Geometrická reprezentace grafů

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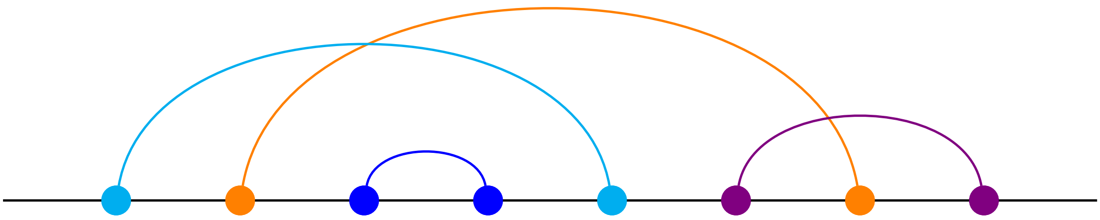

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Information

Many parts of text are taken from the handouts made by Jan Kratochvíl. I also add some of my notes from the lectures and some pictures.

Also there may be some mistakes. If you find some and want to update them, you may find all the sources on the GitHub.

Contents

Part I

Geometrická reprezentace grafů I

1. Introducing

Firstly we will start by the introduction to the main characters – intersection defined graph classes; characterization of chordal graphs.

1.1 Intersection defined graph classes

Definition 1. *The intersection graph of a set family $\mathcal{A}$ is the graph*

$$IG(\mathcal{A}) = (\mathcal{A}, \{ab : a \neq b, a \cap b \neq \emptyset, a, b \in \mathcal{A}\}).$$

Definition 2. *Let $\mathcal{M}$ be a family of sets. We say that a graph G is an intersection graph of (members of) $\mathcal{M}$ if G is isomorphic to the graph $IG(\mathcal{A})$ for some family $\mathcal{A}$ whose all elements belong to $\mathcal{M}$. We write*

$$\mathcal{IG}(\mathcal{M}) = \{\mathcal{IG}(\mathcal{A}) : \mathcal{A} \subseteq \mathcal{M}\}.$$

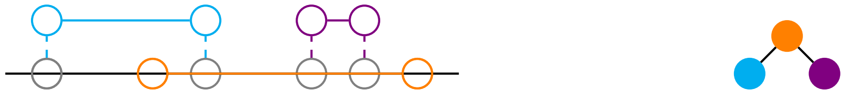
Observation. *For every graph G and every set family $\mathcal{M}, G \in \mathcal{IG}(\mathcal{M})$ if and only if there is a mapping $f : V(G) \to \mathcal{M}$ such that $uv \in E(G)$ iff $f(u) \cap f(v) \neq \emptyset$ holds true for all pairs of distinct vertices u, v of G.*

Observation. *For every family $\mathcal{M}$ (containing at least one nonempty set), it holds that $\mathcal{IG}(\mathcal{M}$ contains all complete graphs and is hereditary (i.e., every induced subgraph of every graph from $\mathcal{IG}(\mathcal{M})$ also belongs to $\mathcal{IG}(\mathcal{M})$).*

1.1.1 Examples

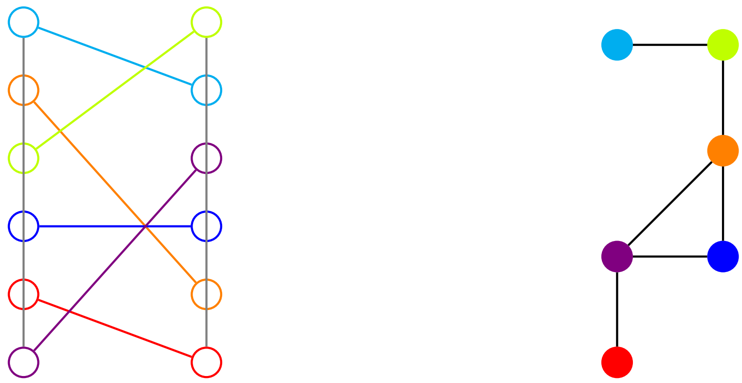
In many cases, the members of $\mathcal{M}$ are defined by their geometric shape. And in most of these cases, the members of $\mathcal{M}$ are arc-connected sets in the plane.

- **Interval graphs** INT $= \mathcal{IG}(\{\text{intervals on a line}\})$
- **Circle graphs** CIR $= \mathcal{IG}(\{\text{chords of a circle}\})$
- **Circular arc graphs** CA $= \mathcal{IG}(\{\text{arcs on a circle}\})$
- **Permutation graphs** PER $= \mathcal{IG}(\{\text{segments connecting two parallel lines}\})$
- **Function graphs** FUN $= \mathcal{IG}(\{\text{curves connecting two parallel lines}\})$
- **Polygon circle graphs** PC $= \mathcal{IG}(\{\text{polygons inscribed in a circle}\})$
- **Segment graphs** SEG $= \mathcal{IG}(\{\text{straight-line segments in the plane}\})$
- **Convex graphs** CONV $= \mathcal{IG}(\{\text{convex sets in the plane}\})$
- **String graphs** STRING $= \mathcal{IG}(\{\text{curves in the plane}\})$



(a) Drawn intervals. (b) Corresponding graph.

Figure 1.1: Example of a graph from INT class.



(a) Drawn permutations. (b) Corresponding graph.

Figure 1.2: Example of a graph from PER class.

1.2 Chordal graphs

Definition 3. *A graph is **chordal** if it does not contain any cycle of length greater than three as an induced subgraph.*

Definition 4. *A vertex u of a graph G is **simplicial** if $G[N_G(u)]$ is a clique.*

Definition 5 (PES)**.** *A **perfect elimination scheme** for a graph G is a linear ordering $u_1, u_2, \ldots, u_n$ of its vertices such that for every i, u_i is simplicial in the induced subgraph $G[\{u_1, u_2, \ldots, u_i\}]$.*

Lemma 1. *Every minimal vertex cut in a chordal graph induces a clique.*

Proof. Let $A \subset V(G)$ be a minimal vertex cut, and suppose u, v be two vertices of A. These vertices are connected by a path in each component of $G \setminus A$. If u and v were not adjacent, a pair of shortest such paths would give rise to an induced cycle of length greater than 3 in G. $\square$

Lemma 2. *Every chordal graph, which is not a complete graph, contains two nonadjacent simplicial vertices.*

Proof. By induction. If G is a complete graph, the claim of the lemma is fulfilled. If G is not complete, it has a vertex cut, say A. Let B be a connected component of $G \setminus A$, and set $G_1 = G[B \cup A]$ and $G_2 = G \setminus B$. By induction hypothesis, each of G_1, G_2 is either complete or has two nonadjacent simplicial vertices. Thus each of them has a simplicial vertex which does not belong to A. Each of these is then also simplicial in entire G, and they are clearly nonadjacent. $\square$

Corollary. Every nonempty chordal graph contains a simplicial vertex.

Definition 6 (Clique-tree decomposition)**.** *A **clique-tree decomposition** of a graph G is a tree $T = (\mathcal{Q}, F)$, with $\mathcal{Q}$ being the set of all maximal cliques of G, such that for every vertex $u \in V(G)$, the subgraph $T[\{Q : u \in Q \in \mathcal{Q}\}]$ is connected.*

Warning!! The vertex set of a clique-tree decomposition of a graph G is uniquely defined, but not necessarily the edge set!!

Theorem 1. *For any graph G, the following statements are equivalent:*

1. *G is chordal,*
2. *G allows a PES.*
3. *G has a clique-tree decomposition, and*
4. *G is an intersection graph of subtrees of a tree.*

Proof. "1. $\Rightarrow$ 2." By induction on the number of vertices, using Lemma 2.

"2. $\Rightarrow$ 3." By induction on the number of vertices again. Suppose $G' = G \setminus v_n$ has a clique-tree $T = (\mathcal{Q}', F')$. If $Q' = N_G(v_n) \in \mathcal{Q}'$, then $Q = N_G[v_n]$ is a maximal clique in G, $\mathcal{Q} = (\mathcal{Q}' \setminus \{Q'\}) \cup \{Q\}$ and $T = (\mathcal{Q}, F)$ is a clique-tree for G, where $F = (F' \setminus \{Q'P : P \in \mathcal{Q}'\}) \cup \{QP : Q'P \in F'\}$. If, on the other hand, $Q' = N_G(v_n) \notin \mathcal{Q}'$, then $\mathcal{Q} = \mathcal{Q}' \cup N_G[v_n]$ and $(\mathcal{Q}, F' \cup N_G[v_n]P)$ is a clique-tree for G for any $P \in \mathcal{Q}'$ such that $N_G(v_n) \subset P$.

"3. $\Rightarrow$ 4." Given a clique-tree decomposition $T = (\mathcal{Q}, F)$, define $T_u = T[\{Q : u \in Q \in \mathcal{Q}\}]$ for $u \in V(G)$. Clearly $V(T_u) \cap V(T_v) \neq \emptyset$ iff u and v belong to the same maximal clique of G, which happens if and only if u and v are adjacent in G.

"4. $\Rightarrow$ 1." Let G be the intersection graph of a collection $\{T_u\}_{u \in V(G)}$ of subtrees of a tree T. Suppose $v_1, v_2, \ldots, v_k$ be an induced cycle in G, with $k > 3$. Then the subtrees T_{v_1} and T_{v_3} are vertex disjoint, and hence there is an edge $e \in E(T)$ which lies on every path connecting T_{v_1} and T_{v_3} in T . This edge separates T into T_1 and T_2 such that T_{v_1} and T_{v_3} belong to different components of $T \setminus e$, say, $T_{v_1} \subseteq T_1$ and $T_{v_3} \subseteq T_2$. One can show by induction on i that for every $i \geq 3$, $T_{v_i} \subseteq T_2$. But then T_{v_k} and T_{v_1} must be disjoint, contradicting the assumption that $v_1 v_k \in E(G)$. □

Corollary. Chordal graphs are perfect (i.e., $\chi(H) = \omega(H)$ for every induced subgraph H of G).

Proof. Consider a PES $u_1, u_2, \ldots, u_n$ for G and color the vertices from u_1 to u_n by the First Fit Method (we try to use minimal color if we cannot use any of them, create a new color). □

2. Interval, permutation and function graphs

2.1 Interval graphs

Definition 7. *A graph is an* ***interval graph*** *if it is isomorphic to the intersection graph of a collection of intervals on a line.*

Observation. *Every interval graph has an interval representation in which all of the intervals are closed.*

Definition 8 (Clique-path decomposition)**.** *A* ***clique-path decomposition*** *of a graph is a clique-tree decomposition in which the underlying tree is a path.*

Theorem 2. *For any graph G, the following statements are equivalent:*

1. *G is an interval graph,*
2. *G has a clique-path decomposition, and*
3. *G is an intersection graph of subpaths of a path.*

Proof. "1. $\Leftrightarrow$ 3." is obvious.

"1. $\Rightarrow$ 2." Assume $I_u, u \in V(G$ is an interval representation of G. Use the fact that intervals on a line have the Helly property, i.e., if any two of a collection of intervals have a nonempty intersection, then all of them have a nonempty intersection. In other words, if $Q_i \in \mathcal{Q}$ is a maximal clique of G, then there exists a point P_i which belongs to $\bigcap_{u \in Q_i} I_u$. Moreover, for every $v \notin Q_i, P_i \notin I_v$, since Q_i is a maximal clique. (E.g., the rightmost of the left endpoints of the intervals $I_u, u \in Q_i$ is a good candidate for P_i.) Order the cliques of Q as $Q_1, Q_2, \ldots, Q_k$ so that $P_1 < P_2 < \cdots < P_k$. Then the path $Q_1 Q_2 \ldots Q_k$ is a clique-path decomposition of G.

"2. $\Rightarrow$ 3." Given a clique-path decomposition $P = (\mathcal{Q}, F)$, define $P_u = P[\{Q : u \in Q \in \mathcal{Q}\}]$ for $u \in V(G)$. Clearly $V(P_u) \cap V(P_v) \neq \emptyset$ iff u and v belong to the same maximal clique of G, which happens if and only u and v are adjacent in G. $\square$

2.2 Comparability graphs

Definition 9. *A graph G is a* ***comparability graph*** *if there exists a partial order $P = (V(G), \leq)$ on the vertex set of G (i.e., an* ***antireflexive***, ***antisymmetric*** *and* ***transitive binary relation****) such that for any two vertices $u, v \in V(G)$, $uv \in E(G)$ if and only if $u \leq v$ or $v \leq u$ (i.e., if u and v are comparable in P). The class of comparability graphs will be denoted by CO.*

Observation. *A graph is a comparability graph if and only if its edges can be transitively oriented.*

Notation. *If $\mathcal{A}$ is a graph class, the symbol $co-\mathcal{A}$ is used to denote the class containing the complements of the graphs in $\mathcal{A}$.*

Observation. *If $A \subseteq B$, then $co - A \subseteq co - B$.*

Theorem 3. *All equivalencies hold:*

1. $FUN = co - CO$
2. $PER = CO \cap co - CO$
3. $INT = CHOR \cap co - CO$

Proof. 1. "FUN $\subseteq$ co-CO": Given a collection of curves joining two vertical parallel lines (and lying in the stripe between them), for any two non-crossing curves, it is uniquely determined which one lies above the other one (this follows from the Jordan curve theorem), and this gives a transitive orientation of the complement of the intersection graph of this collection.

"co-CO $\subseteq$ FUN": Let $G = (V, E)$ be a graph and let $P = (V, \leq)$ be a partial order which corresponds to a transitive orientation of the complement of G. If d is the dimension of P, P is the intersection of d linear orders $L_1, L_2, \ldots, L_d$ of V. In the plane, draw d distinct parallel (vertical) lines $l_1, l_2, \ldots, l_d$, and on each l_i , mark distinct points $P_{iu}, u \in V$ bottom up in the order L_i. Consider piece-wise linear curves $c(u) = P_{1u}P_{2u} \ldots P_{du}$, for $u \in V$. If $uv \in E$, u and v are incomparable in P, and hence there are indices i and j such that $u <_{L_i} v$ and $v <_{L_j} u$, in other words P_{iu} is below P_{iv}, while P_{ju} is above P_{jv}. Hence the curves $c(u)$ and $c(v)$ cross somewhere between l_i and l_j. If, on the other hand, $uv \notin E$, uv is an edge of the complement of G and hence u and v are comparable in P, say, $u \leq v$. But then $u <_{L_i} v$ for every $i = 1, 2, \ldots, d$, and for each $i = 1, 2, \ldots, d - 1$, the curve $c(u)$ lies below the curve $c(v)$ in the stripe between l_i and l_{i+1}. Thus $c(u)$ and $c(v)$ are disjoint.

2. Note first that co-PER $\subseteq$ PER. Indeed, given a permutation representation of a graph, swap the order of the endpoints on one of the bounding lines to obtain a representation of the complement of the given graph. Then PER = co-(co-PER) $\subseteq$ co-PER, and hence PER = co-PER.

"PER $\subseteq$ CO $\cap$ co-CO": Obviously PER $\subseteq$ FUN = co-CO. Then the above small observation implies PER = co-PER $\subseteq$ co-(co-CO) = CO as well.

"CO $\cap$ co-CO $\subseteq$ PER": Suppose both G and its complement can be transitively oriented, say $\overrightarrow{E_1}$ be a transitive orientation of G and $\overrightarrow{E_2}$ a transitive orientation of the complement $-G$ of G. Then $\overrightarrow{E_1} \cup \overrightarrow{E_2}$ is a transitive orientation of the complete graph $K_{V(G)}$ on the vertex set of G, i.e., a linear ordering of the vertices of G. And so is $\overrightarrow{E_1}^{-1} \cup \overrightarrow{E_2}$. Place the vertices of G on two parallel lines, on one of them in the linear order given by $\overrightarrow{E_1} \cup \overrightarrow{E_2}$, on the other one in the order given by $\overrightarrow{E_1}^{-1} \cup \overrightarrow{E_2}$, and connect the two occurrences of a vertex u by a straight-line segment called $s(u)$, for every vertex $u \in V(G)$. If $uv \in E(G)$, then the pair u, v is ordered differently on the two lines (by $\overrightarrow{E_1}$ on one of them and by $\overrightarrow{E_1}^{-1}$ on the other one) and the segments $s(u), s(v)$ cross each other somewhere between the two lines. If $uv \notin E(G)$, the pair u, v is ordered the same way (by $\overrightarrow{E_2}$) on both of the lines, and thus the segments $s(u)$ and $s(v)$ are disjoint. So $\{s(u)\}_{u \in V(G)}$ is a permutation representation of G.

3. "INT $\subseteq$ CHOR $'cap$ co-CO": Let $\{I(u)\}_{u\in V(G)}$ be an interval representation of a graph G. Define a transitive orientation $\overrightarrow{E_2}$ of the non-edges of G by setting $uv \in \overrightarrow{E_2}$ if $\max I(u) < \min I(v)$. Thus $G \in$ co-CO. The fact that $G \in$ CHOR follows from the fact that

$$\text{INT} = \mathcal{IG}(\{\text{connected subgraphs of paths}\}) \subseteq$$

$$\subseteq \mathcal{IG}(\{\text{connected subgraphs of trees}\}) = \text{CHOR}.$$

"CHOR $\cap$ co-CO $\subseteq$ INT": Let G be a chordal graph which allows a transitive orientation $\overrightarrow{E_2}$ of its non-edges. Define a binary relation $<$ on the set $\mathcal{Q}$ of maximal cliques of G by setting

$$Q < Q' \Leftrightarrow \exists u \in Q \exists v \in Q' : uv \in \overrightarrow{E_2}.$$

Claim 4. *The relation $<$ is a partial order on $\mathcal{Q}$.*

Proof of claim. We will show all properties of partial order.

- Antireflexivity: Each $Q \in \mathcal{Q}$ is a clique, so there are no two vertices $u, v \in Q$ that would form a non-edge of G. Hence $Q \not< Q$.
- Antisymmetry: Suppose for the contrary that there are $u \in Q, v \in Q'$ s.t. $uv \in \overrightarrow{E_2}$, and another pair $x \in Q, y \in Q'$ s.t. $yx \in \overrightarrow{E_2}$. First observe that $u \neq x$ and $v \neq y$ (if $u = x$, the transitivity of $\overrightarrow{E_2}$ would imply $yv \in \overrightarrow{E_2}$, which is impossible; if $v = y$, the transitivity of $\overrightarrow{E_2}$ would imply $ux \in \overrightarrow{E_2}$, which is again impossible). Next observe that both uy and xv must be edges of G (if $uy \notin E(G)$, then either $uy \in \overrightarrow{E_2}$, or $yu \in \overrightarrow{E_2}$, yielding $ux \in \overrightarrow{E_2}$ in the former case and $yv \in \overrightarrow{E_2}$ in the letter one, both contradicting the fact that Q and Q' are cliques of G; the case of $xv \notin E(G)$ is analogous). Lastly, we conclude that $G[\{u, v, x, y\}] \simeq C_4$, contradicting the assumption that G is chordal.
- Transitivity: Suppose $Q < Q' < Q''$ and let $u \in Q, v, x \in Q'$ and $y \in Q''$ be vertices such that $uv, xy \in \overrightarrow{E_2}$. If $v = x$, the transitivity of $\overrightarrow{E_2}$ implies $uy \in \overrightarrow{E_2}$, hence $Q < Q''$. If $v \neq x$, one of ux, vy must be a non-edge (otherwise $G[\{u, v, x, y\}]$ would be an induced cycle of length 4, contradicting the assumption that G is chordal). If $ux \notin E(G)$, $ux \in \overrightarrow{E_2}$ and the transitivity of $\overrightarrow{E_2}$ implies $uy \in \overrightarrow{E_2}$. If $vy \notin E(G)$, $vy \in \overrightarrow{E_2}$ and the transitivity of $\overrightarrow{E_2}$ implies $uy \in \overrightarrow{E_2}$. In either case, $Q < Q''$.

$\square$

Claim 5. *The relation $<$ is a linear ordering of $\mathcal{Q}$.*

Proof of claim. Let $Q \neq Q'$ be two different maximal cliques of G. Their maximality implies that none of them is a subset of the other one. Hence there is a vertex $u \in Q$ which does not belong to Q' . If u were adjacent to all vertices of Q', $Q' \cup \{u\}$ would be a clique of G, contradicting the maximality of Q' . Hence there is a $v \in Q'$ such that $uv \notin E(G)$. Then either $uv \in \overrightarrow{E_2}$ or $vu\overrightarrow{E_2}$, thus $Q < Q'$ or $Q' < Q$. $\square$

Claim 6. *Let* $\mathcal{Q} = \{Q_1 < Q_2 < \cdots < Q_k\}$ *be the maximal cliques of* G *ordered by* $<$. *Then* $P_G = (\mathcal{Q}, \{Q_iQ_{i+1} : i = 1, 2, \ldots, k-1\})$ *is a clique-path decomposition of* G, *and hence* $G \in INT$.

Proof of claim. Indeed P_G is a path whose nodes are the maximal cliques of G. It remains to show that vertices of G appear in these cliques consecutively. Suppose $Q < Q' < Q''$ and $u \in Q \cap Q''$. If there were a vertex $v \in Q'$ nonadjacent to u, we would have $uv \in \overrightarrow{E_2}$ because of $Q < Q'$ and $vu \in \overrightarrow{E_2}$ because of $Q' < Q''$, contradicting the antisymmetry of $\overrightarrow{E_2}$. Hence u is adjacent to all vertices of Q' , and thus $u \in Q'$ follows from the maximality of Q'. $\square$

$\square$