Scattering Parameter

Bernd Filipitsch

Relevant Literature
Sophocles J. Orfanidis: *„Electromagnetic Waves and Antennas“*; www.ece.rutgers.edu/~orfanidis/eva

Scattering Parameters

Linear two-port (and multi-port) networks are characterized by a number of equivalent circuit parameters, such as their transfer matrix, impedance matrix, admittance matrix, and scattering matrix. Fig. 12.1.1 shows a typical two-port network.

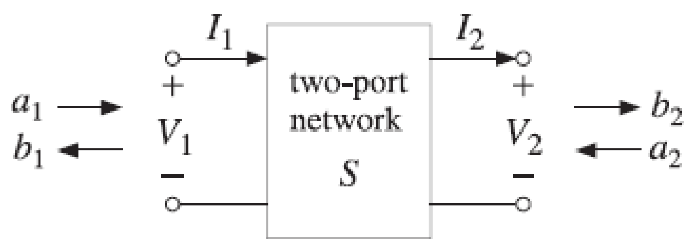

Fig. 12.1.1 Two-port network.

The transfer matrix, also known as the ABCD matrix, relates the voltage and current at port 1 to those at port 2, whereas the impedance matrix relates the two voltages *V1,V2* to the two currents *I1, I2*:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \quad \text{(transfer matrix)}$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ -I_2 \end{bmatrix} \quad \text{(impedance matrix)} \qquad (12.1.1)$$

Scattering Parameters

Thus, the transfer and impedance matrices are the 2×2 matrices:

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \qquad Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \qquad (12.1.2)$$

The admittance matrix is simply the inverse of the impedance matrix, $Y = Z^{-1}$.

The scattering matrix relates the outgoing waves b_1, b_2 to the incoming waves a_1, a_2 that are incident on the two-port:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \quad \text{(scattering matrix)} \qquad (12.1.3)$$

The matrix elements S_{11}, S_{12}, S_{21}, S_{22} are referred to as the *scattering parameters* or the S-parameters. The parameters S_{11}, S_{22} have the meaning of reflection coefficients, and S_{21}, S_{12}, the meaning of transmission coefficients.

Scattering Parameters

The traveling wave variables a1, b1 at port 1 and a2, b2 at port 2 are defined in terms of V_1, I_1 and V_2, I_2 and a real-valued positive reference impedance Z_0 as follows:

$$\boxed{\begin{aligned} a_1 &= \frac{V_1 + Z_0 I_1}{2\sqrt{Z_0}} \\ b_1 &= \frac{V_1 - Z_0 I_1}{2\sqrt{Z_0}} \end{aligned}} \qquad \boxed{\begin{aligned} a_2 &= \frac{V_2 - Z_0 I_2}{2\sqrt{Z_0}} \\ b_2 &= \frac{V_2 + Z_0 I_2}{2\sqrt{Z_0}} \end{aligned}} \qquad \text{(traveling waves)} \qquad (12.1.4)$$

The definitions at port 2 appear different from those at port 1, but they are really the same if expressed in terms of the incoming current $-I_2$:

$$a_2 = \frac{V_2 - Z_0 I_2}{2\sqrt{Z_0}} = \frac{V_2 + Z_0(-I_2)}{2\sqrt{Z_0}}$$

$$b_2 = \frac{V_2 + Z_0 I_2}{2\sqrt{Z_0}} = \frac{V_2 - Z_0(-I_2)}{2\sqrt{Z_0}}$$

Scattering Parameters

The term traveling waves is justified below. Eqs. (12.1.4) may be inverted to express the voltages and currents in terms of the wave variables:

$$\boxed{\begin{aligned} V_1 &= \sqrt{Z_0}(a_1 + b_1) \\ I_1 &= \frac{1}{\sqrt{Z_0}}(a_1 - b_1) \end{aligned}} \qquad \boxed{\begin{aligned} V_2 &= \sqrt{Z_0}(a_2 + b_2) \\ I_2 &= \frac{1}{\sqrt{Z_0}}(b_2 - a_2) \end{aligned}} \qquad (12.1.5)$$

In practice, the reference impedance is chosen to be Z0 = 50 ohm. At lower frequencies the transfer and impedance matrices are commonly used, but at microwave frequencies they become difficult to measure and therefore, the scattering matrix description is preferred. The S-parameters can be measured by embedding the two-port network (the device under test, or, DUT) in a transmission line whose ends are connected to a network analyzer. Fig. 12.1.2 shows the experimental setup.

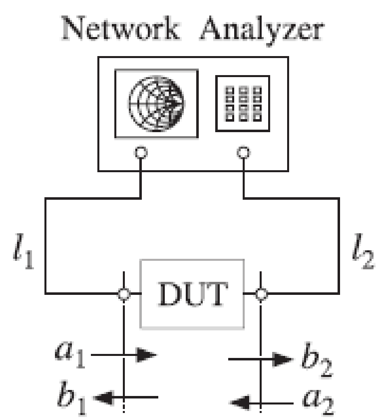

Fig. 12.1.2 Device under test connected to network analyzer.

Scattering Parameters

Fig. 12.1.3 shows more details of the connection. The generator and load impedances are configured by the network analyzer. The connections can be reversed, with the generator connected to port 2 and the load to port 1.

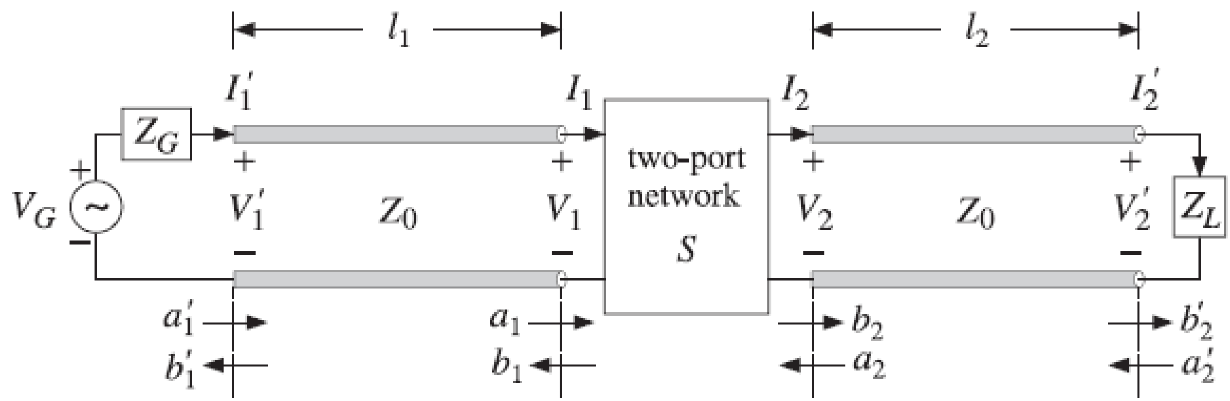

Fig. 12.1.3 Two-port network under test.

Scattering Parameters

The two line segments of lengths l_1, l_2 are assumed to have characteristic impedance equal to the reference impedance Z_0. Then, the wave variables a_1, b_1 and a_2, b_2 are recognized as normalized versions of forward and backward traveling waves.
We have:

$$
\begin{aligned}
a_1 &= \frac{V_1 + Z_0 I_1}{2\sqrt{Z_0}} = \frac{1}{\sqrt{Z_0}} V_{1+} \qquad & a_2 &= \frac{V_2 - Z_0 I_2}{2\sqrt{Z_0}} = \frac{1}{\sqrt{Z_0}} V_{2-} \\
b_1 &= \frac{V_1 - Z_0 I_1}{2\sqrt{Z_0}} = \frac{1}{\sqrt{Z_0}} V_{1-} \qquad & b_2 &= \frac{V_2 + Z_0 I_2}{2\sqrt{Z_0}} = \frac{1}{\sqrt{Z_0}} V_{2+}
\end{aligned}
\tag{12.1.6}
$$

Thus, a_1 is essentially the incident wave at port 1 and b_1 the corresponding reflected wave. Similarly, a_2 is incident from the right onto port 2 and b_2 is the reflected wave from port 2.

Scattering Parameters

The network analyzer measures the waves a_1', b_1' and a_2', b_2' at the generator and load ends of the line segments, as shown in Fig. 12.1.3. From these, the waves at the inputs of the two-port can be determined.

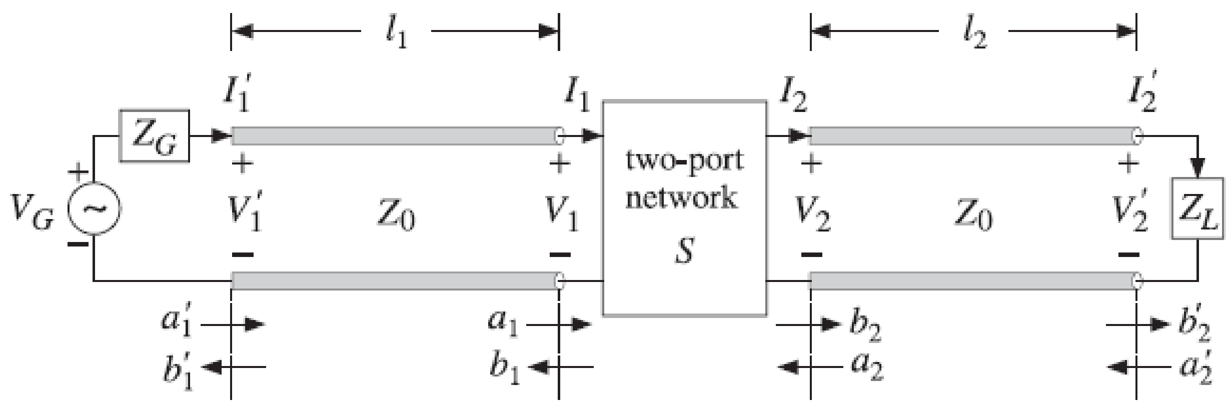

Fig. 12.1.3 Two-port network under test.

Scattering Parameters

$$\begin{bmatrix} V_{1+} \\ V_{1-} \end{bmatrix} = \begin{bmatrix} e^{j\beta l} & 0 \\ 0 & e^{-j\beta l} \end{bmatrix} \begin{bmatrix} V_{2+} \\ V_{2-} \end{bmatrix} \quad \text{(propagation matrix)} \tag{9.7.7}$$

Assuming lossless segments and using the propagation matrices 9.7.7 shown above, we have:

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} e^{-j\delta_1} & 0 \\ 0 & e^{j\delta_1} \end{bmatrix} \begin{bmatrix} a_1' \\ b_1' \end{bmatrix}, \qquad \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} e^{-j\delta_2} & 0 \\ 0 & e^{j\delta_2} \end{bmatrix} \begin{bmatrix} a_2' \\ b_2' \end{bmatrix} \tag{12.1.7}$$

where $\delta_1 = \beta l_l$ and $\delta_2 = \beta l_2$ are the phase lengths of the segments. Eqs. (12.1.7) can be rearranged into the forms:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = D \begin{bmatrix} b_1' \\ b_2' \end{bmatrix}, \quad \begin{bmatrix} a_1' \\ a_2' \end{bmatrix} = D \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad D = \begin{bmatrix} e^{j\delta_1} & 0 \\ 0 & e^{j\delta_2} \end{bmatrix}$$

Scattering Parameters

The network analyzer measures the corresponding S-parameters of the primed variables, that is,

$$\begin{bmatrix} b_1' \\ b_2' \end{bmatrix} = \begin{bmatrix} S_{11}' & S_{12}' \\ S_{21}' & S_{22}' \end{bmatrix} \begin{bmatrix} a_1' \\ a_2' \end{bmatrix}, \quad S' = \begin{bmatrix} S_{11}' & S_{12}' \\ S_{21}' & S_{22}' \end{bmatrix} \quad \text{(measured } S\text{-matrix)} \qquad (12.1.8)$$

The S-matrix of the two-port can be obtained then from:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = D \begin{bmatrix} b_1' \\ b_2' \end{bmatrix} = DS' \begin{bmatrix} a_1' \\ a_2' \end{bmatrix} = DS'D \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \Rightarrow \quad S = DS'D$$

or, more explicitly,

$$\begin{aligned} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} &= \begin{bmatrix} e^{j\delta_1} & 0 \\ 0 & e^{j\delta_2} \end{bmatrix} \begin{bmatrix} S_{11}' & S_{12}' \\ S_{21}' & S_{22}' \end{bmatrix} \begin{bmatrix} e^{j\delta_1} & 0 \\ 0 & e^{j\delta_2} \end{bmatrix} \\ &= \begin{bmatrix} S_{11}' e^{2j\delta_1} & S_{12}' e^{j(\delta_1+\delta_2)} \\ S_{21}' e^{j(\delta_1+\delta_2)} & S_{22}' e^{2j\delta_2} \end{bmatrix} \end{aligned} \qquad (12.1.9)$$

Scattering Parameters

Thus, changing the points along the transmission lines at which the S-parameters are measured introduces only phase changes in the parameters.
Without loss of generality, we may replace the extended circuit of Fig. 12.1.3 with the one shown in Fig. 12.1.4 with the understanding that either we are using the extended two-port parameters S, or, equivalently, the generator and segment l_1 have been replaced
by their Thevenin equivalents, and the load impedance has been replaced by its propagated version to distance l_2.

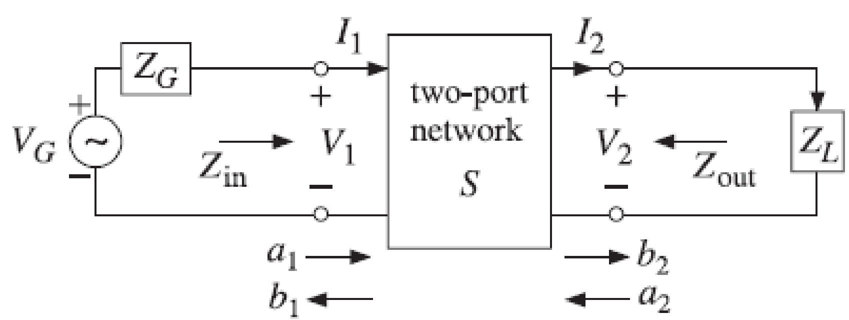

Fig. 12.1.4 Two-port network connected to generator and load.

Scattering Parameters

The actual measurements of the S-parameters are made by connecting to a matched load, $Z_L = Z_0$. Then, there will be no reflected waves from the load, $a_2 = 0$, and the S-matrix equations will give:

$$b_1 = S_{11}a_1 + S_{12}a_2 = S_{11}a_1 \quad \Rightarrow \quad S_{11} = \left.\frac{b_1}{a_1}\right|_{Z_L=Z_0} = \text{reflection coefficient}$$

$$b_2 = S_{21}a_1 + S_{22}a_2 = S_{21}a_1 \quad \Rightarrow \quad S_{21} = \left.\frac{b_2}{a_1}\right|_{Z_L=Z_0} = \text{transmission coefficient}$$

Reversing the roles of the generator and load, one can measure in the same way the parameters S_{12} and S_{22}.

Power Flow

Power flow into and out of the two-port is expressed very simply in terms of the traveling wave amplitudes. Using the inverse relationships (12.1.5), we find:

$$
\begin{aligned}
\frac{1}{2}\,\mathrm{Re}[V_1^* I_1] &= \frac{1}{2}|a_1|^2 - \frac{1}{2}|b_1|^2 \\
-\frac{1}{2}\,\mathrm{Re}[V_2^* I_2] &= \frac{1}{2}|a_2|^2 - \frac{1}{2}|b_2|^2
\end{aligned}
\tag{12.2.1}
$$

The left-hand sides represent the power flow *into* ports 1 and 2. The right-hand sides represent the difference between the power incident on a port and the power reflected from it. The quantity $\mathrm{Re}[V_2^* I_2]/2$ represents the power transferred to the load.

Another way of phrasing these is to say that part of the incident power on a port gets reflected and part enters the port:

$$
\begin{aligned}
\frac{1}{2}|a_1|^2 &= \frac{1}{2}|b_1|^2 + \frac{1}{2}\,\mathrm{Re}[V_1^* I_1] \\
\frac{1}{2}|a_2|^2 &= \frac{1}{2}|b_2|^2 + \frac{1}{2}\,\mathrm{Re}[V_2^* (-I_2)]
\end{aligned}
\tag{12.2.2}
$$

Power Flow

One of the reasons for normalizing the traveling wave amplitudes by $\sqrt{Z_0}$ in the definitions (12.1.4) was precisely this simple way of expressing the incident and reflected powers from a port.

If the two-port is lossy, the power lost in it will be the difference between the power entering port 1 and the power leaving port 2, that is,

$$P_{\text{loss}} = \frac{1}{2}\text{Re}[V_1^* I_1] - \frac{1}{2}\text{Re}[V_2^* I_2] = \frac{1}{2}|a_1|^2 + \frac{1}{2}|a_2|^2 - \frac{1}{2}|b_1|^2 - \frac{1}{2}|b_2|^2$$

Noting that $\mathbf{a}^\dagger\mathbf{a} = |a_1|^2 + |a_2|^2$ and $\mathbf{b}^\dagger\mathbf{b} = |b_1|^2 + |b_2|^2$, and writing $\mathbf{b}^\dagger\mathbf{b} = \mathbf{a}^\dagger S^\dagger S\mathbf{a}$, we may express this relationship in terms of the scattering matrix:

$$P_{\text{loss}} = \frac{1}{2}\mathbf{a}^\dagger\mathbf{a} - \frac{1}{2}\mathbf{b}^\dagger\mathbf{b} = \frac{1}{2}\mathbf{a}^\dagger\mathbf{a} - \frac{1}{2}\mathbf{a}^\dagger S^\dagger S\mathbf{a} = \frac{1}{2}\mathbf{a}^\dagger (I - S^\dagger S)\mathbf{a} \qquad (12.2.3)$$

For a lossy two-port, the power loss is positive, which implies that the matrix $I - S^\dagger S$ must be positive definite. If the two-port is lossless, $P_{\text{loss}} = 0$, the S-matrix will be *unitary*, that is, $S^\dagger S = I$.

Parameter Conversions

It is straightforward to derive the relationships that allow one to pass from one parameter set to another. For example, starting with the transfer matrix, we have:

$$\begin{aligned} V_1 &= AV_2 + BI_2 \\ I_1 &= CV_2 + DI_2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} V_1 &= A\left(\frac{1}{C}I_1 - \frac{D}{C}I_2\right) + BI_2 = \frac{A}{C}I_1 - \frac{AD - BC}{C}I_2 \\ V_2 &= \frac{1}{C}I_1 - \frac{D}{C}I_2 \end{aligned}$$

The coefficients of I_1, I_2 are the impedance matrix elements. The steps are reversible, and we summarize the final relationships below:

$$\begin{aligned} Z &= \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \frac{1}{C}\begin{bmatrix} A & AD - BC \\ 1 & D \end{bmatrix} \\ T &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \frac{1}{Z_{21}}\begin{bmatrix} Z_{11} & Z_{11}Z_{22} - Z_{12}Z_{21} \\ 1 & Z_{22} \end{bmatrix} \end{aligned} \tag{12.3.1}$$

Parameter Conversions

We note the determinants $\det(T) = AD - BC$ and $\det(Z) = Z_{11}Z_{22} - Z_{12}Z_{21}$. The relationship between the scattering and impedance matrices is also straightforward to derive. We define the 2×1 vectors:

$$\boldsymbol{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \quad \boldsymbol{I} = \begin{bmatrix} I_1 \\ -I_2 \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{12.3.2}$$

Then, the definitions (12.1.4) can be written compactly as:

The bold I is a matrix. The cursive I is unit matrix.

$$\begin{aligned} \mathbf{a} &= \frac{1}{2\sqrt{Z_0}}(\boldsymbol{V} + Z_0\boldsymbol{I}) = \frac{1}{2\sqrt{Z_0}}(Z + Z_0 I)\boldsymbol{I} \\ \mathbf{b} &= \frac{1}{2\sqrt{Z_0}}(\boldsymbol{V} - Z_0\boldsymbol{I}) = \frac{1}{2\sqrt{Z_0}}(Z - Z_0 I)\boldsymbol{I} \end{aligned} \tag{12.3.3}$$

where we used the impedance matrix relationship **V** = Z**I** and defined the 2×2 unit matrix I.

Parameter Conversions

It follows then,

$$\frac{1}{2\sqrt{Z_0}}\boldsymbol{I} = (Z + Z_0 I)^{-1}\mathbf{a} \quad \Rightarrow \quad \mathbf{b} = \frac{1}{2\sqrt{Z_0}}(Z - Z_0 I)\boldsymbol{I} = (Z - Z_0 I)(Z + Z_0 I)^{-1}\mathbf{a}$$

Thus, the scattering matrix S will be related to the impedance matrix Z by

$$\boxed{S = (Z - Z_0 I)(Z + Z_0 I)^{-1}} \quad \Leftrightarrow \quad \boxed{Z = (I - S)^{-1}(I + S)Z_0} \qquad (12.3.4)$$

Explicitly, we have:

$$\begin{aligned} S &= \begin{bmatrix} Z_{11} - Z_0 & Z_{12} \\ Z_{21} & Z_{22} - Z_0 \end{bmatrix} \begin{bmatrix} Z_{11} + Z_0 & Z_{12} \\ Z_{21} & Z_{22} + Z_0 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} Z_{11} - Z_0 & Z_{12} \\ Z_{21} & Z_{22} - Z_0 \end{bmatrix} \frac{1}{D_z} \begin{bmatrix} Z_{22} + Z_0 & -Z_{12} \\ -Z_{21} & Z_{11} + Z_0 \end{bmatrix} \end{aligned}$$

where $D_z = \det(Z + Z_0 I) = (Z_{11} + Z_0)(Z_{22} + Z_0) - Z_{12}Z_{21}$. Multiplying the matrix factors, we obtain:

$$S = \frac{1}{D_z}\begin{bmatrix} (Z_{11} - Z_0)(Z_{22} + Z_0) - Z_{12}Z_{21} & 2Z_{12}Z_0 \\ 2Z_{21}Z_0 & (Z_{11} + Z_0)(Z_{22} - Z_0) - Z_{12}Z_{21} \end{bmatrix} \quad (12.3.5)$$

Parameter Conversions

Similarly, the inverse relationship gives:

$$Z = \frac{Z_0}{D_s}\begin{bmatrix} (1+S_{11})(1-S_{22})+S_{12}S_{21} & 2S_{12} \\ 2S_{21} & (1-S_{11})(1+S_{22})+S_{12}S_{21} \end{bmatrix} \qquad (12.3.6)$$

where $D_s = \det(I-S) = (1-S_{11})(1-S_{22})-S_{12}S_{21}$. Expressing the impedance parameters in terms of the transfer matrix parameters, we also find:

$$S = \frac{1}{D_a}\begin{bmatrix} A + \dfrac{B}{Z_0} - CZ_0 - D & 2(AD-BC) \\ 2 & -A + \dfrac{B}{Z_0} - CZ_0 + D \end{bmatrix} \qquad (12.3.7)$$

where $D_a = A + \dfrac{B}{Z_0} + CZ_0 + D$.

This is the connection between S and the transfer matrix.

A connected 2 Port

When the two-port is connected to a generator and load as in Fig. 12.1.4, the impedance and scattering matrix equations take the simpler forms:

$$\boxed{\begin{aligned} V_1 &= Z_{\text{in}} I_1 \\ V_2 &= Z_L I_2 \end{aligned}} \quad \Leftrightarrow \quad \boxed{\begin{aligned} b_1 &= \Gamma_{\text{in}} a_1 \\ a_2 &= \Gamma_L b_2 \end{aligned}} \tag{12.4.1}$$

where Z_{in} is the input impedance at port 1, and Γ_{in}, Γ_L are the reflection coefficients at port 1 and at the load:

$$\Gamma_{\text{in}} = \frac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0}, \quad \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{12.4.2}$$

The input impedance and input reflection coefficient can be expressed in terms of the Z- and S-parameters, as follows:

$$\boxed{Z_{\text{in}} = Z_{11} - \frac{Z_{12} Z_{21}}{Z_{22} + Z_L}} \quad \Leftrightarrow \quad \boxed{\Gamma_{\text{in}} = S_{11} + \frac{S_{12} S_{21} \Gamma_L}{1 - S_{22} \Gamma_L}} \tag{12.4.3}$$

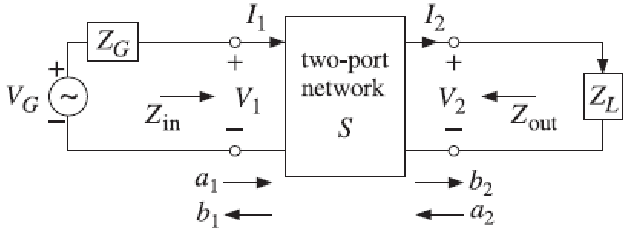

Fig. 12.1.4 Two-port network connected to generator and load.

A connected 2 Port

The equivalence of these two expressions can be shown by using the parameter conversion formulas of Eqs. (12.3.5) and (12.3.6), or they can be shown indirectly, as follows. Starting with $V_2 = Z_L I_2$ and using the second impedance matrix equation, we can solve for I_2 in terms of I_1:

$$V_2 = Z_{21} I_1 - Z_{22} I_2 = Z_L I_2 \quad \Rightarrow \quad I_2 = \frac{Z_{21}}{Z_{22} + Z_L} I_1 \tag{12.4.4}$$

Then, the first impedance matrix equation implies:

$$V_1 = Z_{11} I_1 - Z_{12} I_2 = \left(Z_{11} - \frac{Z_{12} Z_{21}}{Z_{22} + Z_L} \right) I_1 = Z_{\text{in}} I_1$$

Starting again with $V_2 = Z_L I_2$ we find for the traveling waves at port 2:

$$\left.\begin{aligned} a_2 &= \frac{V_2 - Z_0 I_2}{2\sqrt{Z_0}} = \frac{Z_L - Z_0}{2\sqrt{Z_0}} I_2 \\ b_2 &= \frac{V_2 + Z_0 I_2}{2\sqrt{Z_0}} = \frac{Z_L + Z_0}{2\sqrt{Z_0}} I_2 \end{aligned}\right. \quad \Rightarrow \quad a_2 = \frac{Z_L - Z_0}{Z_L + Z_0} b_2 = \Gamma_L b_2$$

A connected 2 Port

Using $V_1 = Z_{\text{in}} I_1$, a similar argument implies for the waves at port 1:

$$a_1 = \frac{V_1 + Z_0 I_1}{2\sqrt{Z_0}} = \frac{Z_{\text{in}} + Z_0}{2\sqrt{Z_0}} I_1$$

$$b_1 = \frac{V_1 - Z_0 I_1}{2\sqrt{Z_0}} = \frac{Z_{\text{in}} - Z_0}{2\sqrt{Z_0}} I_1$$

$$\Rightarrow \quad b_1 = \frac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0} a_1 = \Gamma_{\text{in}} a_1$$

It follows then from the scattering matrix equations that:

$$b_2 = S_{21} a_1 + S_{22} a_2 = S_{22} a_1 + S_{22} \Gamma_L b_2 \quad \Rightarrow \quad \boxed{b_2 = \frac{S_{21}}{1 - S_{22}\Gamma_L} a_1} \qquad (12.4.5)$$

which implies for b_1:

$$b_1 = S_{11} a_1 + S_{12} a_2 = S_{11} a_1 + S_{12} \Gamma_L b_2 = \left(S_{11} + \frac{S_{12} S_{21} \Gamma_L}{1 - S_{22}\Gamma_L} \right) a_1 = \Gamma_{\text{in}} a_1$$

A connected 2 Port

Reversing the roles of generator and load, we obtain the impedance and reflection coefficients from the output side of the two-port:

$$\boxed{Z_{\text{out}} = Z_{22} - \frac{Z_{12}Z_{21}}{Z_{11} + Z_G}} \quad \Leftrightarrow \quad \boxed{\Gamma_{\text{out}} = S_{22} + \frac{S_{12}S_{21}\Gamma_G}{1 - S_{11}\Gamma_G}} \tag{12.4.6}$$

where

$$\Gamma_{\text{out}} = \frac{Z_{\text{out}} - Z_0}{Z_{\text{out}} + Z_0}, \quad \Gamma_G = \frac{Z_G - Z_0}{Z_G + Z_0} \tag{12.4.7}$$

The input and output impedances allow one to replace the original two-port circuit of Fig. 12.1.4 by simpler equivalent circuits. For example, the two-port and the load can be replaced by the input impedance Z_{in} connected at port 1, as shown in Fig. 12.4.1.

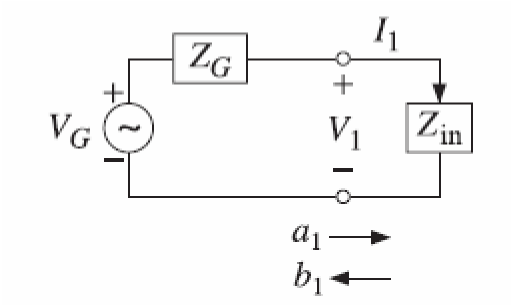

Fig. 12.4.1 Input equivalent circuit

A connected 2 Port

Similarly, the generator and the two-port can be replaced by a Thévenin equivalent circuit connected at port 2. By determining the open-circuit voltage and short-circuit current at port 2, we find the corresponding Thévenin parameters in terms of the impedance parameters:

$$\boxed{V_{\text{th}} = \frac{Z_{21}V_G}{Z_{11}+Z_G}, \quad Z_{\text{th}} = Z_{\text{out}} = Z_{22} - \frac{Z_{12}Z_{21}}{Z_{11}+Z_G}} \tag{12.4.8}$$

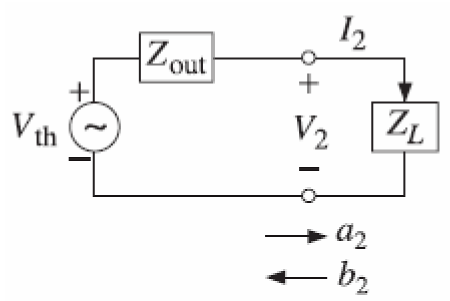

Fig. 12.4.1 output equivalent circuit