Integration

Numerical Methods

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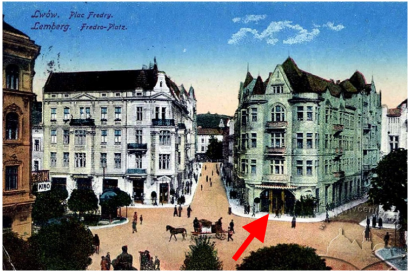
Lecture content

- **Scottish Café in Lviv**. Over glasses of cognac, mathematicians scribbled on the cafe's marble tabletops, and history was made. They were no ordinary guests. These men were Hugo **Steinhaus**, Stefan **Banach** and Stanisław **Ulam** – mathematicians who dreamt big, wrote poems, constructed the atomic bomb and helped organise the first flights to the moon.
  Article: Maths, Madness and the Manhattan Project
- The Scottish Café was the café in Lwów, Poland (now Lviv, Ukraine) where, in the 1930s and 1940s, mathematicians from the Lwów School collaboratively discussed research problems, particularly in **functional analysis** and **topology**.
  Wiki: Scottish Café



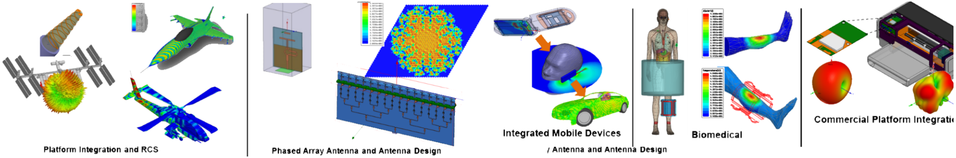
Stanisław Ulam

- In 1939 moved to US
- In 1943 joined the **Manhattan Project**, which was US's wartime effort to create the atomic bomb.
- In the 60s he worked on a **moon flight** programme.
- One of the first, who **used computers** in his projects.
- Discovered the **Monte Carlo** method
- The Monte Carlo method has become a ubiquitous and **standard approach** to computation, and the method has been applied to a vast number of scientific problems. In addition to problems in **physics** and **mathematics**, the method has been applied to **finance**, **social** science, **environmental risk** assessment, **linguistics**, **radiation therapy**, **sports** and many others.

- There are many applications in which **numerical integration** plays **key role** for computation.
- Some well-known functions are defined as integrals. For instance, the **cumulative distribution** of a **bell curve** is defined as:

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \tag{1}$$

- Approximations of $erf(x)$ are needed in **statistical** methods, and one reasonable approach to finding these values is to compute the integral above **numerically**.
- Other times, numerical approximations of integrals are part of a larger systems. For example, approximated solution of the integral equations.



https://www.ozeninc.com/industry-solutions/antenna-design/

- The rendering equation from computer graphics and **ray tracing** is an **integral equation**

$$I(\vec{x},\vec{y}) = g(\vec{x},\vec{y})\left[\varepsilon(\vec{x},\vec{y}) + \int_S \rho(\vec{x},\vec{y},\vec{z}) I(\vec{y},\vec{z})\, d\vec{z}\right]$$

https://youtu.be/J3ue35ago3Y

- Many classical **image processing filters** can be thought of as **convolutions**, given by:

$$(I * g)(x, y) = \iint_{\mathbb{R}^2} I(u, v) g(x - u, y - v)\, du\, dv$$

http://cs.brown.edu/courses/csci2370/2000/1999/cabral.pdf

Quadrature rules - general view

- The problem of **numerical integration** (or **quadrature**) in a single variable can be expressed as: "Given a **sampling** of n points from some function $f(x)$, find an **approximation** of $\int_a^b f(x)dx$".
- Quadrature rule approximating the integral of $f(x)$ on some interval $a \leqslant x \leqslant b$ with a set of n sample points x_i:

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i). \tag{2}$$

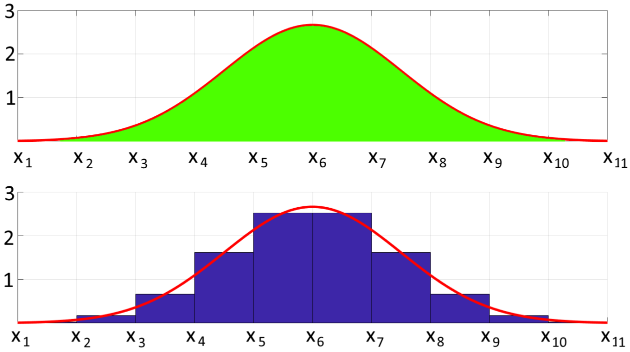
- The choices of x_i and w_i determine a **quadrature rule**.
- For example, the *Riemann integral* presented in introductory calculus:

$$\int_a^b f(x)dx = \lim_{\Delta x_k \to 0} \sum_{k=1}^{n-1} f(\widetilde{x}_k)(x_{k+1} - x_k). \tag{3}$$

- The interval $[a, b]$ is partitioned into pieces $a = x_1 < x_2 ... < x_n = b$, where $\Delta x_k = x_{k+1} - x_k$ and $\widetilde{x}_k$ is any point in $[x_k, x_{k+1}]$.

Riemann integral

$$\int_a^b f(x)dx = \lim_{\Delta x_k \to 0} \sum_{k=1}^{n-1} f(\widetilde{x}_k)(x_{k+1} - x_k).$$



The integral value is given by the **area** of the region (accurate - green and approximated - blue).

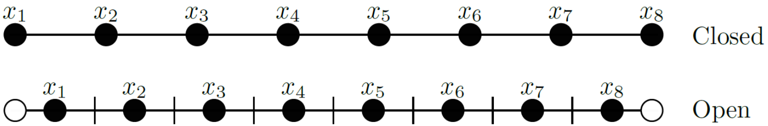
Newton–Cotes formulas

- **Newton–Cotes** formulas are one of the most **frequently** used ones to integrate.
- They assume that x_i's are evenly spaced points in $[a, b]$.
- There are two reasonable choices of evenly spaced samples:
  - **Closed**: x_i's are placed at a and b. In particular, for $k \in \{1...n\}$:

$$x_k = a + \frac{(k-1)(b-a)}{n-1} \tag{4}$$

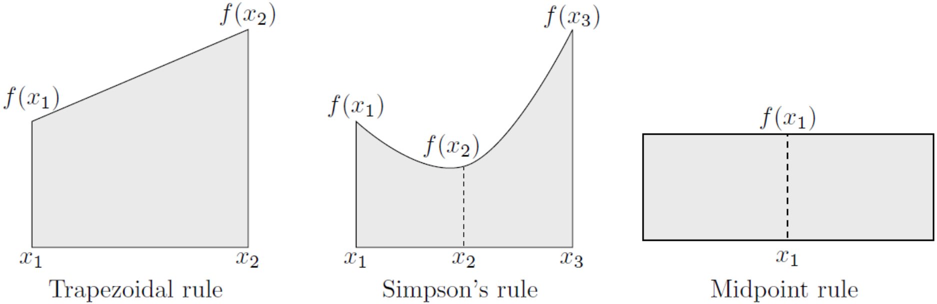
  - **Open**: x_i's are not placed at a or b:

$$x_k = a + \frac{k(b-a)}{n+1} \tag{5}$$



Newton–Cotes formulas

- In the **Newton–Cotes** approach the function $f(x)$ in each of the subregions $[x_k, x_{k+1}]$ is **approximated** using a **low degree**-polynomial:

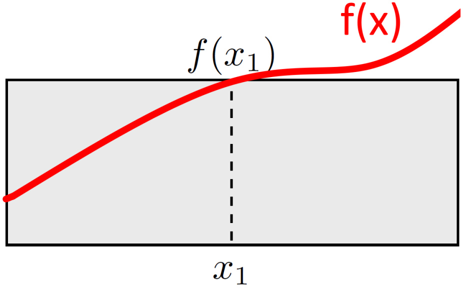


- Then, we will chain together small pieces into **composite** rules when integrating over a large interval $[a, b]$.

Newton–Cotes formulas - Open Rule

- The most common rule for **open** quadrature is the **midpoint rule**, which approximates an integral with the signed area of a **rectangle** through the **midpoint** of the integration interval $[x_k, x_{k+1}]$:

$$\int_{x_i}^{x_{i+1}} f(x)dx \approx \int_{x_i}^{x_{i+1}} f\left(\frac{x_{i+1}+x_i}{2}\right) dx = (x_{i+1}-x_i) f\left(\frac{x_{i+1}+x_i}{2}\right) \tag{6}$$



Midpoint rule

Newton–Cotes formulas - Open Rule

- We usually wish to integrate $f(x)$ with more than one sample point (to **increase** the **accuracy**)
- To do so, we can construct a **composite** rule by summing up smaller **pieces**.
- For example, if we subdivide $[a, b]$ into k intervals, we obtain:

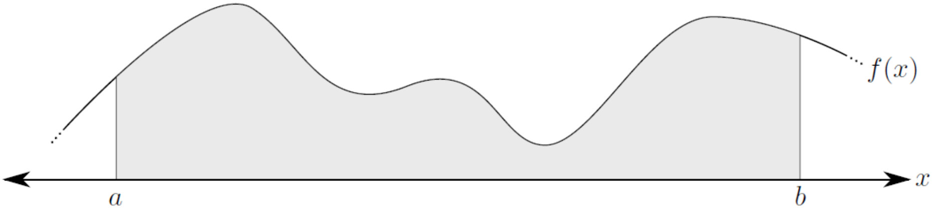
$$\Delta x = \frac{b-a}{k} \tag{7}$$

$$x_i = a + (i-1)\Delta x. \tag{8}$$

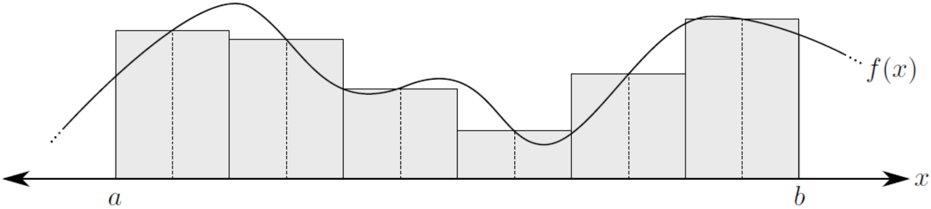
- and the **composite midpoint** rule is:

$$\int_a^b f(x)dx \approx \sum_{i=1}^{k} f\left(\frac{x_i + x_{i+1}}{2}\right)\Delta x \tag{9}$$

Newton–Cotes formulas - Open Rule



Actual integral



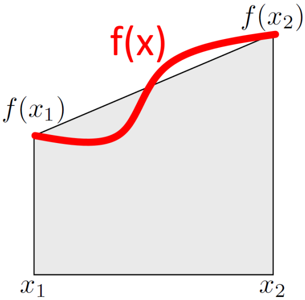
Composite midpoint rule (6 samples)

Newton–Cotes formulas - Closed Rule

An example of the **closed** quadrature is the **trapezoidal rule**, which approximates an integral with the signed area of a **trapeze**, which is constructed by linearly interpolating from $f(x_k)$ to $f(x_{k+1})$ (for the interval $[x_k, x_{k+1}]$):

$$\int_{x_i}^{x_{i+1}} f(x)dx \approx (x_{i+1} - x_i)\left(\frac{f(x_{i+1}) + f(x_i)}{2}\right) \tag{10}$$

At home - derive the above formula.



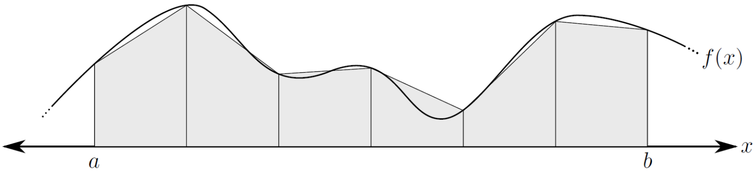
Trapezoidal rule

Newton–Cotes formulas - Closed Rule

- A **composite** rule for the trapezoidal scheme is obtained by summing up smaller **pieces**.
- If we subdivide $[a, b]$ into k intervals, we obtain:

$$\int_a^b f(x)dx \approx \sum_{i=1}^{k} \left(\frac{f(x_i) + f(x_{i+1})}{2} \right) \Delta x = \tag{11}$$

$$= \Delta x \left(\frac{1}{2} f(a) + f(x_2) + \ldots + f(x_k) + \frac{1}{2} f(b) \right)$$

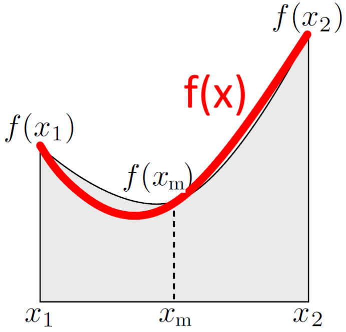


Composite trapezoidal rule (7 samples)

Newton–Cotes formulas - Closed Rule

Another example for the **closed** quadrature is the **Simpson's rule**, which approximates an integral with the signed area enclosed by the parabola that goes through the three points:

$$\int_{x_i}^{x_{i+1}} f(x)dx \approx \frac{(x_{i+1}-x_i)}{6}\left(f(x_i)+4f\left(\frac{x_{i+1}+x_i}{2}\right)+f(x_{i+1})\right) \tag{12}$$

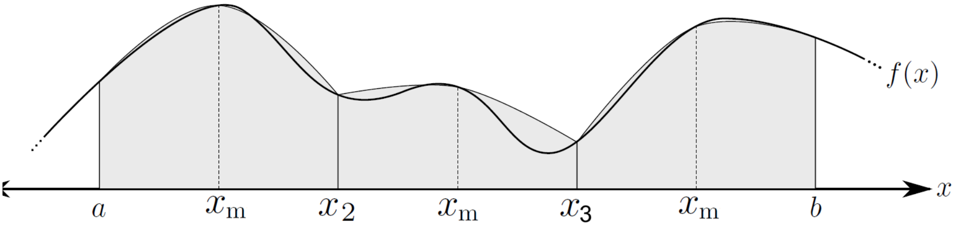


Simpson's rule

Newton–Cotes formulas - Closed Rule

- A **composite** rule for the Simpson's scheme is obtained by summing up smaller **pieces**.
- If we subdivide $[a, b]$ into k intervals, we obtain:

$$\int_a^b f(x)dx \approx \frac{\Delta x}{6} \sum_{i=1}^{k} \left(f(x_i) + 4f\left(\frac{x_{i+1} + x_i}{2}\right) + f(x_{i+1}) \right) \tag{13}$$



Composite Simpson's scheme. 4 samples + 3 samples in the middle-points (x_m).

Monte Carlo integration

- This method was created by the Polish mathematician Stanisław Ulam in order to simulate the behavior of neutrons during the work on the American hydrogen bomb in Los Alamos.
- The name of Monte Carlo refers to the name of the casino in Monaco.
- This method allows to calculate the integral values of functions that can not be determined by standard methods.

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Monte Carlo integration

The next steps of the method are as follows:

- Separate the integration domain using the figure, which area S can be easily computed.
- Select N random points within the selected figure. Let N_1 be the number of points that are under the $f(x)$ curve and N_2 the number of points over the curve (see - fig. below).
- The approximate value of the integral is given by the formula:

$$P = \frac{N_1}{N} \cdot S \tag{14}$$