Q1

$$\begin{aligned}\frac{1}{a+bi} &= \frac{1}{a+bi} * \frac{a-bi}{a-bi}\\ &= \frac{a-bi}{a^2+abi-abi+b^2}\\ &= \frac{a-bi}{a^2+b^2}\\ c+di &= \frac{a}{a^2+b^2} - \frac{bi}{a^2+b^2}\end{aligned}$$

therefore $c = \frac{a}{a^2+b^2}$ and $d = \frac{-b}{a^2+b^2}$

Q2

$$\left(\frac{-1+\sqrt{3}i}{2}\right)^3 = \frac{1}{8} * \left(-1+\sqrt{3}i\right)^3$$

using the identity $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ where $a = -1$ and $b = \sqrt{(}3)i$ we have

$$\begin{aligned}\frac{1}{8} * \left(-1+\sqrt{3}i\right)^3 &= (-1)^3 + 3*(-1)^2*(\sqrt{3}i) + 3*(-1)*(\sqrt{3}i)^2 + (\sqrt{3}i)^3\\ &= \frac{1}{8} * \left(-1 + 3*\sqrt{3}i + 9 - 3*\sqrt{3}i\right)\\ &= \frac{8}{8}\\ &= 1\end{aligned}$$

Q3

$-\sqrt{i}$ and $+\sqrt{i}$ are two distinct square roots of i

Q4

To show:

$$\alpha + \beta = \beta + \alpha \ \forall \alpha, \beta \in \mathbb{C}$$

let $\alpha = a + bi$ and $beta = c + di$
by definition of addition of complex numbers

$$(a+bi) + (c+di) = (a+c) + (b+d)i \tag{1}$$

by definition of complex numbers we know a,b,c,d $\in \mathbb{R}$
so addition is commutative in $\mathbb{R}$

as a consequence $a+b=b+a$ and $c+d=d+c$
there (1) becomes

$$
\begin{aligned}
(a+bi)+(c+di) &= (a+c)+(b+d)i \\
&= (c+a)+(d+b)i \\
&= (c+di)+(a+bi) \text{ (from bidirectionality of how complex number addition is defined)} \\
&= \beta+\alpha
\end{aligned}
$$

■

Q5

To prove:

$$\alpha+(\beta+\gamma)=(\alpha+\beta)+\gamma \ \forall \alpha,\beta,\gamma \in \mathbb{C}$$

Assuming $\alpha=a+bi$, $\beta=c+di$, and $\gamma=e+fi$
evaluating $\alpha+(\beta+\gamma)$

$$
\begin{aligned}
\alpha+(\beta+\gamma) &= (a+bi)+\Big((c+di)+(e+fi)\Big) \\
&= (a+bi)+\Big((c+e)+(d+f)i\Big) \text{ (From addition of complex numbers)} \\
&= \Big(a+(c+e)\Big)+\Big(b+(d+f)\Big)i \text{ (From addition of complex numbers)} \\
&= \Big((a+c)+e\Big)+\Big((b+d)+f\Big)i \text{ (from associativity of real numbers)} \\
&= \Big((a+c)+(b+d)i\Big)+(e+fi) \text{ (Bidirectionality of addition of complex numbers)} \\
&= \Big((a+bi)+(c+di)\Big)+(e+fi) \text{ (Bidirectionality of addition of complex numers)} \\
&= (\alpha+\beta)+\gamma
\end{aligned}
$$

■

Q6

To prove:

$$\alpha\left(\beta\gamma\right)=\left(\alpha\beta\right)\gamma \ \forall \alpha,\beta,\gamma \in \mathbb{C}$$

Assuming

$$
\begin{aligned}
\alpha &= a+bi \\
\beta &= c+di \\
\gamma &= e+fi
\end{aligned}
$$

It will do us well to remember the general formula of complex number multiplication i.e

$$(u+vi)(x+yi)=(ux-vy)+(vx+uy)i$$

Proof:
Evaluating $(\alpha\beta)\gamma$

$$\begin{aligned}
(\alpha\beta)\gamma &= \Big((a+bi)(c+di)\Big)(e+fi) \\
&= \Big((ac-bd)+(ad+cb)i\Big)(e+fi) \\
&= ((ac-bd)e - f(ad+cb)) + ((ac-bd)f + e(ad+cb))i \\
&= (ace - bde - fad - fcb) + (acf - bdf + ead + ecb)i \\
&= (a(ce-fd) - b(de+fc)) + (a(de+fc) + b(ce-fd))i \\
&= (a+bi)((ce-fd)+(de+fc)i) \\
&= (a+bi)\Big((c+di)(e+fi)\Big) \\
&= \alpha(\beta\gamma)
\end{aligned}$$

■

Q7

To prove: $\forall \alpha \in \mathbb{C}$, $\exists \beta \in \mathbb{C}$ such that $\alpha + \beta = 0$
Assumption:

1. if $\alpha \in \mathbb{C}$, then α can be written as $(a + bi)$ where both a and b are real numbers i.e $a, b \in \mathbb{R}$
2. $\forall \alpha \in \mathbb{R}$,$\exists(-\alpha)$ such that $\alpha + (-\alpha) = 0$

Proof:
Let $\alpha \in \mathbb{C}$, so from our assumption it can be written in the form (a + bi)
let $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$, again from our assumption β can be written in the form (x + yi)

$$\begin{aligned}
&(a+bi) + (x+yi) = 0 \\
&(a+x) + (b+y)i = 0 \text{ (from addition of complex numbers)} \\
&(a+x) + (b+y)i = 0 + 0i \text{ (zero of the complex number field)}
\end{aligned}$$

from the last equation above, we equate the real and imaginary parts of both the L.H.S and the R.H.S to each other, we get

$$\begin{aligned}
a + x &= 0 \\
b + y &= 0
\end{aligned}$$

Using assumption (2) we arrive at $x = -a$ and $y = -b$
so $\beta = (-a + (-b)i) = -(a + bi) = -\alpha$

Proof of uniqueness: (Proof by contradiction)
There exists another unique complex number γ such that $\alpha + \gamma = 0$

$$\alpha + \gamma = 0 \text{ (From assumption above)}$$
$$\alpha + \beta = 0 + \beta \text{ (Adding } \beta \text{ to both sides)}$$
$$0 + \gamma = \beta \text{ (From proven result and concept of 0)}$$
$$\gamma = \beta$$

This disproves our assumption, hence β is unique in that $\forall \alpha \in \mathbb{C}$, $\exists \beta \in \mathbb{C}$ such that $\alpha + \beta = 0$ and β is unique ■

Q8

To prove: $\forall \alpha \in \mathbb{C}$, with $\alpha \neq 0$, $\exists \beta \in \mathbb{C}$, $(\alpha\beta = 1) \land (\beta \text{ is unique})$
Assumption:

1. if $\alpha \in \mathbb{C}$, then α can be written as $(a + bi)$ where both a and b are real numbers i.e $a, b \in \mathbb{R}$

Proof:
For a general complex number α which can represented as $a + bi$ (from assumption) we will try to find a complex number β such that $\alpha\beta = 1$ whose components (real and imaginary) can be expressed in terms α's real and imaginary components (i.e a and b)
We take $\beta = c + di$

$$\alpha\beta = 1$$
$$(a + bi)(c + di) = 1$$
$$(ac - bd) + (ad + bc)i = 1 + 0i \text{ from multiplication of two complex numbers}$$

Equating the real and complex parts of the L.H.S to the R.H.S

$$ac - bd = 1 \tag{2}$$
$$ad + bc = 0 \tag{3}$$

Multiplying (1) with -b and (2) with a we have

$$\begin{array}{rrr} & -abc & +b^2d = -b \\ + & abc & +a^2d = 0 \\ \hline \end{array}$$

$$(a^2 + b^2)d = -b$$

and therefore we have $d = \frac{-b}{a^2+b^2}$ and if we plug this into (2) we have

$$\begin{aligned}
ac &= 1 + bd \text{ (from (2))} \\
ac &= 1 + b(\frac{-b}{a^2+b^2}) \\
ac &= 1 - \frac{b^2}{a^2+b^2} \\
c &= \frac{a^2}{a(a^2+b^2)} \\
c &= \frac{a}{a^2+b^2}
\end{aligned}$$

Therefore $\forall \alpha \in \mathbb{C}$, with $\alpha \neq 0$, $\exists \beta \in \mathbb{C}$, $\alpha\beta = 1$ where $\beta = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2} i$
Proof of uniqueness (proof by contradiction):
Let's assume there exists another complex number other than β, γ such that $\alpha\gamma = 1$

$$\begin{aligned}
\alpha\gamma &= 1 \text{ (from assumption)} \\
(\alpha\gamma)\beta &= \beta \text{ (multiplying both sides with } \beta) \\
(\gamma\alpha)\beta &= \beta \text{ (commutativity of multiplication on } \mathbb{C}) \\
\gamma(\alpha\beta) &= \beta \text{ (associativity of multiplication on } \mathbb{C}) \\
\gamma.1 &= \beta \text{ (from above construction of beta such } \alpha\beta = 1) \\
\gamma &= \beta \text{ (multiplicative identity defined on } \mathbb{C})
\end{aligned}$$

disproving our assumption and hence β is unique $\blacksquare$