Statistische Physik im Gleichgewicht

WS 2023/2024 – Blatt 10

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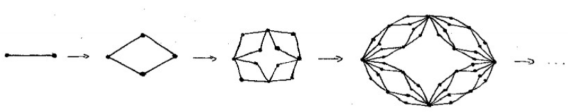
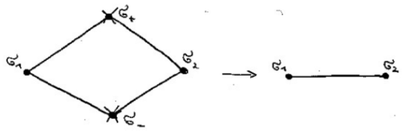
Problem 16: Hierarchical 2D lattice (*7 Points*)





We consider a special lattice in two dimensions, where the form of the Hamiltonian is conserved after renormalization. The lattice is created by consecutively transorming lines into rhombi, as shown in the left figure. We then do this step in reverse for a block transformation, i.e. we replace a rhombus with spins σ_1, σ_2, σ_+, σ_- by a line with spins σ_1 and σ_2. This is shown in the right figure.

The Hamiltonian reads

$$-\beta H = K \sum_{\langle ij \rangle} \sigma_i \sigma_j + H_B \sum_{\langle ij \rangle} (\sigma_i + \sigma_j) + H_S \sum_i \sigma_i.$$

Importantly we use two different parameters H_B and H_S for the external field.

(a) Consider the partition function of one rhombus

$$Z'(\sigma_1, \sigma_2) = \sum_{\sigma_+ = \pm 1} \sum_{\sigma_- = \pm 1} \exp\left(-\beta H(\sigma_1, \sigma_2, \sigma_+, \sigma_-)\right)$$

Show that you can write

$$\ln Z' = K' \sigma_1 \sigma_2 + H'_B(\sigma_1 + \sigma_2) + c' + H'_S(\sigma_1 + \sigma_2)$$

You do not have to calculate K', H'_B, and c' explicitly. However, show that you arrive at a form where $H'_S = H_S$.

(b) Show that you can write the renormalized Hamiltonian as

$$-\beta H' = \sum_{\langle i'j' \rangle} \left(K' \sigma'_i \sigma'_j + H'_B(\sigma'_i + \sigma'_j) + c' \right) + H_S \sum_{i'} \sigma'_i.$$

(c) In fact, it is possible to calculate the renormalized interaction parameters:

$$\begin{aligned} K' &= \frac{1}{2} \ln\left(R_{++} R_{--} / R_{+-}^2\right) \\ H'_B &= \frac{1}{2} \ln\left(R_{++} / R_{--}\right) \\ c' &= 4c + \frac{1}{2} \ln\left(R_{++} R_{--} R_{+-}^2\right). \end{aligned}$$

with $H'_S = H_S$.

Note that in the original, not renormalized, Hamiltonian $c = 0$. Here, we have defined $R_{++} = \exp(2K + 4H_B + H_S) + \exp(-2K - H_S)$, $R_{--} = \exp(-2K + H_S) + \exp(2K - 4H_B - H_S)$ and $R_{+-} = \exp(2H_B + H_S) + \exp(-2H_B - H_S)$. Consider the case $H_B = 0$ and $H_S = 0$. Derive the above relations for this special case. Show that there is exactly one unstable fixed point $0 < K_c < \infty$.

Problem 17: Cumulant method for square lattice (*8 Points*)

Consider a two-dimensional square lattice with

$$-\beta H = K \sum_{\langle ij \rangle} \sigma_i \sigma_j .$$

For the renormalization (semi)group in this system we have $b = 2$.

(a) Create a sketch where you partition a grid with N sites into $N/4$ cells with 4 spins each. Mark the inter- and intra-cell bonds.

The renormalized Hamiltonian is

$$\beta H'(\{\sigma'_\alpha\}) = -\ln Z_0(\{\sigma'_\alpha\}) + \langle V \rangle_0 .$$

(b) Define a renormalized spin from the block transformation $\sigma'_\alpha = \mathrm{sign}(\sigma^1_\alpha + \sigma^2_\alpha + \sigma^3_\alpha + \sigma^4_\alpha)$. Calculate $Z_0(\{\sigma'_\alpha\})$ from the intra-cell probabilities $\exp(-\beta H_0)$ and show that

$$Z_0(\{\sigma'_\alpha\}) = \left(e^{4K} + 6 + e^{-4K}\right)^{N/4} .$$

Hint: In order to do this, you need to divide the space of possible configurations of a block into those that produce $\sigma'_\alpha = +1$ and those that produce $\sigma'_\alpha = -1$. However, for some configurations, the sum $\sum_i \sigma_i = 0$, so it is not clear to which renormalized spin this configuration belongs. Here, we simply assign these configurations to both spins σ'_α, but with only half the weight.

(c) Show that for a spin at a vertex of block α

$$\langle \sigma_{\alpha i} \rangle_0 = \sigma'_\alpha \frac{e^{4K} + 2}{e^{4K} + 6 + e^{-4K}} .$$

Calculate the first cumulant of the interaction term $\langle V \rangle_0$ and write down the total rescaled Hamiltonian.

(d) Find the recursion relation $K'(K)$ and determine the fixed point K^* and the thermal eigenvalue y_t. Compare y_t to the theoretical value $y_t = 1$.

Hint: You can use the substitution $s = e^{4K^*}$.

Feedback:

Roughly how much time did you spend on this problem set?