p.207: 1, 2(a), 4, 5, 6, 7

1. Suppose that R is a rotation in the plane $\mathbb{R}^2$, and let

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

denote its matrix with respect to the standard basis vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

(a) Write the conditions $R^t = R^{-1}$ and $\det(R) = \pm 1$ in terms of equations in a, b, c, d.

With $\det(R) = ad - bc = \pm 1$, we have

$$R^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = (ad - bc)^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = R^{-1}.$$

It follows that

$$\begin{aligned} a(ad - bc) &= d \\ c(ad - bc) &= -b \\ b(ad - bc) &= -c \\ d(ad - bc) &= a, \end{aligned}$$

with $\det(R) = ad - bc = \pm 1$ leading to two different cases (when R is proper or not).

(b) Show that there exists $\varphi \in \mathbb{R}$ such that $a + ib = e^{i\varphi}$.

Proof. Assume the conditions as stated in (a). Then observe that

$$|a + ib|^2 = a^2 + b^2 = a(a) + b(b) = ad \det(R) - bc \det(R) = \det(R)^2 = 1,$$

so that $|a + ib| = 1$. Hence $a + ib \in S^1$ and so take $\varphi = \operatorname{Arg}(a + ib)$. The principal argument $\operatorname{Arg}\colon \mathbb{C} \setminus \{0\} \to (-\pi, \pi]$ is given by taking $\arctan(b/a)$ whenever a is nonzero and adding to this plus or minus π when $a + ib$ lies in the second and third quadrant respectively. When $a + ib$ lies on the negative real axis take the argument as π, and when $a = 0$ take the principal argument to be $\operatorname{sgn}(b)\pi/2$. It follows immediately that $a + ib = \exp(i\varphi)$.

Thus there exists $\varphi \in \mathbb{R}$ such that $a + ib = \exp(i\varphi)$. □

(c) Conclude that if R is proper, then it can be expressed as $z \mapsto ze^{i\varphi}$, and if R is improper, then it takes the form $z \mapsto \overline{z}e^{i\varphi}$, where $\overline{z} = x - iy$.

Proof. Let $x = (x_1\ x_2)^t$. If R is proper, $\det(R) = ad - bc = 1$ so that $c = -b$ and $d = a$. Then

$$Rx = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ ax_2 - bx_1 \end{pmatrix}.$$

View $\mathbb{R}^2$ as $\mathbb{C}$ (by a set bijection), so that $x \mapsto x_1 + ix_2$ and $Rx \mapsto (ax_1 + bx_2) + i(ax_2 - bx_1)$. Observe that $(ax_1 + bx_2) + i(ax_2 - bx_1) = (a - ib)(x_1 + ix_2)$. In part (b) we saw that $a + ib \in S^1$ so $a - ib \in S^1$ as well, so write $a - ib = \exp(-i\varphi)$ for $\varphi \in \mathbb{R}$ as computed from part (a) (we may take $\varphi + 2\pi n$ for any integer n in place of φ as well). Then the action of R on x completely agrees with the action (by multiplication) of $\exp(-i\varphi)$ on $x_1 + ix_2$.

If R is not proper, $\det(R) = ad - bc = -1$ so that $c = b$ and $d = -a$. Then

$$Rx = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ bx_1 - ax_2 \end{pmatrix}.$$

Again we view this action in $\mathbb{C}$ as $Rx \mapsto (ax_1 + bx_2) + i(bx_1 - ax_2) = (a + ib)(x_1 - ix_2) = (a + ib)(\overline{x_1 + ix_2})$. Here the action of R on x agrees with the action of $\exp(i\varphi)$ on $\overline{x_1 + ix_2}$, and so the overall action is given by complex conjugation and then multiplication by $\exp(i\varphi)$.

We can define a group action where the group of orthogonal matrices with determinant -1 or 1 acts on $\mathbb{C}$ in exactly the manner outlined above, where if R is proper $z \mapsto z\exp(-i\varphi)$ and if R is improper $z \mapsto \overline{z}\exp(i\varphi)$. Determine φ from the components a, b in the matrix R in the manner outlined in part (b) up to an additive constant of $2\pi i$. $\square$

2. Suppose that $R \colon \mathbb{R}^3 \to \mathbb{R}^3$ is a proper rotation.

(a) Show that $p(t) = \det(R - tI)$ is a polynomial of degree 3, and prove that there exists $\gamma \in S^2$ (where S^2 denotes the unit sphere in $\mathbb{R}^3$) with

$$R(\gamma) = \gamma.$$

[Hint: Use the fact that $p(0) > 0$ to see that there is $\lambda > 0$ with $p(\lambda) = 0$. Then $R - \lambda I$ is singular, so its kernel is non-trivial.]

Proof. We know that the degree of the characteristic polynomial of an $n \times n$ matrix is n. Then write R and I in their matrix form (with respect to some basis for $\mathbb{R}^3$) and the difference $R - tI$ is then a 3×3 matrix, and by explicit computation of the determinant we have that it is of degree 3. (We have generally that a matrix of the form

$$R - tI = \begin{pmatrix} a - t & b & c \\ d & e - t & f \\ g & h & j - t \end{pmatrix}$$

has a determinant $p(t)$ which is a polynomial of degree 3.)

Then observe that $p(0) = \det(R) = 1$, so that there exists a real root λ (due to the range of cubic polynomials) λ of $p(t)$, so $p(\lambda) = 0$. This means that λ is an eigenvalue for R. Then $R - \lambda I$ is singular, so that the kernel of $R - \lambda I$ is non-trivial. Thus there exists $\gamma \in \mathbb{R}^3$ such that $(R - \lambda I)(\gamma) = R(\gamma) - \lambda\gamma = 0$, so that $R(\gamma) = \lambda\gamma$.

In fact, $\lambda > 0$ because the characteristic polynomial is of degree 3, note that by taking the determinant of the sample matrix given the coefficient of the t^3 term must be -1, so that $\lim_{t\to\infty} p(t) = -\infty$, so because $p(0) = 1$, we must have that $\lambda > 0$ for $p(\lambda) = 0$.

We claim that $\lambda = 1$. The rotation matrix R preserves the inner product and hence the norm. So $|R(\gamma)| = |\lambda\gamma| = \lambda|\gamma| = |\gamma| \implies \lambda = 1$. Hence γ and all of its scalar multiples are fixed by R, that is, the span of γ is the axis fixed by the rotation R. By scaling down $|\gamma|$ to 1 so that $\gamma \in S^2$, we have our desired γ. $\square$

4. Let A_d and V_d denote the area and volume of the unit sphere and unit ball in $\mathbb{R}^d$, respectively.

(a) Prove the formula

$$A_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

so that $A_2 = 2\pi$, $A_3 = 4\pi$, $A_4 = 2\pi^2, \ldots$. Here $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,\mathrm{d}t$ is the Gamma function. [Hint: use polar coordinates and the fact that $\int_{\mathbb{R}^d} e^{-\pi|x|^2}\,\mathrm{d}x = 1$.]

Proof. Directly computing with polar coordinates and making a change of variables, we have

$$\begin{aligned} 1 = \int_{\mathbb{R}^d} \exp\Big(-\pi|x|^2\Big)\,\mathrm{d}x &= \int_{S^{d-1}} \int_0^\infty \exp\Big(-\pi|r\gamma|^2\Big) r^{d-1}\,\mathrm{d}r\,\mathrm{d}\sigma(\gamma) \\ &= \int_0^\infty \exp\big(-\pi r^2\big)\,\big(\pi r^2\big)^{d/2-1}\,\frac{2\pi r}{2\pi^{d/2}}\,\mathrm{d}r \int_{S^{d-1}} \mathrm{d}\sigma(\gamma) \\ &= \frac{A_d}{2\pi^{d/2}} \int_0^\infty \exp(-t) t^{d/2-1}\,\mathrm{d}t \\ &= \frac{A_d\Gamma(d/2)}{2\pi^{d/2}}. \end{aligned}$$

Hence

$$A_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

as desired. $\square$

(b) Show that $\mathrm{d}V_d = A_d$, hence

$$V_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}.$$

In particular $V_2 = \pi$, $V_3 = 4\pi/3, \ldots$.

Proof. The volume of a ball of radius R is given by

$$\begin{aligned} V = \int_{B_R^d} 1\,\mathrm{d}x &= \int_{S^{d-1}} \int_0^R r^{d-1}\,\mathrm{d}r\,\mathrm{d}\sigma(\gamma) \\ &= A_d\left(\frac{r^d}{d}\bigg|_0^R\right) \\ &= \frac{A_d R^d}{d}. \end{aligned}$$

We have that $\mathrm{d}V$ is $A_d R^{d-1}$, which is the surface area of a sphere of radius R. The factor of R^{d-1} is due to the integration of $\int_{S^{d-1}(R)} \mathrm{d}\sigma(R\gamma)$ where $S^{d-1}(R)$ is the sphere of radius R. Expressing the integral as an iterated integral gives the extra R^{d-1} factor, and what remains is A_d. So when $R = 1$, we have $\mathrm{d}V_d = A_d$ as desired.

This means that $V_d = A_d/d = \pi^{d/2}/(d/2\Gamma(d/2))$, but by the definition of the Gamma function, $V_d = \pi^{d/2}/\Gamma(d/2+1)$. □

5. Let A be a $d \times d$ positive definite symmetric matrix with real coefficients. Show that

$$\int_{\mathbb{R}^d} e^{-\pi\langle x, A(x)\rangle}\,\mathrm{d}x = (\det(A))^{-1/2}.$$

This generalizes the fact that $\int_{\mathbb{R}^d} e^{-\pi|x|^2}\,\mathrm{d}x = 1$, which corresponds to the case where A is the identity. [Hint: Apply the spectral theorem to write $A = RDR^{-1}$ where R is a rotation and, D is a diagonal with entries $\lambda_1, \ldots, \lambda_d$, where $\{\lambda_i\}$ are the eigenvalues of A.]

Proof. Positive definite symmetric matrices with real coefficients are diagonalizable by the spectral theorem (their eigenvalues are real and positive). Thus apply the spectral theorem to write $A = RDR^{-1}$ where R is a rotation and D is a diagonal matrix with entries $\lambda_1, \ldots, \lambda_d > 0$, where $\{\lambda_i\}$ are the eigenvalues of A.

It follows that

$$\begin{aligned}
\int_{\mathbb{R}^d} e^{-\pi\langle x, A(x)\rangle}\,\mathrm{d}x &= \int_{\mathbb{R}^d} e^{-\pi\langle x, RDR^{-1}(x)\rangle}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} e^{-\pi x^t RDR^t x}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} e^{-\pi (R^t x)^t DR^t x}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} e^{-\pi y^t D y}|\det(R)|\,\mathrm{d}y \\
&= \int_R \cdots \int_R e^{-\pi(\lambda_1 y_1^2 + \cdots + \lambda_d y_d^2)}\,\mathrm{d}y_1 \cdots \mathrm{d}y_d \\
&= \prod_{i=1}^d \int_R e^{-\pi\lambda_i y_i^2}\,\mathrm{d}y_i \\
&= \prod_{i=1}^d \lambda_i^{-1/2} = (\lambda_1 \cdots \lambda_d)^{-1/2} = (\det(A))^{-1/2},
\end{aligned}$$

since the product of all of the eigenvalues of an operator is its determinant. □

6. Suppose $\psi \in \mathcal{S}(\mathbb{R}^d)$ satisfies $\int |\psi(x)|^2\,\mathrm{d}x = 1$. Show that

$$\left(\int_{\mathbb{R}^d} |x|^2 |\psi(x)|^2\,\mathrm{d}x\right)\left(\int_{\mathbb{R}^d} |\xi|^2 |\hat{\psi}(\xi)|^2\,\mathrm{d}\xi\right) \geq \frac{d^2}{16\pi^2}.$$

This is the statement of the Heisenberg uncertainty principle in d dimensions.

Proof. We have that

$$\begin{aligned}
d = \int_{\mathbb{R}^d} d|\psi(x)|^2 \,\mathrm{d}x &= \int_{\mathbb{R}^d} d\psi(x)\overline{\psi(x)} \,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \nabla(x)\psi(x)\overline{\psi(x)} \,\mathrm{d}x \,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \left(\sum_{i=1}^d \left(\frac{\partial}{\partial x_i} x_i \right) \psi(x)\overline{\psi(x)} \right) \mathrm{d}x \\
&= \sum_{i=1}^d \int_{\mathbb{R}^d} \left(\frac{\partial}{\partial x_i} x_i \right) \psi(x)\overline{\psi(x)} \,\mathrm{d}x \\
&= \sum_{i=1}^d \int_{\mathbb{R}^d} x_i \frac{\partial}{\partial x_i} \Big(\psi(x)\overline{\psi(x)} \Big) \,\mathrm{d}x \quad \text{integrate by parts, } \psi \in \mathcal{S}(\mathbb{R}^d).
\end{aligned}$$

By taking the absolute value,

$$\begin{aligned}
\left| \sum_{i=1}^d \int_{\mathbb{R}^d} x_i \frac{\partial}{\partial x_i} \Big(\psi(x)\overline{\psi(x)} \Big) \,\mathrm{d}x \right| &\leq 2 \int_{\mathbb{R}^d} \left(\sum_{i=1}^d |x_i||\psi(x)| \left| \frac{\partial}{\partial x_i} \psi \right| \right) \mathrm{d}x \\
&= 2 \int_{\mathbb{R}^d} |\psi(x)| \left[(|x_1| \cdots |x_d|) \begin{pmatrix} \left|\frac{\partial}{\partial x_1}\psi\right| \\ \vdots \\ \left|\frac{\partial}{\partial x_d}\psi\right| \end{pmatrix} \right] \mathrm{d}x \\
&\leq 2 \int_{\mathbb{R}^d} |x||\psi(x)||\nabla\psi(x)| \,\mathrm{d}x \quad \text{Cauchy-Schwarz in } \mathbb{R}^d \\
&\leq 2 \left(\int_{\mathbb{R}^d} |x|^2 |\psi(x)|^2 \,\mathrm{d}x \right)^{1/2} \left(\int_{\mathbb{R}^d} |\nabla\psi(x)|^2 \,\mathrm{d}x \right)^{1/2} \quad \text{Cauchy-Schwarz in } L^2(\mathbb{R}) \\
&= 2 \left(\int_{\mathbb{R}^d} |x|^2 |\psi(x)|^2 \,\mathrm{d}x \right)^{1/2} \left(\int_{\mathbb{R}^d} \sum_{i=1}^d \left| \frac{\partial}{\partial x_i} \psi(x) \right|^2 \mathrm{d}x \right)^{1/2} \\
&= 2 \left(\int_{\mathbb{R}^d} |x|^2 |\psi(x)|^2 \,\mathrm{d}x \right)^{1/2} \left(\int_{\mathbb{R}^d} \sum_{i=1}^d (2\pi)^2 |\xi_i|^2 |\hat{\psi}(\xi)|^2 \,\mathrm{d}x \right)^{1/2} \quad \text{Plancherel} \\
&= 4\pi \left(\int_{\mathbb{R}^d} |x|^2 |\psi(x)|^2 \,\mathrm{d}x \right)^{1/2} \left(\int_{\mathbb{R}^d} |\xi|^2 |\hat{\psi}(\xi)|^2 \,\mathrm{d}x \right)^{1/2}.
\end{aligned}$$

It follows that

$$\left(\int_{\mathbb{R}^d} |x|^2 |\psi(x)|^2 \,\mathrm{d}x \right) \left(\int_{\mathbb{R}^d} |\xi|^2 |\hat{\psi}(\xi)|^2 \,\mathrm{d}\xi \right) \geq \frac{d^2}{16\pi^2}$$

as desired. □

7. Consider the time-dependent heat equation in $\mathbb{R}^d$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_d^2}, \quad \text{where } t > 0, \tag{1}$$

with boundary values $u(x,0) = f(x) \in \mathcal{S}(\mathbb{R}^d)$. If

$$\mathcal{H}_t^{(d)}(x) = \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/4t} = \int_{\mathbb{R}^d} e^{-4\pi^2 t|\xi|^2} e^{2\pi i x\cdot\xi} \,\mathrm{d}\xi$$

is the d-dimensional **heat kernel**, show that the convolution

$$u(x,t) = (f * \mathcal{H}_t^{(d)})(x)$$

is indefinitely differentiable when $x \in \mathbb{R}^d$ and $t > 0$. Moreover, u solves (1), and is continuous up to the boundary $t = 0$ with $u(x,0) = f(x)$.

Proof. Let D be any differential operator in the form $D = \sum_i \frac{\partial}{\partial x}^{\alpha_i}$ where α_i are multi-indexes. Then since $f * g = g * f$, we have

$$\begin{aligned} D(f * \mathcal{H}_t^{(d)})(x) &= D \int_{\mathbb{R}^d} f(y) \mathcal{H}_t^{(d)}(x-y) \,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} f(y) D(\mathcal{H}_t^{(d)}(x-y)) \,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} D(f(x-y)) \mathcal{H}_t^{(d)}(y) \,\mathrm{d}y < \infty, \end{aligned}$$

since f is Schwartz. Hence $f * \mathcal{H}_t^{(d)}$ is indefinitely differentiable.

Then take $\triangle u$ from the integral definitions given:

$$\begin{aligned} \triangle u(x,t) &= \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} \int_{\mathbb{R}^d} f(y) \int_{\mathbb{R}^d} e^{-4\pi^2 t|\xi|^2} e^{2\pi i[\sum_{k=1}^d x_k\xi_k]} \,\mathrm{d}\xi \,\mathrm{d}y \\ &= \sum_{i=1}^d \int_{\mathbb{R}^d} f(y) \int_{\mathbb{R}^d} e^{-4\pi^2 t|\xi|^2} \frac{\partial^2}{\partial x_i^2} \left(e^{2\pi i[\sum_{k=1}^d x_k\xi_k]} \right) \mathrm{d}\xi \,\mathrm{d}y \\ &= \sum_{i=1}^d \int_{\mathbb{R}^d} f(y) \int_{\mathbb{R}^d} e^{-4\pi^2 t|\xi|^2} (-4\pi^2 |\xi_i|^2) e^{2\pi i[\sum_{k=1}^d x_k\xi_k]} \,\mathrm{d}\xi \,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} f(y) \int_{\mathbb{R}^d} (-4\pi^2|\xi|^2) e^{-4\pi^2 t|\xi|^2} e^{2\pi i[\sum_{k=1}^d x_k\xi_k]} \,\mathrm{d}\xi \,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} f(y) \int_{\mathbb{R}^d} \frac{\partial}{\partial t} \left(e^{-4\pi^2 t|\xi|^2} \right) e^{2\pi i[\sum_{k=1}^d x_k\xi_k]} \,\mathrm{d}\xi \,\mathrm{d}y = \frac{\partial}{\partial t} u(x,t), \end{aligned}$$

and hence u is a solution to the heat equation.

To show that the solution is continuous, observe that the convolution is continuous wherever $t > 0$. What remains is to show continuity when $t = 0$. To this end, recall that the heat kernel is a good kernel (Gaussian kernel). This means that $(f * \mathcal{H}_t^{(d)})(x) = \int_{\mathbb{R}^d} f(x-y)\mathcal{H}_t^{(d)}(y) \,\mathrm{d}y$ converges uniformly to $f(x)$ as $t \to 0$, so in this manner the limit as $t \to 0$ of $u(x,t)$ is $f(x)$. Thus the solution is continuous on its domain. $\square$