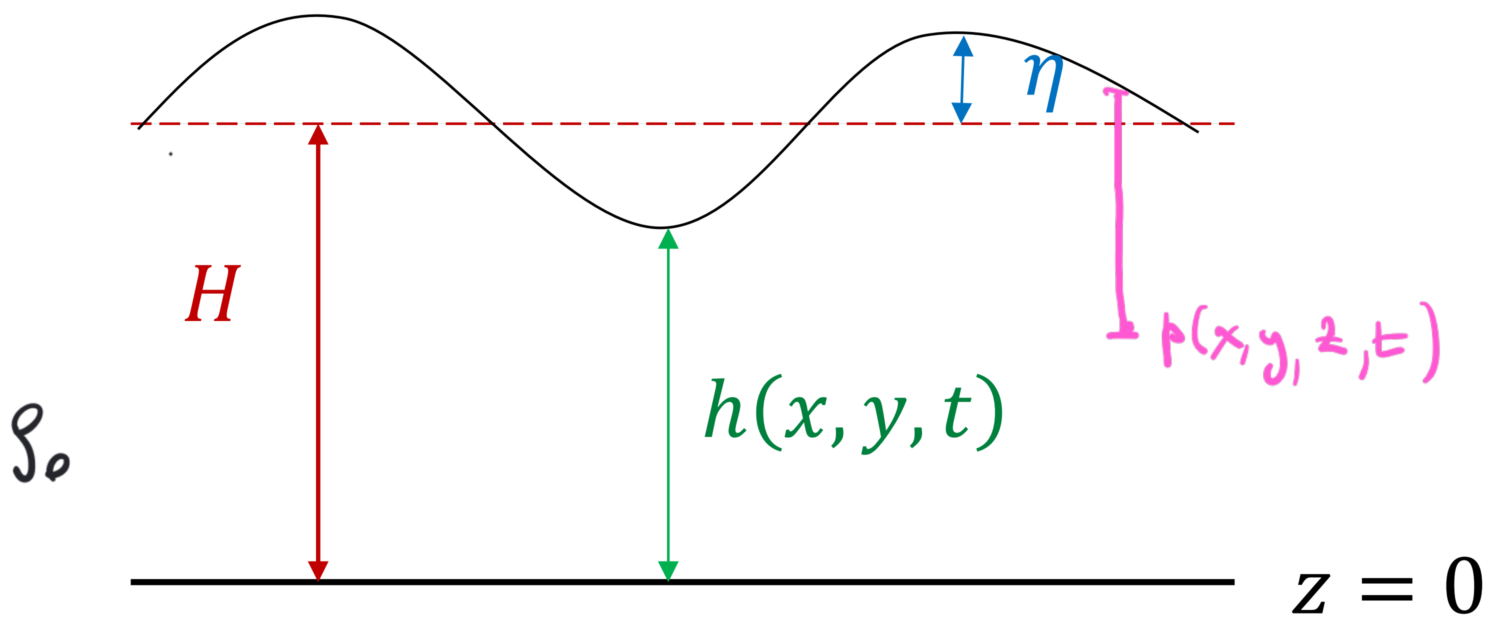
Shallow water model

The shallow water equations describe the dynamics of a thin layer of a constant density fluid in hydrostatic balance, bounded by a rigid surface from below and by a free surface from above. Above this layer, we assume an inert fluid with negligible inertia. We will consider shallow water dynamics in the presence of rotation. This is the simplest model that allows us to study the combined influence of rotation and stratification. It is often referred to as a one and half dimensional model.

One layer SW model

What do we mean by shallow?



$$\frac{H}{L} \ll 1$$

Motion is fully determined by horizontal momentum and continuity equations, because hydrostatic balance holds in the vertical!

Vertical momentum equation

Hydrostatic balance implies that pressure at a given height is equal to the weight of the fluid column just above it

$$p(x, y, z, t) = \rho_0 g[h(x, y, t) - z]$$

Horizontal pressure gradients are hence independent of height!

$$\nabla_h p = \rho_0 q \nabla_h h$$

If flow is initially independent on height, it will remain so. This implies that horizontal velocity is also not dependent on height.

$$\vec{u}(x, y, t) = (u, v)$$

Horizontal momentum equation

$$\frac{D\vec{u}}{Dt} + \vec{f} \times \vec{u} = -\frac{1}{\rho_o}\nabla_h p = -g\nabla_h h$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla_h$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial h}{\partial y}$$

Continuity equation

$\vec{v} = (u, v, w)$

$$\nabla \cdot \vec{v} = 0$$

$$\frac{\partial w}{\partial z} = -\nabla_h \cdot \vec{u}$$

Intergrating from the bottom up to the free surface

$$\int_0^h \frac{\partial w}{\partial z} dz = -\int_0^h \nabla_h \cdot \vec{u}\, dz$$

$$w(z = h) - \cancel{w(z = 0)} = -h\nabla_h \cdot \vec{u}$$

What is the vertical velocity at the free surface?

Continuity equation

A free surface, such as the one at the top of the shallow water layer, is a material surface: it moves with the flow and cannot be crossed by the flow.

$$z - h(x, y, t) = 0 \qquad\qquad \frac{D}{Dt}\left[z - h(x, y, t)\right] = 0$$

At z = h

$$\frac{Dz}{Dt} = w(z = h) = \frac{Dh}{Dt}$$

So continuity equation is a vertically integrated *thickness* equation

$$\frac{Dh}{Dt} + h\nabla \cdot \vec{u} = 0$$

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\vec{u}) = 0$$

SW equations

$$\frac{D\vec{u}}{Dt} + \vec{f} \times \vec{u} = -g\nabla_h h$$

$$\frac{Dh}{Dt} + h\nabla \cdot \vec{u} = 0$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla_h$$

Potential vorticity

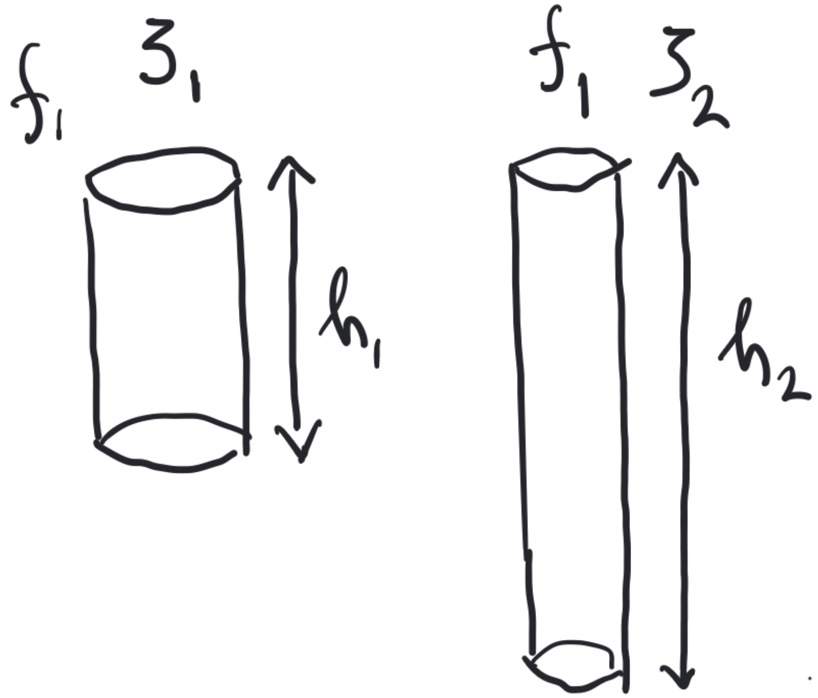
Taking the curl of the momentum equation and combining it with the continuity equation, we obtain the PV equation

$$\frac{D}{Dt}\left[\frac{f+\zeta}{h}\right]=0$$

$$q = \frac{f+\zeta}{h}$$

Because the fluid column can stretch, now the materially conserved quantity is the ratio of the absolute vorticity and the depth of the fluid column.

Conservation of PV



$$\frac{f_1 + \zeta_1}{h_1} = \frac{f_1 + \zeta_2}{h_2}$$

$$h_2 > h_1$$

$$\zeta_2 > \zeta_1$$

Conservation of PV is a consequence of conservation of both mass and angular momentum

Linear SW waves in the f plane

$f = f_0$

We are now going to linearize the SW equation in a state of rest. We are also going to assume constant f (that is, we make the f plane approximation):

$$\overline{u} = \overline{v} = 0$$

$u = \overline{u} + u'$
$v = \overline{v} + v'$

$$u = u'$$

$$v = v'$$

$$h = H + \eta'$$

Linearized momentum equations are

$$\frac{\partial u'}{\partial t} - fv' = -g\frac{\partial \eta'}{\partial x} \qquad (1)$$

$$\frac{\partial v'}{\partial t} + fu' = -g\frac{\partial \eta'}{\partial y} \qquad (2)$$

Linear SW waves in the f plane

Full thickness equation is

$$\frac{\partial \eta'}{\partial t} + u'\frac{\partial \eta'}{\partial x} + v'\frac{\partial \eta'}{\partial y} + (H + \eta')\Big(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\Big) = 0$$

The linearized version is

$$\frac{\partial \eta'}{\partial t} + H\Big(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\Big) = 0 \qquad (3)$$

I am going to drop from now on the prime, with the understanding that we are looking at equations for the small perturbation quantities from the basic state of rest.

Linear SW waves in the f plane

After some manipulations we get

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\frac{\partial}{\partial t}\left[\ \frac{\partial^2}{\partial t^2} + f^2 - gH\nabla^2\right]\eta = 0$$

$$\vec{k} = (k, l)$$

This is a partial differential equation with constant coefficient and we look for linear plane wave solutions of the form

$$(u, v, \eta) = Re\left[(\hat{u}, \hat{v}, \hat{\eta})e^{i(\vec{k}\cdot\vec{x}-\omega t)}\right]$$

Replacing in equation above, we get

$$-i\omega\left[-\omega^2 + f^2 + gHK^2\right]\hat{\eta} = 0$$

with $\quad K^2 = k^2 + l^2$

Linear SW waves in the f plane

For non trivial solutions, we need to have

$$\omega\left[\omega^2 - (f^2 + c^2K^2)\right] = 0 \qquad\qquad c^2 = gH$$

This is the dispersion relation of linear shallow water waves, which has two solutions, or modes.

1. First mode

$$\omega = 0 \qquad\qquad c_p = c_g = 0$$

This is a stationary mode with no phase or energy propagation

$$\eta = \hat{\eta}e^{i(kx+ly)}$$

Linear SW waves in the f plane

This is a geostrophic mode!

$$fv = g\frac{\partial \eta}{\partial x}$$
$$fu = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Being a 2d nondivergent flow, we can introduce a geostrophic streamfunction ψ_g

$$(u, v) = \left(-\frac{\partial \psi_g}{\partial y}, \frac{\partial \psi_g}{\partial x}\right)$$

$$\psi_g = \frac{g}{f}\eta$$

Linear SW waves in the f plane

What about PV?

$$q = \frac{f+\zeta}{h} = \frac{(f+\zeta)}{H\left(1+\frac{\eta}{H}\right)} \sim \frac{f+\zeta}{H}\left(1-\frac{\eta}{H}\right) \sim$$

$$\sim \underbrace{\frac{f}{H}}_{\bar{q}} + \underbrace{\frac{\zeta}{H} - \frac{f\eta}{H^2}}_{q'} + \mathcal{O}(\eta^2)$$

This geostrophic mode has nonzero PV

$$\zeta = \nabla^2 \psi_g$$

$$\eta = \frac{f}{g}\psi_g$$

$$q' = \frac{\nabla^2 \psi_g}{H} - \frac{f^2}{gH^2}\psi_g$$

Linear SW waves in the f plane

2. Second mode

$$\omega^2 = f^2 + c^2 K^2$$

These are Poincarè or inertial-gravity waves.

a. Let's consider long-wave limit

$$k \to 0$$
$$l \to 0$$
$$\lambda \to \infty$$

$$\omega \to \pm f$$

These are inertial waves.

Linear SW waves in the f plane

Their phase speed is:

$$c_{p,x} = \frac{\omega}{K} = \pm\frac{f}{k} \rightarrow \infty$$

$$c_{g,x} = \frac{\partial \omega}{\partial k} = 0$$

$$c_{p,y} = \frac{\omega}{l} = \pm\frac{f}{l} \rightarrow \infty$$

These are very fast waves. For any K (small)

$$|\omega| \geq |f|$$

Linear SW waves in the f plane

b. Let's consider the short-wave limit

$$K \to \infty$$

$$\lambda \to 0$$

$$\omega = \pm cK$$

These are gravity waves!

One can show that

$$\underbrace{|f|}_{10^{-4}\,s^{-1}} \le |\omega| \le \underbrace{N}_{10^{-2}\,s^{-1}}$$

These are waves that have periods ranging from a few minutes to a few hours.

Linear SW waves in the f plane

For any K, we can rewrite the dispersion relation of the inertia-gravity waves as

$$\omega = \pm f \left[1 + \frac{gH}{f^2} K^2\right]^{\frac{1}{2}} = \pm f \left[1 + (L_D K)^2\right]^{\frac{1}{2}}$$

with Rossby radius of deformation $L_D = \frac{\sqrt{gH}}{f}$

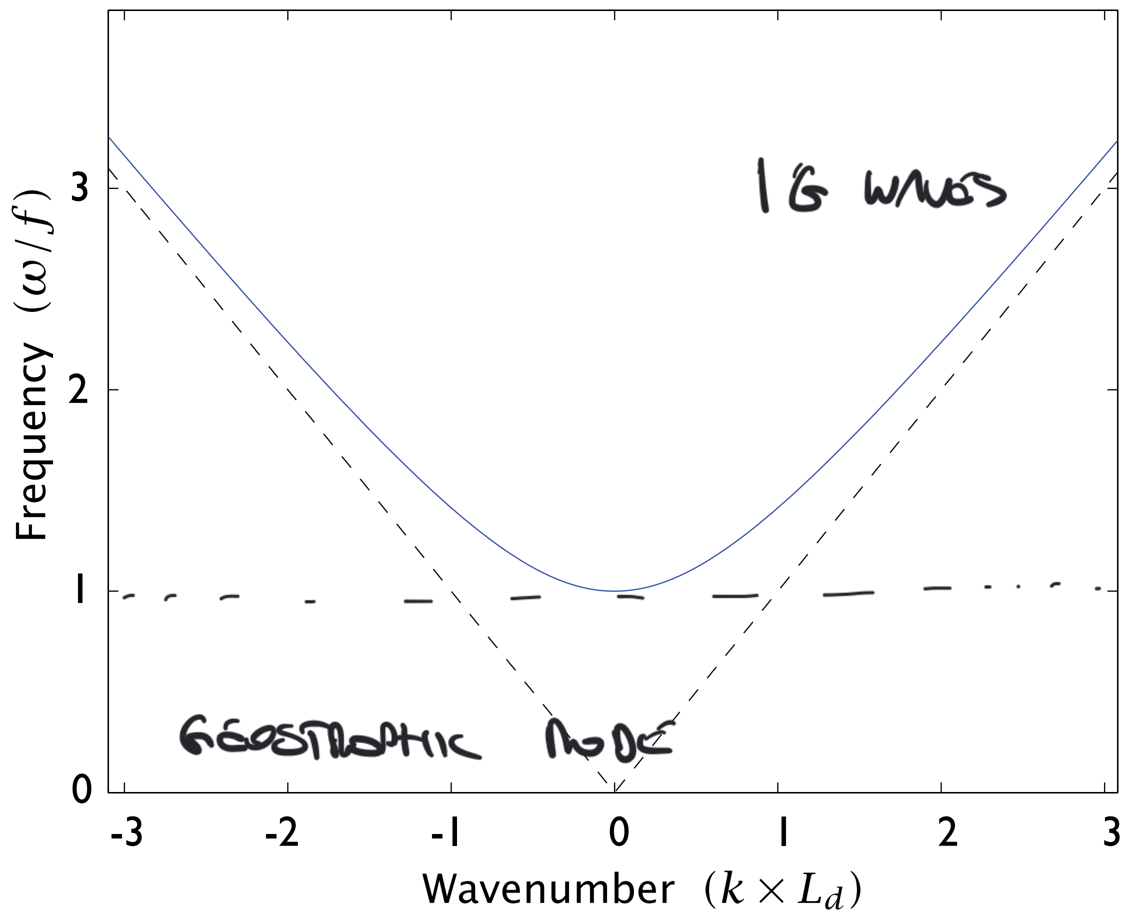
If the wavelength is much larger (shorter) than L_D, we recover the long (short) wave limit.

Linear SW waves in the f plane

The Rossby deformation radius is the length scale at which effects of rotation and stratification are equally important.

Differently from geostrophic mode, inertia-gravity waves are fast, not in geostrophic balance, have nonzero divergence and zero PV.

Linear SW waves in the f plane

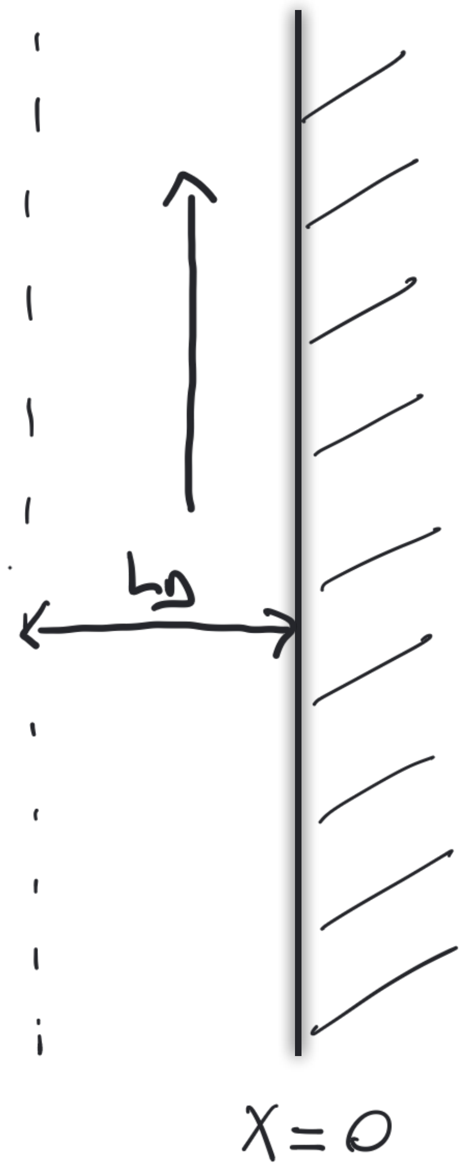


In the midlatitudes, there is a natural separation between fast divergent modes and slow geostrophic modes, that are slowly evolvings and are the ones that make the weather.

Kelvin waves

The Kelvin wave is a particular wave that exists in both presence of rotation and a lateral boundary. To determine the solution, we start from the SW equations and require no normal flow across the boundary.

$f > 0$



$u = 0$

Kelvin waves

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} \qquad (1) \qquad fv = g\frac{\partial \eta}{\partial x}$$

GEOSTROPHIC BALANCE

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} \qquad (2) \qquad \frac{\partial v}{\partial t} = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \qquad (3) \qquad \frac{\partial \eta}{\partial t} + H\frac{\partial v}{\partial y} = 0$$

Combining (2) and (3)

$$\frac{\partial^2 v}{\partial t^2} - c^2\frac{\partial^2 v}{\partial y^2} = 0 \qquad c^2 = gH \qquad \omega = \pm c\, l$$

$$v(x,y,t) = \hat{v}_+ e^{i(ly-\omega t)} F_+(x) + \hat{v}_- e^{i(ly+\omega t)} F_-(x)$$

$$\eta(x,y,t) = \hat{\eta}_+ e^{i(ly-\omega t)} F_+(x) + \hat{\eta}_- e^{i(ly+\omega t)} F_-(x)$$

Kelvin waves

Using Eq. 2 for the northward propagating wave, we get:

$$-\cancel{i}\omega \hat{v}_+ \cancel{e^{i(ly-\omega t)}} \cancel{F_+(x)} = -\cancel{i}l g \hat{\eta}_+ \cancel{e^{i(ly-\omega t)}} \cancel{F_+(x)}$$

$$\hat{\eta}_+ = \frac{\sqrt{gH}}{g} \hat{v}_+$$

For the southward propagating solution, we get:

$$\hat{\eta}_- = -\frac{\sqrt{gH}}{g} \hat{v}_-$$

Kelvin waves

Using Eq. 1 for the northward propagating solution

$$f v = g \frac{\partial \eta}{\partial x}$$

$$f \hat{v}_+ \, e^{i(ly-\omega t)} F_+(x) = g \hat{\eta}_+ e^{i(ly-\omega t)} \frac{dF_+(x)}{dx}$$

$$\frac{d F_+(x)}{F_+} = \frac{f}{g} \frac{\hat{v}_+}{\hat{\eta}_+} dx = \frac{f}{\sqrt{gH}} dx = \frac{dx}{L_D}$$

$$F_+(x) = \exp\left(x / L_D \right)$$

$$v_N(x,y,t) = \hat{v}_+ e^{i(ly-\omega t)} \exp\left(x / L_D \right)$$

Kelvin waves

For the southward propagating wave

$$v_s(x,y,t) = \hat{v}_- \, e^{i(ly-\omega t)} \exp\left(-x/L_D\right)$$

We just retain the solution that decays exponentially away from the boundary, that is the northward propagating solution.

Kelvin waves are waves that are trapped closed to the boundary and propagate along the boundary, leaving the boundary to the right in the NH and to the left in the SH.