Law of total expectation

Assume that we have two random variables X and Y, and that both are discrete and finite, and that $E(X)$ exists.

The law is: $E(X) = E(E(X|Y))$

We show this:

$$E(X) = \sum_{\forall x} x \, P(X=x) = \sum_{\forall x} x \sum_{\forall y} P(X=x, Y=y) = \sum_{\forall x} x \sum_{\forall y} P(X=x|Y=y) \cdot P(Y=y)$$

$$\overset{*}{=} \sum_{\forall y} \underbrace{\left(\sum_{\forall x} x \cdot P(X=x|Y=y) \right)}_{E(X|Y)} P(Y=y)$$

$$= \sum_{\forall y} \underbrace{E(X|Y)}_{\substack{\text{RV} \\ \text{function of } Y}} \cdot P(Y=y) = \underset{Y}{E}\big(\underset{X|Y}{E(X|Y)} \big)$$

What if we have two continuous RV's and that $f(x)$ and $f(y)$ exists, will the derivation also be ok?

$*$ requires that we can change order of $\int_x$ and $\int_y$ which is ok if Fubini's theorem holds (compute a double integral by using an iterated integral).

Law of total venence

$$Var(X) = E(Var(X|Y)) + Var(E(X|Y))$$

We start by remembering that - for the marginal distr. of X

$Var(X) = E(X^2) - (E(X))^2$ from our intro course.

This means that we have the same formula for the conditional distribution of X given Y.

$$Var(X|Y) = E(X^2|Y) - (E(X|Y))^2$$

trick: just take E on both sides

$$E(Var(X|Y)) = \underbrace{E(E(X^2|Y))}_{\substack{E(X^2) \\ \text{using the law of} \\ \text{total } E}} - E(E(X|Y)^2)$$

then get Var into play by $E(X^2) = Var(X) + E(X)^2$, so we add and subtract $E(X)^2$

$$= \underbrace{(E(X^2) - E(X)^2)}_{Var(X)} - \Big(E(E(X|Y)^2) - \underbrace{E(X)^2}_{\substack{[E(E(X|Y))]^2 \\ \text{use the law of tot } E \text{ agen}}}\Big)$$

$$= Var(X) - \Big(\underbrace{E(E(X|Y)^2) - E(E(X|Y))^2}_{Var(E(X|Y)) \text{ by def}}\Big)$$

Rearranging gives — the law of total variance.

$$Var(X) = E(Var(X|Y)) + Var(E(X|Y))$$

Do you also want to look at
The law of total Covariance?

$$Cov(X,Y) = E(Cov(X,Y|Z)) + Cov(E(X|Z), E(Y|Z))$$

see if you may do that by using the definition of covariance?

$$Cov(X,Y) = E(X \cdot Y) - E(X) \cdot E(Y)$$ ← notice this is a "generalization" of the variance $Cov(X,X)$?