Decay of a Coherent State in a Single-Mode Optical Cavity

1 Introduction

In this problem, we investigate the time evolution of a coherent light state $|\alpha\rangle$ in a single-mode optical cavity that leaks photons at a rate κ. The system dynamics are described by the following Lindblad master equation:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] + \mathcal{L}(\rho)$$

where $H = \hbar\omega_c(a^\dagger a + 1/2)$ is the system Hamiltonian and $\mathcal{L}(\rho)$ is the Lindblad superoperator:

$$\mathcal{L}(\rho) = \frac{\kappa}{2}(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a)$$

By moving to a rotating frame at frequency ω_c, the Hamiltonian part is eliminated and the equation simplifies to:

$$\frac{d\rho}{dt} = \frac{\kappa}{2}(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) \tag{1}$$

2 Average Photon Number

The average photon number is defined as $\langle n(t)\rangle = \mathrm{Tr}(\rho(t)a^\dagger a)$. To find the governing differential equation, we use the following rule:

$$\frac{d}{dt}\langle O\rangle = \mathrm{Tr}\left(\frac{d\rho}{dt}O\right)$$

Substituting the Lindblad equation (1) and the operator $O = a^\dagger a = N$, we have:

$$\frac{d}{dt}\langle N\rangle = \mathrm{Tr}\left(\frac{\kappa}{2}(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a)\, a^\dagger a\right)$$

Using the cyclic property of the trace (i.e., $\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA)$), we compute each term separately:

$$\begin{aligned}
\mathrm{Tr}(2a\rho a^\dagger N) &= 2\,\mathrm{Tr}(\rho a^\dagger N a)\\
\mathrm{Tr}(-a^\dagger a\rho N) &= -\,\mathrm{Tr}(\rho N a^\dagger a)\\
\mathrm{Tr}(-\rho a^\dagger a N) &= -\,\mathrm{Tr}(\rho N a^\dagger a)
\end{aligned}$$

Combining these expressions, we obtain:

$$\frac{d}{dt}\langle N\rangle = \frac{\kappa}{2}\operatorname{Tr}\left(\rho\left[2a^\dagger N a - 2Na^\dagger a\right]\right)$$

Now we use the commutation relations. Since $[a, a^\dagger] = 1$, we have:

$$a^\dagger a a = a^\dagger(a^\dagger a + 1) = a^\dagger a^\dagger a + a^\dagger$$

But for simplification, we use the relation $Na = a^\dagger a a = a(a^\dagger a - 1) = a(N-1)$. Therefore:

$$a^\dagger N a = a^\dagger a(N-1) = N(N-1)$$

Also, $Na^\dagger a = N^2$. Substituting these relations, we have:

$$2a^\dagger N a - 2Na^\dagger a = 2N(N-1) - 2N^2 = 2N^2 - 2N - 2N^2 = -2N$$

Thus:

$$\frac{d}{dt}\langle N\rangle = \frac{\kappa}{2}\operatorname{Tr}\left(\rho(-2N)\right) = -\kappa\operatorname{Tr}(\rho N) = -\kappa\langle N(t)\rangle$$

So we arrive at the following differential equation:

$$\frac{d}{dt}\langle n(t)\rangle = -\kappa\langle n(t)\rangle \tag{2}$$

The solution to this equation is an exponential decay:

$$\langle n(t)\rangle = \langle n(0)\rangle e^{-\kappa t}$$

Since the initial state is a coherent state $|\alpha\rangle$, we have $\langle n(0)\rangle = |\alpha|^2$. Therefore, the final solution is:

$$\boxed{\langle n(t)\rangle = |\alpha|^2 e^{-\kappa t}} \tag{3}$$

3 Evolution of the Wigner Function in Phase Space

The Wigner function $W(\beta)$ is a quasi-probability distribution in phase space (with axes x and p) that completely describes the quantum state.

- **At time $t = 0$:** The system state is $|\alpha\rangle$. The Wigner function for a coherent state is a **symmetric** Gaussian distribution:

  $$W_0(\beta) = \frac{2}{\pi}\exp\left(-2|\beta - \alpha|^2\right)$$

  This function is centered at the point α in phase space and has the minimum possible uncertainty (a circle with radius 1/2 in appropriate units).

- **At time $t > 0$:** As can be inferred from solving the Lindblad equation, the system state at any time t remains a **coherent state**, but with an amplitude that decays exponentially:

$$|\psi(t)\rangle = |\alpha e^{-\kappa t/2}\rangle$$

The reason for the exponent $-\kappa t/2$ is that the amplitude α is related to the square root of the average photon number ($|\alpha| = \sqrt{\langle n\rangle}$). From equation (3), we have $\sqrt{\langle n(t)\rangle} = |\alpha|e^{-\kappa t/2}$.

Consequently, the Wigner function at time t will also be a symmetric Gaussian distribution, but its center moves towards the origin:

$$W_t(\beta) = \frac{2}{\pi}\exp\left(-2|\beta - \alpha e^{-\kappa t/2}|^2\right)$$

- **Evolution Properties:**

1. **Center of Distribution:** The center of the Gaussian distribution starts at the point α and moves in a straight line towards the origin ($\beta = 0$): $\alpha(t) = \alpha e^{-\kappa t/2}$.
2. **Width of Distribution:** The width of the Gaussian distribution, which represents the quantum noise of the state, remains **constant** over time. This indicates that photon leakage from the cavity does **not destroy** the quantum coherence of the state; it only reduces its energy (average photon number).
3. **Final State:** At $t \to \infty$, the center of the distribution reaches the origin of phase space, and the state becomes the **vacuum state** $|0\rangle$. The Wigner function of the vacuum state is also a symmetric Gaussian centered at the origin:

$$W_\infty(\beta) = \frac{2}{\pi}\exp\left(-2|\beta|^2\right)$$

4 Conclusion

The decay of a coherent state in a cavity due to photon leakage has two key features:

1. The average photon number (energy) decays exponentially at rate κ: $\langle n(t)\rangle = |\alpha|^2 e^{-\kappa t}$.

2. The quantum state itself preserves its coherence and remains a coherent state with a decaying amplitude: $|\psi(t)\rangle = |\alpha e^{-\kappa t/2}\rangle$. In phase space, this corresponds to the center of the Wigner function moving towards the origin, without any change in its width.