Matroid Theory

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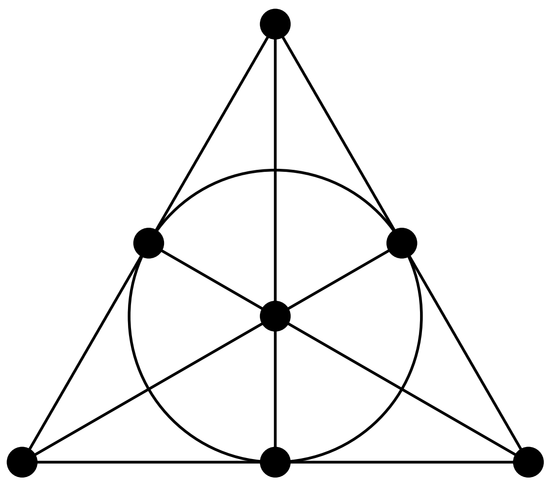

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Information

These are my notes on the course Matroid Theory, which was taught by Ondřej Pangrác in the year 2024.

Keep in mind there may be some mistakes. You may visit GitHub.

1. Basic definitions

Definition 1. ***Matroid*** *$\mathcal{M} = (E,\mathcal{I})$ is for E finite non-empty set and $\mathcal{I} \subseteq 2^E$ (also called as independent sets) satisfying these properties:*

(I1) $\emptyset \in \mathcal{I}$,

(I2) $I \in \mathcal{I} \Rightarrow \forall I' \subseteq I : I' \in \mathcal{I}$,

(I3) $I_1, I_2 \in \mathcal{I}, |I_1| < |I_2| \Rightarrow \exists e \in I_2 \setminus I_1 : I_1 \cup \{e\} \in \mathcal{I}$.

Notation. *For further use and simplification we will sometimes use $I+e$ as a substitution for $I \cup \{e\}$. Similarly also $I - e$ for $I \setminus \{e\}$.*

Example. For a given multi-graph $G = (V,F)$ we will set $E = F$ (or in other words E stands for edges and the set). Independent sets $\mathcal{I}$ will be all acyclic subsets of E. Easily seen (I1) and (I2) is satisfied. For the third one (I3) it is also quite easily seen, because if we have one larger and smaller non-cycles then we can append one edge from the larger to the smaller.

Example. Let E be some elements of a vector space V. If $X \subseteq E$ is independent then it is linearly independent in V.

Definition 2. *Matroid* ***isomorphism*** *for two matroids $\mathcal{M}_i = (E_i,\mathcal{I}_i)$ for $i = 1,2$ is a bijection $f : E_1 \to E_2$ satisfying $\forall X \subseteq E_i : X \in \mathcal{I}_1 \Leftrightarrow f(X) \in \mathcal{I}_2$.*

Definition 3. *$X \subseteq E$ is a* ***circuit*** *if $X \notin \mathcal{I}$ and $\forall x \in X : X - x \in \mathcal{I}$. Also we will denote $\mathcal{C}(\mathcal{M})$ as the set of all circuits of $\mathcal{M}$.*

Lemma 1. *Let $\mathcal{M} = (E,\mathcal{I})$ be a matroid and $\mathcal{C}$ its circuits, then*

(C1) $\emptyset \notin \mathcal{C}$,

(C2) $\forall C_1, C_2 \in \mathcal{C} : C_1 \subseteq C_2 \Rightarrow C_1 = C_2$ *and*

(C3) $C_1, C_2 \in \mathcal{C}, C_1 \neq C_2, e \in C_1 \cap C_2 \Rightarrow C_3 \subseteq (C_1 \cup C_2) - e, C_e \in \mathcal{C}$.

Proof. (C1) and (C2) are easily seen from (I1) and (I2). Now for the third part (C3). □