mathrm{d}x\,.$$

Then for any a the following is true:

$$\int_{-\pi+a}^{\pi+a} f(x)\,\mathrm{d}x = \int_{-\pi+a}^{-\pi} f(x)\,\mathrm{d}x + \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x + \int_{\pi}^{\pi+a} f(x)\,\mathrm{d}x\,.$$

But from (1), we can adjust the bounds of the first integral in the sum, so that

$$\int_{-\pi+a}^{-\pi} f(x)\,\mathrm{d}x = \int_{-\pi+a+2\pi}^{-\pi+2\pi} f(x)\,\mathrm{d}x = \int_{\pi+a}^{\pi} f(x)\,\mathrm{d}x = -\int_{\pi}^{\pi+a} f(x)\,\mathrm{d}x\,.$$

So the summation becomes

$$\int_{-\pi+a}^{\pi+a} f(x)\,\mathrm{d}x = -\int_{\pi}^{\pi+a} f(x)\,\mathrm{d}x + \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x + \int_{\pi}^{\pi+a} f(x)\,\mathrm{d}x = \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x\,.$$

□

2.

(a) *Proof.* The Fourier series for the 2π-periodic function f is written as

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta},$$

where

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,\mathrm{d}\theta\,.$$

For $n \geq 1$,

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2}\left[\hat{f}(n) + \hat{f}(-n) + \hat{f}(n) - \hat{f}(-n)\right]\\ \hat{f}(-n) &= \frac{1}{2}\left[\hat{f}(-n) + \hat{f}(n) + \hat{f}(-n) - \hat{f}(n)\right],\end{aligned}$$

so that

$$\begin{aligned}\hat{f}(n)e^{in\theta} &= \frac{1}{2}\left[\hat{f}(n) + \hat{f}(-n)\right]e^{in\theta} + \frac{1}{2}\left[\hat{f}(n) - \hat{f}(-n)\right]e^{in\theta}\\ \hat{f}(-n)e^{-in\theta} &= \frac{1}{2}\left[\hat{f}(n) + \hat{f}(-n)\right]e^{-in\theta} - \frac{1}{2}\left[\hat{f}(n) - \hat{f}(-n)\right]e^{-in\theta}\\ \hat{f}(n)e^{in\theta} + \hat{f}(-n)e^{-in\theta} &= \left[\hat{f}(n) + \hat{f}(-n)\right]\cos(n\theta) + i\left[\hat{f}(n) - \hat{f}(-n)\right]\sin(n\theta).\end{aligned}$$

Then by pairing up the n and $-n$ terms in the Fourier series for $f(\theta)$,

$$\begin{aligned}f(\theta) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta} &= \hat{f}(0) + \sum_{n\geq 1}\left(\hat{f}(n)e^{in\theta} + \hat{f}(-n)e^{-in\theta}\right)\\ &= \hat{f}(0) + \sum_{n\geq 1}\left(\left[\hat{f}(n) + \hat{f}(-n)\right]\cos(n\theta) + i\left[\hat{f}(n) - \hat{f}(-n)\right]\sin(n\theta)\right).\end{aligned}$$

□

(b) *Proof.* If f is even, then

$$\hat{f}(n) - \hat{f}(-n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} -f(\theta)\left(e^{in\theta} - e^{-in\theta}\right)\mathrm{d}\theta = i\frac{1}{\pi}\int_{-\pi}^{\pi} -f(\theta)\sin(\theta)\,\mathrm{d}\theta = 0,$$

because the product of an even function and an odd function is still odd and so the integral vanishes over the symmetric bounds, so $\hat{f}(n) = \hat{f}(-n)$, and the sine component of the sum above vanishes. Similarly the integrand in $\hat{f}(n) + \hat{f}(-n)$ is an even function, so the integral will not vanish and so the cosine terms will not vanish unless f is the zero function. Thus the series above becomes a cosine series. □

(c) *Proof.* If f is odd, then

$$\hat{f}(n) + \hat{f}(-n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\left(e^{in\theta} + e^{-in\theta}\right)\mathrm{d}\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(\theta)\,\mathrm{d}\theta = 0,$$

because the product of an even function and an odd function is still odd and so the integral vanishes over the symmetric bounds, so $\hat{f}(n) = -\hat{f}(-n)$, and the cosine component of the sum above vanishes. Similarly the integrand in $\hat{f}(n) - \hat{f}(-n)$ is an even function, so the integral will not vanish and so the sine terms will not vanish unless f is the zero function. Thus the series above becomes a sine series. □

(d) *Proof.* For odd n, write $n = 2k+1$ where $k \in \mathbb{Z}$. Then

$$\hat{f}(n) = \hat{f}(2k+1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-i(2k+1)\theta}\,\mathrm{d}\theta\,,$$

and by the change of variables $\phi = \theta - \pi$ and $\mathrm{d}\phi = \mathrm{d}\theta$,

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-i(2k+1)\theta}\,\mathrm{d}\theta = \frac{1}{2\pi}\int_{-2\pi}^{0} -f(\phi+\pi)e^{-i(2k+1)\phi}\,\mathrm{d}\phi = \frac{1}{2\pi}\int_{-2\pi}^{0} -f(\theta)e^{-i(2k+1)\theta}\,\mathrm{d}\theta\,.$$

Note that because f is π-periodic, then it is also 2π periodic, and we can use earlier results to find that

$$\begin{aligned}\frac{1}{2\pi}\int_{-2\pi}^{0} -f(\theta)e^{-i(2k+1)\theta}\,\mathrm{d}\theta &= \frac{1}{2\pi}\int_{-2\pi}^{-\pi} -f(\theta)e^{-i(2k+1)\theta}\,\mathrm{d}\theta + \frac{1}{2\pi}\int_{-\pi}^{0} -f(\theta)e^{-i(2k+1)\theta}\,\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_{0}^{\pi} -f(\theta)e^{-i(2k+1)\theta}\,\mathrm{d}\theta + \frac{1}{2\pi}\int_{-\pi}^{0} -f(\theta)e^{-i(2k+1)\theta}\,\mathrm{d}\theta \\ &= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-i(2k+1)\theta}\,\mathrm{d}\theta = -\hat{f}(n).\end{aligned}$$

So for all odd n, $\hat{f}(n) = -\hat{f}(n)$, which implies that $\hat{f}(n) = 0$. □

(e) *Proof.* Forwards direction. Suppose $\overline{\hat{f}(n)} = \hat{f}(-n)$ for all n. Then

$$\overline{\hat{f}(n)} = \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,\mathrm{d}\theta} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(\theta)}e^{in\theta}\,\mathrm{d}\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{in\theta}\,\mathrm{d}\theta = \hat{f}(-n),$$

and we can rearrange terms to find that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(f(\theta) - \overline{f(\theta)}\right)e^{in\theta}\,\mathrm{d}\theta = \frac{i}{\pi}\int_{-\pi}^{\pi}\mathrm{Im}(f(\theta))e^{in\theta}\,\mathrm{d}\theta = 0.$$

This integral can only vanish when $\mathrm{Im}(f(\theta))$ is simultaneously even and odd, which forces $\mathrm{Im}(f(\theta))$ to be zero. This means $f(\theta)$ is real valued. □

4.

(a)

(b) The Fourier coefficients of f are

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{0}\theta(\pi+\theta)e^{-in\theta}\,\mathrm{d}\theta + \frac{1}{2\pi}\int_{0}^{\pi}\theta(\pi-\theta)e^{-in\theta}\,\mathrm{d}\theta\,,$$

which with a substitution in the first integral (shift theta forwards by π), see that

$$\hat{f}(n) = \frac{1+(-1)^{n+1}}{2\pi}\int_0^\pi \theta(\pi-\theta)e^{-in\theta}\,\mathrm{d}\theta\,.$$

Clearly for all even n the integral vanishes, so we take n to be an odd integer. Then since $\theta(\pi-\theta)$ has roots at 0 and π, it is easy to compute the Fourier coefficients:

$$\hat{f}(n) = \frac{1+(-1)^{n+1}}{2\pi}\int_0^\pi \theta(\pi-\theta)e^{-in\theta}\,\mathrm{d}\theta = \frac{1}{2\pi}\left(\frac{\theta(\pi-\theta)e^{-in\theta}}{-in}\bigg|_0^\pi + \frac{-(\pi-2\theta)e^{-in\theta}}{(-in)^2}\bigg|_0^\pi + \frac{-2e^{-in\theta}}{(-in)^3}\bigg|_0^\pi\right)$$
$$= -\frac{4i}{\pi n^3}.$$

Then the Fourier series for f is

$$\sum_{n\text{ odd}} -\frac{4i}{\pi n^3}e^{in\theta} = \sum_{n\text{ odd}}\left(\frac{4\sin(n\theta)}{\pi n^3} - \frac{4i\cos(n\theta)}{\pi n^3}\right) = \frac{8}{\pi}\sum_{n\text{ odd}\geq 1}\frac{\sin(n\theta)}{n^3},$$

where in the last equality because both sine and the cubing function are odd we may join the negative n sine terms with the positive n sine terms, and similarly, the imaginary part vanishes because the cosine function is even but the cubing function is odd. Observe that the series absolutely converges due to the n^3 term in the denominator, so we may say that

$$f(\theta) = \frac{8}{\pi}\sum_{n\text{ odd}\geq 1}\frac{\sin(n\theta)}{n^3}.$$

5. Compute the Fourier coefficients of f,

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\delta}^0\left(1+\frac{\theta}{\delta}\right)e^{-in\theta}\,\mathrm{d}\theta + \frac{1}{2\pi}\int_0^\delta\left(1-\frac{\theta}{\delta}\right)e^{-in\theta}\,\mathrm{d}\theta\,.$$

In the first integral observe that changing variables from θ to $-\theta$ makes it so that

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^\delta\left(1-\frac{\theta}{\delta}\right)\left(e^{in\theta}+e^{-in\theta}\right)\mathrm{d}\theta = \frac{1}{\pi}\int_0^\delta\left(1-\frac{\theta}{\delta}\right)\cos(n\theta)\,\mathrm{d}\theta\,,$$

and observe that $\hat{f}(0) = \delta/2\pi$. Then

$$\hat{f}(n) = \frac{1}{\pi}\int_0^\delta\left(1-\frac{\theta}{\delta}\right)\cos(n\theta)\,\mathrm{d}\theta = \frac{1}{\pi}\left(\frac{(1-\frac{\theta}{\delta})\sin(n\theta)}{n}\bigg|_0^\delta + \frac{-\cos(n\theta)}{n^2\delta}\bigg|_0^\delta\right) = \frac{1-\cos(n\delta)}{n^2\pi\delta}.$$

The Fourier series for f is given by

$$\frac{\delta}{2\pi} + \sum_{n=-\infty}^{-1}\left(\frac{1-\cos(n\delta)}{n^2\pi\delta}\right)e^{in\theta} + \sum_{n=1}^{\infty}\left(\frac{1-\cos(n\delta)}{n^2\pi\delta}\right)e^{in\theta},$$

where because squaring and the cosine function are even, we rewrite the first sum instead to find the following more convenient form:

$$\frac{\delta}{2\pi} + \sum_{n=1}^{\infty}\left(\frac{1-\cos(n\delta)}{n^2\pi\delta}\right)e^{-in\theta} + \sum_{n=1}^{\infty}\left(\frac{1-\cos(n\delta)}{n^2\pi\delta}\right)e^{in\theta} = \frac{\delta}{2\pi} + 2\sum_{n=1}^{\infty}\left(\frac{1-\cos(n\delta)}{n^2\pi\delta}\right)\cos(n\theta).$$

The sum converges absolutely because of the n^2 term in the denominator, so we may write

$$f(\theta) = \frac{\delta}{2\pi} + 2\sum_{n=1}^{\infty}\left(\frac{1-\cos(n\delta)}{n^2\pi\delta}\right)\cos(n\theta).$$

6.

(a)

(b) The Fourier coefficients of f are

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{0}(-\theta)e^{-in\theta}\,\mathrm{d}\theta + \frac{1}{2\pi}\int_{0}^{\pi}\theta e^{-in\theta}\,\mathrm{d}\theta\,.$$

In the first integral change variables from θ to $-\theta$ to find that

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2\pi}\int_{0}^{\pi}\theta e^{in\theta}\,\mathrm{d}\theta + \frac{1}{2\pi}\int_{0}^{\pi}\theta e^{-in\theta}\,\mathrm{d}\theta = \frac{1}{\pi}\int_{0}^{\pi}\theta\cos(n\theta)\,\mathrm{d}\theta \\ &= \frac{-1+(-1)^n}{\pi n^2} \text{ for } n \neq 0.\end{aligned}$$

Observe that $\hat{f}(0) = \pi/2$ as a result.

(c) Then the Fourier series of f in terms of sines and cosines is

$$f(\theta) \sim \frac{\pi}{2} + \sum_{n=-\infty}^{-1}\left(\frac{-1+(-1)^n}{\pi n^2}\right)e^{in\theta} + \sum_{n=1}^{\infty}\left(\frac{-1+(-1)^n}{\pi n^2}\right)e^{in\theta},$$

and we again flip the first summation to the positive integers by replacing n with $-n$ to see that

$$f(\theta) \sim \frac{\pi}{2} + \sum_{n=1}^{\infty}\left(\frac{-1+(-1)^n}{\pi n^2}\right)e^{-in\theta} + \sum_{n=1}^{\infty}\left(\frac{-1+(-1)^n}{\pi n^2}\right)e^{in\theta} = \frac{\pi}{2} + 2\sum_{n=1}^{\infty}\left(\frac{-1+(-1)^n}{\pi n^2}\right)\cos(n\theta),$$

but the inner fraction vanishes for all even n, so we can sum over the odd n (when n is odd the fraction is $-2/\pi n^2$). Furthermore this sum converges absolutely due to the n^2 term in the denominator, so we may write

$$f(\theta) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd } \geq 1}\frac{\cos(n\theta)}{n^2}.$$

(d) Let $\theta = 0$. Then

$$f(0) = 0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd } \geq 1} \frac{1}{n^2} \implies \frac{\pi^2}{8} = \sum_{n \text{ odd } \geq 1} \frac{1}{n^2}.$$

Furthermore,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n \text{ odd } \geq 1} \frac{1}{n^2} + \sum_{n \text{ even } \geq 1} \frac{1}{n^2},$$

but in the third sum since n is even ($2 \mid n$ yields $4 \mid n^2$), we may factor out $1/4$ from it to find the first sum. So

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n \text{ odd } \geq 1} \frac{1}{n^2} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} \implies \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3} \cdot \frac{\pi^2}{8} = \frac{\pi^2}{6}.$$

7.

(a) *Proof.* Let $\{a_n\}_{n=1}^N, \{b_n\}_{n=1}^N$ be finite sequences of complex numbers as given. Then

$$\begin{aligned}
\sum_{n=M}^{N} a_n b_n &= \sum_{n=M}^{N} a_n (B_n - B_{n-1}) \\
&= \sum_{n=M}^{N} a_n B_n - \sum_{n=M}^{N} a_n B_{n-1} \\
&= a_N B_N + \sum_{n=M}^{N-1} a_n B_n - \sum_{n=M-1}^{N-1} a_{n+1} B_n \\
&= a_N B_N - a_M B_{M-1} + \sum_{n=M}^{N-1} a_n B_n - \sum_{n=M}^{N-1} a_{n+1} B_n \\
&= a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n.
\end{aligned}$$

Thus the summation by parts formula holds. □

(b) *Proof.* Let the partial sums B_n be bounded above by B, and let $\{a_n\}_{n=1}^N$ be a sequence of (positive) real numbers decreasing monotonically to 0. Also require that $N \geq M$. Then

$$\begin{aligned}
\left| \sum_{n=M}^{N} a_n b_n \right| &= \left| a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n \right| \\
&\leq B \left(a_N + a_M + \sum_{n=M}^{N-1} (a_{n+1} - a_n) \right) \\
&= B \left(a_N + a_M + \sum_{n=M}^{N-1} a_n - \sum_{n=M+1}^{N} a_n \right) \\
&= 2Ba_M,
\end{aligned}$$

but because we can choose M, we can make $2Ba_M$ arbitrarily small and so the sequence of partial sums of $\sum a_n b_n$ is a Cauchy sequence as a result, which converges. □

11.

Proof. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of Riemann integrable functions on the interval $[0, 1]$ such that

$$\int_0^1 |f_k(x) - f(x)| \, \mathrm{d}x \to 0 \text{ as } k \to \infty.$$

Then for every $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that for $k > K$, $\int_0^1 |f_k(x) - f(x)| \, \mathrm{d}x < \varepsilon$. So for any ε, take $k > K$ and see that

$$\left|\hat{f}_k(n) - \hat{f}(n)\right| = \left|\int_0^1 (f_k(x) - f(x)) \, e^{-2\pi i n x} \, \mathrm{d}x\right| \leq \int_0^1 |f_k(x) - f(x)| \, \mathrm{d}x < \varepsilon,$$

which means that $\hat{f}_k(n) \to \hat{f}(n)$ uniformly in n as $k \to \infty$. □