27.1 Using definition (27.4) for $\hat{A}$ and $\hat{A}^\dagger$, show that equations (27.5) and (27.6) follow.

Proof. Let $\hat{A} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\frac{\hat{p}_x}{\sqrt{2m\hbar\omega}}$ and $\hat{A}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - i\frac{\hat{p}_x}{\sqrt{2m\hbar\omega}}$ be as given. Then

$$\begin{aligned}\hat{A}^\dagger\hat{A} &= \left(\sqrt{\frac{m\omega}{2\hbar}}\hat{x} - i\frac{\hat{p}_x}{\sqrt{2m\hbar\omega}}\right)\left(\sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\frac{\hat{p}_x}{\sqrt{2m\hbar\omega}}\right)\\ &= \frac{m\omega}{2\hbar}\hat{x}^2 + \frac{i}{2\hbar}i\hbar + \frac{\hat{p}_x^2}{2m\hbar\omega}\\ &= \frac{1}{\hbar\omega}\left(\frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right) - \frac{1}{2} = \frac{\hat{H}}{\hbar\omega} - \frac{1}{2},\end{aligned}$$

and

$$\begin{aligned}\hat{A}\hat{A}^\dagger &= \left(\sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\frac{\hat{p}_x}{\sqrt{2m\hbar\omega}}\right)\left(\sqrt{\frac{m\omega}{2\hbar}}\hat{x} - i\frac{\hat{p}_x}{\sqrt{2m\hbar\omega}}\right)\\ &= \frac{m\omega}{2\hbar}\hat{x}^2 - \frac{i}{2\hbar}i\hbar + \frac{\hat{p}_x^2}{2m\hbar\omega}\\ &= \frac{1}{\hbar\omega}\left(\frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right) + \frac{1}{2} = \frac{\hat{H}}{\hbar\omega} + \frac{1}{2},\end{aligned}$$

so that

$$\left[\hat{A}, \hat{A}^\dagger\right] = \hat{A}\hat{A}^\dagger - \hat{A}^\dagger\hat{A} = \left(\frac{\hat{H}}{\hbar\omega} + \frac{1}{2}\right) - \left(\frac{\hat{H}}{\hbar\omega} - \frac{1}{2}\right) = 1,$$

which is equation (27.5) as desired. Then also

$$\frac{\hbar\omega}{2}\left(\hat{A}^\dagger\hat{A} + \hat{A}\hat{A}^\dagger\right) = \frac{\hbar\omega}{2}\left(\frac{\hat{H}}{\hbar\omega} - \frac{1}{2} + \frac{\hat{H}}{\hbar\omega} + \frac{1}{2}\right) = \hat{H}.$$

It follows that

$$\begin{aligned}\left[\hat{H}, \hat{A}\right] = \left[\frac{\hbar\omega}{2}\left(\hat{A}^\dagger\hat{A} + \hat{A}\hat{A}^\dagger\right), \hat{A}\right] &= \frac{\hbar\omega}{2}\left(\left[\hat{A}^\dagger\hat{A}, \hat{A}\right] + \left[\hat{A}\hat{A}^\dagger, \hat{A}\right]\right)\\ &= \frac{\hbar\omega}{2}\left(\hat{A}^\dagger\left[\hat{A}, \hat{A}\right] + \left[\hat{A}^\dagger, \hat{A}\right]\hat{A} + \hat{A}\left[\hat{A}^\dagger, \hat{A}\right] + \left[\hat{A}, \hat{A}\right]\hat{A}^\dagger\right)\\ &= \frac{\hbar\omega}{2}\left(-1\hat{A} + \hat{A}(-1)\right) = -\hbar\omega\hat{A},\end{aligned}$$

and

$$\begin{aligned}\left[\hat{H}, \hat{A}^\dagger\right] = \left[\frac{\hbar\omega}{2}\left(\hat{A}^\dagger\hat{A} + \hat{A}\hat{A}^\dagger\right), \hat{A}^\dagger\right] &= \frac{\hbar\omega}{2}\left(\left[\hat{A}^\dagger\hat{A}, \hat{A}^\dagger\right] + \left[\hat{A}\hat{A}^\dagger, \hat{A}^\dagger\right]\right)\\ &= \frac{\hbar\omega}{2}\left(\hat{A}^\dagger\left[\hat{A}, \hat{A}^\dagger\right] + \left[\hat{A}^\dagger, \hat{A}^\dagger\right]\hat{A} + \hat{A}\left[\hat{A}^\dagger, \hat{A}^\dagger\right] + \left[\hat{A}, \hat{A}^\dagger\right]\hat{A}^\dagger\right)\\ &= \frac{\hbar\omega}{2}\left(\hat{A}^\dagger 1 + 1\hat{A}^\dagger\right) = \hbar\omega\hat{A},\end{aligned}$$

which is equation (27.6) as desired. □

27.2 The raising and lowering operators of the quantum harmonic oscillator defined in lecture satisfy the equations

$$\hat{A}\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle; \quad \hat{A}^\dagger\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle.$$

(a) Show that the products of $\hat{A}$ and $\hat{A}^\dagger$ satisfy the equations

$$\hat{A}^\dagger\hat{A}\,|n\rangle = n\,|n\rangle\,; \quad \hat{A}\hat{A}^\dagger\,|n\rangle = (n+1)\,|n\rangle\,.$$

Proof. Directly computing, we have

$$\hat{A}^\dagger\hat{A}\,|n\rangle = \hat{A}^\dagger[(\sqrt{n}\,|n-1\rangle)] = \sqrt{n}[\sqrt{n}\,|n\rangle] = n\,|n\rangle$$

and

$$\hat{A}\hat{A}^\dagger\,|n\rangle = \hat{A}[\sqrt{n+1}\,|n+1\rangle] = \sqrt{n+1}[\sqrt{n+1}\,|n\rangle] = (n+1)\,|n\rangle$$

as expected. □

(b) Write the operator $\hat{x}$ in terms of the operators $\hat{A}$ and $\hat{A}^\dagger$.

Write

$$\hat{x} = \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}\left(\hat{A}+\hat{A}^\dagger\right).$$

From the definition of $\hat{A}$ and $\hat{A}^\dagger$, we have

$$\frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}\left(\hat{A}+\hat{A}^\dagger\right) = \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}\left(2\sqrt{\frac{m\omega}{2\hbar}}\hat{x}\right) = \hat{x}.$$

(c) The expectation value of the potential energy in the state $|n\rangle$ is given by

$$\langle V(\hat{x})\rangle = \langle n|\,\frac{1}{2}m\omega^2\hat{x}^2\,|n\rangle\,.$$

Use the result of (b) to rewrite $\hat{x}^2$ in terms of $\hat{A}$ and $\hat{A}^\dagger$. Use results from (a) to show that the expectation value $\langle V(\hat{x})\rangle = \frac{1}{2}E_n$ (half the total energy).

Proof. Find by squaring that

$$\hat{x}^2 = \left(\frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}\left(\hat{A}+\hat{A}^\dagger\right)\right)^2 = \frac{\hbar}{2m\omega}\left(\hat{A}^2+\hat{A}\hat{A}^\dagger+\hat{A}^\dagger\hat{A}+(\hat{A}^\dagger)^2\right).$$

Then

$$\begin{aligned}
\langle n|\,\frac{1}{2}m\omega^2\hat{x}^2\,|n\rangle &= \langle n|\,\frac{1}{2}m\omega^2\left(\frac{\hbar}{2m\omega}\left(\hat{A}^2+\hat{A}\hat{A}^\dagger+\hat{A}^\dagger\hat{A}+(\hat{A}^\dagger)^2\right)\right)|n\rangle \\
&= \langle n|\,\frac{\hbar\omega}{4}\left(\sqrt{n}\sqrt{n-1}\,|n-2\rangle+(n+1)\,|n\rangle+n\,|n\rangle+\sqrt{n+1}\sqrt{n+2}\,|n+2\rangle\right) \\
&= \frac{\hbar\omega}{4}(2n+1) = \frac{1}{2}\left[\left(n+\frac{1}{2}\right)\hbar\omega\right] = \frac{1}{2}E_n
\end{aligned}$$

as desired. □