1. Say $A \subset X$ is dense if $\overline{A} = X$.

   (a) Show that A is dense in X if and only if every nonempty open subset V in X satisfies $V \cap A \neq \emptyset$.

   Proof. Suppose that every nonempty open set V in X satisfies $V \cap A \neq \emptyset$. Then for any $x \in X$, any neighborhood containing x intersects nontrivially with A so that $x \in \overline{A}$, and because x was arbitrary $X \subset \overline{A}$. It is clear that $\overline{A} \subset X$ (since X is closed) so that $\overline{A} = X$ as a result.

   Suppose that $\overline{A} = X$. Then any nonempty open set V in X contains at least one point $x \in X = \overline{A}$ so that necessarily V (an open neighborhood of x) must intersect nontrivially with A.

   Hence A is dense in X if and only if every nonempty open subset V in X satisfies $V \cap A \neq \emptyset$. □

   (b) Assume that X and Y are topological spaces with Y Hausdorff and A is dense in X. Suppose that $f\colon X \to Y$ and $g\colon X \to Y$ are continuous functions with $f(a) = g(a)$ for all $a \in A$. Prove that $f(x) = g(x)$ for all $x \in X$.

   Proof. Suppose by way of contradiction that there is an $x \in X$ such that $f(x) \neq g(x)$. Since Y is Hausdorff, choose neighborhoods U of $f(x)$ and V of $g(x)$ which intersect trivially. Then $x \in f^{-1}(U)$ and $x \in g^{-1}(V)$ with both $f^{-1}(U), g^{-1}(V)$ open in X since f, g are continuous. Because $\overline{A} = X$, the open set $f^{-1}(U) \cap g^{-1}(V)$ intersects nontrivially with A; that is, there exists $a \in A$ with $a \in f^{-1}(U) \cap g^{-1}(V)$. Then by hypothesis $f(a) = g(a)$, but $f(a) \in U$ and $f(a) = g(a) \in V$, which is a contradiction since U and V were chosen to be disjoint.

   Hence $f(x) = g(x)$ for all $x \in X$. □

2. A is a subset of the topological space X.

   (a) Show that $x \in \operatorname{Int}(A)$ if and only if there is an open set U with $x \in U \subset A$.

   Proof. For $x \in X$, suppose that there is an open set U with $x \in U \subset A$. Then by definition of $\operatorname{Int}(A)$ as the union of all open sets contained in A, we have that U is one such open set contained in A and so $x \in U \subset \operatorname{Int}(A)$.

   Conversely, suppose that $x \in \operatorname{Int}(A)$. Then by definition of $\operatorname{Int}(A)$, it follows that x is contained in some open set contained in A. □

   (b) Let the boundary of A be $\operatorname{Bd}(A) = \overline{A} \cap \overline{(X - A)}$. Show that $x \in \operatorname{Bd}(A)$ if and only if every open set V with $x \in V$ contains points of both A and $X - A$.

   Proof. For $x \in X$, suppose that every open set V containing x contains points of both A and $X - A$. Then every open set containing x intersects nontrivially with A, so it follows that $x \in \overline{A}$; similarly every open set containing x intersects nontrivially with $X - A$ so that $x \in \overline{(X - A)}$. Hence $x \in \overline{A} \cap \overline{(X - A)} = \operatorname{Bd}(A)$.

   Conversely, suppose that $x \in \overline{A} \cap \overline{(X - A)} = \operatorname{Bd}(A)$. Then $x \in \overline{A}$ so that every open neighborhood of x intersects nontrivially with A; similarly $x \in \overline{(X - A)}$, from which we have that every open neighborhood of x intersects nontrivially with $X - A$. Then any neighborhood V of x intersects nontrivially with A and also intersects nontrivially with $X - A$ so that V contains points of both A and $X - A$. □

(c) Prove that $\mathrm{Bd}(A) \cap \mathrm{Int}(A) = \emptyset$ and that $\overline{A} = \mathrm{Int}(A) \cup \mathrm{Bd}(A)$.

Proof. Suppose $x \in \mathrm{Bd}(A) \cap \mathrm{Int}(A)$. Then *every* neighborhood of x contains points of $X - A$ since $x \in \mathrm{Bd}(A)$. This is in contradiction with the requirement that $x \in \mathrm{Int}(A)$, which stipulates the existence of a neighborhood of x completely contained in A. Therefore there cannot be any elements x in $\mathrm{Bd}(A) \cap \mathrm{Int}(A)$, meaning $\mathrm{Bd}(A) \cap \mathrm{Int}(A) = \emptyset$.

Suppose $x \in \overline{A}$. Then every neighborhood of x intersects A nontrivially; that is, for any open neighborhood V of x, V contains points of A. What remains is whether or not some V contains points of $X - A$ or not: If some V does not contain points of $X - A$, then V only contains points of A so that $V \subset A$ and so $x \in \mathrm{Int}(A)$. Otherwise *every* V contains both points of A and $X - A$ so that $x \in \mathrm{Bd}(A)$. Hence $x \in \mathrm{Int}(A) \cup \mathrm{Bd}(A)$.

Conversely, suppose that $x \in \mathrm{Int}(A) \cup \mathrm{Bd}(A)$, so that either $x \in \mathrm{Int}(A)$ or $x \in \mathrm{Bd}(A)$ (but not both). If $x \in \mathrm{Int}(A)$ then there exists a neighborhood of x contained in A, from which it follows that $x \in A$ and so every neighborhood containing x necessarily intersects nontrivially with A. In this case $x \in \overline{A}$. In the other case, $x \in \mathrm{Bd}(A)$ so that every neighborhood of x contains points in A as well as points in $X - A$; this is enough to see that every neighborhood of x intersects nontrivially with A so that $x \in \overline{A}$.

Hence $\overline{A} = \mathrm{Int}(A) \cup \mathrm{Bd}(A)$. □

3. Consider $\mathbb{Z}_+$ with the finite complement topology. Determine if the following sequences converge, and if so, to which point or points.

(a) $x_n = 2n + 3$ Converges to every number in the set $\mathbb{Z}_+$.

Proof. Every open set in $\mathbb{Z}_+$ is of the form $\mathbb{Z}_+ - A$ where A is a finite nonempty set of positive integers. To specify a neighborhood $\mathbb{Z}_+ - A$ of some integer m, demand that $m \notin A$.

Take any neighborhood $\mathbb{Z}_+ - A$ of $m \in \mathbb{Z}_+$ (so $m \notin A$). Because the positive integers are well-ordered and $x_{n+1} > x_n$, we can choose N large enough so that x_N is larger than the maximal element of A (one such choice for N is the maximal element of A). Then all but finitely many x_n is in any neighborhood of m for every $m \in \mathbb{Z}_+$. Hence x_n converges to every positive integer. □

(b) $x_n = 3 + (-1)^n$ Does not converge.

Proof. Take the neighborhood of any positive integer $m \neq 2, 4$ of the form $\mathbb{Z}_+ - A$ (with A being a finite nonempty set of positive integers) where $m \notin A$ and $2, 4 \in A$. This neighborhood does not contain x_n for every $n \in \mathbb{Z}_+$, so there is no way for this sequence to converge to m.

Then if $m = 2$ or $m = 4$ consider the neighborhood $\mathbb{Z}_+ - A$ with $m \notin A$ and 2 or 4 in A depending on whichever m is not equal to (so if $m = 2$, then $4 \in A$). This neighborhood does not contain all but finitely many x_n since we can choose n to be even or odd depending on if 2 or 4 is in A and find that an infinite number of elements x_n is not contained in the neighborhood. So in these cases the sequence also cannot converge.

Hence x_n does not converge. $\square$

4. Recall that two topological spaces X and Y are homeomorphic if and only if there is a homeomorphism $h\colon X \to Y$. Suppose that $\{X_\lambda \colon \lambda \in \Lambda\}$ and $\{Y_\lambda \colon \lambda \in \Lambda\}$ are indexed families of topological spaces with X_λ homeomorphic to Y_λ for each $\lambda \in \Lambda$. Prove that $\prod_{\lambda\in\Lambda} X_\lambda$ and $\prod_{\lambda\in\Lambda} Y_\lambda$ are homeomorphic. Use the product topology on the product spaces.

Proof. Let $f_\lambda \colon X_\lambda \to Y_\lambda$ be given homeomorphisms for each $\lambda \in \Lambda$. Then let $h\colon \prod_{\lambda\in\Lambda} X_\lambda \to \prod_{\lambda\in\Lambda} Y_\lambda$ be given by the formula

$$h((x_\lambda)_{\lambda\in\Lambda}) = (f_\lambda(x_\lambda))_{\lambda\in\Lambda};$$

that is, h is just f_λ for every coordinate. It is clear that h is a bijection since each f_λ is a bijection. Define h^{-1} in the natural way by the formula

$$h^{-1}((y_\lambda)_{\lambda\in\Lambda}) = (f_\lambda^{-1}(y_\lambda))_{\lambda\in\Lambda}.$$

We show that h and h^{-1} map open sets to open sets, by showing that they map basis elements to basis elements.

A basis element of $\prod_{\lambda\in\Lambda} X_\lambda$ with the product topology is a product of open sets $\prod_{\lambda\in\Lambda} U_\lambda$ where $U_\lambda = X_\lambda$ for all but finitely many $\lambda \in \Lambda$. Then

$$h\left(\prod_{\lambda\in\Lambda} U_\lambda\right) = (f_\lambda(U_\lambda))_{\lambda\in\Lambda},$$

and since each f_λ is a homeomorphism, it follows that each $f_\lambda(U_\lambda)$ is open (all but finitely many of them will be Y_λ) so that the resulting set is a product of open sets $\prod_{\lambda\in\Lambda} V_\lambda$ where all but finitely many $V\lambda$ are Y_λ. This set is a basis element of $\prod_{\lambda\in\Lambda} Y_\lambda$.

Any basis element of $\prod_{\lambda\in\Lambda}$ is a product of open sets $\prod_{\lambda\in\Lambda} V_\lambda$ where all but finitely many V_λ are Y_λ. We have

$$h^{-1}\left(\prod_{\lambda\in\Lambda} V_\lambda\right) = (f_\lambda^{-1}(V_\lambda))_{\lambda\in\Lambda}.$$

Since each f_λ^{-1} is also a homeomorphism, we have that each $f_\lambda^{-1}(V_\lambda)$ is open (all but finitely many of them will be X_λ), so that the resulting set is a product of open sets $\prod_{\lambda\in\Lambda} U_\lambda$. This set is a basis element of $\prod_{\lambda\in\Lambda} X_\lambda$.

Hence h is a homeomorphism as desired, so that $\prod_{\lambda\in\Lambda} X_\lambda$ and $\prod_{\lambda\in\Lambda} Y_\lambda$ are homeomorphic. $\square$

5. Assume that d and d' are metrics on X and that there are positive constants c_1, c_2 with

$$c_1 d(x,y) \le d'(x,y) \le c_2 d(x,y)$$

for all $x, y \in X$ Show that d and d' induce the same topology.

Proof. We show that the topologies induced by d and d' are mutually finer than each other.

For any $x \in X$ and $\varepsilon > 0$, consider the balls $B_d(x, \varepsilon)$ and $B_{d'}(x, c_1\varepsilon)$. If $y \in B_{d'}(x, c_1\varepsilon)$, then using the inequalities given we have $d(x, y) \leq d'(x, y)/c_1 < \varepsilon$, so that $y \in B_d(x, \varepsilon)$. Since $x \in X$ and $\varepsilon > 0$ were arbitrary, we have that $\mathcal{T}'$ is finer than $\mathcal{T}$.

Similarly, for any $x \in X$ and $\varepsilon > 0$, consider the balls $B_{d'}(x, \varepsilon)$ and $B_d(x, \varepsilon/c_2)$. If $y \in B_d(x, \varepsilon/c_2)$, then using the inequalities given we have $d'(x, y) \leq c_1 d(x, y) < \varepsilon$, so that $y \in B_{d'}(x, \varepsilon)$. Hence $\mathcal{T}$ is finer than $\mathcal{T}'$.

Therefore both metrics induce the same topologies. □

6. We showed in class that on $\mathbb{R}^{\mathbb{Z}_+}$ the box topology is finer than the uniform topology which in turn is finer than the product topology. Give examples that show that the box topology is *strictly* finer than the uniform topology which in turn is *strictly* finer than the product topology. You can use the fact that the product topology is induced by the metric D.

   An example of an open set in the box topology but not on the uniform topology on $\mathbb{R}^{\mathbb{Z}_+}$ is the product $\prod_{n \in \mathbb{Z}_+}(-1, 1)$. Since each factor $(-1, 1)$ is open in $\mathbb{R}$ with the standard topology, the product is open in the box topology on the product space. But on the uniform topology it is impossible to write this set as a union of open sets because some points in this set cannot lie in open neighborhoods from the uniform topology: Pick your favorite sequence x_n from $\mathbb{R}$ converging to 1 where each $x_n \in (-1, 1)$ and form the point $(x_n)_{n \in \mathbb{Z}_+} = (x_1, x_2, \dots)$ (e.g. the sequence $x_n = 1 - 1/n$). Note that in the $\overline{\rho}$ norm this point has distance 1 from zero because the distances $|x_n|$ are as close to 1 as desired, so that any open neighborhood containing 1 necessarily contains points outside of $\prod_{n \in \mathbb{Z}_+}(-1, 1)$. Any open neighborhood containing $(x_n)_{n \in \mathbb{Z}_+}$ contains an open ball of some radius $\varepsilon > 0$ containing $(x_n)_{n \in \mathbb{Z}_+}$, so there are points whose distance to the origin (the point $(0, 0, \dots)$) exceeds 1 in this ball and therefore cannot lie in $\prod_{n \in \mathbb{Z}_+}(-1, 1)$, as every point in this set has distance at most 1 in the $\overline{\rho}$ norm.

   An example of an open set in the uniform topology which is not open on the product topology on the product space is the open 1-ball given by $B_{\overline{\rho}}((0, 0, \dots), 1)$. If this set were open in the product topology then for ever point in the ball there is a basis element contained in the ball. But any basis element of the product topology is of the form $\prod_{n \in \mathbb{Z}_+} U_n$ with all but finitely many $U_n = \mathbb{R}$. We can at most have only finitely many U_n which contain points sufficiently close to the origin (in the $\overline{d}$-norm). All of the other $U_n = \mathbb{R}$ so that in the $\overline{\rho}$ norm we get 1, so that a basis element cannot be contained in the uniform 1-ball.

7. Give $X^{\mathbb{Z}_+}$ the product topology and let $\{\underline{x}_n\}$ be a sequence in $x^{\mathbb{Z}_+}$.

   (a) Show that $\underline{x}_n \to \underline{x}$ if and only if for each $i \in \mathbb{Z}_+$, $\pi_i(\underline{x}_n) \to \pi_i(\underline{x})$. In other words, a sequence converges if and only if all its components converge.

   Proof. If $\underline{x}_n$ converges to $\underline{x}$, then each neighborhood of $\underline{x}$ contains all but finitely many $\underline{x}_n$, but each neighborhood is of the form $\prod_{k \in \mathbb{Z}_+} U_k$ (a union of basis elements) where all but finitely many (finite including zero many) $U_k = X$. Each U_k must contain $\pi_k(\underline{x})$. Since $\prod_{k \in \mathbb{Z}_+} U_k$ contains all but finitely many $\underline{x}_n$, it follows that each U_k must contain all but finitely many $\pi_k(\underline{x}_n)$ (otherwise we reach a contradiction). Since the neighborhood chosen was arbitrary, it follows that $\pi_i(\underline{x}_n) \to \pi_i(\underline{x})$ for each $i \in \mathbb{Z}_+$.

Conversely, suppose that $\pi_i(\underline{x}_n) \to \pi_i(\underline{x})$ for each $i \in \mathbb{Z}_+$. Then open neighborhoods of $\underline{x}$ are in the form $\prod_{k\in\mathbb{Z}_+} U_k$ (a union of basis elements) where all but finitely many $U_k = X$. Without loss of generality, let $U_1, \dots, U_m$ be the finitely many sets U_n not equal to X (if they exist). For each $\pi_j(\underline{x})$, there is N_j such that for $n \geq N_j$, $\pi_j(\underline{x}_n) \in U_j$. So take N to be the maximum element of $\{N_j : 1 \leq j \leq m\}$, so that if $n \geq N$, $\pi_i(\underline{x}_n) \in U_i$ for every $i \in \mathbb{Z}_+$, so that $\underline{x}_n \in \prod_{k\in\mathbb{Z}_+} U_k$. Since $\prod_{k\in\mathbb{Z}_+} U_k$ was arbitrary, $\underline{x}_n \to \underline{x}$. □

(b) Is this result true when we give $X^{\mathbb{Z}_+}$ the box topology? No, because we had to take a maximum of a finite set of positive integers; in the box topology it is possible to start with an open set of $\underline{x}$ (in the same form as before) where every factor U_k is not equal to X. This would generate infinitely many N_k, and so it could be impossible to find a maximal element N of this set of positive integers N_k needed to ensure $\underline{x}_n$ is in the open set when $n \geq N$.

A concrete example might be to take the sequence in $\mathbb{R}^{\mathbb{Z}_+}$ given by

$$\begin{aligned} x_1 &= (0, 0, 0, \dots) \\ x_2 &= (1, 0, 0, 0, \dots) \\ &\vdots \\ x_k &= (\underbrace{1, 1, 1, \dots, 1}_{k \text{ many}}, 0, 0, 0, \dots). \end{aligned}$$

This sequence converges to $(1, 1, \dots)$ in the product topology, but in the box topology it is not possible to find N large enough to ensure that for $n \geq N$, x_n lies in the open set $\prod_{k\in\mathbb{Z}_+}(1/2, 3/2)$ containing $(1, 1, \dots)$.

8. Let (X, d) be a metric space.

(a) Show that $d \colon X \times X \to \mathbb{R}$ is continuous where $X \times X$ is given the product topology.

Proof. Let (a, b) be a nonempty (the preimage under d of the empty set is empty) open interval in $\mathbb{R}$ with the standard topology. Then the preimage under d of (a, b) is given by

$$d^{-1}((a, b)) = \{(x, y) \in X \times X : d(x, y) \in (a, b)\}.$$

Let (x, y) be any element of $d^{-1}((a, b))$, so that $d(x, y) = c$ with $c \in (a, b)$. Taking $\varepsilon = \min\{c - a, b - c\}$, we show that there exist open sets U and V containing x and y respectively such that $d(U \times V) \subset (c - \varepsilon, c + \varepsilon) \subset (a, b)$.

Since X is a metric space, pick $U = B_d(x, \varepsilon/4)$ and $V = B_d(y, \varepsilon/4)$. Then for any $x' \in U$ and $y' \in V$, we have by the triangle inequality that

$$d(x', y') \leq d(x', x) + d(x, y) + d(y, y') < c + \varepsilon/2 < b$$

and

$$a < c - \varepsilon/2 = d(x,y) - \varepsilon/4 - \varepsilon/4 \leq d(x',y') + (d(x',x) - \varepsilon/4)) + (d(y,y') - \varepsilon/4) < d(x',y'),$$

where in the last inequality we used the fact that both $(d(x',x) - \varepsilon/4)), (d(y,y') - \varepsilon/4) < 0$. It follows that $d(x',y') \in (a,b)$, so that $d(U \times V) \subset (a,b)$ as desired. Hence d is continuous. □

(b) If the sequences $x_n \to x$ and $y_n \to y$ converge in X show that the sequence of real numbers $d(x_n, y_n) \to d(x,y)$.

Proof. Let $x_n \to x$ and $y_n \to y$ be convergent sequences in X as given.

Since d is continuous, for every convergent sequence $\underline{x}_n \to \underline{x}$ of $X \times X$, the sequence $d(\underline{x}_n) \to d(\underline{x})$. All that is required to show is that the sequence $\underline{x}_n = (x_n, y_n)$ converges to $\underline{x} = (x,y)$. From a previous problem (note the box and product topologies agree for finite-product product spaces) we know that this must be true since the component sequences are convergent.

Hence $d(x_n, y_n) \to d(x,y)$. □

9. Given metric spaces (X_i, d_i) for $i = 1, \ldots, n$ show that

$$\rho(x,y) = \max\{d_1(x_1,y_1), \ldots, d_n(x_n,y_n)\}$$

is a metric on $\prod_{i=1}^n X_i$.

Proof. Let $x, y, z \in \prod_{i=1}^n X_i$ be arbitrary.

The maximum of a finite set of nonnegative numbers is a nonnegative number, so because each d_i maps into the nonnegative reals, ρ does also. If $x = y$, then every $d_i(x_i, y_i) = 0$ so that the maximum is 0, and the converse is true as well. Hence $\rho(x,y) \geq 0$ for all $x, y \in \prod_{i=1}^n X_i$.

We have

$$\rho(x,y) = \max\{d_i(x_i,y_i)\} = \max\{d_i(y_i,x_i)\} = \rho(y,x)$$

since each d_i is a metric.

Since each d_i is a metric, we have $d_i(x_i,y_i) \leq d_i(x_i,z_i) + d_i(z_i,y_i)$. Then

$$\max\{d_i(x_i,y_i)\} \leq \max\{d_i(x_i,z_i) + d_i(z_i,y_i)\} \leq \max\{d_i(x_i,z_i)\} + \max\{d_i(z_i,y_i)\},$$

where in the last equality we used the fact that the maximum of sums is less than or equal to the sum of maxima. It follows that $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$, which is the triangle inequality.

Hence ρ is a metric on $\prod_{i=1}^n X_i$. □