Q1

$$\begin{aligned}\frac{1}{a+bi} &= \frac{1}{a+bi} * \frac{a-bi}{a-bi}\\ &= \frac{a-bi}{a^2+abi-abi+b^2}\\ &= \frac{a-bi}{a^2+b^2}\\ c+di &= \frac{a}{a^2+b^2} - \frac{bi}{a^2+b^2}\end{aligned}$$

therefore $c = \frac{a}{a^2+b^2}$ and $d = \frac{-b}{a^2+b^2}$

Q2

$$\left(\frac{-1+\sqrt{3}i}{2}\right)^3 = \frac{1}{8} * \left(-1+\sqrt{3}i\right)^3$$

using the identity $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ where $a = -1$ and $b = \sqrt(3)i$ we have

$$\begin{aligned}\frac{1}{8} * \left(-1+\sqrt{3}i\right)^3 &= (-1)^3 + 3*(-1)^2*(\sqrt{3}i) + 3*(-1)*(\sqrt{3}i)^2 + (\sqrt{3}i)^3\\ &= \frac{1}{8} * \left(-1 + 3*\sqrt{3}i + 9 - 3*\sqrt{3}i\right)\\ &= \frac{8}{8}\\ &= 1\end{aligned}$$

Q3

$-\sqrt{i}$ and $+\sqrt{i}$ are two distinct square roots of i

Q4

To show:

$$\alpha + \beta = \beta + \alpha \ \forall \alpha, \beta \in \mathbb{C}$$

let $\alpha = a + bi$ and $beta = c + di$
by definition of addition of complex numbers

$$(a+bi) + (c+di) = (a+c) + (b+d)i \tag{1}$$

by definition of complex numbers we know a,b,c,d $\in \mathbb{R}$
so addition is commutative in $\mathbb{R}$

as a consequence $a+b=b+a$ and $c+d=d+c$
there (1) becomes

$$\begin{aligned}(a+bi)+(c+di) &= (a+c)+(b+d)i \\ &= (c+a)+(d+b)i \\ &= (c+di)+(a+bi) \text{ (from bidirectionality of how complex number addition is defined)} \\ &= \beta+\alpha\end{aligned}$$

■

Q5

To prove:

$$\alpha+(\beta+\gamma)=(\alpha+\beta)+\gamma\ \forall\alpha,\beta,\gamma\in\mathbb{C}$$

Assuming $\alpha=a+bi$, $\beta=c+di$, and $\gamma=e+fi$
evaluating $\alpha+(\beta+\gamma)$

$$\begin{aligned}\alpha+(\beta+\gamma) &= (a+bi)+\Big((c+di)+(e+fi)\Big) \\ &= (a+bi)+\Big((c+e)+(d+f)i\Big) \text{ (From addition of complex numbers)} \\ &= \Big(a+(c+e)\Big)+\Big(b+(d+f)\Big)i \text{ (From addition of complex numbers)} \\ &= \Big((a+c)+e\Big)+\Big((b+d)+f\Big)i \text{ (from associativity of real numbers)} \\ &= \Big((a+c)+(b+d)i\Big)+(e+fi) \text{ (Bidirectionality of addition of complex numbers)} \\ &= \Big((a+bi)+(c+di)\Big)+(e+fi) \text{ (Bidirectionality of addition of complex numers)} \\ &= (\alpha+\beta)+\gamma\end{aligned}$$

■

Q6

To prove:

$$\alpha\,(\beta\gamma)=(\alpha\beta)\,\gamma\ \forall\alpha,\beta,\gamma\in\mathbb{C}$$

Assuming

$$\begin{aligned}\alpha &= a+bi \\ \beta &= c+di \\ \gamma &= e+fi\end{aligned}$$

It will do us well to remember the general formula of complex number multiplication i.e

$$(u+vi)(x+yi)=(ux-vy)+(vx+uy)i$$

Proof:
Evaluating $(\alpha\beta)\gamma$

$$\begin{aligned}
(\alpha\beta)\gamma &= \Big((a+bi)(c+di)\Big)(e+fi) \\
&= \Big((ac-bd)+(ad+cb)i\Big)(e+fi) \\
&= ((ac-bd)e-f(ad+cb))+((ac-bd)f+e(ad+cb))i \\
&= (ace-bde-fad-fcb)+(acf-bdf+ead+ecb)i \\
&= (a(ce-fd)-b(de+fc))+(a(de+fc)+b(ce-fd))i \\
&= (a+bi)((ce-fd)+(de+fc)i) \\
&= (a+bi)\Big((c+di)(e+fi)\Big) \\
&= \alpha(\beta\gamma)
\end{aligned}$$

■

Q7

To prove: $\forall\alpha \in \mathbb{C}$, $\exists\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$
Assumption:

1. if $\alpha \in \mathbb{C}$, then α can be written as $(a + bi)$ where both a and b are real numbers i.e $a, b \in \mathbb{R}$
2. $\forall\alpha \in \mathbb{R}$,$\exists(-\alpha)$ such that $\alpha + (-\alpha) = 0$

Proof:
Let $\alpha \in \mathbb{C}$, so from our assumption it can be written in the form (a + bi)
let $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$, again from our assumption β can be written in the form (x + yi)

$$\begin{aligned}
&(a+bi)+(x+yi) = 0 \\
&(a+x)+(b+y)i = 0 \text{ (from addition of complex numbers)} \\
&(a+x)+(b+y)i = 0+0i \text{ (zero of the complex number field)}
\end{aligned}$$

from the last equation above, we equate the real and imaginary parts of both the L.H.S and the R.H.S to each other, we get

$$\begin{aligned}
a+x &= 0 \\
b+y &= 0
\end{aligned}$$

Using assumption (2) we arrive at $x = -a$ and $y = -b$
so $\beta = (-a + (-b)i) = -(a + bi) = -\alpha$

Proof of uniqueness: (Proof by contradiction)
There exists another unique complex number γ such that $\alpha + \gamma = 0$

$$\alpha + \gamma = 0 \text{ (From assumption above)}$$
$$\alpha + \beta = 0 + \beta \text{ (Adding } \beta \text{ to both sides)}$$
$$0 + \gamma = \beta \text{ (From proven result and concept of 0)}$$
$$\gamma = \beta$$

This disproves our assumption, hence β is unique in that $\forall \alpha \in \mathbb{C}$, $\exists \beta \in \mathbb{C}$ such that $\alpha + \beta = 0$ and β is unique ■

Q8

To prove: $\forall \alpha \in \mathbb{C}$, with $\alpha \neq 0$, $\exists \beta \in \mathbb{C}$, $(\alpha\beta = 1) \wedge (\beta \text{ is unique})$
Assumption:

1. if $\alpha \in \mathbb{C}$, then α can be written as $(a + bi)$ where both a and b are real numbers i.e $a, b \in \mathbb{R}$

Proof:
For a general complex number α which can represented as $a + bi$ (from assumption) we will try to find a complex number β such that $\alpha\beta = 1$ whose components (real and imaginary) can be expressed in terms α's real and imaginary components (i.e a and b)
We take $\beta = c + di$

$$\alpha\beta = 1$$
$$(a + bi)(c + di) = 1$$
$$(ac - bd) + (ad + bc)i = 1 + 0i \text{ from multiplication of two complex numbers}$$

Equating the real and complex parts of the L.H.S to the R.H.S

$$ac - bd = 1 \tag{2}$$
$$ad + bc = 0 \tag{3}$$

Multiplying (1) with -b and (2) with a we have

$$\begin{array}{rrr} & -abc & +b^2d = -b \\ + & abc & +a^2d = 0 \\ \hline \end{array}$$

$$(a^2 + b^2)d = -b$$

and therefore we have $d = \frac{-b}{a^2+b^2}$ and if we plug this into (2) we have

$$\begin{aligned} ac &= 1 + bd \text{ (from (2))} \\ ac &= 1 + b(\frac{-b}{a^2+b^2}) \\ ac &= 1 - \frac{b^2}{a^2+b^2} \\ c &= \frac{a^2}{a(a^2+b^2)} \\ c &= \frac{a}{a^2+b^2} \end{aligned}$$

Therefore $\forall \alpha \in \mathbb{C}$, with $\alpha \neq 0$, $\exists \beta \in \mathbb{C}$, $\alpha\beta = 1$ where $\beta = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$
Proof of uniqueness (proof by contradiction):
Let's assume there exists another complex number other than β, γ such that $\alpha\gamma = 1$

$$\begin{aligned} \alpha\gamma &= 1 \text{ (from assumption)} \\ (\alpha\gamma)\beta &= \beta \text{ (multiplying both sides with } \beta) \\ (\gamma\alpha)\beta &= \beta \text{ (commutativity of multiplication on } \mathbb{C}) \\ \gamma(\alpha\beta) &= \beta \text{ (associativity of multiplication on } \mathbb{C}) \\ \gamma.1 &= \beta \text{ (from above construction of beta such } \alpha\beta = 1) \\ \gamma &= \beta \text{ (multiplicative identity defined on } \mathbb{C}) \end{aligned}$$

disproving our assumption and hence β is unique ■

Q9

To prove: $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$, $\forall \lambda, \beta, \gamma \in \mathbb{C}$
Assumption:

1. if $\alpha \in \mathbb{C}$, then α can be written as $(a + bi)$ where both a and b are real numbers i.e $a, b \in \mathbb{R}$

Proof:
Let

$$\begin{aligned} \alpha &= a + bi \\ \beta &= c + di \\ \lambda &= x + yi \end{aligned}$$

Evaluation $\lambda(\alpha+\beta)$

$$\begin{aligned}
\lambda(\alpha+\beta) &= (x+yi)\Big((a+bi)+(c+di)\Big) \\
&= (x+yi)\Big((a+c)+(b+d)i\Big) \text{ (from addition defined on } \mathbb{C}) \\
&= \Big(x(a+c)-y(b+d)\Big)+\Big(x(b+d)+y(a+c)\Big)i \text{ (from multiplication defined on } \mathbb{C}) \\
&= (xa+cd-yb-yd)+(xb+xd+ya+yc)i \\
&= (xa-yb)+(xb+ya)i+(xc-yd)+(xd+yc)i \\
&= (x+yi)(a+bi)+(x+yi)(c+di) \\
&= \lambda\alpha+\lambda\beta
\end{aligned}$$

■

Q10

Find: x $\in \mathbb{R}^4$ such that $(4,-3,1,7)+2x=(5,9,-6,8)$
Solution:
because x $\in \mathbb{R}^4$ we can assume x to be of form (a,b,c,d) and we get the equation

$$(4,-3,1,7)+2(a,b,c,d)=(5,9,-6,8)$$

we get 4 equations

$$\begin{aligned}
4+2a&=5\\
3+2b&=9\\
1+2c&=-6\\
7+2d&=8
\end{aligned}$$

and

$$\begin{aligned}
a&=\frac{1}{2}\\
b&=6\\
c&=-\frac{7}{2}\\
d&=\frac{1}{2}
\end{aligned}$$

Q11

To show $\lambda \notin \mathbb{C}$ such that $\lambda(2-3i,5+4i,-6+7i)=(12-5i,7+22i,-32-9i)$
Assumption:

1. λ is of form (a+bi) because $lambda \in \mathbb{C}$

Because we have scalar multiplication defined on $\mathbb{F}^n$
$\lambda(2-3i, 5+4i, -6+7i) = (12-5i, 7+22i, -32-9i)$ becomes
$\Big(\lambda(2-3i), \lambda(5+4i), \lambda(-6+7i)\Big) = (12-5i, 7+22i, -32-9i)$
so because of that we three equations

$$\lambda(2-3i) = 12-5i \tag{4}$$
$$\lambda(5+4i) = 7+22i \tag{5}$$
$$\lambda(-6+7i) = -32-9i \tag{6}$$

From (4) we have

$$\lambda(2-3i) = 12-5i$$
$$(a+bi)(2-3i) = 12-5i \text{ (definition of } \lambda)$$
$$(2a+3b)+(-3a+2b)i = 12+5i \text{ (multiplication defined on } \mathbb{C})$$

Equating the real and imaginary parts of the last equation we get

$$2a+3b = 12 \tag{7}$$
$$-3a+2b = -5 \tag{8}$$

Multiplying (7) with 3 and (8) with 2 we have

$$\begin{array}{rl} & 6a \ +9b = 36 \\ + & -6a+4b = -10 \\ \hline \end{array}$$

$$13b = 26$$

hence b = 2 , a=3 and $\lambda = (2+3i)$
substituting the value of λ in (6) we have

$$\begin{aligned}(2+3i)(-6+7i) &= (-12-21)+(14-18)i \\ &= -33-4i \neq -32+9i\end{aligned}$$

hence there cannot be $\lambda \notin \mathbb{C}$ such that $\lambda(2-3i, 5+4i, -6+7i) = (12-5i, 7+22i, -32-9i)$ ■