1. Prove that for any sets A and B,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Proof. For any x, the statement $x \in A \cup (B \cap C)$ is written using logical symbols as $x \in A \vee (x \in B \wedge x \in C)$. Observe that the truth table for this statement coincides with the truth table for the statement $(x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$, which is equivalent to saying $x \in (A \cup B) \cap (A \cup C)$:

$x \in A$	$x \in B$	$x \in C$	$x \in A \vee (x \in B \wedge x \in C)$	$(x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

Hence $x \in A \cup (B \cap C)$ is logically equivalent to $x \in (A \cup B) \cap (A \cup C)$, and because x was arbitrary the two sets $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$ are mutually included in each other and hence equal. □

2. Determine if the following statements are true or false.

(a) For $f\colon A \to B$ and $A_0, A_1 \subset A$,

$$f(A_0 \cup A_1) = f(A_0) \cup f(A_1).$$

TRUE

Proof. We will show that both sets are mutually included in each other. In the case that A_0 or A_1 is empty, the statement holds trivially, since the image of the empty set under f is the empty set. So suppose A_0, A_1 are nonempty.

Then let $x \in f(A_0 \cup A_1)$, so that there exists $a \in A_0 \cup A_1$ such that $f(a) = x$. Thus either $a \in A_0$ or $a \in A_1$ (or both), so that $x = f(a) \in f(A_0) \cup f(A_1)$, and since x was arbitrary, $f(A_0 \cup A_1) \subset f(A_0) \cup f(A_1)$.

Then let $x \in f(A_0) \cup f(A_1)$, so that $x \in f(A_0)$ or $x \in f(A_1)$ (or both). So either there exists $a_0 \in A_0$ such that $x = f(a_0)$ or there exists $a_1 \in A_1$ such that $x = f(a_1)$ (or both), and since $a_0, a_1 \in A_0 \cup A_1$, it follows that $x \in f(A_0 \cup A_1)$. Since x was arbitrary the reverse inclusion $f(A_0 \cup A_1) \supset f(A_0) \cup f(A_1)$ holds.

Hence $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$. □

(b) For any sets A and B,

$$A - (B - A) = A - B.$$

TRUE

Proof. The set $A-(B-A)$ is given by $\{s \mid s \in A \wedge s \notin (B-A)\}$. Decompose the statement $s \notin (B-A)$ into $\neg(s \in B \wedge s \notin A)$, which is equivalent to $s \notin B \vee s \in A$.

Then $s \in A \wedge s \notin (B-A)$ is logically equivalent to $s \in A \wedge (s \notin B \vee s \in A)$, which is equivalent to $(s \in A \wedge s \notin B) \vee (s \in A \wedge s \in A)$. This last statement is logically equivalent to $s \in A \wedge s \notin B$, and so the set $A-(B-A)$ can be written as $\{s \mid s \in A \wedge s \notin B\}$, which is exactly the definition of $A-B$.

Hence $A-(B-A) = A-B$. $\square$

(c) For any sets V_1, W_1, V_2, W_2,

$$(V_1 \times W_1) \cup (V_2 \times W_2) = (V_1 \cup V_2) \times (W_1 \cup W_2).$$

FALSE

Proof. When V_1, W_1, V_2, W_2 are nonempty, let $v_1 \in V_1$ and $w_2 \in W_2$ be given. Then the ordered pair (v_1, w_2) is an element of $(V_1 \cup V_2) \times (W_1 \cup W_2)$ but cannot be an element of $(V_1 \times W_1) \cup (V_2 \times W_2)$. In this case we have $(V_1 \times W_1) \cup (V_2 \times W_2) \subsetneq (V_1 \cup V_2) \times (W_1 \cup W_2)$. $\square$

(d) If $A \times B$ is finite then both A and B are finite. **FALSE**

Proof. Taking A to be an infinite set and B to be the empty set, the cartesian product $A \times B$ is also the empty set because there are no elements $b \in B$ which we could use to form an element $(a, b) \in A \times B$ for any element $a \in A$.

Hence $A \times B$ is finite while A is infinite. $\square$

(e) If A and B are nonempty and $A \times B$ is finite then both A and B are finite. **TRUE**

Proof. Let A, B be nonempty sets. Then we prove the contrapositive. To that end, suppose either A or B is an infinite set. Without loss of generality let A be an infinite set. Note that since A, B are nonempty, $A \times B$ is nonempty.

Choose one $b \in B$ (the set B is nonempty). Then there are an infinite number of elements of the form (a, b) for $a \in A$ in $A \times B$ since the number of options for a is not finite, as A is an infinite set.

Thus $A \times B$ is necessarily an infinite set. By the contrapositive, when A, B are nonempty, if $A \times B$ is finite then both A and B are finite. $\square$

3. Suppose that X is a set and $\{A_\alpha \colon \alpha \in I\}$ is an indexed family of sets (which contains at least one set). Prove that

$$X - \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (X - A_\alpha).$$

Proof. For any s, the statement $s \in X - \bigcup_{\alpha \in I} A_\alpha$ is equivalently written as $s \in X \wedge (s \notin \bigcup_{\alpha \in I} A_\alpha)$, which is equivalent to

$$s \in X \wedge \left[\bigwedge_{\alpha \in I} s \notin A_\alpha \right].$$

Distributing, we have that $\bigwedge_{\alpha\in I}(s \in X \wedge s \notin A_\alpha)$, which is equivalent to saying that $s \in \bigcap_{\alpha\in I}(X - A_\alpha)$. Hence $X - \bigcup_{\alpha\in I} A_\alpha = \bigcap_{\alpha\in I}(X - A_\alpha)$. □

4. Define a relation S on the set of real numbers $\mathbb{R}$ by

$$S = \{(a,b) \in \mathbb{R}\times\mathbb{R} \colon a - b \text{ is an integer}\}.$$

Prove that S is an equivalence relation on $\mathbb{R}$.

Proof. Reflexivity: Let $x \in \mathbb{R}$. Then $x - x = 0 \in \mathbb{Z}$ so that $(x,x) \in S$. Hence S is reflexive.

Symmetry: Let $x, y \in \mathbb{R}$, and suppose that $(x,y) \in S$; that is, $x-y \in \mathbb{Z}$. Then observe that $y-x = -(x-y)$, and since $x - y \in \mathbb{Z}$, then $-(x-y) = y - x \in \mathbb{Z}$ also since $\mathbb{Z}$ is closed under addition (and subtraction). Hence $(y,x) \in S$, and so S is symmetric.

Transitivity: Let $x, y, z \in \mathbb{Z}$ with $(x,y), (y,z) \in S$ so that $x - y, y - z \in \mathbb{Z}$. Then observe that $x - z = (x-y) + (y-z)$ and again by closure of addition in $\mathbb{Z}$, $x - z \in \mathbb{Z}$ so that $(x,z) \in \mathbb{Z}$. Hence S is transitive also.

It follows that S is an equivalence relation on $\mathbb{R}$. □

5. Suppose that $<_A$ is a strict linear order on A, and $<_B$ is a strict linear order on B. Prove that the dictionary order relation on $A \times B$ is a strict linear order on $A \times B$.

Proof. Let $(a_1,b_1), (a_2,b_2), (a_3,b_3)$ be arbitrary elements of $A \times B$, and endow $A \times B$ with the dictionary order $<$.

Comparability: Suppose that $(a_1,b_1) \neq (a_2,b_2)$. We show that either $(a_1,b_1) < (a_2,b_2)$ (so that either $a_1 <_A a_2$ or $a_1 = a_2 \wedge b_1 <_B b_2$), or $(a_2,b_2) < (a_1,b_1)$ (so that either $a_2 <_A a_1$ or $a_2 = a_1 \wedge b_2 <_B b_1$).

This yields three cases: $a_1 \neq a_2$, or $b_1 \neq b_2$, or both. In the cases where $a_1 \neq a_2$ or both $a_1 \neq a_2$ and $b_1 \neq b_2$, either $a_1 <_A a_2$ or $a_2 <_A a_1$ so that from the definition of the dictionary order on $A \times B$ given comparability holds. When $b_1 \neq b_2$ (and $a_1 = a_2$), again also either $b_1 <_B b_2$ or $b_2 <_B b_1$ so that comparability follows from the definition of the dictionary order on $A \times B$. Hence comparability holds in general for the dictionary order on $A \times B$.

Reflexivity never holds: For some (a_1,b_1) if we suppose by way of contradiction that $(a_1,b_1) < (a_1,b_1)$, then by the definition of the dictionary order on $A \times B$, either $a_1 <_A a_1$ or $a_1 = a_1 \wedge b_1 <_B b_1$, which is always in contradiction to the consequences of the total orders on A and B. Hence by contradiction reflexivity never holds for the dictionary order on $A \times B$.

Transitivity: Suppose $(a_1,b_1) < (a_2,b_2)$, so that $a_1 <_A a_2$ or $a_1 = a_2 \wedge b_1 <_B b_2$; suppose also that $(a_2,b_2) < (a_3,b_3)$, so that $a_2 <_A a_3$ or $a_2 = a_3 \wedge b_2 <_B b_3$. These statements yield four cases, each of which imply transitivity:

(1) Suppose that $a_1 <_A a_2$ and $a_2 <_A a_3$, so that due to the total ordering on A, $a_1 <_A a_3$. It follows that $(a_1, b_1) < (a_3, b_3)$.

(2) Suppose that $a_1 <_A a_2$ and $a_2 = a_3 \wedge b_2 <_B b_3$, which again from the total ordering on A imply that $a_1 <_A a_3$. It follows that $(a_1, b_1) < (a_3, b_3)$.

(3) Suppose that $a_1 = a_2 \wedge b_1 <_B b_2$ and $a_2 <_A a_3$, so that $a_1 <_A a_3$. Again $(a_1, b_1) < (a_3, b_3)$.

(4) Suppose that $a_1 = a_2 \wedge b_1 <_B b_2$ and $a_2 = a_3 \wedge b_2 <_B b_3$, so that by the total ordering on B, $b_1 <_B b_3$. It also follows that $a_1 = a_3$, and so it follows that $(a_1, b_1) < (a_3, b_3)$.

Thus the dictionary order on $A \times B$ is transitive.

Hence the dictionary order on $A \times B$ is a total ordering. □

6. Suppose that $<_A$ is a strict linear order on A, and $<_B$ is a strict linear order on B and there exists a bijection $\phi\colon A \to B$ so that

$$a <_A a' \text{ implies } \phi(a) <_B \phi(a').$$

Show that

$$\phi(a) <_B \phi(a') \text{ implies } a <_A a'.$$

Proof. We prove the contrapositive. Suppose that $a <_A a'$ is false; that is, either $a = a'$ or $a' <_A a$. Since ϕ is a bijection which preserves order in one direction, it follows that either $\phi(a) = \phi(a')$ or $\phi(a') <_B \phi(a)$. This is equivalent to saying that $\phi(a) <_B \phi(a')$ is false, so by the contrapositive the statement

$$\phi(a) <_B \phi(a') \text{ implies } a <_A a'$$

is true. □

7. Assuming that the set of real numbers $\mathbb{R}$ has the least upper bound property, show that it has the greatest lower bound property.

Proof. Suppose that $\mathbb{R}$ has the least upper bound property. Then take any nonempty subset A of $\mathbb{R}$ which is bounded below, and let $\alpha \in \mathbb{R}$ be one such lower bound. So the set of lower bounds of A, call it L, is a nonempty subset of $\mathbb{R}$ as it contains α; furthermore L is bounded above since A is nonempty and contains some $a \in \mathbb{R}$ such that for any lower bound $\gamma \in \mathbb{R}$ of A, $\gamma \leq a$.

Because $\mathbb{R}$ has the least upper bound property, the set L has a least upper bound in $\mathbb{R}$; without loss of generality let α be the least upper bound of L. We claim that α is the greatest lower bound of A.

Let $\varepsilon > 0$ be given. Then $\alpha - \varepsilon < \alpha$, and it follows that $\alpha - \varepsilon$ is not an upper bound of L and is also not in A since there exists some $\ell \in L$ such that $\alpha - \varepsilon < \ell \leq a$ for any $a \in A$. Then since $\alpha - \varepsilon < \ell \leq \alpha < \alpha + \varepsilon$ for any $\varepsilon > 0$, it follows that $\alpha \in L$ (ℓ depends on ε, and as ε is taken to be as small as we wish then ℓ must eventually take on the value of α, and ℓ is always an element of L). So α is indeed a lower bound for A.

Then α is also the greatest lower bound for A. For given $\varepsilon > 0$, $\alpha + \varepsilon$ is no longer a lower bound for A since α was the lowest upper bound for L; that is, $\alpha + \varepsilon \notin L$. Hence α is necessarily the greatest lower bound of A. □

8. Suppose that A and B are sets. Suppose that A is countable, and there is a surjective function $f\colon A \to B$. Prove that B is countable.

Proof. Because A is countable, it is either finite or countably infinite. In the case that A is the empty set, the image of A under f (which is B since f is surjective) is also the empty set so that B is the empty set, which is finite. In this case B is countable.

So let A be nonempty and countable, so that there exists a surjective map $g\colon \mathbb{Z}_+ \to A$. Then the composition $f \circ g\colon \mathbb{Z}_+ \to B$ is surjective since the composition of surjective maps is surjective, and so in all cases B is countable. □

9. Prove that the set of rational numbers is countable.

Proof. Since $\mathbb{Z} \times \mathbb{Z}$ is countable (as finite products of countable sets is countable, and $\mathbb{Z}$ is countable), there exists a surjection $g\colon \mathbb{Z}_+ \to \mathbb{Z} \times \mathbb{Z}$. Then take the surjection $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ given by $f((n, m)) = n/m$, and compose it with g to form the composite map $f \circ g\colon \mathbb{Z}_+ \to \mathbb{Q}$, which is surjective because the composition of two surjective functions is surjective. Hence $\mathbb{Q}$ is countable. □

10. A real number is said to be algebraic if and only if it satisfies some polynomial equation of positive degree of the form
$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0,$$
where each a_i is an integer. Assuming that a degree n polynomial has at most n distinct roots, prove that the set of all algebraic numbers is countable.

Proof. Since a countable union of countable sets is countable, we consider polynomials of degree n for each $n \in \mathbb{Z}_+$, and determine that the set of roots of polynomials of degree n is countable for each $n \in \mathbb{Z}_+$.

For any given positive integer n, we first determine the cardinality of the set of polynomials of degree n. A polynomial of degree n evaluated at $x \in \mathbb{R}$ is of the form
$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
where each $a_i \in \mathbb{Z}$ (though we may want to say that $a_n \neq 0$ since the polynomial is of degree n specifically). Polynomials of degree n are completely determined in the natural way by the $(n+1)$-tuples $(a_n, \ldots, a_0) \in \mathbb{Z}^{n+1}$, and because $\mathbb{Z}^{n+1}$ is a finite product of the countable set $\mathbb{Z}$, it is also countable. Thus there are countably many polynomials of degree n for each positive integer n.

Since every polynomial of degree n has at most n roots, it follows from the finiteness of n that the cardinality of the set of roots of polynomials of degree n is countable: Each polynomial of degree n corresponds to a finite set of at most n real numbers, and we can take the countable union of these finite sets (since there are countably many polynomials of degree n). Thus there are countably many real numbers which are roots of polynomials of degree n.

Then take the countable union over every positive integer n of each of these countable sets of real numbers which are roots of polynomials of degree n, and so it follows that the set of algebraic numbers is countable. $\square$