A Technique for Orthogonal Frequency Division Multiplexing Frequency Offset Correction

Paul H. Moose, *Member, IEEE*

Abstract—This paper discusses the effects of frequency offset on the performance of orthogonal frequency division multiplexing (OFDM) digital communications. The main problem with frequency offset is that it introduces interference among the multiplicity of carriers in the OFDM signal. It is shown, and confirmed by simulation, that to maintain signal-to-interference ratios of 20 dB or greater for the OFDM carriers, offset is limited to 4% or less of the intercarrier spacing. Next, the paper describes a technique to estimate frequency offset using a repeated data symbol. A maximum likelihood estimation (MLE) algorithm is derived and its performance computed and compared with simulation results. Since the intercarrier interference energy and signal energy both contribute coherently to the estimate, the algorithm generates extremely accurate estimates even when the offset is far too great to demodulate the data values. Also, the estimation error depends only on total symbol energy so it is insensitive to channel spreading and frequency selective fading. A strategy is described for initial acquisition in the event of uncertainty in the initial offset that exceeds 1/2 the carrier spacing, the limit of the MLE algorithm.

I. INTRODUCTION

THE TECHNIQUE described in this paper has been developed to correct frequency offset errors in digital communications systems employing orthogonal frequency division multiplexing (OFDM) as the method of modulation. The aim of the paper is twofold; to show the effect offset errors have on the signal-to-noise ratio of the OFDM carriers and to present an algorithm to estimate offset so that it may be removed prior to demodulation.

OFDM is a bandwidth efficient signalling scheme for digital communications that was first proposed by Chang [1]. The main difference between frequency division multiplexing (FDM) and OFDM, is that in OFDM the spectrum of the individual carriers mutually overlap, giving therefore an optimum spectrum efficiency (asymptotically Q b/Hz for 2^Q-ary modulation of each carrier). Nevertheless, the OFDM carriers exhibit orthogonality on a symbol interval if synthesized such that they are spaced in frequency exactly at the reciprocal of the symbol interval. Fortunately, this synthesis can be accomplished perfectly, in principle, utilizing the discrete Fourier transform (dft) as first described by Darlington [2] and later, for data modems, by Weinstein and Ebert [3]. With the recent evolution of integrated circuit digital signal processing (dsp) chips, OFDM has become practical to implement and

is being suggested as an efficient modulation for applications ranging from modems [4], to digital audio broadcast [5].

One of the principal advantages of OFDM is its utility for transmission at very nearly optimum performance in unequalized channels and in multipath channels. As described in [3]–[5], intersymbol interference (ISI) and intercarrier interference (ICI) can be entirely eliminated by the simple expedient of inserting between symbols a small time interval known as a guard interval. The length of the guard interval is made equal to or greater than the time spread of the channel. If, into this guard interval, the symbol signal waveform is extended periodically, orthogonality of the carriers is maintained over the symbol period, thus eliminating ICI. Also, since successive symbols do not overlap because of the guard interval, ISI is eliminated, too.

One of the principal disadvantages of OFDM is sensitivity to frequency offset in the channel. For example, the coded OFDM system developed by CCETT (Centre Commun. d'Etudes de Telediffusion et Telecommuications) for digital sound broadcasting to mobile receivers incorporates an AFC (automatic frequency control) loop in the receiver to reduce frequency offset caused by tuning oscillator inaccuracies and doppler shift [6].

There are two deleterious effects caused by frequency offset; one is the reduction of signal amplitude in the output of the filters matched to each of the carriers and the second is introduction of ICI from the other carriers which are now no longer orthogonal to the filter. Because, in OFDM, the carriers are inherently closely spaced in frequency compared to the channel bandwidth, the tolerable frequency offset becomes a very small fraction of the channel bandwidth. Maintaining sufficient open loop frequency accuracy can become difficult in links, such as satellite links with multiple frequency translations or, as mentioned previously, in mobile digital radio links that may also introduce significant Doppler shift. The effects of frequency offset are presented in Section II.

In Section III, we present an algorithm to estimate frequency offset from the demodulated data signals in the receiver. The algorithm extends to OFDM, with important differences, a method described by Simon and Divsalar [7] for single carrier MPSK. The technique involves repetition of a data symbol and comparison of the phases of each of the carriers between the successive symbols. Since the modulation phase values are not changed, the phase shift of each of the carriers between successive repeated symbols is due to the frequency offset. The frequency offset is estimated using a maximum likelihood estimate (MLE) algorithm. Performance of the algorithm as

a function E_c/N_o (individual OFDM carrier energy to one-sided spectral density of additive white Gaussian noise) and frequency offset is included in Section III.

In the event that the frequency offset exceeds $\pm 1/2$ the intercarrier spacing, the maximum limits of the algorithm, a strategy is required for initial acquisition. One such strategy is described in Section IV.

II. OFDM TRANSLATION IN A CHANNEL WITH FREQUENCY OFFSET

An OFDM transmission symbol is given by the N point complex modulation sequence

$$x_n = (1/N) \sum_{k=-K}^{K} X_k e^{2\pi jnk/N};$$
$$n = 0, 1, 2, \ldots N-1; \quad N \geq 2K+1. \quad (1)$$

It consists of $2K+1$ complex sinusoids which have been modulated with $2K+1$ complex modulation values $\{X_k\}$. We note that the individual sinusoids are orthogonal on the symbol interval, that is

$$\sum_{n=0}^{N-1} x_{nk} x_{nl}^* = (1/N)|X_k|^2 \delta_{kl} \quad (2)$$

where $x_{nk} = (1/N) X_k e^{2\pi jnk/N}$.

We also note that the N point discrete Fourier transform (dft) of (1) is the N point sequence

$$\text{DFT}_N\{x_n\}$$
$$= \left\{ \sum_{n=0}^{N-1} x_n e^{-2\pi jnk/N} \right\}$$
$$= \{X_0, X_1, \ldots X_K, 0, 0, \ldots 0, 0, X_{-K} \ldots X_{-2}, X_{-1}\} \quad (3)$$

of modulation values. Equation (1) is the inverse discrete Fourier transform (IDFT) of (3) and defines a practical modulation-carrier synthesis technique for generating OFDM with perfect orthogonality.

After passing through a bandpass channel, the complex envelope of the received sequence can be expressed as

$$y_n = (1/N) \left[\sum_{k=-K}^{K} X_k H_k e^{2\pi jn(k+\epsilon)/N} \right] + w_n;$$
$$n = 0, 1, 2, \ldots, N-1 \quad (4)$$

where H_k is the transfer function of the channel at the frequency of the kth carrier, ϵ is the relative frequency offset of the channel (the ratio of the actual frequency offset to the intercarrier spacing), and w_n is the complex envelope of additive white Gaussian noise (AWGN). Let the actual symbol transmitted be the $N + N_g$ point sequence

$$\{x_{N-Ng}, \ldots, x_{N-2}, x_{N-1}, x_0, x_1, \ldots, x_{N-1}\} \quad (5)$$

with N_g greater than or equal to the time spread of the channel. The N_g point precursor signal allows the received symbol sequence to reach steady state by $n = 0$ (we assume

synchronization at this stage of the receiver) leading to a received sequence as given by (4). It is assumed that the impulse response of the channel does not change (much) during the symbol plus guard interval (this corresponds to "slow-fading" in a radio frequency channel).

The insertion of guard intervals renders the received carriers orthogonal on the N point symbol interval. However, the demodulation process, which is implemented with a dft (the dft is equivalent to matched filter reception in the absence of frequency offset) is affected by frequency offset. That is,

$$Y_k = \sum_{n=0}^{N-1} y_n e^{-2\pi jkn/N}, \quad (6)$$

the kth element of the dft sequence, consists of three components;

$$Y_k = (X_k H_k)\{(\sin \pi\epsilon)/N \sin (\pi\epsilon/N)\}$$
$$\cdot e^{j\pi\epsilon(N-1)/N} + I_k + W_k. \quad (7)$$

The first component is the modulation value X_k modified by the channel transfer function. This component experiences an amplitude reduction and phase shift due to the frequency offset. Since N is always much greater than $\pi\epsilon$, $N \sin (\pi\epsilon/N)$ may be replaced by $\pi\epsilon$.

The second term is the ICI caused by the frequency offset and is given by

$$I_k = \sum_{\substack{l=-K \\ l \neq k}}^{K} (X_l H_l)\{(\sin \pi\epsilon)/(N \sin (\pi(l - k + \epsilon)/N))\}$$
$$\cdot e^{j\pi\epsilon(N-1)/N} e^{-j\pi(l-k)/N}. \quad (8)$$

In order to evaluate the statistical properties of the ICI, some further assumptions are necessary. Specifically, it will be assumed that $E[X_k] = 0$ and $E[X_k X_l^*] = |X|^2 \delta_{lk}$, that is, the modulation values have zero mean and are uncorrelated. With this provision $E[I_k] = 0$, and

$$E[|I_k|^2] = |X|^2 \sum_{\substack{l=-K \\ l \neq k}}^{K} E\{|H_l|^2\}$$
$$\cdot \{\sin \pi\epsilon\}^2 / \{N \sin (\pi(l - k + \epsilon)/N)\}^2. \quad (9)$$

The average channel gain, $E\{|H_l|^2\} = |H|^2$, is constant so it can be separated from the sum and (9) becomes

$$E[|I_k|^2] = |X|^2 |H|^2 \{\sin \pi\epsilon\}^2$$
$$\cdot \sum_{\substack{p=-K-k \\ p \neq 0}}^{K-k} 1/\{N \sin (\pi(p + \epsilon)/N)\}^2. \quad (10)$$

The sum in (10) can be bounded for $\epsilon = 0$. It consists of $2K$ positive terms. The interval of the sum is contained within the longer interval $-2K \leq p \leq 2K$, its location dependent on k. Recall that $2K \leq N-1$. Also note the following; the argument of the sum is periodic with period N, it is an even function of p, and it is even about $p = N/2$. Thus the $2K$ terms of the sum
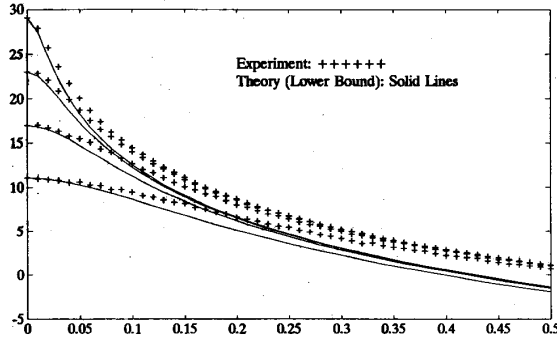
Fig. 1.   SNR versus relative frequency offset.

are a subset of the N terms in the intervals $-N/2 \le p \le -1$ and $1 \le p \le N/2$ for every k. Consequently,

$$\sum_{\substack{p=-K-k \\ p \ne 0}}^{K-k} 1/(N \sin \pi p/N)^2 < 2 \sum_{p=1}^{N/2} 1/(N \sin (\pi p/N))^2. \quad (11)$$

Observe that $(\sin \pi p/N)^2 \ge (2p/N)^2$ for $|p| \le N/2$. Therefore,

$$2 \sum_{p=1}^{N/2} 1/(N \sin (\pi p/N))^2$$

$$< 2 \sum_{p=1}^{N/2} 1/(2p)^2 < \frac{1}{2} \sum_{p=1}^{\infty} 1/p^2 = \pi^2/12 = 0.882 \quad (12)$$

upper bounds the sum for small ϵ. Numerically, we have determined that the sum in (10) is bounded by 0.5947 for $\epsilon < 0.5$ so that

$$E[|I_k|^2] \le 0.5947|X|^2|H|^2(\sin \pi \epsilon)^2; \quad |\epsilon| \le 0.5 \quad (13)$$

upper bounds the variance of the ICI for values of frequency offset up to plus or minus one half the carrier spacing.

Equation (13) may be used to give a lower bound for the SNR at the output of the dft for the OFDM carriers in a channel with AWGN and frequency offset. Thus,

$$\text{SNR} \ge |X|^2|H|^2\{\sin \pi \epsilon/(\pi \epsilon)\}^2/$$
$$\{0.5947|X|^2|H|^2(\sin \pi \epsilon)^2 + E[|W_k|^2]\}. \quad (14)$$

It is easily established that $|X|^2|H|^2/E[|W_k|^2] = E_c/N_o$ where E_c is the averaged received energy of the individual carriers and $N_o/2$ is the power spectral density of the AWGN in the bandpass transmission channel. Therefore, (14) may be more conveniently expressed as

$$\text{SNR} \ge \{E_c/N_o\}\{\sin \pi \epsilon/(\pi \epsilon)\}^2/$$
$$\{1 + 0.5947(E_c/N_o)(\sin \pi \epsilon)^2\} \quad (15)$$

with equality at $\epsilon = 0$. Equation (15) is plotted in Fig. 1 versus frequency offset ϵ for values of E_c/N_o equal to 11, 17, 23, and 29 dB. Simulation results for $N = 256$, $K = 96$, and 8 PSK modulation in a flat channel are included for validation. It can be seen that the bound of (15) is quite tight for small values of ϵ but is about 3 dB too low at $\epsilon = 0.5$.

III. FREQUENCY OFFSET ESTIMATION

If an OFDM transmission symbol is repeated, one receives, in the absence of noise, the $2N$ point sequence

$$r_n = (1/N)\left[\sum_{k=-K}^{K} X_k H_k e^{2\pi j n (k+\epsilon)/N}\right];$$
$$n = 0, 1, \ldots, 2N - 1. \quad (16)$$

The kth element of the N point dft of the first N points of (16) is

$$R_{1k} = \sum_{n=0}^{N-1} r_n e^{-2\pi j n k/N}; \quad k = 0, 1, 2, \ldots, N - 1, \quad (17)$$

and the kth element of the dft of the second half of the sequence is

$$R_{2k} = \sum_{n=N}^{2N-1} r_n e^{-2\pi j n k/N}$$
$$= \sum_{n=0}^{N-1} r_{n+N} e^{-2\pi j n k/N}; \quad k = 0, 1, \ldots, N - 1. \quad (18)$$

But from (16),

$$r_{n+N} = r_n e^{2\pi j \epsilon} \to R_{2k} = R_{1k} e^{2\pi j \epsilon}. \quad (19)$$

Including the AWGN one obtains

$$\begin{aligned} Y_{1k} &= R_{1k} + W_{1k} \\ Y_{2k} &= R_{1k} e^{2\pi j \epsilon} + W_{2k} \end{aligned}; \quad k = 0, 1, 2, \ldots, N - 1. \quad (20)$$

Observe that between the first and second DFT's, both the ICI and the signal are altered in exactly the same way, by a phase shift proportional to frequency offset. Therefore, if offset ϵ is estimated using observations (20) it is possible to obtain accurate estimates even when the offset is too large for satisfactory data demodulation.

It is shown in the Appendix that the maximum likelihood estimate (MLE) of ϵ is given by

$$\hat{\epsilon} = (1/2\pi)\tan^{-1}$$
$$\cdot \left\{ \left(\sum_{k=-K}^{K} \text{Im}[Y_{2k}Y_{1k}^*]\right) \middle/ \left(\sum_{k=-K}^{K} \text{Re}[Y_{2k}Y_{1k}^*]\right)\right\}. \quad (21)$$

This is an intuitively satisfying result since, in the absence of noise, the angle of $Y_{2k}Y_{1k}^*$ is $2\pi\epsilon$ for each k. Fig. 2 shows simulation results for the estimate of ϵ obtained using (21) versus ϵ for values of E_c/N_o corresponding to 17 and 5 dB.

A. Statistical Properties of the Estimate

The conditional mean and variance of $\hat{\epsilon}$ given ϵ and $\{R_k\}$ can be approximated as follows. Consider the complex products $Y_{2k}Y_{1k}^*$ from which we estimate ϵ. For a given ϵ, subtract the corresponding phase, $2\pi\epsilon$, from each product to obtain the tangent of the phase error

$$\tan[2\pi(\hat{\epsilon} - \epsilon)] = \left\{\left(\sum_{k=-K}^{K} \text{Im}[Y_{2k}Y_{1k}^* e^{-2\pi j \epsilon}]\right) \middle/ \right.$$
$$\left. \cdot \left(\sum_{k=-K}^{K} \text{Re}[Y_{2k}Y_{1k}^* e^{-2\pi j \epsilon}]\right)\right\}. \quad (22)$$
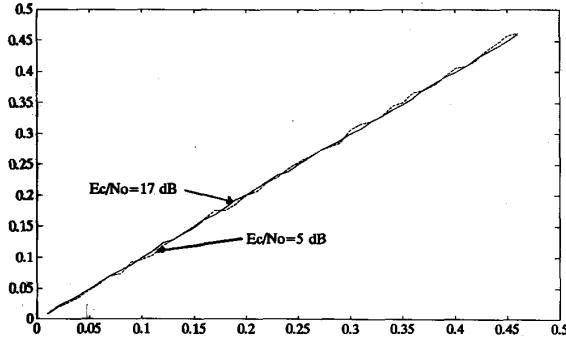
Fig. 2. MLE relative frequency offset estimate versus relative frequency offset.

Fig. 3. MLE relative frequency offset estimate error standard deviation versus E_s/N_o.

For $|\hat{\epsilon} - \epsilon| \ll 1/2\pi$, the tangent can be approximated by its argument so that

$$\hat{\epsilon} - \epsilon \approx (1/2\pi)$$
$$\cdot \left\{ \left(\sum_{k=-K}^{K} \mathrm{Im}[(R_{1k} + W_{2k}e^{-2\pi j\epsilon})(R_{1k}^* + W_{1k}^*)] \right) \Big/ \right.$$
$$\left. \left(\sum_{k=-K}^{K} \mathrm{Re}[(R_{1k} + W_{2k}e^{-2\pi j\epsilon})(R_{1k}^* + W_{1k}^*)] \right) \right\}.$$
$$(23)$$

At high signal-to-noise ratios, a condition compatible with successful communications signaling, (23) may be approximated by

$$\hat{\epsilon} - \epsilon \approx (1/2\pi)$$
$$\cdot \left\{ \left(\sum_{k=-K}^{K} \mathrm{Im}[W_{2k}R_{1k}^* e^{-2\pi j\epsilon} + R_{1k}W_{1k}^*] \right) \Big/ \right.$$
$$\left. \left(\sum_{k=-K}^{K} |R_{1k}|^2 \right) \right\}$$
$$(24)$$

from which we find that

$$E[\hat{\epsilon} - \epsilon \mid \epsilon, \{R_k\}] = 0. \qquad (25)$$

Therefore, for small errors, the estimate is conditionally unbiased.

The conditional variance of the estimate is easily determined for (24).

$$\mathrm{Var}[\hat{\epsilon} \mid \epsilon, \{R_k\}] = 1/\{(2\pi)^2(E_s/N_o)\} \qquad (26)$$

where

$$E_s = (T/N) \sum_{n=0}^{N-1} |r_n|^2$$

is the total symbol energy. Since the total energy is the sum of the energies of the $2K + 1$ carriers, the error variance of the offset estimate will in practice be very low. Fig. 3 shows the sample standard deviation of the error in the relative frequency offset estimate for 100 simulation trials of (21) versus E_s/N_o for $2K + 1 = 193$ carriers and for offsets of
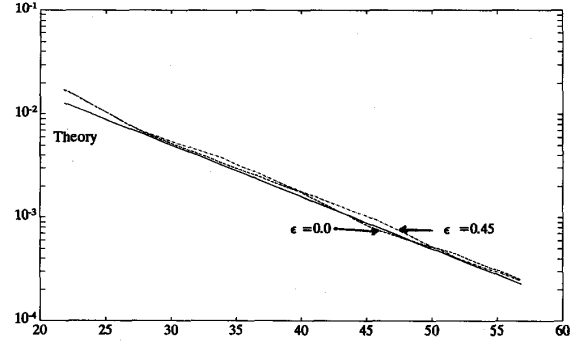
$\epsilon = 0$ and 0.45. The theoretical standard deviation from (25) is plotted for comparison. We conclude that (21) will give very accurate estimates of the relative frequency offset ϵ. Under normal conditions for communications signalling, the accuracy is sufficient to correct ϵ to well within tolerances (see (14) and Fig. 1) for negligible losses due any residual error in the offset estimate.

The limits of accurate estimation by (21) are $|\epsilon| \leq 0.5$, that is, $\pm 1/2$ the intercarrier spacing. As $\epsilon \to 0.5$, $\hat{\epsilon}$ may, due to noise and the discontinuity of the arctangent, jump to -0.5. When this happens the estimate is no longer unbiased and, in practice, it becomes useless. Thus, for frequency offsets exceeding one half the carrier spacing, an initial acquisition strategy must be prescribed. One such strategy is discussed in Section IV.

B. Frequency Offset Estimation in a Multipath Channel

It is evident from (25) and (26) that the mean and variance of the offset estimate do not depend on the actual received frequency coefficients $\{R_k\}$. Furthermore, if the symbol pair has been received through an unknown multipath channel, and as described in Section I it has been preceded by a periodic precursor of length $N_g \geq N_m$ the time spread of the channel then the carriers remain at their steady state values throughout the duration of both symbols because the modulation values are repeated. Thus, as no guard interval is required between the symbol pair, the algorithm of (21) can be used without modification.

Fig. 4 shows six amplitude responses of a channel with five multipaths whose arrival times have been uniformly distributed over an interval $T_m = T/16$. The paths have equal weight and random phases so that the overall channel exhibits frequency selective Rayleigh fading as is evident from the figure. Fig. 5 shows estimates of ϵ from (21) with $N_g = N/16$ for the same conditions as Fig. 2. It can be seen that the estimate is unaffected by the multipath.

IV. ACQUISITION

In the event that the frequency offset is greater than $\pm 1/2$ the carrier spacing, a strategy for initial acquisition to bring the offset within the limits of the algorithm must be developed.
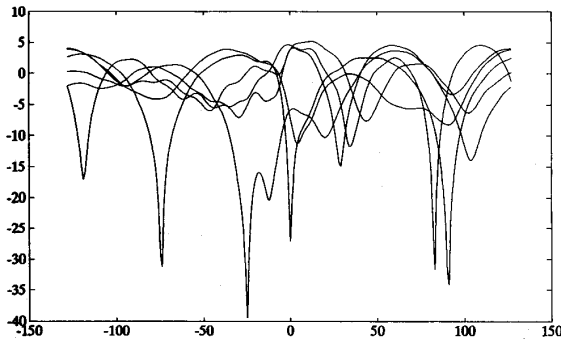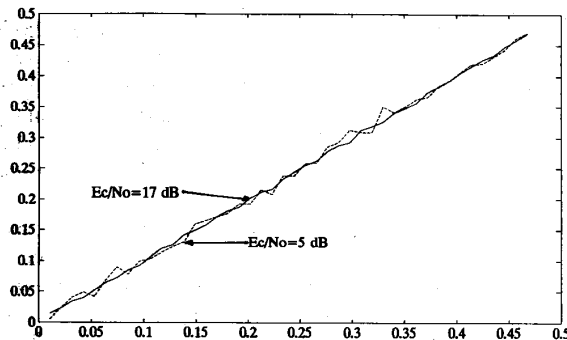
Fig. 4. Multipath transmission channel.

Fig. 5. MLE relative frequency offset estimate versus relative frequency offset in a multipath channel.

We envision that, if continuous, the OFDM symbol stream will be punctuated at appropriate intervals with repeated symbols. A continuous symbol stream occurs in applications such as digital audio broadcast [5]. A second possibility is that OFDM modulation is used in session oriented digital data or voice communications such as digital radio [8]. Here, we envision that the session initiation interval will include one or more repeated symbols.

The basic strategy for initial frequency offset acquisition, in either case, is to shorten the dft's and use larger carrier spacings such that the phase shift does not exceed $\pm\pi$. The frequency offset in Hz is $\delta = \epsilon/T = \epsilon\Delta f$ where Δf is the intercarrier spacing and T is the symbol interval. Let us assume that the initial frequency offset is no greater than $\pm\delta_{\max}$. Then

$$\Delta f_{\text{initial}} \geq 2\delta_{\max} \tag{27}$$

determines the minimum initial carrier spacing, and corresponding dft lengths. If the average power of the shortened symbols is kept the same, the variance of the estimate of $\epsilon_{\text{initial}}$ will be larger than for the longer data symbols since there is less symbol energy. Also, the offset estimate error for the shortened symbols, since it estimates the fraction of carrier spacing, corresponds to a proportionately larger fractional offset for the longer data symbols. However, the MLE estimate is so accurate that in practice the initial estimate still may be adequate. If not, it is refined by following the shortened

repeated symbols by a repetition of the first full length data symbol or by the use of an AFC loop as shown in [6].

To illustrate, consider the following example for a digital audio broadcasting service. Assume an intercarrier spacing for the data stream of 1 kHz, and a frequency offset uncertainty in the system dominated by the long term accuracy of the oscillators in the receiver that heterodyne the received signal to IF and quadrature demodulate to obtain the complex envelope. Assume VHF radio transmission at 150 MHz and overall oscillator uncertainty (long term stability) of 1 part in 10^5. Thus, $\delta_{\max} = 1500$ Hz and $\Delta f_{\text{initial}}$ must be greater than 3000 Hz. At regular intervals in the data stream of 1 ms (plus guard interval) symbols insert a short symbol of length 250 μs (4 kHz carrier spacing) and repeat it once. From this repeated shortened symbol estimate ϵ. Assume E_c/N_o is 11 dB for the data symbols and that there are 200 carriers, so that E_s/N_o is 34 dB. The shortened symbols have 1/4th energy so E_s/N_o is only 28 dB for the initial estimate of ϵ. From (25) we find that $\sigma_{\epsilon\text{initial}} = 0.0063$ so that $\sigma_{\epsilon\text{data}} = 0.025$. From (15) we see that residual offset errors of this magnitude cause ICI resulting in a signal-to-interference ratio of 24.3 dB in the data symbols for only 0.2 dB loss in overall SNR at E_c/N_o of 11 dB.

This example illustrates a situation for which the initial offset acquisition estimate with shortened data symbols is of sufficient accuracy that refinement of the estimate with longer symbols is not necessary.

V. DISCUSSION AND CONCLUSIONS

We have seen that, as expected, frequency offset in OFDM causes serious loss of SNR of the dft outputs due primarily to ICI. A lower bound for SNR has been derived and simulation results show that the bound is quite accurate for small offsets, but about 3 dB too pessimistic as the offset approaches 1/2 the carrier spacing.

An algorithm for maximum likelihood estimate (MLE) of frequency offset using the dft values of a repeated data symbol has been presented. It has been shown that for small error in the estimate, the estimate is conditionally unbiased and is consistent in the sense that the variance is inversely proportional to the number of carriers in the OFDM signal. Furthermore, both the signal values and the ICI contribute coherently to the estimate so that it is possible to obtain very accurate estimates even when the offset is too great, that is there is too much ICI, to demodulate the data values. Since the estimation error depends only on total symbol energy, the algorithm works equally well in multipath spread channels. However, it is required that the frequency offset as well as the channel impulse response be constant for a period of two symbols.

The accuracy required of frequency offset correction depends on how much residual offset can be tolerated. Offset induced ICI can be treated quite satisfactorily as additional AWGN since its source is the multiplicity of other OFDM carriers that are zero mean and uncorrelated random processes. Note that the SNR defined in Section II [see (14) and (15)] is just E_c/N_o in the absence of offset. Thus we may interpret (15) as the effective E_c/N_o of the carriers, or if divided

by the number of bits encoded in each of the carriers, their effective E_b/N_o. Required E_b/N_o of course depends upon the modulation constellation, the fading statistics of the channel the forward error control coding, if any, employed in the OFDM system and the desired BER (see, for example, [5, Figs. 11–13]).

The acquisition range of the algorithm presented here is $\pm 1/2$ the intercarrier spacing of the repeated symbol. It is independent of the modulation constellations chosen for the carriers and whether the symbols are coherently or differentially encoded. The AFC loop shown in [6] does not require a repeated symbol. However its acquisition range is only $\pm 1/2m$ of the intercarrier spacing for m-ary PSK. The initial frequency offset at the time of the initiation of the communication session may be greater than 1/2 the intercarrier spacing and thus even outside the range of the MLE algorithm. In this event, a strategy is required for initial acquisition. We propose to use a pair of shortened data symbols whose carrier spacing is sufficiently large to insure that the algorithm will operate within its range. Due to the low variance of the initial estimate, further refinement will normally not be required. It may be advantageous to use shortened repeated symbols for tracking offset variations too, instead of an AFC loop, because this reduces the time during which the channel must be stable.

APPENDIX
MAXIMUM LIKELIHOOD ESTIMATE OF DIFFERENTIAL PHASE

Let M complex values $\{Z_k\}$ be represented by a length $2M$ row vector

$$Z = [Z_{1R} \quad Z_{2R} \cdots Z_{MR} \quad Z_{1I} \quad Z_{2I} \cdots Z_{MI}]$$
$$= [Z_R \quad Z_I]. \tag{A.1}$$

Consider the random vectors

$$Y_1 = R_1 + W_1 \tag{A.2}$$
$$Y_2 = R_1 H(\Theta) + W_2 \tag{A.3}$$

where

$$H(\Theta) = \begin{bmatrix} C & S \\ C & S \end{bmatrix}, \quad C = \cos(\Theta)I \quad \& \quad S = \sin(\Theta)I \tag{A.4}$$

is a $2M \times 2M$ rotation matrix. The maximum likelihood estimate of the parameter Θ, given the observations Y_1 and Y_2 (see, for example, Sage and Melsa, [9, p. 196]) is the value of Θ that maximizes the conditional joint density function of the observations. That is

$$\hat{\Theta} = \max_{\Theta}[f(Y_1, Y_2 \mid \Theta)] \tag{A.5}$$

which can be written as

$$\hat{\Theta} = \max_{\Theta}[f(Y_2 \mid \Theta, Y_1)f(Y_1 \mid \Theta)]. \tag{A.6}$$

But Θ gives no information about Y_1, that is

$$f(Y_1 \mid \Theta) = f(Y_1) \tag{A.7}$$

so that

$$\hat{\Theta} = \max_{\Theta}[f(Y_2 \mid \Theta, Y_1)]. \tag{A.8}$$

To find the conditional density function in (A.8), note that

$$Y_2 = (Y_1 - W_1)H(\Theta) + W_2 \tag{A.9}$$

so that

$$Y_2 = Y_1 H(\Theta) + W_2 - W_1 H(\Theta). \tag{A.10}$$

If W_1 and W_2 are Gaussian, zero mean white random vectors with variance σ^2, then the conditional density function in (A.6) is multivariate Gaussian with mean value vector $Y_1 H(\Theta)$ and $2M \times 2M$ covariance matrix

$$K = E[(W_2 - W_1 H(\Theta))^t (W_2 - W_1 H(\Theta))] = 2\sigma^2 I. \tag{A.11}$$

We note that K is independent of Θ, therefore,

$$\hat{\Theta} = \max_{\Theta}[f(Y_2 \mid \Theta, Y_1)] = \min_{\Theta}[J(\Theta)] \tag{A.12}$$

with

$$J(\Theta) = (Y_2 - Y_1 H(\Theta))(Y_2 - Y_1 H(\Theta))^t. \tag{A.13}$$

Using the fact that

$$H(\Theta)[dH(\Theta)/d\Theta]^t + [dH(\Theta)/d\Theta]H^t(\Theta) = 0 \tag{A.14}$$

we can find that

$$dJ(\Theta)/d\Theta = -Y_2[dH(\Theta)/d\Theta]^t Y_1^t - Y_1[dH(\Theta)/d\Theta]Y_2^t. \tag{A.15}$$

Using (A.4), it follows directly that (A.15) is identically zero when $\hat{\Theta} = \Theta$ such that

$$\sin(\hat{\Theta})[Y_{2R}Y_{1R}^t + Y_{2I}Y_{1I}^t] = \cos(\hat{\Theta})[Y_{2I}Y_{1R}^t - Y_{2R}Y_{1I}^t]. \tag{A.16}$$

Therefore,

$$\hat{\Theta} = \tan^{-1}[(Y_{2I}Y_{1R}^t - Y_{2R}Y_{1I}^t)/(Y_{2R}Y_{1R}^t + Y_{2I}Y_{1I}^t)]$$
$$= \tan^{-1}\left\{ \left(\sum_{k=1}^{M} \text{Im}[Y_{2k}Y_{1k}^*]\right)\left(\sum_{k=1}^{M}\text{Re}[Y_{2k}Y_{1k}^*]\right)\right\} \tag{A.17}$$

is the maximum likelihood estimate (MLE) of Θ.

REFERENCES

[1] R. W. Chang, "Synthesis of band-limited orthogonal signals for multichannel data transmission," *Bell Syst. Tech. J.*, vol. 45, pp. 1775–1796, Dec. 1966.
[2] S. Darling, "On digital single-sideband modulators," *IEEE Trans. Circuit Theory*, vol. CT-17, pp. 409–414, Aug. 1970.
[3] S. B. Weinstein and P. M. Ebert, "Data transmission by frequency-division multiplexing using the discrete Fourier transform," *IEEE Trans. Commun. Technol.*, vol. COM-19, pp. 628–634, Oct. 1971.
[4] J. A. C. Bingham, "Multicarrier modulation for data transmission: An idea whose time has come," *IEEE Commun. Mag.*, vol. 28, pp. 17–25, Mar. 1990.

[5] M. Alard and R. Halbert, "Principles of modulation and channel coding for digital broadcasting for mobile receivers," *EBU Rev.*, no. 224, pp. 3–25, Aug. 1987.

[6] B. LeFloch, R. Halbert-Lassalle, and D. Castelain, "Digital sound broadcasting to mobile receivers," *IEEE Trans. Consum. Elec.*, vol. 35, no. 3, pp. 493–503, Aug. 1989.

[7] M. K. Simon and D. Divsalar, "Doppler-corrected differential detection of MPSK," *IEEE Trans. Commun.*, vol. 37, no. 2, pp. 99–109, Feb. 1989.

[8] D. C. Cox, W. S. Gifford, and H. Sherry, "Low-power digital radio as a ubiquitous subscriber loop," *IEEE Commun. Mag.*, vol. 29, pp. 92–95, Mar. 1991.

[9] A. P. Sage and J. L. Melsa, *Estimation Theory with Applications to Communications and Control.* New York: McGraw-Hill, 1971.

Paul M. Moose (M'79) was born in Omaha, NE, on July 22, 1938. He received the B.S., M.S., and Ph.D. degrees in electrical engineering in 1960, 1966, and 1970, respectively, from the University of Washington, Seattle.

Since 1977 he has been with the Department of Electrical and Computer Engineering, Naval Postgraduate School, Monterey, CA, where he is an Associate Professor of Electrical and Computer Engineering. His research interests are in digital communications. He is a co-founder of and consultant for Mercury Digital Communications Inc., Monterey, CA.