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Section 7.9

13. $w'' + w = \delta(t-\pi)$ for $w(0) = 0,\ w'(0) = 0$ Solve.

Take the Laplace transform of both sides to find:

$$s^2 W(s) + W(s) = e^{-\pi s} \to W(s) = \frac{1}{1+s^2} e^{-\pi s}$$

From the translation property $\mathcal{L}^{-1}[e^{-as}F(s)]](t) = f(t-a)u(t-a)$ to find a solution for $w(t)$:

$$w(t) = \sin(t-\pi)u(t-\pi)$$

31. Since the system has zero initial conditions it is convenient when computing the transfer function $H(s)$ and the impulse response function $h(t)$:

$$ay'' + by' + cy = \delta(t) \to \left(as^2 + bs + c\right) Y(s) = 1 \to \frac{Y(s)}{(1)} = H(s) = \frac{1}{as^2 + bs + c}$$

Give the roots of the polynomial $s^2 + \frac{b}{a}s + \frac{c}{a}$ as r_1 and r_2. Then proceed:

$$H(s) = \frac{1}{as^2 + bs + c} = \frac{1}{a(s-r_1)(s-r_2)} \to H(s) = \frac{1}{a}\left[\frac{\frac{1}{r_1 - r_2}}{s - r_1} - \frac{\frac{1}{r_1 - r_2}}{s - r_2}\right] \to h(t) = \frac{e^{r_1 t} - e^{r_2 t}}{a\left(r_1 - r_2\right)}$$

The impulse response function $h(t)$ involves exponentials which as $t \to \infty$ should converge, in order to bound $h(t)$ from above. This will only happen when $\mathrm{Re}(r_1) \leq 0$ and $\mathrm{Re}(r_2) \leq 0$, and so the linear system governed by $ay'' + by' + cy = \delta(t)$ is made stable.

Section 8.3

11. The point $x = 0$ is an ordinary point for the function $p(x) = x + 2$, so we may continue with our series expansion methods. Give $y(x)$ and $y'(x)$ as generic power series:

$$y(x) = \lim_{n\to\infty} \sum_{k=0}^{n} a_k x^k,\ y'(x) = \lim_{n\to\infty} \sum_{k=0}^{n} k a_k x^{k-1}$$

$$y' + (x+2)y = 0 \to \lim_{n\to\infty} \sum_{k=0}^{n} k a_k x^{k-1} + (x+2) \lim_{n\to\infty} \sum_{k=0}^{n} a_k x^k = 0$$

$$\to \lim_{n\to\infty} \sum_{k=0}^{n} k a_k x^{k-1} + \lim_{n\to\infty} \sum_{k=0}^{n} a_k x^{k+1} + \lim_{n\to\infty} \sum_{k=0}^{n} 2 a_k x^k = 0$$

$$\to \lim_{n\to\infty} \sum_{k=-1}^{n} \left(k+1\right) a_{k+1} x^k + \lim_{n\to\infty} \sum_{k=1}^{n} a_{k-1} x^k + \lim_{n\to\infty} \sum_{k=0}^{n} 2 a_k x^k = 0$$

$$\rightarrow \lim_{n\to\infty}\sum_{k=1}^{n}(k+1)\,a_{k+1}x^k + a_1 + \lim_{n\to\infty}\sum_{k=1}^{n}a_{k-1}x^k + \lim_{n\to\infty}\sum_{k=1}^{n}2a_kx^k + 2a_0 = 0$$

$$\rightarrow 2a_0 + a_1 + \lim_{n\to\infty}\sum_{k=1}^{n}\left((k+1)\,a_{k+1} + a_{k-1} + 2a_k\right)x^k = 0$$

Then the following recurrence relation is formed:

$$2a_0 + a_1 = 0,\ \ (k+1)\,a_{k+1} + a_{k-1} + 2a_k = 0$$

Generate four more terms in terms of a_0 by finding a general formula for a_{k+1}, then using the first value given by $-2a_0 = a_1$ to find subsequent terms.

$$a_{k+1} = \frac{-1}{k+1}\left(2a_k + a_{k-1}\right) \rightarrow a_2 = \frac{3a_0}{2},\ \ a_3 = \frac{-a_0}{3}$$

The first four terms of $y(x)$ as given by the generic series above in terms of a_0 are:

$$y(x) \approx \sum_{k=0}^{3}a_kx^k = a_0 - 2a_0x + \frac{3a_0}{2}x^2 - \frac{a_0}{3}x^3$$

17. The point $x = 0$ is an ordinary point for the function $p(x) = -x^2$ and $q(x) = 1$, so we may continue with our series expansion methods. Give $w(x)$, $w'(x)$, and $w''(x)$ as generic power series:

$$w(x) = \lim_{n\to\infty}\sum_{k=0}^{n}a_kx^k,\ \ w'(x) = \lim_{n\to\infty}\sum_{k=0}^{n}ka_kx^{k-1},\ \ w''(x) = \lim_{n\to\infty}\sum_{k=0}^{n}k(k-1)a_kx^{k-2}$$

$$w'' - x^2w' + w = 0 \rightarrow \lim_{n\to\infty}\sum_{k=0}^{n}k(k-1)a_kx^{k-2} - \lim_{n\to\infty}\sum_{k=0}^{n}ka_kx^{k+1} + \lim_{n\to\infty}\sum_{k=0}^{n}a_kx^k = 0$$

$$\rightarrow \lim_{n\to\infty}\sum_{k=1}^{n}(k+2)(k+1)a_{k+2}x^k + 2a_2 - \lim_{n\to\infty}\sum_{k=1}^{n}(k-1)a_{k-1}x^k + \lim_{n\to\infty}\sum_{k=1}^{n}a_kx^k + a_0 = 0$$

$$\rightarrow a_0 + 2a_2 + \lim_{n\to\infty}\sum_{k=1}^{n}\left((k+2)(k+1)a_{k+2} - (k-1)a_{k-1} + a_k\right)x^k = 0$$

Then the following recurrence relation is formed, which we use to form more terms (in terms of a_0 and a_1 since there are two initial conditions not given):

$$a_2 = -\frac{1}{2}a_0,\ \ (k+2)(k+1)a_{k+2} - (k-1)a_{k-1} + a_k = 0$$

$$\rightarrow a_{k+2} = \frac{(k-1)a_{k-1} - a_k}{(k+2)(k+1)} \rightarrow a_3 = \frac{-a_1}{6}$$

Then an approximation for $w(x)$ to four terms is given by:

$$w(x) \approx \sum_{k=0}^{3}a_kx^k = a_0 + a_1x - \frac{1}{2}a_0x^2 - \frac{1}{6}a_1x^3$$

19. The point $x = 0$ is an ordinary point for the function $p(x) = -2x$, so we may continue with our series expansion methods. Give $y(x)$ and $y'(x)$ as generic power series:

$$y(x) = \lim_{n\to\infty} \sum_{k=0}^{n} a_k x^k,\ y'(x) = \lim_{n\to\infty} \sum_{k=0}^{n} k a_k x^{k-1}$$

$$y' - 2xy = 0 \to \lim_{n\to\infty} \sum_{k=0}^{n} k a_k x^{k-1} - 2 \lim_{n\to\infty} \sum_{k=0}^{n} a_k x^{k+1} = 0$$

$$\to \lim_{n\to\infty} \sum_{k=1}^{n} (k+1) a_{k+1} x^k + a_1 - 2 \lim_{n\to\infty} \sum_{k=1}^{n} a_{k-1} x^k = 0$$

$$\to a_1 + \lim_{n\to\infty} \sum_{k=1}^{n} \left((k+1) a_{k+1} - 2a_{k-1}\right) x^k = 0$$

The following recurrence relation is found, with its general formula:

$$a_1 = 0,\ (k+1) a_{k+1} - 2a_{k-1} = 0 \to a_{k+1} = \frac{2a_{k-1}}{k+1} \to a_k = \frac{2a_{k-2}}{k}$$

Because $a_1 = 0$, any subsequent a_k for odd k are going to be 0. Then also notice that since the recurrence relation only works for even numbers, change the indexing variable into $k = 2c$ for natural numbers c. Then find that the recurrence relation can be written as $a_{2c} = \frac{a_{2(c-1)}}{c}$ (sort of suspicious notation, but ignoring the '2' as part the literal index of the sequence, treat these terms as adjacent since we iterate over integers c). Taking a_0 to be the first element of this sequence, we can find that the explicit formula for a_k can be $a_k = \frac{a_0}{c!}$ where $k = 2c$. Then we can rewrite the series (where we sum over c instead) for y as:

$$y(x) = \lim_{n\to\infty} \sum_{c=0}^{n} a_0 \frac{x^{2c}}{c!} = a_0 e^{x^2}$$