Environmental Fluid Mechanics, *M.Sc. Environmental Meteorology* – 2022/23

Barotropic waves:
Kelvin, Poincaré and Rossby waves

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(Marco Adami, *Scirocco*, 2008)

References

Benoit Cushman-Roisin and Jean-Marie Beckers, **Introduction to geophysical fluid dynamics : physical and numerical aspects**, 2nd ed., Academic Press, 2011.
https://webapps.unitn.it/Biblioteca/it/Web/LibriElettroniciDettaglio/117963

- 9. Barotropic Waves
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    - 9.2 The Kelvin Wave
    - 9.3 Inertia-Gravity Waves (Poincaré Waves)
    - 9.4 Planetary Waves (Rossby Waves)
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Pijush K. Kundu, Ira M. Cohen, David R. Dowling, **Fluid mechanics**, 5th ed., Academic Press, Elsevier, 2012.
https://www-sciencedirect-com.ezp.biblio.unitn.it/book/9780123821003/fluid-mechanics

- 13. Geophysical Fluid Dynamics
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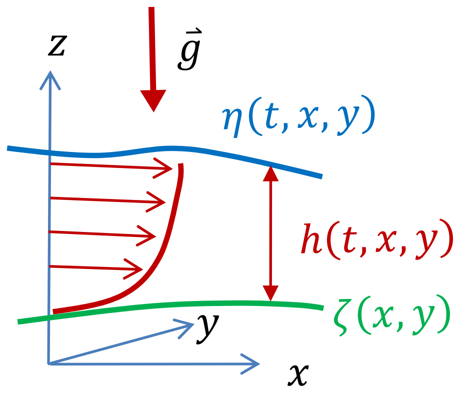
    - 13.12. Kelvin Wave
    - 13.15. Rossby Wave

EQUATIONS FOR BAROTROPIC WAVES

The equations for the barotropic waves



Navier-Stokes equations

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} + 2\vec{\Omega}\times\vec{u} = \vec{g} - \frac{1}{\rho}\nabla p + \nu\nabla^2\vec{u}$$

Assumptions:

1. constant density
2. shallow water approximation

$$\frac{\partial p}{\partial z} = -\rho g$$

→ hydrostatic vertical pressure distribution $\quad p = \rho g(\eta - z)$

Momentum equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -g\frac{\partial \eta}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial y^2} + \nu\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -g\frac{\partial \eta}{\partial y} + \nu\frac{\partial^2 v}{\partial x^2} + \nu\frac{\partial^2 v}{\partial y^2} + \nu\frac{\partial^2 v}{\partial z^2}$$

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

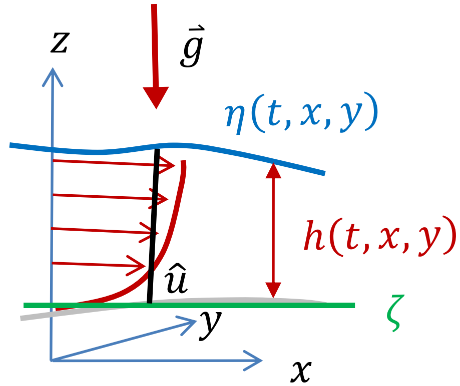
+ kinematic boundary condition at the free surface

$$\frac{\partial \eta}{\partial t} + u\Big|_\eta \frac{\partial \eta}{\partial x} + v\Big|_\eta \frac{\partial \eta}{\partial y} - w\Big|_\eta = 0$$

Depth-integrated continuity equation

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(\hat{u}h) + \frac{\partial}{\partial y}(\hat{v}h) = 0 \qquad h = \eta - \zeta \qquad \hat{u}h = \int_\zeta^\eta u\,dz \qquad \hat{v}h = \int_\zeta^\eta v\,dz$$

The linearized equations



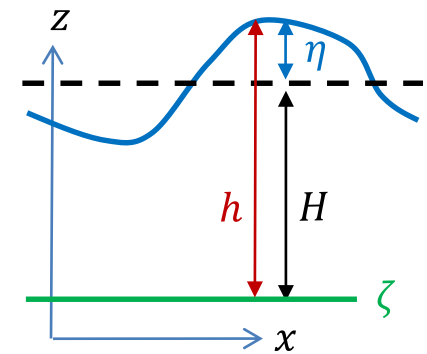
Assumptions:

- horizontal bottom
- advective acceleration terms are negligible ($Ro \ll 1$ for GFD)
- friction is negligible ($Ek \ll 1$ for GFD)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -g\frac{\partial \eta}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial y^2} + \nu\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -g\frac{\partial \eta}{\partial y} + \nu\frac{\partial^2 v}{\partial x^2} + \nu\frac{\partial^2 v}{\partial y^2} + \nu\frac{\partial^2 v}{\partial z^2}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(\hat{u}h) + \frac{\partial}{\partial y}(\hat{v}h) = 0$$



- free surface oscillation are small compared with the reference depth $\quad h = H + \eta \qquad \eta \sim \Delta H \ll H$

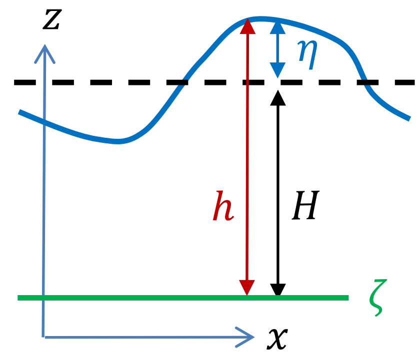
$$\frac{\partial \eta}{\partial t} + \hat{u}\frac{\partial \eta}{\partial x} + (H+\eta)\frac{\partial \hat{u}}{\partial x} + \hat{v}\frac{\partial \eta}{\partial y} + (H+\eta)\frac{\partial \hat{v}}{\partial y} = 0$$

- we study waves that travel with celerity much larger than the flow velocity $\quad c \sim C = \frac{L}{T} \gg U$

$$\underset{\substack{\frac{\Delta H}{T} \\ \frac{C\,\Delta H}{U\,H}}}{\frac{\partial \eta}{\partial t}} + \underbrace{\hat{u}\frac{\partial \eta}{\partial x} + \hat{v}\frac{\partial \eta}{\partial y}}_{\substack{U\frac{\Delta H}{L} \\ \frac{\Delta H}{H} \ll 1}} + \underbrace{H\frac{\partial \hat{u}}{\partial x} + H\frac{\partial \hat{v}}{\partial y}}_{\substack{H\frac{U}{L} \\ 1}} = 0$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y}\right) = 0$$

The linearized depth-averaged equations



$$\frac{\partial u}{\partial t} - f v = -g\frac{\partial \eta}{\partial x} \qquad \frac{\partial v}{\partial t} + f u = -g\frac{\partial \eta}{\partial y}$$

depth-averaged

$$\frac{\partial \hat{u}}{\partial t} - f \hat{v} = -g\frac{\partial \eta}{\partial x} \qquad \frac{\partial \hat{v}}{\partial t} + f \hat{u} = -g\frac{\partial \eta}{\partial y} \qquad \frac{\partial \eta}{\partial t} + H\left(\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y}\right) = 0$$

In the following $\hat{u} \rightarrow u$ to simply the notation / strictly valid if velocity is uniform along the vertical

$$\frac{\partial u}{\partial t} - f v = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + f u = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

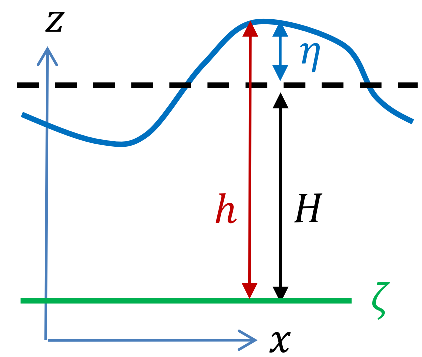
Set of three linear equations in three unknowns (u, v, η)

Basis for linear barotropic (i.e., surface) waves

The simplest case: one-directional surface gravity waves without rotation

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} \qquad \frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$



$$\begin{cases} \dfrac{\partial u}{\partial t} + g\dfrac{\partial \eta}{\partial x} = 0 \\ \dfrac{\partial \eta}{\partial t} + H\dfrac{\partial u}{\partial x} = 0 \end{cases}$$

$$\frac{\partial^2 u}{\partial t^2} + g\frac{\partial^2 \eta}{\partial t \partial x} = 0$$

$$\frac{\partial^2 \eta}{\partial t \partial x} + H\frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial^2 u}{\partial t^2} - gH\frac{\partial^2 u}{\partial x^2} = 0$$

Wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}$ celerity $c = \sqrt{gH}$

Solution for the velocity $u(x,t) = F_1(x + c\,t) + F_2(x - c\,t)$

Using one of the two equations, e.g. $\dfrac{\partial \eta}{\partial t} + H\dfrac{\partial u}{\partial x} = 0$

Solution for the free surface

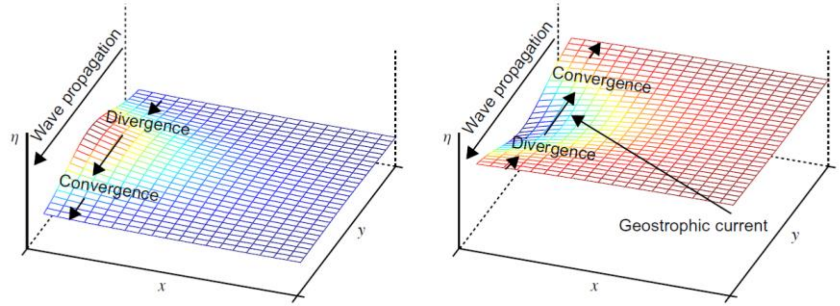
$$\eta(x,t) = -\sqrt{H/g}\,[F_1(x + c\,t) - F_2(x - c\,t)]$$

Kelvin waves

Kelvin waves along a solid boundary

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} \qquad \frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$



(Cushman-Roisin & Beckers, 2011)

Assumption: $u = 0$

$$\frac{\partial v}{\partial t} + g\frac{\partial \eta}{\partial y} = 0 \rightarrow \frac{\partial^2 v}{\partial t^2} + g\frac{\partial^2 \eta}{\partial t \partial y} = 0$$

$$\frac{\partial \eta}{\partial t} + H\frac{\partial v}{\partial y} = 0 \rightarrow \frac{\partial^2 \eta}{\partial t \partial x} + H\frac{\partial^2 v}{\partial y^2} = 0$$

$$fv - g\frac{\partial \eta}{\partial x} = 0$$

Wave equation $\quad \frac{\partial^2 v}{\partial t^2} - c^2\frac{\partial^2 v}{\partial y^2} = 0$

$$c = \sqrt{gH}$$

Solution

$$v(x,y,t) = F_1(x, y + c\,t) + F_2(x, y - c\,t)$$

$$\eta(x,y,t) = -\sqrt{H/g}\,[F_1(x, y + c\,t) - F_2(x, y - c\,t)]$$

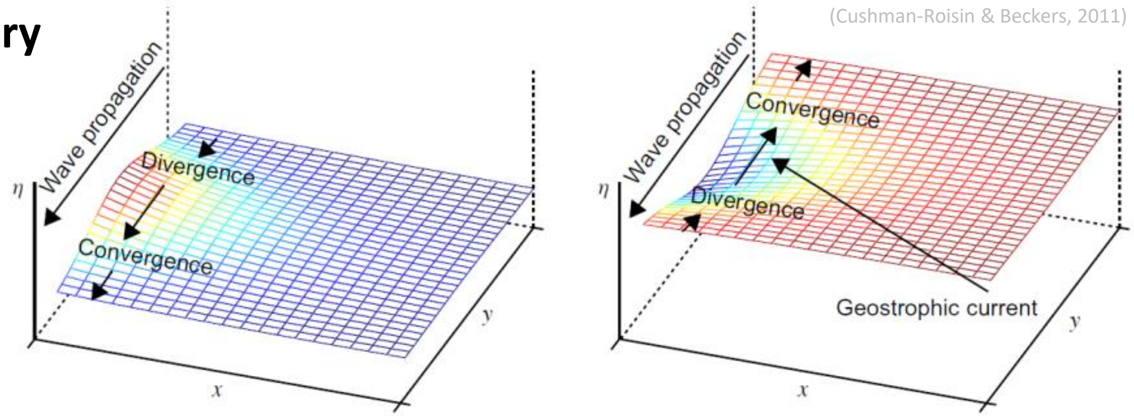
$$fF_1 + \sqrt{gH}\frac{\partial F_1}{\partial x} = 0 \qquad \frac{\partial F_1}{\partial x} = -\frac{f}{c}F_1 \qquad F_1 = F_{10}(y + c\,t)\exp\left(-\frac{x}{R}\right)$$

$$fF_2 - \sqrt{gH}\frac{\partial F_2}{\partial x} = 0 \qquad \frac{\partial F_2}{\partial x} = \frac{f}{c}F_2 \qquad F_2 = F_{20}(y - c\,t)\exp\left(\frac{x}{R}\right)$$

Rossby radius of deformation

$$R = \frac{c}{f} = \frac{\sqrt{gH}}{f}$$

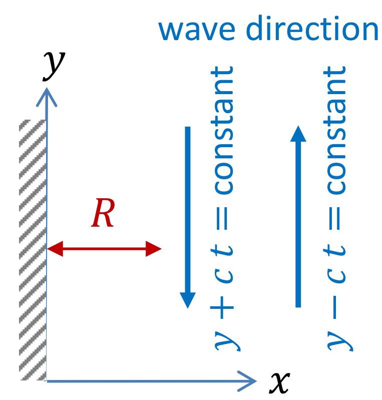
Kelvin waves along a solid boundary



$$
\begin{cases}
\dfrac{\partial v}{\partial t} + g\dfrac{\partial \eta}{\partial y} = 0 \\
\dfrac{\partial \eta}{\partial t} + H\dfrac{\partial v}{\partial y} = 0 \\
f v - g\dfrac{\partial \eta}{\partial x} = 0
\end{cases}
$$

General solution

$$R = \frac{c}{f} = \frac{\sqrt{gH}}{f}$$



$$u = 0$$

$$v = F_{10}(y + c\,t)\exp\left(-\frac{x}{R}\right) + F_{20}(y - c\,t)\exp\left(\frac{x}{R}\right)$$

$$\eta = -\sqrt{H/g}\left[F_{10}(y + c\,t)\exp\left(-\frac{x}{R}\right) - F_{20}(y - c\,t)\exp\left(\frac{x}{R}\right)\right]$$

Which direction is the wave travelling?
(northern hemisphere)

$\exp\left(\frac{x}{R}\right) \to \infty$ far away from the solid boundary on the left

Actual solution

$$u = 0$$

$$v = F(y + c\,t)\exp\left(-\frac{x}{R}\right)$$

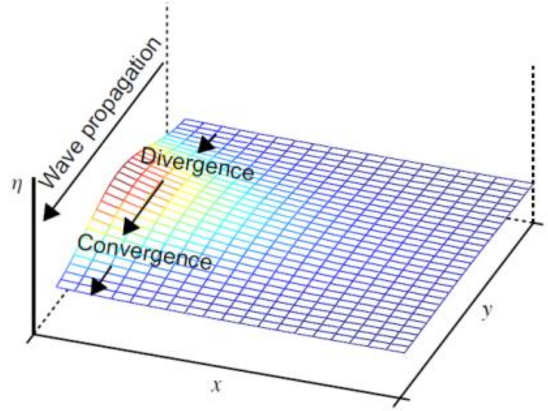
$$\eta = -\sqrt{H/g}\,F(y + c\,t)\exp\left(-\frac{x}{R}\right)$$

The wave is trapped laterally (along x) and travels with the coast on its right ($y = y_0 - c\,t$)

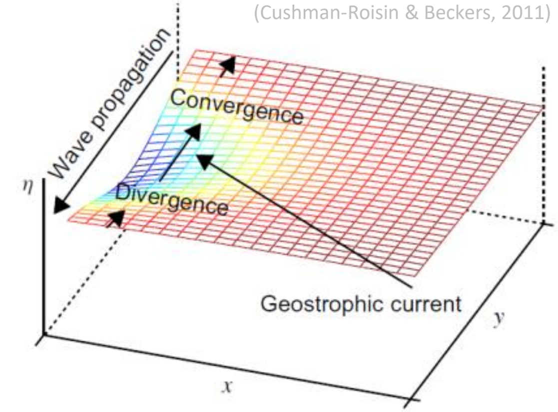
$F(y_0)$ to be determined with the initial conditions

Kelvin waves

$$
\begin{cases}
\dfrac{\partial v}{\partial t} + g\dfrac{\partial \eta}{\partial y} = 0 \\
\dfrac{\partial \eta}{\partial t} + H\dfrac{\partial v}{\partial y} = 0 \\
fv - g\dfrac{\partial \eta}{\partial x} = 0
\end{cases}
$$



(Cushman-Roisin & Beckers, 2011)



Properties:

- trapping distance $R = c/f$ increases with reduced rotation
- wave travels with the coast on its right (northern hemisphere)

  but local velocity direction follow the geostrophic rule $v = \dfrac{g}{f}\dfrac{\partial \eta}{\partial x}$
- the wave is non-dispersive (celerity $c = \sqrt{gH}$ is not a function of wavenumber)

Example: *English Channel*
The tidal range decays fron
French to the English coast

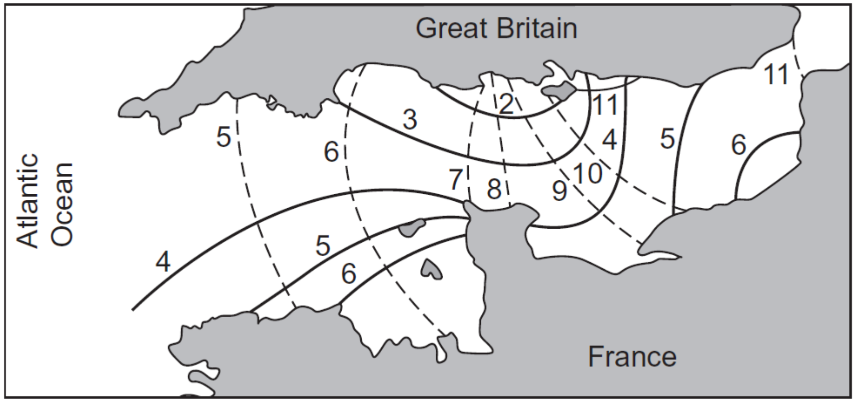
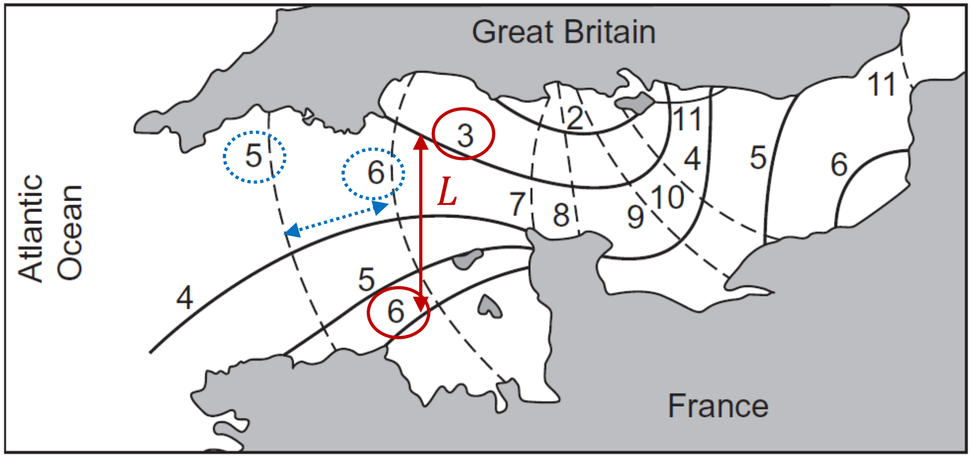


FIGURE 9.2 Cotidal lines (dashed) with time in lunar hours for the M2 tide in the English Channel showing the eastward progression of the tide from the North Atlantic Ocean. Lines of equal tidal range (solid, with value in meters) reveal larger amplitudes along the French coast, namely to the right of the wave progression in accordance with Kelvin waves. (*From Proudman, 1953, as adapted by Gill, 1982*)

(Cushman-Roisin & Beckers, 2011)



$H \sim 40$ m

$$c = \sqrt{gH} \sim 20 \text{ m/s}$$

$$R = \frac{c}{f} \sim 200 \text{ km}$$

Difference in tidal amplitude on the two coasts

$L \sim 100$ km

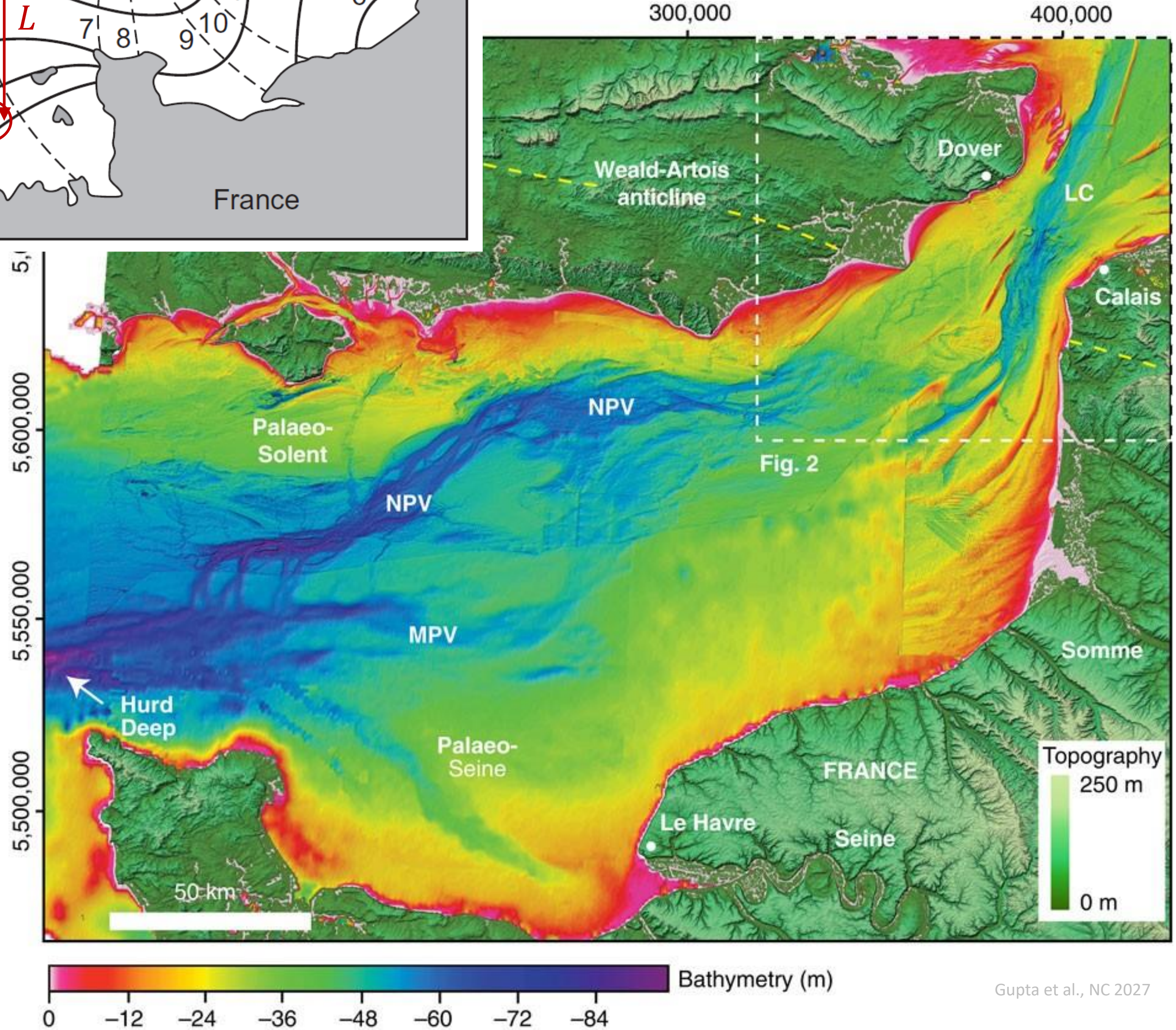
$$\exp\left(-\frac{L}{R}\right) \sim 0.6$$

Check of tidal wave celerity

$\Delta x \sim 50$ km

$\Delta t = 1$ hr = 3600 s

$$c = \frac{\Delta x}{\Delta t} \sim 15 \text{ m/s}$$



Gupta et al., NC 2027

Kelvin wave

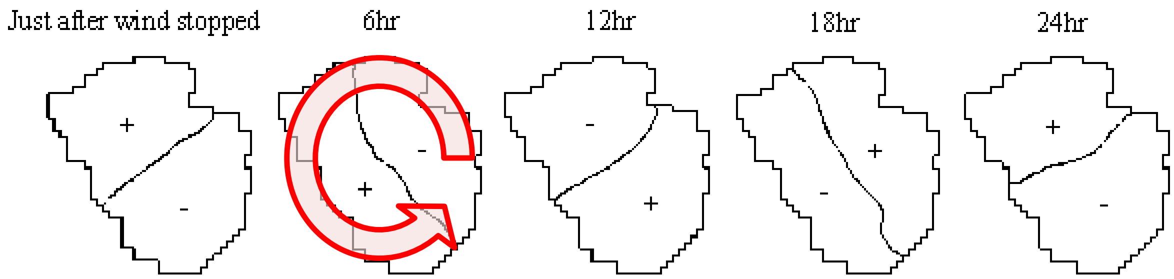
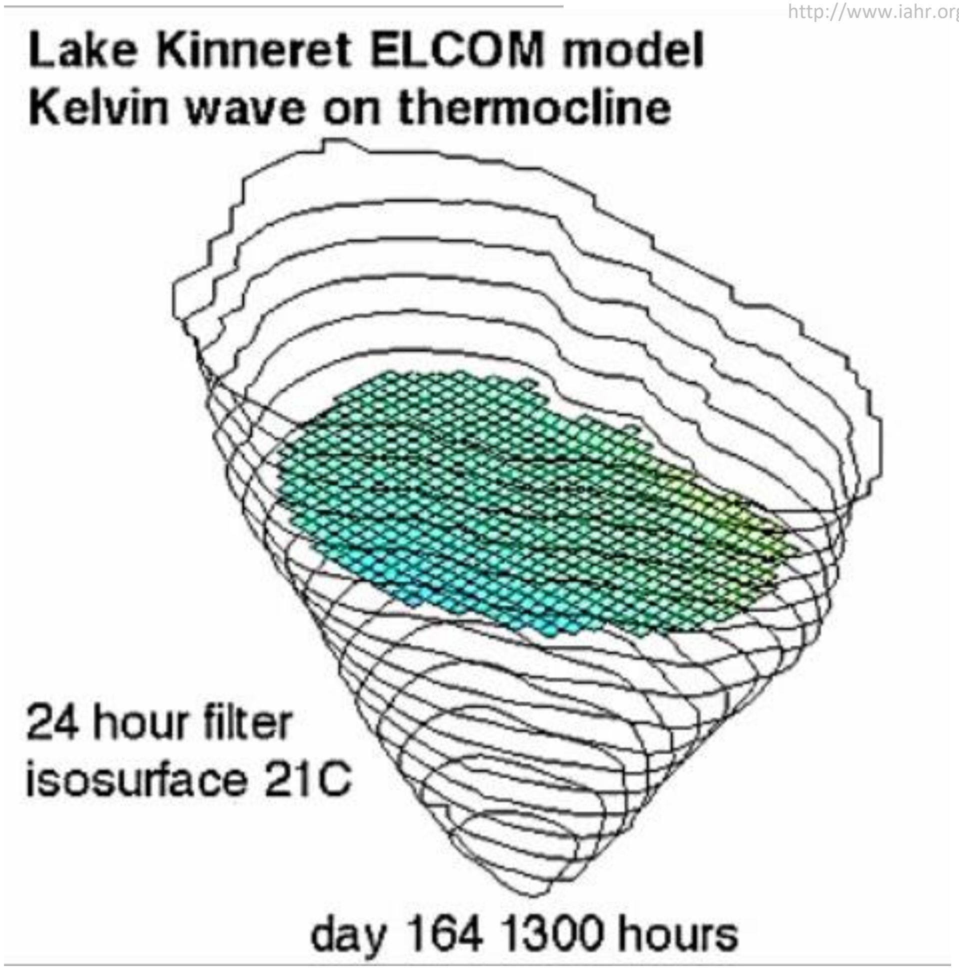


Fig. 8 Spatial interface elevation pattern after NW5m/s wind stopped

http://www.iahr.org/e-library/beijing_proceedings/Theme_B/A%20STUDY%20ON%20INTERNAL%20SEICHE.html



Kelvin waves:
counter-clockwise propagation
(northern hemisphere)

(internal) Kelvin wave

Lake Kinneret **thermocline** motion that follows a typical **Kelvin wave** pattern. Visualization is produced by the ELCOM 3D hydrodynamic model. Created by Ben R. Hodges, University of Texas at Austin. Field data for developing the model provided by J. Imberger, Centre for Water Research, University of Western Australia.

http://www.youtube.com/watch?v=SZlix47Jq4A

POINCARÉ

WAVES

Poincaré (inertia-gravity) waves

$$\begin{cases} \dfrac{\partial u}{\partial t} - fv + g\dfrac{\partial \eta}{\partial x} = 0 \\ \dfrac{\partial v}{\partial t} + fu + g\dfrac{\partial \eta}{\partial y} = 0 \\ \dfrac{\partial \eta}{\partial t} + H\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\right) = 0 \end{cases}$$

Constant coefficients:

- flat bottom (constant H)
- f-plane (constant f)

frequency

wavenumbers

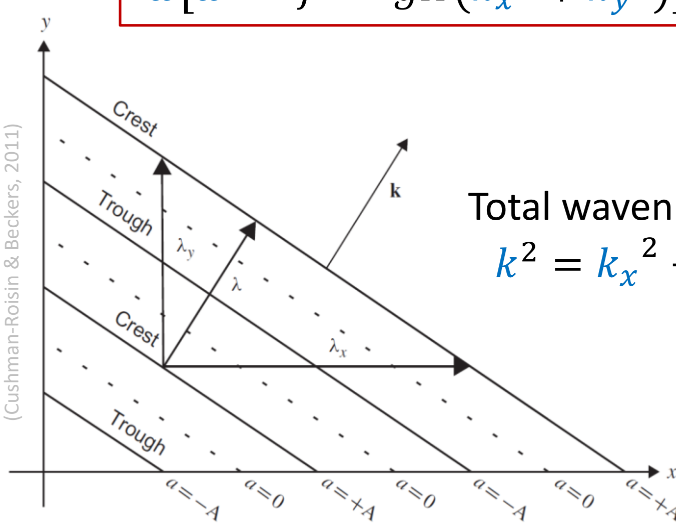
It is possible to look for a Fourier-mode solution:

$$\begin{pmatrix} \eta \\ u \\ v \end{pmatrix} = real\left\{\begin{pmatrix} A \\ U \\ V \end{pmatrix} \exp\left[i\left(k_x x + k_y y - \omega t\right)\right]\right\}$$

$$\begin{cases} -i\omega U - fV + igk_x A = 0 \\ -i\omega V + fU + igk_y A = 0 \\ -i\omega A + H\left(ik_x U + ik_y V\right) = 0 \end{cases}$$

A solution different from the trivial one ($A = U = V = 0$) exists only if the determinant of matrix of the coefficients vanishes → dispersion relation

$$\boxed{\omega\left[\omega^2 - f^2 - gH\left({k_x}^2 + {k_y}^2\right)\right] = 0}$$



(Cushman-Roisin & Beckers, 2011)

Total wavenumber:

$$k^2 = {k_x}^2 + {k_y}^2$$

Roots of the dispersion relation

$$\omega\left[\omega^2 - f^2 - gHk^2\right] = 0$$

- $\omega = 0$
  → steady geostrophic flows
- $\omega = \pm\sqrt{f^2 + gHk^2}$
  → travelling waves

Poincaré waves (inertia-gravity waves)

$$k = \frac{2\pi}{L} \qquad \omega = \frac{2\pi}{T}$$

Wave celerity (phase speed): $c = \frac{\Delta x}{\Delta t} = \frac{\omega}{k}$

Radius of deformation $R = \frac{\sqrt{gH}}{f}$

Dispersion relation
(frequency and wavenumber are related)

$$\omega = \sqrt{f^2 + gHk^2}$$

gravity waves

no rotation ($f = 0$)
or short and fast waves

$$k^2 \gg \frac{f^2}{gH} = \frac{1}{R^2}$$

$$\omega = k\sqrt{gH}$$

$$c = \sqrt{gH}$$

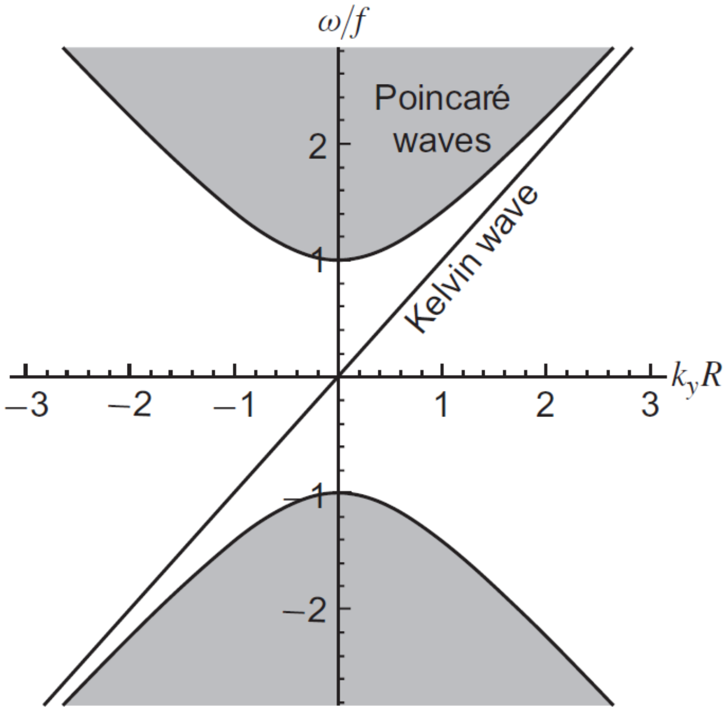
non-dispersive
(like Kelvin waves)

non-hydrostatic?
(short and fast)

intermediate cases

$$\omega \geq f$$

dispersive waves



(Cushman-Roisin & Beckers, 2011)

super-inertial waves

long waves

$$k^2 \ll \frac{f^2}{gH} = \frac{1}{R^2}$$

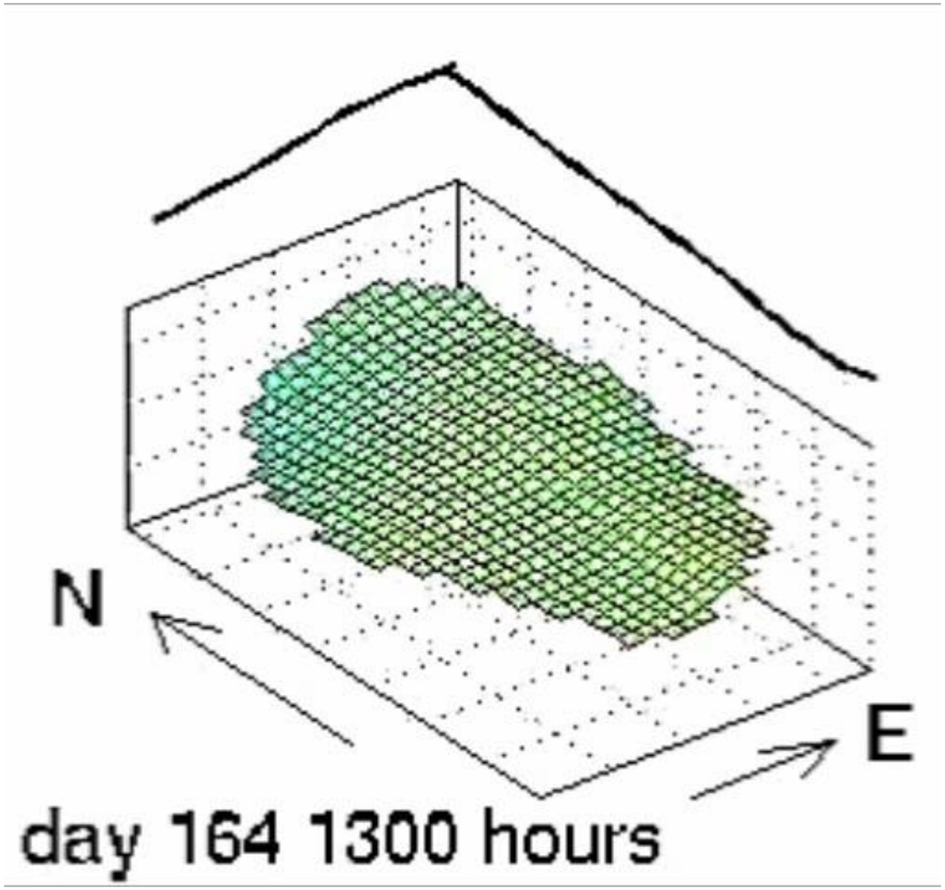
$$\omega \cong f$$

$$c \cong \frac{f}{k}$$

dispersive waves
(modes separate from each other)

inertial waves

Poincaré wave



Internal Poincare wave on the thermocline of Lake Kinneret. From 3D hydrodynamic model. See Hodges, Imberger, Saggio and Winters (2000) in Limnology and Oceanography
http://www.youtube.com/watch?v=8mdAazUspAs

Velocity field

$$\begin{cases} -i\omega U - fV + igk_x A = 0 \\ -i\omega V + fU + igk_y A = 0 \\ -i\omega A + H\left(ik_x U + ik_y V\right) = 0 \end{cases}$$

Fourier-mode solution:

$$\begin{pmatrix} \eta \\ u \\ v \end{pmatrix} = real\left\{\begin{pmatrix} A \\ U \\ V \end{pmatrix} \exp\left[i\left(k_x x + k_y y - \omega t\right)\right]\right\}$$

If the dispersion relation is satisfied, $\omega[\omega^2 - f^2 - gHk^2] = 0$, one equation is redundant

$$\begin{cases} -i\omega U - fV + igk_x A = 0 \\ -i\omega V + fU + igk_y A = 0 \end{cases} \Rightarrow \quad U = \frac{gA\left(k_x\omega + ik_y f\right)}{\omega^2 - f^2} \qquad V = \frac{gA\left(-ik_x + k_y\omega\right)}{\omega^2 - f^2}$$

or equivalent, with $\omega^2 - f^2 = gHk^2$: $\quad U = \dfrac{gA\left(k_x\omega + ik_y f\right)}{gHk^2} \qquad V = \dfrac{gA\left(-ik_x + k_y\omega\right)}{gHk^2}$

1D ($k_y = 0$), no rotation ($f = 0$) $\quad U = \dfrac{gAk_x\omega}{\omega^2} = gA\dfrac{k_x}{\omega} = gA\dfrac{1}{c} = \dfrac{gA}{\sqrt{gH}} = A\sqrt{g/H}$

Solution for 1D wave without rotation

$$u(x,t) = u_1 F_1(x + c\,t) + u_2 F_2(x - c\,t)$$

$$\eta(x,t) = -\sqrt{H/g}\,u_1 F_1(x + c\,t) + \sqrt{H/g}\,u_2 F_2(x - c\,t)$$
$$= A[F_1(x + c\,t) + F_2(x - c\,t)]$$

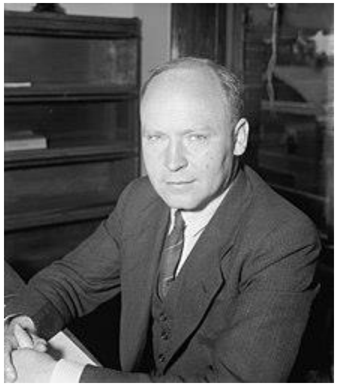
$$A = -u_1\sqrt{H/g} \qquad A = u_2\sqrt{H/g}$$

$$u_1 = -A\sqrt{g/H} \qquad u_2 = A\sqrt{g/H}$$

$$u(x,t) = A\sqrt{\frac{g}{H}}[-F_1(x + c\,t) + F_2(x - c\,t)]$$

ROSSBY WAVES

Planetary waves – Rossby waves



Carl-Gustaf Rossby
(1898-1957)

$$\begin{cases} \dfrac{\partial u}{\partial t} - f v + g\dfrac{\partial \eta}{\partial x} = 0 \\ \dfrac{\partial v}{\partial t} + f u + g\dfrac{\partial \eta}{\partial y} = 0 \\ \dfrac{\partial \eta}{\partial t} + H\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\right) = 0 \end{cases}$$

Very long and slow waves (quasi-geostrophic)

Variable coefficients:
- flat bottom (constant H)
- β-plane (small variations of f)

f variations in a β-plane $\quad f = 2\Omega \sin\varphi \quad \varphi = \varphi_0 + \dfrac{y}{a} \quad a = 6371$ km (Earth's radius)

$$f = 2\Omega \sin\varphi_0 + 2\Omega\cos\varphi_0 \frac{y}{a}$$

$$f_0 = 2\Omega \sin\varphi_0 \qquad \beta_0 = \frac{2\Omega}{a}\cos\varphi_0$$

$$f = f_0 + \beta_0 y$$

(dimensionless) planetary number

$$\beta = \frac{\beta_0 L}{f_0} \ll 1$$

Fourier-mode analysis (linear equation with constant coefficients) $\quad \cos(k_x x + k_y y - \omega t)$

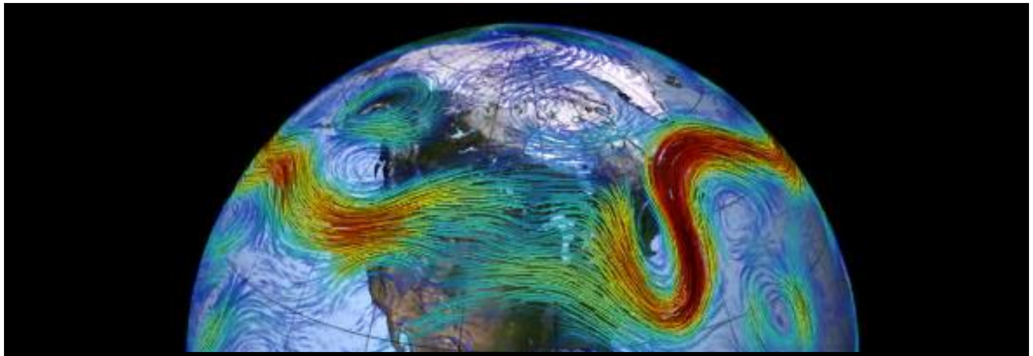
→ dispersion relation

with $R = \sqrt{gH}/f_0$

$$\omega = -\beta_0 R^2 \frac{k_x}{1 + R^2({k_x}^2 + {k_y}^2)} = -\frac{\beta_0\, k_x}{R^{-2} + k^2}$$

Total wavenumber:
$k^2 = {k_x}^2 + {k_y}^2$

Rossby waves: Information from online resources



https://oceanservice.noaa.gov/facts/rossby-wave.html

Chapter 3: Rossby waves and instability

Parcel displacements and the conservation of potential vorticity

The Rossby wave dispersion relation

Topographic Rossby waves, baroclinic Rossby waves and vertical modes

https://www.youtube.com/watch?v=pwV54L-NXzM

ROSSBY WAVES AND EXTREME WEATHER

https://www.youtube.com/watch?v=MzW5Isbv2A0

Rossby waves: Dispersion relation

$$\omega = -\beta_0 R^2 \frac{k_x}{1 + R^2\left({k_x}^2 + {k_y}^2\right)}$$

Cushman-Roisin & Beckers's eq. (9.27)

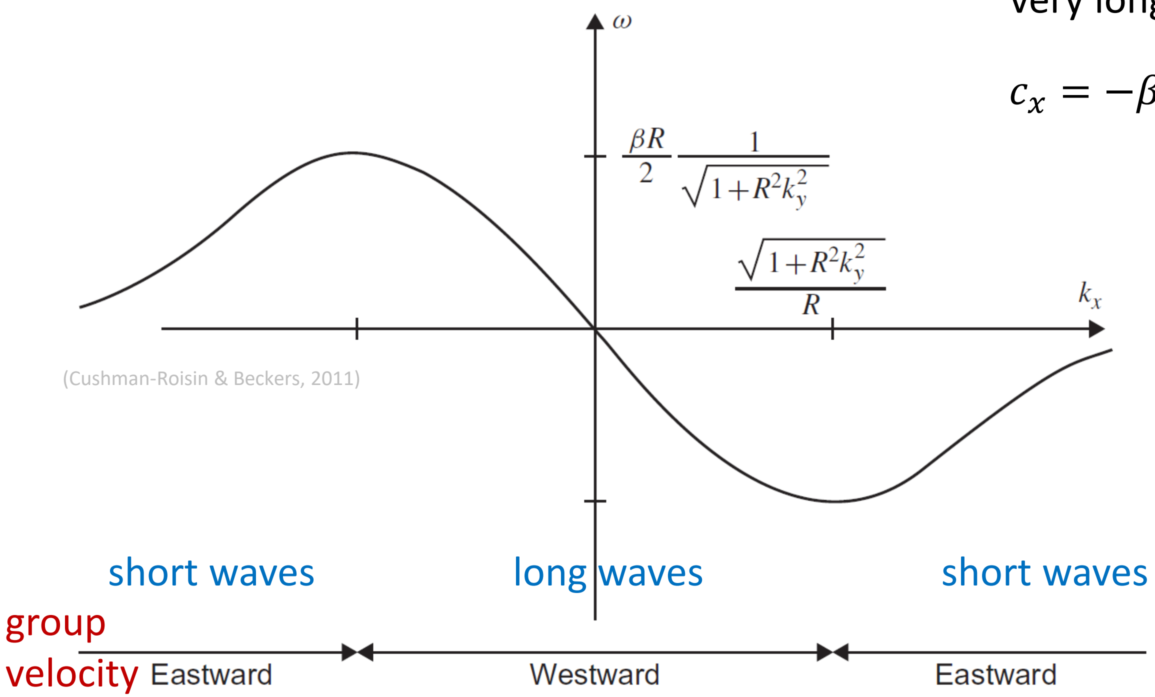
zonal phase speed

$$c_x = \frac{\omega}{k_x} = -\frac{\beta_0 R^2}{1 + R^2\left({k_x}^2 + {k_y}^2\right)} < 0$$

(westward)

very long waves (${k_x}^{-1} \gg R \rightarrow k_x R \ll 1$)

$c_x = -\beta_0 R^2$ (largest wave speed)



if $R^2k^2 \gg 1$ then

$$\omega \cong -\beta_0 \frac{k_x}{\left({k_x}^2 + {k_y}^2\right)}$$

Nick Hall's YouTube video	$\omega = Ul - \frac{\beta l}{l^2 + m^2}$	$= Uk_x - \frac{\beta_0 k_x}{{k_x}^2 + {k_y}^2}$

Interaction with the jet stream

https://en.wikipedia.org/wiki/Jet_stream

Jet streams:

- polar jets, at 9–12 km above sea level
- subtropical jets at 10–16 km (higher altitude and somewhat weaker)

Typical speed $\sim O(100 \text{ km/h})$

Eastward direction (West → East) > 0

Rossby wave's phase speed

Rigid lid
$$c_x = \frac{\omega}{k_x} = U - \frac{\beta_0}{{k_x}^2 + {k_y}^2}$$

Variable thickness
$$c_x = \frac{\omega}{k_x} = U - \frac{U + \beta_0 R^2}{1 + R^2\left({k_x}^2 + {k_y}^2\right)}$$

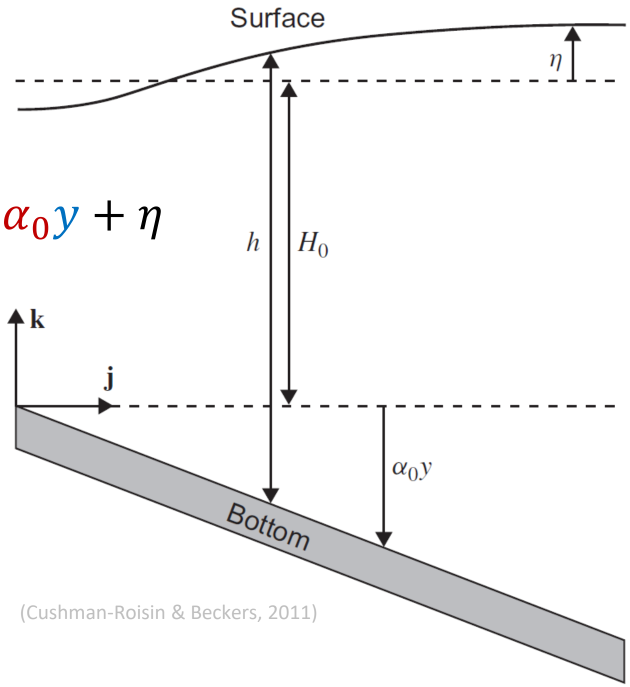
Topographic Rossby waves

$$\begin{cases} \dfrac{\partial u}{\partial t} - fv + g\dfrac{\partial \eta}{\partial x} = 0 \\ \dfrac{\partial v}{\partial t} + fu + g\dfrac{\partial \eta}{\partial y} = 0 \\ \dfrac{\partial \eta}{\partial t} + \dfrac{\partial}{\partial x}(uh) + \dfrac{\partial}{\partial y}(vh) = 0 \end{cases}$$

Variable coefficients:

- sloping bottom (small variation of reference depth H) in any direction (e.g., y, but not only)
- constant f

$$h = H_0 + \alpha_0 y + \eta$$



(Cushman-Roisin & Beckers, 2011)

Assumption $\alpha = \dfrac{\alpha_0 L}{H_0} \ll 1$

Linearized continuity equation

$$\frac{\partial \eta}{\partial t} + H_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \alpha_0 v = 0$$

Dispersion relation $\omega = \alpha_0 \dfrac{g}{f} \dfrac{k_x}{1 + R^2\left({k_x}^2 + {k_y}^2\right)}$

Phase speed $c_x = \dfrac{\alpha_0 g}{f} \dfrac{1}{1 + R^2\left({k_x}^2 + {k_y}^2\right)}$

Group velocity $c_g = \dfrac{d\omega}{dk}$

Rossby waves: Conservation of potential vorticity

$$\begin{cases} \dfrac{\partial u}{\partial t} - fv + g\dfrac{\partial \eta}{\partial x} = 0 \\ \dfrac{\partial v}{\partial t} + fu + g\dfrac{\partial \eta}{\partial y} = 0 \\ \dfrac{\partial \eta}{\partial t} + \dfrac{\partial}{\partial x}(uh) + \dfrac{\partial}{\partial y}(vh) = 0 \end{cases}$$

Variable coefficients:

- β-plane (small variations of f with latitude, i.e. y)
- sloping bottom (small variation of reference depth H) in any direction (e.g., y, but not only)

Vorticity $\zeta = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$

Potential vorticity $q = \dfrac{\zeta + f}{h}$

$$\boxed{\frac{D}{Dt} q = 0}$$

(material derivative)

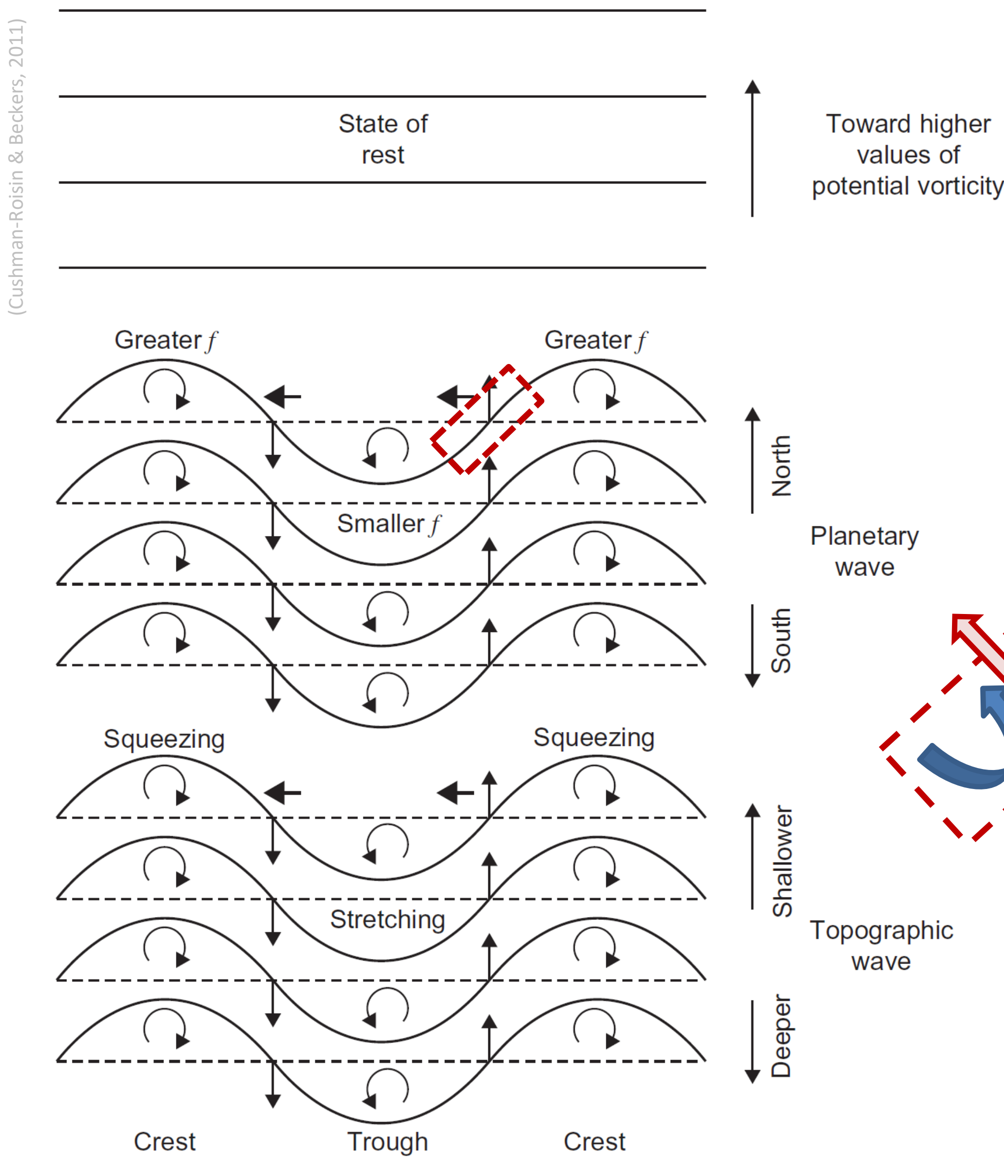
$$f = f_0 + \beta_0 y \qquad h = H_0 + \alpha_0 y + \eta$$

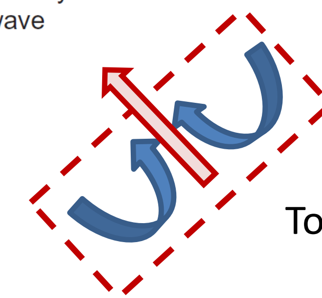
$$q = \frac{\zeta + f_0 + \beta_0 y}{H_0 + \alpha_0 y + \eta} \cong \frac{1}{H_0}\left(\zeta + f_0 + \beta_0 y - \frac{\alpha_0 f_0}{H_0} y - \frac{f_0}{H_0}\eta\right)$$

two formally similar mechanisms

Similarity between planetary and topographic Rossby waves

$$\boxed{q = \frac{\zeta + f}{h}}$$





Referring to null relative vorticity

$$\frac{\zeta + f_0 + \beta_0 y}{h_0 + \alpha_0 y} = \frac{f_0}{h_0}$$

Planetary waves with constant h

$$\zeta + f_0 + \beta_0 y = f_0$$

$$\zeta = -\beta_0 y$$

Topographic waves with constant f

$$\frac{\zeta + f_0}{h_0 + \alpha_0 y} = \frac{f_0}{h_0}$$

$$\frac{\zeta + f_0}{h_0}\left(1 - \frac{\alpha_0 y}{h_0}\right) \cong \frac{f_0}{h_0}$$

$$\zeta \cong \frac{f_0}{h_0}\alpha_0 y$$

Rossby waves

Summary of Rossby waves' properties:

• They are dispersive (the phase speed varies with wavelength), although waves much longer than the Rossby Radius, LD, are non-dispersive, $\sigma \cong -\beta k R^2$, and these propagate straight westward.

• They are anisotropic (the phase speed and frequency vary with direction of propagation, even for a fixed wavelength).

• They have wavecrests with phase speed moving **westward** along latitude circles, $cp = \sigma/k < 0$. This also describes the topographic waves calculate above, where the topography slopes upward to the north; adjust accordingly for other orientations of the slope, so that the wavecrests always move with shallower water to their right (reverse this in the southern hemisphere).

• Energy propagation, with group velocity can occur in any direction. The group velocity, gc, is the gradient of the surface $\sigma(k,l)$, hence is perpendicular to the contours of constant frequency, pointing toward higher values of σ.

• For east-west propagation, the group velocity is directed eastward for shorter waves and westward for longer waves. This is evident in the 'Green function' solution for waves generated by a point source, oscillating at a single frequency.

https://www.ocean.washington.edu/courses/oc512/rossby-waves-gfd109-lec5a-07.pdf