Phy HW : Problem 16

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a). With scaling factor b, we want to relate eff. coupling constant K' b/w S_1 and S_{b+1} with the original Hamiltonian.

Compare term, by tracing out $S_2, S_3, \dots S_b$

$$A e^{K' S_1 S_{b+1}} \leftrightarrow \sum_{S_2, \dots S_b} e^{K \sum_{i=1}^{b} S_i S_{i+1}}$$

Hence, we calculate the sum, ~~...~~,

$$\sum_{S_2 \dots S_b} e^{K \sum_{i=1}^{b} S_i S_{i+1}} \quad (I)$$

Note that, $e^{xn} = \cosh x (1 + n \tanh x)$, $n = \pm 1$; $\Rightarrow$ Compare $A \cosh K' (1 + S_1 S_{b+1} \tanh K')$

Then,

$$(I) = \sum_{S_3, \dots} e^{K \sum_{i=3}^{b} S_i S_{i+1}} \sum_{S_2} e^{K(S_1 S_2 + S_2 S_3)}$$

$$= \sum_{S_3, \dots} e^{K \sum_{i=3}^{b} S_i S_{i+1}} \cosh^2 K \sum_{S_2} (1 + S_1 S_2 \tanh K)(1 + S_2 S_3 \tanh K)$$

$$= \text{——}\, '' \,\text{——} \cosh^2 K [(1 + S_1 \tanh K)(1 + S_3 \tanh K) + (1 - S_1 \tanh K)(1 - S_3 \tanh K)]$$

All terms linear in S_3 will be constant terms and that in S_1 will not contribute to K'.

$$\Rightarrow (I) \propto \sum_{S_3, \dots} e^{K \sum_{i=3}^{b} S_i S_{i+1}} (1 + S_1 S_3 \tanh^2 K)$$

$$= \sum_{S_4, \dots} e^{K \sum_{i=4}^{b} S_i S_{i+1}} \sum_{S_3} e^{K S_3 S_4} (1 + S_1 S_3 \tanh^2 K)$$

$$\cancel{= \text{——}\, '' \,\text{——} [(1 + S_4 \tanh K) + (1 - S_4 \tanh K)] \times [}$$

$$= \text{——}\, '' \,\text{——} \cosh K [(1 + S_4 \tanh K)(1 + S_1 \tanh^2 K) + (1 - S_4 \tanh K)(1 - S_1 \tanh^2 K)]$$

$$\propto \sum_{S_4, \dots} e^{K \sum_{i=4}^{b} S_i S_{i+1}} (1 + S_1 S_4 \tanh^3 K)$$

~~Recursively~~ we have, iteratively

$$(\mathrm{I}) \propto \sum_{S_b} e^{KS_bS_{b+1}} (1 + S_1 S_b \tanh^{b-1} K)$$

$$= \cosh K \left[(1 + S_{b+1} \tanh K)(1 + S_1 \tanh^{b-1} K) + (1 - S_{b+1} \tanh K)(1 - S_1 \tanh^{b-1} K) \right]$$

$$\propto (1 + S_1 S_{b+1} \tanh^b K) \quad (*)$$

Recall, we need to compare with,

$$A e^{K' S_1 S_{b+1}} = A \cosh K' (1 + S_1 S_{b+1} \tanh K').$$

$$\propto (1 + S_1 S_{b+1} \tanh K'). \quad (**)$$

$(*)$ and $(**)$ $\Rightarrow$ $\tanh K' = \tanh^b K \Rightarrow u' = u^b$.

b). K incorporates the inverse temperature $\frac{1}{T}$, so $T > 0 \Rightarrow K$ decreases. Therefore $\tanh K$ also decreases., i.e $u < 1$.

Since $u' = u^b$, if $u < 1$, u' will be even smaller, hence it is not a fixed point. and $T = 0$ is also an unstable fixed point.

c). $\xi(u') = \frac{1}{b} \xi(u)$ needs to hold.

Hence

$$\xi(\tanh^b K) = \frac{1}{b} \xi(\tanh K)$$

This can be done by extracting scaling factor using logarith.

i.e.

$$\xi(\tanh^b K) = \frac{1}{\ln \tanh^b K} = \frac{1}{b \ln \tanh K} = \frac{1}{b} \xi(\tanh K).$$

Therefore $\boxed{\xi(u) = \frac{1}{\ln u}}$

Low. temp limit, $K \to \infty$, $\tanh K \simeq 1 - 2e^{-2K}$

$$\xi(\tanh K) = \frac{1}{\ln(1 - 2e^{-2K})}$$

Again use $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$, since $1 - 2e^{-2K} < 1$,

$$\xi(K) \approx \frac{1}{-2e^{-2K}} = -\frac{1}{2} e^{2K} \propto e^{2K}$$

The corelation length diverges.