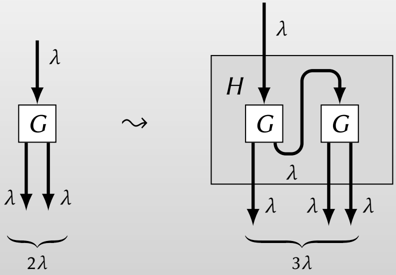
Extending the stretch of a PRG:

$\underline{H(s):}$
  $x := G(s)$
  $y := G(x_{\text{right}})$
  return $x_{\text{left}} \| y$



Claim:

If G is a secure length-doubling PRG then H is a secure length-*tripling* PRG. That is, $\mathcal{L}^{H}_{\text{prg-real}} \equiv \mathcal{L}^{H}_{\text{prg-rand}}$.

Overview:

Want to show:

$\mathcal{L}^H_{\text{prg-real}}$		$\mathcal{L}^H_{\text{prg-rand}}$
$\underline{\text{QUERY}():}$ $s \leftarrow \{0,1\}^\lambda$ return $H(s)$	$\approx$	$\underline{\text{QUERY}():}$ $z \leftarrow \{0,1\}^{3\lambda}$ return z

Standard hybrid technique:

- Starting with $\mathcal{L}^H_{\text{prg-real}}$, make a sequence of small modifications
- Each modification has *negligible* effect on calling program
- Sequence of modifications ends with $\mathcal{L}^H_{\text{prg-rand}}$

Overview:

Want to show:

$$\begin{array}{|l|}\hline \mathcal{L}^{H}_{\text{prg-real}} \\ \hline \underline{\textsc{query}():} \\ \quad s \leftarrow \{0,1\}^{\lambda} \\ \quad \text{return } H(s) \\ \hline \end{array} \approx \begin{array}{|l|}\hline \mathcal{L}^{H}_{\text{prg-rand}} \\ \hline \underline{\textsc{query}():} \\ \quad z \leftarrow \{0,1\}^{3\lambda} \\ \quad \text{return } z \\ \hline \end{array}$$

The proof will **use** the fact G is a secure PRG. In other words,

$$\begin{array}{|l|}\hline \mathcal{L}^{G}_{\text{prg-real}} \\ \hline \underline{\textsc{query}():} \\ \quad s \leftarrow \{0,1\}^{\lambda} \\ \quad \text{return } G(s) \\ \hline \end{array} \approx \begin{array}{|l|}\hline \mathcal{L}^{G}_{\text{prg-rand}} \\ \hline \underline{\textsc{query}():} \\ \quad z \leftarrow \{0,1\}^{2\lambda} \\ \quad \text{return } z \\ \hline \end{array}$$

Security proof

$$\mathcal{L}^{H}_{\text{prg-real}}$$

$\underline{\text{QUERY}()\text{:}}$
  $s \leftarrow \{0,1\}^\lambda$
  return $H(s)$

Starting point is $\mathcal{L}^{H}_{\text{prg-real}}$.

$$\begin{array}{|l|}\hline \quad \mathcal{L}^{H}_{\text{prg-real}} \\ \hline \underline{\textsc{query}():} \\ \quad s \leftarrow \{0,1\}^{\lambda} \\ \quad \text{return } \boxed{H(s)} \\ \hline \end{array}$$

Starting point is $\mathcal{L}^{H}_{\text{prg-real}}$. Fill in details of H

Security proof

$\mathcal{L}^{H}_{\text{prg-real}}$

$\underline{\text{QUERY}():}$

$s \leftarrow \{0,1\}^\lambda$

$x := G(s)$

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  $s \leftarrow \{0,1\}^\lambda$
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These statements appear in $\mathcal{L}^{G}_{\text{prg-real}}$.

Security proof

$$\begin{array}{|l|}\hline \textsc{query}():\\ \quad x := \textsc{query}'()\\ \quad y := G(x_{\text{right}})\\ \quad \text{return } x_{\text{left}} \| y \\ \hline \end{array} \diamond \begin{array}{|l|}\hline \quad \mathcal{L}^G_{\text{prg-real}}\\ \hline \textsc{query}'():\\ \quad s \leftarrow \{0,1\}^\lambda\\ \quad \text{return } G(s)\\ \hline \end{array}$$

Factor out in terms of $\mathcal{L}^G_{\text{prg-real}}$.

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Security of PRG allows to replace $\mathcal{L}^{G}_{\text{prg-real}}$ with $\mathcal{L}^{G}_{\text{prg-rand}}$.

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Inline call to QUERY′.

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$x \leftarrow \{0,1\}^{2\lambda}$
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Sampling 2λ uniform bits is the same as sampling λ and then λ more.

Security proof

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These statements appear in $\mathcal{L}^{G}_{\text{prg-real}}$.

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Factor out in terms of $\mathcal{L}^G_{\text{prg-real}}$.

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Security of PRG allows to replace $\mathcal{L}^{G}_{\text{prg-real}}$ with $\mathcal{L}^{G}_{\text{prg-rand}}$.

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Security of PRG allows to replace $\mathcal{L}^{G}_{\text{prg-real}}$ with $\mathcal{L}^{G}_{\text{prg-rand}}$.

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\hline
\underline{\textsc{query}():} \\
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\quad \text{return } x_{\text{left}} \| y \\
\hline
\end{array}
\diamond
\begin{array}{|l|}
\hline
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\hline
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\end{array}
$$

Inline the call to QUERY′.

Security proof

$$\begin{array}{l} \underline{\text{QUERY}():} \\ \quad x_{\text{left}} \leftarrow \{0,1\}^{\lambda} \\ \quad y \leftarrow \{0,1\}^{2\lambda} \\ \quad \text{return } x_{\text{left}} \| y \end{array}$$

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$\quad x_{\text{left}} \leftarrow \{0,1\}^{\lambda}$
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Inline the call to $\textsc{query}'$.

$$
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\textsc{query}():\\
\quad x_{\text{left}} \leftarrow \{0,1\}^{\lambda}\\
\quad y \leftarrow \{0,1\}^{2\lambda}\\
\quad \text{return } x_{\text{left}} \| y
\end{array}}
$$

Uniform 2λ bits concatenated with λ bits = Uniform 3λ bits.

Security proof

```
QUERY():
  z ← {0,1}^{3λ}
  return z
```

2λ uniform bits concatenated with λ uniform bits = 3λ uniform bits.

QUERY():
$z \leftarrow \{0,1\}^{3\lambda}$
return z

2λ uniform bits concatenated with λ uniform bits = 3λ uniform bits.

$\mathcal{L}^{H}_{\text{prg-rand}}$
$\underline{\text{QUERY}():}$ $\quad z \leftarrow \{0,1\}^{3\lambda}$ $\quad$ return z

This is just $\mathcal{L}^{H}_{\text{prg-rand}}$.

Summary

We showed:

$$\begin{array}{|l|}\hline \mathcal{L}^{H}_{\text{prg-real}} \\ \hline \underline{\textsc{query}():} \\ \quad s \leftarrow \{0,1\}^{\lambda} \\ \quad \text{return } H(s) \\ \hline \end{array} \approx \ldots \approx \begin{array}{|l|}\hline \mathcal{L}^{H}_{\text{prg-rand}} \\ \hline \underline{\textsc{query}():} \\ \quad z \leftarrow \{0,1\}^{3\lambda} \\ \quad \text{return } z \\ \hline \end{array}$$

So H is a secure PRG when G is a secure PRG.

A question

H contains two calls to G. We applied the security of G (replacing $\mathcal{L}^G_{\text{prg-real}}$ with $\mathcal{L}^G_{\text{prg-rand}}$) separately to each call to G.

Does the proof still work if we apply security in the other order?

Attempted security proof

$\mathcal{L}^{H}_{\text{prg-real}}$

$\underline{\textsc{query}():}$
$s \leftarrow \{0,1\}^{\lambda}$
$x := G(s)$
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Starting point.

Attempted security proof

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Starting point. Can we write this call to G in terms of $\mathcal{L}^{G}_{\text{prg-real}}$?

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Argument to G must be chosen *uniformly*

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Argument to G must be chosen *uniformly* but x_{right} is not!