Barevnost grafů a kombinatorických struktur

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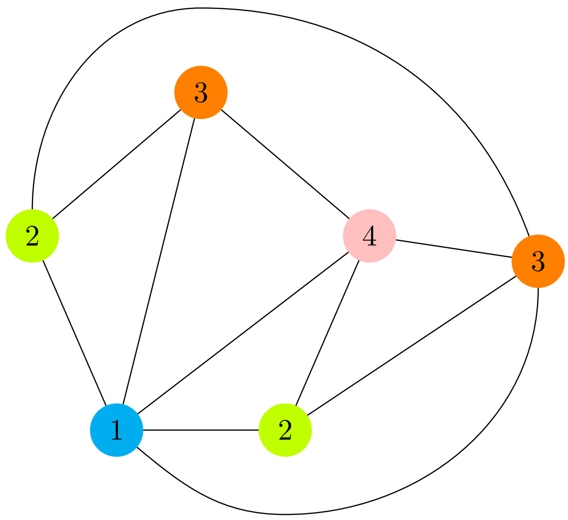

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Information

Also there may be some mistakes. If you find some and want to update them, you may find all the sources on the GitHub.

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1. Revision and introduction

One can already know some basics to the graph coloring and also some of the theorems. Therefore we will briefly introduce some basics and revisit some of the known theorems and perhaps some we will show throughout this document.

Notation. *First of all some basic notations.*

- $\chi(G)$ *is minimal vertex coloring of a graph* G*.*
- $\Delta(G)$ *is the maximal degree of a graph* G*.*
- $\delta(G)$ *is the minimal degree of a graph* G*.*
- $\chi_e(G)$ *is minimal edge coloring of a graph* G*.*

Theorem 1 (Four colors)**.** *Every planar graph* G *has* $\chi(G) \leq 4$*.*

Theorem 2 (Brooks)**.** *If* G *is a connected graph and* $G \neq K_n$ *and* $G \neq C_{2n+1}$ *then* $\chi(G) \leq \Delta(G)$*.*

Theorem 3 (Vizing)**.** $\Delta(G) \leq \chi_e(G) \leq \Delta(G) + 1$

Theorem 4 (General Euler's formula)**.** *If* G *can be drawn on a surface of Euler genus* g *then* $|E| \leq |V| + |F| - g$*. Where* $|F|$ *is for the number of faces of the drawing.*

From Euler's formula one can see that if $|V| \geq 3$ then $|E| \leq 3|V| + 3g - 6$. Therefore the average degree is $\frac{2|E|}{|V|} = \frac{6(g-2)}{|V|}$.

Theorem 5 (Heawood's formula)**.** *If* G *can be drawn on a surface of Euler genus* g *then*

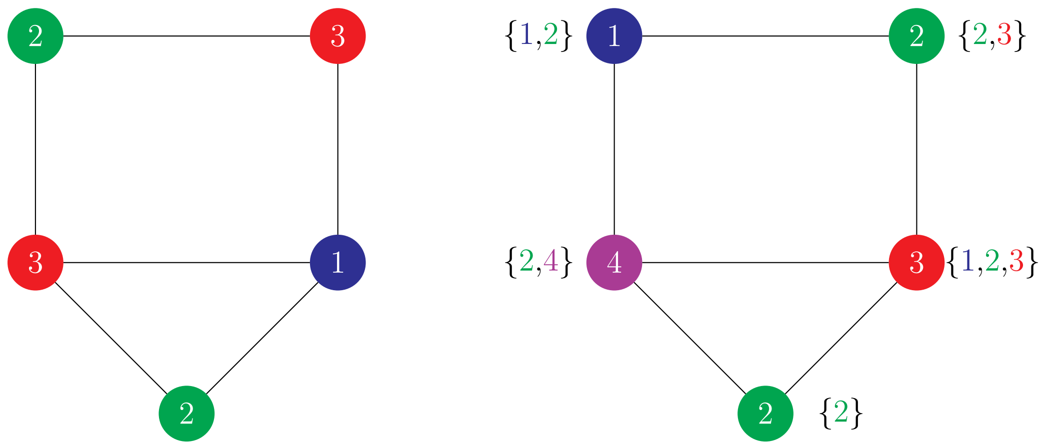
$$\chi(G) \leq \frac{7 + \sqrt{24g + 1}}{2}$$

Where this formula is tight for Klein's bottle. Then we will show us that deciding if a planar graph is 3-colorable is NP-hard problem. But on the other hand we have this theorem.

Theorem 6 (Grötsch)**.** *Every planar graph without triangle is 3-colorable.*

2. List coloring

Firstly we can see a graph G and its normal coloring which is depicted on a picture 2.1a. Whereas the list coloring shown on picture 2.1b is that each vertex has a assigned list for which we can choose colors. Otherwise the coloring is the same.



(a) Normal coloring of a graph G.　　(b) List coloring of a graph G.

Figure 2.1: Difference between basic and list coloring of a graph G.

Definition 1. *k-**list-assignment** is for an assignment for all vertices of size k.*

Definition 2. *The graph is k-**list-colorable** if G can be colored by every k-list-assignment.*

Definition 3. ***List chromatic number** of a graph G is denoted as $\chi_l(G)$. That is the min k s.t. G is k-list-colorable.*

Observation. $\chi(G) \leq \chi_l(G)$

Now one could prove that the Heawood's formula works pretty much the same, thus $\chi_l(G) \leq \frac{7+\sqrt{24g+1}}{2}$, but only if $g > 0$. On the other hand one can find a planar graph which has $\chi_l(G) > 4$, but it can be shown that for every planar graph G the $\chi_l(G) \leq 5$ which was proven by Thomassen. Vizing's theorem will be shown and the Brooks' theorem remains the same.

Definition 4. *L is a **degree-list-assignment** for graph G if $|L(v)| \geq \deg(v)$ $\forall v \in V$. And G is a **degree-list-colorable** if G is colorable from every degree-list-assignment.*

Definition 5 (Gallai tree)**.** ***Gallai tree** is a graph where each 2-connected block is either a clique or an odd cycle.*

Observation. *Gallai trees are not degree-list-colorable.*

Theorem 7 (Brooks)**.** *If G is connected and G is not a gallai tree then G is degree-list-colorable.*

Proof.

Lemma 1. *G is connected, L is a degree-list-assignment $(\exists v \in V(G))$ $|L(v)| > \deg(v) \Rightarrow$ G is L-colorable.*

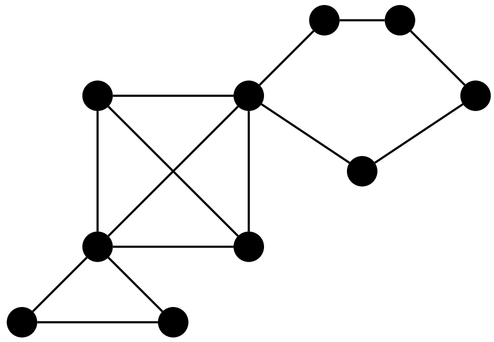


Figure 2.2: Example of a gallai tree.

Lemma 2. *G is connected, L is a degree-list-assignment, $uv \in E(G)$, u is not a cut vertex, G not L-colorable then $L(u) \subseteq L(v)$.*

Corollary. G connected, L is a degree-list-assignment, $uv \in E(G)$, u,v not cut vertices, G not L-colorable then $L(u) = L(v)$.

Corollary. G is 2-connected, L is degree-list-assignment, G is not L-colorable then $\forall u,v \in V(G) : L(u) = L(v)$ and G is a clique or an odd cycle.

□

TODO: From this part it is missing.

3. Critical graphs

Exercise. *If G is k-critical, then it is 2-connected.*