Bureau
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The relation between the numbers M r ii(M)_r B j ii(B)_j

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The relation between the numbers M r ii(M)_r and B j ii(B)_j Contents

1. The relation between the numbers M_r and B_j

In our recent study [1] on alternate sums of powers of integers, i.e. of sums of the type

$$ {}_rZ_n = \sum_{j=1}^{n} (-1)^j j^r, \text{ with } r = 1, 2, \ldots, \tag{1}$$

it was shown that they can be expressed in the form

$$ {}_rZ_n = \sum_{j=1}^{r-1} {}_r\mu_j t^j + \frac{1}{2}(-1)^n (2t)^r, \tag{2}$$

where $t = \left[\frac{n+1}{2}\right]$ is given by the number of terms in Equation (1), while ${}_r\mu_j$ are coefficients that have been tabulated.

We found that these coefficients can be expressed by the "basic" integers $M_r \equiv {}_r\mu_1$, since

$$ {}_r\mu_j = \frac{2^{j-1} r!}{j!(r-j+1)!} M_{r-j+1}, \text{ for } r > j \tag{3}$$

The alternate sum Equation (1), written in powers of t, thus begins for r even with

$$ {}_rZ_n = M_r t + {}_r\mu_2 t^2 + \ldots, \tag{4}$$

whereas for r odd there is no term proportional to t.

It appears that the numbers M_r play a role similar to that of the well-known Bernoulli numbers B_j which occur in the study of the sums

$$ {}_rS_n \equiv \sum_{j=1}^{n} j^r. \tag{5}$$

Some similarities between the numbers B_j and M_r have already been noted in [1], but it has not been possible to establish a clear link between the two series. It is the purpose of the present Note to make up for this deficiency. Some familiarity with [1] will make the reading easier.

The idea is to use the fact that the numbers M_r are the coefficients of t, as shown in Equation (4). If we can subdivide ${}_rZ_n$ into a sum of contributions of type ${}_rS_n$ and determine the coefficients of the linear terms t (in which the Bernoulli numbers occur), it must be possible to obtain the required relation by comparison of the respective coefficients.

Let us begin by decomposing the alternate sum Equation (1) in a suitable way—the very rearrangement we tried to avoid in [1]. Since n is even, we can put $n/2 = t$. Then

$$ \begin{aligned} {}_rZ_n &= \sum_{j=1}^{t} (2j)^r - \sum_{j=1}^{t} (2j-1)^r \\ &= 2^r \sum_{j=1}^{t} j^r - 2^r \left(\frac{1}{2}\right)^r \sum_{j=1}^{t} \sum_{k=0}^{r} \binom{r}{k} (-2)^k j^k \\ &= 2^r {}_rS_n - \sum_{k=0}^{r} \binom{r}{k} (-2)^k {}_kS_n \end{aligned} \tag{6}$$

$$= -\sum_{k=0}^{r-1} \binom{r}{k}(-2)^k {}_kS_n.$$

Let us now look at the development of a Bernoullian sum ${}_rS_n$. From relation (23) given in [1] we conclude that, for $r \geq 2$ and even,

$$_rS_n = \,...\, + \frac{1}{r}\binom{r}{r-1}B_r t^{r+1-r} = \,...\, + B_r t. \tag{7}$$

We note in passing that ${}_rS_n$ with r odd has no terms proportional to t as the development stops at t^2, exactly as for ${}_rZ_n$.

The two cases with $r = 0$ and $r = 1$ appearing in Equation (6) have to be treated separately. One finds

$$_0S_n = t \tag{8}$$

and

$$_1S_n = \frac{1}{2}t + \frac{1}{2}t^2.$$

A look at Equation (4) shows that the value of M_r can be obtained from Equation (6) by assembling the coefficients of t appearing in ${}_kS_n$. Writing Equation (6) as

$$_rZ_n = -{}_0S_n + 2r{}_1S_n - \sum_{k=2}^{r-1} \binom{r}{k}(-2)^k {}_kS_n, \tag{9}$$

we find, with Equation (7) and Equation (8),

$$M_r = -1 + 2r\frac{1}{2} - \sum_{j=2}^{r-1} \binom{r}{j}(-2)^j B_j \tag{10}$$

$$= r - 1 - \sum_{j=2,\ (\text{even})}^{r-2} 2^j \binom{r}{j} B_j,$$

This is the relation looked for. It shows that the two series of numbers M_r and B_j are indeed closely linked, and relation Equation (10) is even somewhat reminiscent of the recurrence formula (18) found previously in [1]. It may be worthwhile noting that the sum in Equation (10) yields an even integer.

Let us check (10) with three practical applications.

- For = 8:

$$M_8 = 8 - 1 - \sum_{j=2}^{6} 2^j \binom{8}{j} B_j$$

$$= 7 - \left[2^2\binom{8}{2}B_2 + 2^4\binom{8}{4} + 2^6\binom{8}{8}B_6\right]$$

$$= 7 - \left[\frac{428}{6} - \frac{1670}{30} + \frac{6428}{42}\right] = -17;$$

- for $r = 10$:

$$M_{10} = 10 - 1 - \sum_{j=2}^{8} 2^j \binom{10}{j} B_j = \,...\, = 155;$$

- for $r = 12$:

$$M_{12} = 11 - \sum_{j=2}^{10} 2^j \binom{12}{j} B_j = \ \dots \ = -2\ 73$$

All these results agree with the numerical values given in [1].

Obviously, the existence of the new relation Equation (10) does not me an that the numbers M_r become superfluous; their practical usefulness is obvious in [1]. In any case, Equation (10) is a very useful tool for their numerical evaluation, since it is a simple relation making use only of the Bernoulli numbers, which are readily available in tabular form, for example up to B_{60} in [2].

References

[1] J.W. Müller: “Sums of alternate powers — an empirical approach”, Rapport BIPM-94/14 (1994)

[2] “Handbook of Mathematical Functions”, eg. .. by M. Abramowitz and I.A. Stegun, NBS, AMS 55 (GPO, Washington, 1964)

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