Problem 17 Arya Prasetya / Fabian Steger

a) Partition funct. of one rhombus,

$$Z'(\sigma_1,\sigma_2) = \sum_{\sigma_+=\pm1}\sum_{\sigma_-=\pm1} \exp\left(K\sum_{\langle ij\rangle}\sigma_i\sigma_j + H_B\sum_{\langle ij\rangle}(\sigma_i+\sigma_j) + H_s\sum_i \sigma_i\right)$$

Seperate into different exponents,

$$Z'(\sigma_1,\sigma_2) = \overbrace{\sum_{\sigma_+=\pm1}\sum_{\sigma_-=\pm1} \exp\left(K\sum_{\langle ij\rangle}\sigma_i\sigma_j\right)}^{(I)} \cdot \overbrace{\sum_{\sigma_+=\pm1}\sum_{\sigma_-=\pm1} \exp\left(H_B\sum_{\langle ij\rangle}(\sigma_i+\sigma_j)\right)}^{(II)} \cdot$$

$$\times \underbrace{\sum_{\sigma_+=\pm1}\sum_{\sigma_-=\pm1} \exp\left(H_s\sum_i\sigma_i\right)}_{(III)}.$$

$$(I) = \sum_{\sigma_+=\pm1}\sum_{\sigma_-=\pm1} \exp\left(K(\sigma_+\sigma_1+\sigma_+\sigma_2+\sigma_-\sigma_1+\sigma_-\sigma_2)\right)$$

$$= \sum_{\sigma_+=\pm1} \exp\left(K(\sigma_+\sigma_1+\sigma_+\sigma_2)\right)\cdot\sum_{\sigma_-=\pm1}\exp\left(K(\sigma_-\sigma_1+\sigma_-\sigma_2)\right).$$

$$= \left[\exp(K(\sigma_1+\sigma_2)) + \exp(-K(\sigma_1+\sigma_2))\right]^2$$

Recall, $e^{nx} = (1+n\tanh x)\cosh x$.

$$\Rightarrow (I) = \left[\cosh^2K(1+\sigma_1\tanh K)(1+\sigma_2\tanh K) + \cosh^2K(1-\sigma_1\tanh K)(1-\sigma_2\tanh K)\right]^2$$

$$= \left[\cosh^2K\left(1+\cancel{\sigma_1\tanh K}+\cancel{\sigma_2\tanh K}+\sigma_1\sigma_2\tanh^2K+1-\cancel{\sigma_1\tanh K}-\cancel{\sigma_2\tanh K}\right.\right.$$
$$\left.\left.+\sigma_1\sigma_2\tanh^2K\right)\right]^2$$

$$= \left[2\cosh^2K(1+\sigma_1\sigma_2\tanh^2K)\right]^2$$

with appropriate choice of K', we can write,

$$= \underbrace{4\cosh^4K}_{e^{c_1'}}\ \underbrace{\left[(1+\sigma_1\sigma_2\tanh^2K)\right]^2}_{e^{K'\sigma_1\sigma_2}}$$

$$(II) = \sum_{\sigma_+=\pm1}\sum_{\sigma_-=\pm1}\exp\left(H_B(\sigma_++\sigma_1+\sigma_++\sigma_2+\sigma_-+\sigma_1+\sigma_-+\sigma_2)\right)$$

$$= \underbrace{\exp(2H_B(\sigma_1+\sigma_2))}_{e^{H_B'(\sigma_1+\sigma_2)}}\cdot\underbrace{\sum_{\sigma_+=\pm1}\sum_{\sigma_-=\pm1}\exp(2H_B(\sigma_++\sigma_-))}_{e^{c_2'}\text{ - const value!}}$$

Similarly

$$(III) = \underbrace{\exp(H_s(\sigma_1+\sigma_2))}_{e^{H_s'(\sigma_1+\sigma_2)}}\cdot\underbrace{\sum_{\sigma_+=\pm1}\sum_{\sigma_-=\pm1}\exp(H_s(\sigma_++\sigma_-))}_{e^{c_3'}\text{ - again constant}}$$

Therefore,

$$Z' = (\mathrm{I}) \cdot (\mathrm{II}) \cdot (\mathrm{III})$$

$$= \exp(c_1') \exp(K'\sigma_1\sigma_2) \exp(c_2') \exp(H_B'(\sigma_1+\sigma_2)) \exp(c_3') \exp(H_S'(\sigma_1+\sigma_2))$$

$$= \exp\left(K'\sigma_1\sigma_2 + H_B'(\sigma_1+\sigma_2) + H_S'(\sigma_1+\sigma_2) + \underbrace{c_1' + c_2' + c_3'}_{c'} \right)$$

$$\Rightarrow \ln Z' = K'\sigma_1\sigma_2 + H_B'(\sigma_1+\sigma_2) + H_S'(\sigma_1+\sigma_2) + c'$$

Moreover, since $\exp(H_S(\sigma_1+\sigma_2)) = \exp(H_S'(\sigma_1+\sigma_2))$

$$\Rightarrow \quad H_S = H_S'.$$

b) Since interblock interaction is a simple on-site interaction, non-existent.

$\cancel{+ H_S \sum_i \sigma_i}$

We can write the renormalized Hamiltonian as,

$$-\beta H' = \sum_{\langle i'j' \rangle} \underbrace{\ln Z'(\sigma_i', \sigma_j')}_{\text{intra block interaction}} = \sum_{\langle i',j' \rangle} \left(K'\sigma_i'\sigma_j' + H_B'(\sigma_i'+\sigma_j') + c'\right) + \sum_i \sigma_i'.$$

c) Case $H_B = 0$ and $H_S = 0$:

$$Z' = \sum_{\sigma_3=\pm1} \sum_{\sigma_4=\pm1} \cdots = 4\cosh^4 K \left[(1+\sigma_1\sigma_2 \tanh^2 K)\right]^2$$

$$= 4\cosh^4 K \left[1 + 2\sigma_1\sigma_2 \tanh^2 K + \tanh^4 K\right]$$