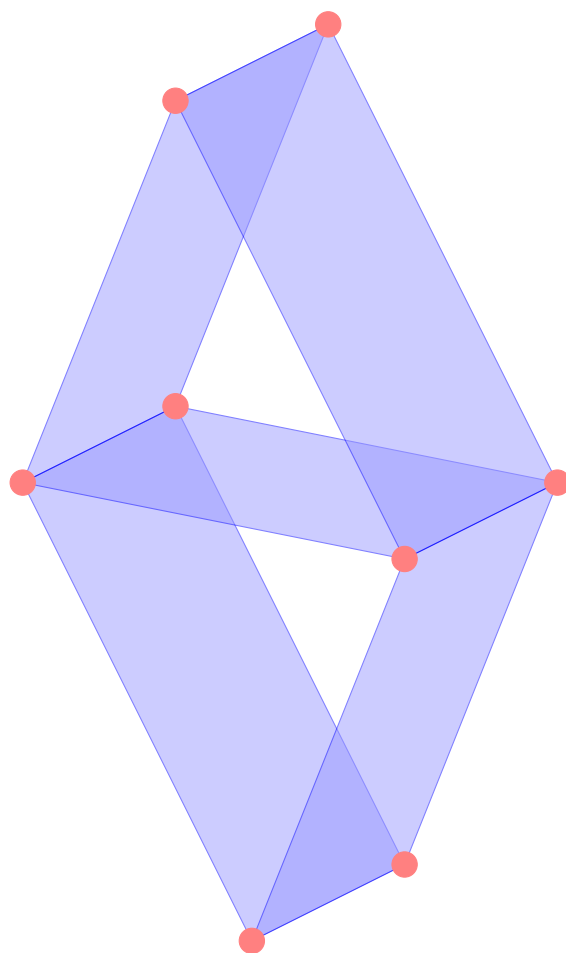


# Matroid Theory

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March 22, 2024

## Information

*These are my notes on the course Matroid Theory, which was taught by  
Ondřej Pangrác in the year 2024.*

Keep in mind there may be some **mistakes**. You may visit [GitHub](#).

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# 1. Basic definitions

**Definition 1. Matroid**  $\mathcal{M} = (E, \mathcal{I})$  is for  $E$  finite non-empty set and  $\mathcal{I} \subseteq 2^E$  (also called as independent sets) satisfying these properties:

(I1)  $\emptyset \in \mathcal{I}$ ,

(I2)  $I \in \mathcal{I} \Rightarrow \forall I' \subseteq I : I' \in \mathcal{I}$ ,

(I3)  $I_1, I_2 \in \mathcal{I}, |I_1| < |I_2| \Rightarrow \exists e \in I_2 \setminus I_1 : I_1 \cup \{e\} \in \mathcal{I}$ .

**Notation.** For further use and simplification we will sometimes use  $I+e$  as a substitution for  $I \cup \{e\}$ . Similarly also  $I-e$  for  $I \setminus \{e\}$ .

*Example.* For a given multi-graph  $G = (V, F)$  we will set  $E = F$  (or in other words  $E$  stands for edges and the set). Independent sets  $\mathcal{I}$  will be all acyclic subsets of  $E$ . Easily seen (I1) and (I2) is satisfied. For the third one (I3) it is also quite easily seen, because if we have one larger and smaller non-cycles then we can append one edge from the larger to the smaller.

*Example.* Let  $E$  be some elements of a vector space  $V$ . If  $X \subseteq E$  is independent then it is linearly independent in  $V$ .

**Definition 2. Matroid isomorphism** for two matroids  $\mathcal{M}_i = (E_i, \mathcal{I}_i)$  for  $i = 1, 2$  is a bijection  $f : E_1 \rightarrow E_2$  satisfying  $\forall X \subseteq E_i : X \in \mathcal{I}_1 \Leftrightarrow f(X) \in \mathcal{I}_2$ .

## 1.1 Circuits

**Definition 3.**  $X \subseteq E$  is a **circuit** if  $X \notin \mathcal{I}$  and  $\forall x \in X : X - x \in \mathcal{I}$ . Also we will denote  $\mathcal{C}(\mathcal{M})$  as the set of all circuits of  $\mathcal{M}$ .

**Lemma 1.** Let  $\mathcal{M} = (E, \mathcal{I})$  be a matroid and  $\mathcal{C}$  its collection of circuits, then

(C1)  $\emptyset \notin \mathcal{C}$ ,

(C2)  $\forall C_1, C_2 \in \mathcal{C} : C_1 \subseteq C_2 \Rightarrow C_1 = C_2$  and

(C3)  $C_1, C_2 \in \mathcal{C}, C_1 \neq C_2, e \in C_1 \cap C_2 \Rightarrow C_3 \subseteq (C_1 \cup C_2) - e, C_e \in \mathcal{C}$ .

*Proof.* (C1) and (C2) are easily seen from (I1) and (I2). Now for the third part (C3). So for contradiction let  $C_1, C_2, e$  be as mentioned in the first part, but  $(C_1 \cup C_2) - e \in \mathcal{I}$ . Then  $\exists f \in C_2 \setminus C_1 : C_2 - f \in \mathcal{I}$ . Now find  $I \in \mathcal{I}$  max s.t.  $C_2 \setminus \{f\} \subseteq I \subseteq C_1 \cup C_2$ . If  $f \notin I$  then it would contain  $C_2$  which is dependent and  $\exists g \in C_1 \setminus C_2 : g \notin I$  otherwise it would contain  $C_1$  which is dependent. Therefore

$$|I| \leq |C_1 \cup C_2| - 2 < |(C_1 \cup C_2) - e|$$

and now we may use the third axiom (I3) that is  $\exists x \in |(C_1 \cup C_2) - e| \setminus I$  s.t.  $I + x \in \mathcal{I}$  (this cannot be otherwise  $I$  contains the whole  $C_2$ ). Now  $I + x$  contradicts the maximality of  $I$ .  $\square$

**Claim 1.** Let  $E$  and  $\mathcal{C} \subseteq 2^E$  satisfying all (C1), (C2) and (C3). Then set  $\mathcal{I} = \{X \subseteq E \mid \forall C \in \mathcal{C} : C \not\subseteq X\}$  and  $\mathcal{M} = (E, \mathcal{I})$  is a matroid.

*Proof.* We have to show all properties of matroid. That is (I1) is trivially satisfied and (I2) also trivially holds. For the last (I3) we use a contradiction. For that we have  $I_1, I_2 \in \mathcal{I}$ , then  $\forall e \in I_2 \setminus I_1 : I_1 + e \notin \mathcal{I}$ . Let  $I_3 \subseteq I_1 \cup I_2$  s.t.  $|I_3| > |I_1|$  and  $|I_1 \setminus I_3|$  is minimal. If  $|I_1 \setminus I_3|$  would be empty then (I3) will hold, therefore assume it is non-empty.

Fix  $e \in I_1 \setminus I_3$ . Let  $I_k = |I_3 - f| + e$  for  $(f \in I_3 \setminus I_1)$ . This cannot be independent ( $\notin \mathcal{I}$ ) therefore  $\exists C_k \subseteq T_k : C_k \in \mathcal{C}$  and  $f \notin C_k, e \in C_k$ .

$(I_3 \setminus I_1) \cap C_k = \emptyset$  hence  $C_k \subseteq T_k \setminus (I_3 \setminus I_1) = (I_1 \cap I_3) + e \subseteq I_1$  this is not possible so it must be non-empty. Then  $\exists g \in (I_3 \setminus I_1) \cap C_k \Rightarrow C_k, C_g \in \mathcal{C}, e \in C_k \cap C_g, f \notin C_k, g \notin C_g$  but  $(C_k \cup C_g) - e \subseteq I_3$  which is contradiction with (C3).  $\square$

## 1.2 Basis

**Definition 4.** Let  $\mathcal{M} = (E, \mathcal{I})$  be a matroid. Then  $B$  is a **basis** iff  $B \in \mathcal{I}, \forall x \in E \setminus B : B + x \notin \mathcal{I}$ .

**Proposition 2.** Let  $B_1, B_2$  be bases of  $\mathcal{M}$ , then  $|B_1| = |B_2|$ .

*Proof.* If  $|B_1| < |B_2|$  then by (I3)  $\exists x \in B_2 \setminus B_1 : B_1 + x \in \mathcal{I}$ .  $\square$

**Definition 5.** Let  $\mathcal{B}(\mathcal{M}) = \{B \subseteq E, B \text{ is a basis}\}$  be a collection of basis satisfying

(B1)  $\mathcal{B} \neq \emptyset$  and

(B2)  $B_1, B_2 \in \mathcal{B}, e \in B_1 \setminus B_2 \Rightarrow \exists f \in B_2 \setminus B_1 : (B_1 - e) + f \in \mathcal{B}$ .

One can see that (B2) can be proven using  $I_1 - e =: B_1$  and  $I_2 = B_2$ . And alternatively we may exchange the second property for the following.

(B2')  $B_1, B_2 \in \mathcal{B}, \forall x \in B_2 \setminus B_1, \exists y \in B_1 \setminus B_2$  s.t.  $(B_1 - y) + x \in \mathcal{B}$ .

*Proof.* To proof that indeed we can exchange these properties we see that  $B_1 + x$  has to be dependent therefore  $\exists C \in \mathcal{C}, C \subseteq B_1 + x$ . Now we show that  $C$  is unique, otherwise  $C_1, C_2 \subseteq B_1 + x, x \in C_1 \cap C_2, \exists C_3 \subseteq (C_1 \cup C_2) - x \subseteq B_1$  which is a contradiction. Also this is sometimes called **fundamental circuit w.r.t.  $B_1$  and  $x$** . See that  $C \setminus B_2 \neq \emptyset$  so  $\exists y \in C \setminus B_2 \subseteq B_1 \setminus B_2$ ; hence  $(B_1 + x) - y$  does not contain a circuit and  $|(B_1 + x) - y| = |B_1| \Rightarrow (B_1 + x) - y \in \mathcal{B}$ .  $\square$

**Proposition 3.** Let  $E \neq \emptyset$  finite set and  $\mathcal{B} \subseteq 2^E$  satisfying (B1) and (B2). Let  $\mathcal{I} = \{X \subseteq E : \exists B \in \mathcal{B} : X \subseteq B\}$  then  $\mathcal{M} = (E, \mathcal{I})$  is a matroid.

*Proof.* (I1) and (I2) are trivial. For (I3) use the following lemma.

**Lemma 2.** Let  $\mathcal{B}$  be such that it satisfies (B1) and (B2). Then  $\forall B_1, B_2 \in \mathcal{B} : |B_1| = |B_2|$ .

*Proof.* By contradiction suppose  $|B_1| > |B_2|$  with minimal  $|B_1 \setminus B_2|$ . Then  $e \in B_1 \setminus B_2 \Rightarrow \exists f \in B_2 \setminus B_1 : (B_1 - e) + f \in \mathcal{B}$  and also  $|(B_1 - e) + f| = |B_1|$  which leads to  $|((B_1 - e) + f) \setminus B_2| < |B_1 \setminus B_2|$  which is a contradiction with the minimality.  $\square$

$\square$

### 1.3 Rank function

**Definition 6.** For a matroid  $\mathcal{M} = (E, \mathcal{I})$  define a **rank function**  $r : 2^E \rightarrow \mathbb{Z}_0^+$ , such that  $r(X) = \max_{I \subseteq X, I \in \mathcal{I}} |I|$  and  $r(\mathcal{M}) = r(E)$ .

**Claim 4.** Rank function has the following properties:

$$(R1) \ X \subseteq E : 0 \leq r(X) \leq |X|,$$

$$(R2) \ X \subseteq Y \subseteq E \Rightarrow r(X) \leq r(Y) \text{ and}$$

$$(R3) \ X, Y \subseteq E : r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y) \text{ (which is called **submodularity**)}.$$

*Proof of the properties.* While (R1) and (R2) are obvious and now we will show that (R3) also holds. Let  $I_1$  be the max independent in  $X \cap Y$  and  $I_2$  be an extension  $I_2 \supseteq I_1$  and max independent in  $X \cup Y$ . Now  $r(X \cup Y) + r(X \cap Y) = |I_2| + |I_1|$  and also  $|I_2 \cap X| \leq r(X)$  and  $|I_2 \cap Y| \leq r(Y)$ . We apply simple rule  $|A| + |B| = |A \cup B| + |A \cap B|$  and get

$$r(X) + r(Y) \geq |I_2 \cap X| + |I_2 \cap Y| = |I_2| + |I_1| = r(X \cup Y) + r(X \cap Y)$$

□

**Theorem 5.** For  $E \neq \emptyset$  finite set and  $r : 2^E \rightarrow \mathbb{Z}_0^+$  satisfying (R1), (R2) and (R3). Then  $\mathcal{I} = \{X \subseteq E \mid r(X) = |X|\}$  and  $\mathcal{M} = (E, \mathcal{I})$  is a matroid.

**Lemma 3.** For  $E \neq \emptyset$  finite set and  $r : 2^E \rightarrow \mathbb{Z}_0^+$  satisfying (R1), (R2) and (R3). It holds that if  $X, Y \subseteq E \ \forall y \in Y : r(X) = r(X + y)$  then  $r(X) = r(X \cup Y)$ .

*Proof of lemma 3.* Let  $Y \setminus X = \{y_1, y_2, \dots, y_k\}$  and now we will prove it by induction on  $k$ . For  $k = 1$  it obviously holds. For  $k \geq 2$  we use the submodularity.

$$\begin{aligned} r(X) + r(X) &= r(X \cup \{y_1, y_2, \dots, y_{k-1}\}) + r(X + y_k) \geq r(X \cup \{y_1, y_2, \dots, y_k\}) + r(X) \\ &\quad \text{(by induction hypothesis)} \\ r(X) &= \geq r(X \cup \{y_1, y_2, \dots, y_k\}) \\ r(X) &\geq r(X \cup Y) \end{aligned}$$

For the other inequality we use (R2) and hence we obtain equality. □

*Proof of theorem 5.* Again we have to prove all the properties of matroids. For (I1)  $\emptyset \in \mathcal{I}$  we use (R1) to get that  $0 \leq r(X) \leq |\emptyset| = 0$  therefore it is satisfied. For (I2)  $I \in \mathcal{I}, I' \subseteq I \Rightarrow I' \in \mathcal{I}$  we have that  $r(I) = |I|$  and  $I' \subseteq I$  so  $r(I') + r(I \setminus I') \geq r(I) + r(\emptyset) = |I| + 0$  and also by (R1)  $r(I') + r(I \setminus I') \leq |I'| + |I \setminus I'| = |I|$  so all inequalities are actually equalities.

Lastly the third (I3). Let  $I_1, I_2 \in \mathcal{I}, |I_1| < |I_2| \Rightarrow \exists e \in I_2 \setminus I_1$  s.t.  $I_1 + e \in \mathcal{I}$ . For this we will use Lemma 3. We have that  $r(I_i) = |I_i|$  for  $i = 1, 2$  where  $|I_1| < |I_2|$  for contradiction assume that  $\forall e \in I_2 \setminus I_1 : I_1 + e \notin \mathcal{I}$  therefore  $|I_1| = r(I_1) \leq r(X + e) < |I_1| + 1$ . This means that the  $\leq$  is  $=$  instead. Now use the lemma

$$|I_2| = r(I_2) \leq r(I_1 \cup (I_2 \setminus I_1)) = r(I_1) = |I_1|$$

where  $I_1 \cup (I_2 \setminus I_1) \supseteq I_2$  and we get that  $|I_2| \leq |I_1|$  which is a contradiction. □

For this part we could use any field  $F$  but as of now  $\mathbb{R}$  is sufficient enough. Let  $A \in \mathbb{R}^{n \times n}$ . Where  $n$ -columns are the elements of matroid. We usually take the multi-set of the columns or indexes, since they may be the same (parallel) columns. Also observe that for every operation as in Gauss elimination it will preserve the matroid same up to isomorphism. So we may use the Gauss-Jordan elimination. By other operations we may have matrix  $I$  of  $r(E)$  rows and number of basis columns. This format is often called *the standard form*. An example is when  $A = (0, 0, \dots, 0)$  where this is a special case when  $|\mathcal{I}| = 1$  and  $\mathcal{I} = \{\emptyset\}$ .

## 1.4 Uniform matroids

**Definition 7.** For  $0 \leq r \leq n \neq 0, |E| = n$  and  $\mathcal{I} = \{X \subseteq E : |X| \leq r\}$  is **Uniform matroid**  $U_{r,n} = (E, \mathcal{I})$ .

All the properties should be formally proven, but one can already see that all (I1), (I2) and (I3) are really satisfied. Now we will show us some examples.

*Example.* The matroid  $A = (0, 0, \dots, 0)$  is actually  $\mathcal{M}(A) \cong U_{0,n}$ .

*Example.* Now remind ourselves the example in the beginning, that is the graphic matroid. If we take graph as a trees then  $r(\mathcal{M}) = r(E) =$  the size of spanning tree. For a tree with  $n$  vertices we obtain  $U_{n,n}$  matroid. Note that we can now see that every operation that does not effect circuits neither effects the matroid.

*Example.* For other graphic matroids see:

- $\mathcal{M}(C_n) = U_{n-1,n}$ ;
- $U_{0,n}$  is for a graph with  $n$ -loops;
- $U_{1,n}$  is for  $n$ -parallel edge;
- $U_{2,n}$  is more interesting, because  $U_{2,2}$  is a  $P_3$  and  $U_{2,3}$  is  $C_3$ , but for  $n \geq 4$  there is no (multi-)graph representing it.

## 1.5 Visualization of matroids

From the previous part we see that some matroids cannot be visualized by graphs. So the question is whether we can visualize it in some other way. Lets take  $U_{2,4}$  for an example, for which we construct the matrix:

$$\begin{pmatrix} 1 & 0 & \textcolor{red}{1} & \textcolor{blue}{1} \\ 0 & 1 & \textcolor{red}{1} & \textcolor{blue}{-1} \end{pmatrix}$$

See that the **red numbers** are forced to be ones and then the **blue ones** are forced to use some other element. This also implies that the field has to have at least three elements. This way we can visualize the  $U_{2,4}$  by depicting the vectors in the plane.

Alternatively we may use affine spaces. For them we know that one element is always independent and so are two points which also forms a line. To add dependent points we have to put them on the line. Therefore four points on the line also visualize  $U_{2,4}$ . Also for  $U_{1,3}$  we use single coordinate for all points.

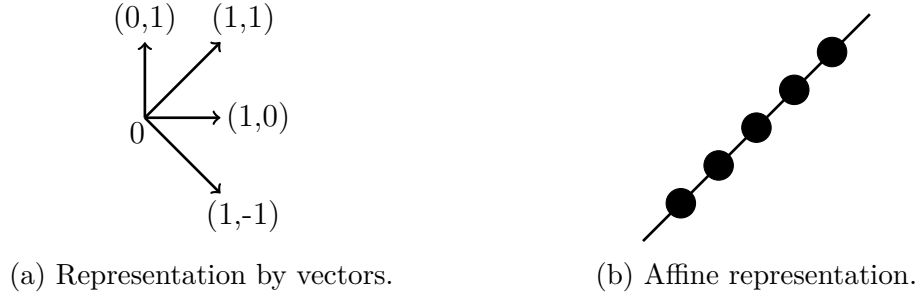


Figure 1.1: Some different ways of visualization of  $U_{2,4}$ .

Also when using affine spaces we can easily visualize  $U_{3,n}$  by simply putting  $n$  points in general position.

Some additional notes are for when we use some other field one can use the Fano's plane which specifically represents a Fano's matroid  $F_7$ . Which cannot be represented by reals.

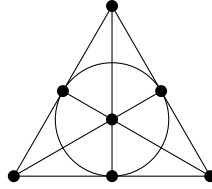


Figure 1.2: Fano's plane.

## 1.6 (Direct) Sum of matroids (also disjoint union)

**Definition 8.** We have two matroids  $\mathcal{M}_i = (E_i, \mathcal{I}_i)$  for  $i = 1, 2$ , then the (direct) sum  $\mathcal{M}_1 \oplus \mathcal{M}_2$  is defined as a matroid  $\mathcal{M} = (E, \mathcal{I})$  where  $E = E_1 \dot{\cup} E_2$  and  $\mathcal{I} = \{X \subseteq E, X \cap E_i \in \mathcal{I}_i, i = 1, 2\}$ .

**Observation.** Lets see the basis and circuits:

$$\mathcal{B}(\mathcal{M}_1 \oplus \mathcal{M}_2) = \{B_1 \cup B_2, B_i \in \mathcal{B}_i, i = 1, 2\}$$

$$\mathcal{C}(\mathcal{M}_1 \oplus \mathcal{M}_2) = \mathcal{C}_1 \cup \mathcal{C}_2$$

$$X \subseteq E : r(X) = r_1(X \cap E_1) + r_2(X \cap E_2)$$



## 2. Duals & Minors

### 2.1 Duals

Firstly we will take a look at duals of matroids.

**Claim 6.** Let  $\mathcal{M} = (E, \mathcal{I})$  be a matroid and  $\mathcal{B}$  its bases. We set  $\mathcal{B}^* = \{E \setminus B : B \in \mathcal{B}\}$  then  $\mathcal{B}^*$  satisfies (B1) and (B2).

*Proof.* Firstly (B1) is easy. For (B2)  $(E \setminus B_1), (E \setminus B_2) \in \mathcal{B}^*, \forall e \in (E \setminus B_1) \setminus (E \setminus B_2) = B_2 \setminus B_1, \exists f \in B_1 \setminus B_2 = (E \setminus B_2) \setminus (E \setminus B_1)$ . Now we apply (B2') and get that  $(B_1 - f) + e \in \mathcal{B}$  then  $E \setminus ((B_1 - f) + e) = ((E \setminus B_1) - e) + f$ .  $\square$

Therefore  $\exists \mathcal{M}^*$  matroid with  $\mathcal{B}(\mathcal{M}^*) = \mathcal{B}^*$  which will be called a **dual**. Generally we will denote duals  $\mathcal{M}$  as  $\mathcal{M}^*$ . Also we will call such elements with a prefix co-. For example  $\mathcal{B}^*$  will be a co-bases and so on.

**Observation.**  $(\mathcal{M}^*)^* = \mathcal{M}$

**Proposition 7.** There are two basic observations:

1.  $r(\mathcal{M}) + r(\mathcal{M}^*) = |E|$
2.  $\forall X \subseteq E : r^*(X) = |X| - r(\mathcal{M}) + r(E \setminus X)$

*Proof.* 1. is obvious from the definitions. 2. we denote  $I^*$  as the max independent subset of  $X$  in  $\mathcal{M}^*$  and  $I$  as the max independent subset of  $E \setminus X$  in  $\mathcal{M}$ . Then  $\exists B \in \mathcal{B}, B \cap I^* = \emptyset$ . We may assume  $I \subseteq B \Rightarrow r(B) = |B| = r(\mathcal{M})$ .

$$\begin{aligned} B &\subseteq E \setminus I^* \\ r(B) &\leq r(E \setminus I^*) \leq r(\mathcal{M}) \text{ therefore we get equality} \\ r(B) &= r(E \setminus I^*) \end{aligned}$$

Let  $B^* = E \setminus B$  be the basis of  $\mathcal{M}^*$  and  $I^* \subseteq B^*$ , moreover  $I^* = B^* \cap X$ . Also  $I = B \cap (E \setminus X)$ .

$$|X| = |X \cap B| + |X \cap B^*| = |B| - |I| + |I^*| = r(\mathcal{M}) - r(E \setminus X) + r^*(X)$$

$\square$

**Definition 9.** We say that  $H \subseteq E$  is a **hyperplane** in  $\mathcal{M}$  if  $H$  is  $\subseteq$ -max subset with  $r(H) < r(\mathcal{M})$ . Or in other words

$$\forall e \in E \setminus H : r(H) < r(H + e) < r(\mathcal{M}) \text{ so } r(H) = r(\mathcal{M}) - 1$$

**Lemma 4.** For a matroid  $\mathcal{M} = (E, \mathcal{I})$  the following holds:

$$\forall C^* \subseteq E, C^* \in \mathcal{C}^*(\mathcal{M}) \Leftrightarrow E \setminus C^* \text{ is a hyperplane of } \mathcal{M}.$$

*Proof.* " $\Rightarrow$ " Suppose  $C^* \in \mathcal{C}^*$  by the dual of proposition 7 we get

$$\begin{aligned} r(X) &= |X| - r^*(\mathcal{M}) + r^*(E \setminus X) \\ r(E \setminus C^*) &= |E \setminus C^*| - r^*(\mathcal{M}) + r^*(C^*) \\ &= |E| - |C^*| - r^*(\mathcal{M}) + |C^*| - 1 \\ &= r(\mathcal{M}) - 1 \end{aligned}$$

then this is maximal with this property, therefore it is a hyperplane.

" $\Leftarrow$ " is analogous.  $\square$

Lets now diverge to the graphic matroids, where when we decrease  $r(\mathcal{M})$  by 1 this will result in increasing components of connectivity; hence co-circuits are edge-cuts.

**Proposition 8.** *Let  $\mathcal{M} = (E, \mathcal{I})$  be a matroid,  $C \in \mathcal{C}, C^* \in \mathcal{C}^*$  then  $|C \cap C^*| \neq 1$ .*

*Proof.* By contradiction assume that  $C \cap C^* = \{e\}$ . Lets define  $H = E \setminus C^*$  a hyperplane for which  $e \notin H$ . Now we compute the following

$$\begin{aligned} r(C) + r(E) &= r(C - e) + r(H + e) \\ &= r(C \cap H) + r(C \cup H) \\ &\leq r(C) + r(H) \\ &= r(H) + r(E) - 1, \text{ which is a contradiction.} \end{aligned}$$

$\square$

Now we take a look at how does a direct sum of duals look like.

$$\begin{aligned} \mathcal{B}((\mathcal{M}_1 \oplus \mathcal{M}_2)^*) &= \{E \setminus B, B \in \mathcal{B}(\mathcal{M}_1 \oplus \mathcal{M}_2)\} \\ &= \{E \setminus (B_1 \cup B_2), B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\} \\ &= \{(E_1 \cup E_2) \setminus (B_1 \cup B_2), B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\} \\ &= \{(E_1 \setminus B_1) \cup (E_2 \setminus B_2), B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\} \\ &= \{B_1^* \cup B_2^*, B_1^* \in \mathcal{B}^*(\mathcal{M}_1), B_2^* \in \mathcal{B}^*(\mathcal{M}_2)\} \\ &= \mathcal{B}(\mathcal{M}_1^* \oplus \mathcal{M}_2^*) \end{aligned}$$

Therefore we may say that  $(\mathcal{M}_1 \oplus \mathcal{M}_2)^* = \mathcal{M}_1^* \oplus \mathcal{M}_2^*$ .

## 2.2 Minors

Reader may already know minors in graph theory. They are made by two operations. Deletions of vertices and edges and contractions of edges. For minors it will work similarly, therefore we define such operations.

**Definition 10** (Deletion). *For a matroid  $\mathcal{M} = (E, \mathcal{I})$  for  $T \subseteq E$  we define  $\mathcal{M} \setminus T = (E \setminus T, \mathcal{I}')$  where  $\mathcal{I}' = \{I \in \mathcal{I}, I \subseteq E \setminus T\}$ .*

**Definition 11** (Contraction).  $\mathcal{M}/T = (\mathcal{M}^* \setminus T)^*$

**Proposition 9.** *For  $\mathcal{M} = (E, \mathcal{I}), T \subseteq E$  assume  $X \subseteq E \setminus T$ , then*

1.  $r_{\mathcal{M} \setminus T}(X) = r_{\mathcal{M}}(X)$  and

$$2. \ r_{\mathcal{M}/T}(X) = r_{\mathcal{M}}(X \cup T) - r_{\mathcal{M}}(T).$$

*Proof.* 1. can be easily seen. For 2. we do the following computation.

$$\begin{aligned} r_{\mathcal{M}/T}(X) &= r_{\mathcal{M}^* \setminus T}^*(X) \\ &= |X| - r_{\mathcal{M}^* \setminus T}(E \setminus T) + r_{\mathcal{M}^* \setminus T}((E \setminus T) \setminus X) \\ &= |X| - r^*(E \setminus T) + r^*(E \setminus (T \cup X)) \\ &= |X| - (|E \setminus T| - r(E) + r(T)) + |E \setminus (T \cup X)| - r(E) + r(E \setminus (E \setminus (T \cup X))) \\ &= |X| - |E| + |T| + |E| - |T| - |X| + r(E) - r(E) - r(T) + r(T \cup X) \\ &= r(T \cup X) - r(T) \end{aligned}$$

□

**Proposition 10.** *We can prove that  $\mathcal{M}/T = (E \setminus T, \mathcal{I}')$  where  $\mathcal{I}' = \{I \subseteq E \setminus T \mid I \cup B_T \in \mathcal{I}\}$ , where  $B_T$  is max independent subset of  $T$ .*

*Proof.* In one direction let's have  $I \subseteq E \setminus T$  s.t.  $I \cup B_T \in \mathcal{I}$  now its rank has to be equal to its size.

$$\begin{aligned} r_{\mathcal{M}/T}(I) &= r(I \cup T) - r(T) \\ &= r(I \cup B_T) - r(B_T) \\ &= |I \cup B_T| - |B_T| \\ &= |I| + |B_T| - |B_T| = |I| \end{aligned}$$

Now the other direction. For that let's have  $X \in \mathcal{I}'$  then the following holds.

$$\begin{aligned} |X| &= r_{\mathcal{M}/T}(X) = r(X \cup T) - r(T) = r(X \cup T) - r(B_T) = r(X \cup B_T) - |B_T| \\ |X \cup B_T| &= |X| + |B_T| = r(X \cup B_T) \Rightarrow X \cup B_T \in \mathcal{I} \end{aligned}$$

□

Therefore we may say that for dual it is true that:  $B' \in \mathcal{B}(\mathcal{M}/T) \Leftrightarrow \exists B_T \in \mathcal{B}(\mathcal{M}|T)$  s.t.  $B' \cup B_T \in \mathcal{B}(\mathcal{M})$  where existential kvantifikator can be replaced by for all. Also on notation  $\mathcal{M}|T = \mathcal{M} \setminus (E \setminus T)$  called the *restriction*.

**Proposition 11.**  $\mathcal{C}(\mathcal{M}/T)$  are minimal non-empty members  $\{C \setminus T \mid C \in \mathcal{C}(\mathcal{M})\}$ .

*Proof.* For " $\Rightarrow$ " consider  $C_1 \in \mathcal{C}(\mathcal{M}/T)$  and  $B_T$  max independent in  $\mathcal{M}|T$ .  $C_1 \cup B_T \notin \mathcal{I}, \forall e \in C_1 : (C_1 - e) \cup B_T \in \mathcal{I}$  and therefore  $\exists C \in \mathcal{C}(\mathcal{M})$  s.t.  $C_1 \subseteq C \subseteq C_1 \cup B_T \Rightarrow C_1 = C \setminus B_T = C \setminus T$ .

Now for " $\Leftarrow$ " let  $D \setminus T$  be min non-empty member  $\{C \setminus T \mid C \in \mathcal{C}(\mathcal{M})\}$  so  $D \cap T \subsetneq D \Rightarrow D \cap T \in \mathcal{I} \Rightarrow \exists B_T$  max independent in  $\mathcal{M}|T$  s.t.  $D \cap T \subseteq B_T$  and now  $D \subseteq D \cup B_T = (D \setminus T) \cup B_T \notin \mathcal{I}$  because it contains a circuit  $\Rightarrow (D \setminus T) \notin \mathcal{I}(\mathcal{M}/T)$  therefore  $\exists D' \in \mathcal{C}(\mathcal{M}/T) : D' \subseteq D \setminus T$  if  $D' \subsetneq D \setminus T$  then it would contradict the minimality. □

**Claim 12.** For  $\mathcal{M} = (E, \mathcal{I}), T_1, T_2 \subseteq E, T_1 \cap T_2 = \emptyset$  we have:

1.  $(\mathcal{M} \setminus T_1) \setminus T_2 = (\mathcal{M} \setminus T_2) \setminus T_1 = \mathcal{M} \setminus (T_1 \cup T_2);$
2.  $(\mathcal{M}/T_1)/T_2 = (\mathcal{M}/T_2)/T_1 = \mathcal{M}/(T_1 \cup T_2);$

$$3. (\mathcal{M} \setminus T_1)/T_2 = (\mathcal{M}/T_2) \setminus T_1.$$

*Proof.* Lets go through all statements.

1. This is obviously true, since we only delete sets which is same.

2. For this see

$$\begin{aligned} (\mathcal{M}/T_1)/T_2 &= (\mathcal{M}^* \setminus T_1)^*/T_2 = ((\mathcal{M}^* \setminus T_1) \setminus T_2)^* \\ &= (\mathcal{M}^* \setminus (T_1 \cup T_2))^* = \mathcal{M}/(T_1 \cup T_2). \end{aligned}$$

3. Lets have  $X \subseteq E \setminus (T_1 \cup T_2)$ , then:

$$\begin{aligned} r_{(\mathcal{M}/T_2) \setminus T_1}(X) &= r_{\mathcal{M}/T_2}(X) = r_{\mathcal{M}}(X \cup T_2) - r_{\mathcal{M}}(T_2) \\ &= r_{\mathcal{M} \setminus T_1}(X \cup T_2) - r_{\mathcal{M} \setminus T_1}(T_2) = r_{(\mathcal{M} \setminus T_1)/T_2}(X). \end{aligned}$$

□

*Corollary.*  $\mathcal{M}$  has minor  $\mathcal{N}$ , then  $\mathcal{M}^*$  has minor  $\mathcal{N}^*$ . Namely  $\mathcal{N} = (\mathcal{M} \setminus T_1)/T_2$  then  $\mathcal{N}^* = (\mathcal{M}^* \setminus T_2)/T_1$ .

## 2.3 Duals and minors of vector matroids

**Theorem 13.** Let  $\mathcal{M} = \mathcal{M}(A)$  be a matroid over matrix  $A$  s.t.  $\mathcal{M} \not\cong U_{0,n}$  and  $\mathcal{M} \not\cong U_{n,n}$ . We may consider standard form which is  $A = (I_n | D)$  and  $\mathcal{M}(A) = \mathcal{M}((I_n | D))$  and  $r = r(\mathcal{M})$  then  $\mathcal{M}^* = \mathcal{M}(D^T | I_{n-r})$ .

*Proof.* See that  $B \in \mathcal{B}(\mathcal{M}) \Leftrightarrow B = \{e_1, e_2, \dots, e_n\}$  which all  $e_i$  are linearly independent in  $(I_n | D)$ . Now take a look at the matrix:

$$\left[ \begin{array}{c|cc} & D_{11} & D_{12} \\ I_n & & \\ & D_{21} & D_{22} \end{array} \right]$$

Consider that first  $s$  columns are taken as first columns from  $I_n$  and the rest is from  $D_{11}$ . Which also means that  $r(D_{21}) = r - s$ . Now take the transposed matrix.

$$\left[ \begin{array}{cc|c} D_{11}^T & D_{21}^T & \\ D_{12}^T & D_{22}^T & \\ & & I_{n-r} \end{array} \right]$$

There the rank  $r(D_{21}^T) = r - s$  which means that all of  $E \setminus B$  are linearly independent in  $(D^T | I_n)$ . □

Now we take a moment to explore the deletion. For that we just erase the columns which are about to be deleted. So for contraction is also nicely working from the dual and deletion. All in all vector matroids are closed under both **minors** and **duals**. Also note that if  $D$  would be symmetric (i.e.  $D = D^T$ ) then  $\mathcal{M} \cong \mathcal{M}^*$  also called **self-dual**.

## 2.4 Duals and minors of graphic matroids

Firstly consider a deletion in a graphic matroid. This will correspond to simple deletion of all such edges. Now for contractions we will be looking at it just for one single element.

**Proposition 14.**  $\mathcal{M} = \mathcal{M}(G), e \in E$  which is an element of  $\mathcal{M}$  and also an edge. Then  $\mathcal{M}(G)/\{e\} = \mathcal{M}(G/e)$ .

*Proof.* If  $e$  is a loop, then we can simply delete it in matroid and also in the graph. Otherwise  $U \subseteq E - e$  is acyclic in  $G/e$  if and only if  $I \cup \{e\}$  is acyclic in  $G$  if and only if  $I \in \mathcal{M}(G/e) \iff I \cup \{e\} \in \mathcal{I}$ .  $\square$

From this proposition and the easy observation we obtain the fact that graphic matroids are closed under **minor** operations. Let now consider duals of graphic matroids for a moment. The duals are quite nice since it follows from graph duals which are fairly known. That is we have one-to-one correspondence between  $E(G)$  and  $E(G^*)$ . Also  $(G^*)^* = G$  and  $\mathcal{M}^*(G) = \mathcal{M}(G^*)$ . Moreover having a circuit in  $G$  corresponds to min cut of the dual hence the duality works. There the duality works.

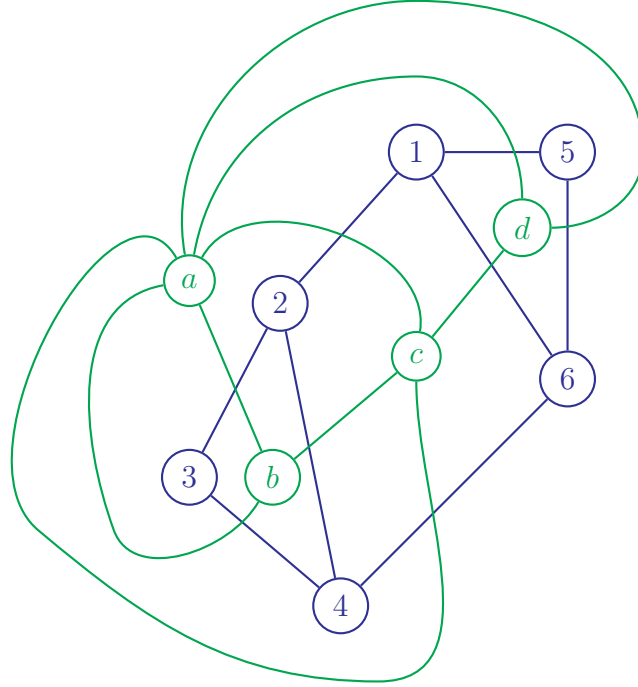


Figure 2.1: A primal graph  $G$  and its dual  $G^*$ .

Also duals in Matroids (and also minors) were inspired from graphs. Now we have encountered only planar graphs. But one can ask what about non-planar ones. By Kuratowski and the fact that graphic matroids are closed under minor operations we get the following proposition.

**Proposition 15.**  $\mathcal{M}^*(K_5)$  (and  $\mathcal{M}^*(K_{3,3})$ ) are not graphic.

*Proof.* By a contradiction:  $\exists G : \mathcal{M}(G) = \mathcal{M}^*(K_5)$ . By that  $|E| = 10, r(\mathcal{M}^*(K_5)) = 6 = 10 - 4$ . Also  $|V(G)| = 7$  so average degree is  $\frac{20}{7} < 3$ . Which means that there exists a vertex  $v$  s.t.  $\deg(v) = 1$  or  $2$ . This leads to at most 2 edges therefore the size of min edge cut is 2 and so is the size of co-circuit of  $\mathcal{M}^*(K_5)$  hence the size of circuit in  $\mathcal{M}(K_5)$  is 2 which is a contradiction. Similar argument can be made for  $K_{3,3}$ .  $\square$

From these facts if we have a non-planar graph its dual is **not graphic**. Therefore we get the following diagram.

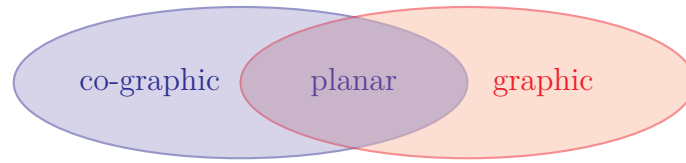


Figure 2.2: Diagram of graphic duals.

## 3. Algorithms

In this section we will walk through some of the known matroid algorithms for solving some problems. Also we may derive some more information about the matroids themselves by looking at the solution of the given algorithms.

### 3.1 Greedy algorithm

The main algorithm is greedy one. One can already know the greedy algorithm for min spanning tree which is gradually putting lightest edges not forming cycle into the set. Now it will be pretty much the same only we will be maximizing and not minimizing. That is for minimizing we create a constant  $C$  and subtract every value from it and still look for maximum which will indeed be a minimum with the original values.

First question is how to save all values for given matroid. Because the size of  $\mathcal{I}$  can be exponentially large. Therefore we will be working with so called **oracle**. That is for  $X \subseteq E$  we ask if  $X$  is independent  $\in \mathcal{I}$  or not.

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**Algorithm 1** Greedy algorithm.

---

**Require:** Matroid  $\mathcal{M} = (E, \mathcal{I})$  and weight function  $w : E \rightarrow \mathbb{R}_0^+$ .

**Ensure:**  $I \in \mathcal{I}$  with  $\max w(I) = \sum_{e \in I} w(e)$ .

- 1: Sort  $E$  s.t.  $w(e_1) \geq w(e_2) \geq \dots \geq w(e_n)$ .
  - 2:  $E_0 = \emptyset$
  - 3: **for**  $i = 1, 2, \dots, n$  **do**
  - 4:     **if**  $E_{i-1} + e_i \in \mathcal{I}$  **then**
  - 5:          $E_i = E_{i-1} + e_i$
  - 6:     **else**
  - 7:          $E_i = E_{i-1}$
  - 8:     **end if**
  - 9: **end for**
  - 10: **return**  $E_n$
- 

#### 3.1.1 Correctness

Firstly we can clearly see that  $\forall i : E_i \subseteq E$  and  $E_i \in \mathcal{I} \Rightarrow E_n \in \mathcal{I}$  which also means that  $E_n \in \mathcal{I}$  and  $E_n \in \mathcal{B}$ .

Now look at maximality. For contradiction  $\exists E' \in \mathcal{B}$  s.t.  $w(E') > w(E_n)$ . See that  $E_n$  is formed by  $e_{i_1}, e_{i_2}, \dots, e_{i_r}$  if  $r = r(\mathcal{M})$  ( $i_1 < i_2 < \dots < i_r$ ) and also  $E'$  is formed by  $e_{j_1}, e_{j_2}, \dots, e_{j_r}$  ( $j_1 < j_2 < \dots < j_r$ ), where the lower index the higher weight. Therefore  $\exists k$  s.t.  $w(e_{i_k}) > w(e_{j_k})$  and let's take such smallest one. Now let's have  $I_1 = \{e_{i_1}, \dots, e_{i_{k-1}}\}$  and  $I_2 = \{e_{j_1}, \dots, e_{j_k}\}$  and apply (I3) because  $|I_1| < |I_2|$  therefore  $\exists l : I_1 + e_{j_l} \in \mathcal{I}$  and  $w(e_{j_l}) \geq w(e_{j_k}) > w(e_{i_k})$  and  $e_{j_l} \notin I_1$  which is a contradiction because algorithm would have chosen  $e_{j_l}$  instead of  $e_{i_k}$ .

### 3.1.2 Time complexity

Firstly sorting  $E$  is in  $O(n \log n)$ . Then the loop has  $n$  repetitions where in every step we call an oracle. Lets say that the oracle has time complexity  $t$  so the entire loop take  $nt$  time. So altogether we obtain  $O(n \log n + nt)$ .

Lastly note that non-negative weight function is important for the algorithm to work. But we may see that this procedure can be used to define the matroid. Which we will talk about now.

### 3.1.3 Defining matroid by greedy algorithm

**Proposition 16.** *We have  $E$  finite non-empty set, let  $\mathcal{F} \subseteq 2^E, \emptyset \in \mathcal{F}$  s.t.  $\forall w : E \rightarrow \mathbb{R}_0^+$  greedy algorithm find max weighted member of  $\mathcal{F}$ , then  $\mathcal{F}$  satisfies (I1), (I2) and (I3).*

*Proof.* Firstly (I1) is easy, since  $\emptyset \in \mathcal{F}$ .

Now take a look at (I2). By contradiction  $I' \subseteq I$  and  $I \in \mathcal{F}$  but  $I' \notin \mathcal{F}$ . WLOG  $|I| = |I'| + 1$ . We will define the weight function like this:

$$w(e) = \begin{cases} 2 & \text{if } e \in I' \\ 1 & \text{if } e \in I \setminus I' \\ 0 & \text{otherwise} \end{cases}.$$

So  $w(I) = 2 \cdot |I'| + 1$  and greedy algorithm find  $|I'|$ , but at least one element has to be skipped. So the weight of the result is  $\leq 2(|I'| - 1) + 1 = 2 \cdot |I'| - 1 < w(I)$  which is a contradiction.

Finally (I3) which is  $I_1, I_2 \in \mathcal{F}, |I_1| < |I_2| : \exists e \in I_2 \setminus I_1 : I_1 + e \in \mathcal{F}$  and WLOG  $|I_2| = |I_1| + 1$ . By contradiction  $I_1, I_2 \in \mathcal{F}, |I_1| = |I_2| - 1$  and  $\forall e \in I_2 \setminus I_1 : I_1 + e \notin \mathcal{F}$ . Denote  $k = |I_1|$ . Lets define the weight function.

$$w(e) = \begin{cases} k + 2 & \text{if } e \in I_1 \\ k + 1 & \text{if } e \in I_2 \setminus I_1 \\ 0 & \text{otherwise} \end{cases}.$$

Where  $w(I_2) \geq |I_2| \cdot (k + 1) = (k + 1)^2$ . The greedy algorithm firstly takes all  $e \in I_1$  so after  $k$  steps  $E_k = I_1$ , but then non-zero elements are skipped ( $I_2 \setminus I_1$ ) so the result has  $w = I_1 \cdot (k + 2) = k(k + 2)$  but  $k(k + 2) = k^2 + 2k < k^2 + 2k + 1 = (k + 1)^2$  so we got the contradiction.  $\square$

## 3.2 Matroid intersection problem

Firstly we must define this problem. As an **input** we have  $\mathcal{M}_i = (E, \mathcal{I}_i)$  for  $i = 1, 2$  and **output** is to find  $\max I \in \mathcal{I}_1 \cap \mathcal{I}_2$ . Firstly observe that there always exists a solution  $\emptyset$ .

**Theorem 17.** *Having  $\mathcal{M}_i = (E, \mathcal{I}_i)$  for  $i = 1, 2$  then*

$$\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| = \min_{E_1 \cup E_2 = E} r_1(E_1) + r_2(E_2).$$

*Proof.* For " $\leq$ " we have  $I \in \mathcal{I}_1 \cap \mathcal{I}_2, E_1 \cup E_2 = E$ . We may consider it is disjoint since  $E_2 = E \setminus E_1$  might only decrease the rank.



$$\begin{array}{rcl}
|I \cap E_1| & = & r_1(I \cap E_1) \leq r_1(E_1) \\
|I \cap E_2| & = & r_2(I \cap E_2) \leq r_2(E_2) \quad (\text{sum both together}) \\
|I| & & \leq r_1(E_1) + r_2(E_2)
\end{array}$$

Now " $\geq$ " for which we will state the algorithm. Firstly start in  $I = \emptyset$ . Then  $I \in \mathcal{I}_1 \cap \mathcal{I}_2$  construct  $H$  bipartite graph with  $I, X = E \setminus I$  parts. Create arc  $y \rightarrow x$  if  $(I - y) + x \in \mathcal{I}_1$  and arc  $y \leftarrow x$  if  $(I - y) + x \in \mathcal{I}_2$ . Then define  $X_1 = \{x \in X \mid I + x \in \mathcal{I}_1\}$  and  $X_2 = \{x \in X \mid I + x \in \mathcal{I}_2\}$ .

1. If  $X_1 \cap X_2 = \emptyset$  we just add such element from intersection to  $I$ .
2. Otherwise find  $X_1 \rightarrow X_2$  path if there is one, and choose shortest one. That is  $x_0 y_1 x_1 y_2 x_2 \dots y_k x_k$  this path where  $x_0 \in X_1$  and  $x_k \in X_2$ . Now we update  $I$  like this  $I' = (I \setminus \{y_1, y_2, \dots, y_k\}) \cup \{x_0, x_1, x_2, \dots, x_k\}$ . Clearly the size is bigger but it is independent? We will sketch the proof of this. We have to show that  $I' \in \mathcal{I}_1 \cap \mathcal{I}_2$ .

For  $I' \in \mathcal{I}_1$  we proceed by induction on  $k$ . Lets have  $I \in \mathcal{I}$  and  $(I - y_k) + x_k$  because  $y_k \rightarrow x_k$  by the definition of arc it implies that it is  $\in \mathcal{I}_1$ . Lets denote  $I_l = (I \setminus \{y_l, \dots, y_k\}) \cup \{x_l, \dots, x_k\} = (((I \setminus \{y_{l+1}, \dots, y_k\}) \cup \{x_{l+1}, \dots, x_k\}) - y_l) + x_l$  and  $I_{l+1}$  is  $\in \mathcal{I}_1$  by induction. So  $I_l = (I_{l+1} - y_l) + x_l$ ,  $I_l + x_l \in \mathcal{I}_1$  and  $y_l \rightarrow x_l$  therefore  $(I - y_l) + x_l \in \mathcal{I}_1$ . We obtained two independent sets where  $|I_l - x_l| = |(I - y_l) + x_l| - 1$  so by (I3)  $\exists z \in (I - y_l) + x_l$  s.t.  $(I_l - x_l) + z \in \mathcal{I}_1$ .

If  $z = x_l$  we are done, otherwise if  $z \neq x_l$  we set  $(I - y_l + x_l) \setminus (I_l - x_l) = \{x_l, y_{l+1}, \dots, y_k\}$  so  $\exists i : z = y_i$   $(I_l - x_l) + y_i \in \mathcal{I}_1$  and again apply (I3) so  $\exists z' \in ((I_l - x_l) + y_i) \setminus (I - y_l)$  s.t.  $(I - y_l) + z' \in \mathcal{I}_1$  and  $z' = \{x_{l+1}, \dots, x_k\}$ . This implies that  $y_l \rightarrow z'$  which contradicts the shortest path. Hence we may remove  $y_1, \dots, y_k$  and add  $x_1, \dots, x_k$ . Now we only need to show we may also add  $x_0$ . For showing  $I' \in \mathcal{I}_2$  we would proceed similarly.

3. Next step is what if the path does not exists? We want to show a partitioning  $E_1$  and  $E_2$ .

□