Exercise 4

Consider the frequency distribution for t_i

$$p(t_i \mid \sigma_{ti}, T) = \frac{1}{\sqrt{2\pi\sigma_{ti}^2}}\, e^{-\frac{(t_i - T)^2}{2\sigma_{ti}^2}}$$

Therefore, the likelyhood will be

$$\mathcal{L} = \prod_{i=1}^{N} p(t_i \mid \sigma_{ti}, T) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma_{ti}^2}}\, e^{-\frac{(t_i - T)^2}{2\sigma_{ti}^2}}$$

Taking the logarithm

$$\ln \mathcal{L} = \ln\left(\prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma_i^2}}\, e^{-\frac{(t_i - T)^2}{2\sigma_{ti}^2}}\right)$$

$$= \sum_{i=1}^{N} \ln\left[\frac{1}{\sqrt{2\pi\sigma_i^2}}\, e^{-\frac{(t_i - T)^2}{2\sigma_{ti}^2}}\right]$$

$$= \sum_{i=1}^{N} \left\{\ln\left(\frac{1}{\sqrt{2\pi\sigma_{ti}^2}}\right) + \ln\left(e^{-\frac{(t_i - T)^2}{2\sigma_{ti}^2}}\right)\right\}$$

$$= -\left\{\underbrace{\sum_{i=1}^{N} \ln\left(\sqrt{2\pi\sigma_{ti}^2}\right)}_{K.} + \frac{(t_i - T)^2}{2\sigma_{ti}^2}\right\}$$

Thus $$\ln \mathcal{L} = K - \sum_{i=1}^{N} \frac{(t_i - T)^2}{2\sigma_{ti}^2}$$

Taking the derivative with respect to T

$$\frac{d}{dT}(\ln \mathcal{L}) = -\sum_{i=1}^{N} \frac{-\cancel{2}(t_i - T)}{\cancel{2}\sigma_{ti}^2} = 0 \Rightarrow T = \sum_{i=1}^{N} \frac{t_i}{\sigma_{ti}^2} \Big/ \sum_{i=1}^{N} \frac{1}{\sigma_{ti}^2}$$

$$\frac{d^2}{dT^2}(\ln \mathcal{L}) = \sum_{i=1}^{N} \frac{-1}{\sigma_{ti}^2} < 0$$, therefore

T is a maximum, and its value is.

$$T = \frac{\sum_{i=1}^{N} \frac{t_i}{\sigma_{ti}^2}}{\sum_{i=1}^{N} \frac{1}{\sigma_{ti}^2}}$$ which corresponds to the weighted meen.

Exercise 5

From Eq. 7 we know that

$$\chi^2 = [Y - AX]^T C^{-1} [Y - AX]$$

We need to take the derivative with respect to x.
First, lets expand χ^2

$$\chi^2 = [Y^T - X^T A^T][C^{-1}Y - C^{-1}AX]$$

$$\chi^2 = Y^T C^{-1} Y - X^T A^T C^{-1} Y - Y^T C^{-1} AX + X^T A^T C^{-1} AX$$

Now,

$$\frac{d\chi^2}{dX} = 0 \Rightarrow -A^T C^{-1} Y - Y^T C^{-1} A + A^T C^{-1} AX + X^T A^T C^{-1} A = 0$$

$$(-A^T C^{-1} (Y - AX) - (Y - AX)^T C^{-1} A = 0$$

Solving for X and X^T separately

$$-A^T C^{-1} Y + A^T C^{-1} AX = 0$$

$$X = [A^T C^{-1} A]^{-1} [A^T C^{-1} Y]$$

$$-Y^T C^{-1} A + X^T A^T C^{-1} A = 0$$

$$X^T = (A^T C^{-1} A)^{-1} (Y^T C^{-1} A)$$

The results coincide with eq. (5)