14.1 Show that

(a) Product of two orthogonal matrices is orthogonal.

Proof. Let A, B be two orthogonal matrices; that is, $A^T A = I$ and $B^T B = B$. Then $(AB)^T(AB) = B^T A^T AB = B^T IB = B^T B = I$ and the same is true for BA by symmetry. □

(b) Trace of a matrix remains invariant under a similarity transformation.

Proof. Let A, S be matrices and let S be invertible. Then $\mathrm{Tr}(S^{-1}AS) = \mathrm{Tr}(S^{-1}(AS)) = \mathrm{Tr}((AS)S^{-1}) = \mathrm{Tr}(A(SS^{-1})) = \mathrm{Tr}(A)$. □

(c) A Hermitian matrix remains Hermitian under unitary transformation.

Proof. Let H be a Hermitian matrix and U be a unitary matrix. Then $U^{-1}HU = U^\dagger HU$, so that $(U^\dagger HU)^\dagger = U^\dagger H^\dagger (U^\dagger)^\dagger = U^\dagger HU$. Hence H under a unitary transformation is still Hermitian. □

14.2 (a) Show that if $|v'\rangle = U|v\rangle$ where $|v\rangle$ is complex and $\langle v|v\rangle = \langle v'|v'\rangle$, then U must be unitary.

Proof. Let U, $|v\rangle$ be as given, with $|v'\rangle = U|v\rangle$. Suppose that $\langle v|v\rangle = \langle v'|v'\rangle$. Then by definition,

$$\begin{aligned}
\langle v|v\rangle = v_i^* v_i &= \langle v'|v'\rangle \\
&= (U_{ij}v_j)^*(U_{ik}v_k) \\
&= U_{ij}^* U_{ik} v_j v_k \\
&= U_{ji}^\dagger U_{ik} v_j v_k \\
&= (U^\dagger U)_{jk} v_j v_k.
\end{aligned}$$

So by enforcing the equality we must have that $(U^\dagger U)_{jk} = \delta_{jk}$, that is $U^\dagger U = I$. Hence U is unitary (its adjoint is its inverse). □

(b) Two matrices U and H are related by $U = e^{i\alpha H}$ where α is real and H is independent of α. Show that if H is Hermitian, then U is unitary.

Proof. Let U, H be given with $U = e^{i\alpha H}$ and H independent of α. For any positive integer power, $(H^n)^\dagger = (H^\dagger)^n$, as H commutes with itself. Then with the definition of the exponential function, the fact that the adjoint is a linear transformation, and the above fact, we find that $U^\dagger = (\exp(i\alpha H))^\dagger = \exp\big(-i\alpha H^\dagger\big)$:

$$U^\dagger = (\exp(i\alpha H))^\dagger = \left(\sum_{n=0}^{\infty} \frac{(i\alpha H)^n}{n!}\right)^\dagger = \sum_{n=0}^{\infty} \frac{(-i\alpha H^\dagger)^n}{n!} = \exp\Big(-i\alpha H^\dagger\Big)$$

But because H is Hermitian, $U^\dagger = \exp(-i\alpha H)$. Again using the fact that H commutes with itself, $U^\dagger U = \exp(i\alpha H)\exp(-i\alpha H) = \exp(0H) = I$. Hence U is unitary. □

14.3 Consider the matrix $L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(a) Evaluate L^2.

$$L^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I.$$

(b) Show that $e^{\theta L}$ can be written as a two-dimensional rotation matrix.

Using the fact that $L^2 = -I$, it follows that

$$\begin{aligned} \exp(\theta L) = \sum_{n=0}^{\infty} \frac{\theta^n L^n}{n!} &= L \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} + I \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} \\ &= \begin{pmatrix} 0 & -\sin(\theta) \\ \sin(\theta) & 0 \end{pmatrix} + \begin{pmatrix} \cos(\theta) & 0 \\ 0 & \cos(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \end{aligned}$$

which is a rotation of vectors through θ degrees counterclockwise. (This is consistent with $e^{i\theta} \cdot z$ rotating z through θ degrees counterclockwise in the complex plane; the algebra is the same due to an isomorphism where $i \mapsto L$.)