Let G be a group and let A be a nonempty set.

1. (DF4.1.1) Let G act on the set A. Prove that if $a, b \in A$ and $b = g \cdot a$ for some $g \in G$, then $G_b = gG_ag^{-1}$. Deduce that if G acts transitively on A then the kernel of the action is $\cap_{g\in G}\, gG_ag^{-1}$.

   Proof. Let G act on A with $b = g \cdot a$ for $a, b \in A$ for some $g \in G$. We also have that $g^{-1} \cdot b = g^{-1} \cdot (g \cdot a) = (g^{-1}g) \cdot a = 1 \cdot a = a$.

   With $gG_ag^{-1} = \{gxg^{-1} \mid x \in G_a\}$, observe that any element $gxg^{-1} \in gG_ag^{-1}$ satisfies

   $$(gxg^{-1}) \cdot b = (gx) \cdot (g^{-1} \cdot b) = g \cdot (x \cdot a) = g \cdot a = b,$$

   so that $gxg^{-1} \in G_b$. Hence $gG_ag^{-1} \subseteq G_b$.

   Similarly, observe that for any $y \in G_b$, we may find $ghg^{-1} \in gG_ag^{-1}$ such that $y = ghg^{-1}$. Choose $h = g^{-1}yg$, where indeed $h = g^{-1}yg \in G_a$ because

   $$h \cdot a = (g^{-1}yg) \cdot a = (g^{-1}y) \cdot b = g^{-1} \cdot b = a.$$

   Then $y \in gG_ag^{-1}$, and hence $G_b \subseteq gG_ag^{-1}$.

   If G acts transitively on A; that is, there is only one orbit and so for any $a, c \in A$, there is some $g \in G$ such that $a = g \cdot c$. We may obtain the kernel of this action by finding $\cap_{c\in A}\, G_c$, but because this action is transitive on A, we may use the above result to rewrite this set intersection.

   For $a, c \in A$, there exists $g \in G$ such that $G_c = gG_ag^{-1}$; as a result, if we fix a and let g take on every element in G, then the sets gG_ag^{-1} take on every G_c for $c \in A$. Hence $\cap_{c\in A}\, G_c = \cap_{g\in G}\, gG_ag^{-1}$, which is the kernel of the transitive action of G on A. □

2. (DF4.1.4) Let S_3 act on the set Ω of ordered pairs: $\{(i, j) \mid 1 \le i, j \le 3\}$ by $\sigma((i, j)) = (\sigma(i), \sigma(j))$. Find the orbits of S_3 on Ω. For each $\sigma \in S_3$ find the cycle decomposition of σ under this action (i.e., find its cycle decomposition when σ is considered as an element of S_9 — first fix a labelling of these nine ordered pairs). For each orbit $\mathcal{O}$ of S_3 acting on these nine points pick some $a \in \mathcal{O}$ and find the stabilizer of a in S_3.

   The orbit of S_3 containing $a \in \Omega$ takes on the form $\{\sigma(a) \mid \sigma \in S_3\}$, and we know that the group action will partition A into disjoint orbits of this form. We find the orbits by taking the six permutations of S_3 and applying them to $(1, 1)$ and $(1, 2)$; we need not try any others since after this point we find all of the elements in Ω. The following table exhibits this method:

σ	$\sigma((1,1))$	$\sigma((1,2))$
1	$(1,1)$	$(1,2)$
$(1\,2)$	$(2,2)$	$(2,1)$
$(2\,3)$	$(1,1)$	$(1,3)$
$(1\,3)$	$(3,3)$	$(3,2)$
$(1\,2\,3)$	$(2,2)$	$(2,3)$
$(1\,3\,2)$	$(3,3)$	$(3,1)$

So the two orbits that form are $\{(c,c) \mid 1 \leq c \leq 3\}$ (the first column) and $\{(a,b),(b,a) \mid a \neq b, 1 \leq a,b \leq 3\}$ (the second column). Notice they are disjoint and their union forms Ω as expected.

We use a suggestive notation to simplify forming the cycle decomposition of σ under this group action. Using the matrices below we can establish a labelling of the elements of Ω:

$$\begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \\ \mathbf{7} & \mathbf{8} & \mathbf{9} \end{pmatrix} = \begin{pmatrix} (1,1) & (1,2) & (1,3) \\ (2,1) & (2,2) & (2,3) \\ (3,1) & (3,2) & (3,3) \end{pmatrix}$$

Then by tracking how each element in S_3 permutes $\{\mathbf{1},\dots,\mathbf{9}\}$, we can find the following cycle decompositions (viewing them as elements of S_9):

σ	cycle decomposition for σ
1	1
$(1\,2)$	$(\mathbf{1}\,\mathbf{5})(\mathbf{2}\,\mathbf{4})(\mathbf{3}\,\mathbf{6})(\mathbf{7}\,\mathbf{8})(\mathbf{9})$
$(2\,3)$	$(\mathbf{2}\,\mathbf{3})(\mathbf{4}\,\mathbf{7})(\mathbf{5}\,\mathbf{9})(\mathbf{6}\,\mathbf{8})(\mathbf{1})$
$(1\,3)$	$(\mathbf{1}\,\mathbf{9})(\mathbf{2}\,\mathbf{8})(\mathbf{3}\,\mathbf{7})(\mathbf{4}\,\mathbf{6})(\mathbf{5})$
$(1\,2\,3)$	$(\mathbf{1}\,\mathbf{5}\,\mathbf{9})(\mathbf{2}\,\mathbf{6}\,\mathbf{7})(\mathbf{3}\,\mathbf{4}\,\mathbf{8})$
$(1\,3\,2)$	$(\mathbf{1}\,\mathbf{9}\,\mathbf{5})(\mathbf{2}\,\mathbf{7}\,\mathbf{6})(\mathbf{3}\,\mathbf{8}\,\mathbf{4})$

It is clear from these cycle decompositions that for $a \in \{(c,c) \mid 1 \leq c \leq 3\}$ (the first orbit), the stabilizer of a in S_3 is $S_{3a} = \{1, (x\,y) \mid x, y \neq a, 1 \leq x, y \leq 3\}$; for example, $(1\,2)((3,3)) = (3,3)$ since 3 is not found in the cycle $(1\,2)$. Then for $b \in \{(a,b),(b,a) \mid a \neq b, 1 \leq a,b \leq 3\}$ (the second orbit), the stabilizer of b in S_3 is $S_{3b} = \{1\}$, since the only 1-cycles present in any of the cycle decompositions above are those that fix elements from the first orbit.

3. (DF4.1.10) Let H and K be subgroups of the group G. For each $x \in G$ define the HK *double coset* of x in G to be the set
$$HxK = \{hxk \mid h \in H, k \in K\}.$$
   (a) Prove that HxK is the union of the left cosets $x_1K, \dots, x_nK$ where $\{x_1K, \dots, x_nK\}$ is the orbit containing xK of H acting by left multiplication on the set of left cosets of K.

   (b) Prove that HxK is a union of right cosets of H.

   (c) Show that HxK and HyK are either the same set or are disjoint for all $x, y \in G$. Show that the set of HK double cosets partitions G.

   (d) Prove that $|HxK| = |K| \cdot \left|H \colon H \cap xKx^{-1}\right|$.

   (e) Prove that $|HxK| = |H| \cdot \left|K \colon K \cap x^{-1}Hx\right|$.

4. Q4. Let G be a finite group and H a subgroup. Consider the partition of G into double cosets HgH as in problem 10.

(a) Prove that every left coset contained in a given double coset has nonempty intersection with every right coset contained in the same double coset.

(b) Deduce that if $n = |G : H|$ then there exist elements $g_1, \cdots g_n$ in G that belong to distinct left cosets and to distinct right cosets.

This means that G is the disjoint union of the Hg_i and also the disjoint union of the g_iH.