Artificial Intelligence
Exercise Set 4

5p. 1. Consider two medical tests, A and B, for a virus. Test A is 95% effective at recognizing the virus when it is present, but has a 10% false positive rate (indicating that the virus is present, when it is not). Test B is 90% effective at recognizing the virus, but has a 5% false positive rate. The two tests use independent methods of identifying the virus. The virus is carried by 1% of all people. Say that a person is tested for the virus using only one of the tests, and that test comes back positive for carrying the virus. Which test returning positive is more indicative of someone really carrying the virus? **Tip**: Which posterior is larger:

$$P(CarryVirus|TestA) \text{ or } P(CarryVirus|TestB)?$$

$P(CarryVirus) = 0.01.$
$P(TestA|CarryVirus) = 0.95.$
$P(TestA|\neg CarryVirus) = 0.10.$
$P(TestB|CarryVirus) = 0.90.$
$P(TestB|\neg CarryVirus) = 0.05.$

$$P(CarryVirus|TestA) = \frac{P(TestA|CarryVirus)P(CarryVirus)}{P(TestA)} = \frac{0.95 \times 0.01}{0.1085} = 0.0875.$$

$P(TestA) =$
$= P(TestA|CarryVirus)\, P(CarryVirus)\, P(TestA|\neg CarryVirus)\, P(\neg CarryVirus)$
$= 0.95 \times 0.01 + 0.10 \times 0.99 = 0.0095 + 0.099 = 0.1085.$

Similar calculation for $P(CarryVirus|TestB)$ *yields* 0.1538.

Given the calculations above, B is more indicative of carrying the virus.

6p. 2. Given two **independent** Boolean variables A and B, determine x and y in the joint distribution of $\mathbf{P}(A;B)$ shown below:

$$P(A = T; B = T) = 0{,}15$$
$$P(A = T; B = F) = 0{,}45$$
$$P(A = F; B = T) = x$$
$$P(A = F; B = F) = y$$

	$B = T$	B = F
$A = T$	0.15	0.45
$A = F$	x	y

$\mathrm{P}(A = T) = 0.6$
$P(B = T) = 0.15 + x$
$P(A = T; B = T) = P(A = T)P(B = T)(A \text{ and } B \text{ are independent})$

$$0.6(0.15 + x) = 0.15 \implies x = \frac{0.15}{0.6} - 0.15 = 0.1$$
$y = 0.4 - x = 0.3$

3p 3. Prove the conditionalized version of the chain rule, *e* = fixed background evidence:

$$P(A, B|\, e) = P(A|B, e)\, P(B|e).$$

$$P(A, B|\, e) = \frac{P(A, B, e)}{P(e)} \qquad \text{definition of cond. prob.}$$

$$= P(A\,\big|\,B, e)P(B, e)/\,P(e) \qquad \text{chain rule}$$

$$= P(A|B, e)P(B|e). \qquad \text{definition of cond. prob.}$$

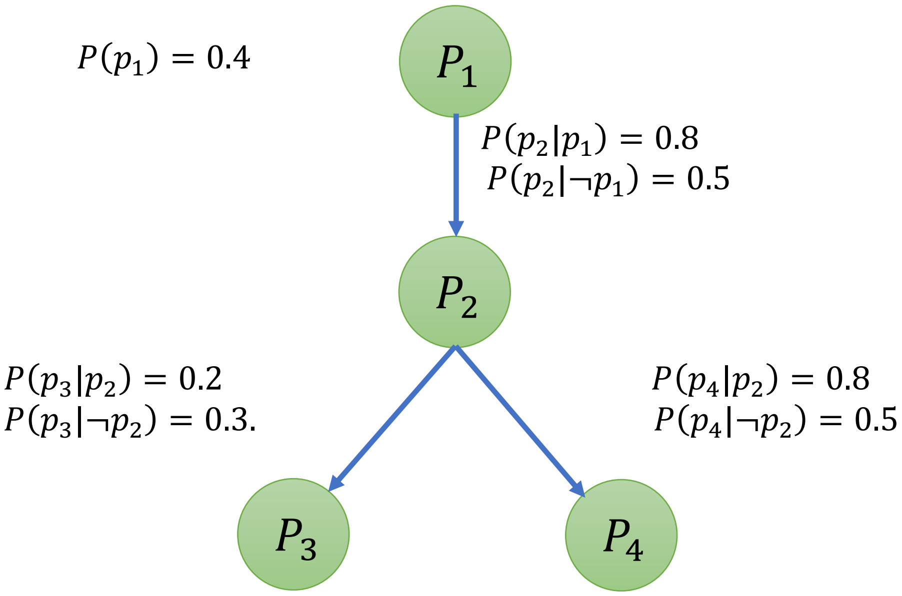
3p 4. Give a proof or a counterexample for the assertion: the independence of A and B implies the independence of $\neg A$ and B, **i.e.**

$$P(A|B) = P(A) \implies P(\neg A|B) = P(\neg A).$$

Given: $P(A|B) = P(A)$.

$P(\neg A|B) = 1 - P(A|B) = 1 - P(A) = P(\neg A)$.

13p 5. Given the network in the figure below, calculate the following probabilities using *inference by enumeration*. **Tip**: similar to the example in class.



2p a. $P(\neg p_3)$
3p b. $P(p_2|\neg p_3)$
4p c. $P(p_1|p_2, \neg p_3)$
4p d. $P\left(p_1|\neg p_{3,} p_4\right)$

a. 2p

$P(\neg p_3) = \sum_{P_1,P_2,P_4} P(P_1, P_2, \neg p_3, P_4) = \sum_{P_1,P_2,P_4} P(P_1)\, P(P_2|P_1)P(\neg p_3|P_2)P(P_4|P_2) =$

$= P(p_1)P(p_2|p_1)P(\neg p_3|p_2)P(p_4|p_2)$
$+ P(p_1)P(p_2|p_1)P(\neg p_3|p_2)P(\neg p_4|p_2)$
$+P(p_1)P(\neg p_2|p_1)P(\neg p_3|\neg p_2)P(p_4|\neg p_2)$
$+ P(p_1)P(p_2|p_1)P(\neg p_3|p_2)P(p_4|p_2)$
$+P(p_1)P(\neg p_2|p_1)P(\neg p_3|\neg p_2)P(\neg p_4|p_2)$
$+P(\neg p_1)P(p_2|\neg p_1)P(\neg p_3|p_2)P(p_4|p_2)$
$+P(\neg p_1)P(p_2|\neg p_1)P(\neg p_3|p_2)P(\neg p_4|p_2)$
$+ P(\neg p_1)P(\neg p_2|\neg p_1)P(\neg p_3|\neg p_2)P(p_4|\neg p_2)$
$+ P(\neg p_1)P(\neg p_2|\neg p_1)P(\neg p_3|\neg p_2)P(\neg p_4|\neg p_2) =$

$= 0{,}4 * 0{,}8 * 0{,}8 * 0{,}8 + 0{,}4 * 0{,}8 * 0{,}8 * 0{,}2 + 0{,}4 * 0{,}2 * 0{,}7 * 0{,}5 + 0{,}4 * 0{,}2 * 0{,}7 * 0{,}5 + 0{,}6 * 0{,}5 * 0{,}8 * 0{,}8 + 0{,}6 * 0{,}5 * 0{,}8 * 0{,}2 + 0{,}6 * 0{,}5 * 0{,}7 * 0{,}5 + 0{,}6 * 0{,}5 * 0{,}7 * 0{,}5 = 0.762.$

OBS!! One can ease your calculations in this case (for this net) by eliminating P_4, this was not covered in the course, but someone brought it up:

$\sum_{P_1,P_2,P_4} P(P_1)\, P(P_2|P_1)P(\neg p_3|P_2)P(P_4|P_2) =$
$\sum_{P_1,P_2} P(P_1)\, P(P_2|P_1)P(\neg p_3|P_2) \times \sum_{P_4} P(P_4|P_2) =$
$\sum_{P_1,P_2} P(P_1)\, P(P_2|P_1)P(\neg p_3|P_2) \times 1.$

Explanation below:

Inference by enumeration sums the joint probabilities of elementary events by definition. Inference by enumeration itself does not take into account the conditional independence of variables, recall the chain rule for Bayesian nets, in lecture 5. However, after you write the sum as you did on the previous page, you can expand the calculation and write those sums as above, and you should be getting the exact same result. There is also a variable elimination algorithm, and other tricks you can apply that we have not covered during this course, but the point of this exercise was to simply apply inference by enumeration as such.

b. 3p

$$P(p_2|\neg p_3) = \frac{P(p_2, \neg p_3)}{P(\neg p_3)} .$$

$$\begin{aligned} P(p_2, \neg p_3) &= \sum_{P_1,P_4} P(P_1, p_2, \neg p_3, P_4) = \sum_{P_1,P_4} P(P_1)P(p_2|P_1)P(\neg p_3|p_2)P(P_4|p_2) \\ &= P(p_1)P(p_2|p_1)P(\neg p_3|p_2)P(p_4|p_2) \\ &+ P(p_1)P(p_2|p_1)P(\neg p_3|p_2)P(\neg p_4|p_2) \\ &+ P(\neg p_1)P(p_2|\neg p_1)P(\neg p_3|p_2)P(p_4|p_2) \\ &+ P(\neg p_1)P(p_2|\neg p_1)P(\neg p_3|p_2)P(\neg p_4|p_2) \\ &= 0.4 * 0.8 * 0.8 * 0.8 + 0.4 * 0.8 * 0.8 * 0.2 + 0.6 * 0.5 * 0.8 * 0.8 \\ &+ 0.6 * 0.5 * 0.8 * 0.2 = 0.496. \end{aligned}$$

$$P(p_2|\neg p_3) = \frac{P(p_2, \neg p_3)}{P(\neg p_3)} = \frac{0.496}{0.762} = 0.650.$$

c. 4p

$$P(p_1|p_2, \neg p_3) = \frac{P(p_1, p_2, \neg p_3)}{P(p_2, \neg p_3)} .$$

$$P(p_1, p_2, \neg p_3) = \sum_{P_4} P(p_1, p_2, \neg p_3, P_4) = \sum_{P_4} P(p_1)P(p_2|p_1)P(\neg p_3|p_2)P(P_4|p_2)$$
$$= P(p_1)P(p_2|p_1)P(\neg p_3|p_2)P(p_4|p_2)$$
$$+ P(p_1)P(p_2|p_1)P(\neg p_3|p_2)P(\neg p_4|p_2)$$
$$= 0.4 * 0.8 * 0.8 * 0.8 + 0.4 * 0.8 * 0.8 * 0.2 = 0.256.$$

$$P(p_1|p_2, \neg p_3) = \frac{\mathrm{P}(p_1, p_2, \neg p_3)}{P(p_2, \neg p_3)} = \frac{0.256}{0.496} = 0.516.$$

d. 4p

$$P(p_1|\neg p_{3,} p_4) = \frac{P(p_1, \neg p_3, p_4)}{P(\neg p_3, p_4)}$$

$$P(p_1, \neg p_3, p_4) = \sum_{P_2} P(p_1, P_2, \neg p_3, p_4) = \sum_{P_2} P(p_1)P(P_2|p_1)P(\neg p_3|P_2)P(p_4|P_2) =$$
$$= P(p_1)P(p_2|p_1)P(\neg p_3|p_2)P(p_4|p_2) + P(p_1)P(\neg p_2|p_1)P(\neg p_3|\neg p_2)P(p_4|\neg p_2) =$$
$$= 0.4 * 0.8 * 0.8 * 0.8 + 0.4 * 0.2 * 0.7 * 0.5 = 0.232.$$

$$P(\neg p_3, p_4) = \textstyle\sum_{P_1,P_2} P(P_1, P_2, \neg p_3, p_4) = P(p_1, \neg p_3, p_4) + P(\neg p_1, \neg p_3, p_4)$$

$$P(\neg p_1, \neg p_3, p_4) = \sum_{P_2} P(\neg p_1, P_2, \neg p_3, p_4) =$$
$$= \sum_{P_2} P(\neg p_1)P(P_2|\neg p_1)P(\neg p_3|P_2)P(p_4|P_2) =$$
$$= P(\neg p_1)P(p_2|\neg p_1)P(\neg p_3|p_2)P(p_4|p_2) +$$
$$+ P(\neg p_1)P(\neg p_2|\neg p_1)P(\neg p_3|\neg p_2)P(p_4|\neg p_2) =$$

$$= 0.6 * 0.5 * 0.8 * 0.8 + 0.6 * 0.5 * 0.7 * 0.5 = 0.297.$$

$$P(p_1|\neg p_{3,} p_4) = \frac{P(p_1, \neg p_3, p_4)}{P(\neg p_3, p_4)} = \frac{0.232}{0.232 + 0.297} = \frac{0.232}{0.529} = 0.438.$$