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Optimal Solution of The Two-Stage Kalman Estimator *

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Abstract

The optimal solution of estimating a set of dynamic state in the presence of a random bias employing a two-stage Kalman estimator is addressed. It is well known that, under an algebraic constraint, the optimal estimate of the system state can be obtained from a two-stage Kalman estimator. Unfortunately, this algebraic constraint is seldom satisfied for practical systems. This paper proposes a general form of the optimal solution of the two-stage estimator, in which the algebraic constraint is removed. Furthermore, it is shown that, by applying the adaptive process noise covariance concept, the optimal solution of the two-stage Kalman estimator is composed of a modified bias-free filter and an bias-compensating filter, which can be viewed as a generalized form of the conventional two-stage Kalman estimator.

1 Introduction

Consider the problem of estimating the state of a dynamic system in the presence of a random bias that influence the system dynamics and/or the observations [5]. It is common to use the Kalman filtering technique to deal with this problem. It is also known that, when using a Kalman filter, an accurate model of the process dynamics and observations is required. Therefore, ignoring the bias vector may lead to an unacceptably large error and even divergence of the filter. Hence, a common practice is to treat the bias as part of the system state and then estimate the bias as well as the system state. This leads to an *augmented Kalman estimator* whose computational cost and inaccuracy increase with the bias vector dimension. To maintain the computational cost at a lower level and avoid the numerical inaccuracies introduced by computations of large vectors and matrices, Friedland [1] proposed to employ the two-stage Kalman estimator. His idea is to decouple the augmented filter into two parallel filters. The first filter, the "bias-free" filter, is based on the assumption that the bias is nonexisting. The second filter, the "bias" filter, produces the remaining states, as well as the output of the bias-free filter, in order to reconstruct the original system states.

Here, we review some previous works on this issue. In [1], Friedland augments the bias state to the system state and shows that the resulting variance equation, using transformation technique, decomposes so that the decomposition for system states exists. This derivation is algebraic in nature and unfortunately cannot be applied to more general dynamical systems. In [2], Mendel assumes existence of a decomposition for the system state and a specific structure for the bias at the outset. Then, he determines the conditions under which the assumed structure produces an optimal estimator so that the decomposition of the system state is a valid one. He also generalizes Friedland's decomposition to allow variable dimension in the bias. However, the bias state is constant. In [3], Ignagni utilizes estimation theory to rederive and extend Friedland's result to the problems where the bias state is constant and the system state and the bias state may be initially correlated. In [4], Ignagni considered the case of a bias driven by a white noise which is uncorrelated with the system noise. However, the result he obtained is suboptimal. In [5], Alouani considered a white Gaussian bias noise that is correlated with the system noise. It was proved that under an algebraic constraint on the correlation between the system noise and bias noise, the proposed two-stage Kalman estimator is optimal. Since almost all practical systems may not satisfy this algebraic constraint, he also concluded that all two-stage estimators are suboptimal.

The purpose of this paper is to find the optimal solution of the two-stage Kalman estimator problem, in which a white Gaussian bias noise is correlated with the system noise without any constraint. Our method proceeds as follows. First, we ignore the bias term to form the bias-free filter. Then, we define a optimal bias-compensating filter to compensate the bias-free filter in order to reconstruct the original state. To derive the optimal bias-compensating filter, we partition the augmented Kalman gain into a linear combination of the bias-free filter's gain, the bias filter's gain and a compensating gain. Next, in order to calculate the compensating gain, the compensating covariance matrices are defined and recursively calculated. By a proper choice of the sensitivity matrices, the compensating covariance matrices and gain can be efficiently computed. Finally, it is shown that the optimal bias-compensating filter can be obtained as a linear combination of a bias filter and a compensating filter. Furthermore, by applying the adaptive process noise covariance concept, the compensating filter can be obtained implicitly by using a modified bias-free filter. In summary, the proposed optimal solution is a linear combination of a modified bias-free filter and a bias filter, which is structurely similar to a conventional two-stage Kalman filter.

This paper is organized as follows. In section 2, we state the problem of interest. In section 3, the optimal two-stage Kalman estimator is derived for state estimation in the presence of a random bias without any constraint. Application of this new result to solve a suboptimal problem of an earlier paper is also given. Section 4 is the conclusion. Detailed proof is provided in the appendix.

2 Statement of The Problem

The problem of interest is described by the discretized equation set

$$\begin{aligned} X_{k+1} &= A_k X_k + B_k \gamma_k + W_k^x, & (1)\\ \gamma_{k+1} &= C_k \gamma_k + W_k^\gamma, & (2)\\ Y_k &= H_k X_k + D_k \gamma_k + \eta_k, & (3) \end{aligned}$$

where $X_k \in R^n$ is the system state, $\gamma_k \in R^p$ is the vector bias, $Y_k \in R^m$ is the measurement vector and the quantities W_k^x, W_k^γ and η_k are zero-mean white Gaussian sequences with the following variances: $E[W_k^x(W_l^x)'] = Q_k^x\delta_{kl}$, $E[W_k^\gamma(W_l^\gamma)'] = Q_k^\gamma\delta_{kl}$, $E[W_k^x(W_l^\gamma)'] = Q_k^{x\gamma}\delta_{kl}$, $E[\eta_k(\eta_l)'] = R_k\delta_{kl}$, and $E[W_k^x(\eta_l)'] = E[W_k^\gamma(\eta_l)'] = 0$, where $'$ denotes transpose. This system model (1) - (2) may represent the dynamics of a maneuvering target, for example. In this case, X represents the Cartesian coordinates of the target position and velocity, while γ represents the target acceleration(see [6]).

Treating X_k and γ_k as the augmented system state, an augmented Kalman estimator may be used to produce the optimal augmented state estimates:

$$\begin{aligned} X_{k|k-1} &= A_{k-1}X_{k-1|k-1} + B_{k-1}\gamma_{k-1|k-1}, & (4)\\ \gamma_{k|k-1} &= C_{k-1}\gamma_{k-1|k-1}, & (5)\\ X_{k|k} &= X_{k|k-1} + K_k^x(Y_k - H_kX_{k|k-1} - D_k\gamma_{k|k-1}), & (6)\\ \gamma_{k|k} &= \gamma_{k|k-1} + K_k^\gamma(Y_k - H_kX_{k|k-1} - D_k\gamma_{k|k-1}), & (7) \end{aligned}$$

$$\begin{aligned} K_k^x &= (P_{k|k-1}^x H_k' + P_{k|k-1}^{x\gamma} D_k')M_k^{-1}\\ &= (P_{k|k}^x H_k' + P_{k|k}^{x\gamma} D_k')R_k^{-1}, & (8)\\ K_k^\gamma &= ((P_{k|k-1}^{x\gamma})' H_k' + P_{k|k-1}^{\gamma} D_k')M_k^{-1}\\ &= ((P_{k|k}^{x\gamma})' H_k' + P_{k|k}^{\gamma} D_k')R_k^{-1}, & (9) \end{aligned}$$

with

$$\begin{aligned} P_{k|k-1}^x &= A_{k-1}P_{k-1|k-1}^x A_{k-1}' + B_{k-1}P_{k-1|k-1}^\gamma B_{k-1}' +\\ &\quad B_{k-1}(P_{k-1|k-1}^{x\gamma})'A_{k-1}' + A_{k-1}P_{k-1|k-1}^{x\gamma}B_{k-1}'\\ &\quad +Q_{k-1}^x, & (10)\\ P_{k|k-1}^{x\gamma} &= A_{k-1}P_{k-1|k-1}^{x\gamma}C_{k-1}' + B_{k-1}P_{k-1|k-1}^{\gamma}C_{k-1}'\\ &\quad +Q_{k-1}^{x\gamma}, & (11)\\ P_{k|k-1}^{\gamma} &= C_{k-1}P_{k-1|k-1}^{\gamma}C_{k-1}' + Q_{k-1}^\gamma, & (12)\\ P_{k|k}^x &= (I - K_k^xH_k)P_{k|k-1}^x - K_k^xD_k(P_{k|k-1}^{x\gamma})', & (13)\\ P_{k|k}^{x\gamma} &= (I - K_k^xH_k)P_{k|k-1}^{x\gamma} - K_k^xD_kP_{k|k-1}^{\gamma}, & (14)\\ (P_{k|k}^{x\gamma})' &= (I - K_k^\gamma D_k)(P_{k|k-1}^{x\gamma})' - K_k^\gamma H_kP_{k|k-1}^x, & (15)\\ P_{k|k}^{\gamma} &= (I - K_k^\gamma D_k)P_{k|k-1}^{\gamma} - K_k^\gamma H_kP_{k|k-1}^{x\gamma}, & (16)\\ M_k &= H_kP_{k|k-1}^xH_k' + D_k(P_{k|k-1}^{x\gamma})'H_k' +\\ &\quad H_kP_{k|k-1}^{x\gamma}D_k' + D_kP_{k|k-1}^{\gamma}D_k' + R_k. & (17) \end{aligned}$$

As mentioned previously, the computational cost, memory and inaccuracy increase with the augmented state dimension. Hence, the filter model (4) - (17) is impractical to implement. The reason for this computational complexity is the extra computations of $P^{x\gamma}$ terms. Therefore, if these $P^{x\gamma}$ terms can be eliminated, we can reduce the complexity from implementation point of view. This is the main idea of using decoupled filter to deal with this problem. In next section, an equivalent implementation of the above filter using decoupled two-stage Kalman estimator is proposed.

3 Derivation of the Optimal Two-Stage Kalman Estimator

The design of a two-stage Kalman estimator consists of two steps. First, form the bias-free filter by ignoring the bias term. Second, a bias-compensating filter is designed to compensate the bias-free filter in order to reconstruct the original filter.

3.1 Bias-Free Filter

If the bias term is ignored($\gamma = 0$), the bias-free filter is just a Kalman filter based on the model (1) and (3) with different statistics for W_k^x(i.e. $\bar{Q}_k^x$ is used instead of Q_k^x). Hence, the bias-free filter is given by

$$\begin{aligned} \bar{X}_{k|k-1} &= A_{k-1}\bar{X}_{k-1|k-1}, & (18)\\ \bar{X}_{k|k} &= \bar{X}_{k|k-1} + \bar{K}_k^x(Y_k - H_k\bar{X}_{k|k-1}), & (19)\\ \bar{K}_k^x &= \bar{P}_{k|k-1}^x H_k'[H_k\bar{P}_{k|k-1}^x H_k' + R_k]^{-1} \end{aligned}$$

$$
\begin{aligned}
&= \bar{P}^x_{k|k} H'_k R_k^{-1}, &(20)\\
\bar{P}^x_{k|k-1} &= A_{k-1}\bar{P}^x_{k-1|k-1}A'_{k-1} + \bar{Q}^x_{k-1}, &(21)\\
\bar{P}^x_{k|k} &= (I - \bar{K}^x_k H_k)\bar{P}^x_{k|k-1}, &(22)
\end{aligned}
$$

where $\bar{X}$ represents the estimate of state process when the bias is ignored and $\bar{P}$ is the error covariance of $\bar{X}$. In order for the bias-free filter to be stable, it is a common practice to let $\bar{Q}^x_{k-1}$ remain positive semi-definite for all time.

3.2 Optimal Bias-Compensating Filter

The bias-free filter $\bar{X}$ can be corrected by adding a bias-compensating filter to reconstruct the original filter. A bias-compensating filter X^γ is said to be optimal, denoted by $X^{\gamma o}$, if it satisfies

$$
\begin{aligned}
X_{k|k-1} &= \bar{X}_{k|k-1} + X^{\gamma o}_{k|k-1}, &(23)\\
X_{k|k} &= \bar{X}_{k|k} + X^{\gamma o}_{k|k}, &(24)
\end{aligned}
$$

where $X_{k|k}$ and $\bar{X}_{k|k}$ are the output of the augmented filter and the bias-free filter, respectively. Substituting (18), (19), (4) and (6) into (23) and (24), we obtain the optimal bias-compensating filter as follows:

$$
\begin{aligned}
X^{\gamma o}_{k|k-1} &= A_{k-1}X^{\gamma o}_{k-1|k-1} + B_{k-1}\gamma_{k-1|k-1}, &(25)\\
X^{\gamma o}_{k|k} &= (I - \bar{K}^x_k H_k)X^{\gamma o}_{k|k-1} - \bar{K}^x_k D_k \gamma_{k|k-1} + \\
&\quad (K^x_k - \bar{K}^x_k)(Y_k - H_k\bar{X}_{k|k-1} - \\
&\quad D_k\gamma_{k|k-1} - H_k X^{\gamma o}_{k|k-1}). &(26)
\end{aligned}
$$

Unfortunately, equation (26) involves an unavailable term K^x_k which is the original augmented Kalman gain to be avoided. Therefore, we must seek another computable form of K^x_k. In order to do this, we express K^x_k as the following form

$$
K^x_k = \bar{K}^x_k + V_k K^\gamma_k + \tilde{K}^x_k, \qquad (27)
$$

where V_k is a sensitivity matrix to be defined and $\tilde{K}^x_k$ is a compensating gain to make (27) hold. In (27), the bias-free filter's gain $\bar{K}^x_k$ and the bias filter's gain K^γ_k are available. In order to calculate $\tilde{K}^x_k$, we first define the following compensating covariance matrices:

$$
\begin{aligned}
\tilde{P}^x_{k|k-1} &= P^x_{k|k-1} - \bar{P}^x_{k|k-1} - U_k(P^{x\gamma}_{k|k-1})', &(28)\\
\tilde{P}^{x\gamma}_{k|k-1} &= P^{x\gamma}_{k|k-1} - U_k P^\gamma_{k|k-1}, &(29)\\
\tilde{P}^x_{k|k} &= P^x_{k|k} - \bar{P}^x_{k|k} - V_k(P^{x\gamma}_{k|k})', &(30)\\
\tilde{P}^{x\gamma}_{k|k} &= P^{x\gamma}_{k|k} - V_k P^\gamma_{k|k}, &(31)
\end{aligned}
$$

where U_k and V_k are sensitivity matrices to be defined. Using the definitions (28) through (31) and the expression (27), the compensating covariance matrices can be calculated recursively via

$$
\begin{aligned}
\tilde{P}^x_{k|k-1} &= A_{k-1}\tilde{P}^x_{k-1|k-1}A'_{k-1} + A_{k-1}\tilde{P}^{x\gamma}_{k-1|k-1}B'_{k-1}\\
&\quad +Q^x_{k-1} - \bar{Q}^x_{k-1} - U_k(Q^{x\gamma}_{k-1})' + (\bar{U}_k - U_k)\\
&\quad C_{k-1}(A_{k-1}\tilde{P}^{x\gamma}_{k-1|k-1} + \bar{U}_k C_{k-1}P^\gamma_{k-1|k-1})', &(32)\\
\tilde{P}^{x\gamma}_{k|k-1} &= A_{k-1}\tilde{P}^{x\gamma}_{k-1|k-1}C'_{k-1} + Q^{x\gamma}_{k-1} - U_k Q^\gamma_{k-1} + \\
&\quad (\bar{U}_k - U_k)C_{k-1}P^\gamma_{k-1|k-1}C'_{k-1} +, &(33)\\
\tilde{P}^x_{k|k} &= (I - \bar{K}^x_k H_k - \tilde{K}^x_k H_k)\tilde{P}^x_{k|k-1} - \tilde{K}^x_k H_k \bar{P}^x_{k|k-1}\\
&\quad +((I - \bar{K}^x_k H_k)U_k - \bar{K}^x_k D_k - \tilde{K}^x_k S_k - V_k)\\
&\quad (\tilde{P}^{x\gamma}_{k|k-1} + U_k P^\gamma_{k|k-1})', &(34)\\
\tilde{P}^{x\gamma}_{k|k} &= (I - \bar{K}^x_k H_k - \tilde{K}^x_k H_k)\tilde{P}^{x\gamma}_{k|k-1} + ((I - \bar{K}^x_k H_k)\\
&\quad U_k - \bar{K}^x_k D_k - \tilde{K}^x_k S_k - V_k)P^\gamma_{k|k-1}, &(35)
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{U}_k &= (A_{k-1}V_{k-1} + B_{k-1})C^{-1}_{k-1}, &(36)\\
S_k &= H_k U_k + D_k. &(37)
\end{aligned}
$$

The Eqs. (32) through (35) are derived in the Appendix A. Using the recursive form of $\tilde{P}^x_{k|k}$ and $\tilde{P}^{x\gamma}_{k|k}$, expressed in (34) and (35), respectively, the compensating gain $\tilde{K}^x_k$, appeared in (27), can be calculated via

$$
\begin{aligned}
\tilde{K}^x_k &= (((I - \bar{K}^x_k H_k)U_k - \bar{K}^x_k D_k - V_k)(H_k\tilde{P}^{x\gamma}_{k|k-1} + \\
&\quad S_k P^\gamma_{k|k-1})' + (I - \bar{K}^x_k H_k)(\tilde{P}^x_{k|k-1}H'_k + \\
&\quad \tilde{P}^{x\gamma}_{k|k-1}D'_k))M_k^{-1}, &(38)
\end{aligned}
$$

where

$$
\begin{aligned}
M_k &= H_k\bar{P}^x_{k|k-1}H'_k + R_k + S_k(H_k\tilde{P}^{x\gamma}_{k|k-1} + S_k P^\gamma_{k|k-1})'\\
&\quad +H_k(\tilde{P}^x_{k|k-1}H'_k + \tilde{P}^{x\gamma}_{k|k-1}D'_k). &(39)
\end{aligned}
$$

The Eq. (38) is derived in the Appendix B.

Notice that the sensitivity matrices U_k and V_k are left to be defined. In other words, the general form of $\tilde{K}^x_k$ as described above is valid for any combination of U_k and V_k. However, the Eqs. (32) through (35) as well as $\tilde{K}^x_k$ are too complicated to implement if U_k and V_k are not properly defined. In order to simplify the computation, we find that the $\tilde{P}^{x\gamma}$ terms can be eliminated if the U_k and V_k are selected as

$$
\begin{aligned}
U_k &= \bar{U}_k + (Q^{x\gamma}_{k-1} - \bar{U}_k Q^\gamma_{k-1})(P^\gamma_{k|k-1})^{-1}, &(40)\\
V_k &= (I - \bar{K}^x_k H_k)U_k - \bar{K}^x_k D_k - \tilde{K}^x_k S_k, &(41)
\end{aligned}
$$

with $\tilde{P}^{x\gamma}_{0|0}$ set to be 0. With these choices of the U_k, V_k and $\tilde{P}^{x\gamma}_{0|0}$, the compensating covariance matrices are simplified to

$$
\begin{aligned}
\tilde{P}^{x\gamma}_{k|k-1} &= \tilde{P}^{x\gamma}_{k|k} = 0, &(42)\\
\tilde{P}^x_{k|k-1} &= A_{k-1}\tilde{P}^x_{k-1|k-1}A'_{k-1} + Q^x_{k-1} - \bar{Q}^x_{k-1}\\
&+ U_k Q^\gamma_{k-1}\bar{U}'_k - Q^{x\gamma}_{k-1}\bar{U}'_k - U_k(Q^{x\gamma}_{k-1})', &(43)\\
\tilde{P}^x_{k|k} &= (I - \bar{K}^x_k H_k - \tilde{K}^x_k H_k)\tilde{P}^x_{k|k-1} - \\
&\quad \tilde{K}^x_k H_k \bar{P}^x_{k|k-1}, &(44)
\end{aligned}
$$

and the simplified compensating gain $\tilde{K}_k^x$ can be calculated simply as

$$\begin{aligned}\tilde{K}_k^x &= (I - \bar{K}_k^x H_k)\tilde{P}_{k|k-1}^x H_k' \\ &\quad [H_k(\bar{P}_{k|k-1}^x + \tilde{P}_{k|k-1}^x)H_k' + R_k]^{-1}. \quad (45)\end{aligned}$$

With the above computable form of $\tilde{K}_k^x$, the optimal bias-compensating filter can be expressed as a linear combination of the bias filter and a compensating filter, denoted by $\tilde{X}^\gamma$. The result, proved in the Appendix C, is

Optimal bias-compensating filter:

$$\begin{aligned}X_{k|k-1}^{\gamma o} &= U_k \gamma_{k|k-1} + \tilde{X}_{k|k-1}^\gamma, \quad (46)\\ X_{k|k}^{\gamma o} &= V_k \gamma_{k|k} + \tilde{X}_{k|k}^\gamma \quad (47)\end{aligned}$$

with

$$\begin{aligned}\tilde{X}_{k|k-1}^\gamma &= A_{k-1}\tilde{X}_{k-1|k-1}^\gamma + \\ &\quad (\bar{U}_k - U_k)C_{k-1}\gamma_{k-1|k-1}, \quad (48)\\ \tilde{X}_{k|k}^\gamma &= (I - \bar{K}_k^x H_k)\tilde{X}_{k|k-1}^\gamma + \\ &\quad \tilde{K}_k^x(Y_k - H_k\bar{X}_{k|k-1} - H_k\tilde{X}_{k|k-1}^\gamma) \quad (49)\end{aligned}$$

Notice that in order for (45) to be calculated stably, the matrix $(\bar{P}_{k|k-1}^x + \tilde{P}_{k|k-1}^x)$ should be positive semi-definite. However, if this situation is not satisfied, then U_k and V_k must be reselected. This is achieved if we redefine U_k and V_k as

$$\begin{aligned}U_k &= \bar{U}_k + (A_{k-1}\tilde{P}_{k-1|k-1}^{x\gamma}C_{k-1}' + Q_{k-1}^{x\gamma} - \\ &\quad \bar{U}_k Q_{k-1}^\gamma)(P_{k|k-1}^\gamma)^{-1}, \quad (50)\\ V_k &= (I - \bar{K}_k^x H_k)U_k - \bar{K}_k^x D_k. \quad (51)\end{aligned}$$

With these choices of the U_k and V_k, the compensating covariance matrices are obtained as

$$\begin{aligned}\tilde{P}_{k|k-1}^{x\gamma} &= 0, \quad (52)\\ \tilde{P}_{k|k-1}^x &= A_{k-1}\tilde{P}_{k-1|k-1}^x A_{k-1}' + A_{k-1}\tilde{P}_{k-1|k-1}^{x\gamma}B_{k-1}' \\ &\quad + Q_{k-1}^x - \bar{Q}_{k-1}^x + (\bar{U}_k - U_k)P_{k|k-1}^\gamma U_k' \\ &\quad - \bar{U}_k(Q_{k-1}^{x\gamma})', \quad (53)\\ \tilde{P}_{k|k}^x &= (I - \bar{K}_k^x H_k)\tilde{P}_{k|k-1}^x - \tilde{K}_k^x \\ &\quad (H_k(\bar{P}_{k|k-1}^x + \tilde{P}_{k|k-1}^x) + S_k P_{k|k-1}^\gamma U_k'), \quad (54)\\ \tilde{P}_{k|k}^{x\gamma} &= -\tilde{K}_k^x S_k P_{k|k-1}^\gamma, \quad (55)\end{aligned}$$

the compensating gain $\tilde{K}_k^x$ is calculated as

$$\begin{aligned}\tilde{K}_k^x &= (I - \bar{K}_k^x H_k)\tilde{P}_{k|k-1}^x H_k'[H_k(\bar{P}_{k|k-1}^x + \tilde{P}_{k|k-1}^x)H_k' \\ &\quad + R_k + S_k P_{k|k-1}^\gamma S_k']^{-1}, \quad (56)\end{aligned}$$

and the optimal bias-compensating filter is calculated as

$$\begin{aligned}X_{k|k-1}^{\gamma o} &= U_k \gamma_{k|k-1} + \tilde{X}_{k|k-1}^\gamma, \quad (57)\\ X_{k|k}^{\gamma o} &= V_k \gamma_{k|k} + \tilde{X}_{k|k}^\gamma, \quad (58)\end{aligned}$$

with

$$\begin{aligned}\tilde{X}_{k|k-1}^\gamma &= A_{k-1}\tilde{X}_{k-1|k-1}^\gamma + (\bar{U}_k - U_k)C_{k-1}\gamma_{k-1|k-1}, (59)\\ \tilde{X}_{k|k}^\gamma &= (I - \bar{K}_k^x H_k)\tilde{X}_{k|k-1}^\gamma + \tilde{K}_k^x(Y_k - H_k\bar{X}_{k|k-1} - \\ &\quad S_k\gamma_{k|k-1} - H_k\tilde{X}_{k|k-1}^\gamma). \quad (60)\end{aligned}$$

Notice that (56) is calculated stablely because the matrix, in inverse form, is equivalent to M_k, expressed by (17), which is positive definite as a basic assumption of a Kalman filter. Using (29), (23), (46) and the identity $\tilde{P}_{k|k-1}^{x\gamma} = 0$, the Eqs. (7), (9) and (16) of the bias filter should have the following new form:

$$\begin{aligned}\gamma_{k|k} &= \gamma_{k|k-1} + K_k^\gamma(Y_k - H_k\bar{X}_{k|k-1} - \\ &\quad S_k\gamma_{k|k-1} - H_k\tilde{X}_{k|k-1}^\gamma), \quad (61)\\ K_k^\gamma &= P_{k|k-1}^\gamma S_k'[H_k(\bar{P}_{k|k-1}^x + \tilde{P}_{k|k-1}^x)H_k' \\ &\quad + R_k + S_k P_{k|k-1}^\gamma S_k']^{-1}, \quad (62)\\ P_{k|k}^\gamma &= (I - K_k^\gamma S_k)P_{k|k-1}^\gamma. \quad (63)\end{aligned}$$

3.3 Optimal Two-Stage Kalman Estimator

From the previous derivations, It is straightforward to define the general form of the optimal two-stage Kalman estimator as

The optimal two-stage Kalman estimator:

$$\begin{aligned}X_{k|k}^o &= \bar{X}_{k|k} + X_{k|k}^{\gamma o} \\ &= \bar{X}_{k|k} + V_k\gamma_{k|k} + \tilde{X}_{k|k}^\gamma, \quad (64)\end{aligned}$$

where $\bar{X}_{k|k}$ is the output of the bias-free filter, $\gamma_{k|k}$ is the output of the bias filter, $\tilde{X}_{k|k}^\gamma$ is the compensating filter, defined in (49) or (60), and V_k is the sensitivity matrix, defined in (41) or (51).

Notice that $\bar{Q}_{k-1}^x$ can be set to any positive semi-definite value as a design parameter at this moment. However, if we adopt the adaptive process noise covariance concept as [5], then we can also obtain similar structure as the conventional two-stage Kalman estimator. The result is described as follows. Owing to the similarity to the bias-free filter, the compensating filter, defined by (48), (49) and (45), can be calculated implicitely by using a bias-free filter as follows. Combining (18), (19), (20), (21) and (22) with (48), (49), (45), (43) and (44), respectively, we obtain

$$\begin{aligned}\bar{X}_{k|k-1} + \tilde{X}_{k|k-1}^\gamma &= A_{k-1}(\bar{X}_{k-1|k-1} + \tilde{X}_{k-1|k-1}^\gamma) \\ &\quad + (\bar{U}_k - U_k)C_{k-1}\gamma_{k-1|k-1}, \quad (65)\\ \bar{X}_{k|k} + \tilde{X}_{k|k}^\gamma &= (\bar{X}_{k|k-1} + \tilde{X}_{k|k-1}^\gamma) + (\bar{K}_k^x + \tilde{K}_k^x) \\ &\quad (Y_k - H_k(\bar{X}_{k|k-1} + \tilde{X}_{k|k-1}^\gamma)), (66)\\ \bar{K}_k^x + \tilde{K}_k^x &= (\bar{P}_{k|k-1}^x + \tilde{P}_{k|k-1}^x)H_k'[H_k(\bar{P}_{k|k-1}^x \\ &\quad + \tilde{P}_{k|k-1}^x)H_k' + R_k]^{-1}, \quad (67)\end{aligned}$$

$$\bar{P}^x_{k|k-1} + \tilde{P}^x_{k|k-1} = A_{k-1}(\bar{P}^x_{k-1|k-1} + \tilde{P}^x_{k-1|k-1})A'_{k-1} + Q^x_{k-1} + U_k Q^\gamma_{k-1}\bar{U}'_k - Q^{x\gamma}_{k-1}\bar{U}'_k - U_k(Q^{x\gamma}_{k-1})', \quad (68)$$

$$\bar{P}^x_{k|k} + \tilde{P}^x_{k|k} = (I - (\bar{K}^x_k + \tilde{K}^x_k)H_k)(\bar{P}^x_{k|k-1} + \tilde{P}^x_{k|k-1}), \quad (69)$$

Based on the above calculations, if we augment the bias-free filter to include an external input, i.e., $(\bar{U}_k - U_k)C_{k-1}\gamma_{k-1|k-1}$, and applying the adaptive covariance concept, i.e.,

$$\bar{Q}^x_{k-1} = Q^x_{k-1} + U_k Q^\gamma_{k-1}\bar{U}'_k - Q^{x\gamma}_{k-1}\bar{U}'_k - U_k(Q^{x\gamma}_{k-1})', \quad (70)$$

then $\bar{X}_{k|k}$ and $\tilde{X}^\gamma_{k|k}$ can be obtained at the same time by using a modified form of the bias-free filter, named as modified bias-free filter which is to be defined. As mentioned in the last section, in order for this modified bias-free filter to be stable, it is required that the matrix $(\bar{P}^x_{k|k-1} + \tilde{P}^x_{k|k-1})$ should remain positive semi-definite. This requirement is guaranteed if $\bar{Q}^x_{k-1}$, expressed as (70), is positive semi-definite for all time. If this is the case and if we define the following parameters:

$$\bar{X}^m_{k|k-1} = \bar{X}_{k|k-1} + \tilde{X}^\gamma_{k|k-1}, \quad (71)$$

$$\bar{X}^m_{k|k} = \bar{X}_{k|k} + \tilde{X}^\gamma_{k|k}, \quad (72)$$

$$\bar{K}^m_k = \bar{K}^x_k + \tilde{K}^x_k, \quad (73)$$

$$\bar{P}^m_{k|k-1} = \bar{P}^x_{k|k-1} + \tilde{P}^x_{k|k-1}, \quad (74)$$

$$\bar{P}^m_{k|k} = \bar{P}^x_{k|k} + \tilde{P}^x_{k|k}, \quad (75)$$

then the optimal two-stage Kalman estimator can be implemented by the following:

$$X^o_{k|k} = \bar{X}^m_{k|k} + V_k\gamma_{k|k}, \quad (76)$$

where $\bar{X}^m_{k|k}$ is the modified bias-free filter calculated via

$$\bar{X}^m_{k|k-1} = A_{k-1}\bar{X}^m_{k-1|k-1} + (\bar{U}_k - U_k)C_{k-1}\gamma_{k-1|k-1} \quad (77)$$

$$\bar{X}^m_{k|k} = \bar{X}^m_{k|k-1} + \bar{K}^m_k(Y_k - H_k\bar{X}^m_{k|k-1}), \quad (78)$$

$$\bar{K}^m_k = \bar{P}^m_{k|k-1}H'_k[H_k\bar{P}^m_{k|k-1}H'_k + R_k]^{-1}, \quad (79)$$

$$\bar{P}^m_{k|k-1} = A_{k-1}\bar{P}^m_{k-1|k-1}A'_{k-1} + \bar{Q}^x_{k-1}, \quad (80)$$

$$\bar{P}^m_{k|k} = (I - \bar{K}^m_k H_k)\bar{P}^m_{k|k-1}, \quad (81)$$

and $\gamma_{k|k}$ is the bias-free filter calculated via

$$\gamma_{k|k-1} = C_{k-1}\gamma_{k-1|k-1}, \quad (82)$$

$$\gamma_{k|k} = \gamma_{k|k-1} + K^\gamma_k(Y_k - H_k\bar{X}^m_{k|k-1} - S_k\gamma_{k|k-1}) \quad (83)$$

$$P^\gamma_{k|k-1} = C_{k-1}P^\gamma_{k-1|k-1}C'_{k-1} + Q^\gamma_{k-1}, \quad (84)$$

$$K^\gamma_k = P^\gamma_{k|k-1}S'_k[H_k\bar{P}^m_{k|k-1}H'_k + R_k + S_kP^\gamma_{k|k-1}S'_k]^{-1}, \quad (85)$$

$$P^\gamma_{k|k} = (I - K^\gamma_k S_k)P^\gamma_{k|k-1}. \quad (86)$$

However, if $\bar{Q}^x_{k-1}$, expressed as (70), is not positive semi-definite for all time, then the above simplified form of the optimal two-stage Kalman estimator can not be used and the general form, expressed by (64) and (50 - (60, of the optimal two-stage Kalman estimator must be used. Notice that if U_k, obtained by (40), equals $\bar{U}_k$, then the external input of the modified bias-free filter will be vanished. Therefore, the above optimal two-stage Kalman estimator will be identical to the conventional two-stage Kalman estimator proposed by [1]. Besides, the required constraint $U_k = \bar{U}_k$ is equivalent to $Q^{x\gamma}_{k-1} = \bar{U}_kQ^\gamma_{k-1}$ which is the algebraic constraint of [5].

3.4 Example

At last, we demonstrate the application of the proposed optimal two-stage Kalman estimator to solve the suboptimal problem which appeared in [4]. The problem is described as follows. If the bias term undergo some random variation with time and the process noise is uncorrelated with the bias noise, then $Q^\gamma_{k-1} \neq 0$ and $Q^{x\gamma}_{k-1} = 0$. Let the conventional two-stage Kalman estimator, $X^c_{k|k}$, be defined as the following (used by[4]):

$$U_k = \bar{U}_k, \quad (87)$$

$$V_k = (I - \bar{K}^x_k H_k)\bar{U}_k - \bar{K}^x_k D_k, \quad (88)$$

$$\bar{Q}^x_{k-1} = Q^x_{k-1} + B_{k-1}Q^\gamma_{k-1}B'_{k-1}, \quad (89)$$

$$X^c_{k|k} = \bar{X}_{k|k} + V_k\bar{\gamma}_{k|k}, \quad (90)$$

where $\bar{\gamma}_{k|k}$ is the bias filter calculated via

$$\bar{\gamma}_{k|k-1} = C_{k-1}\bar{\gamma}_{k-1|k-1}, \quad (91)$$

$$\bar{\gamma}_{k|k} = \bar{\gamma}_{k|k-1} + \bar{K}^\gamma_k(Y_k - H_k\bar{X}_{k|k-1} - S_k\bar{\gamma}_{k|k-1}), \quad (92)$$

$$\bar{P}^\gamma_{k|k-1} = C_{k-1}\bar{P}^\gamma_{k-1|k-1}C'_{k-1} + Q^\gamma_{k-1}, \quad (93)$$

$$\bar{K}^\gamma_k = \bar{P}^\gamma_{k|k-1}S'_k[H_k\bar{P}^x_{k|k-1}H'_k + R_k + S_k\bar{P}^\gamma_{k|k-1}S'_k]^{-1} \quad (94)$$

$$\bar{P}^\gamma_{k|k} = (I - \bar{K}^\gamma_k S_k)\bar{P}^\gamma_{k|k-1}, \quad (95)$$

then the filter is not an optimal solution of the two-stage Kalman estimator. We can prove this fact as follows. First, we define the error of the filter and the optimal one as $\varepsilon_k = V_k(\gamma_{k|k} - \bar{\gamma}_{k|k}) + \tilde{X}^\gamma_{k|k}$. Then, we can express ε_k recursively as

$$\varepsilon_k = (I - (\bar{K}^x_k + V_k\bar{K}^\gamma_k)H_k)A_{k-1}\varepsilon_{k-1} + (B_{k-1} - (\bar{K}^x_k + V_k\bar{K}^\gamma_k)(H_kB_{k-1} + D_kC_{k-1}))(\gamma_{k-1|k-1} - \bar{\gamma}_{k-1|k-1}) + (K^x_k - (\bar{K}^x_k + V_k\bar{K}^\gamma_k))(Y_k - H_k\bar{X}_{k|k-1} - S_k\gamma_{k|k-1} - H_k\tilde{X}^\gamma_{k|k-1}). \quad (96)$$

Notice that, in this problem, the last two terms of (96) are nonzero and cannot be cancelled each other for all time. Therefore, ε_k will not be vanished for all time. In other words, the filter is not an optimal solution as expected. However, if U_k and V_k are calculated by (40) and (41), respectively, then $\bar{Q}^x_{k-1}$ is calculated as

$$\bar{Q}^x_{k-1} = Q^x_{k-1} + \bar{U}_k(Q^\gamma_{k-1} - Q^\gamma_{k-1}[P^\gamma_{k|k-1}]^{-1}Q^\gamma_{k-1})\bar{U}'_k. \quad (97)$$

Since $\tilde{Q}^x_{k-1}$, calculated in (97), is positive semi-definite, hence the simplified form of the optimal two-stage Kalman estimator (76) through (86) will provide the optimal solution of the proposed problem.

As the above example depicts, it seems that if the system becomes more complicated, then also the optimal two-stage Kalman estimator becomes. In the complicated case, the conventional two-stage Kalman estimator may not be an optimal solution and the optimal two-stage Kalman estimator will provide the solution.

4 Conclusions

The contributions of this paper are twofold. The first contribution is to derive the optimal two-stage Kalman estimator. This is an optimal solution for state estimation in the presence of a general random bias without any constraint. Applying the adaptive process noise covariance technique, the proposed optimal two-stage Kalman estimator can be implemented with a modified bias-free filter and a bias filter. The second contribution is that, with the proposed optimal two-stage Kalman estimator, we can formally assess the suboptimality of the conventional two-stage Kalman estimator in a practical system. However, the proposed optimal two-stage Kalman estimator may be complicated for some applications. Therefore, how to derive a suboptimal solution to reduce the computational burden is also under investigation.

5 Appendix

A. Proof of the Compensating Covariance Matrices

The proof is omitted.

B. Proof of the General Compensating Gain

The proof is omitted.

C. Proof of the Decomposition of the Optimal Bias-Compensating Filter

The proof is omitted.

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