Statistische Physik im Gleichgewicht

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Problem 8: Landau theory with external field   (*2 Points*)

Analyze the graph of the Landau free energy:

$$f(m) = am^2 + bm^4 - hm.$$

with an external field $h \neq 0$. Without loss of generality, you can assume $h > 0$.

(a) Here, we choose $a = kt$ with $t = T - T_c$ and $k, b > 0$. Examine possible values for a and b and their effect on the free energy. In particular, investigate the equilibrium states for different parameter values by finding local and global extrema.

(b) Sketch the free energy above, below and exactly at the critical temperature

Problem 9: Landau free energy for liquid crystals   (*4 Points*)

Consider a Landau free energy where the cubic term is allowed:

$$f(Q) = f_0 + \frac{1}{2}aQ^2 + \frac{1}{4}bQ^4 - \frac{1}{3}cQ^3.$$

with $a = k(T - T^*)$ for a yet unspecified temperature T^*, $b, c, k > 0$. This is the free energy that describes the phase transition from isotropic to nematic phase for a liquid crystal. The liquid crystal is described here via its scalar order parameter Q with $Q \geq 0$ which reflects the change from random orientations of the constituent molecules ($Q = 0$) to a nematic order ($Q > 0$). With this background information we now analyze the free energy above.

(a) Determine the equilibrium value of Q depending on temperature. The equilibrium value is given by the global minimum of the free energy. Which other extrema do you find?

(b) Find the three characteristic temperatures where the properties of the system change. Sketch the free energy for all different states of the system.

(c) Show that the free energy given above represents a first-order phase transition.

Problem 10: Critical exponent of the van der Waals problem   (*4 Points*)

Compute explicitly the critical exponents α, α'.

(a) Show that the internal energy for a single phase state of the van-der-Waals gas is:

$$u(T, V) = u_{\text{ideal}}(T) - \frac{a}{v}$$

with $u_{\text{ideal}}(T) = \frac{3}{2}k_B T$ (for monoatomic gases). It is useful to recall that:

$$p(T, V) = \frac{k_B T}{v - b} - \frac{a}{v^2},$$

where $v = V/N$ is the volume per particle, a is a measure of the average attraction between particles, and b is the volume excluded from v by one particle.

Use this to calculate the heat capacity above the critical temperature and determine α.

Suggestion: Find an expression for $\left(\frac{\partial U}{\partial V}\right)_T$ and integrate it over V. In order to arrive there, assume $U = U(T, V)$ and substitute it into the total differential $\mathrm{d}S$ of the entropy $S(U, V)$. You can use the property of the exact differential from the second lecture.

(b) Below the critical temperature, the system is in a coexistence region of liquid and gas phases. In order to calculate the internal energy, we will use the so-called lever rule.

   (i) Derive a formula for the total volume per particle v in the mixture: starting from the total volume $V = V_L + V_G$ and particle number $N = N_L + N_G$, show that $v = \theta v_L + (1 - \theta)v_G$, where $\theta = N_L/N$ is the relative amount of atoms in the liquid state. Here, $v_{L,G} = V_{L,G}/N$.

   (ii) Transform this equation into an equation for Ψ, Ψ_L, Ψ_G and show that $\theta = \frac{\Psi_G}{\Psi_G - \Psi_L}$ along the critical isochore. *Hint: We defined in the lecture the dimensionless distance from the critical value $\Psi = v/v_c - 1$.*

   (iii) Calculate the heat capacity (to leading order) and the exponent α' via the internal energy:

$$u = \theta u_L + (1 - \theta)u_G.$$

Hint: You can substitute the result from exercise (a). Also, recall that $\frac{1}{1+x} \approx 1 - x + x^2$.

Feedback:

Roughly how much time did you spend on this problem set?