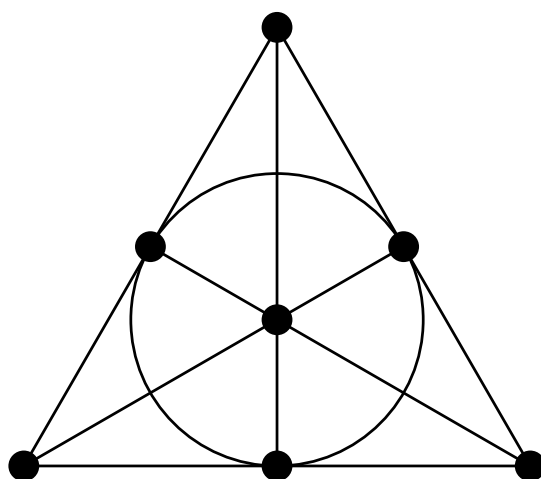


Matroid Theory

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Information

*These are my notes on the course Matroid Theory, which was taught by
Ondřej Pangrác in the year 2024.*

Keep in mind there may be some **mistakes**. You may visit [GitHub](#).

1. Basic definitions

Definition 1. Matroid $\mathcal{M} = (E, \mathcal{I})$ is for E finite non-empty set and $\mathcal{I} \subseteq 2^E$ (also called as independent sets) satisfying these properties:

(I1) $\emptyset \in \mathcal{I}$,

(I2) $I \in \mathcal{I} \Rightarrow \forall I' \subseteq I : I' \in \mathcal{I}$,

(I3) $I_1, I_2 \in \mathcal{I}, |I_1| < |I_2| \Rightarrow \exists e \in I_2 \setminus I_1 : I_1 \cup \{e\} \in \mathcal{I}$.

Notation. For further use and simplification we will sometimes use $I+e$ as a substitution for $I \cup \{e\}$. Similarly also $I-e$ for $I \setminus \{e\}$.

Example. For a given multi-graph $G = (V, F)$ we will set $E = F$ (or in other words E stands for edges and the set). Independent sets \mathcal{I} will be all acyclic subsets of E . Easily seen (I1) and (I2) is satisfied. For the third one (I3) it is also quite easily seen, because if we have one larger and smaller non-cycles then we can append one edge from the larger to the smaller.

Example. Let E be some elements of a vector space V . If $X \subseteq E$ is independent then it is linearly independent in V .

Definition 2. Matroid isomorphism for two matroids $\mathcal{M}_i = (E_i, \mathcal{I}_i)$ for $i = 1, 2$ is a bijection $f : E_1 \rightarrow E_2$ satisfying $\forall X \subseteq E_i : X \in \mathcal{I}_1 \Leftrightarrow f(X) \in \mathcal{I}_2$.

1.1 Circuits

Definition 3. $X \subseteq E$ is a **circuit** if $X \notin \mathcal{I}$ and $\forall x \in X : X - x \in \mathcal{I}$. Also we will denote $\mathcal{C}(\mathcal{M})$ as the set of all circuits of \mathcal{M} .

Lemma 1. Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and \mathcal{C} its collection of circuits, then

(C1) $\emptyset \notin \mathcal{C}$,

(C2) $\forall C_1, C_2 \in \mathcal{C} : C_1 \subseteq C_2 \Rightarrow C_1 = C_2$ and

(C3) $C_1, C_2 \in \mathcal{C}, C_1 \neq C_2, e \in C_1 \cap C_2 \Rightarrow C_3 \subseteq (C_1 \cup C_2) - e, C_e \in \mathcal{C}$.

Proof. (C1) and (C2) are easily seen from (I1) and (I2). Now for the third part (C3). So for contradiction let C_1, C_2, e be as mentioned in the first part, but $(C_1 \cup C_2) - e \in \mathcal{I}$. Then $\exists f \in C_2 \setminus C_1 : C_2 - f \in \mathcal{I}$. Now find $I \in \mathcal{I}$ max s.t. $C_2 \setminus \{f\} \subseteq I \subseteq C_1 \cup C_2$. If $f \notin I$ then it would contain C_2 which is dependent and $\exists g \in C_1 \setminus C_2 : g \notin I$ otherwise it would contain C_1 which is dependent. Therefore

$$|I| \leq |C_1 \cup C_2| - 2 < |(C_1 \cup C_2) - e|$$

and now we may use the third axiom (I3) that is $\exists x \in |(C_1 \cup C_2) - e| \setminus I$ s.t. $I + x \in \mathcal{I}$ (this cannot be otherwise I contains the whole C_2). Now $I + x$ contradicts the maximality of I . \square

Claim 1. *Lets have E and $\mathcal{C} \subseteq 2^E$ satisfying all (C1), (C2) and (C3). Then set $\mathcal{I} = \{X \subseteq E | \forall C \in \mathcal{C} : C \not\subseteq X\}$ and $\mathcal{M} = (E, \mathcal{I})$ is a matroid.*

Proof. We have to show all properties of matroid. That is (I1) is trivially satisfied and (I2) also trivially holds. For the last (I3) we use a contradiction. For that we have $I_1, I_2 \in \mathcal{I}$, then $\forall e \in I_2 \setminus I_1 : I_1 + e \notin \mathcal{I}$. Let $I_3 \subseteq I_1 \cup I_2$ s.t. $|I_3| > |I_1|$ and $|I_1 \setminus I_3|$ is minimal. If $|I_1 \setminus I_3|$ would be empty then (I3) will hold, therefore assume it is non-empty.

Fix $e \in I_1 \setminus I_3$. Let $I_k = |I_3 - f| + e$ for $(f \in I_3 \setminus I_1)$. This cannot be independent ($\notin \mathcal{I}$) therefore $\exists C_k \subseteq T_k : C_k \in \mathcal{C}$ and $f \notin C_k, e \in C_k$.

$(I_3 \setminus I_1) \cap C_k = \emptyset$ hence $C_k \subseteq T_k \setminus (I_3 \setminus I_1) = (I_1 \cap I_3) + e \subseteq I_1$ this is not possible so it must be non-empty. Then $\exists g \in (I_3 \setminus I_1) \cap C_k \Rightarrow C_k, C_g \in \mathcal{C}, e \in C_k \cap C_g, f \notin C_k, g \notin C_g$ but $(C_k \cup C_g) - e \subseteq I_3$ which is contradiction with (C3). \square

1.2 Basis

Definition 4. *Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. Then B is a **basis** iff $B \in \mathcal{I}, \forall x \in E \setminus B : B + x \notin \mathcal{I}$.*

Proposition 2. *Let B_1, B_2 be bases of \mathcal{M} , then $|B_1| = |B_2|$.*

Proof. If $|B_1| < |B_2|$ then by (I3) $\exists x \in B_2 \setminus B_1 : B_1 + x \in \mathcal{I}$. \square

Definition 5. *Let $\mathcal{B}(\mathcal{M}) = \{B \subseteq E, B \text{ is a basis}\}$ be a collection of basis satisfying*

(B1) $\mathcal{B} \neq \emptyset$ and

(B2) $B_1, B_2 \in \mathcal{B}, e \in B_1 \setminus B_2 \Rightarrow \exists f \in B_2 \setminus B_1 : |B_1 - e| + f \in \mathcal{B}$.

One can see that (B2) can be proven using $I_1 - e =: B_1$ and $I_2 = B_2$.

Proposition 3. *Let $E \neq \emptyset$ finite set and $\mathcal{B} \subseteq 2^E$ satisfying (B1) and (B2). Let $\mathcal{I} = \{X \subseteq E : \exists B \in \mathcal{B} : X \subseteq B\}$ then $\mathcal{M} = (E, \mathcal{I})$ is a matroid.*

Proof. (I1) and (I2) are trivial. For (I3) use the following lemma.

Lemma 2. *Let \mathcal{B} be such that it satisfies (B1) and (B2). Then $\forall B_1, B_2 \in \mathcal{B} : |B_1| = |B_2|$.*

Proof. By contradiction suppose $|B_1| > |B_2|$ with minimal $|B_1 \setminus B_2|$. Then $e \in B_1 \setminus B_2 \Rightarrow \exists f \in B_2 \setminus B_1 : (B_1 - e) + f \in \mathcal{B}$ and also $|(B_1 - e) + f| = |B_1|$ which leads to $|((B_1 - e) + f) \setminus B_2| < |B_1 \setminus B_2|$ which is a contradiction with the minimality. \square

\square

1.3 Rank function

Definition 6. *For a matroid $\mathcal{M} = (E, \mathcal{I})$ define a **rank function** $r : 2^E \rightarrow \mathbb{Z}_0^+$, such that $r(X) = \max_{I \subseteq X, I \in \mathcal{I}} |I|$ and $r(\mathcal{M}) = r(E)$.*