Angular momentum budget and transport

Angular momentum is a basic physical quantity in a rotating system. For a closed system, AM is conserved. For a planet, the total AM is the sum of the AM of each component, including the solid Earth, the atmosphere and the ocean. Change of AM in one component will have to be balanced by change of AM in an another component.

We already defined the atmospheric AM (making the thin shell approximation) as

$$M = (\Omega a \cos\theta + u) a \cos\theta$$

Primitive equations

Before writing an equation for the AM budget, let's review the primitive equations. These are the equations of motion with the following approximations:

1. The hydrostatic approximation for the vertical momentum equation;

2. The thin-shell approximation;

3. The traditional approximation, in which Coriolis and metric terms involving the vertical velocity are neglected.

Primitive equations

$$\frac{Du}{Dt} - 2\Omega \sin\theta v - \frac{uv}{a}\tan\theta = -\frac{1}{a\rho\cos\theta}\frac{\partial p}{\partial \lambda} \quad + F_\lambda \tag{1}$$

$$\frac{Dv}{Dt} + 2\Omega \sin\theta u + \frac{u^2}{a}\tan\theta = -\frac{1}{a\rho}\frac{\partial p}{\partial \theta} \tag{2}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{3}$$

where

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t} + \frac{u}{a\cos\theta}\frac{\partial}{\partial \lambda} + \frac{v}{a}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}\right) \tag{4}$$

Continuity equation is

$$\frac{\partial \rho}{\partial t} + \frac{1}{a\cos\theta}\frac{\partial(u\rho)}{\partial \lambda} + \frac{1}{a\cos\theta}\frac{\partial}{\partial \theta}(v\rho\cos\theta) + w\frac{\partial(w\rho)}{\partial z} = 0 \tag{5}$$

AM budget and zonal momentum equation

$$\frac{DM}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial \lambda} + F_\lambda a\cos\theta \qquad \vec{v} = (u, v, w)$$

Expanding material derivative

$$a\cos\theta\left(\frac{\partial u}{\partial t} + \frac{u}{a\cos\theta}\frac{\partial u}{\partial \lambda} + w\frac{\partial u}{\partial z}\right) + \frac{v}{a}\frac{\partial}{\partial \theta}\left(\Omega a^2\cos^2\theta + ua\cos\theta\right) = -\frac{1}{\rho}\frac{\partial p}{\partial \lambda} + F_\lambda a\cos\theta$$

which reduces to

$$a\cos\theta\frac{Du}{Dt} - uv\sin\theta - 2\Omega av\cos\theta\sin\theta = -\frac{1}{\rho}\frac{\partial p}{\partial \lambda} + F_\lambda a\cos\theta$$

Dividing by $a\cos\theta$

$$\frac{Du}{Dt} - \frac{uv}{a}\tan\theta - 2\Omega v\sin\theta = -\frac{1}{\rho a\cos\theta}\frac{\partial p}{\partial \lambda} + F_\lambda$$

which is just the zonal momentum equation

AM budget

Torque is force times the moment arm

AM budget

One important consequence of this budget is that AM is conserved in a frictionless axisymmetric atmosphere. Axisymmetric means that there are no variations from the zonal mean.

$$\frac{D[M]}{Dt} = -\frac{1}{\rho}\cancel{\left[\frac{\partial p}{\partial \lambda}\right]} + \cancel{F_\lambda a \cos\vartheta} = 0$$

Vector invariant form of momentum equation

The momentum equation can also be written in vector form

$$\frac{D\vec{v}}{Dt} + 2\vec{\Omega} \times \vec{v} = -\frac{1}{\rho}\nabla p - \nabla\phi \qquad \phi = gz$$

$$\frac{D\vec{v}}{Dt} = \frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}$$

We can rewrite the advection as

$$\vec{v}\cdot\nabla\vec{v} = \nabla\Big(\frac{\vec{v}^2}{2}\Big) - \vec{v}\times\vec{\omega}$$

with relative vorticity $\vec{\omega} = \nabla \times \vec{v}$ whose vertical component in cartesian coordinates is

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Vector invariant form of momentum equation

$$\frac{\partial \vec{v}}{\partial t} + (2\vec{\Omega} + \vec{\omega}) \times \vec{v} = -\frac{1}{\rho}\nabla p - \nabla(\phi + \frac{1}{2}\vec{v}^{\,2})$$

where we have introduced the absolute vorticity $\vec{\omega}_a = 2\vec{\Omega} + \vec{\omega}$

with vertical component $f + \zeta$

We can then rewrite the horizontal momentum equations as

$$\frac{\partial u}{\partial t} - (f+\zeta)v + w\frac{\partial u}{\partial z} = -\frac{1}{a\cos\theta}\left(\frac{1}{\rho}\frac{\partial p}{\partial \lambda} + \frac{1}{2}\frac{\partial \vec{u}^2}{\partial \lambda}\right)$$

$$\frac{\partial v}{\partial t} + (f+\zeta)u + w\frac{\partial v}{\partial z} = -\frac{1}{a}\left(\frac{1}{\rho}\frac{\partial p}{\partial \theta} + \frac{1}{2}\frac{\partial \vec{u}^2}{\partial \theta}\right)$$

with horizontal velocity vector $\vec{u} = (u, v)$

AM budget and sfc winds

AM budget

$$\frac{DM}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial \lambda} + F_\lambda a \cos\theta$$

$$\rho\frac{DM}{Dt} = -\frac{\partial p}{\partial \lambda} + \rho F_\lambda a \cos\theta \qquad \text{Eq. (1)}$$

Continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

$$M\frac{\partial \rho}{\partial t} + M\nabla \cdot (\rho \vec{v}) = 0 \qquad \text{Eq. (2)}$$

Take (1) + (2)

AM budget and sfc winds

$$\frac{\partial}{\partial t}(\rho M) + \nabla \cdot (\rho M \vec{v}) = -\frac{\partial p}{\partial \lambda} + \rho a \cos\theta F_\lambda$$

Integrate in zonal and vertical direction

$$\frac{\partial}{\partial t}\int\int \rho M d\lambda dz + \nabla \cdot \int\int \rho M \vec{v} d\lambda dz = \int\int \rho a \cos\theta F_\lambda d\lambda dz$$

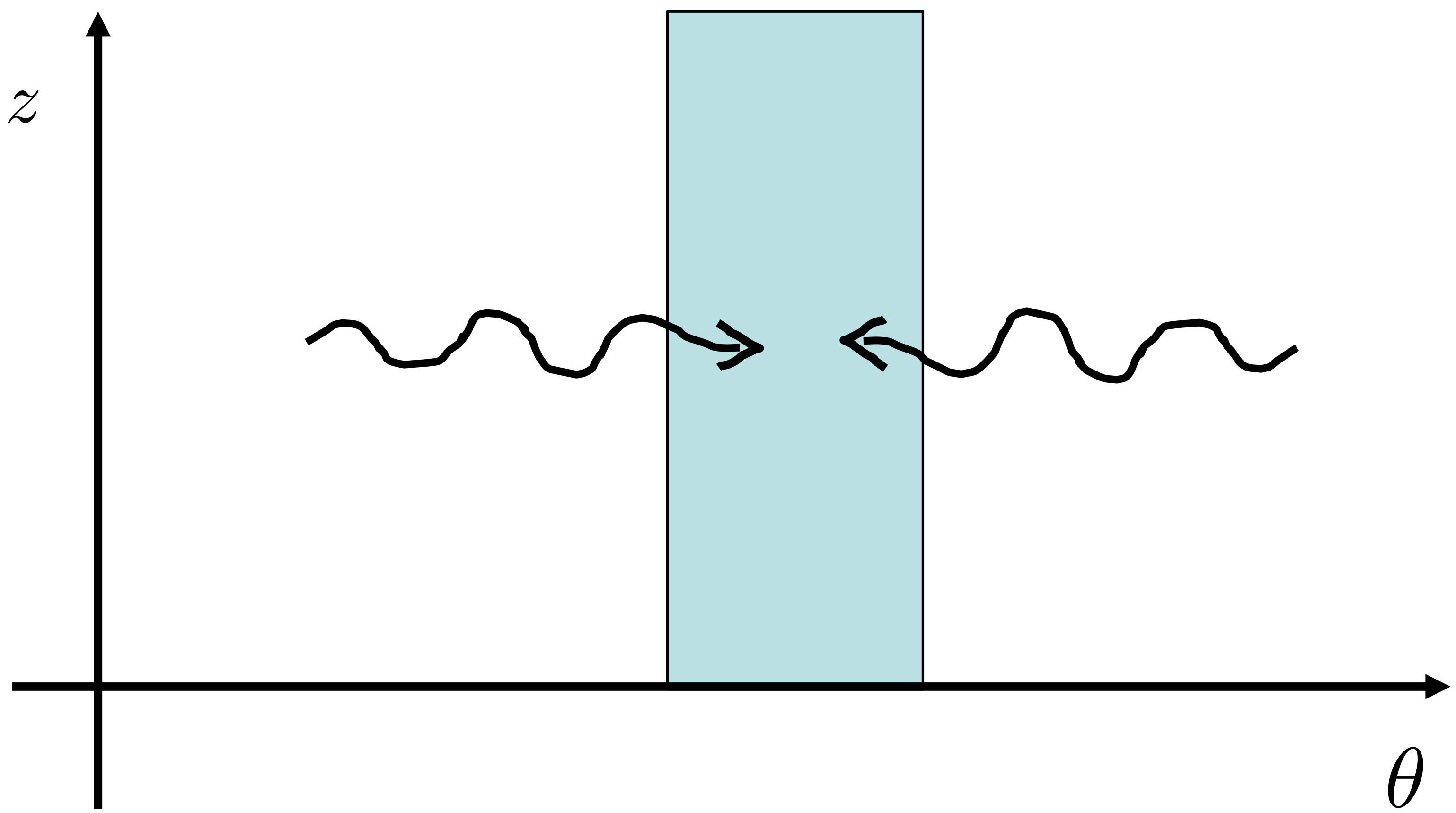
Assuming steady state

$$\nabla \cdot \int\int \rho M \vec{v} d\lambda dz = \int\int \rho a \cos\theta F_\lambda d\lambda dz$$

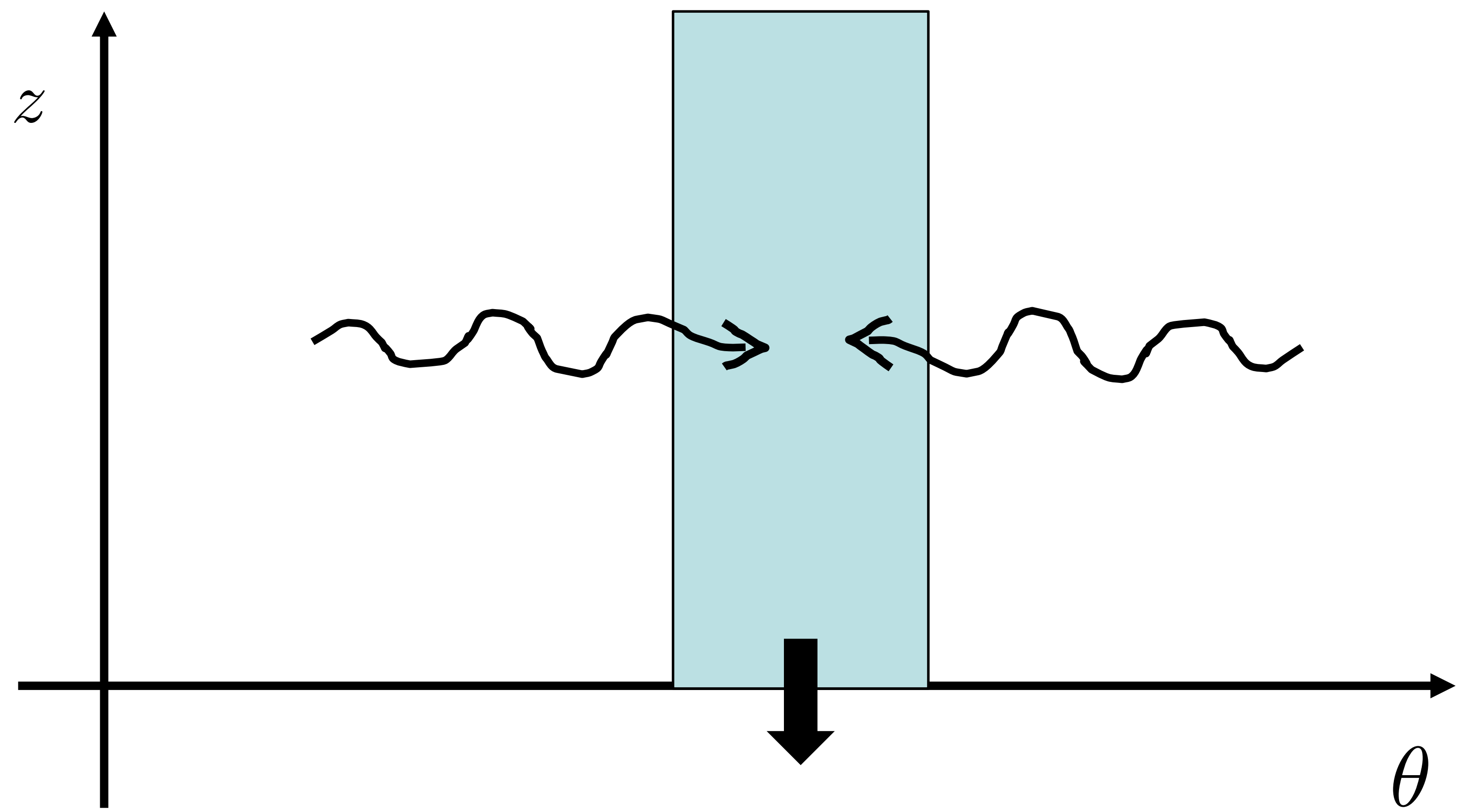
Convergence/divergence of AM flux into the atmospheric column

Removal/addition of AM from the atmospheric column by friction

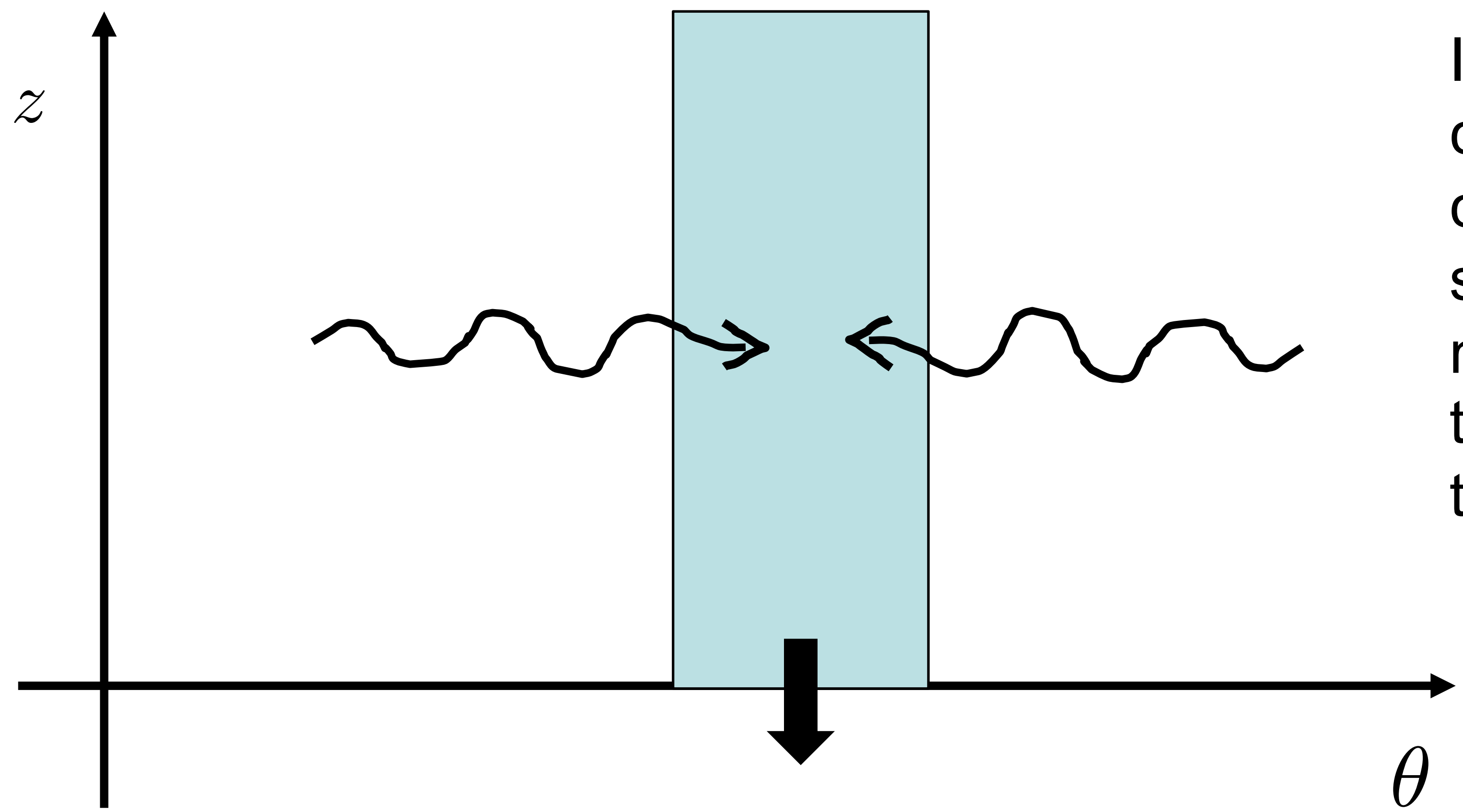
AM budget and sfc winds



AM budget and sfc winds

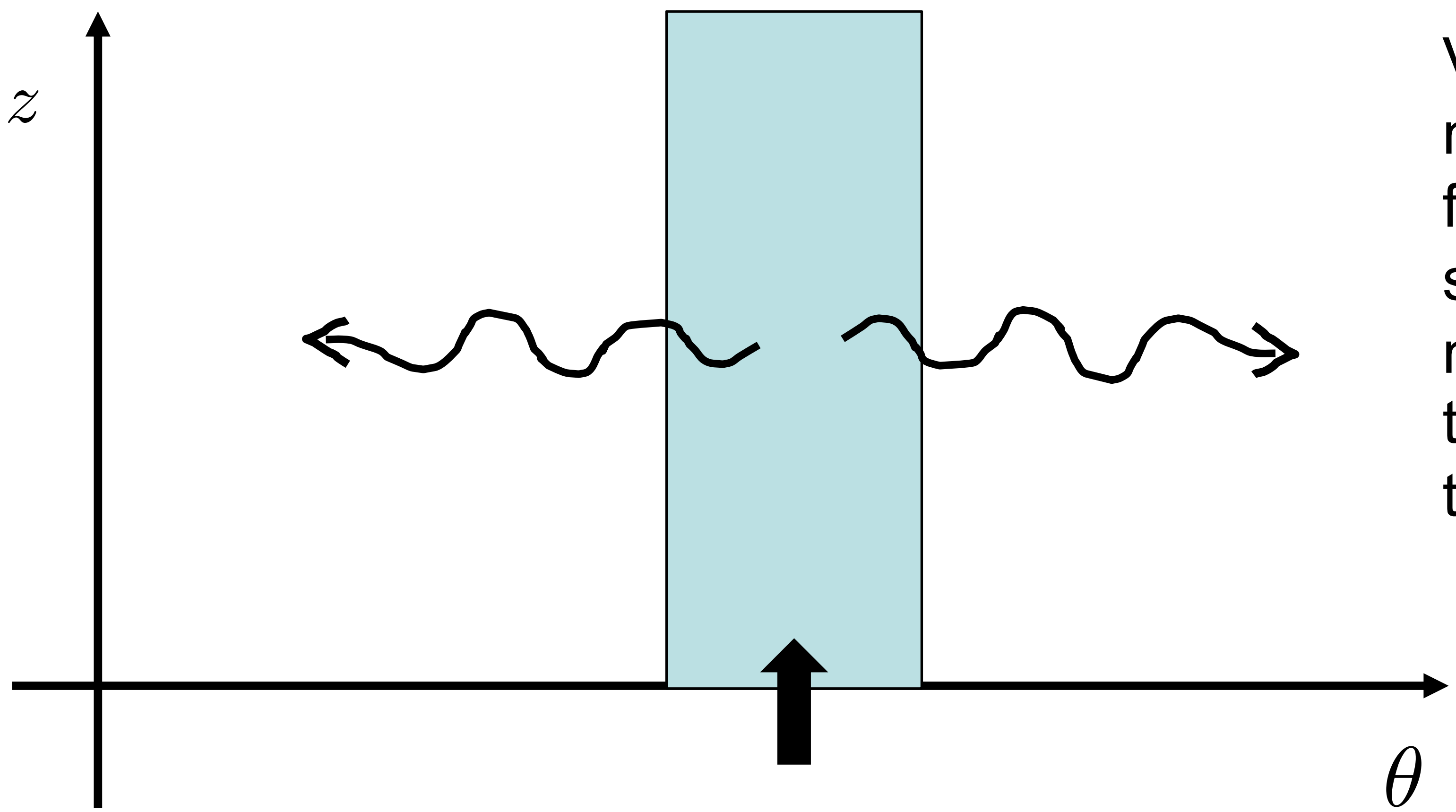


AM budget and sfc winds



If large-scale motions converge AM flux in a column, to achieve steady state friction needs to remove it through frictional drag at the surface.

AM budget and sfc winds



Viceversa, if large-scale motions diverge AM flux from a column, to achieve steady state friction needs to acquire it through frictional drag at the surface.

AM budget and sfc winds

To link AM flux divergence/convergence to the surface winds, we are going to parameterize friction as Rayleigh drag, that is we are going to assume that friction is proportional to but opposing surface winds

$$F_\lambda = -ku_{sfc}$$

Hence, the AM budget becomes

$$\nabla \cdot \int \int \rho M \vec{v} d\lambda dz \propto -ku_{sfc}$$

which shows that

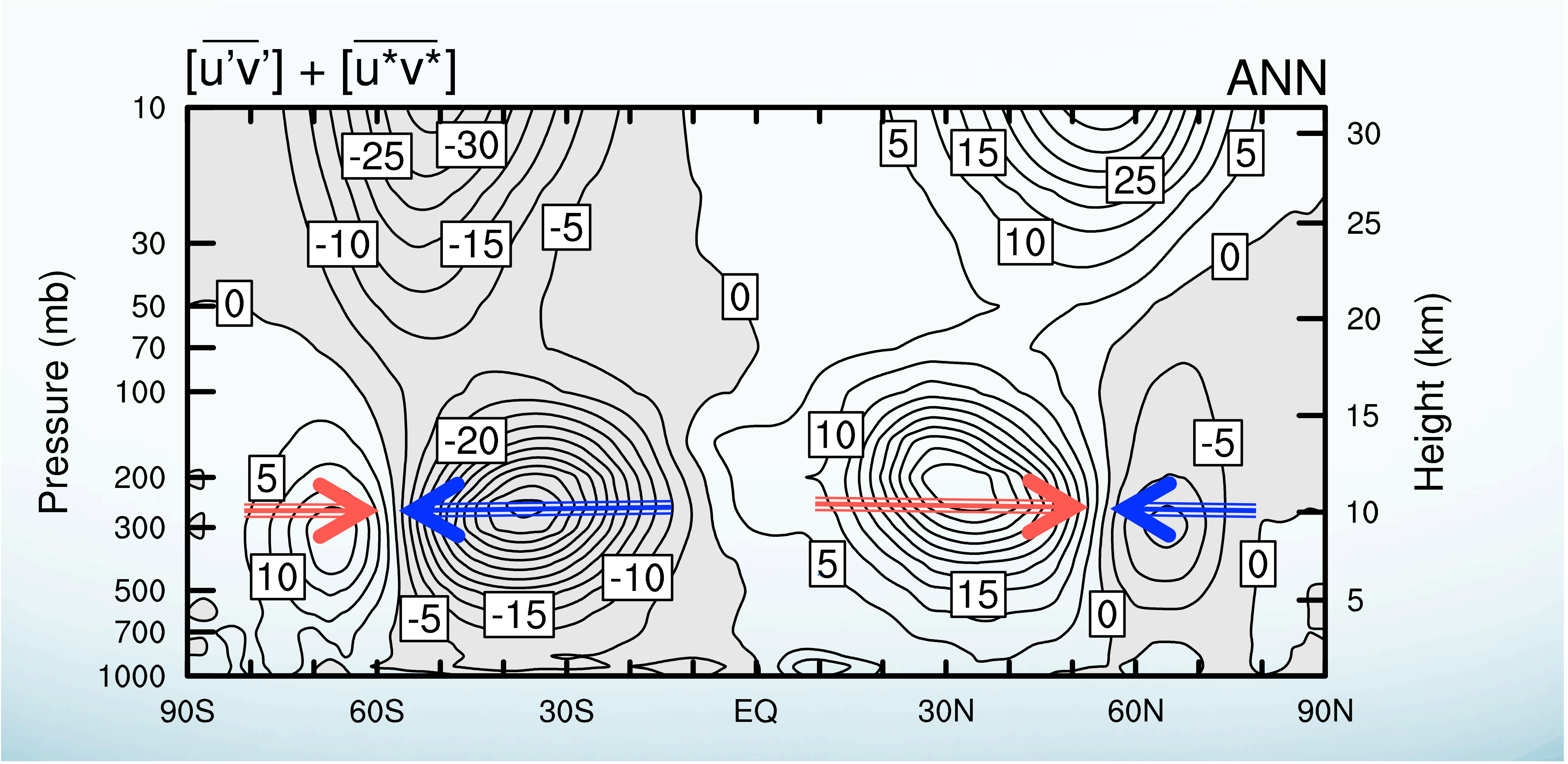
$$\begin{cases} u_{sfc} > 0 & \text{where AM converges} \\ u_{sfc} < 0 & \text{where AM diverges} \end{cases}$$

AM budget and sfc winds

If AM is converging into a column, steady state can be achieved only if AM is transferred from atmosphere into the surface.
If AM is diverging from a column, column needs to acquire AM from the SFC. We already saw, somewhat empirically, that AM diverges away from tropics and converges in the midlatitudes. This explains why we have sfc easterlies in the near equatorial region and sfc westerlies in the midlatitudes.

As we will show, one of the fundamental properties of large-scale waves is that they converge zonal momentum in their source region. This is a consequence of the Coriolis parameter increasing with latitude.

Momentum transport by eddies



Hartmann (1994)

Rossby waves

Simplest model to study Rossby wave dynamics is barotropic dynamics on $\beta -$ plane. This means a 2D flow

$$\vec{u} = (u, v)$$

$$f = 2\Omega \sin\theta = f_o + \beta y$$

$$y = a(\vartheta - \vartheta_o)$$

$$\beta = \left.\frac{df}{dy}\right|_o = \frac{2\Omega \cos\theta_o}{a}$$

Momentum equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - (f_o + \beta y)v = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{(a)}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + (f_o + \beta y)u = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \text{(b)}$$

Rossby waves

Taking

$$\frac{\partial}{\partial x}(\mathrm{b}) - \frac{\partial}{\partial y}(\mathrm{a})$$

we obtain a vorticity equation

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + \beta v = 0 \qquad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Also using continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Vorticity equation is a statement of conservation of absolute vorticity

$$\frac{Dq}{Dt} = 0 \qquad q = f + \zeta$$

Rossby waves

Given that the flow is nondivergent and 2D, we can define a streamfunction ψ

$$(u, v) = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x} \right) \qquad \zeta = \nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}$$

So we can rewrite the vorticity equation as

$$\frac{\partial \zeta}{\partial t} - \frac{\partial \psi}{\partial y}\frac{\partial \zeta}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial \zeta}{\partial y} + \frac{df}{dy}\frac{\partial \psi}{\partial x} = 0$$

$$\frac{\partial \zeta}{\partial t} + J(\psi, f + \zeta) = 0$$

J is just the Jacobian and it allows to write in a compact way the advection terms.

$$J(A, B) = \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x} \qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + J(\psi,)$$

Rossby waves

We are going to linearize vorticity equation about a mean state with uniform zonal wind

$$u = \overline{u} + u' \qquad \overline{u} = -\frac{\partial \overline{\psi}}{\partial y} \qquad \overline{\psi} = -\overline{u}y \qquad \overline{\zeta} = \frac{\partial \overline{v}}{\partial x} - \frac{\partial \overline{u}}{\partial y} = 0$$

$$v = v' \qquad \overline{v} = \frac{\partial \overline{\psi}}{\partial x} = 0$$

Vorticity equation for the perturbed state is

$$\frac{\partial \zeta'}{\partial t} + J(\overline{\psi} + \psi', f + \zeta') = 0$$

$$\frac{\partial \zeta'}{\partial t} + J(\overline{\psi}, \zeta') + J(\psi', f) = 0$$

$$\frac{\partial \zeta'}{\partial t} + \overline{u}\frac{\partial \zeta'}{\partial x} + v'\beta = 0$$

Rossby waves

$$\frac{\partial \zeta'}{\partial t} + \overline{u}\frac{\partial \zeta'}{\partial x} + v'\beta = 0$$

$$v' = \frac{\partial \psi'}{\partial x}$$

$$u' = -\frac{\partial \psi'}{\partial y}$$

$$\zeta' = \nabla^2 \psi'$$

These are linear equations with constant coefficients and solutions are linear plane waves

$$\psi'(x, y, t) = Re\left\{\hat{\psi} e^{i(kx+ly-\omega t)}\right\}$$

$$\zeta'(x, y, t) = Re\left\{\hat{\zeta} e^{i(kx+ly-\omega t)}\right\}$$

$$\hat{\zeta} = -(k^2 + l^2)\hat{\psi}$$

Properties of linear plane waves

Wave amplitude $\hat{\psi}$

Frequency ω

Zonal and meridional wavenumber k, l

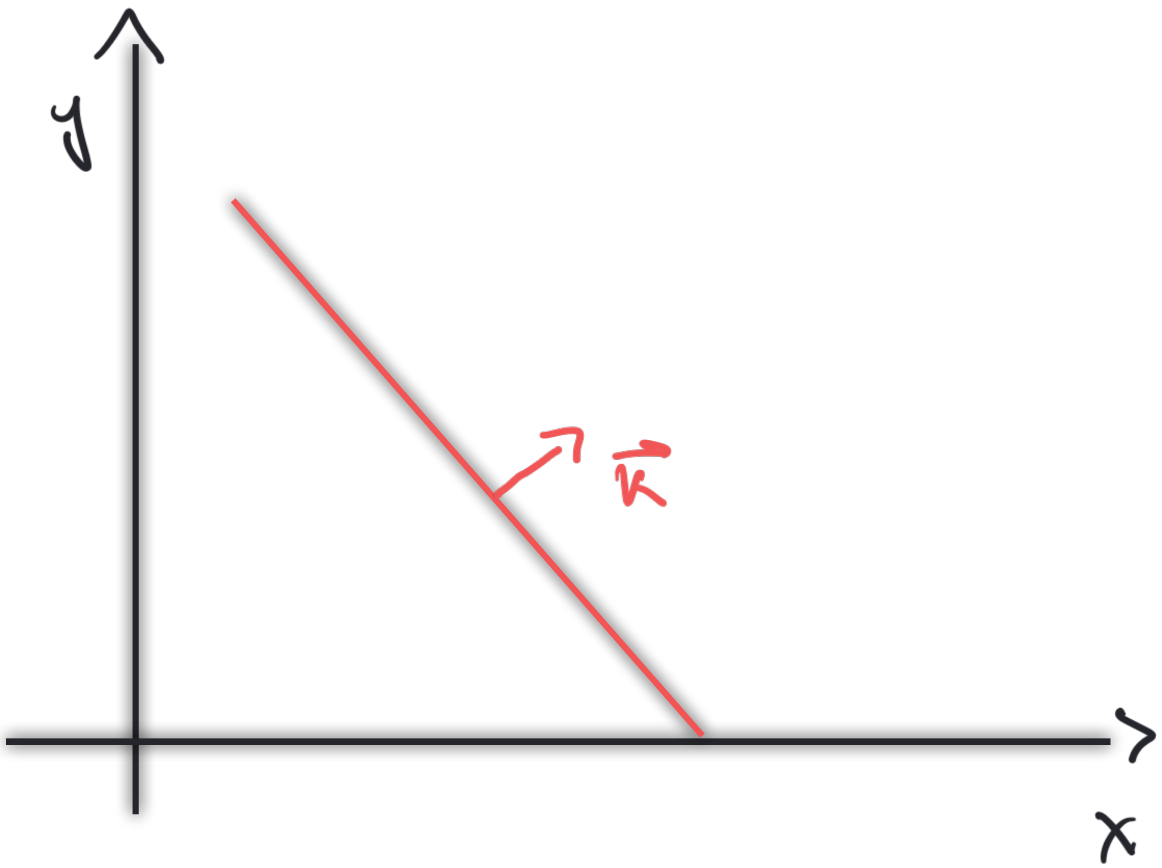
Wave vector $\vec{K} = (k, l)$

Phase of the wave

$$\phi(x, y, t) = kx + ly - \omega t$$

The wave has constant value on lines of constant phase

Properties of linear plane waves



$$\phi = kx + ly - \omega t$$

Assuming constant ϕ

$$y = -\frac{k}{l}x + \text{const}$$

$$k, l > 0$$

Lines of constant phase propagate in the direction perpendicular to the line of constant phase, which defines the direction of the wave vector

$$\vec{k} = \nabla\phi = \left(\frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y}\right)$$

Properties of linear plane waves

$$\phi = \vec{k}\cdot\vec{x} - \omega t = \mathbb{K}s - \omega t \qquad \mathbb{K} = (k^2 + l^2)^{1/2}$$

Waves propagate in such a way that the value attained at s is the same attained at $s + \lambda$

$$e^{i\mathbb{K}s} = e^{i\mathbb{K}(s+\lambda)} = e^{i(\mathbb{K}s + 2\pi)} \qquad \lambda = \frac{2\pi}{\mathbb{K}}$$

$$\lambda_x = \frac{2\pi}{k} \qquad \lambda_y = \frac{2\pi}{l}$$

Properties of linear plane waves

Phase speed: speed at which lines of constant phase propagate

$$\phi = \mathbb{K}s - \omega t = kx + ly - \omega t$$

$$c_p = \left.\frac{\partial s}{\partial t}\right)_\phi = \frac{\omega}{\mathbb{K}} \qquad c_{p,x} = \left.\frac{\partial x}{\partial t}\right)_{\phi,y} = \frac{\omega}{k} \qquad c_{p,y} = \left.\frac{\partial y}{\partial t}\right)_{\phi,x} = \frac{\omega}{l}$$

Group velocity: velocity at which the wave group propagates. This is also the velocity of energy propagaton.

Waves are called **non dispersive** if their phase speed does not depend on wave vector. This means that a wave packet given by the superposition of waves with different wavenumbers will retain its shape as it propagates.

Properties of linear plane waves

Waves are called **dispersive** if their phase speed is a function of the wave vector.

When this is the case, the individual wave in a wave packet will propagate with different phase speeds and the packet will change its shape (will disperse).

$$\vec{c}_g = \frac{\partial \omega}{\partial \vec{k}} = \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l} \right)$$

Properties of Rossby waves

Going back to the barotropic perturbation vorticity equation

$$\frac{\partial}{\partial t}\nabla^2\psi' + \overline{u}\frac{\partial}{\partial x}\nabla^2\psi' + \beta\frac{\partial\psi'}{\partial x} = 0$$

and replacing into it our wavy solution $\psi'(x,y,t) = \hat{\psi}e^{i(kx+ly-\omega t)}$

$$-(k^2+l^2)[-i\omega + ik\overline{u}]\hat{\psi}e^{i(kx+ly-\omega t)} + \beta ik\hat{\psi}e^{i(kx+ly-\omega t)} = 0$$

$$\big[-(k^2+l^2)(-i\omega + ik\overline{u}) + \beta ik\big]\hat{\psi} = 0$$

Non trivial solutions will exist if

$$\omega = \overline{u}k - \frac{\beta k}{k^2+l^2}$$

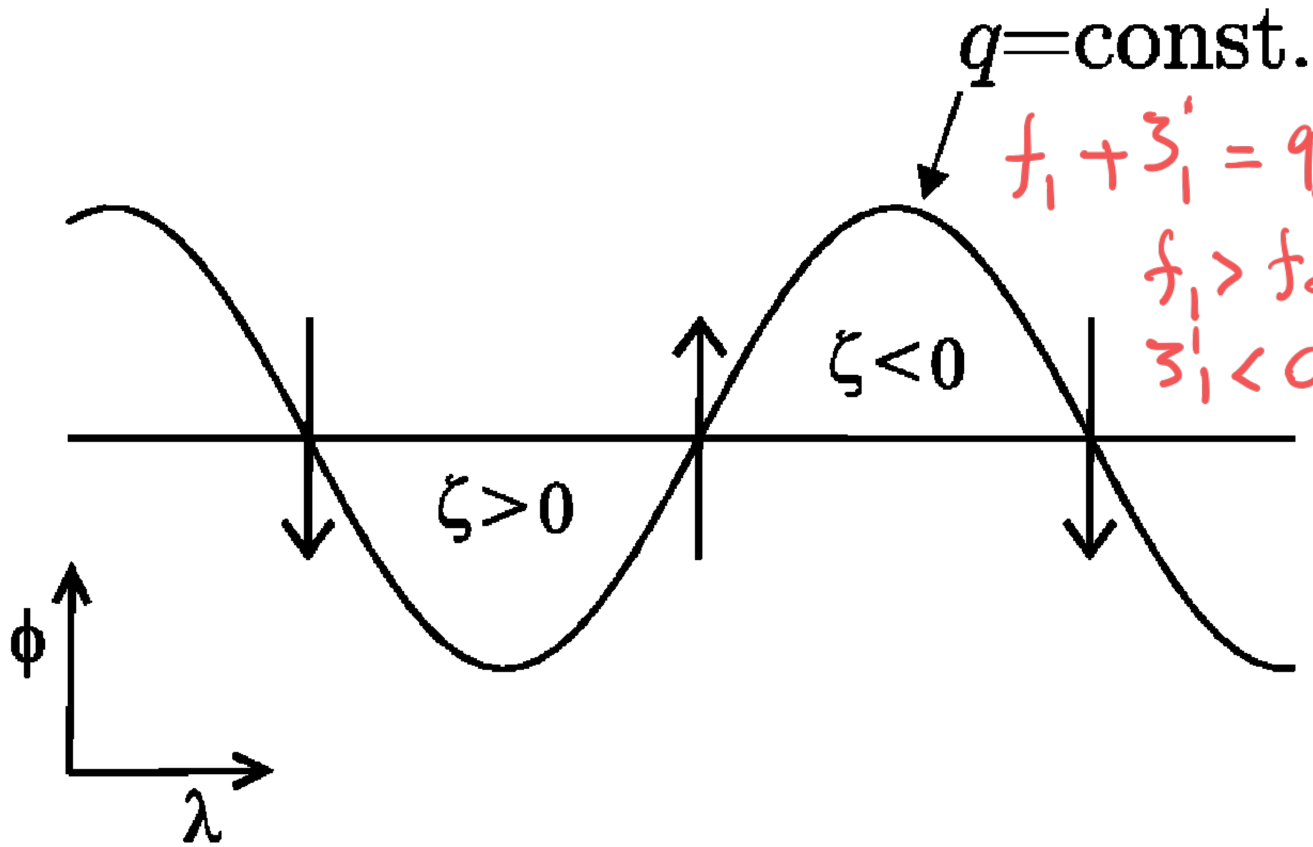
Dispersion relation of Rossby waves

These are waves with westward phase speed relative to the mean flow

$$c_{p,x} = \frac{\omega}{k} = \overline{u} - \frac{\beta}{k^2+l^2}$$

Properties of Rossby waves

a) Propagation mechanism of Rossby waves relies on conservation of absolute vorticity. NH



$q_e = f_0$

$f_1 + \zeta_1' = q_e = f_0$

$f_1 > f_0$

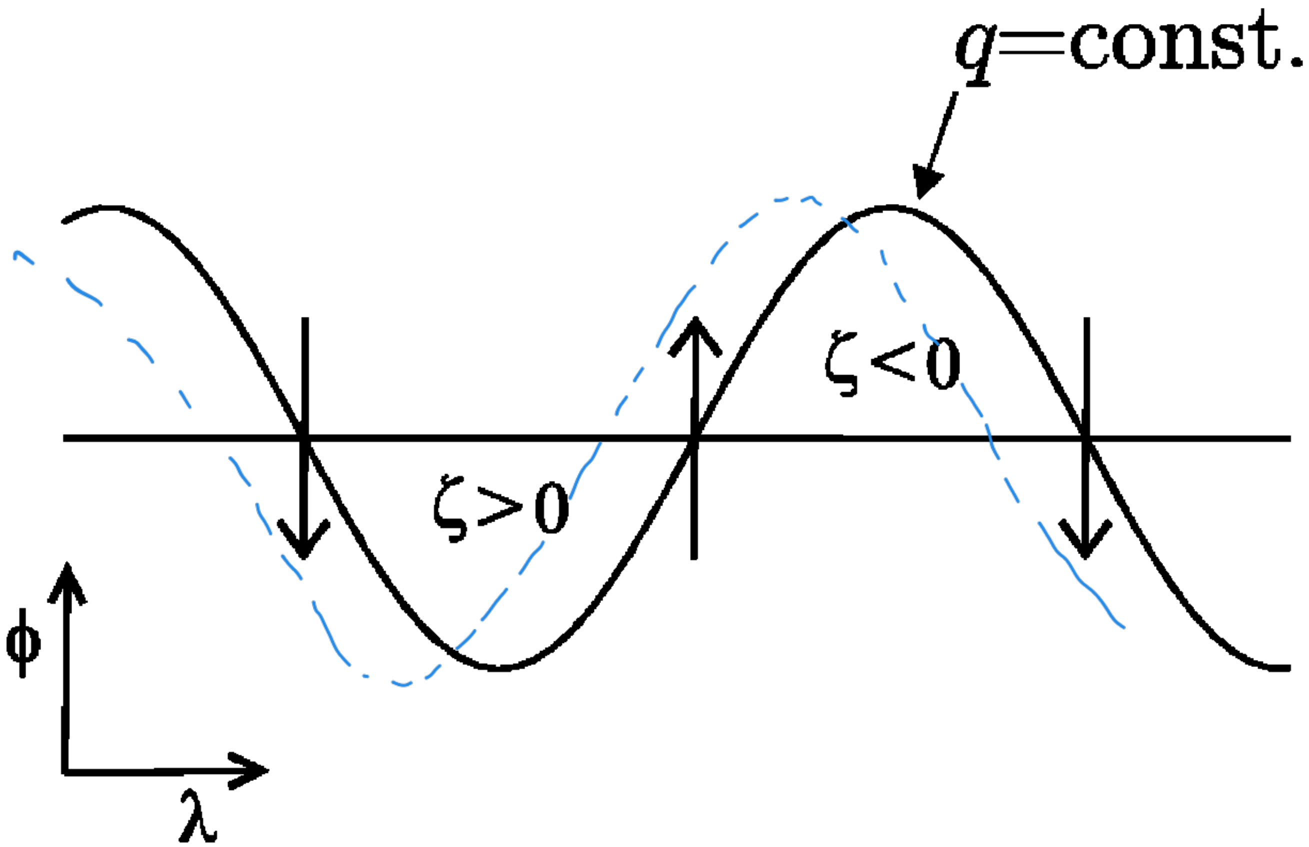
$\zeta_1' < 0$

For parcels displaced poleward of their initial position, conservation of q implies a negative relative vorticity perturbation

For parcels displaced equatorward, conservation of q implies a positive relative vorticity perturbation

Properties of Rossby waves

a) Propagation mechanism of Rossby waves relies on conservation of absolute vorticity.



Impact on the resulting meridional velocity will cause the vorticity perturbation to propagate to the west with time:

$$\frac{\partial \zeta'}{\partial t} = -\beta v' \begin{cases} <0 & v'>0 \\ >0 & v'<0 \end{cases}$$

Overall effect is westward phase propagation relative to the mean flow.

Properties of Rossby waves

b) Zonal phase speed

$$c_{p,x} = \frac{\omega}{k} = \bar{u} - \frac{\beta}{k^2 + l^2}$$

c) Meridional group velocity

$$c_{g,y} = \frac{\partial \omega}{\partial l} = \frac{2\beta k l}{(k^2 + l^2)^2}$$

Sign of the group velocity will depend on the sign of kl

Properties of Rossby waves

For any k and ω

$$l = \pm \left[\frac{\beta}{\bar{u} - c} - k^2 \right]^{1/2}$$

So we have a sign ambiguity in the meridional wavenumber

Meridional propagation of Rossby waves

Which direction will Rossby waves propagate?

Sign of meridional group velocity depends on kl and we have a sign ambiguity. How do we resolve this? That is to say, given that we assume $k > 0$, how do we determine in which meridional direction energy propagates?

We are going to resolve this ambiguity using a commonly used approach in problems related to wave propagation.

WE ADD A SMALL AMOUNT OF FRICTION TO THE PROBLEM

Radiation condition and Rayleigh friction

Solution with small amount of friction can tell us which solution in the inviscid problem (no friction) is physically plausible.

Going back to barotropic vorticity equation:

$$\frac{\partial \zeta'}{\partial t} + \bar{u}\frac{\partial \zeta'}{\partial x} + \beta v' = -r\zeta' \qquad r > 0$$

Looking for the same wavy solutions

$$+i\omega(k^2+l^2)\hat{\psi} - \bar{u}ik(k^2+l^2)\hat{\psi} + i\beta k\hat{\psi} - r(k^2+l^2)\hat{\psi} = 0$$

Radiation condition and Rayleigh friction

Dispersion relation now is

$$\omega_r = \bar{u}k - \frac{\beta k}{k^2+l^2} - ir$$

$$\omega_r = \omega_R - ir$$

Replacing this complex frequency in our wavy solutions

$$\psi'(x,y,t) = \mathrm{Re}\left\{ \hat{\psi}\, e^{i(kx+ly-\omega_R t)}\, e^{-rt} \right\}$$

$$e^{i\phi} = \cos\phi + i\sin\phi$$

Radiation condition and Rayleigh friction

Now, we can approach the same problem by assuming that the solution with friction will have a real frequency, but a meridional wavenumber with an imaginary part.

$$l_r = l + l'$$

Doing a first order Taylor expansion

$$\omega_R(k, l_r) \approx \omega_r + ir =$$

$$\approx \cancel{\omega_R(k, l)} + \underbrace{\left.\frac{\partial \omega_R}{\partial l_r}\right|_{l}}_{c_{g,y}} l' = \cancel{\omega_r} + ir$$

$$l' = \frac{ir}{c_{g,y}}$$

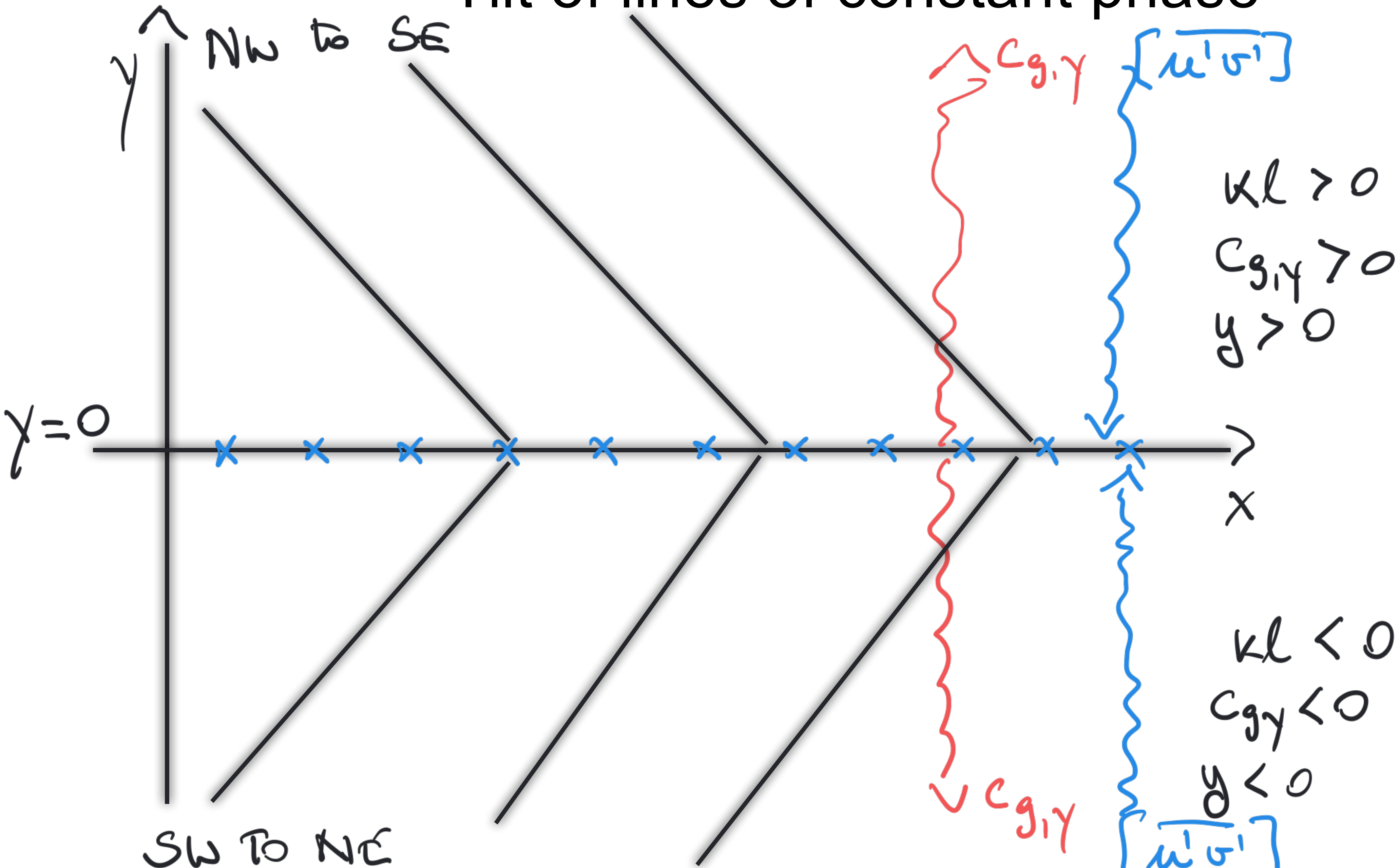
Radiation condition and Rayleigh friction

Solutions will be

$$\psi' = \mathrm{Re}\left\{ \hat{\psi}\, e^{i(kx+ly-\omega_R t)}\, e^{-\frac{ry}{c_{g,y}}} \right\}$$

Tilt of lines of constant phase

$$y = a(\vartheta - \vartheta_0)$$

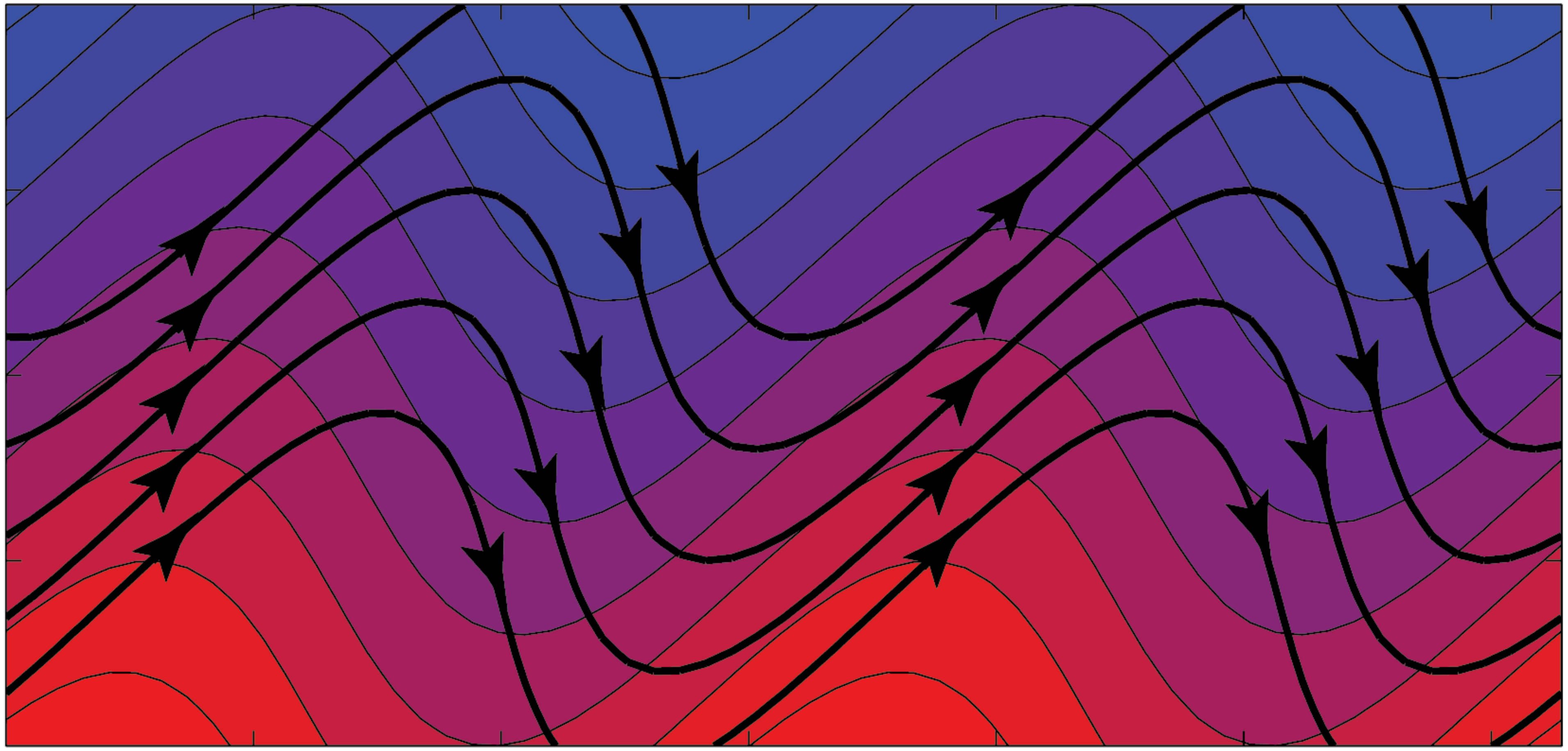


Lines of constant phase

$$kx + ly = \text{constant}$$

$$y = -\frac{k}{l}x + \text{const}$$

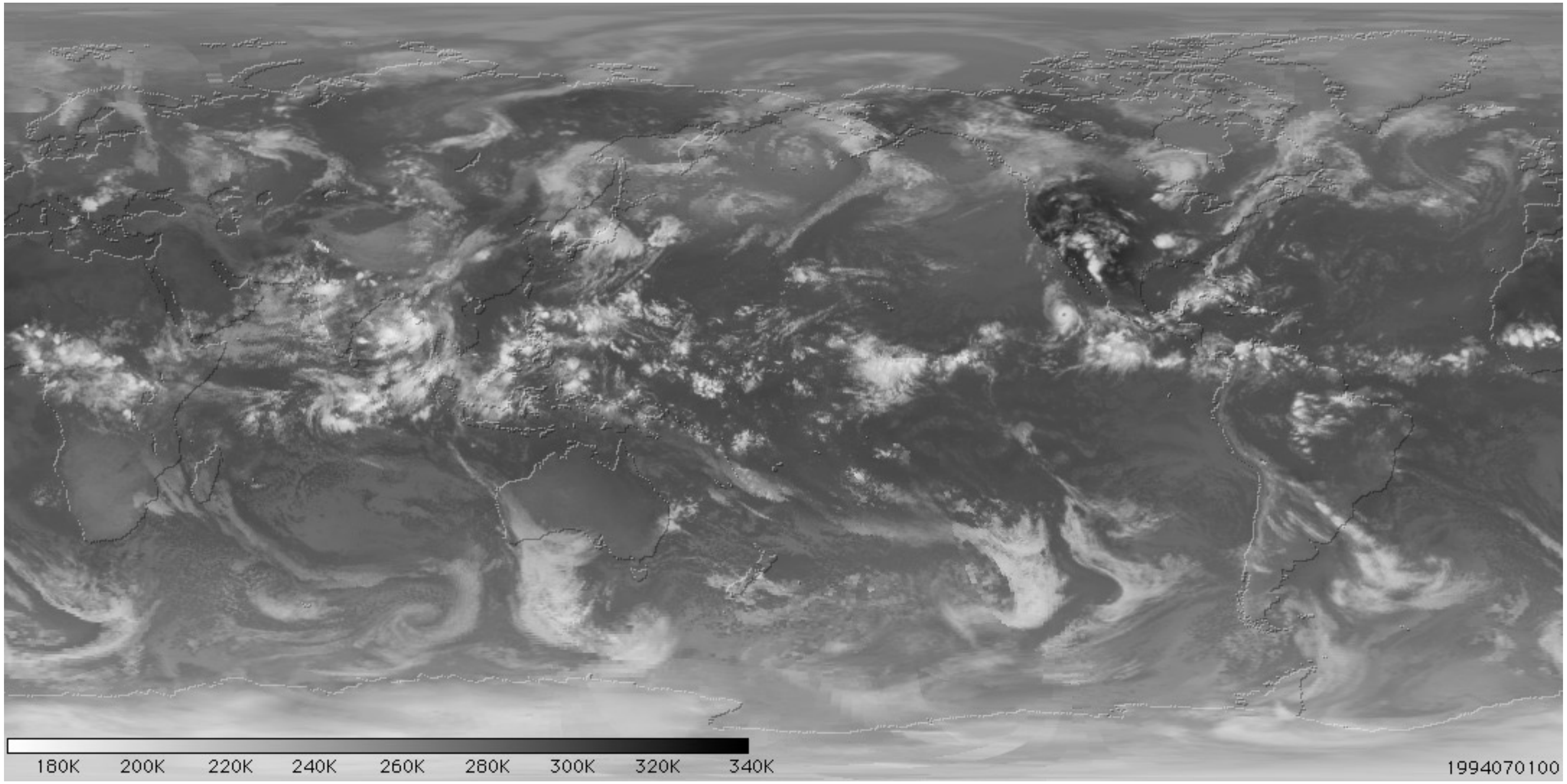
Tilt of lines of constant phase



u*>0	u*<0	u*>0	u*<0
v*>0	v*<0	v*>0	v*<0
T*>0	T*<0	T*>0	T*<0

ENERGY PROPAGATES AWAY FROM THE SOURCE REGION!

Tilt of lines of constant phase



The tilt of barotropic Rossby waves phase lines indicates the direction of the group velocity (and energy propagation).

Connection to the momentum fluxes

$$[\overline{u'v'}]$$

$$\psi' = Re\left\{\hat{\psi}\, e^{i(kx+ly-\omega t)}\right\}$$

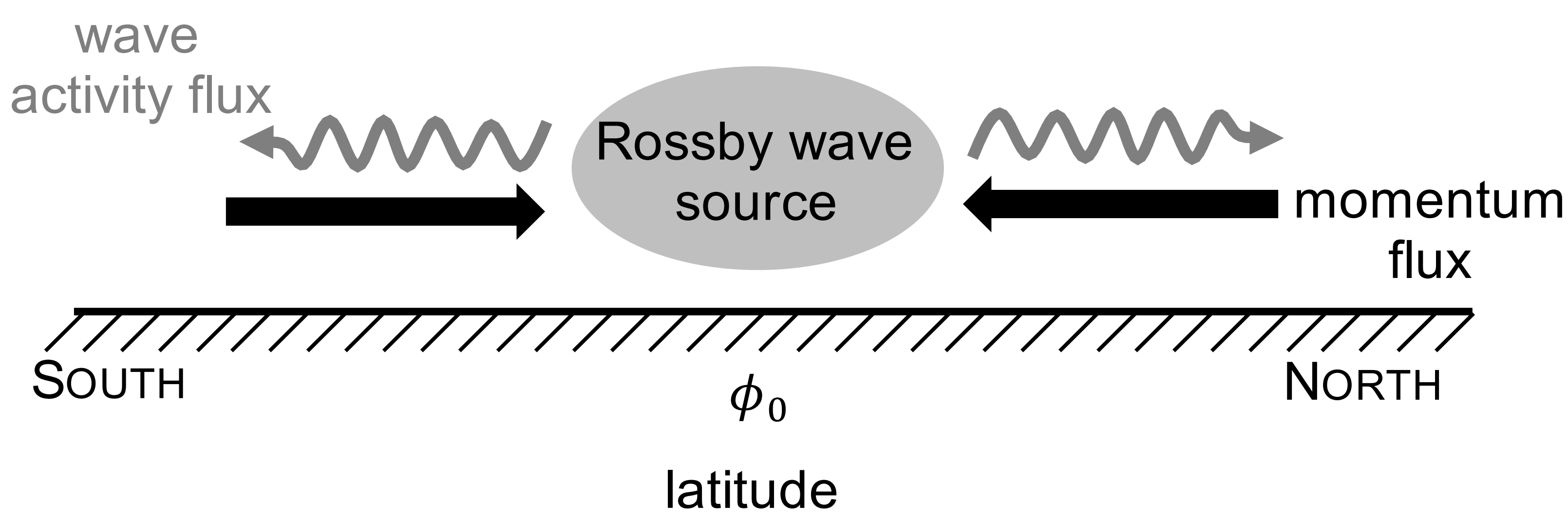
$$u' = -\frac{\partial\psi'}{\partial y}$$

$$v' = \frac{\partial\psi'}{\partial x}$$

$$[\overline{u'v'}] = \frac{1}{2\pi}\,\frac{1}{\Delta t}\int_0^{2\pi}\int_0^{\Delta t} u'v'\, d\lambda\, dt = -\frac{kl}{2}\,|\hat{\psi}|^2$$

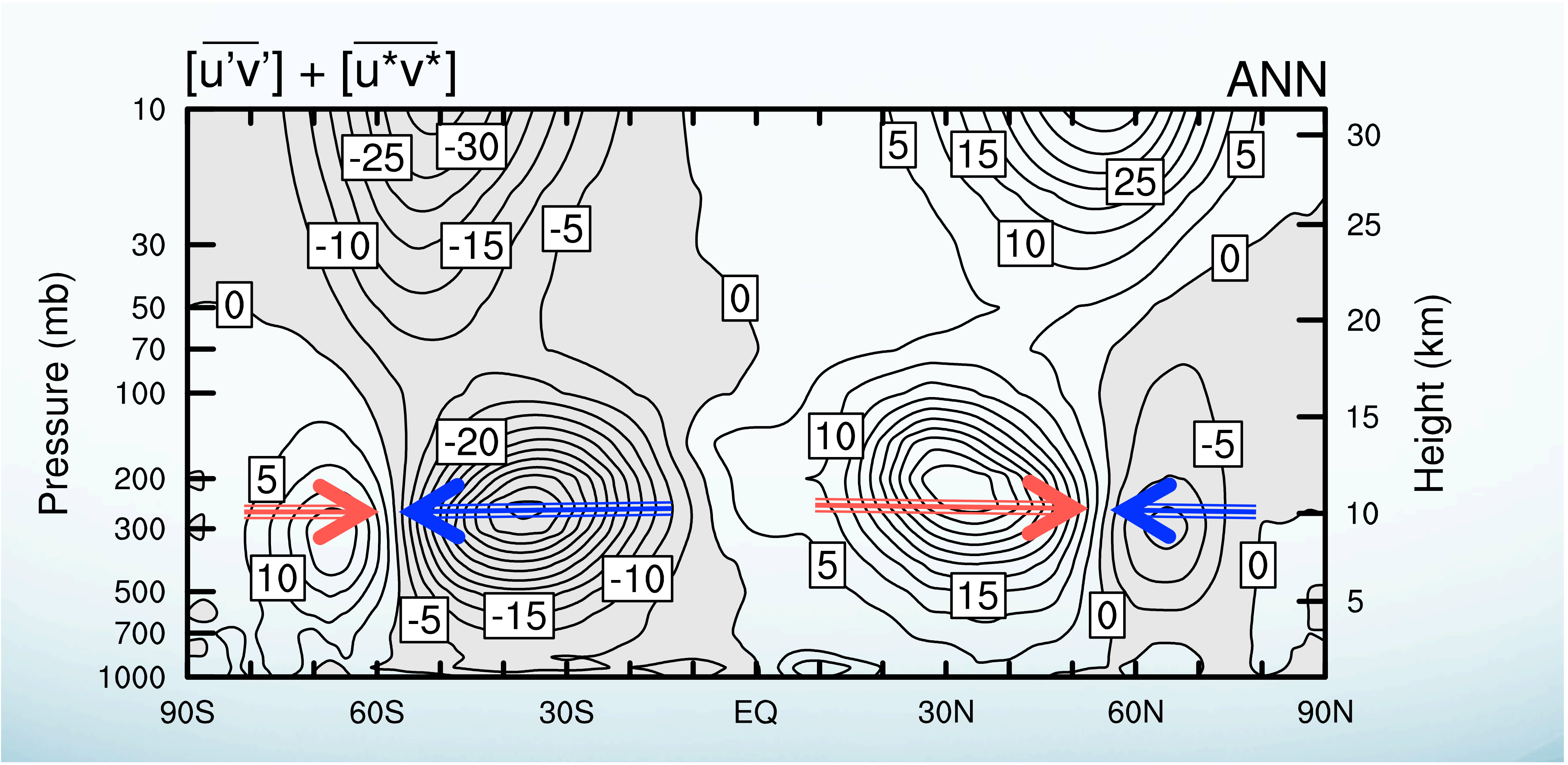
Connection to the momentum fluxes

Barotropic Rossby waves converge (westerly) momentum into source region!



Courtesy of Marty Singh

Connection to the momentum fluxes



Hartmann (1994)

Propagation of Rossby waves

Let's consider again the dispersion relation of Rossby waves:

$$c_{p,x} = c = \bar{u} - \frac{\beta}{k^2 + l^2}$$

$$l^2 = \frac{\beta}{\bar{u} - c} - k^2$$

When $l^2 > 0$, the solution is wave like. When $l^2 < 0$, the solution is evanescent. Hence, meridional propagation of Rossby waves requires $l^2 > 0$.

Propagation of Rossby waves

One necessary condition for Rossby wave propagation is

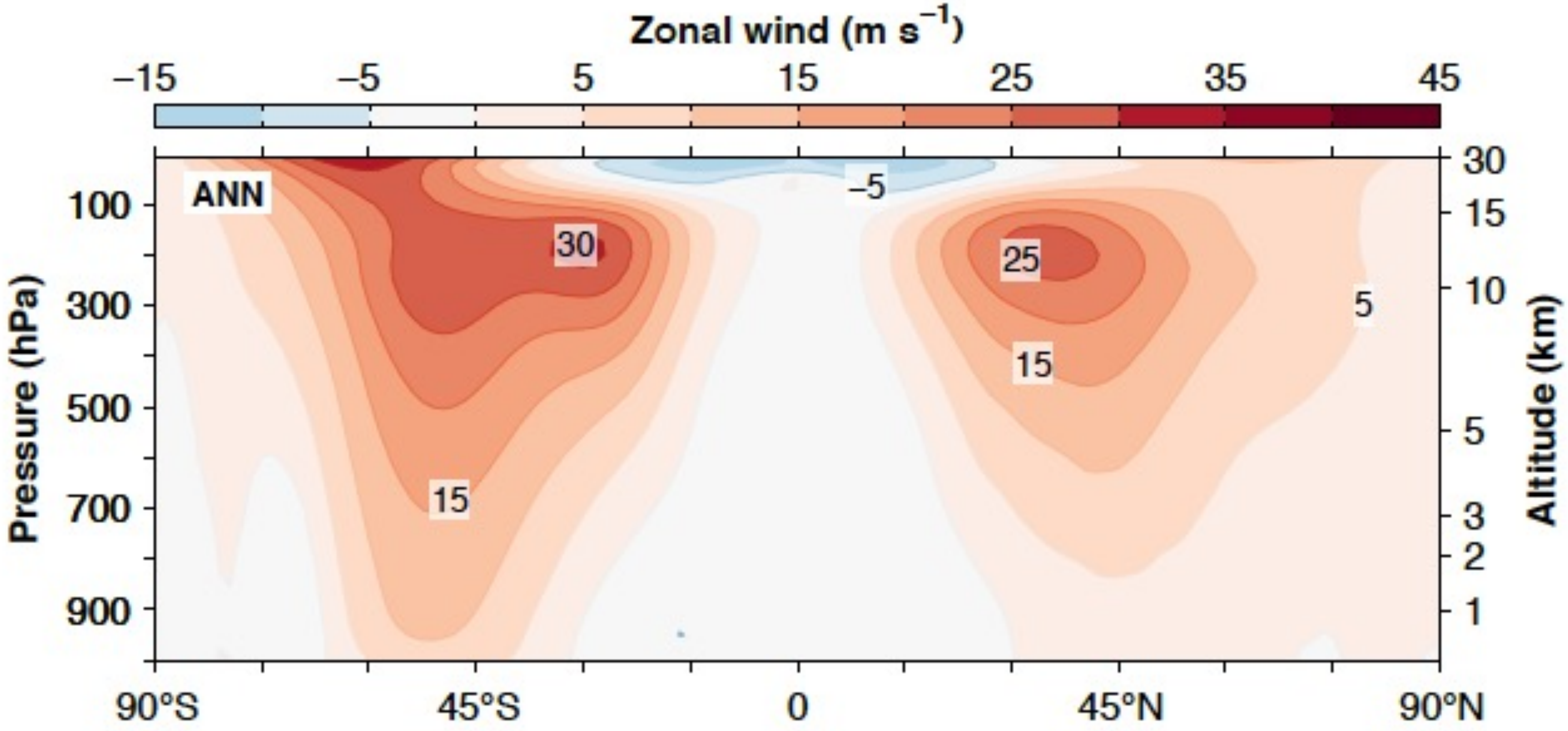
$$\bar{u} - c > 0$$

That is to say, the phase speed of the waves needs to be westward relative to the mean flow. While we derived this condition for constant background wind, it can be shown that this result carries over to the case of a slowly varying zonal wind.

As $\bar{u} \to c$ the wave group velocity slows down and the wave cannot propagate any longer. The linearity assumption breaks down and nonlinear wave breaking occurs.

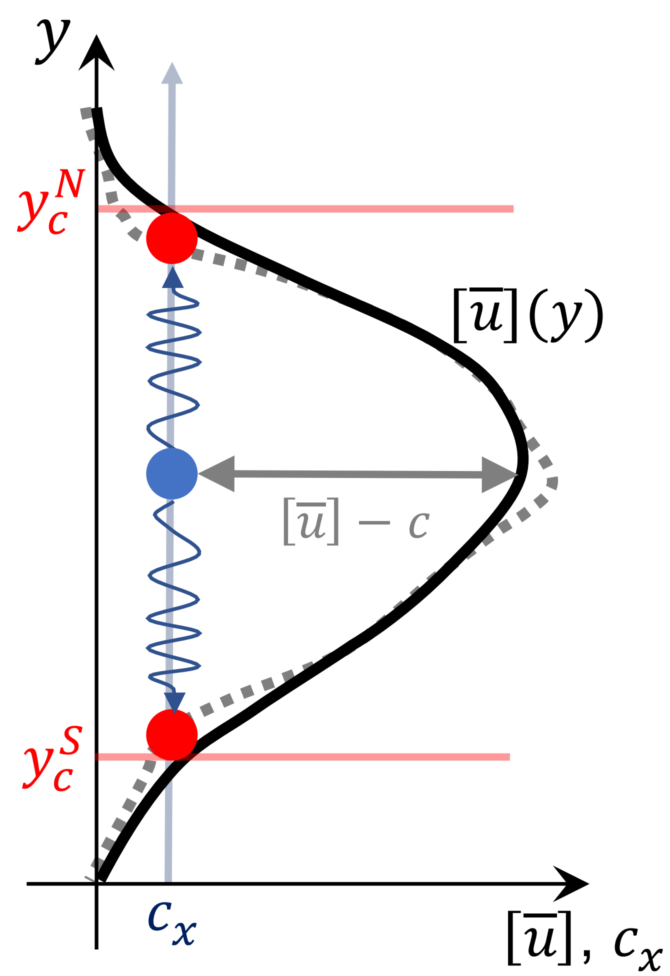
Location where this happens are called critical latitudes. Waves that approach their critical latitudes are expected to dissipate through nonlinear breaking and drag.

Propagation of Rossby waves



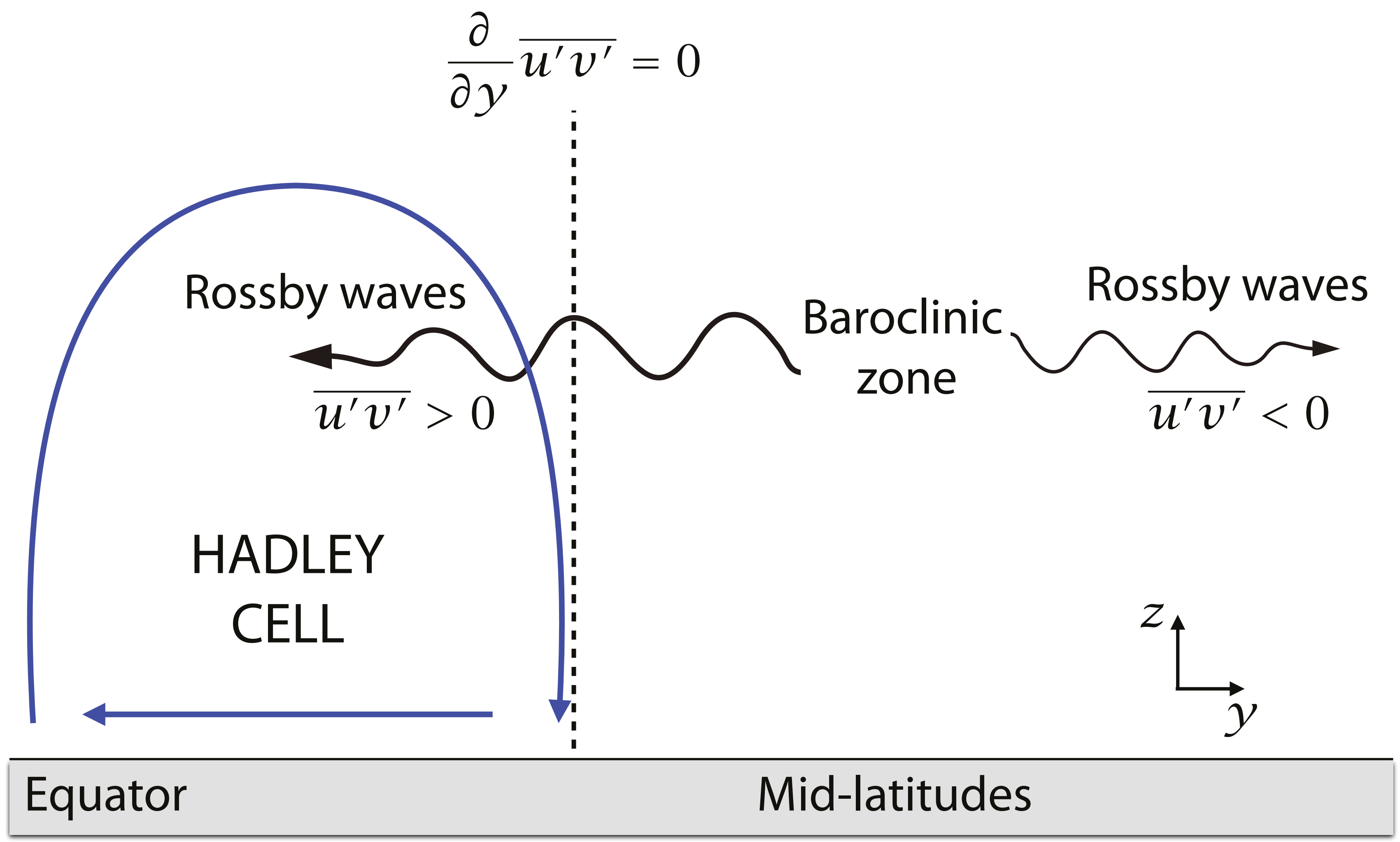
Propagation of Rossby waves

Important conclusion is that Rossby waves propagate away from their source region until they approach their critical latitude where they break. There is very little propagation past the zero wind line. The wave breaking results in eddy momentum flux divergence.

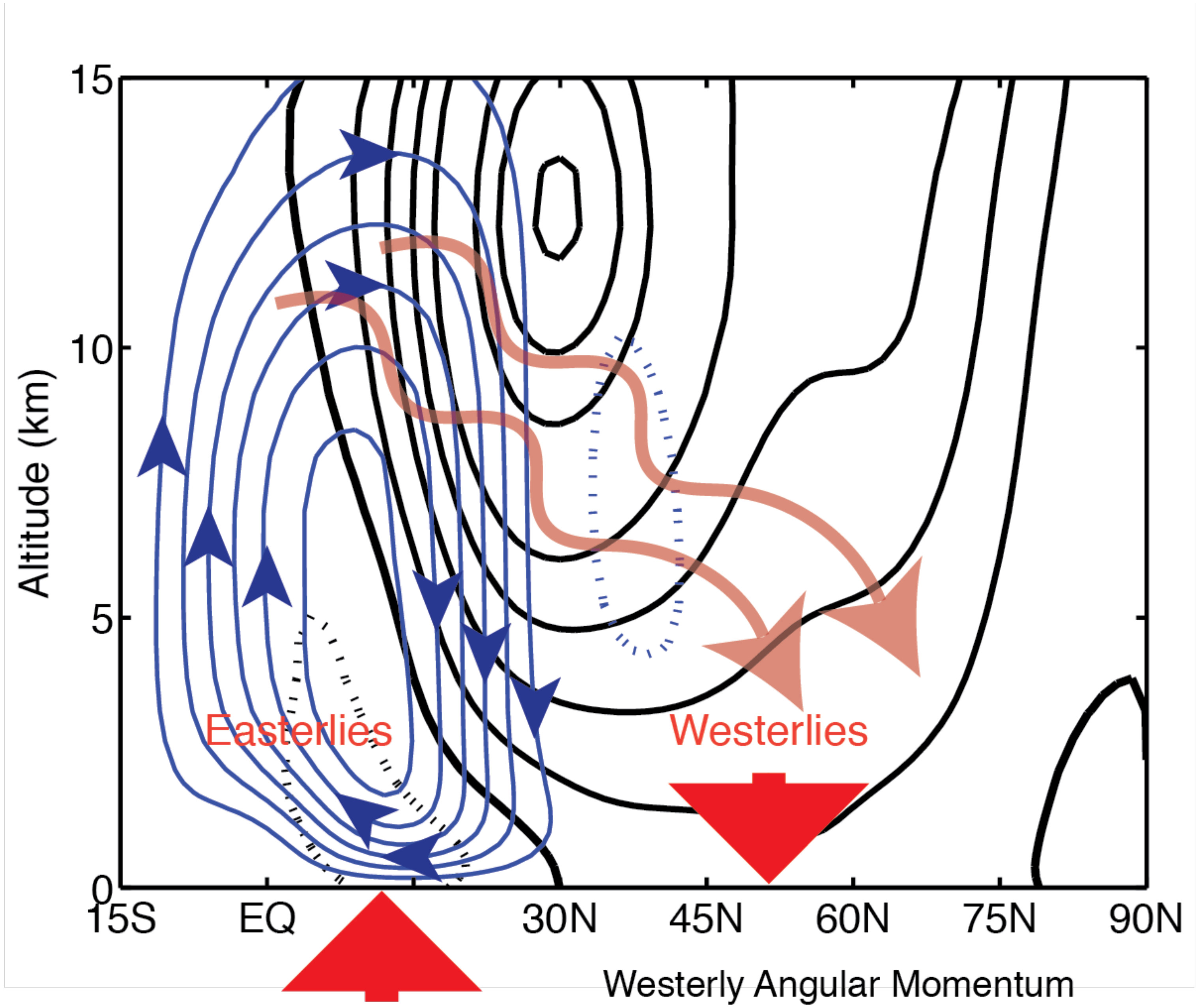


Courtesy of Marty Singh

Going back to the global angular momentum budget



Going back to the global angular momentum budget



Westerly Angular Momentum to Atmosphere in Easterlies

Westerly Angular Momentum to Earth in Westerlies

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2. Held, 2000: The General Circulation of the Atmosphere, Lecture Notes from Woods Hole Summer School https://www.gfdl.noaa.gov/wp-content/uploads/files/user_files/ih/lectures/woods_hole.pdf)