Environmental Fluid Mechanics, *M.Sc. Environmental Meteorology* – 2022/23

Stratified flows:
Equilibrium, instability and turbulence

Marco Toffolon
(University of Trento)

(Marco Adami, *Scirocco*, 2008)

References

Benoit Cushman-Roisin and Jean-Marie Beckers, **Introduction to geophysical fluid dynamics : physical and numerical aspects**, 2nd ed., Academic Press, 2011.
https://webapps.unitn.it/Biblioteca/it/Web/LibriElettroniciDettaglio/117963

- 11. Stratification
  - 11.1 Introduction
  - 11.2 Static Stability
- 14. Turbulence in Stratified Fluids
  - 14.1 Mixing in Stratified Fluids

Pijush K. Kundu, Ira M. Cohen, David R. Dowling, **Fluid mechanics**, 5th ed., Academic Press, Elsevier, 2012.
https://www-sciencedirect-com.ezp.biblio.unitn.it/book/9780123821003/fluid-mechanics

- 1. Introduction
  - 1.10. Stability of Stratified Fluid Media
- 11. Instability
  - 11.3. Kelvin-Helmholtz Instability
- 12. Turbulence
  - 12.11. Turbulence in a Stratified Medium

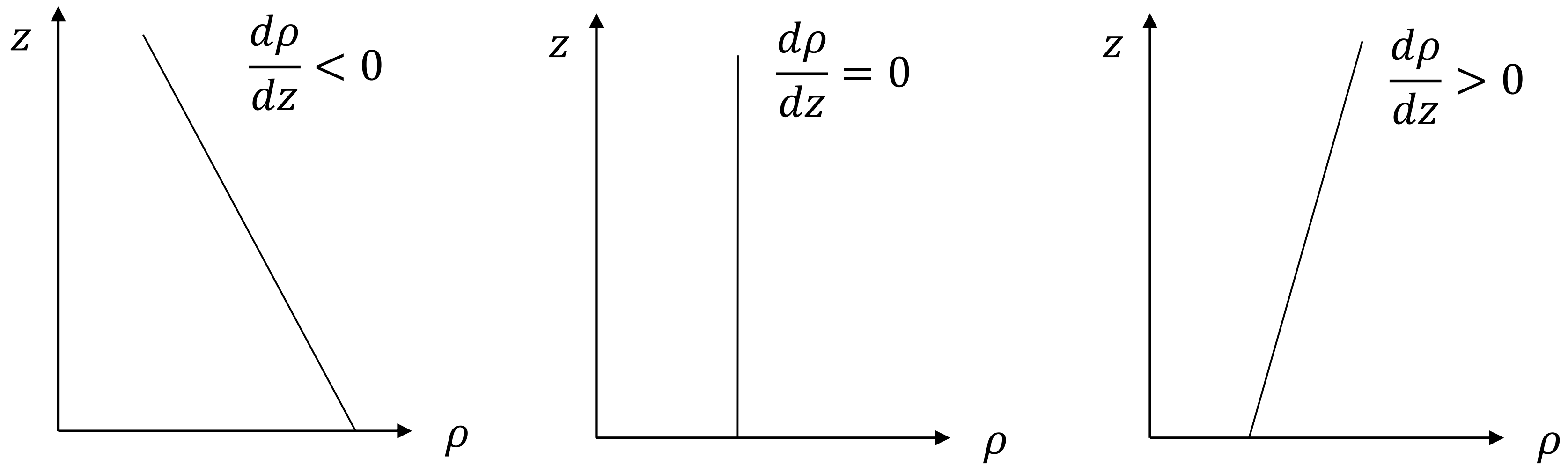
Scott A. Socolofsky, **Mixing and Transport Processes in the Environment, Lecture notes – part 2**.

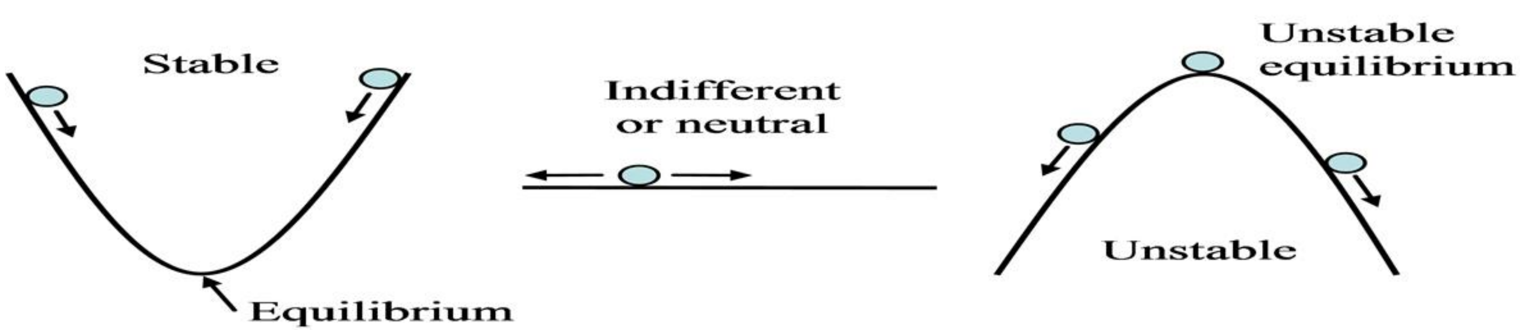
- ch. 8 – Concepts, Definitions, and Governing Equations
- ch. 11 – Instability

Density stratification and gravitational equilibrium

Equilibrium (hp. neglecting the effect of pressure on density)

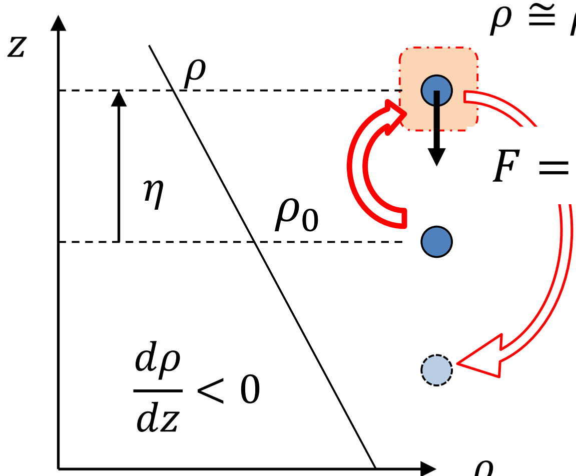
stable neutral unstable





Oscillations around stability: a simple model

$$\rho \cong \rho_0 + \frac{d\rho}{dz}\eta$$



stable equilibrium

$$F = -g(\rho_0 - \rho)\delta V$$

second Newton's law

$$F = ma = \rho_0 \delta V \frac{d^2\eta}{dt^2}$$

$$\frac{d^2\eta}{dt^2} - \frac{g}{\rho_0}\frac{d\rho}{dz}\eta = 0$$

$$\frac{d^2\eta}{dt^2} + N^2\eta = 0$$

$$N = \sqrt{-\frac{g}{\rho}\frac{d\rho}{dz}}$$

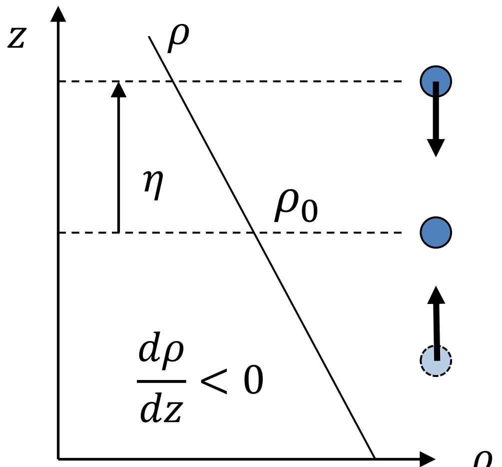
(buoyancy frequency)

solution: oscillating motion
around the equilibrium position

$$\eta = \eta_0 \cos(Nt)$$

oscillation period $$T = \frac{2\pi}{N}$$

Oscillations around stability: analogy



stable equilibrium

$$\boxed{N = \sqrt{-\frac{g}{\rho}\frac{d\rho}{dz}}}$$


David Brunt
(1886-1965)

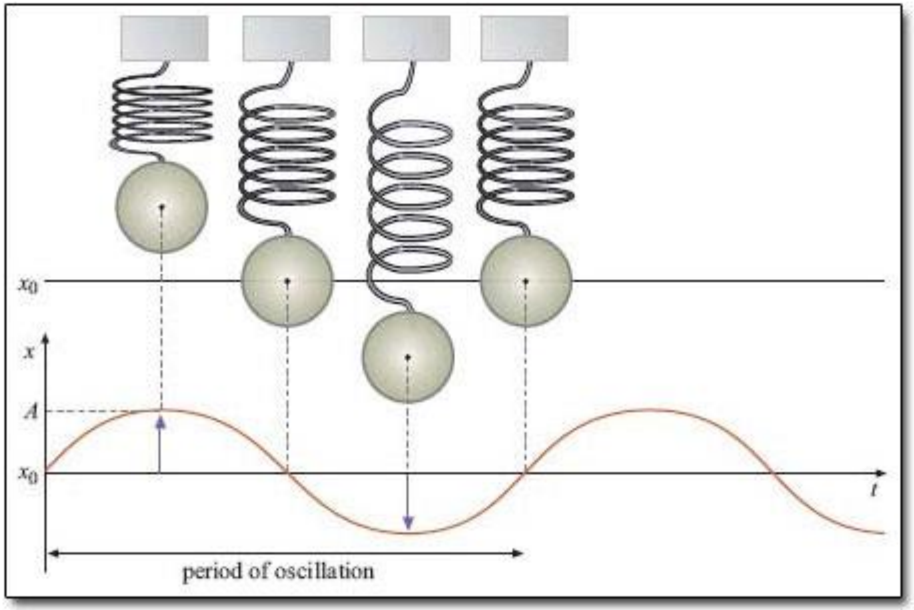

Vilho Väisälä
(1889-1969)

Brunt-Väisälä buoyancy frequency N [s^{-1}]
(physical interpretation: “spring oscillations”)

$$\frac{d^2\eta}{dt^2} + N^2\eta = 0$$

$$\eta = \eta_0 \cos(Nt)$$

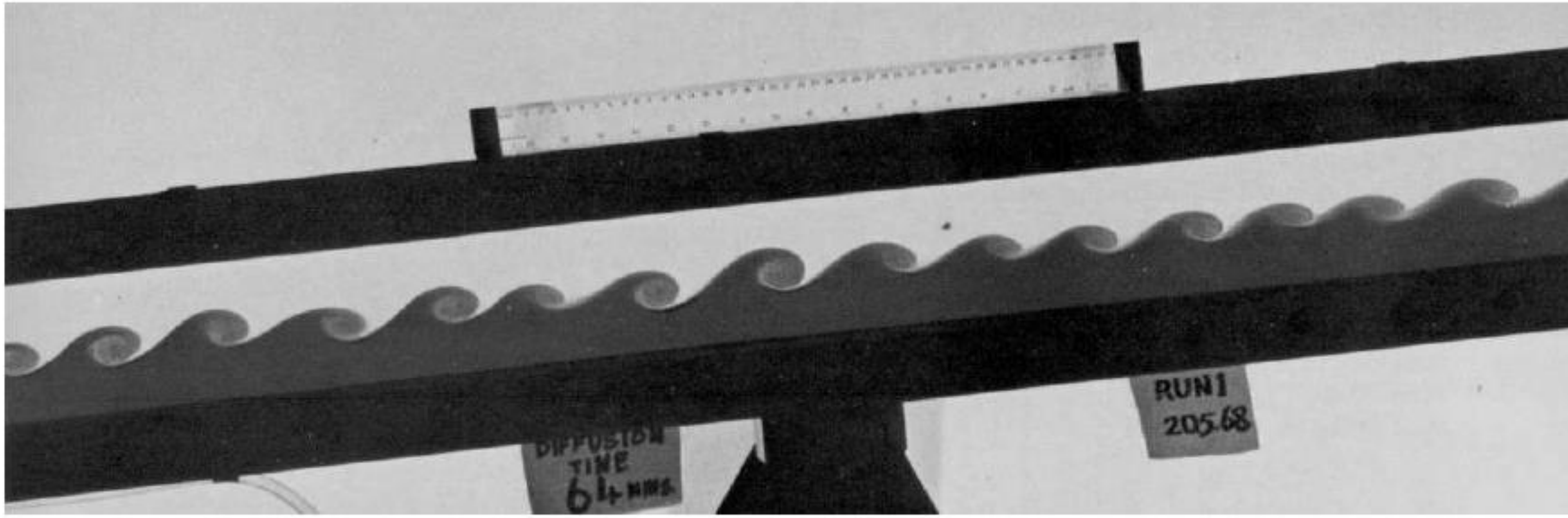
$$T = \frac{2\pi}{N}$$

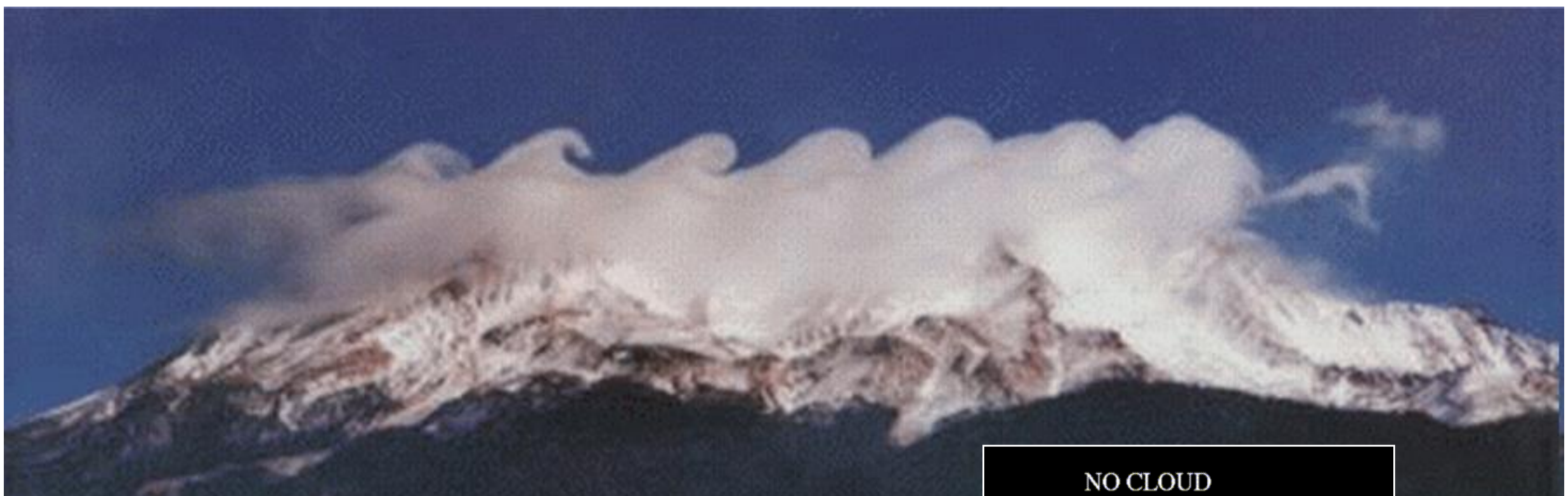


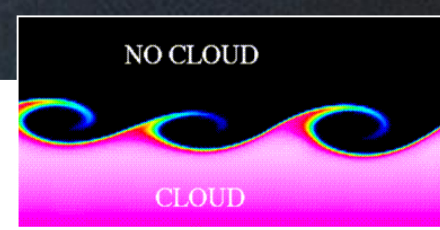
http://dspace.jorum.ac.uk/xmlui/bitstream/handle/10949/1022/Items/T356_1_section20.html

Instability across density interfaces

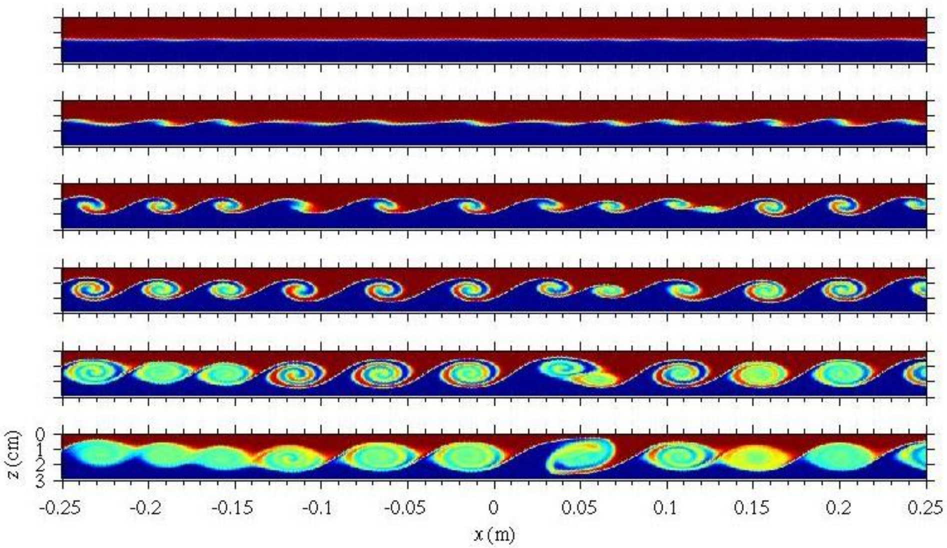
Examples of the Kelvin-Helmholtz instability







http://www.efluids.com/efluids/gallery/gallery_pages/cloud_instability_2.jsp

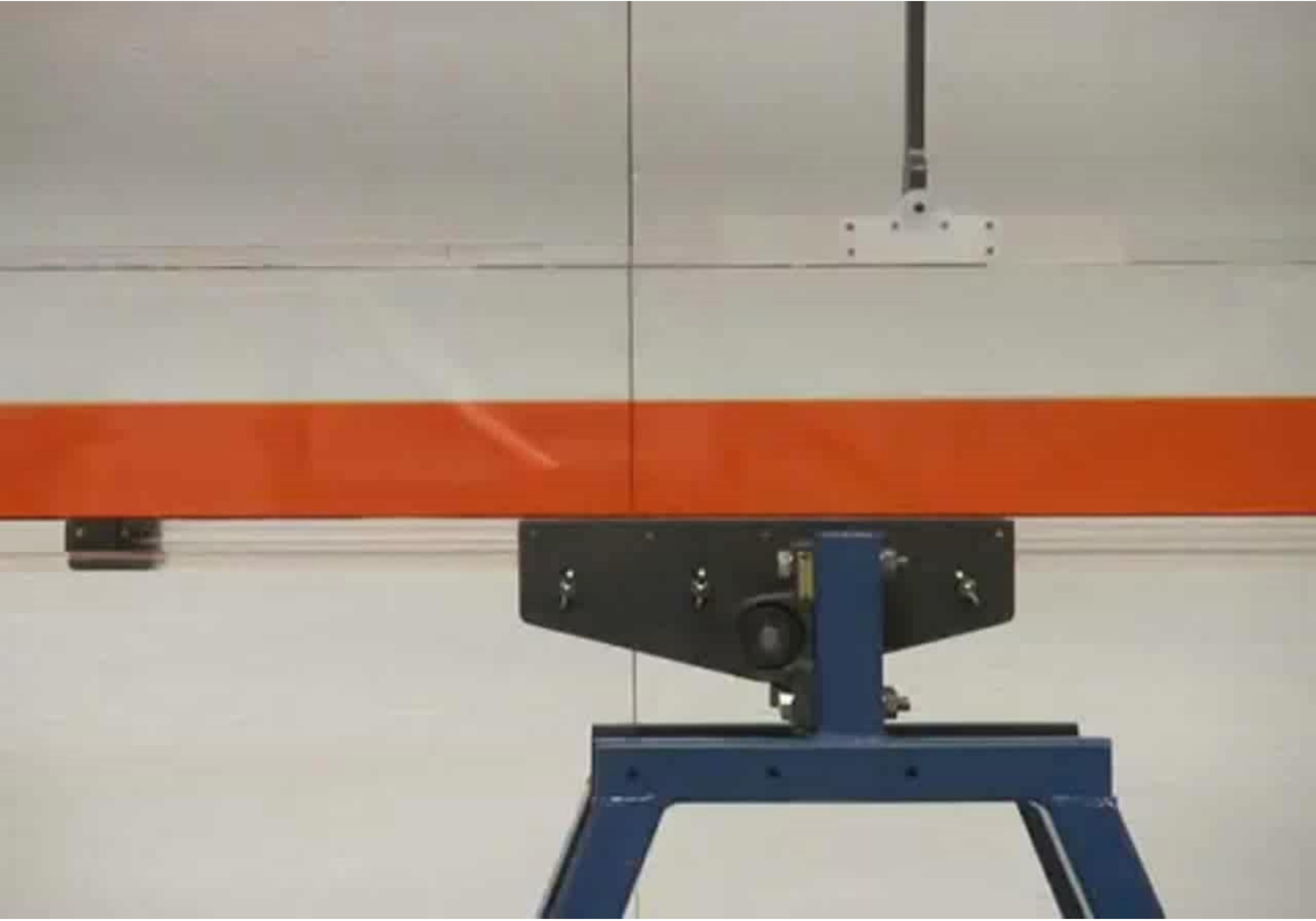


http://www.physics.mun.ca/~danielb/research/aestus/aestus.html

http://www.efluids.com/efluids/gallery/gallery_pages/kelvin_helm_page.jsp

Examples of the Kelvin-Helmholtz instability – in Trentino

Photo: Sebastiano Piccolroaz



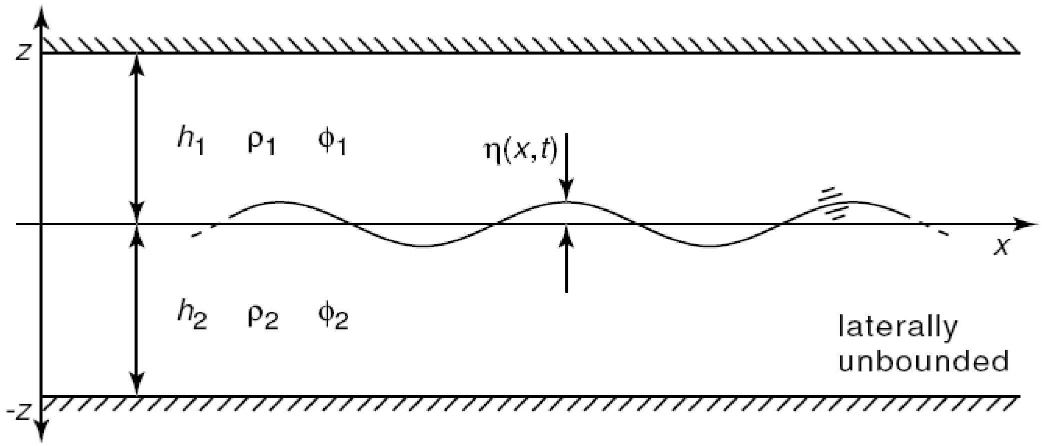
Kelvin-Helmholtz instability: an enlightening experiment

https://www.youtube.com/watch?v=UbAfvcaYr00

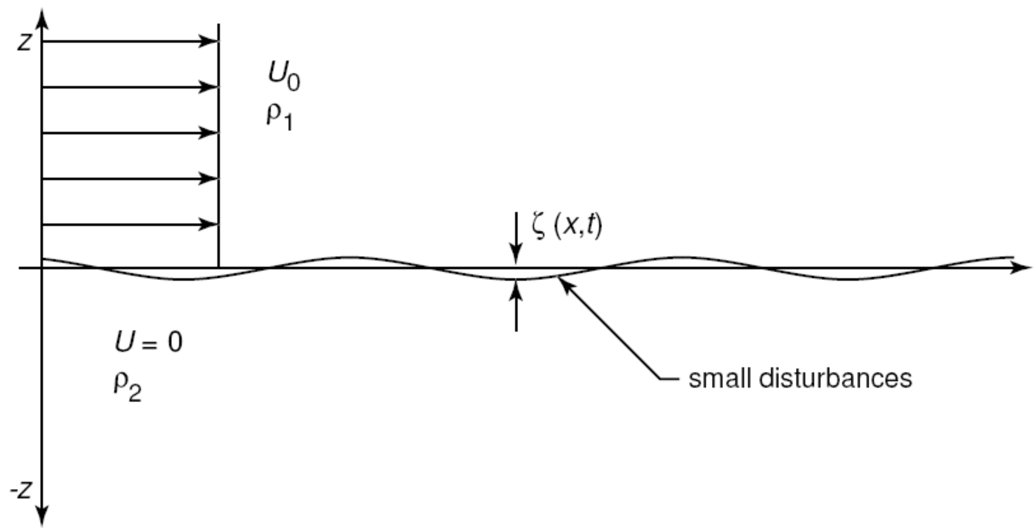
Instability in stratified flows: the Kelvin-Helmholtz instability

Hp: two layers with different densities; immiscible fluids, inviscid motion.

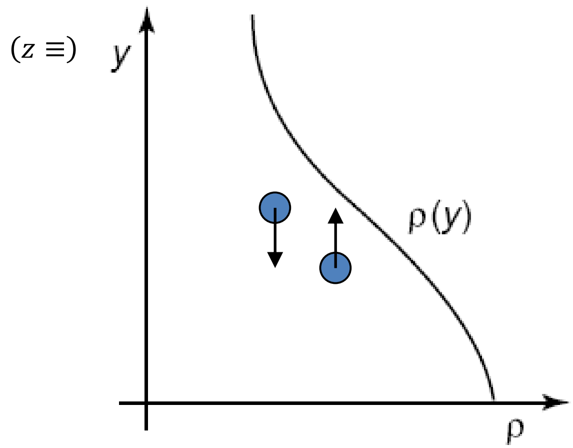
No overall motion: interfacial waves



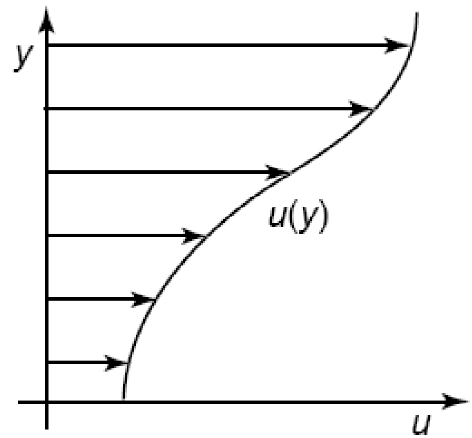
Relative motion between layers: disturbances grow or decay?



Heuristic explanation



vertical gradient of density

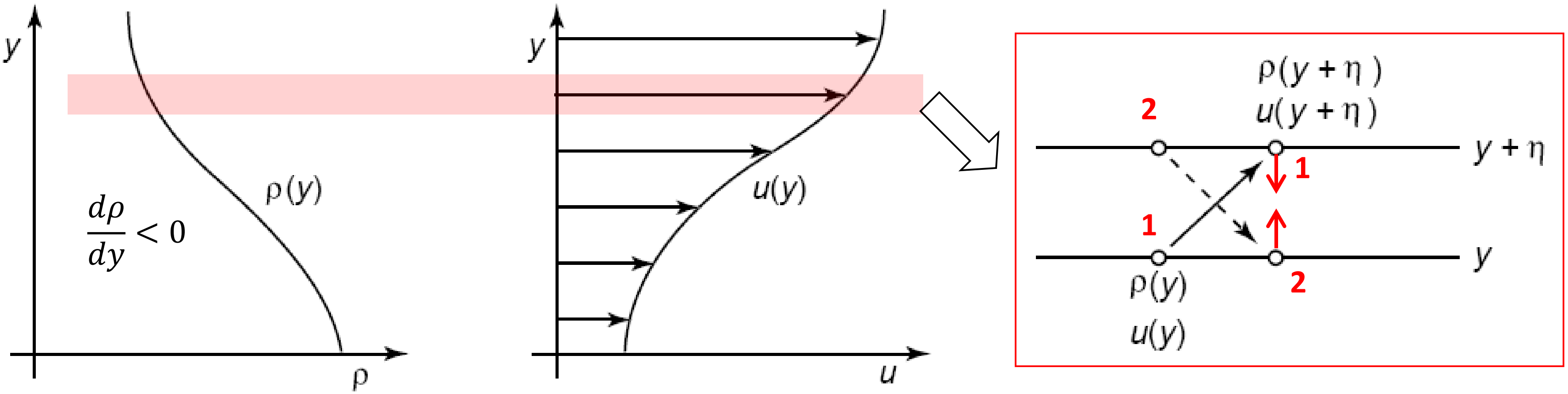


vertical gradient of velocity

instability: when the dissipated kinetic energy is larger than the work required by the buoyancy forces to move fluid particles

(without viscosity)

Mechanical work of the buoyancy forces



buoyancy force

$$\Delta B = -g\rho(y) + g\rho(y+\eta) = -g\rho(y) + g\left[\rho(y) + \frac{d\rho}{dy}\eta + O(\eta^2)\right] \cong g\frac{d\rho}{dy}\eta$$

work

water particle **1**

$$W_{B1} = \int_0^{\delta y} g\frac{d\rho}{dy}\eta d\eta = g\frac{d\rho}{dy}\frac{(\delta y)^2}{2}$$

$$\Delta B = -g\rho(y+\eta) + g\rho(y) \cong -g\frac{d\rho}{dy}\eta$$

water particle **2**

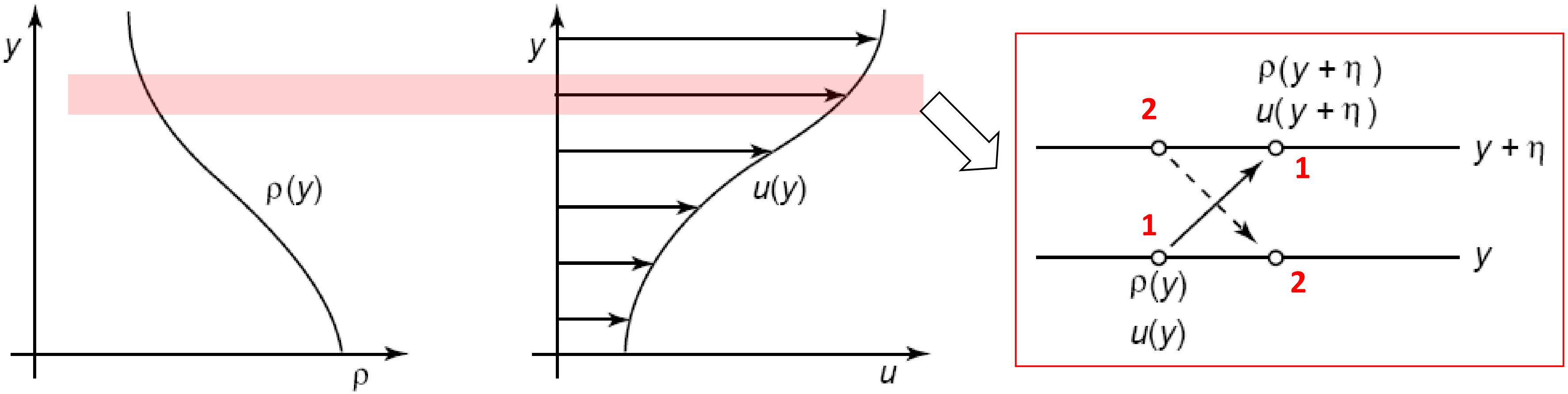
$$W_{B2} = \int_{\delta y}^{0} -g\frac{d\rho}{dy}\eta d\eta = g\frac{d\rho}{dy}\frac{(\delta y)^2}{2}$$

total work

$$W_B = W_{B1} + W_{B2} = g\frac{d\rho}{dy}(\delta y)^2 < 0$$

(energy is required to exchange the particles)

Variation of kinetic energy



before $$E(t_a) = \frac{\rho_0 u^2}{2} + \frac{\rho_0 (u + \delta u)^2}{2}$$

(1) **(2)**

after $$E(t_b) \cong 2\frac{\rho_0}{2}\left(\frac{u + (u + \delta u)}{2}\right)^2$$

hp. average velocity

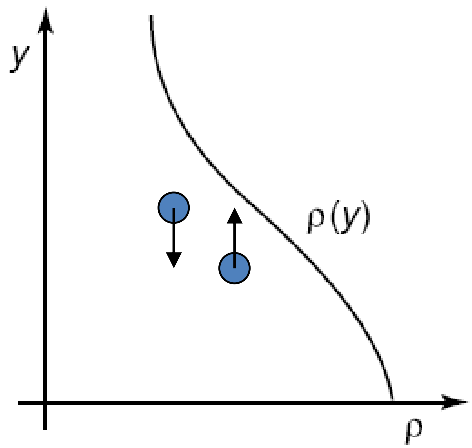
variation of kinetic energy

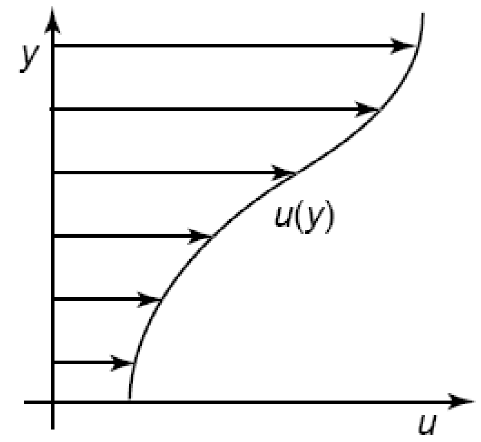
$$\Delta E = E(t_b) - E(t_a) = -\frac{\rho_0}{4}(\delta u)^2$$

(after the exchange, the kinetic energy is smaller)

Threshold for instability

instability: when the dissipated kinetic energy is larger than the work required by the buoyancy forces to move fluid particles





$$-\Delta E > -W_B$$

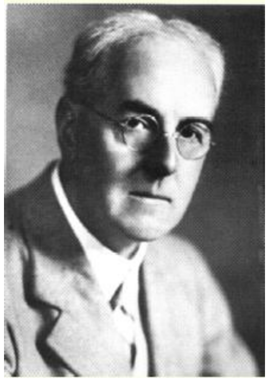
$$\frac{\rho_0}{4}(\delta u)^2 > -g\frac{d\rho}{dy}(\delta y)^2$$

(without viscosity)

$\Rightarrow$

Richardson number

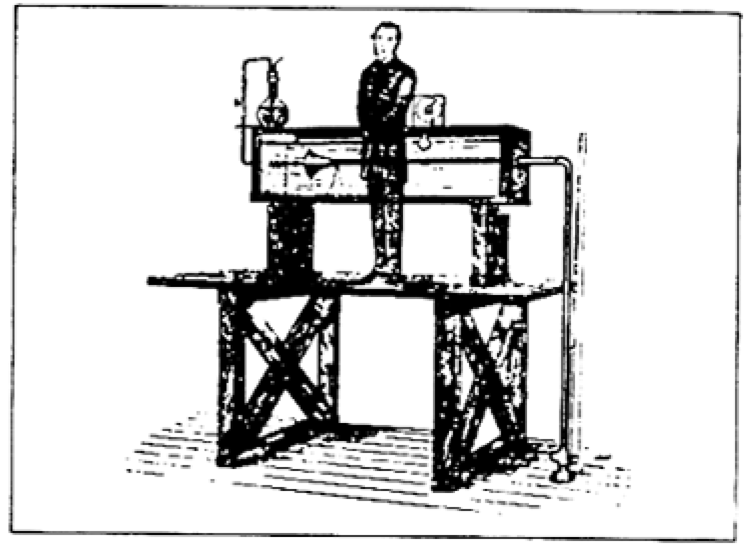
$$Ri = -\frac{g}{\rho_0}\frac{d\rho}{dy}\left(\frac{du}{dy}\right)^{-2} < \frac{1}{4}$$



Lewis Fry Richardson (1881-1953)

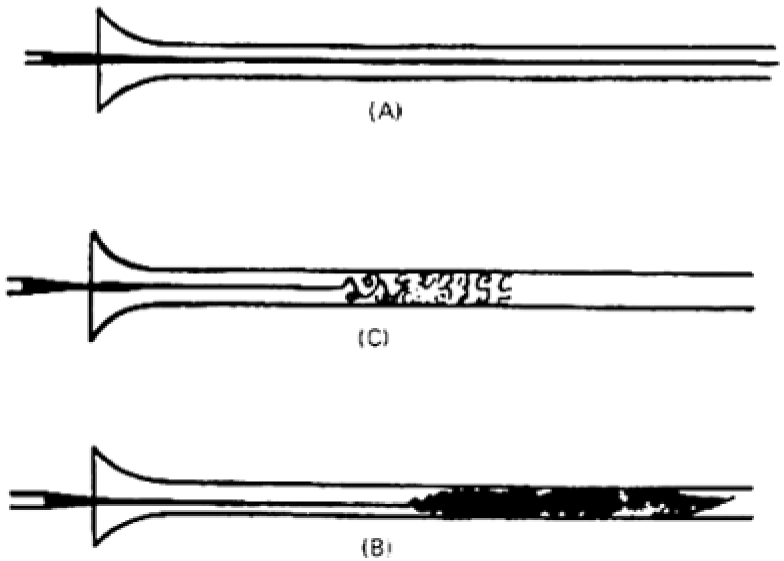
$$\boxed{N^2 = -\frac{g}{\rho_0}\frac{d\rho}{dy}}$$ square of Brunt-Väisälä frequency [T^{-2}]

Analogies with instability in turbulence



"Classical" turbulence:
Reynolds experiment

laminar flow: disturbances decay



turbulent flow: disturbances grow

$$\mathrm{Re} = \frac{UL}{\nu}$$

stabilizing factor: viscosity

Stratified flows

stabilizing factor: density difference

$$Ri = \frac{-\frac{g}{\rho}\frac{d\rho}{dz}}{\left(\frac{du}{dz}\right)^2}$$

Relevant dimensionless number

Richardson number:

- gradient
$$Ri = N^2 \left(\frac{du}{dz}\right)^{-2} = \frac{-\frac{g}{\rho}\frac{d\rho}{dz}}{\left(\frac{du}{dz}\right)^2}$$

- "bulk"
$$Ri_b = \frac{g' H}{U^2} = \frac{1}{F_d^2}$$
$$g' = \frac{\rho'}{\rho_0} g$$

- flux
$$Ri_f = \frac{g\langle \tilde{\rho}\tilde{w}\rangle}{\bar{\rho} u_*^2 \frac{du}{dz}}$$
(turbulence) ratio between energy removal rate due to buoyancy forces and production due to shear

densimetric Froude number

$$F_d = \frac{U}{\sqrt{g' H}} = \frac{1}{\sqrt{Ri_b}}$$

Reynolds number

$$\mathrm{Re} = \frac{UL}{\nu}$$

Prandtl number

$$\mathrm{Pr} = \frac{\nu}{\kappa}$$

Turbulent mixing suppression in stratified flows

Turbulence reduction due to stratification

Simple empirical laws

Effect of stratification
(depending on Richardson number)

$$Ri = -\frac{g}{\rho}\frac{d\rho}{dz}\left(\frac{du}{dz}\right)^{-2}$$

$$D^T_{z,strat} = D^T_{z,0}(1 + a\ Ri)^b$$

$$Ri_b = -\frac{g\Delta\rho\ H}{\bar{\rho}\ \Delta U^2}$$ (bulk)

Autore	a	b
(1) Munk and Anderson (1948)	10/3	-3/2
(2) Officer (1976)	1	-2
(3) Schiller amd Sayre (1975)	10/3	-5/4

Some references:

Munk, WH, ER Anderson (1948), Notes on a theory of the thermocline, *J. Marine Res.*, 7:276–95.
Large, WG, JC McWilliams, SC Doney (1994), Oceanic vertical mixing: A review and a model with a nonlocal boundary layer parameterization, *Reviews of Geophysics*, 32:363–403.

Otherwise:

- specific single point second order **turbulence models** (k-ε, Mellor-Yamada, etc.)

- LES (Large Eddy Simulation)
- DNS (Direct Numerical Simulation) eddy-resolving methods

Examples of empirical laws

Pacanowski and Philander (JPO1981)

$$\nu_z = \nu_0(1+\alpha Ri)^{-n} + \nu_{bg}$$ viscosity

$$D_z = \nu_z(1+\alpha Ri)^{-1} + D_{bg}$$ diffusivity

background viscosity/diffusivity

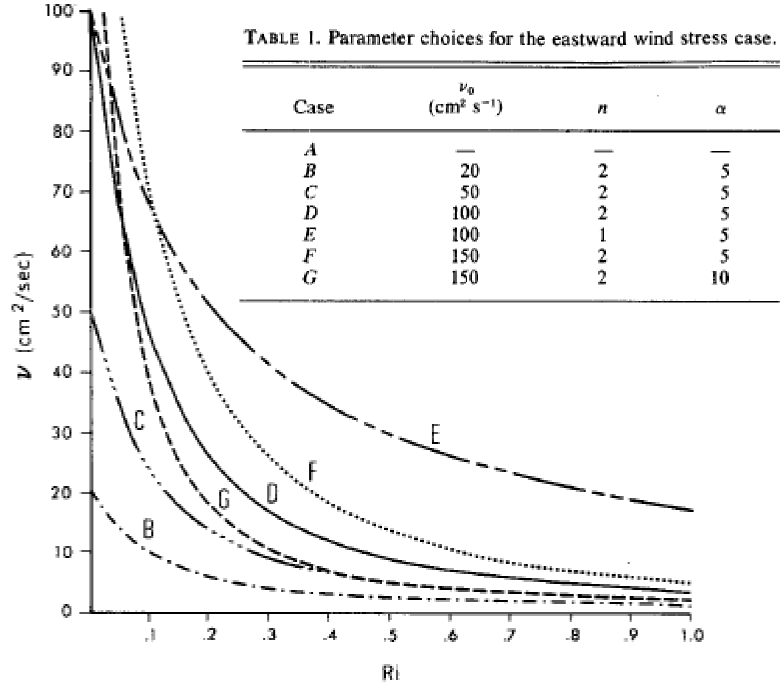


TABLE 1. Parameter choices for the eastward wind stress case.

Case	ν_0 ($cm^2 s^{-1}$)	n	α
A	—	—	—
B	20	2	5
C	50	2	5
D	100	2	5
E	100	1	5
F	150	2	5
G	150	2	10

Large and Gent (JPO1999)

$$\nu_z = \nu_s + \nu_w + \nu_d$$

viscosity, diffusivity (for velocity, temperature, salinity, etc.)

resolved vertical shear

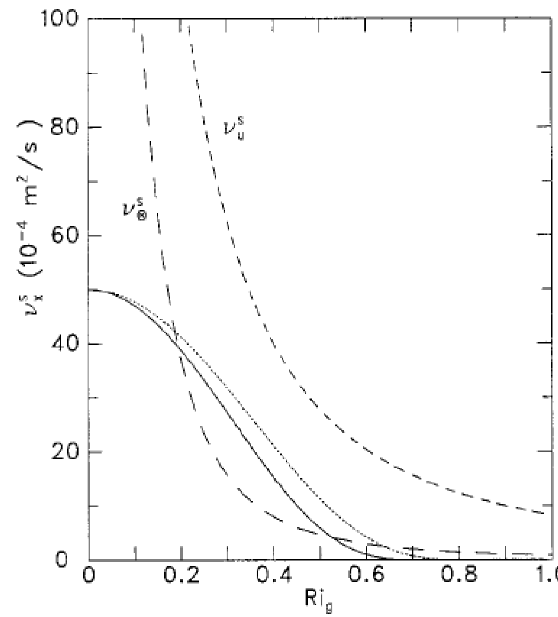
unresolved internal wave breaking

double diffusion

$$\nu_{w,u} = 10^{-4}\, m^2 s^{-1}$$

$$\nu_{w,T} = 10^{-5}\, m^2 s^{-1}$$

$$\frac{\nu_s}{\nu_0} = \begin{cases} 1 & Ri < 0 \\ \left[1-\left(\frac{Ri}{Ri_0}\right)^2\right]^3 & 0 < Ri < Ri_0 \\ 0 & Ri > Ri_0 \end{cases}$$



Estimating the parameters

Elements for the dynamic reconstruction of the reduced vertical eddy diffusivity:

- reference viscosity/diffusivity (possibly vertical profile)
- background viscosity/diffusivity (unresolved processes)
- gradient Richardson number:
  - buoyancy frequency (depends on temperature profile)
  - shear frequency (depends on flow field)

$$Ri = \frac{N^2}{S^2}$$

$$N^2 = -\frac{g}{\rho}\frac{\partial \rho}{\partial z} = \beta g \frac{\partial T}{\partial z}$$

β thermal expansibility of water

$$\rho = \rho_o[1 - \beta(T - T_o)]$$

$$S^2 = \left(\frac{\partial U}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2$$

Estimates for models that do not resolve hydrodynamics

$$S^2 = {S_{lw}}^2 + {S_{iw}}^2 + {S_{bg}}^2$$

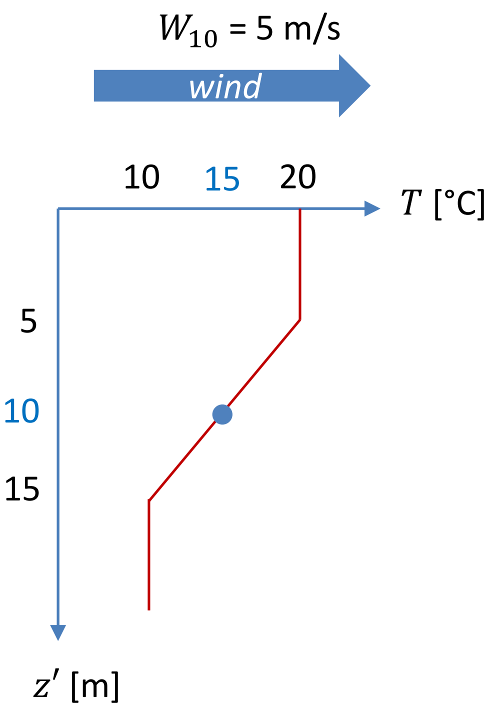
wind stress (τ_w) contribution
$\sim$ law of the wall

$$S_{lw} = \frac{\partial U_0}{\partial z} \sim \frac{u_*}{\kappa\, z} = \frac{\sqrt{\tau_w/\rho}}{\kappa\, z}$$

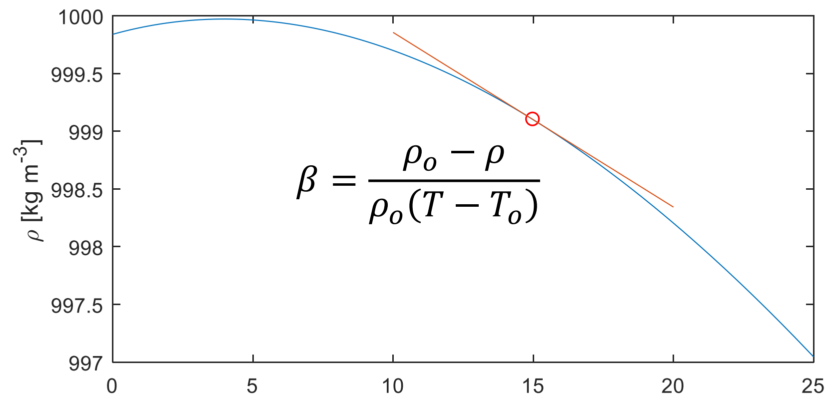
background
$2 \cdot 10^{-7}$ $m^2 s^{-1}$ (Lawrence et al., 2002)

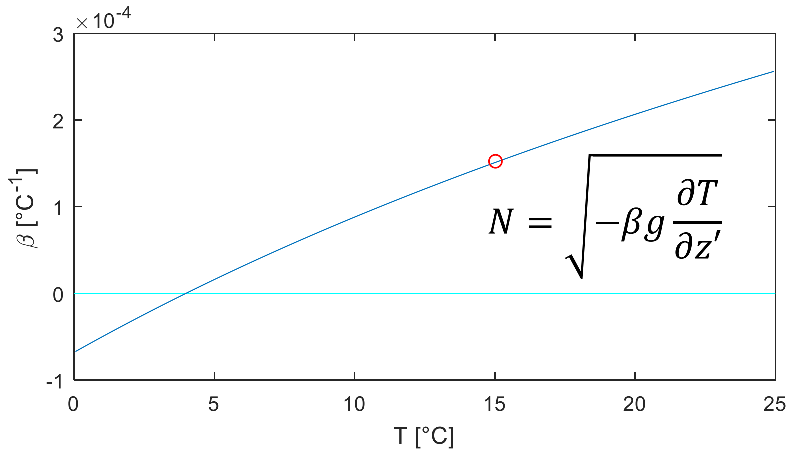
internal wave contribution
$\gamma = 0.7$ (Mellor et al., 1989)

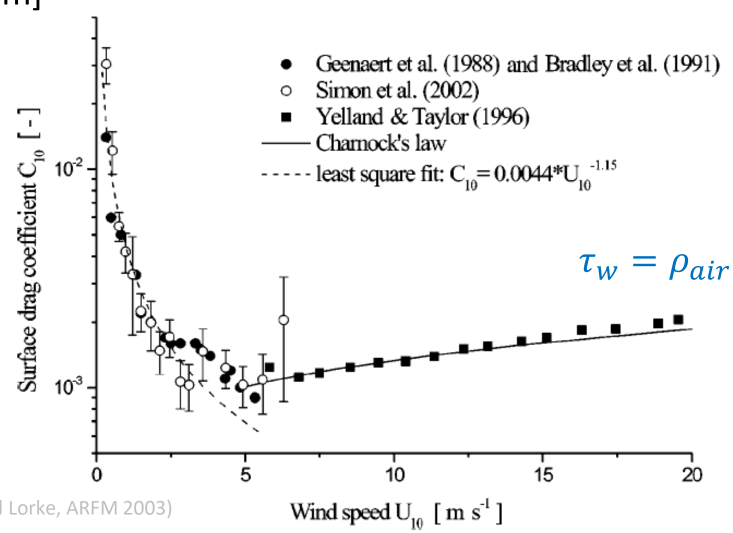
An example of an estimate of Ri for a wind-driven flow in a lake



T_0 = 15 °C
ρ_0 = 999.1 kg m^{-3}
β_0 = 1.5e-4 °C^{-1}
τ_w = 0.030 N m^{-2}
u_* = 0.0055 m s^{-1}
N = 0.039 s^{-1}
S = 0.0014 s^{-1}
S^2 = 1.9e-6 s^{-2}
Ri = 792
$\nu_{z,0}^T$ = 2.2e-2 m^{2}s^{-1}
$\nu_{z,strat}^T$ = 1.6e-7 m^{2}s^{-1}
$\Delta\rho$ = 1.5 kg m^{-3}







(Wüest and Lorke, ARFM 2003)

$$\tau_w = \rho_{air} C_D {W_{10}}^2$$

$$u_* = \sqrt{\tau_w/\rho}$$

$$S \cong \frac{u_*}{\kappa\, z}$$

$$\nu_{z,strat}^T = \nu_{z,0}^T \left(1 + \frac{10}{3} Ri\right)^{-3/2}$$

$$Ri = \frac{N^2}{S^2}$$