AARHUS UNIVERSITY

# Methods 4 - 1

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BSc Programme in Cognitive Science

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# Methods 4 – Overview

- Wraps up the series of methods courses
- Modelling from a birds-eye view
- Advanced concepts
- Revisiting methods introduced in earlier methods courses
- Comprehensive Bayesian perspective

# Methods 4 – Overview

**Advanced concepts:**

- Causal reasoning using directed acyclic graphs (DAGs)
- Mixture models
- Gaussian processes
- Measurement error
- Missing data
- [Stan programming]

# Methods 4 – Overview

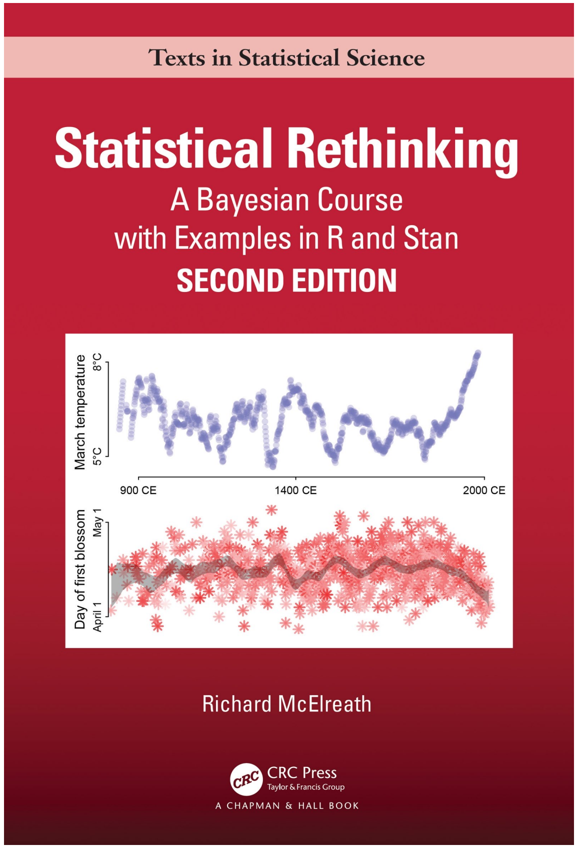
**Revisit:**

- Regression modelling
- Generalized linear models
- Multilevel modelling
- Markov chain Monte Carlo sampling, Hamiltonian Monte Carlo sampling
- Learning to fit models using probabilistic programming

# Resources

**Textbook:**

[1]McElreath, R. (2020). *Statistical Rethinking: A Bayesian Course with Examples in R and Stan (2nd ed.).* Chapman and Hall/CRC. doi:10.1201/9780429029608



# Schedule

| Course week | Week of year | Topics and readings |
|---|---|---|
| 1 | 7 | Statistical models (chapters 1,2) |
| 2 | 8 | Distributions and sampling (chapters 2,3) |
| 3 | 10 | Gaussian models and linear regression (chapter 4) |
| 4 | 11 | Several predictors, directed acyclic graphs (chapters 5) |
| 5 | 14 | Causal inference (chapter 6) |
| 6 | 15 | Model comparison (chapter 7) |
| 7 | 16 | Interactions (chapter 8) |
| 8 | 17 | Markov chain Monte Carlo, maximum entropy (chapters 9, 10) |
| 9 | 18 | Generalized linear models (chapters 11) |
| 10 | 19 | Mixture models, ordered categorical outcomes/predictors (chapter 12) |
| 11 | 20 | Multilevel models (chapter 13) |

# Resources

**Author’s current videos:**

- 2023 lectures: https://www.youtube.com/playlist?list=PLDcUM9US4XdPz-KxHM4XHt7uUVGWWVSus

**Author’s older videos:**

- 2022 lectures: https://www.youtube.com/playlist?list=PLDcUM9US4XdMROZ57-OIRtIK0aOynbgZN
- 2019 lectures: https://www.youtube.com/playlist?list=PLDcUM9US4XdNM4Edgs7weiyIguLSToZRI

# Resources

**Code:**

- This course's main repository:
  https://github.com/methods-4-f24/methods-4-resources

- R package (please install this!):
  https://github.com/rmcelreath/rethinking

# Exam

[https://kursuskatalog.au.dk/en/course/122677/Methods-4-Bayesian-Computational-Modeling](https://kursuskatalog.au.dk/en/course/122677/Methods-4-Bayesian-Computational-Modeling)

**“Ordinary examination and re-examination:**

The exam consists of a portfolio containing a number of assignments. The total length of the portfolio is: 3-7 assignments.

Their form and length will be announced on Brightspace by the teacher at the start of the semester. The portfolio may include products. Depending on their length, and subject to the teacher’s approval, these products can replace some of the standard pages in the portfolio.

It must be possible to carry out an individual assessment. So if some parts of the portfolio have been produced by a group, it must be stated clearly which parts each student is responsible for, and which parts the group as a whole is responsible for.

The complete portfolio must be submitted for assessment in the Digital Exam system. Each student submits a portfolio.”

# Exam

- Portfolio consisting of 3 assignments
- Each assignment will require you to create an R Markdown notebook consisting of a mix of text and code.
- Due
  1. End of week 10 (Sunday 17 March, 23:59)
  2. End of week 15 (Sunday 21 April, 23:59)
  3. End of week 18 (Sunday 12 May, 23:59)

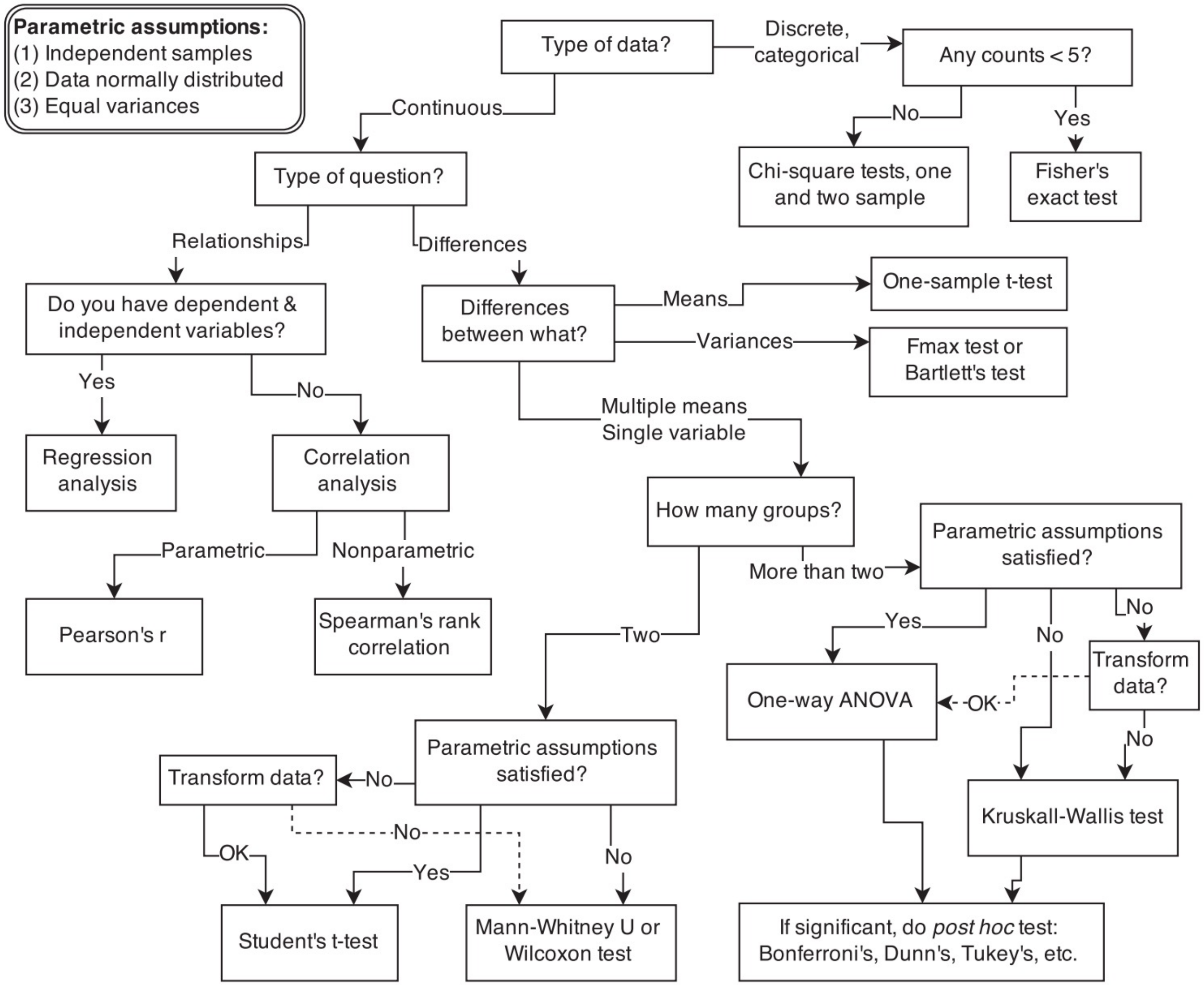
You will receive a (short) feedback message from us on your portfolio assignments that you can use for improvements before finalizing your hand-ins.

# How do science and statistics relate to each other?

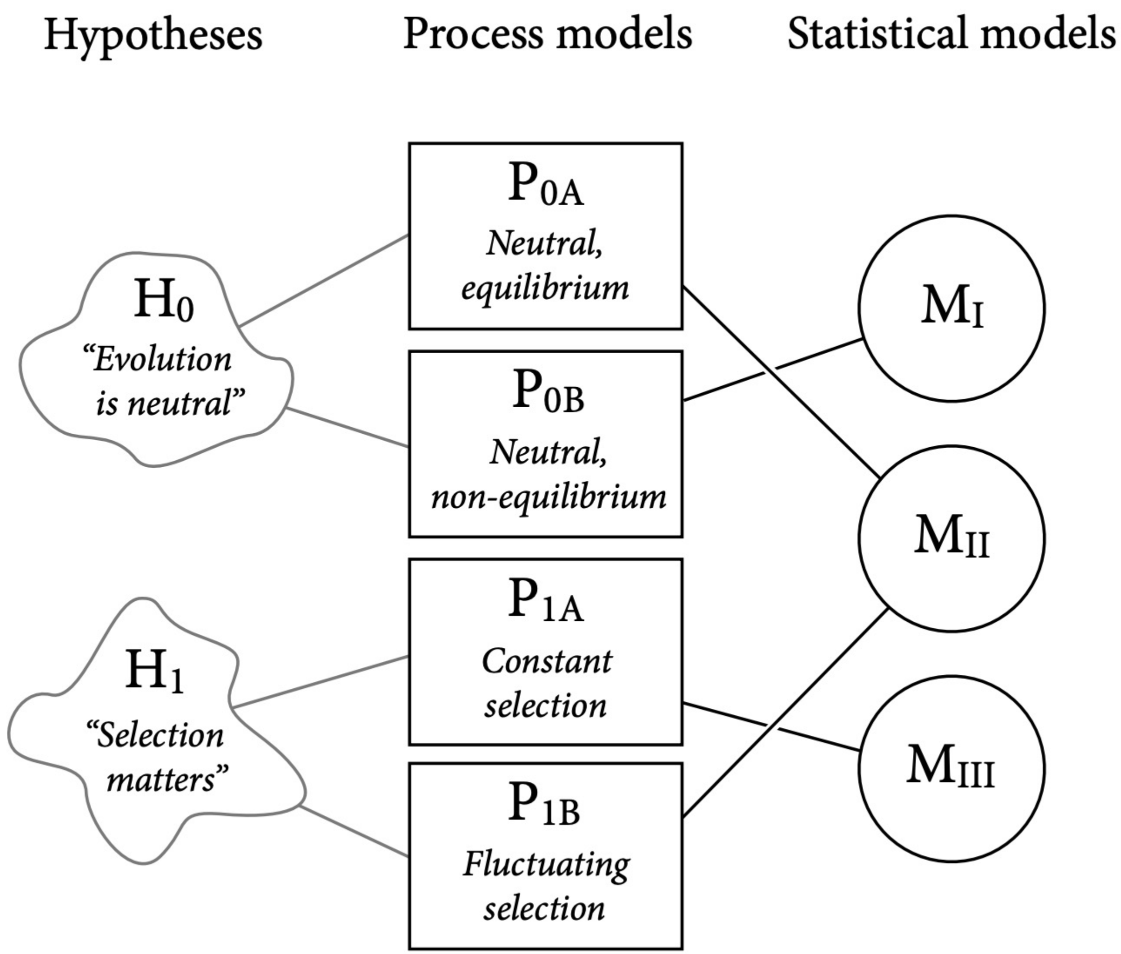
- Your take on this?
- What's the difference between a **statistical** model and a **mechanistic**/process model?
- Example: nutrition science based on
    - Statistical analysis of the outcomes associated with observed/manipulated eating habits (what could go wrong?)
    - Interventions based on knowledge of human physiology
- Karl Popper: science advances by **falsification**.
- Falsification of what?
- Not falsification of null hypotheses. What can go wrong when we try to falsify **null hypotheses**?
- Further: what's the additional problem with **point** null hypotheses?
- Answer: they **can't** be true, so you can always falsify them. You just have to collect enough data.

# Statistical analysis by flowchart

Why not?



# Hypotheses and models



# Bayesian Inference

# A surprising piece of information

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**19 November 2012** Last updated at 18:19

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## Does chocolate make you clever?

**By Charlotte Pritchard**
BBC News

**Eating more chocolate improves a nation's chances of producing Nobel Prize winners - or at least that's what a recent study appears to suggest. But how much chocolate do Nobel laureates eat, and how could any such link be explained?**

# A surprising piece of information

Messerli, F. H. (2012). Chocolate Consumption, Cognitive Function, and Nobel Laureates. *New England Journal of Medicine, 367*(16), 1562–1564.

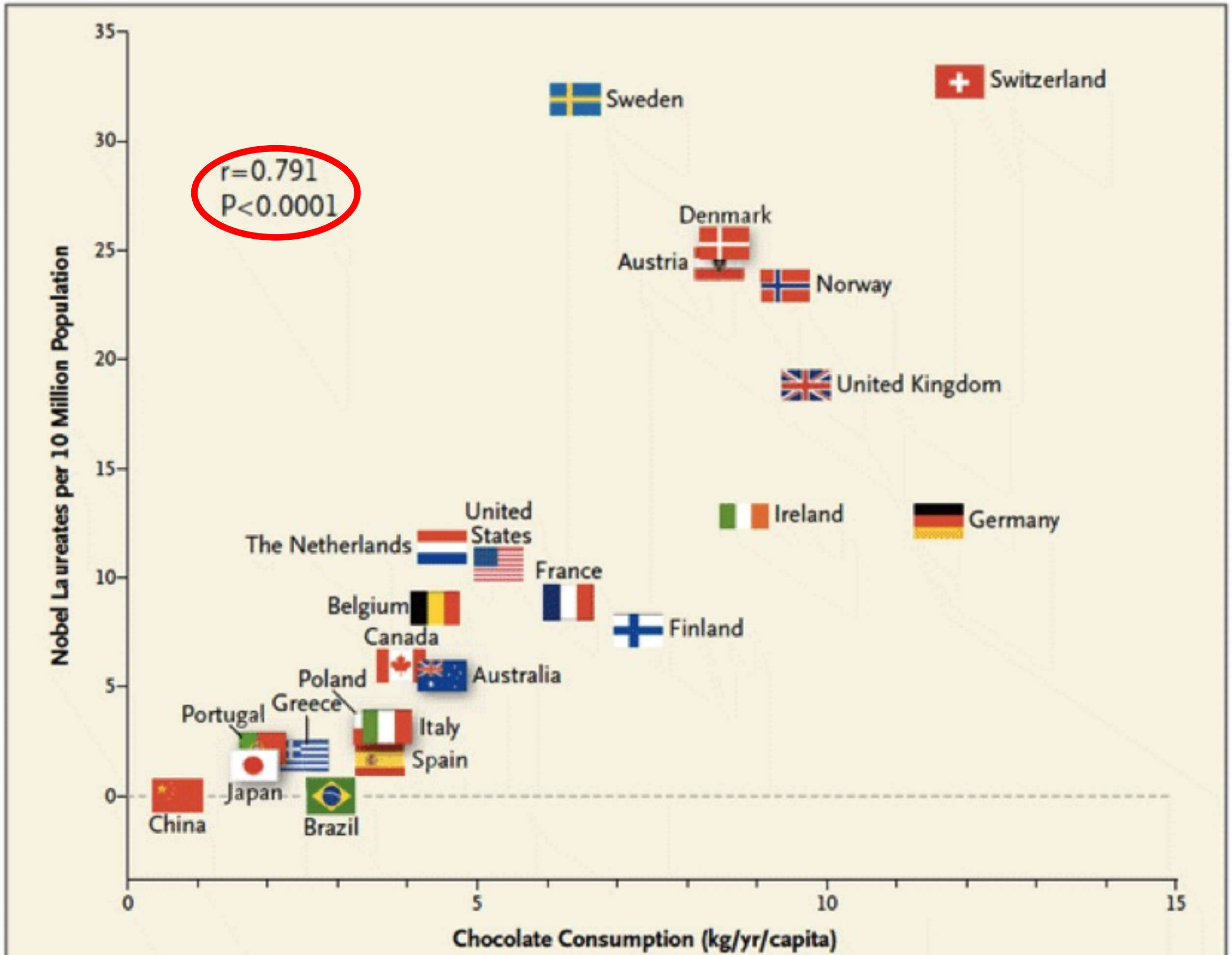



**Figure 1.** Correlation between Countries' Annual Per Capita Chocolate Consumption and the Number of Nobel Laureates per 10 Million Population.

# So will I win the Nobel prize if I eat lots of chocolate?

This is a question referring to **uncertain quantities**. Like almost all scientific questions, it cannot be answered by deductive logic. *Nonetheless, quantitative answers can be given – **but they can only be given in terms of probabilities***.

Our question here can be rephrased in terms of a conditional probability:

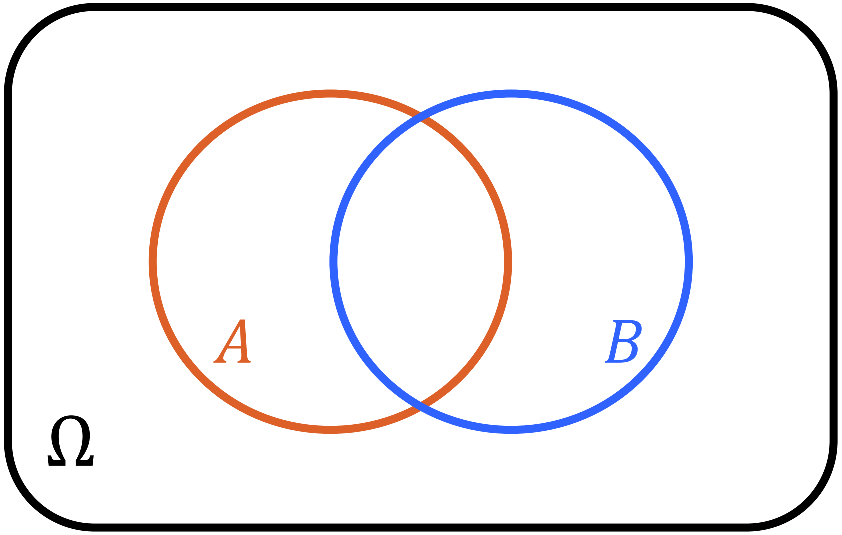
$$p(Nobel \mid amount\ of\ chocolate) = ?$$

To answer it, we have to learn to calculate such quantities. The tool for this is **Bayesian inference**.

***However:** note that no amount of statistical analysis will tell you anything about the causal mechanism behind this if you don't have **a hypothesis about that mechanism and a causal scientific model of it**!*

# Calculating with probabilities: the setup

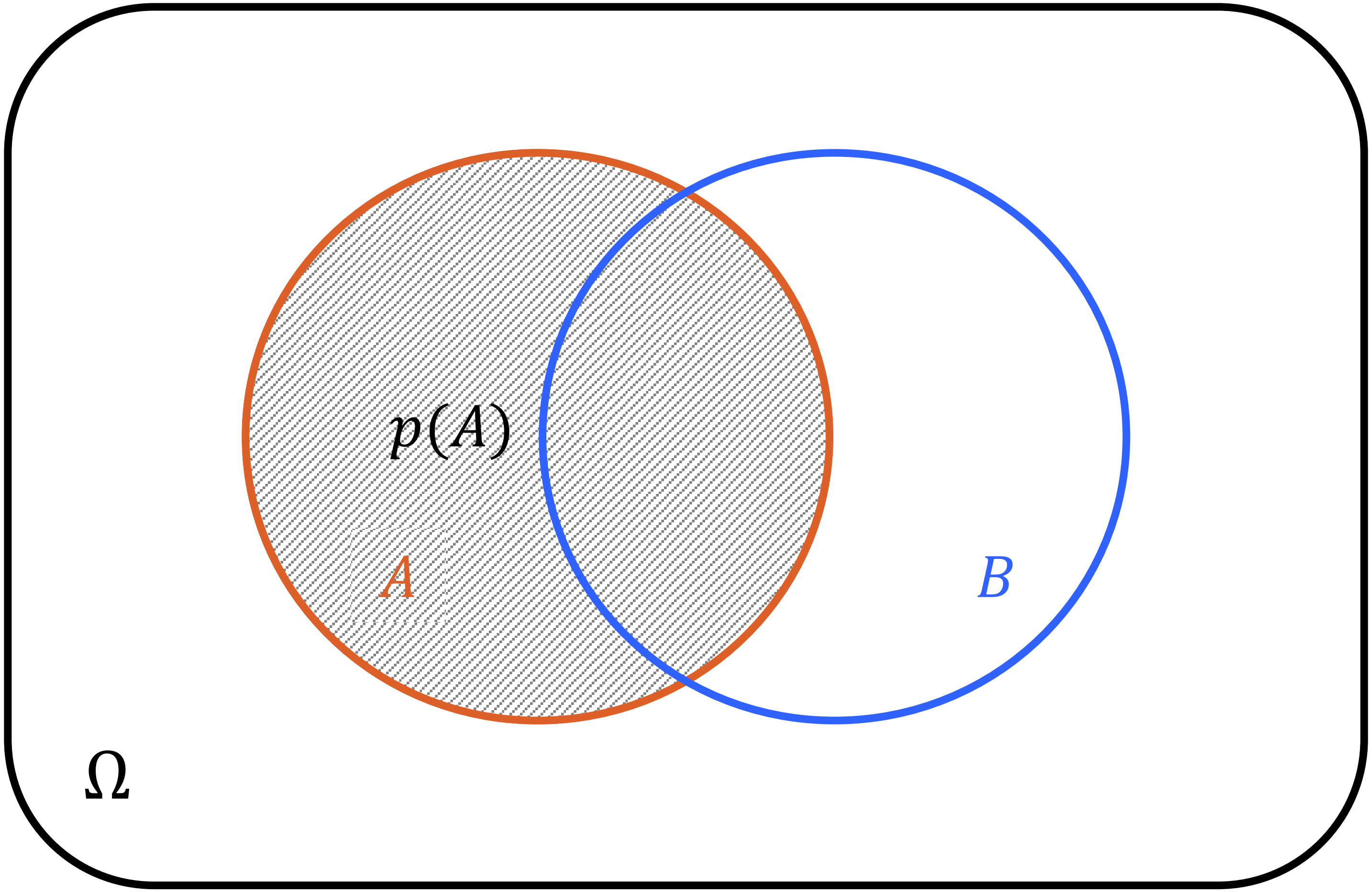
We assume a probability space $\Omega$ with subsets $A$ and $B$



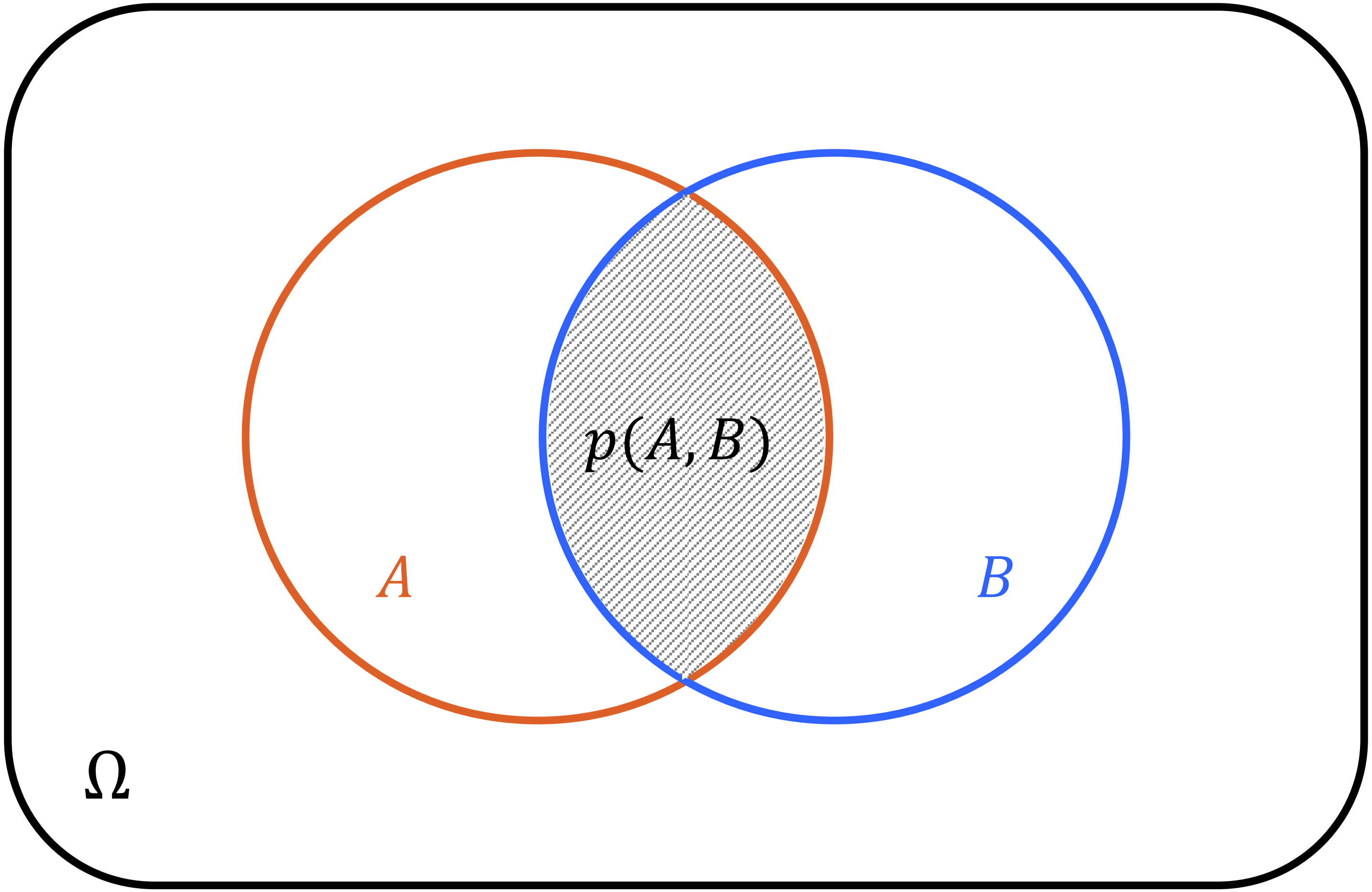

In order to understand *the rules of probability*, we need to understand **three kinds of probabilities**

- *Marginal* probabilities like $p(A)$
- *Joint* probabilities like $p(A, B)$
- *Conditional* probabilities like $p(B|A)$

# Marginal probabilities



# Joint probabilities



# What is ‘marginal’ about marginal probabilities?

- Let $A$ be the statement ‘the sun is shining’
- Let $B$ be the statement ‘it is raining’
- $\bar{A}$ negates $A$, $\bar{B}$ negates $B$

Consider the following table of joint probabilities:

| | $B$ | $\bar{B}$ | Marginal probabilities |
|---|---|---|---|
| $A$ | $p(A, B) = 0.1$ | $p(A, \bar{B}) = 0.5$ | $p(A) = 0.6$ |
| $\bar{A}$ | $p(\bar{A}, B) = 0.2$ | $p(\bar{A}, \bar{B}) = 0.2$ | $p(\bar{A}\ ) = 0.4$ |
| Marginal probabilities | $p(B) = 0.3$ | $p(\bar{B}) = 0.7$ | Sum of all probabilities $\sum p(\cdot,\cdot) = 1$ |

*Marginal probabilities* get their name from being at the margins of tables such as this one.

# Conditional probabilities

- In the previous example, what is the probability that the sun is shining given that it is not raining?
- This question refers to a conditional probability: $p(A|\bar{B})$
- You can find the answer by asking yourself: out of all times where it is not raining, which proportion of times will the sun be shining?

| | $B$ | $\bar{B}$ | Marginal probabilities |
|---|---|---|---|
| $A$ | $p(A,B) = 0.1$ | $p(A,\bar{B}) = 0.5$ | $p(A) = 0.6$ |
| $\bar{A}$ | $p(\bar{A},B) = 0.2$ | $p(\bar{A},\bar{B}) = 0.2$ | $p(\bar{A}) = 0.4$ |
| Marginal probabilities | $p(B) = 0.3$ | $p(\bar{B}) = 0.7$ | Sum of all probabilities $\sum p(\cdot,\cdot) = 1$ |

- This means we have to divide the joint probability of 'sun shining, not raining' by the sum of all joint probabilities where it is not raining:

$$p(A|\bar{B}) = \frac{p(A,\bar{B})}{p(A,\bar{B}) + p(\bar{A},\bar{B})} = \frac{p(A,\bar{B})}{p(\bar{B})} = \frac{0.5}{0.7} \approx 0.71$$

# The rules of probability

Considerations like the ones above led to the following definition of the **rules of probability:**

1. $\sum_a p(a) = 1$ (*Normalization*)

2. $p(B) = \sum_a p(a, B)$ (*Marginalization* – the **sum rule**)

3. $p(A, B) = p(A|B)p(B) = p(B|A)p(A)$ (*Conditioning* – the **product rule**)

These are **axioms**, ie they are assumed to be true. Therefore, we cannot test them the way we could test a theory. However, we can see if they turn out to be useful.

# The rules of probability

R. T. Cox showed in 1946 that the rules of probability theory can be derived from *three basic desiderata*:

1. Representation of degrees of plausibility by real numbers
2. Qualitative correspondence with common sense (in a well-defined sense)
3. Consistency

By mathematical proof (i.e., by *deductive* reasoning) the three desiderata as set out by Cox imply the rules of probability (i.e., the rules of *inductive* reasoning).

*This means that anyone who accepts the desiderata must accept the rules of probability.*

«Probability theory is nothing but common sense reduced to calculation.»

— Pierre-Simon Laplace, 1819

# Bayes' rule

- The product rule of probability states that

$$p(A|B)p(B) = p(B|A)p(A)$$

- If we divide by $p(B)$, we get **Bayes' rule:**

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)} = \frac{p(B|A)p(A)}{\sum_a p(B|a)p(a)}$$

- The last equality comes from unpacking $p(B)$ according to the product and sum rules:

$$p(B) = \sum_a p(B,a) = \sum_a p(B|a)p(a)$$

# Bayes' rule: what problem does it solve?

- Why is Bayes' rule important?
- It allows us to invert conditional probabilities, ie to pass from $p(B|A)$ to $p(A|B)$:

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)}$$

- In other words, it allows us to update our belief about $A$ in light of observation $B$

# Bayes' rule: the chocolate example

In our example, it is immediately clear that $P(Nobel|chocolate)$ is very different from $P(chocolate|Nobel)$. While the first is hopeless to determine directly, the second is much easier to find out: ask Nobel laureates how much chocolate they eat. Once we know that, we can use Bayes' rule:

likelihood model

prior

$$p(Nobel|chocolate) = \frac{p(chocolate|Nobel)P(Nobel)}{p(chocolate)}$$

posterior

evidence

***However:*** *note that no amount of statistical analysis will tell you anything about the causal mechanism behind this if you don't have* ***a hypothesis about that mechanism and a causal scientific model of it****!*

# A simple example of Bayesian inference

**(adapted from Jaynes (1976))**

Two manufacturers, *A* and *B*, deliver the same kind of components that turn out to have the following lifetimes (in hours):

**A:**

```
59.5814
37.3953
47.5956
40.5607
48.6468
36.2789
31.5110
31.3606
45.6517
```

**B:**

```
48.8506
48.7296
59.1971
51.8895
```

Assuming prices are comparable, from which manufacturer would you buy?

# A simple example of Bayesian inference

## How do we compare such samples?

- By comparing their arithmetic means

## Why do we take means?

- If we take the mean as our estimate, the error in our estimate is the mean of the errors in the individual measurements
- Taking the mean as maximum-likelihood estimate implies a **Gaussian error distribution**
- A Gaussian error distribution appropriately reflects our **prior** knowledge about the errors whenever we know nothing about them except perhaps their variance

# A simple example of Bayesian inference

## What next?

- Let's do a *t*-test (but first, let's compare variances with an *F*-test):

```
>> [fh,fp,fci,fstats] = vartest2(xa,xb)

fh =          fp =          fci =          fstats =

     0             0.3297        0.2415         fstat: 3.5114
                                19.0173           df1: 8
                                                  df2: 3
```

Variances not significantly different!

```
>> [h, p, ci, stats]= ttest2(xa,xb)

h =           p =           ci =           stats =

     0            0.0665       -21.0191          tstat: -2.0367
                                  0.8151            df: 11
                                                    sd: 8.2541
```

Means not significantly different!

Is this satisfactory? No, so what can we learn by turning to probability theory (i.e., Bayesian inference)?

# A simple example of Bayesian inference

## The procedure in brief:

- Determine your question of interest («What is the probability that...?»)
- Specify your statistical model (likelihood and prior)
- Calculate the posterior using Bayes' theorem
- Ask your question of interest of the posterior

## All you need is the rules of probability theory.

(Sometimes you'll encounter a nasty integral. But that's only a technical difficulty, not a conceptual one, and software packages like SPM will solve it for you – normally).

# A simple example of Bayesian inference

The question:

- What is the probability that the components from manufacturer $B$ have a longer lifetime than those from manufacturer $A$?
- More specifically: given how much more expensive they are, how much longer do I require the components from $B$ to live.
- Example of a *decision rule:* **if the components from *B* live 3 hours longer than those from *A* with a probability of at least 80%, I will choose those from *B*.**

# A simple example of Bayesian inference

The model:

Likelihood (Gaussian):

$$p(\{y_i\}|\mu,\lambda) = \prod_{i=1}^{n} \left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}} \exp\left(-\frac{\lambda}{2}(y_i-\mu)^2\right)$$

This is the probability of making observations $\{y_i\}_{i=1,\ldots,n}$ if the **mean** of the sampling distribution is $\mu$ and its **precision** is $\lambda$.

Prior (Gaussian-gamma):

$$p(\mu,\lambda|\mu_0,\kappa_0 a_0,b_0) = \mathcal{N}(\mu|\mu_0,(\kappa_0\lambda)^{-1})\mathrm{Gam}(\lambda|a_0,b_0)$$

This is our assumption about the realistic range in which we expect to find $\mu$ and $\lambda$, determined by the **hyperparameters** $\mu_0, \kappa_0, a_0$, and $b_0$.

# A simple example of Bayesian inference

- Applying Bayes' rule gives us the **posterior hyperparameters** $\mu_n, \kappa_n, a_n$ and $b_n$

- If we choose prior hyperparameters $\kappa_0 = 0$, $a_0 = 0$, $b_0 = 0$, the posterior hyperparameters are:

$$\mu_n = \bar{y} \qquad \kappa_n = n \qquad a_n = \frac{n}{2} \qquad b_n = \frac{n}{2} s^2$$

- This means that all we need is $n$, the number of data points; $\bar{y}$, their mean; and $s^2$, their variance.

- If we choose different prior hyperparameters, the equations for the posterior hyperparameters look a bit more complicated, but in any case they can easily be calculated for our example model.

- In many applications of Bayesian inference, the posterior cannot be calculated analytically and written in terms of a function determined by hyperparameters. In these cases, **approximate Bayesian inference** has to be used, using for example *Monte Carlo sampling* or *variational calculus*.

# A simple example of Bayesian inference

The joint posterior distributions of lifetimes $\mu_A$ of products from manufacturer $A$ and $\mu_B$ are $p(\mu_A|\{y_i\}_A)$ and $p(\mu_B|\{y_k\}_B)$, respectively.

We can now use them to answer our question: what is the probability that parts from $B$ live at least 3 hours longer than parts from $A$?

$$p(\mu_B - \mu_A > 3) = \int_{-\infty}^{\infty} p(\mu_A|\{y_i\}_A) \int_{\mu_A+3}^{\infty} p(\mu_B|\{y_k\}_B)\mathrm{d}\mu_B \, \mathrm{d}\mu_A = 0.9501$$

Note that **the classical procedure with the *t*-test told us that there was «no significant difference»** even though according to our Bayesian calculation **there is a >95% probability that the parts from *B* will last at least 3 hours longer than those from *A***.

# Bayesian inference

The procedure in brief:

- Determine your question of interest («What is the probability that...?»)
- Specify your statistical model (likelihood and prior)
- Ask your question of interest of the posterior

All you need is the rules of probability theory

[– and a causal model if you want to do science and not just statistics.]

## From now on: flipped classroom ☺

The procedure in brief:

- You've done the reading (hopefully…)
- You may also have watched the videos

Then we go through the content again and explain it to each other.