Q1

$$\begin{aligned}
\frac{1}{a+bi} &= \frac{1}{a+bi} * \frac{a-bi}{a-bi} \\
&= \frac{a-bi}{a^2+abi-abi+b^2} \\
&= \frac{a-bi}{a^2+b^2} \\
c+di &= \frac{a}{a^2+b^2} - \frac{bi}{a^2+b^2}
\end{aligned}$$

therefore $c = \frac{a}{a^2+b^2}$ and $d = \frac{-b}{a^2+b^2}$

Q2

$$\left(\frac{-1+\sqrt{3}i}{2}\right)^3 = \frac{1}{8} * \left(-1+\sqrt{3}i\right)^3$$

using the identity $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ where $a = -1$ and $b = \sqrt(3)i$ we have

$$\begin{aligned}
\frac{1}{8} * \left(-1+\sqrt{3}i\right)^3 &= (-1)^3 + 3*(-1)^2*(\sqrt{3}i) + 3*(-1)*(\sqrt{3}i)^2 + (\sqrt{3}i)^3 \\
&= \frac{1}{8} * \left(-1 + 3*\sqrt{3}i + 9 - 3*\sqrt{3}i\right) \\
&= \frac{8}{8} \\
&= 1
\end{aligned}$$

Q3

$-\sqrt{i}$ and $+\sqrt{i}$ are two distinct square roots of i

Q4

To show:

$$\alpha + \beta = \beta + \alpha \ \forall \alpha, \beta \in \mathbb{C}$$

let $\alpha = a + bi$ and $beta = c + di$
by definition of addition of complex numbers

$$(a+bi) + (c+di) = (a+c) + (b+d)i \tag{1}$$

by definition of complex numbers we know a,b,c,d $\in \mathbb{R}$
so addition is commutative in $\mathbb{R}$

as a consequence $a+b=b+a$ and $c+d=d+c$
there (1) becomes

$$\begin{aligned}(a+bi)+(c+di) &= (a+c)+(b+d)i \\ &= (c+a)+(d+b)i \\ &= (c+di)+(a+bi) \text{ (from bidirectionality of how complex number addition is defined)} \\ &= \beta+\alpha\end{aligned}$$

$\blacksquare$

Q5

To prove:

$$\alpha+(\beta+\gamma)=(\alpha+\beta)+\gamma \ \forall \alpha,\beta,\gamma \in \mathbb{C}$$

Assuming $\alpha=a+bi$, $\beta=c+di$, and $\gamma=e+fi$
evaluating $\alpha+(\beta+\gamma)$

$$\begin{aligned}\alpha+(\beta+\gamma) &= (a+bi)+\Big((c+di)+(e+fi)\Big) \\ &= (a+bi)+\Big((c+e)+(d+f)i\Big) \text{ (From addition of complex numbers)} \\ &= \Big(a+(c+e)\Big)+\Big(b+(d+f)\Big)i \text{ (From addition of complex numbers)} \\ &= \Big((a+c)+e\Big)+\Big((b+d)+f\Big)i \text{ (from associativity of real numbers)} \\ &= \Big((a+c)+(b+d)i\Big)+(e+fi) \text{ (Bidirectionality of addition of complex numbers)} \\ &= \Big((a+bi)+(c+di)\Big)+(e+fi) \text{ (Bidirectionality of addition of complex numers)} \\ &= (\alpha+\beta)+\gamma\end{aligned}$$

$\blacksquare$

Q6

Q7

To prove: $\forall \alpha \in \mathbb{C}$, $\exists \beta \in \mathbb{C}$ such that $\alpha+\beta=0$
Assumption:

1. if $\alpha \in \mathbb{C}$, then α can be written as $(a+bi)$ where both a and b are real numbers i.e $a,b \in \mathbb{R}$
2. $\forall \alpha \in \mathbb{R}, \exists(-\alpha)$ such that $\alpha+(-\alpha)=0$

Proof:
Let $\alpha \in \mathbb{C}$, so from our assumption it can be written in the form (a + bi)
let $\beta \in \mathbb{C}$ such that $\alpha+\beta=0$, again from our assumption β can be written in

the form (x + yi)

$$(a+bi)+(x+yi)=0$$
$$(a+x)+(b+y)i=0 \text{ (from addition of complex numbers)}$$
$$(a+x)+(b+y)i=0+0i \text{ (zero of the complex number field)}$$

from the last equation above, we equate the real and imaginary parts of both the L.H.S and the R.H.S to each other, we get

$$a+x=0$$
$$b+y=0$$

Using assumption (2) we arrive at $x=-a$ and $y=-b$
so $\beta=(-a+(-b)i)=-(a+bi)=-\alpha$

Proof of uniqueness: (Proof by contradiction)
There exists another unique complex number γ such that $\alpha+\gamma=0$

$$\alpha+\gamma=0 \text{ (From assumption above)}$$
$$\alpha+\beta=0+\beta \text{ (Adding } \beta \text{ to both sides)}$$
$$0+\gamma=\beta \text{ (From proven result and concept of 0)}$$
$$\gamma=\beta$$

This disproves our assumption, hence β is unique in that $\forall\alpha\in\mathbb{C}$, $\exists\beta\in\mathbb{C}$ such that $\alpha+\beta=0$ and β is unique