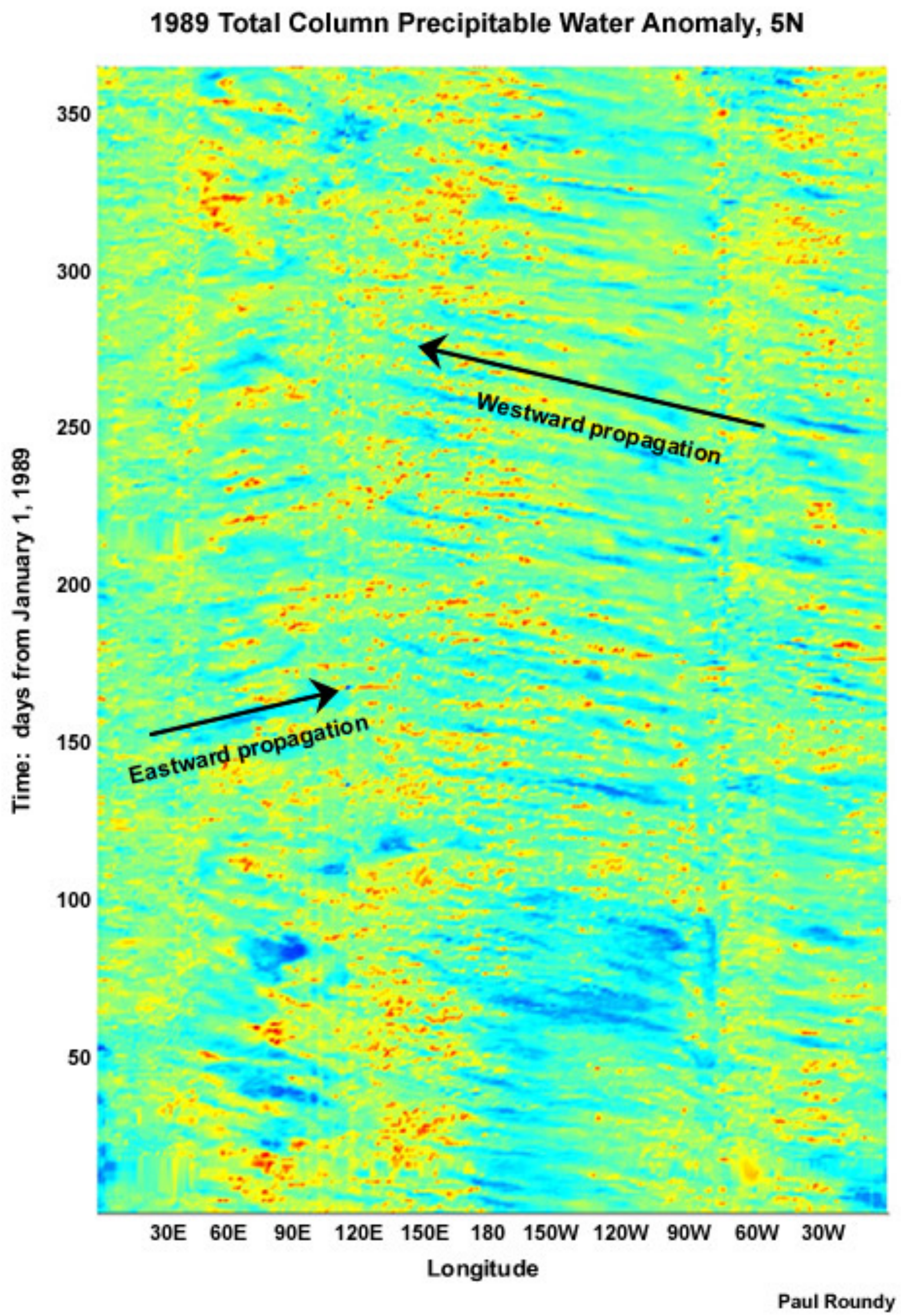
Equatorial waves

Theory of equatorial waves extends back to Matsuno (1966), but at that time it was impossible to diagnose the existence of these waves in observations. Gill (1980) extended the Matsuno wave theory, but a fundamental question remained: ARE THESE THEORETICAL WAVES PRESENT AND IMPORTANT IN THE TROPICAL ATMOSPHERE?

We have finally been able to detect these waves thanks to satellite data. Usually these observations are based on Hovmoller diagrams of appropriately filtered fields (such as OLR or precipitable water).

Evidence of equatorial waves



Equatorial waves

Some waves are observed to be coupled to convection, while others are not. In general, and we will discuss why, convectively-coupled waves propagate more slowly than the dry counterparts.

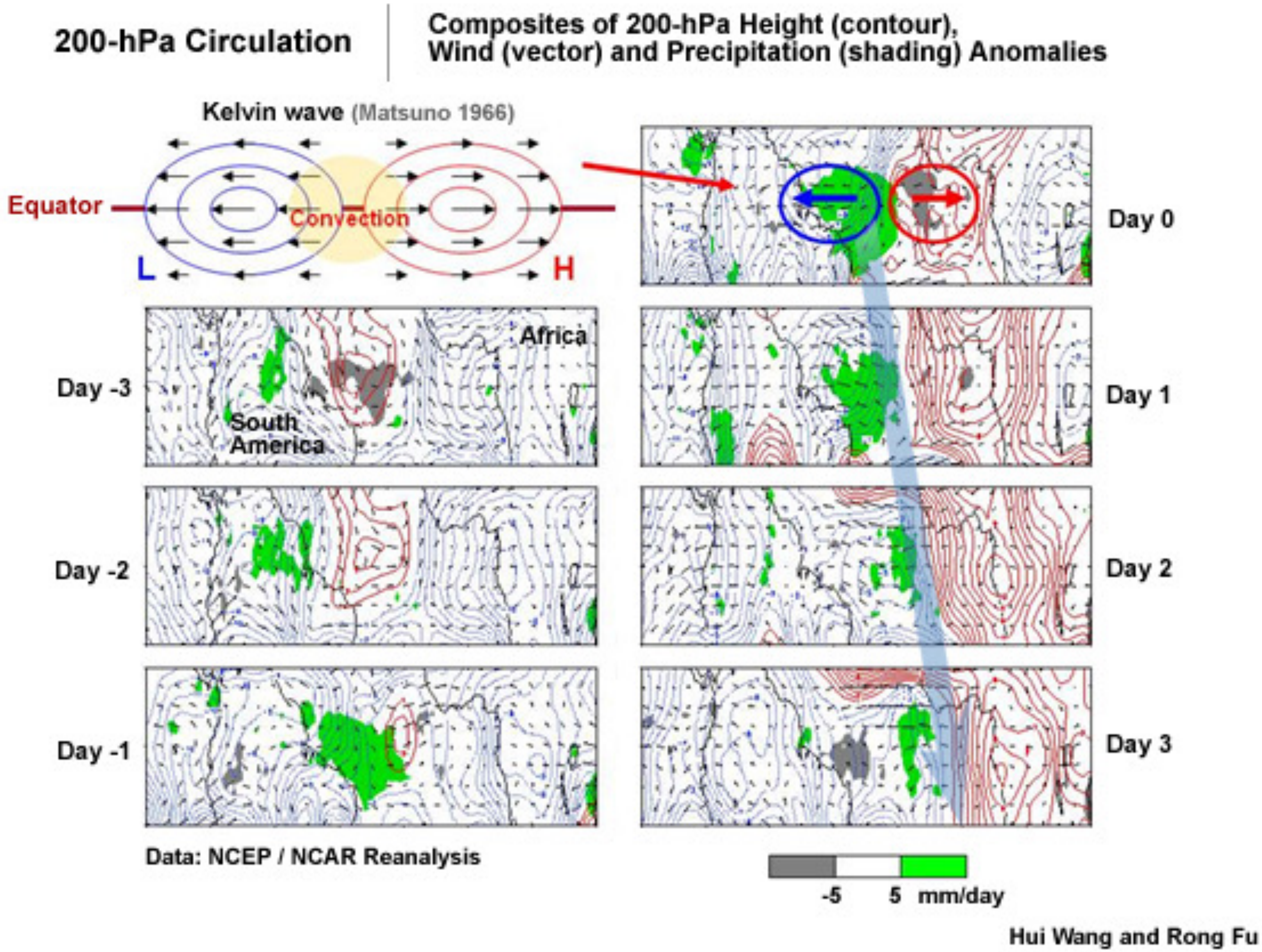
We will start with some observational evidence, to then move to the theory, which is based on dry dynamics, and we will then consider how convection modifies the dry dynamics.

Kelvin waves

Kelvin waves have long been known to fluid dynamicists interested in the atmosphere. They were first identified by Lord Kelvin in the 19th century. Broadly speaking, Kelvin waves are large-scale waves that are trapped around a boundary and propagate along it. The boundary can be a physical boundary, such as a coastline or a mountain range. But in the tropical region, the equator also represents a boundary, as this is where the Coriolis parameter changes sign.

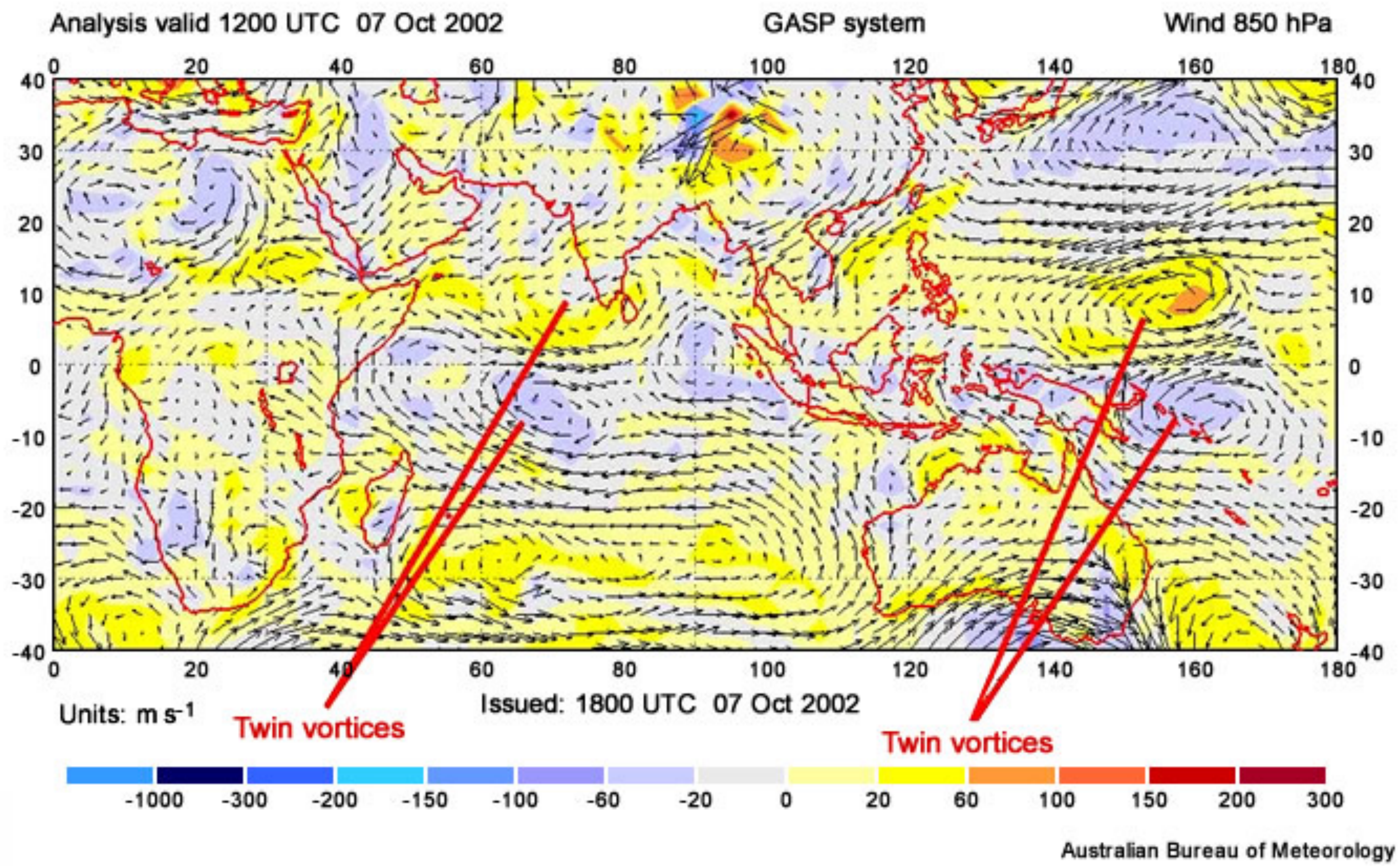
Convectively-coupled Kelvin waves have a typical period of 6 – 7 days, and a phase speed of 12 – 25 m/s. In the Indian ocean, they propagate more slowly at a speed of about 12 – 15 m/s.

Kelvin waves



Rossby waves

Just as in the extratropics, Rossby waves exist because of PV conservation on a varying PV field.



Rossby waves

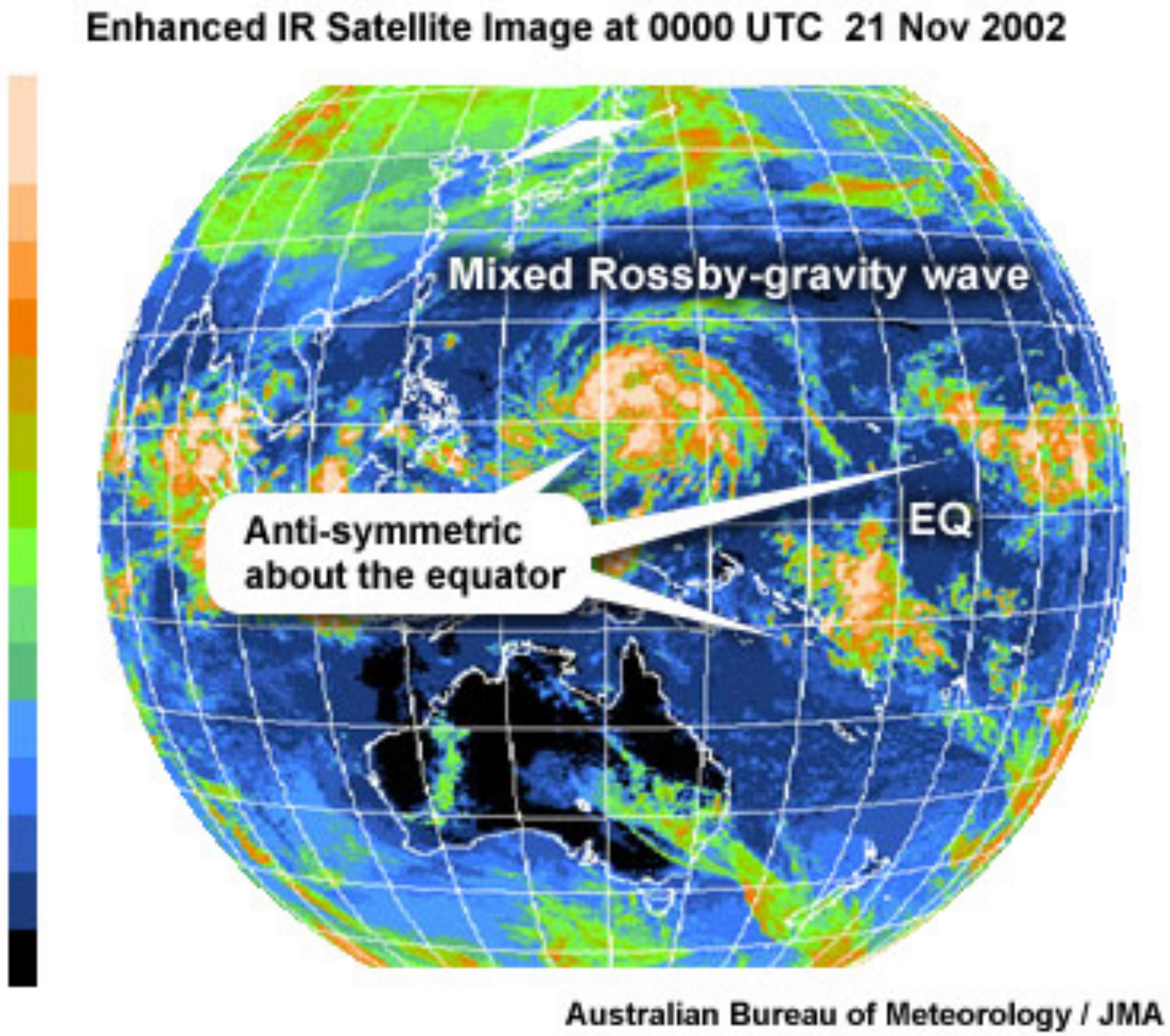
Long Rossby waves typically range in size from about 4000 to 10000 km. The meridional distance between the two twin vortices is about 20 degrees of latitude. Their lifetime is long, spanning from a few days to weeks. They propagate westward with speeds of the order of 10 – 20 m/s for dry waves, 5 – 7 m/s for convectively-coupled waves (and oceanic Rossby waves propagate at a speed of about 1 m/s). Given that the equatorial Pacific is about 17,760 Km across, an atmospheric Rossby wave would travel across the Pacific in about 18 days, while an oceanic Rossby wave takes about 210 days.

Mixed Rossby-Gravity (MRG) waves

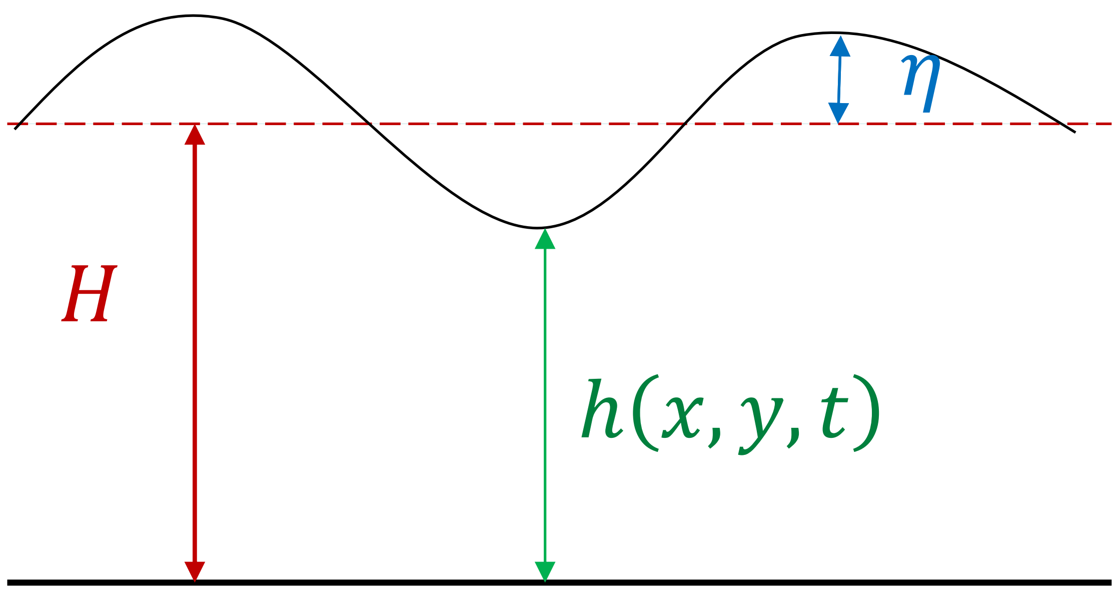
These waves are common in the deep tropics. They are usually forced by clusters of thunderstorms, which makes them very important for short-term forecasts. Longitudinal scales of these waves is about 1,000 – 4,000 km, with periods of 4 – 5 days, moving at typical speeds of 8 – 10 m/s.

We will see how these waves are a distinc feature of the tropical atmosphere and they are subject to two competing restoring forces, PV conservation and buoyancy. As such, they are characterized by divergent PV centers. Such waves are hence expected to be strongly modulated by convection. They occur most frequently over the western and central Pacific and during NH summer and fall.

Mixed Rossby-Gravity (MRG) waves



Shallow water waves in β-plane



$$y = a\theta$$

$$f(y) = f(y=0) + \left.\frac{df}{dy}\right|_0 y = \beta y$$

$$\beta = \frac{2\Omega\cos\theta_0}{a} \sim 2.3\cdot 10^{-11}\mathrm{m}^{-1}\mathrm{s}^{-1}$$

$$z = 0$$

As we did for the SW system on an f plane, we are now going to linearize the equations around a mean state at rest.

Shallow water waves in β-plane

The linearized SW waves on the beta plane are

$$\frac{\partial u}{\partial t} - \beta y v = -\frac{\partial \phi}{\partial x} \qquad (1) \qquad \phi = g\eta$$

$$\frac{\partial v}{\partial t} + \beta y u = -\frac{\partial \phi}{\partial y} \qquad (2) \qquad c^2 = gH$$

$$\frac{\partial \phi}{\partial t} + c^2 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0 \qquad (3)$$

Shallow water waves in β-plane

Based on dimensional analysis, we can introduce the following scales

$$L_{eq} = \sqrt{\frac{c}{\beta}} \qquad T = \frac{1}{\sqrt{c\beta}}$$

L_{eq} is the equatorial deformation radius. We will see how this is the natural length scale over which equatorial waves decay.

Notice how the Rossby deformation radius in midlatitudes has a differen power dependence on f.

$$L_D = \frac{c}{f} = \frac{\sqrt{gH}}{f}$$

Shallow water waves in β-plane

However, if we take $f = \beta y \sim \beta L_{eq}$

$$L_{eq} = \frac{c}{\beta L_{eq}} \qquad L_{eq}^2 = \frac{c}{\beta} \qquad L_{eq} = \sqrt{\frac{c}{\beta}}$$

For c = 25 m/s, $L_{eq} \sim 1000$ km and T $\sim$ ½ day

For an ocean wave, c = 2 m/s, $L_{eq} \sim 300$ km and T ~ 1.7 day

Shallow water waves in β-plane

We now want to reduce these three equations in three unknowns into one equation in one unknown. First of all, we are going to obtain a vorticity equation:

$$\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \beta v + \beta y \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

ζ

Combining this with the continuity equation, we obtain a linearized PV equation:

$$\frac{\partial}{\partial t}\left(\zeta - \frac{\beta y \phi}{c^2}\right) + \beta v = 0 \qquad (4)$$

Shallow water waves in β-plane

We now want to obtain one equation in one unknown. To to this, we are going to take:

$$\frac{\beta y}{c^2}\frac{\partial}{\partial t}(1)$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(2)$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t \partial y}(3)$$

$$\frac{\partial}{\partial x}(4)$$

Shallow water waves in β-plane

And the resulting equations are

$$u_{tt} = \frac{\partial^2 u}{\partial t^2}$$

$$\frac{f}{c^2}u_{tt} - \frac{f^2}{c^2}v_t = -\frac{f}{c^2}\phi_{xt} \qquad (a)$$

$$\frac{1}{c^2}v_{ttt} + \frac{f}{c^2}u_{tt} = -\frac{1}{c^2}\phi_{ytt} \qquad (b)$$

$$\frac{1}{c^2}\phi_{tty} + (u_{xyt} + v_{yyt}) = 0 \qquad (c)$$

$$v_{xxt} - u_{xyt} - \frac{f}{c^2}\phi_{xt} + \beta v_x = 0 \qquad (d)$$

We now take a – (b + c + d)

Shallow water waves in β-plane

$$\frac{1}{c^2}\frac{\partial^3 v}{\partial t^3} + \frac{(\beta y)^2}{c^2}\frac{\partial v}{\partial t} - \frac{\partial}{\partial t}\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}\right) - \beta\frac{\partial v}{\partial x} = 0$$

And we now look for wavy solutions in the zonal direction, but the amplitude of the wave will be a function of y:

$$v(x,y,t) = \hat{v}(y)\, e^{i(kx-\omega t)}$$

$\omega > 0$

$k > 0$ Eastward propagation

$k < 0$ Westward propagation

$$\frac{\omega^2}{c^2}\hat{v} - \frac{\beta^2 y^2}{c^2}\hat{v} + \frac{d^2\hat{v}}{dy^2} - k^2\hat{v} - \frac{\beta k}{\omega}\hat{v} = 0$$

$$\frac{d^2\hat{v}}{dy^2} + \left(\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega} - \frac{\beta^2 y^2}{c^2}\right)\hat{v} = 0$$

Shallow water waves in β-plane

We can also write this equation in nondimensional form, by scaling time and length scale with

$$\tilde{\omega} = \omega T = \frac{\omega}{(\beta c)^{1/2}} \qquad \tilde{y} = \frac{y}{L_{eq}} \qquad \tilde{k} = k L_{eq}$$

$$\frac{d^2 \tilde{v}}{d\tilde{y}^2} + \left(\tilde{\omega}^2 - \tilde{k}^2 - \frac{\tilde{k}}{\tilde{\omega}} - \tilde{y}^2 \right) \tilde{v} = 0$$

Shallow water waves in β-plane

Before looking in detail at the full solutions, let's consider some interesting limits.

1. $H \rightarrow \infty$. This is equivalent to considering non divergent motions.

$$\frac{d^2 \hat{v}}{dy^2} + \left(-k^2 - \frac{k}{\omega}\beta\right)\hat{v} = 0$$

We can look for wavy solutions in y $\qquad \hat{v}(y) = e^{ily}$

$$\left(-l^2 - k^2 - \frac{k}{\omega}\beta\right)\hat{v} = 0$$

For non trivial solutions

$$\omega = -\frac{\beta k}{k^2 + l^2}$$

Westward propagating Rossby waves

Shallow water waves in β-plane

2. $\beta \to 0.$

$$\frac{d^2\hat{v}}{dy^2} + \left(\frac{\omega^2}{c^2} - k^2\right)\hat{v} = 0$$

$$\left(-l^2 - k^2 + \frac{\omega^2}{c^2}\right)\hat{v} = 0$$

$$\omega^2 = c^2\left(k^2 + l^2\right)$$

Look for wavy solutions in y

$$\hat{v}(y) \propto e^{ily}$$

On equatorial beta plane, both Rossby waves and gravity waves exist. But while in the extratropics there is a clear separation between the two classes of waves, in the tropics they are not as clearly separated.

Shallow water waves in β-plane

Let's look at another special class of solutions, those with no meridional motion.

LOOK FOR SOLUTIONS

$$(u, \phi) = (\hat{u}(y), \hat{\phi}(y))\, e^{i(kx - \omega t)}$$

$$\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} \qquad -i\omega\hat{u} = -ik\hat{\phi} \qquad \omega\hat{u} = k\hat{\phi} \qquad \mathrm{I}$$

$$\beta y u = -\frac{\partial \phi}{\partial y} \qquad \beta y \hat{u} = -\frac{d\hat{\phi}}{dy} \qquad \beta y k\hat{\phi} + \omega\frac{d\hat{\phi}}{dy} = 0 \qquad \mathrm{II}$$

$$\frac{\partial \phi}{\partial t} + c^2\frac{\partial u}{\partial x} = 0 \qquad -i\omega\hat{\phi} + c^2 i\hat{u}k = 0 \qquad (\omega^2 - c^2k^2)\hat{\phi} = 0 \qquad \mathrm{III}$$

For non trivial solutions $\omega^2 = c^2k^2 \qquad \omega = \pm ck$

Shallow water waves in β-plane

Let's look at another special class of solutions, those with no meridional motion.

Replacing this dispersion relation in Eq. II

$$\beta y k \hat{\phi} + ck \frac{d\hat{\phi}}{dy} = 0$$

For $\omega = ck$

$$\frac{d\hat{\phi}}{dy} + \frac{\beta y}{c}\hat{\phi} = 0$$

For $\omega = -ck$

$$\frac{d\hat{\phi}}{dy} - \frac{\beta y}{c}\hat{\phi} = 0$$

$$\hat{\phi}(y) \propto e^{\mp y^2/2L_{eq}^2}$$

Shallow water waves in β-plane

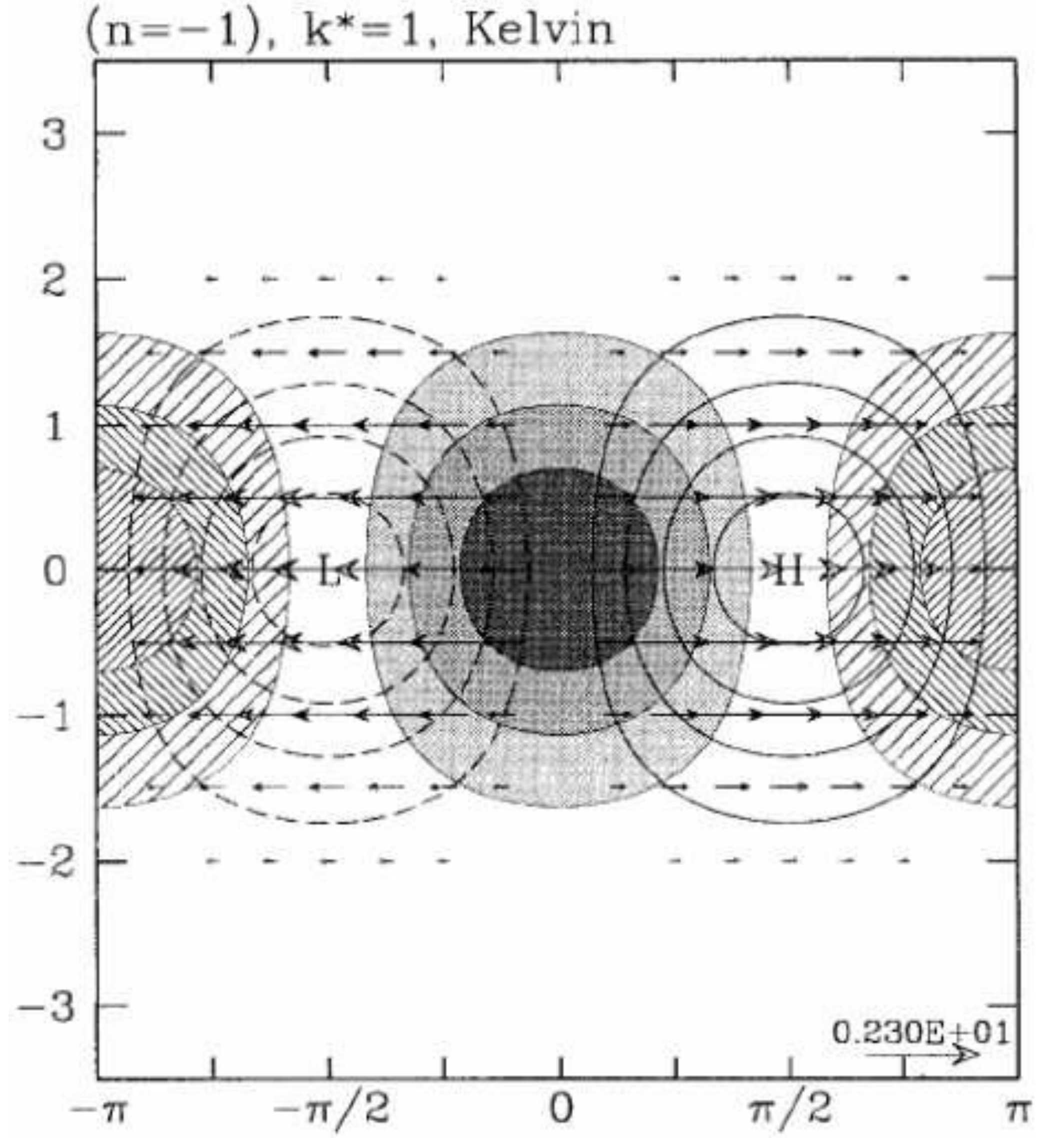
These are solutions that either decay or grow exponentially away from the equator. The only physically plausible solution is the one that decays exponentially with distance from the equator. This corresponds to the eastward propagating wave:

$$\hat{\phi}(y) \propto e^{-y^2/2L_{eq}^2} \qquad \boxed{\omega = ck}$$

While the decay with distance from the equator is specifically true for waves with no meridional motion, the exponential decay away from the equator is a property of all solutions on the beta plane. These are waves that are trapped about the equator!

The wave with no meridional motion is an equatorially trapped Kelvin wave.

Kelvin wave



Wheeler et al. (2000)

General case

Let's now consider the general case. The equation for the meridional structure of the wave is:

$$\frac{d^2\tilde{v}}{d\tilde{y}^2} + \left(\tilde{\omega}^2 - \tilde{k}^2 - \frac{\tilde{k}}{\tilde{\omega}} - \tilde{y}^2\right)\tilde{v} = 0$$

We can rewrite this equation by expressing solutions in the following form:

$$\tilde{v}(\tilde{y}) = \psi(\tilde{y})\, e^{-\tilde{y}^2/2}$$

$$\frac{d^2\psi}{d\tilde{y}^2} - 2\tilde{y}\frac{d\psi}{d\tilde{y}} + \lambda\psi = 0 \qquad \lambda = \tilde{\omega}^2 - \tilde{k}^2 - \frac{\tilde{k}}{\tilde{\omega}} - 1$$

General case

We do this because this is a well-known equation, called the Hermite's equation, which is an eigenvalue equation, which has solutions only if

$$\lambda = 2m \qquad m = 0, 1, 2, \ldots$$

$$\tilde{\omega}^2 - \tilde{k}^2 - \frac{\tilde{k}}{\tilde{\omega}} = 2m + 1$$

The solutions are Hermite's polynomials

$$\psi(\tilde{y}) = H_m(\tilde{y})$$

$$H_m(\tilde{y}) = (-1)^m e^{\tilde{y}^2} \frac{d^m}{d\tilde{y}^m}\left(e^{-\tilde{y}^2}\right)$$

Hermite's polynomials

For $m = 0$

$$H_0 = 1$$

For $m = 1$

$$H_1 = 2\tilde{y}$$

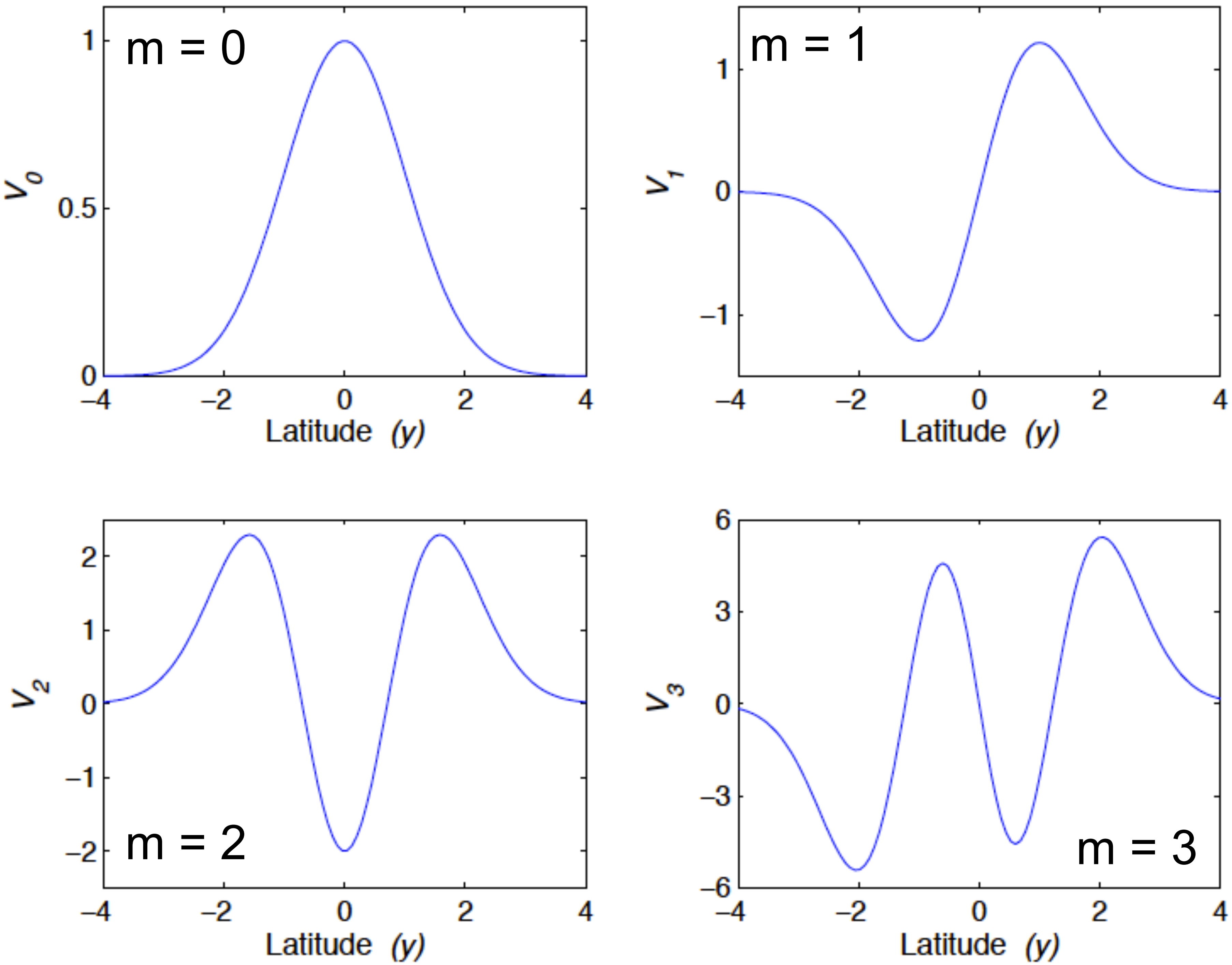
For $m = 2$

$$H_2 = 4\tilde{y}^2 - 2$$

For $m = 3$

$$H_3 = 8\tilde{y}^3 - 12\tilde{y}$$

General case



General case

Hence, our solutions are

$$\tilde{v}(\tilde{x},\tilde{y},\tilde{t}) = CH_m(\tilde{y})e^{-\tilde{y}^2/2}e^{i(\tilde{k}\tilde{x}-\tilde{\omega}\tilde{t})}$$

with

$$\tilde{\omega}^2 - \tilde{k}^2 - \frac{\tilde{k}}{\tilde{\omega}} = 2m+1$$

These solutions are waves in the zonal direction, have a meridional structure described by the Hermite's polynomials, mith *m* being the number of nodes in the meridional direction and the exponential factor guarantees that the boundary condition is verified. For *m* even, *v* is symmetric about the equator, while for *m* odd, *v* is antisymmetric about the equator (opposite for zonal wind and height/pressure).

General case

The dispersion relation is cubic in the frequency:

$$\tilde{\omega}^2 - \tilde{k}^2 - \frac{\tilde{k}}{\tilde{\omega}} = 2m + 1$$

For every k and m, there are three ω solutions.

Limiting and special cases

To further explore the solutions in the general case, we will consider special cases of the dispersion relation. We will first partition the waves by frequency and consider separetely the high frequency gravity waves and the low frequency planetary waves. We first consider the case for m ≥ 1 because the m = 0 case can be treated exactly. We will then also look at the m = –1 case, which we will see describes Kelvin waves.

Limiting and special cases

Let's consider m $\neq 0$ and let's consider the high and low frequency limits.

a. High frequency waves

$$\tilde{\omega}^2 = \tilde{k}^2 + (2m+1)$$

$$\omega^2 = c^2 k^2 + (2m+1)\beta c$$

$$\omega = \pm\sqrt{c^2 k^2 + (2m+1)\beta c}$$

These are eastward and westward propagating inertia-gravity waves.

Limiting and special cases

Remeber that in the f-plane

$$\omega^2 = f^2 + c^2 K^2$$

On the beta plane

$$f = \beta y \qquad\qquad y = L_{eq} = \sqrt{\frac{c}{\beta}}$$

$$f^2 = \beta^2 L_{eq}^2 = \beta c$$

These are called equatorially trapped Poincarè o inertia-gravity waves.

Limiting and special cases

b. Low frequency waves

$$-\frac{\tilde{k}}{\tilde{\omega}} = (2m+1) + \tilde{k}^2$$

$$\tilde{\omega} = -\frac{\tilde{k}}{(2m+1)+\tilde{k}^2}$$

In dimensional form

$$\omega = -\frac{\beta k}{k^2 + \frac{\beta}{c}(2m+1)}$$

These are equatorially trapped Rossby waves

Limiting and special cases

We can further consider two limits for these Rossby waves: the short and long waves

a. Short-wave limit $K \to \infty$

$$\omega = -\frac{\beta}{K}$$

$$c_{p,x} = \frac{\omega}{k} = -\frac{\beta}{K^2}$$

$$c_{g,x} = \frac{\partial \omega}{\partial k} = \frac{\beta}{K^2}$$

Short Rossby waves have eastward energy propagation

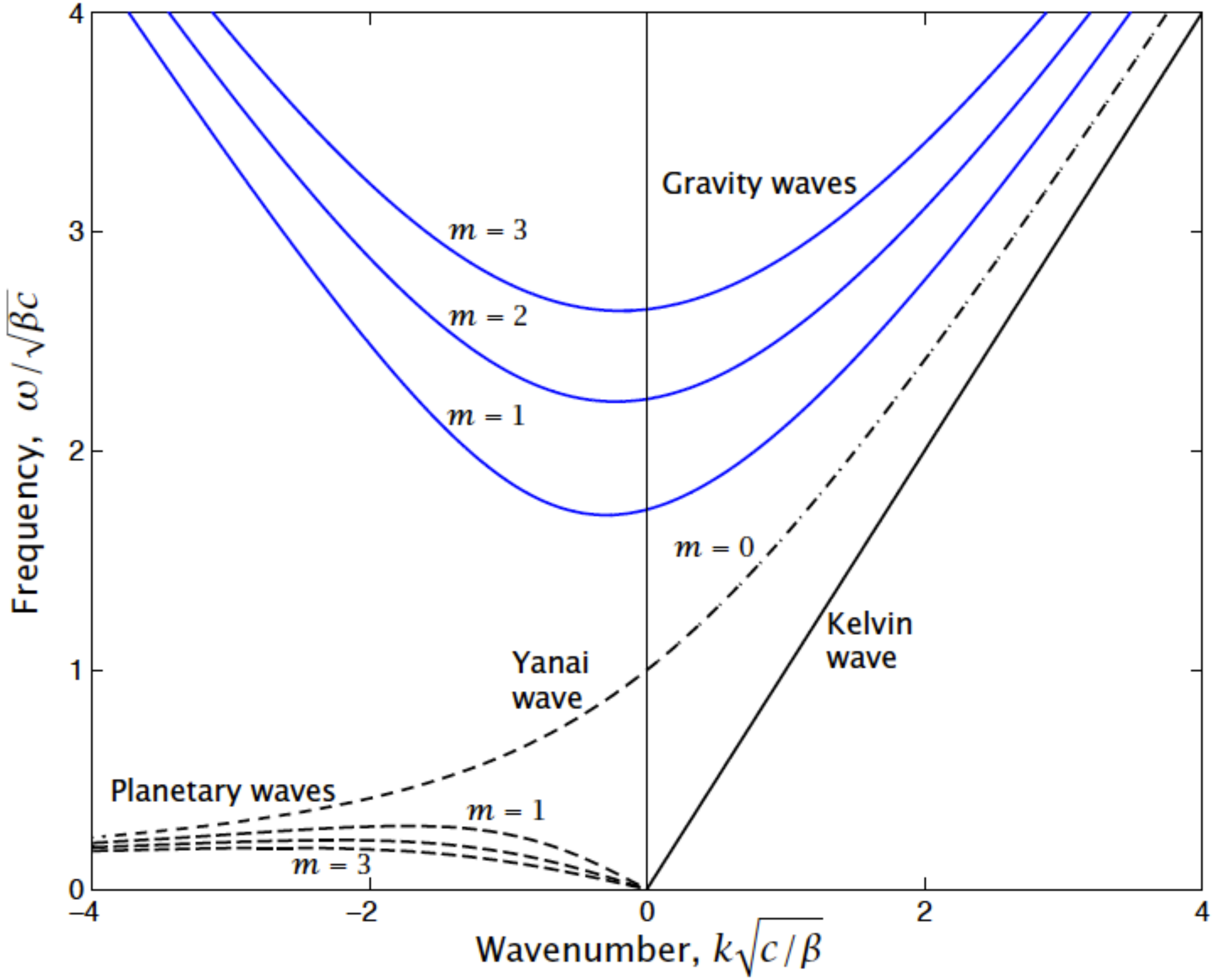
b. Long wave limit $K \to 0$

$$\omega = -\frac{ck}{(2m+1)}$$

$$c_{p,x} = \frac{\omega}{k} = c_{g,x} = \frac{\partial \omega}{\partial k} = -\frac{c}{2m+1}$$

Long Rossby waves have westward energy propagation

Dispersion relation



Special values of m

In addition to the limiting cases of low and high frequency, there are two other cases in which the dispersion relation can be solved.

1. m = 0. These waves are called Yanai waves or mixed-Rossby gravity waves, because they span the two types of waves (rossby and gravity).

The dispersion relation in this case can be written as:

$$\tilde{\omega}^2 - \tilde{k}^2 - \frac{\tilde{k}}{\tilde{\omega}} = 1$$

$$(\tilde{\omega} + \tilde{k})(\tilde{\omega}^2 - \tilde{k}\tilde{\omega} - 1) = 0$$

with 3 possible solutions

Special values of m

1. A westward propagating gravity wave. This is not physically plausible as it is a wave whose amplitude increases exponentially away from the equator.

$$\tilde{\omega} = -\tilde{k} \qquad\qquad \omega = -ck$$

2. The other solutions are

$$\tilde{\omega} = \frac{\tilde{k}}{2} \pm \sqrt{\left(\frac{\tilde{k}}{2}\right)^2 + 1}$$

$$\omega = \frac{kc}{2} \pm \frac{1}{2}\sqrt{k^2c^2 + 4\beta c}$$

Special values of m

For $k = 0$ $\quad \omega = \sqrt{\beta c}$

Notice how this is the limit of the high frequency waves for m = 0 and k = 0

For $k \to \pm\infty$

$$\tilde{\omega} = \frac{\tilde{k}}{2} \pm \frac{\tilde{k}}{2}\left[1 + \left(\frac{2}{\tilde{k}}\right)^2\right]^{1/2} \simeq \frac{\tilde{k}}{2} \pm \frac{\tilde{k}}{2}\left[1 + \frac{1}{2}\frac{4}{\tilde{k}^2}\right] \sim$$

$$\sim \frac{\tilde{k}}{2} \pm \frac{\tilde{k}}{2} \pm \frac{1}{\tilde{k}}$$

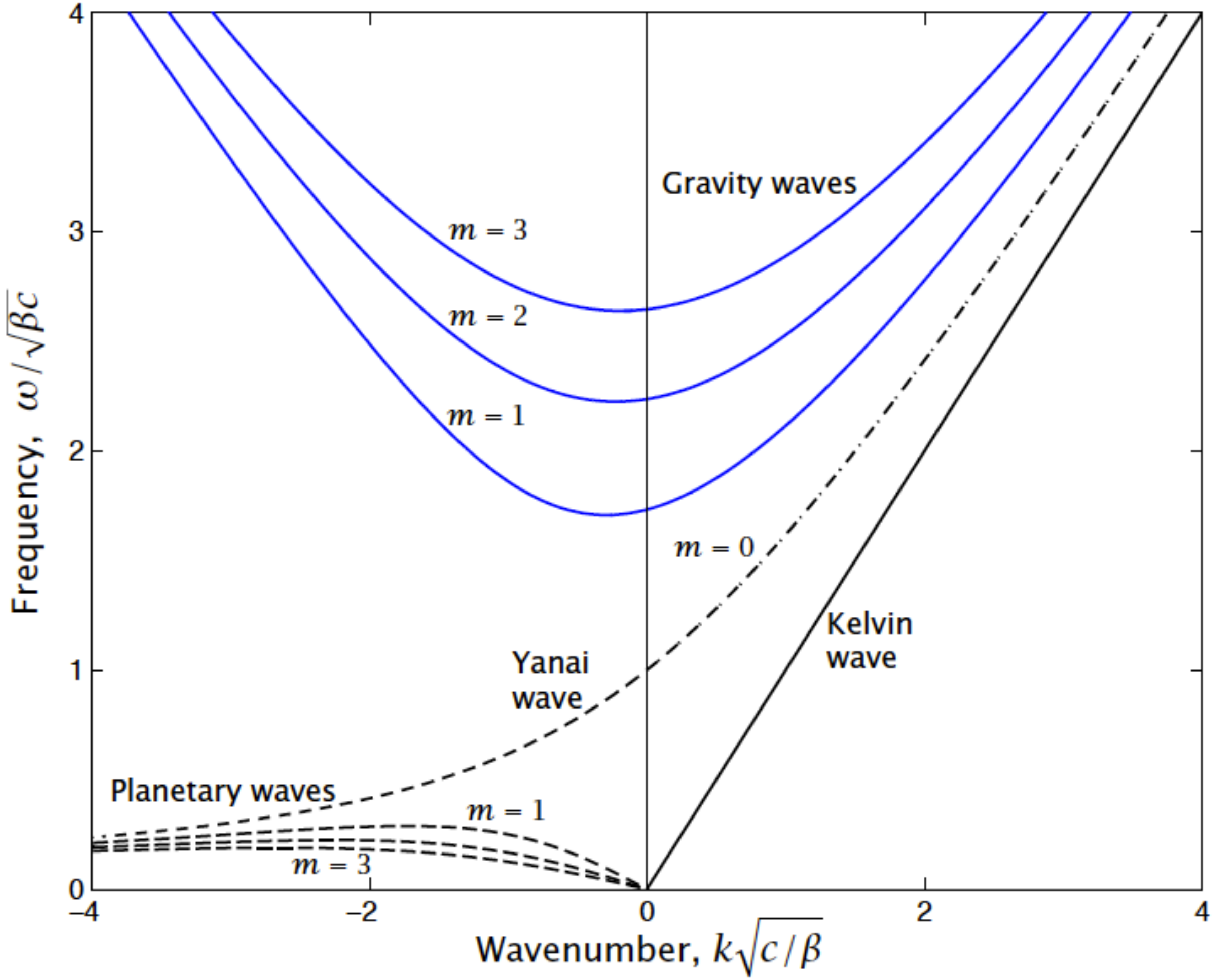
+ solution $\quad \tilde{\omega} = \tilde{k}$, $\quad \omega = ck$ $\quad$ Eastward propagating gravity wave

− solution $\quad \tilde{\omega} = -\frac{1}{\tilde{k}}$, $\quad \omega = -\beta/k$ $\quad$ Westward propagating Rossby wave

Special values of m

This $m = 0$ is a mode that does not exist in the extratropics. In particular, it connects two families of modes that in the extratropics are spectrally separated. One can show that the mixed-Rossby gravity waves have an eastward group velocity.

Dispersion relation



Special values of m

2. m = –1. While the Hermite's equation has solutions for m positive integer or zero, there is a class of waves that happen to satisfy the dispersion relation with m = –1.

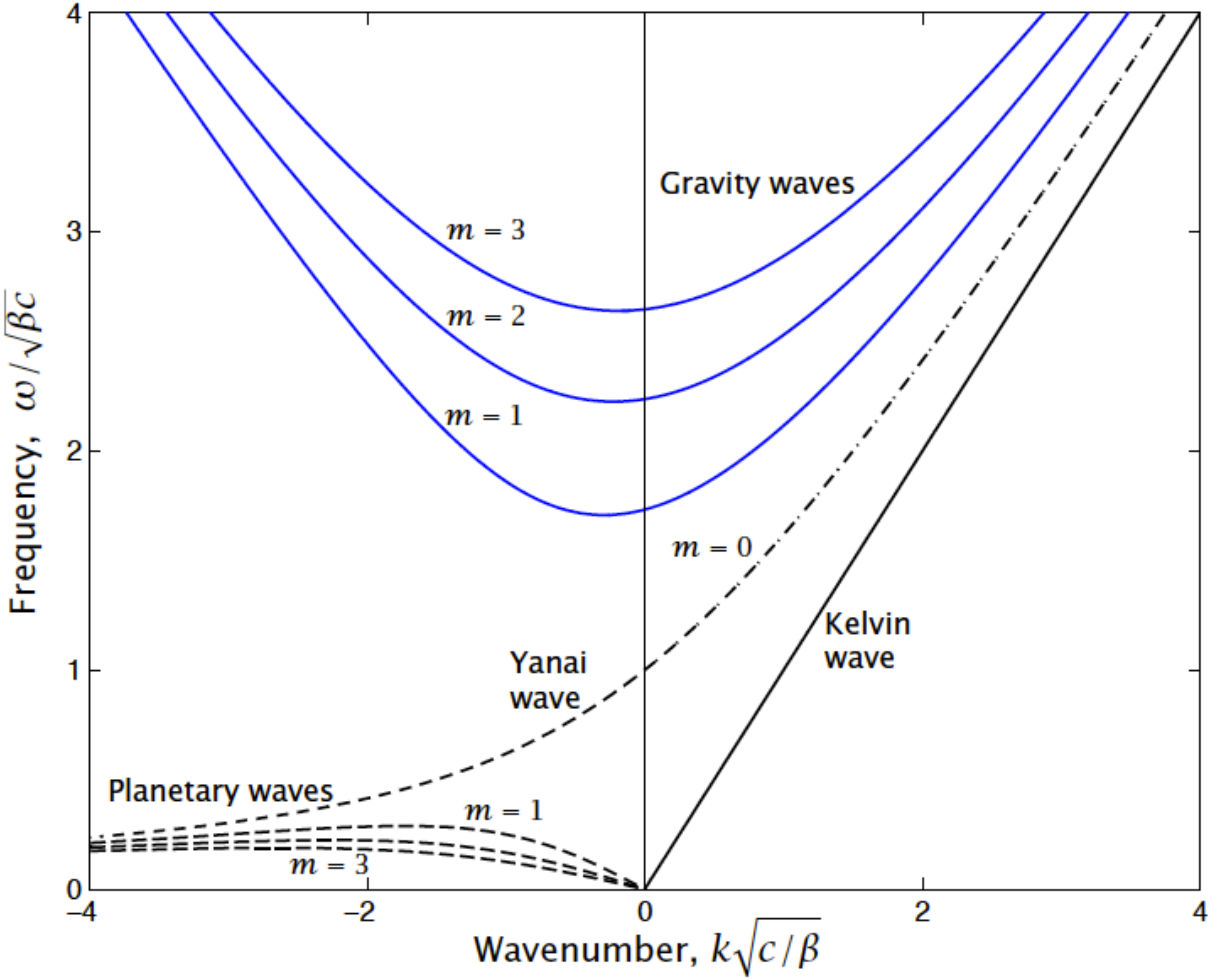
$$\tilde{\omega}^2 - \hat{k}^2 - \frac{\hat{k}}{\tilde{\omega}} = -1$$

$$(\tilde{\omega} - \hat{k})(\tilde{\omega} + \hat{k}\tilde{\omega} + 1) = 0$$

$$\tilde{\omega} = \hat{k} \qquad \omega = ck$$

This is the Kelvin wave that we have already discussed. It is equatorially trapped and propagates eastward.

Dispersion relation



To summarize

The equation for the meridional structure of the SW waves on the equatorial beta plane is:

$$\frac{d^2\tilde{v}}{d\tilde{y}^2} + (\tilde{\omega}^2 - \tilde{k}^2 - \frac{\tilde{k}}{\tilde{\omega}} - \tilde{y}^2)\tilde{v} = 0$$

Solutions exist if

$$\tilde{\omega}^2 - \tilde{k}^2 - \frac{\tilde{k}}{\tilde{\omega}} = 2m + 1 \qquad\qquad m = 0, 1, 2, ...$$

And are of the form

$$\tilde{v}(\tilde{x}, \tilde{y}, \tilde{t}) = CH_m(\tilde{y})e^{-\tilde{y}^2/2}e^{i(\tilde{k}\tilde{x} - \tilde{\omega}\tilde{t})}$$

To summarize

Hm are Hermite's polynomials of order m. m is the number of nodes in the meridional direction. For $m \geq 1$, for each k and m we have two classes of solutions:

1. High frequency eastward and westward propagating inertia-gravity (IG) waves.

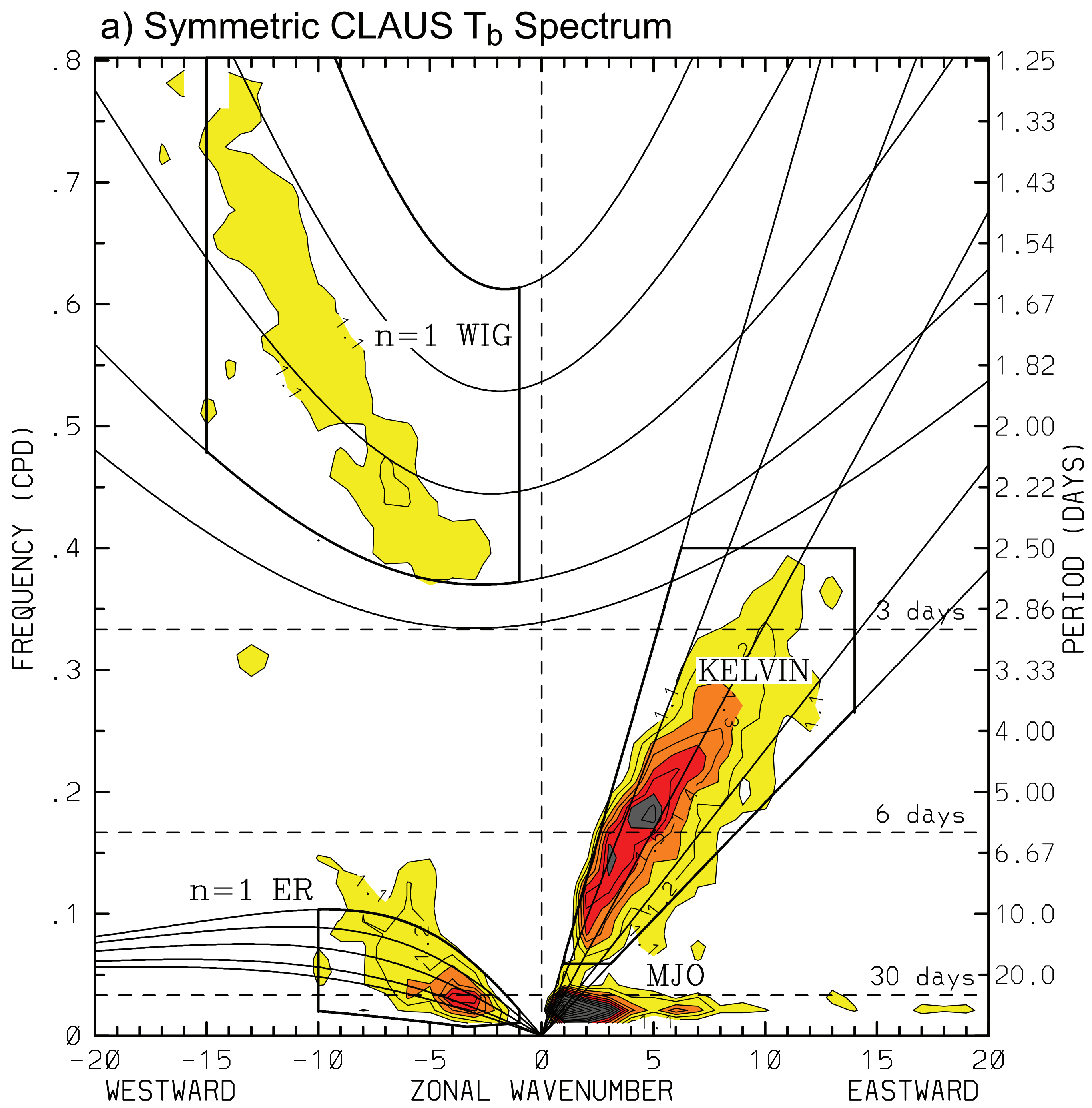
2. Low-frequency westward propagating equatorial Rossby (ER) waves.

Fo m = 0, we have Yanai or mixed Rossby-gravity waves (MRG)

For m = -1, we have an eastward propagrating Kelvin Wave.

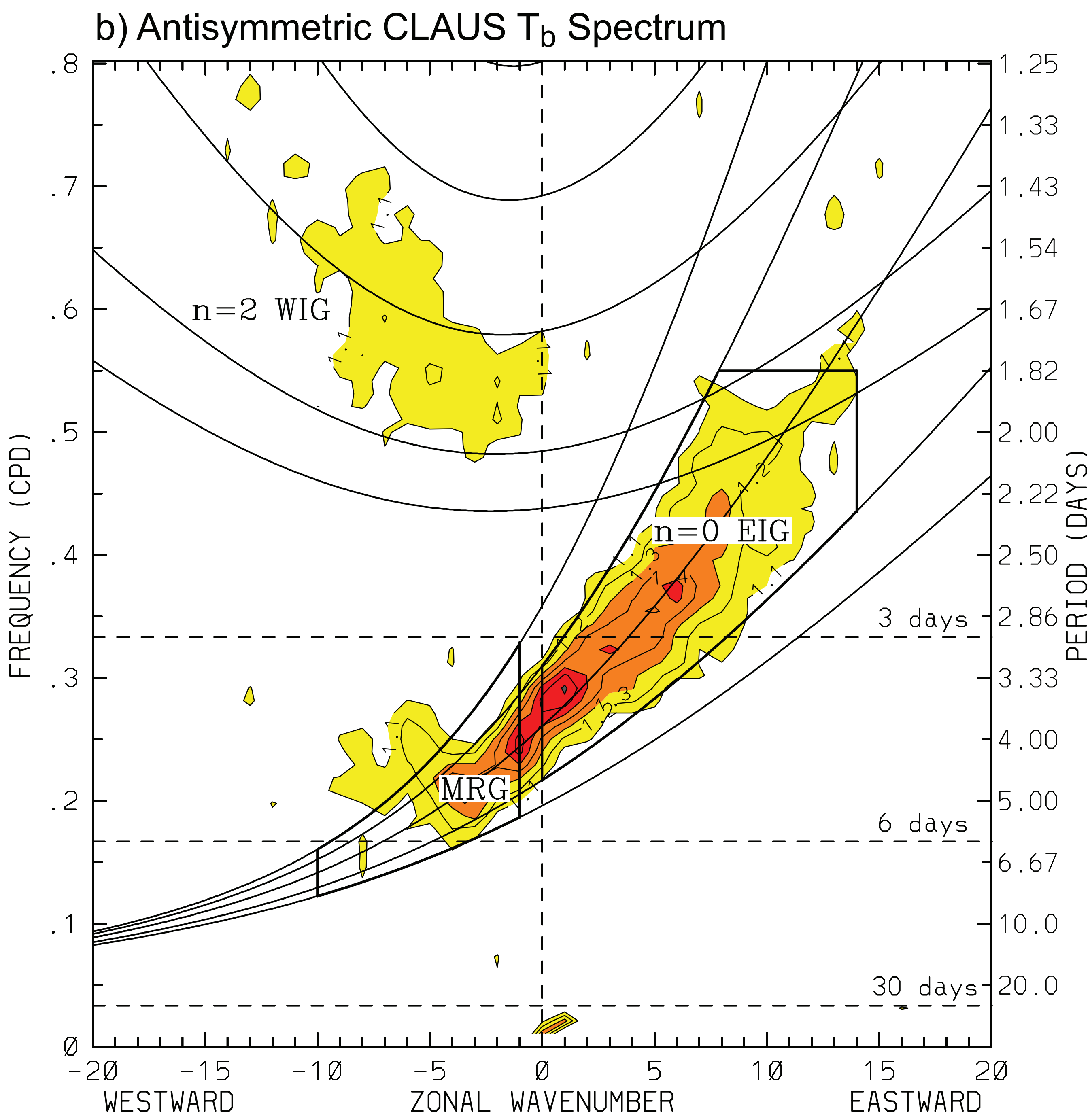
IG and Kelvin waves are more divergent, whereas the ER and MRG waves are more rotational. For m even, these modes are symmetric about the equator in v, but antisymmetric in term of u and pressure. Opposite for m odd.

Do we see any spectral evidence of these waves?



Kiladis et al. (2009) adapted from Wheeler and Kiladis (1999)

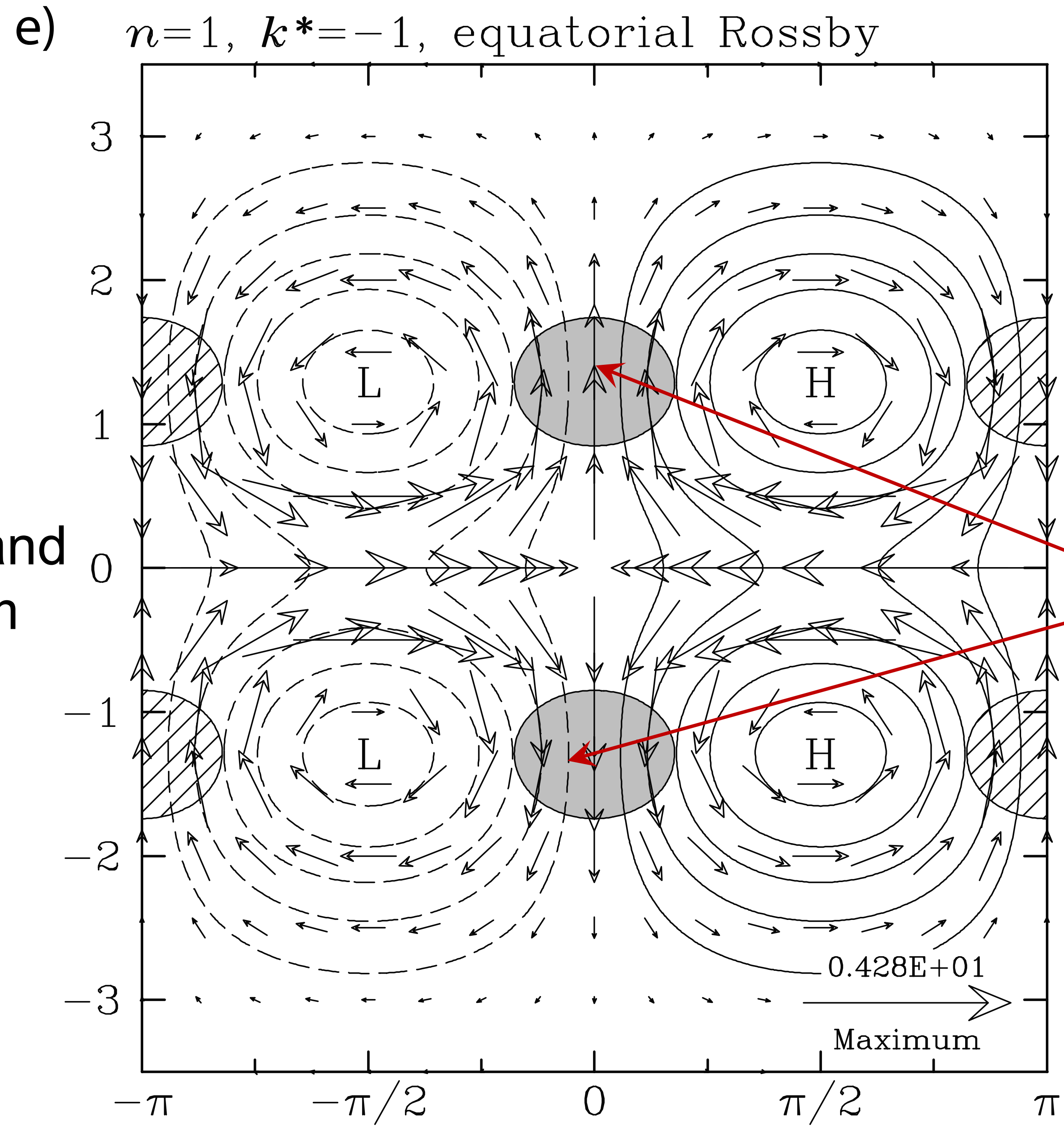
Do we see any spectral evidence of these waves?



Kiladis et al. (2009) adapted from Wheeler and Kiladis (1999)

Equatorial Rossby waves

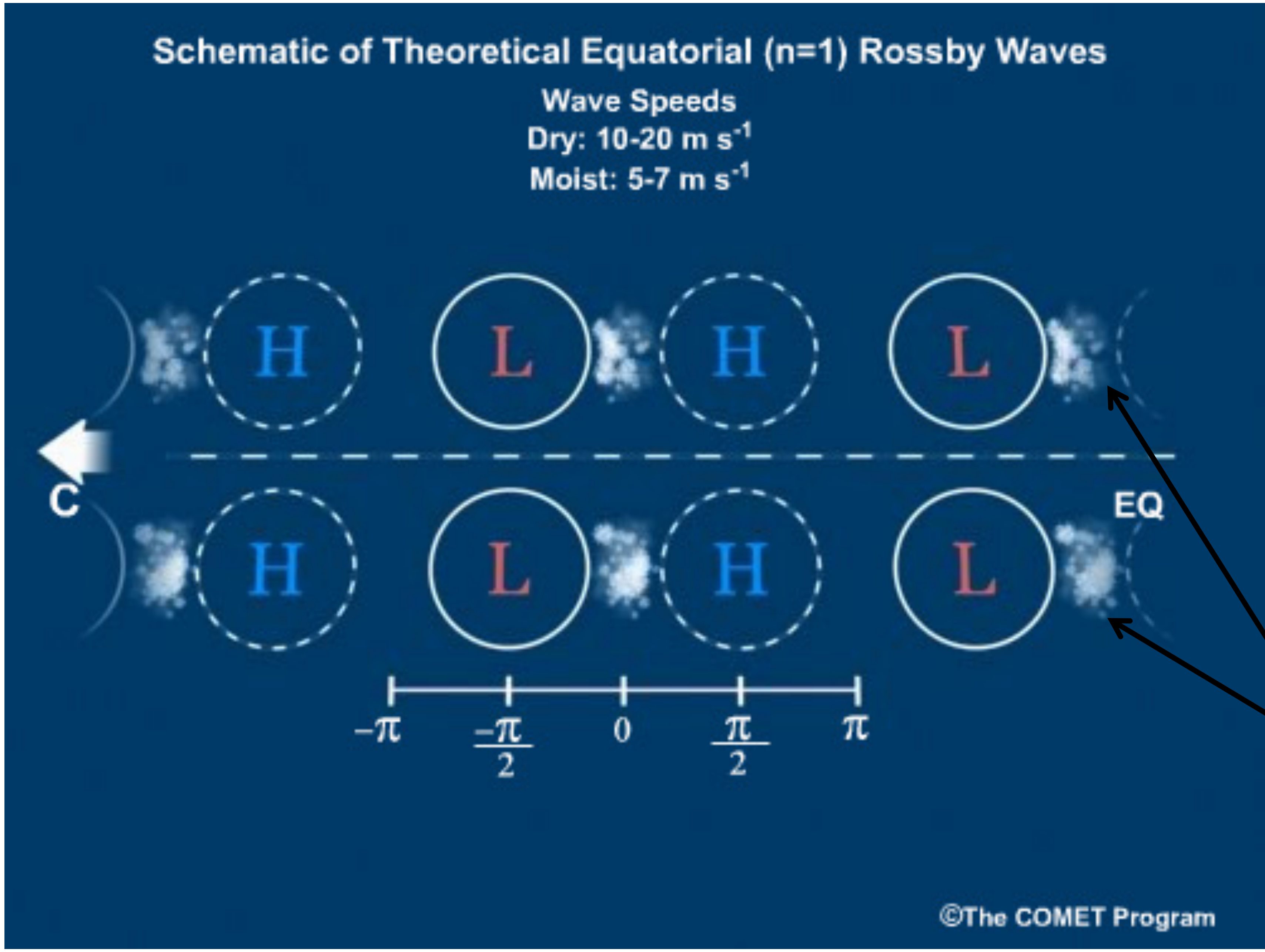
m = 1



Winds are largest close to equator and decay rapidly from there

Maximum convergence east of the low

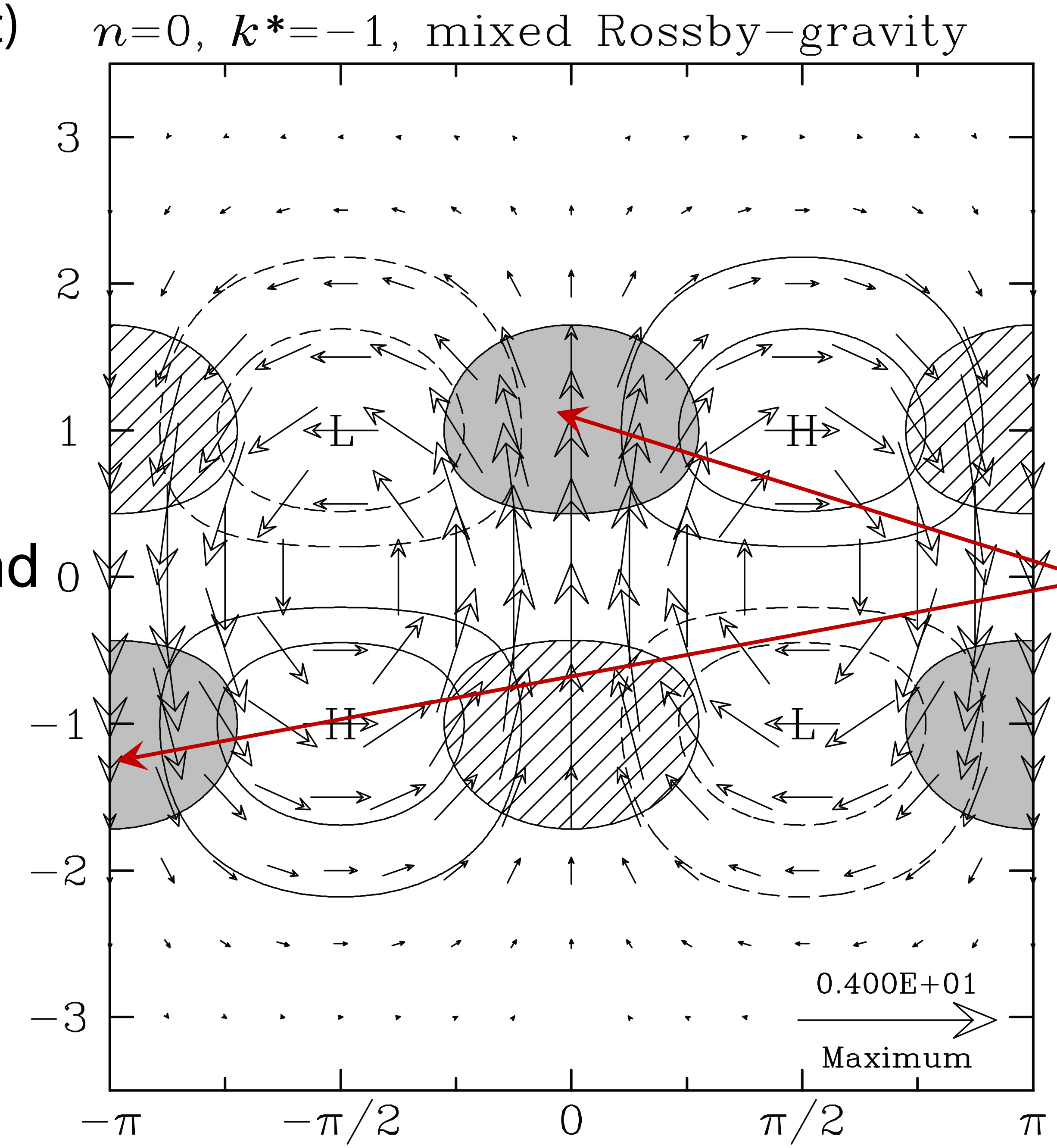
Equatorial Rossby waves



Diabatic generation of cyclonic PV rearward of lower pressures acts as a brake on wave propagation

Mixed Rossby-gravity waves

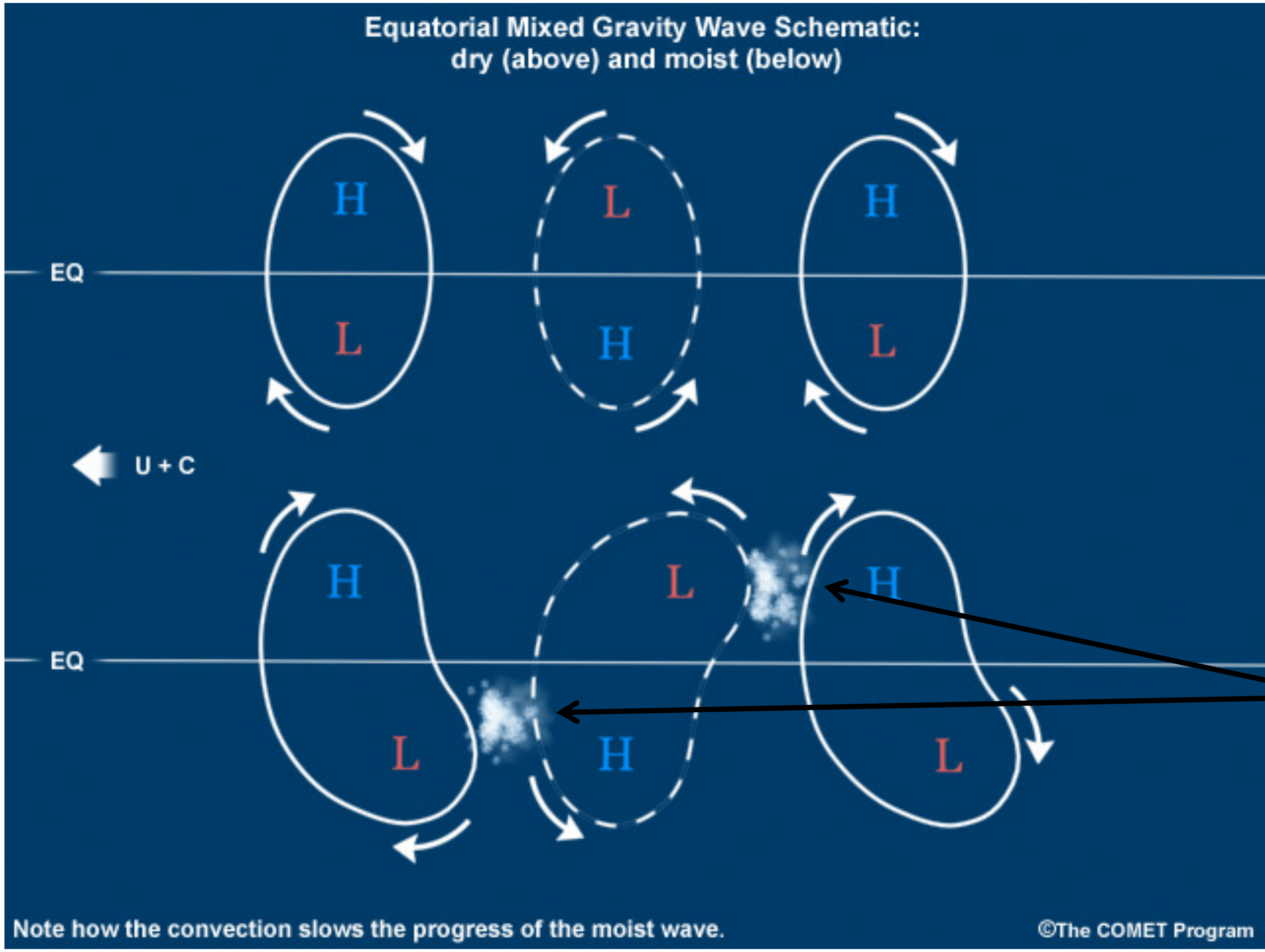
c)



Winds are largest close to equator and decay rapidly from there

Maximum convergence east of the low

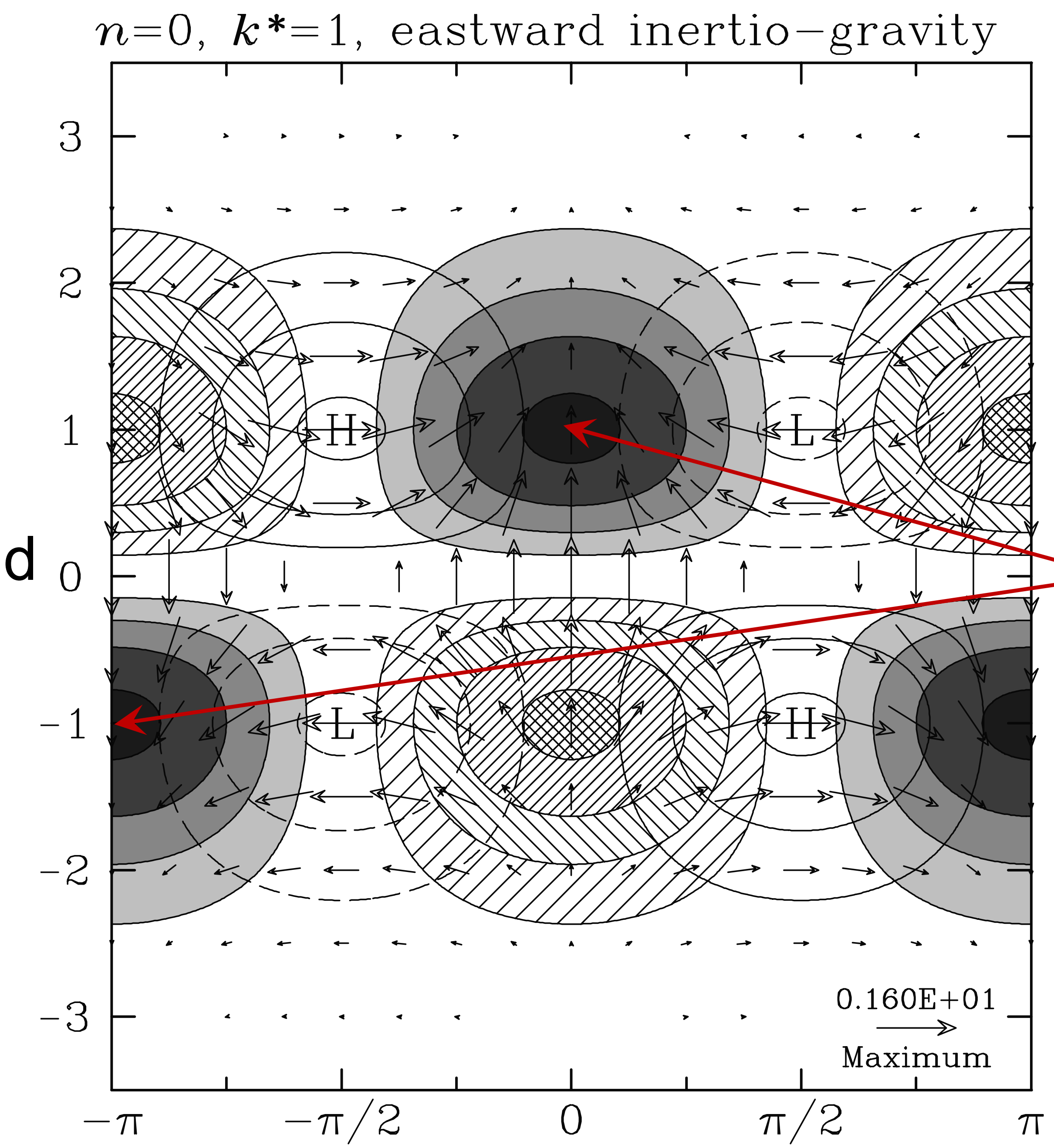
Mixed Rossby-gravity waves



Diabatic generation of cyclonic PV rearward of lower pressures acts as a brake on wave propagation

Eastward inertia-gravity waves

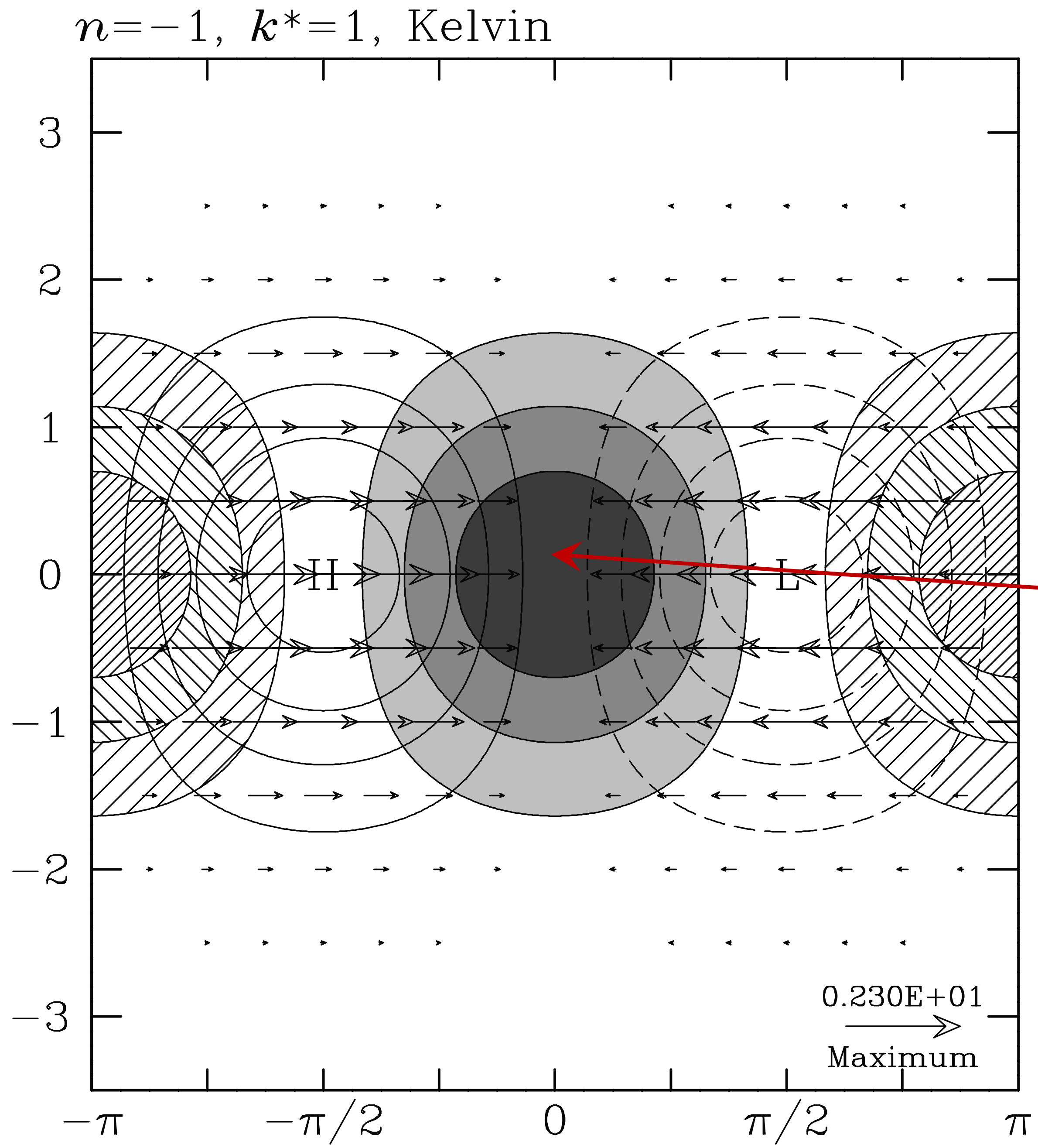
Winds are largest close to equator and decay rapidly from there



Maximum convergence west of the low

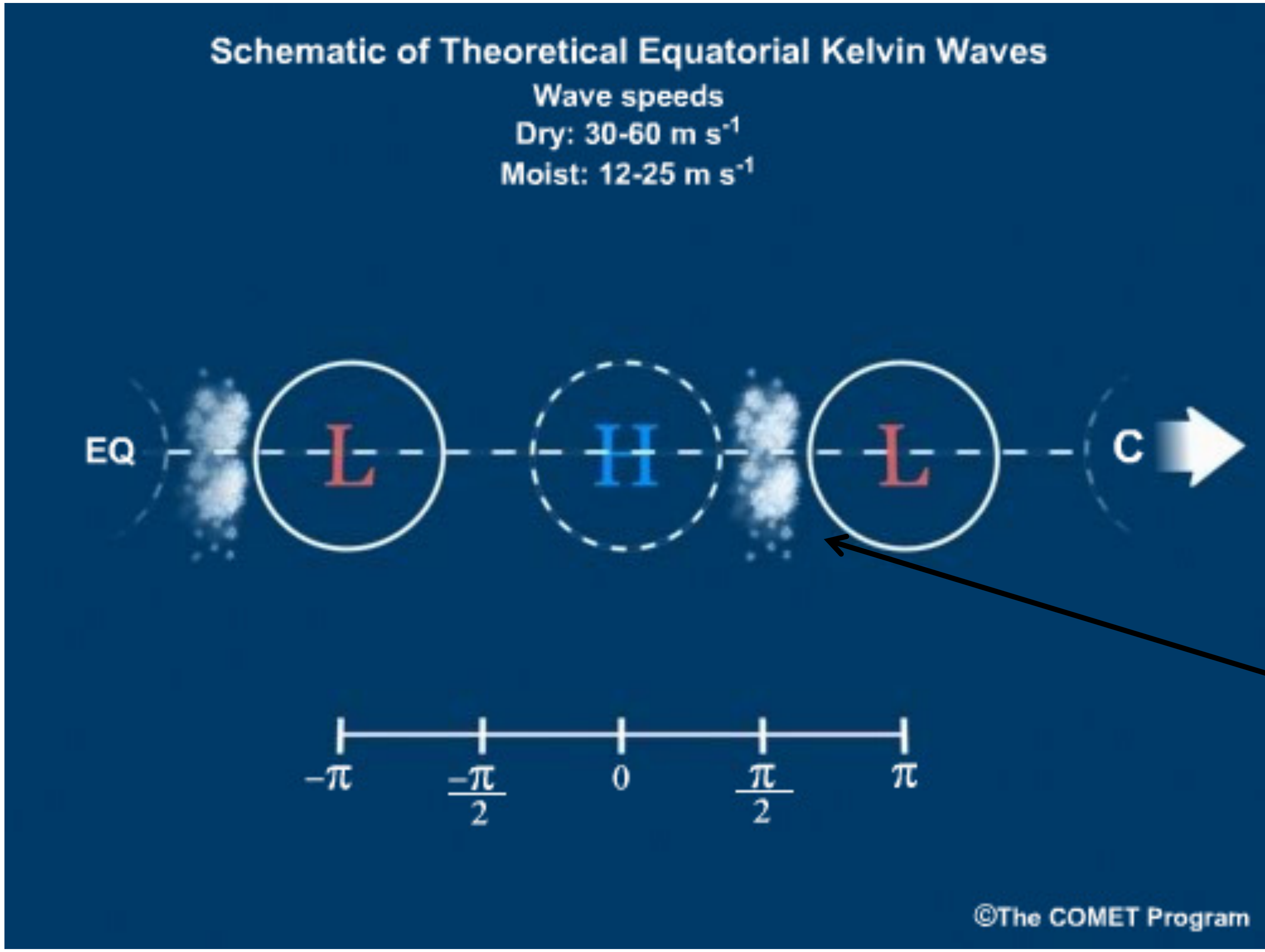
Kelvin waves

Zonal winds and height in phase.



Maximum convergence west of the low

Kelvin waves



Diabatic generation of cyclonic PV rearward of lower pressures acts as a brake on wave propagation

Vertical wave structure

To what extent are these waves relevant for the 3D atmosphere? And what can we say about the vertical structure of equatorial waves? We can linearize the primitive equations about a state at rest:

$$\frac{\partial u}{\partial t} - \beta y v = -\frac{\partial \phi}{\partial x}$$

$$\frac{\partial v}{\partial t} + \beta y u = -\frac{\partial \phi}{\partial y}$$

$$\frac{\partial \phi}{\partial z_*} = \frac{RT}{H_0}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho_0}\frac{\partial}{\partial z_*}(\rho_0 w_*) = 0$$

$$\frac{\partial T}{\partial t} + w_* S = 0$$

$$S = \frac{N^2 H_0}{R}$$

Vertical wave structure

From hydrostatic balance, we can express T in terms of the vertical gradient of the geopotential and replace in the TD equation to obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial z_*}\right) + N^2 w_* = 0$$

We can then combine continuity and the TD equation to obtain:

$$\frac{\partial}{\partial t} L\Phi - N^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

with

$$L = \frac{1}{\rho_0}\frac{\partial}{\partial z_*}\left(\rho_0 \frac{\partial}{\partial z_*}\right)$$

Vertical wave structure

Let's now suppose that the geopotential has a vertical structure that is an eigenfunction of L, which means

$$L\Phi_v = -\lambda\Phi_v$$

With constant λ

$$\frac{\partial \phi}{\partial t} + \frac{N^2}{\lambda}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

$$c^2 = \frac{N^2}{\lambda} = gh_e$$

$$\frac{\partial u}{\partial t} - \beta y v = -\frac{\partial \phi}{\partial x}$$

$$\frac{\partial v}{\partial t} + \beta y u = -\frac{\partial \phi}{\partial y}$$

Vertical wave structure

These are isomorphic to the SW equations. We hence can construct solutions that are SW solutions and whose vertical structures are eigenfunctions of L. This seems easy, but there are complications. The first complication is that, even for dry waves, there are different functions that are eigenfunctions of L, and these give either vertically uniform structures or slanted ridges and troughs. The first is the most relevant for the troposphere, the second for the stratosphere.

For the troposphere, the vertical structure is of the form

$$\exp(z/2H)\sin(nz+b)$$

with vertical wavenumber n

Vertical wave structure

One can show that there exists a well define relation between the vertical wavelength and the equivalent depth of the corresponding SW wave. For N constant, vertical modes are sinusoidal in z and one can obtain

$$n = \frac{2\pi}{L_z} = \left(\frac{N^2}{gh_e} - \frac{1}{4H_0}\right)$$

n is inversely proportional to h_e. Larger equivalent depth, smaller n, larger vertical wavelength. For non-constant N, the modes are only quasi-sinusoidal, with a vertical wavelength that is small in regions of large N (stratosphere) and larger in regions of smaller N.

Vertical wave structure

TABLE 1. Vertical Structure Values for Dry Waves in a Constant *N* Atmosphere[a]

h_e	L_z (km)	$\sqrt{gh_e}$ (m s^{-1})	R_e (Degrees Latitude)
	$H = 7.3$ km, $dT_0/dz = -7.0$ K km^{-1} (Troposphere)		
10	6.0	9.9	6.0
20	8.5	14.0	7.1
50	13.4	22.1	9.0
100	19.2	31.3	10.7
200	27.9	44.3	12.7
500	47.5	70.0	15.9
	$H = 6.1$ km, $dT_0/dz = +2.5$ K km^{-1} (Lower Stratosphere)		
10	2.6	9.9	6.0
20	3.7	14.0	7.1
50	5.8	22.1	9.0
100	8.3	31.3	10.7
200	11.8	44.3	12.7
500	18.9	70.0	15.9

Vertical wave structure

One of the most important questions is what best matches observations. In general one would expect that the wavelength will depend on the vertical structure of the heating that generates these waves.

For instance, deep heating will more strongly excite a vertical mode with half a wavelength being equal to the depth of the troposphere

$$\frac{L_z}{2} = 15 \text{ Km}$$

$$h_e = 200 \text{ m}$$

$$c = 44 - 50 \text{ m/s}$$

Vertical wave structure

Dry Kelvin waves with this speed are observed, but waves coupled to convection show dispersive characteristics corresponding to much shallower equivalent depths. Explaining the mismatch between the implied equivalent depth of CCEWs and that of most energetically forced waves in a dry atmosphere with realistic stratification is one of the major challenges for the theory of these waves.

How do we include moisture

Can the inclusion of condensational heating and cooling explain the discrepancy between observed CCEWs and the dry theory?

$$\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial z_*}\right) + N^2 w_* = Q$$

We are now going to suppose that the heating perturbations are proportional to the upward motion perturbations.

$$Q = \alpha w_* N^2 \qquad \frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial z_*}\right) + N^2 w_*(1-\alpha) = 0$$

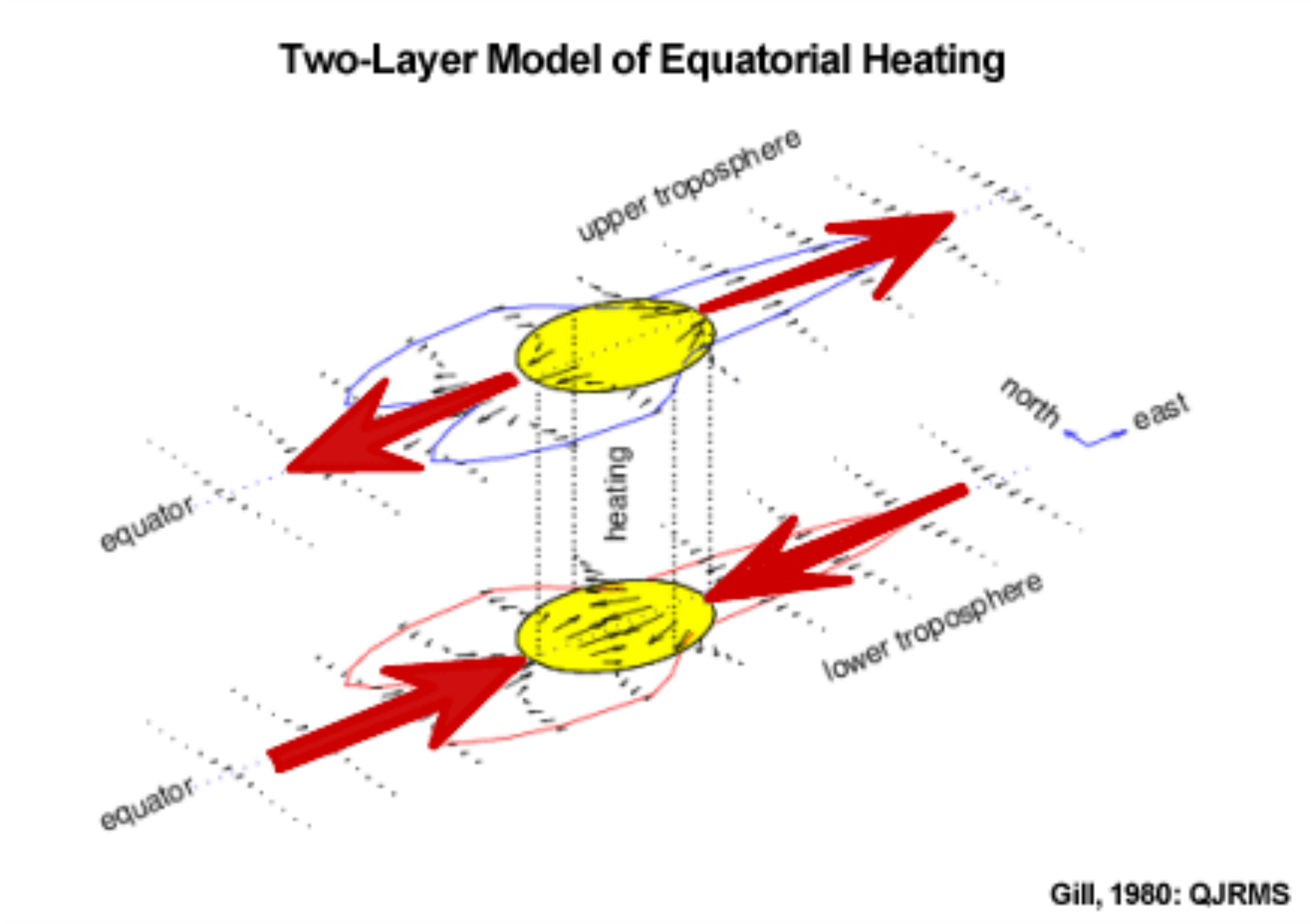
So we see that any positive α reduces the stability of the atmosphere and we can choose a value that brings theoretical predictions of waves in lines with those observed in CCEWs. But discrepancies still exist and a lot of work is still being done.

The first baroclinic mode

The «gravest» mode, which extends through the depth of the troposphere, is also known as first baroclinic mode. This mode is triggered by deep heating maximizing in the mid troposphere, with vertical velocities also maximizing in the mid troposphere and being zero close to the surface and the tropopause. By continuity, these vertical velocities result in horizontal velocities and pressure perturbations that maximize at the surface and the tropopause and have opposite sign.

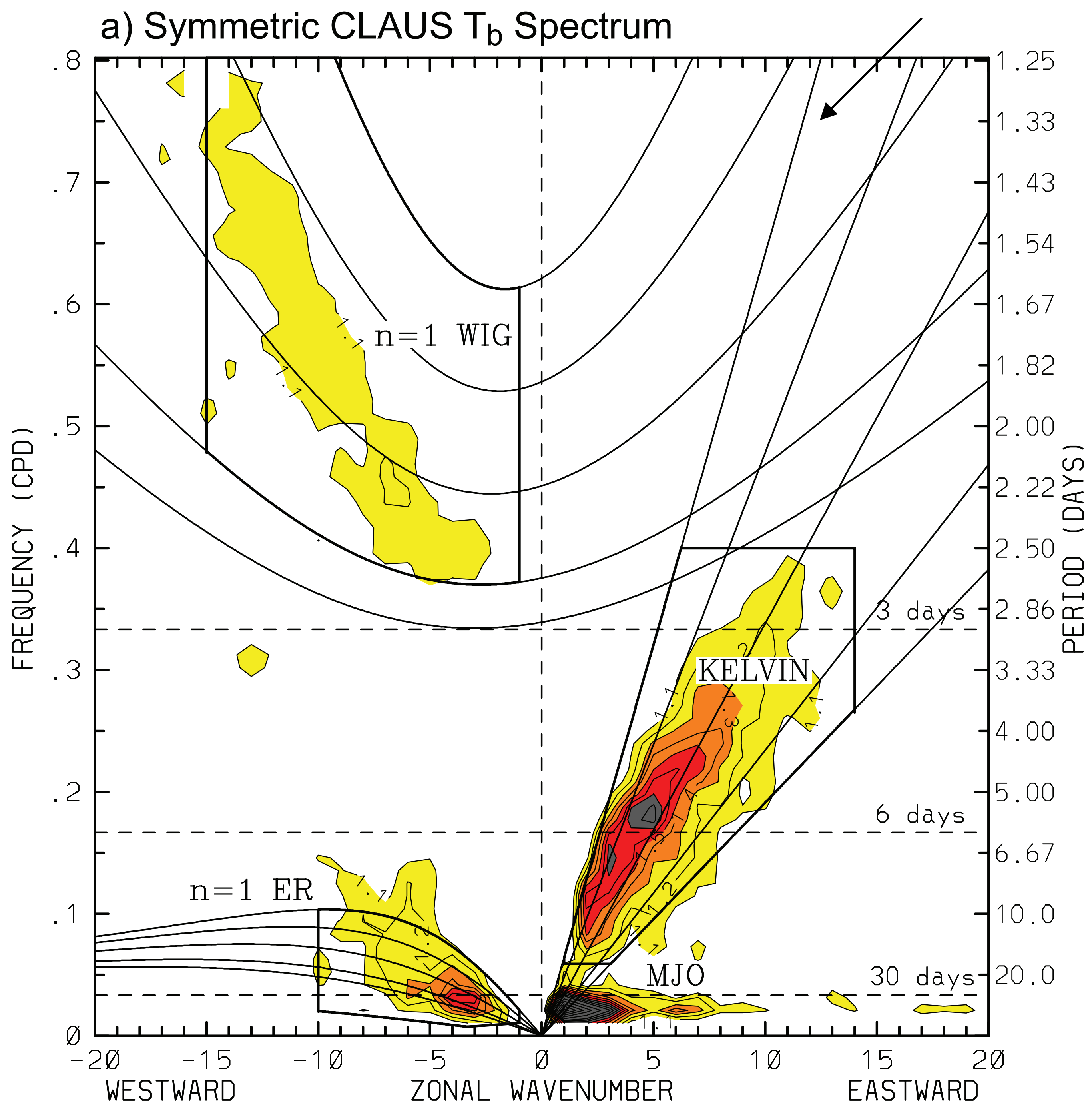
Thus, lower tropospheric convergence corresponds to low surface pressure and mid-tropospheric upward vertical motion, and upper tropospheric divergence and a high pressure. Patterns of lower tropospheric convergence (divergence) and upper tropospheric divergence (convergence) observed in the tropics are in reasonably close correspondence, reflecting this simple vertical structure for large-scale motions in the tropics as expected for regions in which convective heating dominates.

The first baroclinic mode



Observed spectra

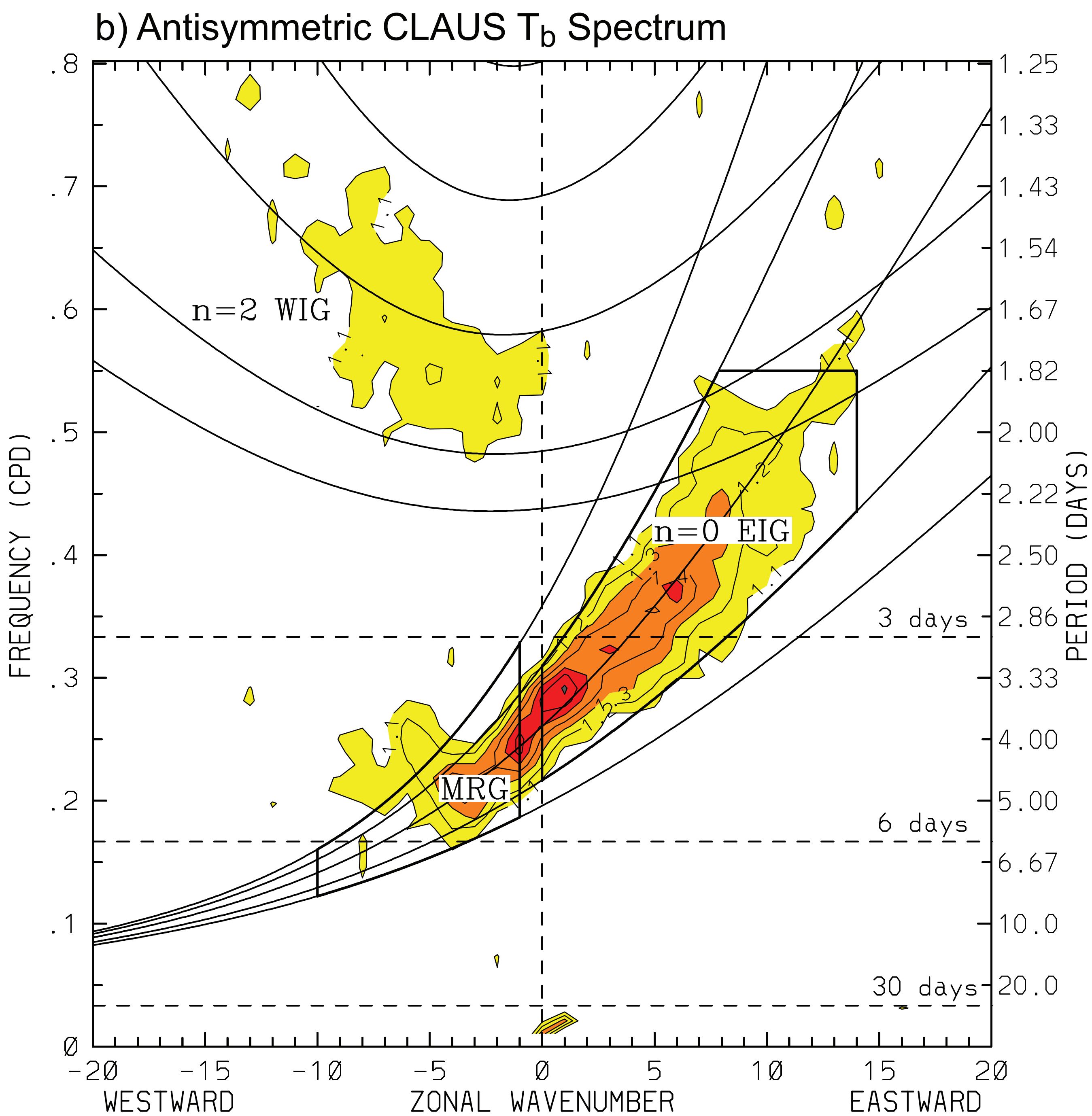
90 m



The black lines correspond to theoretical curves for equivalent depth of 8, 12, 25, 50 and 90 m

Kiladis et al. (2009) adapted from Wheeler and Kiladis (1999)

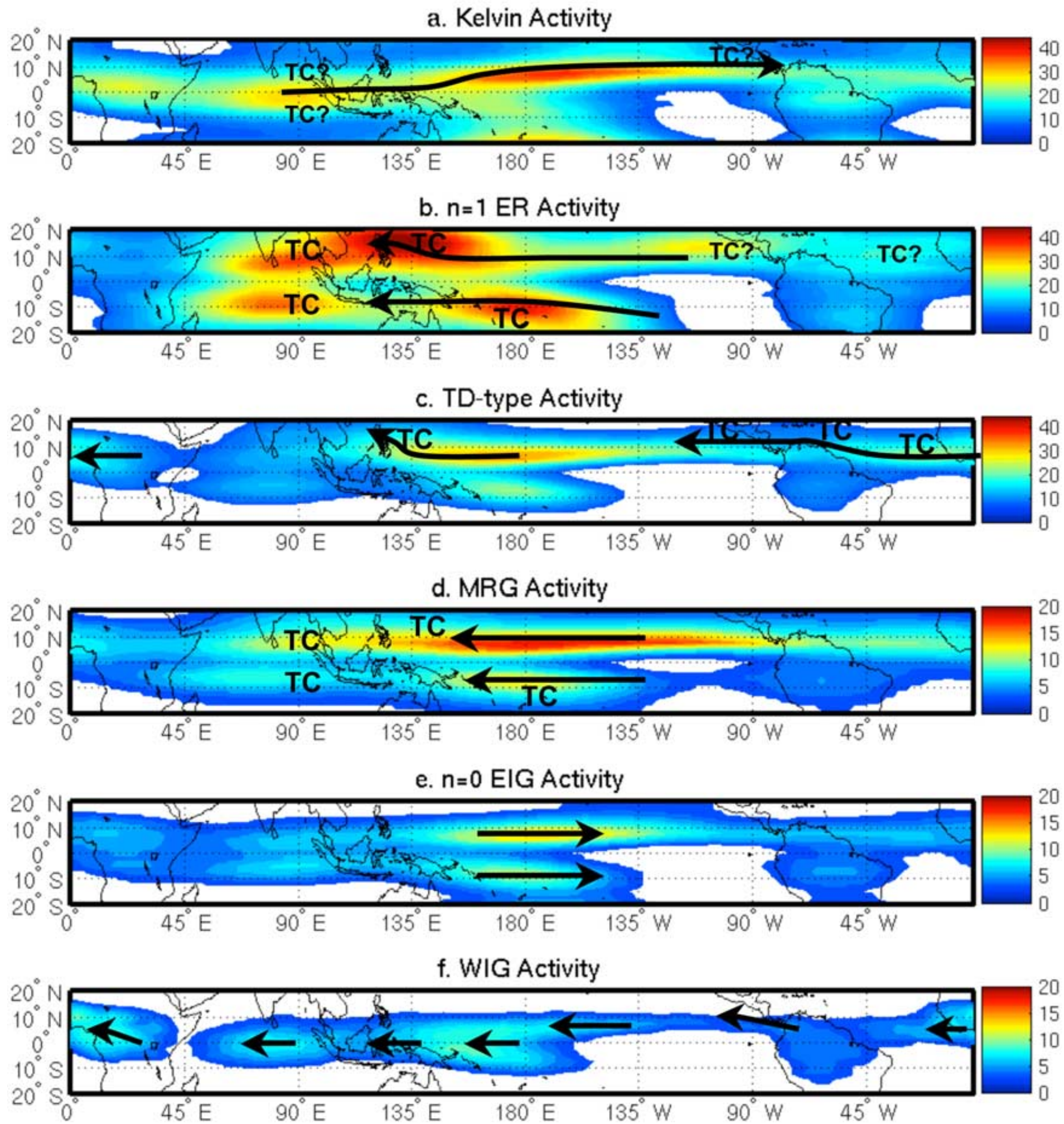
Observed spectra



The black lines correspond to theoretical curves for equivalent depth of 8, 12, 25, 50 and 90 m

Kiladis et al. (2009) adapted from Wheeler and Kiladis (1999)

Wave activity



Kiladis et al. (2009)

Equatorial wave monitoring

Each of the equatorial wave modes described above propagates at a unique velocity and in a unique direction. Each equatorial wave mode is also associated with a unique kinematic and, in particular, convective structure and typically lasts for a unique length of time. As a consequence, monitoring and forecasting of equatorial waves is fairly straightforward given appropriate spatial and temporal filtering of anomalous outgoing longwave radiation (modulated by clouds) and upper/lower tropospheric wind fields. This allows for the isolation of a given wave mode from a set of observations or forecast fields.

Equatorial wave forecasting

Unfortunately, there is limited skill associated with forecasts of equatorial wave activity whether such forecasts are derived from persistence (e.g., extrapolation of ongoing conditions into the future) or numerical model forecasts. Typical forecast skill extends out to 1-5 days, or up to half of a given equatorial wave's life span (or period). As a result, improving the skill of forecasts of equatorial waves and associated meteorological phenomena is an active, vibrant area of research in tropical meteorology.

Reading material

1. Matsuno, 1966: Quasi-geostrophic motions in the equatorial area. J. of Meteor. Soc. Japan, 44, 15 – 43.
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3. Vallis, 2017: Atmospheric and Oceanic Fluid Dynamics, Cambridge Press (Chap. 8)
4. Wheeler and Kiladis, 1999: Convectively Coupled Equatorial Waves: Analysis of Clouds and Temperature in the Wavenumber–Frequency Domain. J. Atmos. Sci., 56, 374 –399.
5. Kiladis et al., 2009: Convectively-coupled equatorial waves. Reviews of Geophysics, 47, RG2003/2009.