Artificial Intelligence - Solutions
Exercise Set 3

9p 1. Which of the following are correct?

a. False $\vDash$ True. TRUE – False has no models so it entails any sentence, and True is true in all models so it entails every sentence.
b. True $\vDash$ False. FALSE
c. $(A \wedge B) \vDash (A \Leftrightarrow B)$. TRUE – left-hand side has one model; right-hand side has 2 models, one of which is the model on the left-hand side.
d. $A \Leftrightarrow B \vDash A \vee B$. FALSE – left-hand side has one model with both A and B false, which is not satisfying the right-hand side.
e. $A \Leftrightarrow B \vDash \neg A \vee B$. TRUE – right-hand side is essentially $A \Rightarrow B$ which is one side of the left-hand side.
f. $(A \wedge B) \Rightarrow C \vDash (A \Rightarrow C) \vee (B \Rightarrow C)$. TRUE – right-hand side is false only when both $(A \Rightarrow C)$ and $(B \Rightarrow C)$ are false, i.e. when A and B are true and C is false, which makes left-hand side false.
g. $(C \vee (\neg A \wedge \neg B)) \equiv ((A \Rightarrow C) \wedge (B \Rightarrow C))$. TRUE – apply distributivity.
h. $(A \vee B) \wedge (\neg C \vee \neg D \vee E) \vDash (A \vee B)$. TRUE – removing a conjunct allows for more models.
i. $(A \vee B) \wedge (\neg C \vee \neg D \vee E) \vDash (A \vee B) \wedge (\neg D \vee E)$. FALSE – removing a disjunct allows for fewer models.

3p 2. Either prove or find a counterexample to each of the following assertions:

a. If $\alpha \vDash \gamma$ or $\beta \vDash \gamma$ (or both) then $(\alpha \wedge \beta) \vDash \gamma$. TRUE

   If more info is added to the knowledge base, then the set of entailed sentences can only increase. This property is also called monotonicity.

b. If $\alpha \vDash (\beta \wedge \gamma)$ then $\alpha \vDash \beta$ and $\alpha \vDash \gamma$. TRUE

   If $(\beta \wedge \gamma)$ is true in every model of α then β and γ are also true in every model of α.

c. If $\alpha \vDash (\beta \vee \gamma)$ then $\alpha \vDash \beta$ or $\alpha \vDash \gamma$ (or both). FALSE

   For β = A and $\gamma = \neg A$, it doesn't hold.

3p 3. According to some political pundits, a person who is radical (R) is electable (E) if they are conservative (C), but otherwise is not electable. Which of the following are correct representations of this assertion? Explain why the representation is correct or why it is not.

a. $(R \wedge E) \Leftrightarrow C$
b. $R \Rightarrow (E \Leftrightarrow C)$
c. $R \Rightarrow ((C \Rightarrow E) \vee \neg E)$

a. No. First sentence asserts, that all conservatives are radical.

b. Yes. If a person is a radical, then they are electable if and only if they are conservative.

c. No. This is ALWAYS true. A statement that is always true is also know as a tautology.

5p 4. For each of the following statements, either prove it is true or otherwise give a counterexample.

a. $P(a|b \wedge a) = 1$. 2p
b. If $P(a|b, c) = P(b|a, c)$, then $P(a|c) = P(b|c)$. 1p
c. If $P(a|b,c) = P(a)$, then $P(b|c) = P(b)$. 1p
d. If $P(a|b) = P(a)$, then $P(a|b, c) = P(a|c)$. 1p

a. True.

$P(a|b \wedge a) = \frac{P(a \wedge (b \wedge a))}{P(b \wedge a)} = \frac{P(b \wedge a)}{P(b \wedge a)} = 1.$

b. True.

$P(a, b, c) = P(a|b, c)\,P(b,c) = P(b|a, c)\,P(a,c)$ by the product rule $\xRightarrow{P(a|b,c) = P(b|a,c)}$ $P(b,c) = P(a,c) \Rightarrow P(b|c)P(c) = P(a,c)P(c) \Rightarrow P(b|c) = P(a,c).$

c. False

$P(a|b,c) = P(a)$ states that a is independent of b and c, it makes no claim regarding the dependence of b and c.

E.g. counter-example: a and b are the results of two independent coin flips, and c = b.

e. False

$P(a|b) = P(a)$ states that a is independent of b, nothing regarding the conditionality of a and b and c.

E.g. counter-example: a and b are the results of two independent coin flips, and c equals the XOR of a and b.

4p 5. Consider the full joint distribution in Table 1, calculate the following:

	$tootache$		$\neg toothache$	
	$change$	$\neg change$	$change$	$\neg change$
$cavity$	0.108	0.012	0.072	0.008
$\neg cavity$	0.016	0.064	0.144	0.576

Table 1. Full joint distribution of three Boolean variables: $\mathrm{Toothache, \ Cavity, \ ChangeOnXRay.}$

1p a. $\mathbf{P}(Cavity) =< 0.2, 0.8 >.$

1p b. $\mathbf{P}(Toothache \mid cavity) = < \frac{0.108 + 0.012}{0.2}, \frac{0.072 + 0.008}{0.2} > = < 0.6, 0.4 >.$

2p c. $\mathbf{P}(Cavity \mid toothache \lor change).$

$P(toothache \lor change) = 0.108 + 0.012 + 0.016 + 0.064 + 0.072 + 0.144 = 0.416.$

$\mathbf{P}(Cavity \mid toothache \lor change) =$
$= < \frac{0.108 + 0.012 + 0.072}{0.416}, \frac{0.016 + 0.064 + 0.144}{0.416} > = < 0.4615, 0.5384 >$

6p 6. After Mr. Willoughby's yearly check-up, the doctor has some bad news: Mr. Willoughby tested positive for a rare disease and the test is 99% accurate (i.e., the probability of testing positive when one has the disease is 0.99, as is the probability of testing negative when one doesn't). The good news is that this disease affects only 1 in 10000 people of his age. Why is it good news that the disease is rare?

What are the chances Mr. Willoughby actually has the disease? Calculate.

Given:
$P(test|disease) = 0.99.$
$P(\neg test|\neg disease) = 0.99.$
$P(disease) = 0.0001$

Bayes: $P(disease|test) = \frac{P(test|disease)P(disease)}{P(test)} = \frac{0.000099}{0{,}010098} = 0.0098039.$

$P(test) = P(test|disease)P(disease) + P(test|\neg disease)P(\neg disease) = 0.99 \times 0.0001 + 0.01 \times 0.9999 = 0.000099 + 0.009999.$

That means the disease is much rarer than the test accuracy is. A positive test result does not mean the disease is likely.