Environmental Fluid Mechanics, *M.Sc. Environmental Meteorology* – 2022/23

Stratified flows: Interfacial waves

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(Marco Adami, *Scirocco*, 2008)

References

Benoit Cushman-Roisin and Jean-Marie Beckers, **Introduction to geophysical fluid dynamics : physical and numerical aspects**, 2nd ed., Academic Press, 2011.
https://webapps.unitn.it/Biblioteca/it/Web/LibriElettroniciDettaglio/117963

- 9. Stratification
  - 9.3 Inertia-Gravity Waves (Poincaré Waves)
- 12. Layered Models
  - 12.2 Layered Models
  - 12.4 Two-Layer Models
  - 12.5 Wind-Induced Seiches in Lakes

Pijush K. Kundu, Ira M. Cohen, David R. Dowling, **Fluid mechanics**, 5th ed., Academic Press, Elsevier, 2012.
https://www-sciencedirect-com.ezp.biblio.unitn.it/book/9780123821003/fluid-mechanics

- 7. Gravity Waves
  - 7.4. Standing Waves

Scott A. Socolofsky, **Mixing and Transport Processes in the Environment, Lecture notes – part 2**.

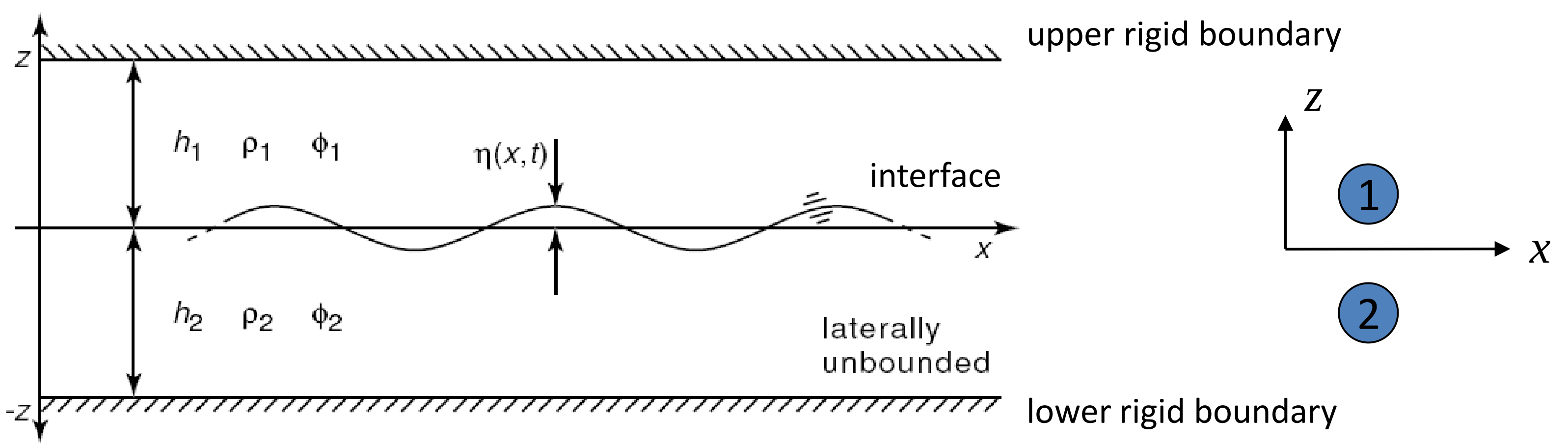
- ch. 9 – Mixing in Lakes and Reservoirs
- ch. 10 – Internal Waves

WAVES IN STRATIFIED FLOWS

Interfacial waves

Assumptions:
- immiscible fluids
- rigid lid upper and lower boundaries
- plane flow (2D)
- inviscid flow (null viscosity, large Re)
- irrotational flow in each layer
- basic flow at rest ($U_1 = U_2 = 0$)
- small amplitude waves → perturbation method

Aim:
- not to study the stability (neutrally stable case),
- but propagation



Equations

Velocity potential ϕ
(irrotational flow)

$$u = \frac{\partial \phi}{\partial x} \qquad w = \frac{\partial \phi}{\partial z}$$

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

$$\boxed{\nabla^2 \phi_1 = 0}$$ upper layer

$$\boxed{\nabla^2 \phi_2 = 0}$$ lower layer

Momentum equation
(inviscid flow)

$$\rho \frac{d\vec{u}}{dt} = -\nabla p + \rho \vec{g}$$

$$\rho \frac{du}{dt} = -\frac{\partial p}{\partial x} \qquad \rho \frac{dw}{dt} = -\frac{\partial p}{\partial z} - \rho g$$

Unsteady Bernoulli theorem

$$\rho \frac{\partial \phi}{\partial t} + p + \rho g z = c_0$$

(constant along a streamline)

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z}$$

(plane flow *x-z*)

Boundary conditions

Interface: $F = z - \eta = 0$

Kinematic condition $\left.\frac{dF}{dt}\right|_{\eta} = 0$ $\left.\left(w - \frac{\partial\eta}{\partial t} - u\frac{\partial\eta}{\partial x}\right)\right|_{\eta} = \left.\left(\frac{\partial\phi}{\partial z} - \frac{\partial\eta}{\partial t} - \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x}\right)\right|_{\eta} = 0$

Dynamic condition $\left.p_1\right|_{\eta} = \left.p_2\right|_{\eta}$ (tangential shear stress are null for inviscid flow)

Upper (z=h_1) and lower (z=-h_2) boundaries: $\left.w\right|_{h_1} = \left.\frac{\partial\phi}{\partial z}\right|_{h_1} = 0$ $\left.w\right|_{-h_2} = \left.\frac{\partial\phi}{\partial z}\right|_{-h_2} = 0$

Wave: periodic in space (x) and time (t) $f(kx - \omega t)$

k wavenumber
ω frequency

Linearized equations

Continuity equations (two layers)

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial z^2} = 0 \qquad \frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial z^2} = 0$$

Interface conditions (z=0)

$$\frac{\partial \phi_1}{\partial z} - \frac{\partial \eta}{\partial t} = 0 \qquad \frac{\partial \phi_2}{\partial z} - \frac{\partial \eta}{\partial t} = 0 \qquad \rho_1 \frac{\partial \phi_1}{\partial t} + \rho_1 g \eta = \rho_2 \frac{\partial \phi_2}{\partial t} + \rho_2 g \eta$$

Boundary conditions

$$\frac{\partial \phi_1}{\partial z} = 0 \quad (z = h_1) \qquad \frac{\partial \phi_2}{\partial z} = 0 \quad (z = -h_2)$$

Structure of the solution

$$\eta(x,t) = A \exp[i(kx - \omega t)] = A[\cos(kx - \omega t) + i \sin(kx - \omega t)]$$

$$\phi_j(x,z,t) = Z_j(z) \exp[i(kx - \omega t)]$$ (complex notation)

Linearized governing equations

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial z^2} = 0$$

$$\frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial z^2} = 0$$

$$\phi_1 = Z_1(z) \exp[i(kx - \omega t)]$$

$$\frac{\partial^2 \phi_1}{\partial x^2} = -k^2 Z_1 \exp[i(kx - \omega t)]$$

$$\frac{\partial^2 \phi_1}{\partial z^2} = \frac{\partial^2 Z_1}{\partial z^2} \exp[i(kx - \omega t)]$$

$$\frac{\partial^2 Z_1}{\partial z^2} - k^2 Z_1 = 0$$

$$\frac{\partial^2 Z_2}{\partial z^2} - k^2 Z_2 = 0$$

$$Z_1 = C_1 \exp(kz) + C_2 \exp(-kz)$$

$$Z_2 = D_1 \exp(kz) + D_2 \exp(-kz)$$

Linearized boundary conditions

$$\phi_1 = Z_1(z)\exp[i(kx-\omega t)]$$
$$\eta = A\exp[i(kx-\omega t)]$$

$$Z_1 = C_1\exp(kz) + C_2\exp(-kz)$$

$(z = h_1)$ $\quad \dfrac{\partial \phi_1}{\partial z} = 0 \qquad \dfrac{\partial Z_1}{\partial z} = 0 \qquad kC_1\exp(kh_1) - kC_2\exp(-kh_1) = 0$

$(z = 0)$ $\quad \dfrac{\partial \phi_1}{\partial z} - \dfrac{\partial \eta}{\partial t} = 0 \qquad \dfrac{\partial Z_1}{\partial z} + i\omega A = 0 \qquad kC_1 - kC_2 + i\omega A = 0$

$$C_1, C_2 = A\,f(k,\omega,h_1)$$

The same boundary conditions
for the lower layer: $Z_2 = D_1\exp(kz) + D_2\exp(-kz)$ $\longrightarrow$ $D_1, D_2 = A\,f(k,\omega,h_2)$

$(z = 0)$ Dynamic boundary condition

$$\rho_1\frac{\partial \phi_1}{\partial t} + \rho_1 g\eta = \rho_2\frac{\partial \phi_2}{\partial t} + \rho_2 g\eta$$

$$-i\omega\rho_1 Z_1 + \rho_1 gA = -i\omega\rho_2 Z_2 + \rho_2 gA$$

$$-i\omega\rho_1(C_1 + C_2) + \rho_1 gA = -i\omega\rho_2(D_1 + D_2) + \rho_2 gA$$

Dispersion relation

$$f(k,\omega,h_1,h_2,g,\rho_1,\rho_2) = 0$$

Solution

$$Z_j = C_1 \exp(kz) + C_2 \exp(-kz)$$

$$\phi_j = Z_j \exp[i(kx - \omega t)]$$

$$Z_1 = \frac{iA\omega}{k} \frac{\cosh[k(z - h_1)]}{\sinh(kh_1)}$$

$$Z_2 = -\frac{iA\omega}{k} \frac{\cosh[k(z + h_2)]}{\sinh(kh_2)}$$

potential:

$$\phi_1 = \frac{iA\omega}{k} \frac{\cosh[k(z - h_1)]}{\sinh(kh_1)} \exp[i(kx - \omega t)] \qquad \phi_2 = -\frac{iA\omega}{k} \frac{\cosh[k(z + h_2)]}{\sinh(kh_2)} \exp[i(kx - \omega t)]$$

→ velocity in the two layers:

$$u_1 = \frac{\partial \phi_1}{\partial x} = -A\omega \frac{\cosh[k(z - h_1)]}{\sinh(kh_1)} \exp[i(kx - \omega t)] \qquad u_2 = \frac{\partial \phi_2}{\partial x} = A\omega \frac{\cosh[k(z + h_2)]}{\sinh(kh_2)} \exp[i(kx - \omega t)]$$

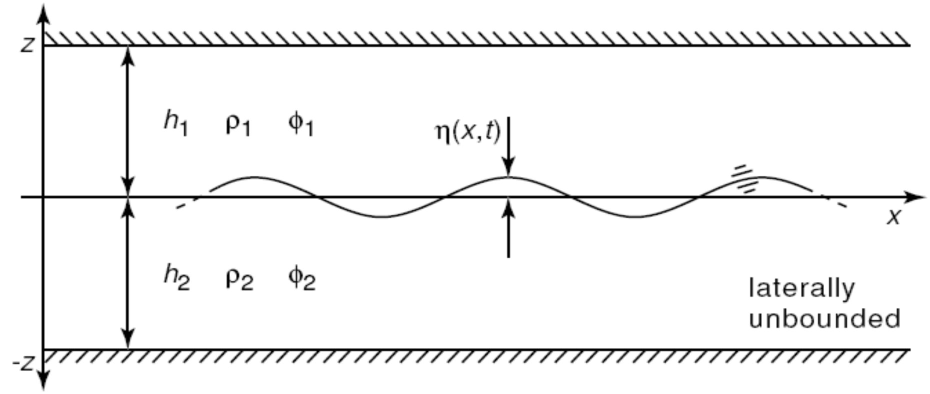
$$w_1 = \frac{\partial \phi_1}{\partial z} = iA\omega \frac{\sinh[k(z - h_1)]}{\sinh(kh_1)} \exp[i(kx - \omega t)] \qquad w_2 = \frac{\partial \phi_2}{\partial z} = -iA\omega \frac{\sinh[k(z + h_2)]}{\sinh(kh_2)} \exp[i(kx - \omega t)]$$

interface position: $\eta = A \exp[i(kx - \omega t)]$

$$\boxed{\text{amplitudes cannot be determined by means of a linear analysis}}$$

$$\boxed{\sinh(\alpha) = \frac{\exp(\alpha) - \exp(-\alpha)}{2} \qquad \cosh(\alpha) = \frac{\exp(\alpha) + \exp(-\alpha)}{2}}$$

Dispersion relationship



Linearized solution → relationship between frequency and wavenumber

$$\eta = A\cos(kx - \omega t)$$

frequency
period

$$\omega = \frac{2\pi}{T}$$

wave number
wave length

$$k = \frac{2\pi}{\lambda}$$

$$\frac{\omega^2}{k}\left(\frac{\rho_1}{\tanh(kh_1)} + \frac{\rho_2}{\tanh(kh_2)}\right) = (\rho_2 - \rho_1)g$$

to be specified for particular cases

$$\eta = A\cos[k(x - c\,t)]$$

celerity of wave propagation

$$c = \frac{dx}{dt} = \frac{\omega}{k}$$

$$\tanh(\alpha) = \frac{\exp(\alpha) - \exp(-\alpha)}{\exp(\alpha) + \exp(-\alpha)}$$

Particular case 1: Unlimited domain (deep water)

$$\boxed{\frac{\omega^2}{k}\left(\frac{\rho_1}{\tanh(kh_1)}+\frac{\rho_2}{\tanh(kh_2)}\right)=(\rho_2-\rho_1)g}$$

$k\,h_1 \to \infty \qquad \tanh(kh_1) \to 1$

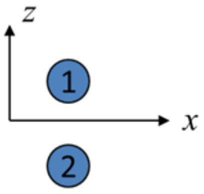
$k\,h_2 \to \infty \qquad \tanh(kh_2) \to 1$

frequency $\omega = \pm\sqrt{gk\dfrac{(\rho_2-\rho_1)}{(\rho_2+\rho_1)}}$ celerity $c = \pm\sqrt{\dfrac{g}{k}\dfrac{(\rho_2-\rho_1)}{(\rho_2+\rho_1)}}$

celerity depends on k → different wavelengths separate from each other (dispersive behaviour)

<u>free surface waves</u> $\rho_1 \cong 0$ (air)

frequency $\omega = \sqrt{gk}$ celerity $c = \pm\sqrt{\dfrac{g}{k}}$



<u>Boussinesq waves</u> $\rho_1 \cong \rho_2$ $g' = \dfrac{(\rho_2-\rho_1)}{\rho_0}g$

frequency $\omega = \sqrt{\dfrac{g'k}{2}}$ celerity $c = \pm\sqrt{\dfrac{g'}{2k}}$

Particular case 2: Shallow water

$$\boxed{\frac{\omega^2}{k}\left(\frac{\rho_1}{\tanh(kh_1)}+\frac{\rho_2}{\tanh(kh_2)}\right)=(\rho_2-\rho_1)g}$$

$kh_1 \to 0 \qquad \tanh(kh_1) \to kh_1$

$kh_2 \to 0 \qquad \tanh(kh_2) \to kh_2$

frequency $\omega = k\sqrt{g\dfrac{h_1h_2(\rho_2-\rho_1)}{(\rho_2h_1+\rho_1h_2)}}$ celerity $c = \pm\sqrt{g\dfrac{h_1h_2(\rho_2-\rho_1)}{(\rho_2h_1+\rho_1h_2)}}$

celerity is independent of k → different wavelengths travel together

<u>free surface waves</u> $\rho_1 \cong 0$ (air)

frequency $\omega = k\sqrt{gh}$ celerity $c = \pm\sqrt{gh}$

<u>Boussinesq waves</u> $\rho_1 \cong \rho_2$

frequency $\omega = k\sqrt{\dfrac{g'h_1h_2}{h_1+h_2}}$ celerity $c = \pm\sqrt{\dfrac{g'h_1h_2}{h_1+h_2}}$

Free surface effect

surface boundary condition

$$\rho \frac{\partial \phi}{\partial t} + p + \rho g z = c_0$$

dispersion relationship

$$\omega^4 = f(k, h_1, h_2, \rho_1, \rho_2)$$

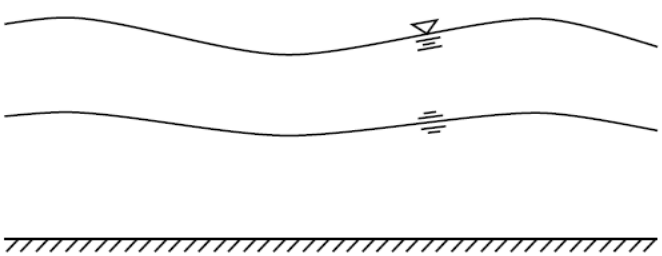
Boussinesq long waves (shallow water): two simplified solutions

$$c = \sqrt{g(h_1 + h_2)} = \sqrt{gH}$$

external mode - fast
(free surface waves)

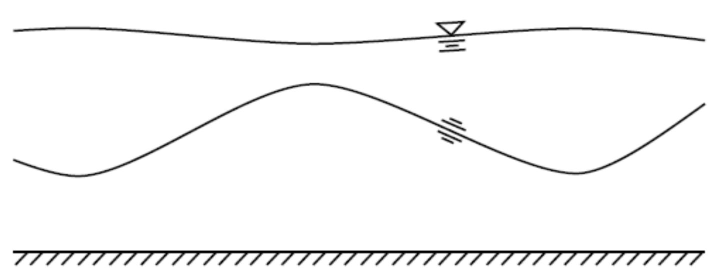
$$c = \sqrt{g' \frac{h_1 h_2}{(h_1 + h_2)}} = \sqrt{g' \frac{h_1 h_2}{H}}$$

internal mode - slow
(interfacial waves)



barotropic motion

∇p parallel to $\nabla \rho$



baroclinic motion

∇p inclined with respect to $\nabla \rho$

Water density

Water density

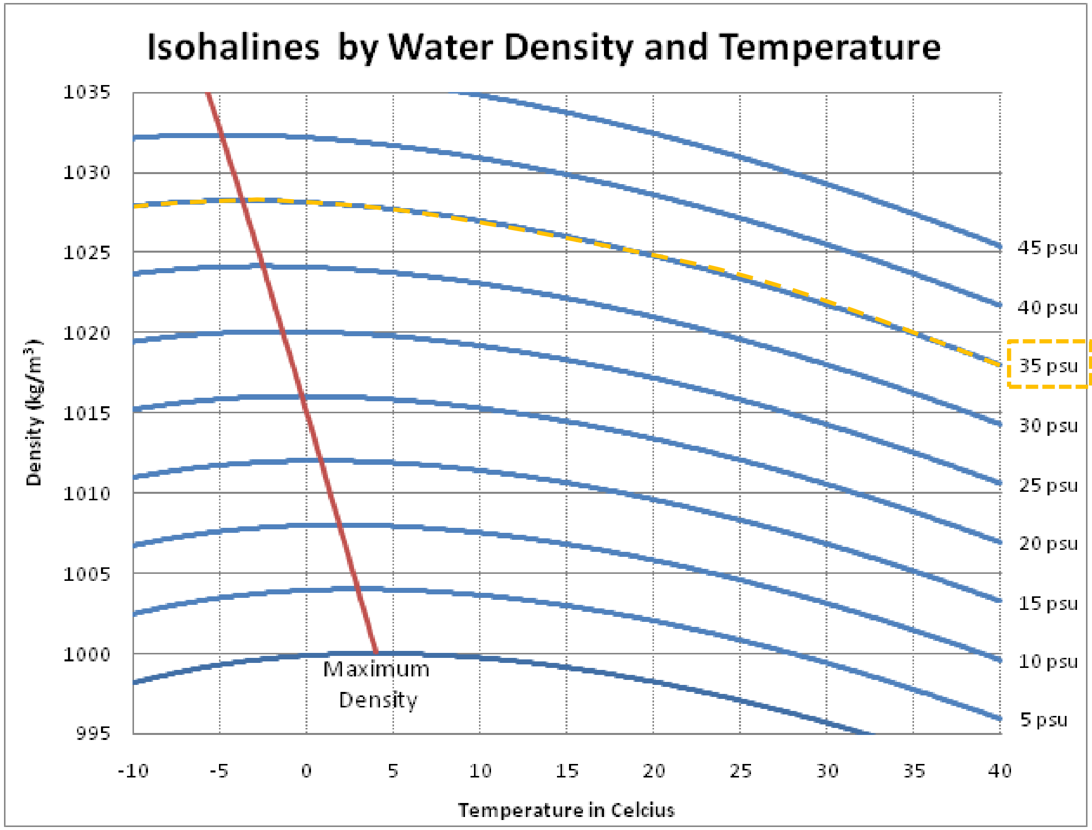
$\rho_{water} \sim 1000\ kg/m^3$ $(\rho_{air} \sim 1.2\ kg/m^3)$

$\rho\,(x, y, z, t)$ depends on temperature, dissolved phases, suspended matter, pressure

ρ as a function of temperature T and salinity S

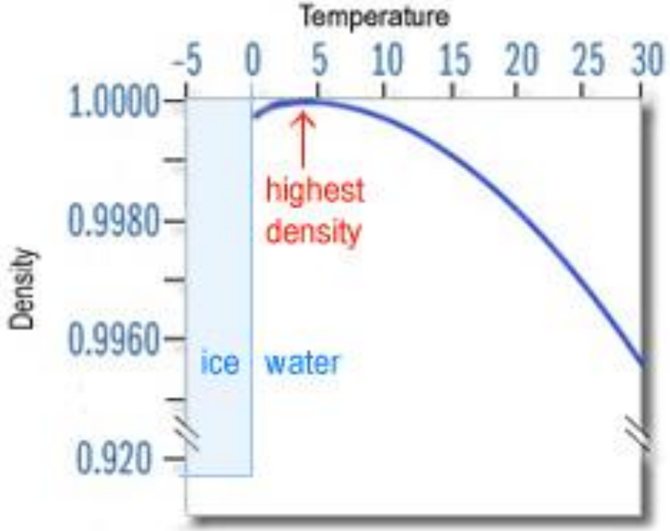
For water motion, density is relevant only with gravity

Salinity (S) is defined as the concentration of salts and measured in PSU (Practical Salinity Units) = g / kg (invariant with temperature).
Sea water has an average salinity of 35 PSU ($\rho_{water} \sim 1025\ kg/m^3$).



http://linkingweatherandclimate.com/ocean/waterdensity.php

Remember that ice is much lighter than liquid water



http://www.rmbel.info/reports/Static/stratification.aspx

Effects of water compressibility

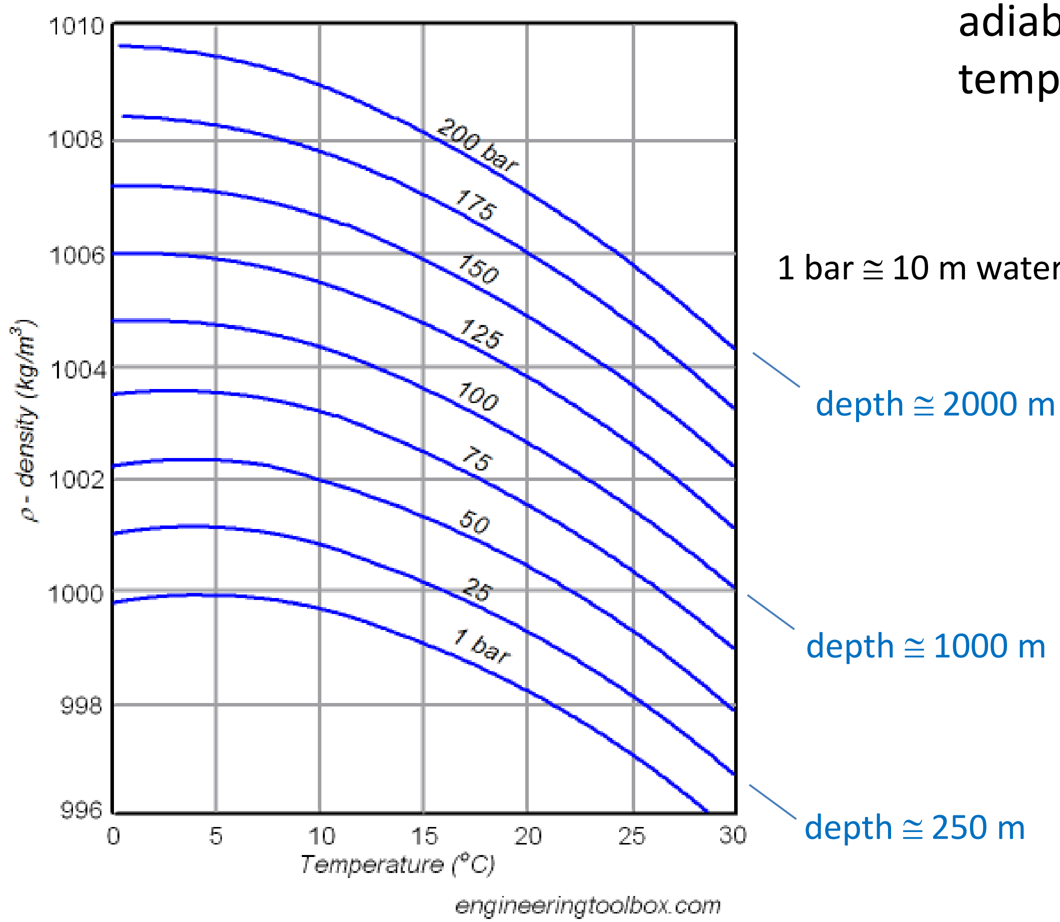
$$\rho^P = \rho^0(1 - P/K)^{-1}$$

(Chen and Millero, LO 1986)

ρ^P: density (T, S, P) ρ^0: density (T, S) P: pressure $K(T)$

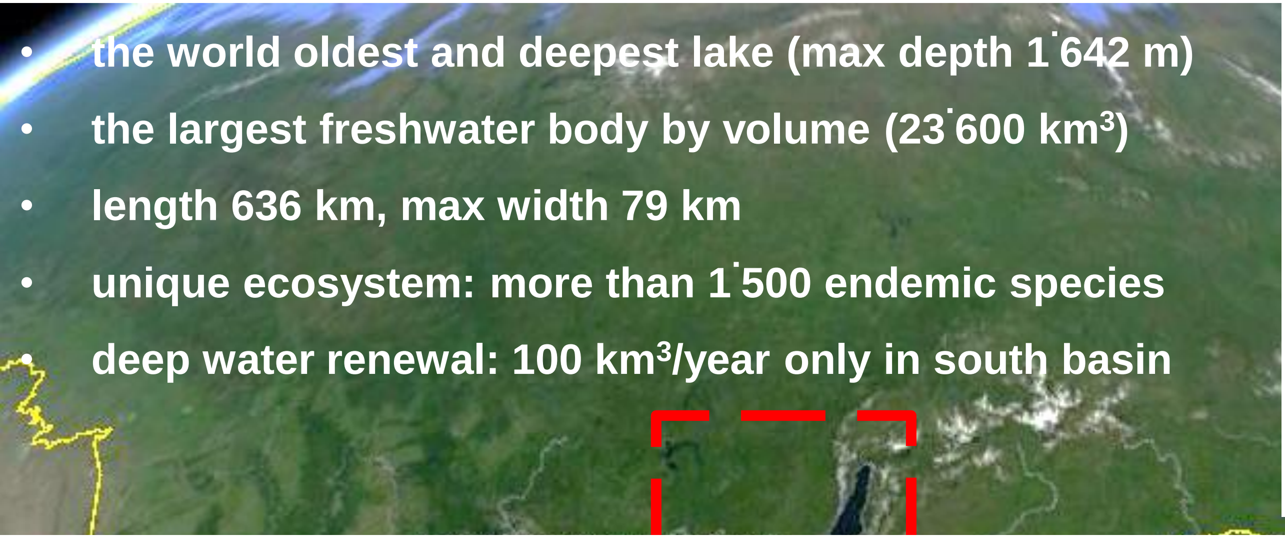
water density ρ depends on pressure P, too!

→ deepening a water parcel adiabatically increases the temperature (as for air)

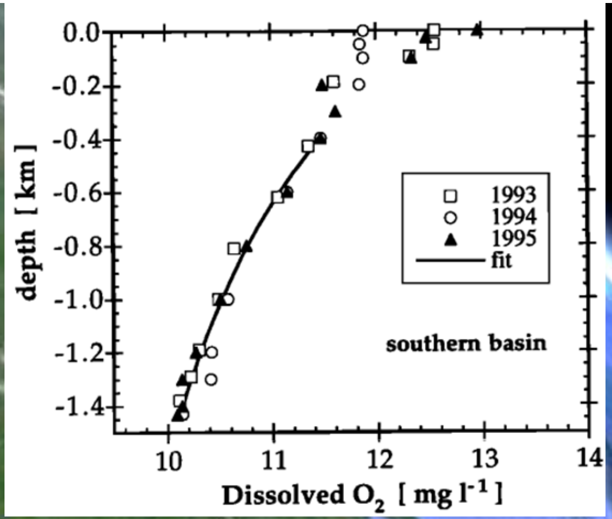


thermobaricity is important only in deep lakes (>300 m)

Lake Baikal (Siberia): an extraordinary deepwater renewal

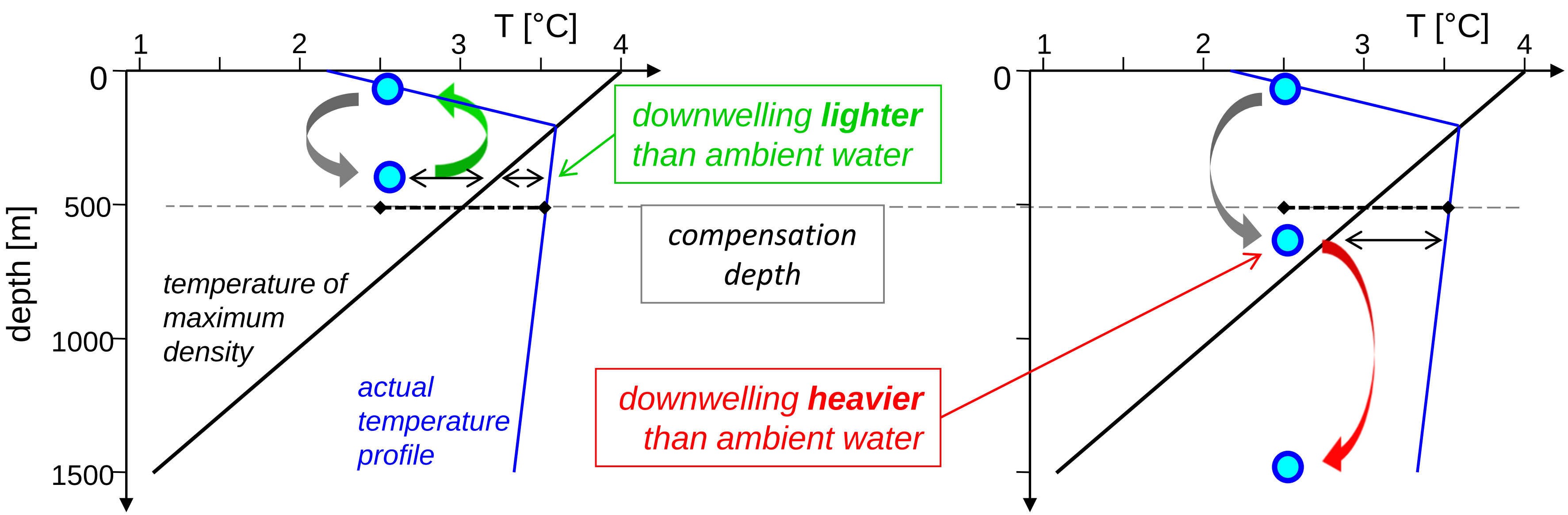


- the world oldest and deepest lake (max depth 1'642 m)
- the largest freshwater body by volume (23'600 km³)
- length 636 km, max width 79 km
- unique ecosystem: more than 1'500 endemic species
- deep water renewal: 100 km³/year only in south basin



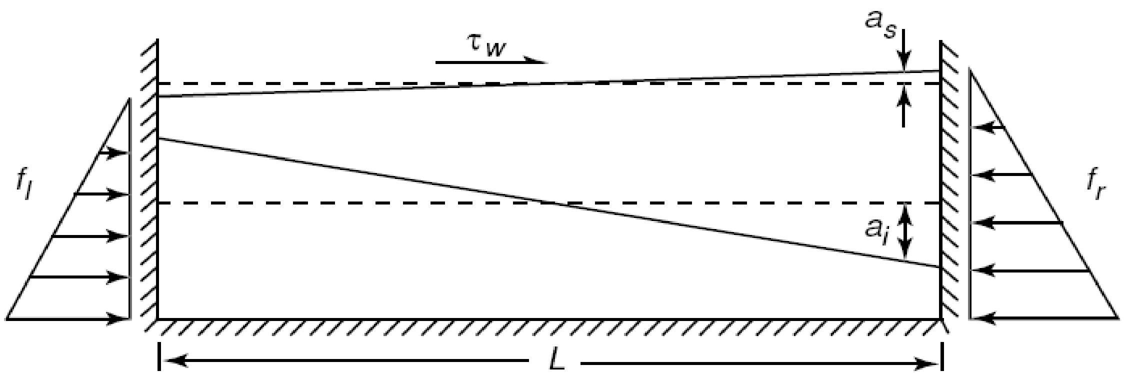
weak wind forcing: small depth of downwelling → back to initial position

strong wind forcing: large depth of downwelling → down to the bottom



Standing waves: Seiches

Wind-induced set-up



wind → stationary flow
→ seiching waves
($n = 1$ in simple basins)

hydrostatic equilibrium while wind blows $\Sigma F_x = F_l - F_r + \tau_w L = 0$

$$F_l \cong \frac{1}{2}\rho_0 g (H - a_s)^2$$

$$F_r \cong \frac{1}{2}\rho_0 g (H + a_s)^2$$

$$-2\rho_0 g a_s H + \tau_w L = 0$$

free surface set-up $a_s = \dfrac{\tau_w L}{2\rho_0 g H}$

pressure balance $p_l = p_r$

$$p_l = \rho_1 g(h_1 - a_s - a_i) + \rho_2 g(h_2 + a_i)$$

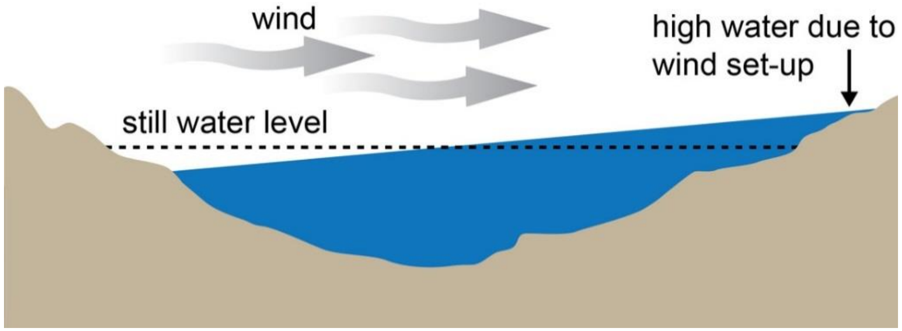
$$p_r = \rho_1 g(h_1 + a_s + a_i) + \rho_2 g(h_2 - a_i)$$

interface set-up $a_i = \dfrac{\rho_0}{(\rho_2 - \rho_1)} a_s$

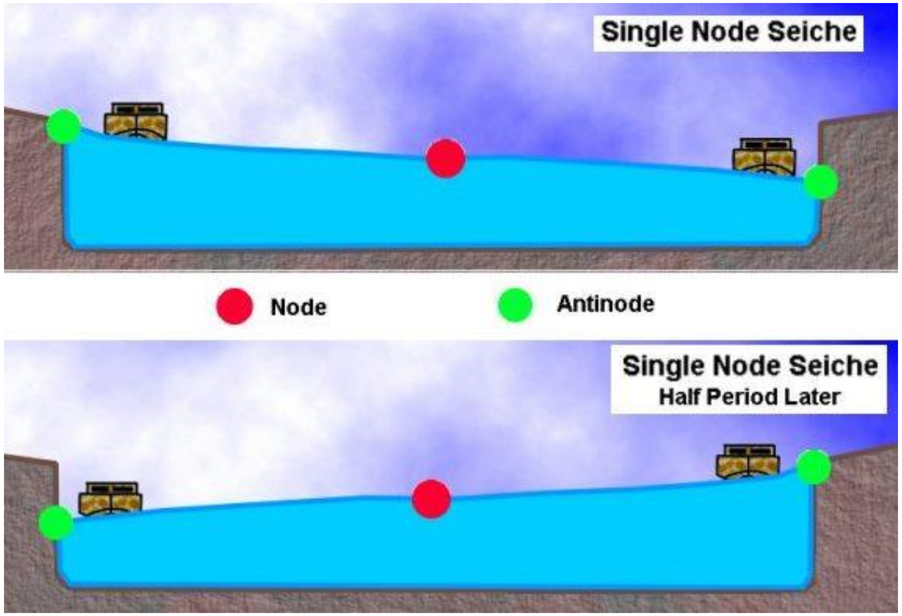
Waves – *seiche*

The word *seiche* is believed to have a Latin derivation from *siccus*, meaning *dry* or *exposed* and has been used now for several centuries to describe a phenomenon of tide-like rise and fall of water, sometimes occurring for hours or days at the narrow end of Lake of Geneva […] In Europe the first record of this rhythmic movement of water is attributed to de Duillier in 1730, but the phenomenon was mentioned much earlier in a chronicle by Schulthaiss in 1549 as occurring in Lake Constance (Switzerland). That it was a feature of many lakes appears to have been noticed first by Vaucher in 1803 […]

B.W. Wilson (1972), Seiches, *Advances in Hydroscience, 8*, 1-94.
(https://www.sciencedirect.com/science/article/pii/B9780120218080500061)

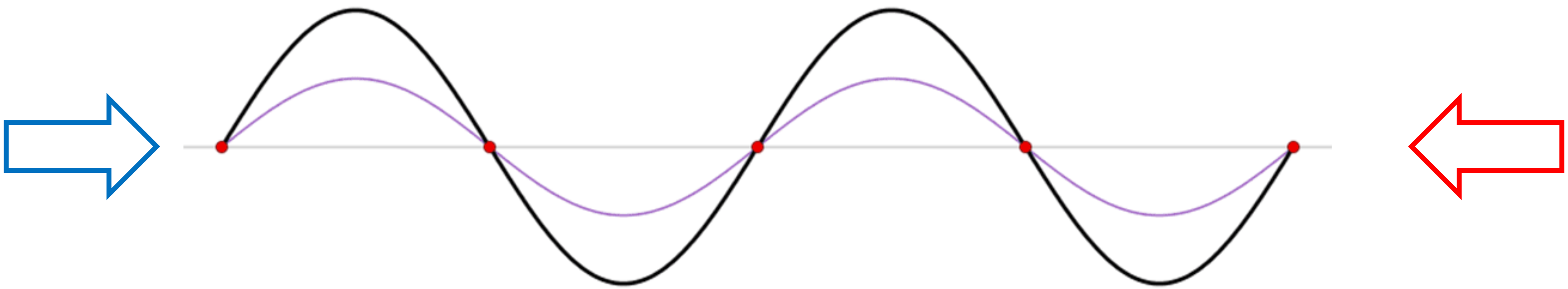


Credit: NOAA Great Lakes Environmental Research Laboratory.
(https://www.watershedcouncil.org/seiches.html)



(http://agatelady.blogspot.com/2013/08/seiches-and-wind-set-up-on-lake-superior.html)

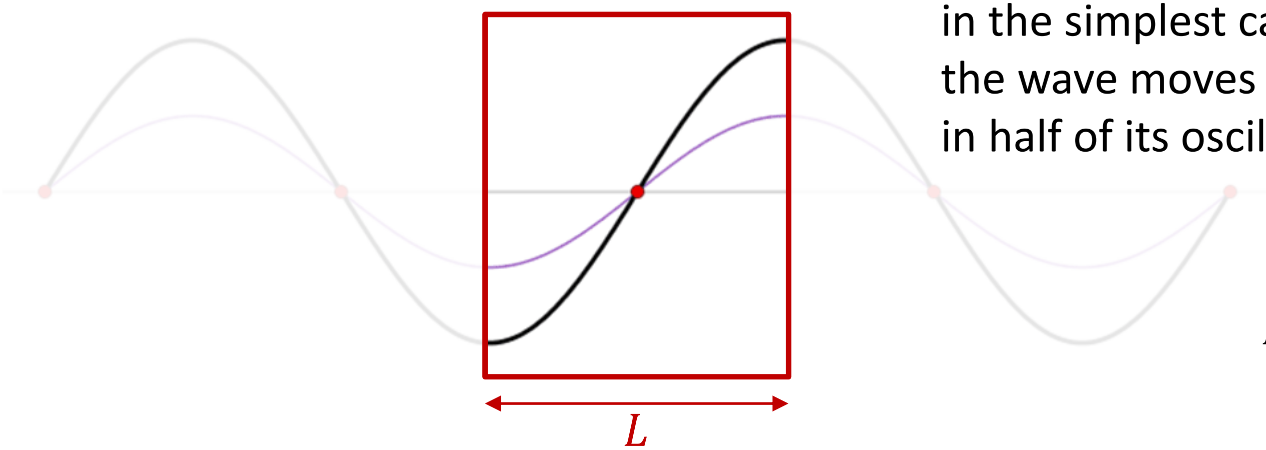
Seiches: standing waves



(https://oceanservice.noaa.gov/facts/seiche.html)

phase speed of waves in a shallow lake of depth h
(with gravity acceleration g)

$$c = \sqrt{g\,h}$$



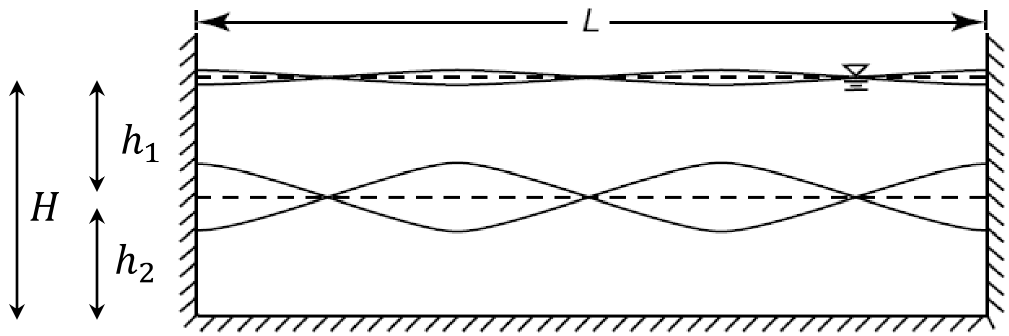
in the simplest case, a uni-nodal *seiche*,
the wave moves from one to the other coast
in half of its oscillation period T

$$L = c\,\frac{T}{2} = \frac{\lambda}{2}$$

Merian equation (1828)
for the period of a *seiche*

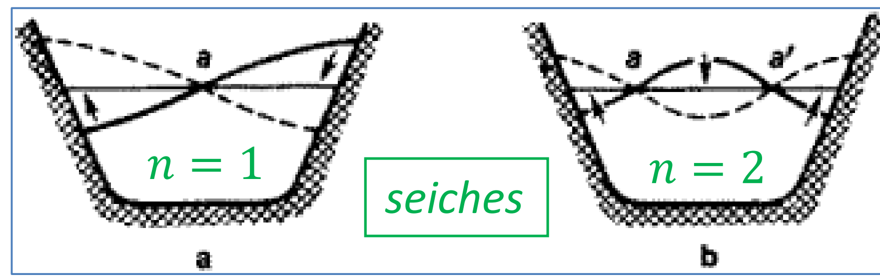
$$T = \frac{2L}{\sqrt{g\,h}}$$

Stationary waves (closed basin)



finite length of the basin → integer number n of half-wavelengths

$$L = n\frac{\lambda}{2}$$



http://encyclopedia2.thefreedictionary.com/Seiching

wavenumber $k = \frac{2\pi}{\lambda} = \frac{n\,\pi}{L}$

celerity $c = \frac{2\pi}{k\,T} = \frac{2L}{n\,T}$

period $T = \frac{2L}{n\,c}$ $\xrightarrow{\text{external mode}}$ $\boxed{T = \frac{2L}{n\sqrt{gH}}}$ *(fast)*

internal mode

$$\boxed{T = \frac{2L}{n\sqrt{g'\frac{h_1 h_2}{H}}}}$$

Merian formula

(slow)

$$g' = g\frac{\Delta\rho}{\rho_0}$$

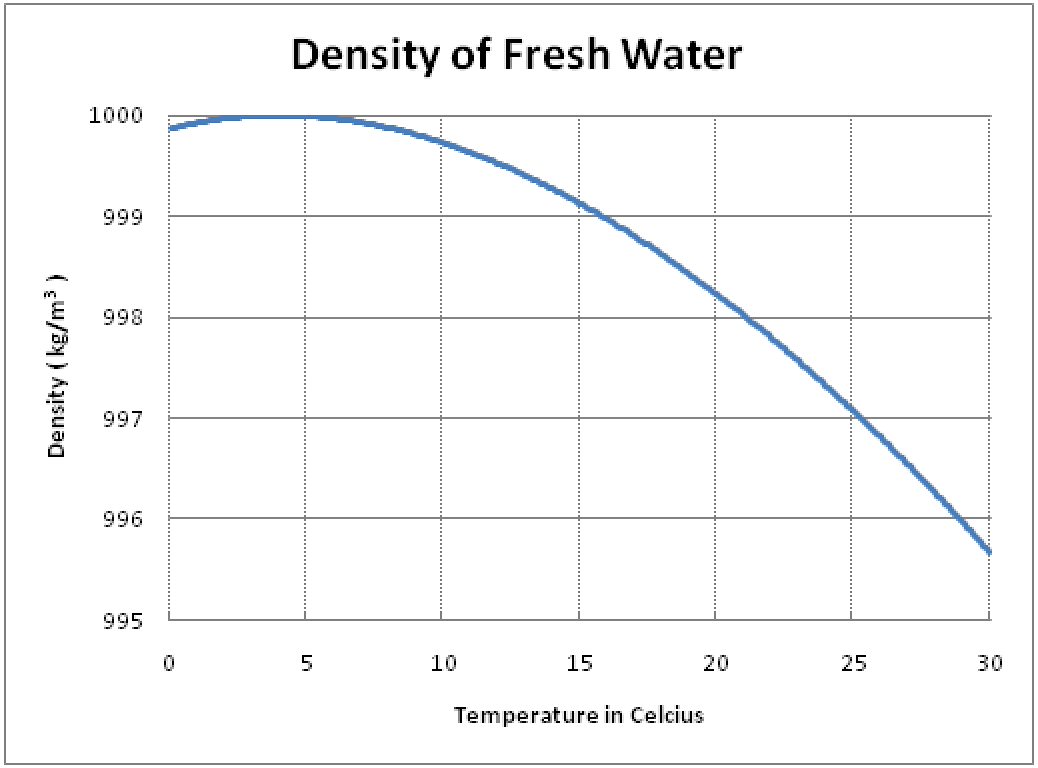
Example: period of seiches

external mode

$$T = \frac{2L}{n\sqrt{g\, H}}$$

internal mode

$$T = \frac{2L}{n\sqrt{g' \frac{h_1 h_2}{H}}}$$



Water density according to Unesco formula (1981)

(https://link.springer.com/content/pdf/bbm:978-3-319-18908-6/1.pdf)

$$\rho = \frac{\rho(S,T,0)}{1 - \boxed{\frac{p}{K(S,T,p)}}}$$

effect of compressibility

$$\rho(S,T,0) = \rho_{SMOW} + B_1 S + C_1 S^{1.5} + d_0\, S^2$$

- calculation of the SMOW density (*Standard Mean Ocean Water*):

$$\rho_{SMOW} = a_0 + a_1 T + a_2 T^2 + a_3 T^3 + a_4 T^4 + a_5 T^5$$

where:

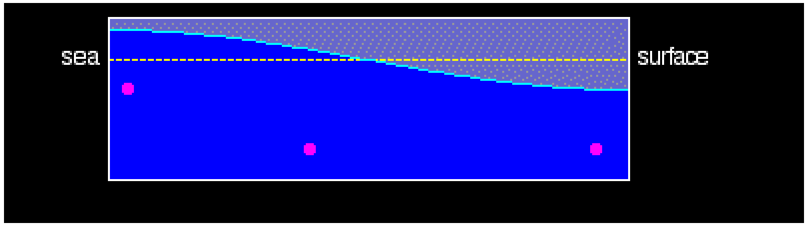
$$\left.\begin{aligned} a_0 &= 999.842\,594 \\ a_1 &= 6.793\,953 \times 10^{-2} \\ a_2 &= -9.095\,290 \times 10^{-3} \\ a_3 &= 1.001\,685 \times 10^{-4} \\ a_4 &= -1.120\,083 \times 10^{-6} \\ a_5 &= 6.536\,332 \times 10^{-9} \end{aligned}\right\}$$

e.g., online calculator

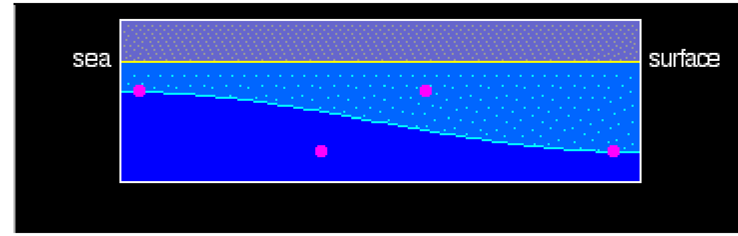
http://www.phys.ocean.dal.ca/~kelley/seawater/d

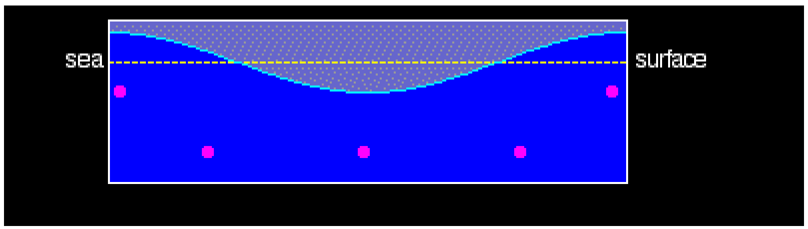
External (surface) waves

Internal waves

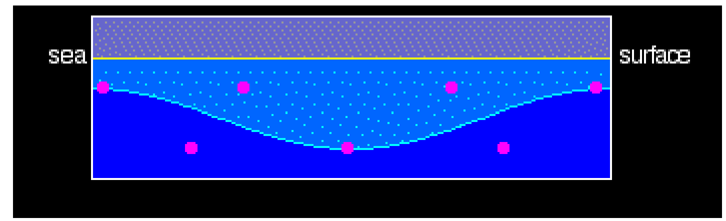


first mode seiche

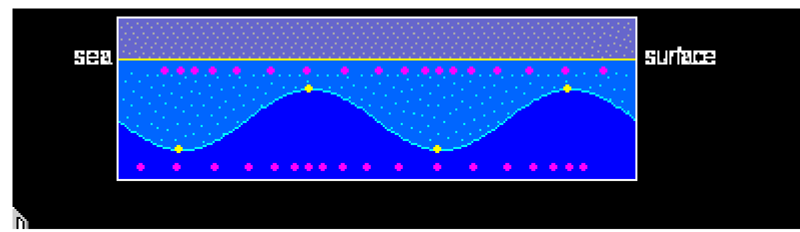




second mode seiche



propagating internal wave



http://www.es.flinders.edu.au/~mattom/IntroOc/

Example: lake Caldonazzo (TN)

$$T = \frac{2L}{n\sqrt{gH}}$$

$$T = \frac{2L}{n\sqrt{g'\frac{h_1 h_2}{H}}}$$

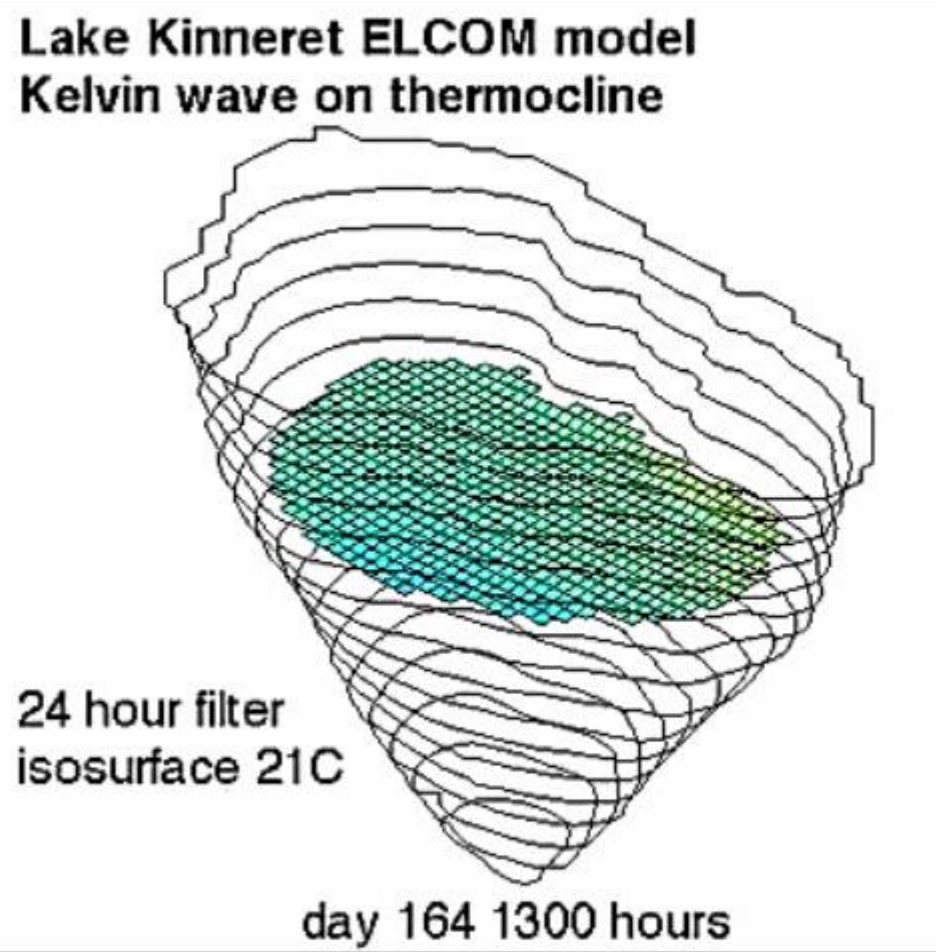
L = 4 km
H = 40 m
h1 = 10 m
h2 = 30 m

T1 = 20°C
T2 = 10°C
$\rho 1 = 998.2$ kg m^{-3}
$\rho 2 = 999.7$ kg m^{-3}

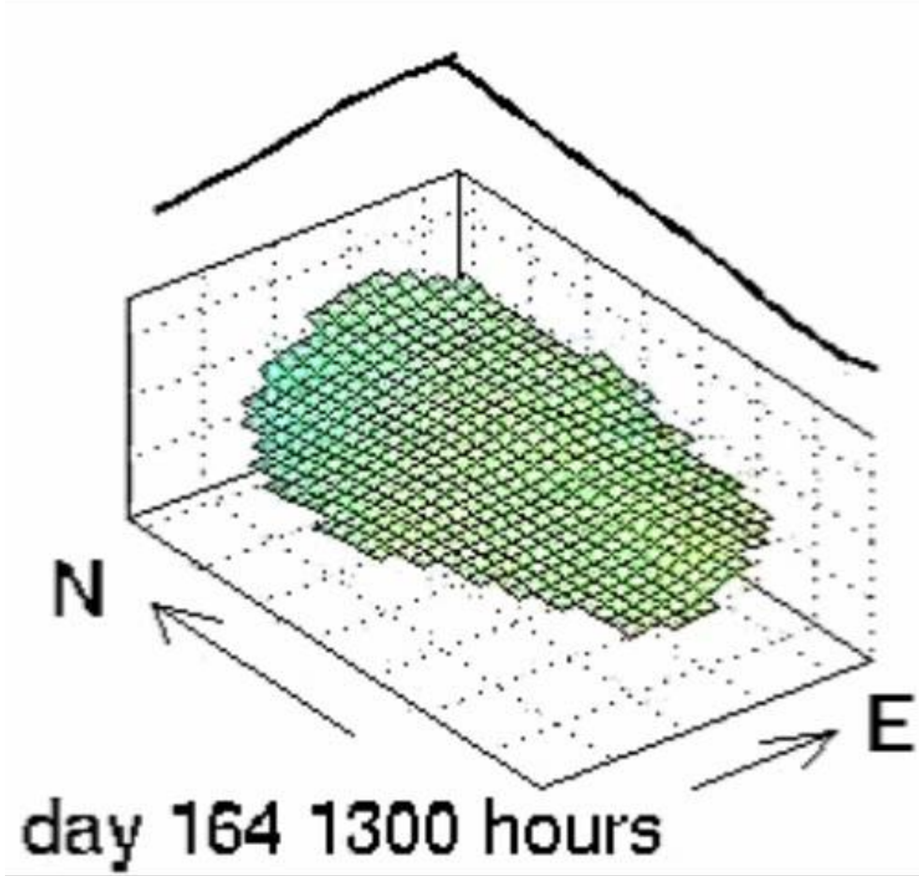
external	internal
g = 9.8 m s^{-2}	g' = 0.015 m s^{-2}
c = 20 m s^{-1}	c = 0.33 m s^{-1}
n = 1	n = 1
T = 400 s	T = 6.7 hours

Effect of Earth rotation (Kelvin, Poincaré waves)

important if the Rossby radius of deformation $R = c/f < L$ (size of the lake)



Lake Kinneret **thermocline** motion that follows a typical **Kelvin wave** pattern. Visualization is produced by the ELCOM 3D hydrodynamic model. Created by Ben R. Hodges, University of Texas at Austin. Field data for developing the model provided by J. Imberger, Centre for Water Research, University of Western Australia.
http://www.youtube.com/watch?v=SZlix47Jq4A



Internal Poincare wave on the thermocline of Lake Kinneret. From 3D hydrodynamic model. See Hodges, Imberger, Saggio and Winters (2000) in Limnology and Oceanography
http://www.youtube.com/watch?v=8mdAazUspAs