Statistische Physik im Gleichgewicht

WS 2023/2024 – Blatt 3

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Problem 5: Critical exponents and Rushbrooke inequality (*2 Points*)

Consider a magnetic system exhibiting a continuous phase transition from a ferromagnetic to a paramagnetic phase. Slightly below the critical point, the order parameter (the magnetization) $M(T, H)$ exhibits an algebraic dependence on the reduced temperature $\tau = \frac{T_c - T}{T_c}$:

$$M(T, 0) \propto \tau^{\beta}.$$

Similar relations can be found for the magnetic susceptibility at constant temperature $\chi_T = \frac{\partial M}{\partial H}|_T$ and the specific heat at constant field $c_H = \frac{T}{N}\frac{\partial S}{\partial T}|_H$:

$$\chi_T \propto \tau^{-\gamma} \qquad\qquad c_H \propto \tau^{-\alpha}.$$

The exponents α, β, and γ are called critical exponents.

The specific heat at constant field c_H can be related to that at constant magnetization c_M by the equation:

$$c_H = c_M + \frac{T}{N}\left(\frac{\partial M}{\partial T}|_H\right)^2 \Big/ \frac{\partial M}{\partial H}|_T.$$

Starting from this, derive the Rushbrooke inequality:

$$\alpha + 2\beta + \gamma \geq 2.$$

In fact, even the equality holds. *Hint: Think about the necessary constraints on c_M, c_H, and χ_T for thermodynamic stability.*

In the following two exercises (6 and 7), you sometimes have to calculate limits. The following Laurent series expansions of hyperbolic functions around the origin may be helpful:

$$\begin{aligned}
1/\sinh x = \operatorname{csch} x &= \frac{1}{x} - \frac{x}{6} + \frac{7x^3}{360} + \mathcal{O}(x^5) \\
1/\cosh x = \operatorname{sech} x &= 1 - \frac{x^2}{2} + \frac{5x^4}{24} + \mathcal{O}(x^6) \\
1/\tanh x = \coth x &= \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \mathcal{O}(x^5)
\end{aligned}$$

Problem 6: Mean-field approximation for the S-spin Ising model (*4 Points*)

We consider the Ising model:

$$H = -J\sum_{\langle ij\rangle} S_i S_j - h\sum_i S_i,$$

with spin values $S_i = -S, -S+1, ..., S-1, S$ (S is an integer or half-integer). Perform the mean-field approximation and show that the partial partition sum for spin S_i is given by:

$$Z_i = \frac{\sinh\left(q(S+\frac{1}{2})\right)}{\sinh\left(\frac{1}{2}q\right)},$$

where $q = \beta(Jmz + h)$ and z is the bond number. Determine the critical temperature of this system. *Hint: Express m as a log-derivative of the partial partition sum. Then proceed analogously to the Ising model in the lecture. In the resulting implicit equation for m, consider the slope of the equation at the origin. You can set $h = 0$ and ignore terms of second order or higher in expansions for finding the critical temperature.*

Problem 7: Mean-field approximation for the Heisenberg model (*4 Points*)

Now we consider the Heisenberg model, where spins can freely reorient in 3 dimensions:

$$H = -J\sum_{\langle ij\rangle} \mathbf{S}_i \cdot \mathbf{S}_j - h\sum_i S_i^z,$$

with unit vectors $\mathbf{S}_i$. The direction of the magnetic field is chosen as the z-direction without loss of generality. Therefore, we can assume that $\langle \mathbf{S}_i \rangle = \mathbf{m} = m_z \mathbf{e}_z$.

(a) Derive the mean-field Hamiltonian of the Heisenberg model and calculate the partition sum. *Hint: Since the spins are now continuous, you need to evaluate an integral.*

(b) Derive a self-consistent equation for m_z, analogous to the Ising model. *Hint: Use a log-derivative of the partition function.*

(c) Calculate the critical point and the exponent β.

Feedback:

Roughly how much time did you spend on this problem set?