Matroid Theory

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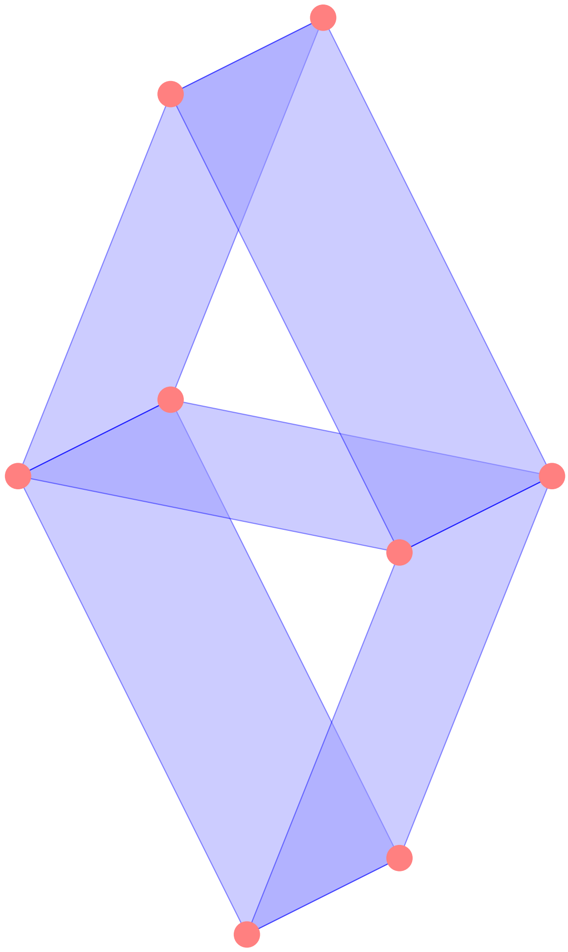

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Information

These are my notes on the course Matroid Theory, which was taught by Ondřej Pangrác in the year 2024.

Keep in mind there may be some mistakes. You may visit GitHub.

1. Basic definitions

Definition 1. ***Matroid*** *$\mathcal{M} = (E,\mathcal{I})$ is for E finite non-empty set and $\mathcal{I} \subseteq 2^E$ (also called as independent sets) satisfying these properties:*

(I1) $\emptyset \in \mathcal{I}$,

(I2) $I \in \mathcal{I} \Rightarrow \forall I' \subseteq I : I' \in \mathcal{I}$,

(I3) $I_1, I_2 \in \mathcal{I}, |I_1| < |I_2| \Rightarrow \exists e \in I_2 \setminus I_1 : I_1 \cup \{e\} \in \mathcal{I}$.

Notation. *For further use and simplification we will sometimes use $I+e$ as a substitution for $I \cup \{e\}$. Similarly also $I - e$ for $I \setminus \{e\}$.*

Example. For a given multi-graph $G = (V,F)$ we will set $E = F$ (or in other words E stands for edges and the set). Independent sets $\mathcal{I}$ will be all acyclic subsets of E. Easily seen (I1) and (I2) is satisfied. For the third one (I3) it is also quite easily seen, because if we have one larger and smaller non-cycles then we can append one edge from the larger to the smaller.

Example. Let E be some elements of a vector space V. If $X \subseteq E$ is independent then it is linearly independent in V.

Definition 2. *Matroid* ***isomorphism*** *for two matroids $\mathcal{M}_i = (E_i,\mathcal{I}_i)$ for $i = 1,2$ is a bijection $f : E_1 \to E_2$ satisfying $\forall X \subseteq E_i : X \in \mathcal{I}_1 \Leftrightarrow f(X) \in \mathcal{I}_2$.*

1.1 Circuits

Definition 3. *$X \subseteq E$ is a* ***circuit*** *if $X \notin \mathcal{I}$ and $\forall x \in X : X - x \in \mathcal{I}$. Also we will denote $\mathcal{C}(\mathcal{M})$ as the set of all circuits of $\mathcal{M}$.*

Lemma 1. *Let $\mathcal{M} = (E,\mathcal{I})$ be a matroid and $\mathcal{C}$ its collection of circuits, then*

(C1) $\emptyset \notin \mathcal{C}$,

(C2) $\forall C_1, C_2 \in \mathcal{C} : C_1 \subseteq C_2 \Rightarrow C_1 = C_2$ and

(C3) $C_1, C_2 \in \mathcal{C}, C_1 \neq C_2, e \in C_1 \cap C_2 \Rightarrow C_3 \subseteq (C_1 \cup C_2) - e, C_e \in \mathcal{C}$.

Proof. (C1) and (C2) are easily seen from (I1) and (I2). Now for the third part (C3). So for contradiction let C_1, C_2, e be as mentioned in the first part, but $(C_1 \cup C_2) - e \in \mathcal{I}$. Then $\exists f \in C_2 \setminus C_1 : C_2 - f \in \mathcal{I}$. Now find $I \in \mathcal{I}$ max s.t. $C_2 \setminus \{f\} \subseteq I \subseteq C_1 \cup C_2$. If $f \notin I$ then it would contain C_2 which is dependent and $\exists g \in C_1 \setminus C_2 : g \notin I$ otherwise it would contain C_1 which is dependent. Therefore

$$|I| \leq |C_1 \cup C_2| - 2 < |(C_1 \cup C_2) - e|$$

and now we may use the third axiom (I3) that is $\exists x \in |(C_1 \cup C_2) - e| \setminus I$ s.t. $I + e \in \mathcal{I}$ (this cannot be otherwise I contains the whole C_2). Now $I + x$ contradicts the maximality of I. $\square$

Claim 1. *Lets have E and $\mathcal{C} \subseteq 2^E$ satisfying all (C1), (C2) and (C3). Then set $\mathcal{I} = \{X \subseteq E | \forall C \in \mathcal{C} : C \nsubseteq X\}$ and $\mathcal{M} = (E, \mathcal{I})$ is a matroid.*

Proof. We have to show all properties of matroid. That is (I1) is trivially satisfied and (I2) also trivially holds. For the last (I3) we use a contradiction. For that we have $I_1, I_2 \in \mathcal{I}$, then $\forall e \in I_2 \setminus I_1 : I_1 + e \notin \mathcal{I}$. Let $I_3 \subseteq I_1 \cup I_2$ s.t. $|I_3| > |I_1|$ and $|I_1 \setminus I_3|$ is minimal. If $|I_1 \setminus I_3|$ would be empty then (I3) will hold, therefore assume it is non-empty.

Fix $e \in I_1 \setminus I_3$. Let $I_k = |I_3 - f| + e$ for $(f \in I_3 \setminus I_1)$. This cannot be independent $(\notin \mathcal{I})$ therefore $\exists C_k \subseteq T_k : C_k \in \mathcal{C}$ and $f \notin C_k, e \in C_k$.

$(I_3 \setminus I_1) \cap C_k = \emptyset$ hence $C_k \subseteq T_k \setminus (I_3 \setminus I_1) = (I_1 \cap I_3) + e \subseteq I_1$ this is not possible so it must be non-empty. Then $\exists g \in (I_3 \setminus I_1) \cap C_k \Rightarrow C_k, C_g \in \mathcal{C}, e \in C_k \cap C_g, f \notin C_k, g \notin C_g$ but $(C_k \cup C_g) - e \subseteq I_3$ which is contradiction with (C3). □

1.2 Basis

Definition 4. *Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. Then B is a **basis** iff $B \in \mathcal{I}, \forall x \in E \setminus B : B + x \notin \mathcal{I}$.*

Proposition 2. *Let B_1, B_2 be bases of $\mathcal{M}$, then $|B_1| = |B_2|$.*

Proof. If $|B_1| < |B_2|$ then by (I3) $\exists x \in B_2 \setminus B_1 : B_1 + x \in \mathcal{I}$. □

Definition 5. *Let $\mathcal{B}(\mathcal{M}) = \{B \subseteq E, B \text{ is a basis}\}$ be a collection of basis satisfying*

(B1) $\mathcal{B} \neq \emptyset$ and

(B2) $B_1, B_2 \in \mathcal{B}, e \in B_1 \setminus B_2 \Rightarrow \exists f \in B_2 \setminus B_1 : |B_1 - e| + f \in \mathcal{B}$.

One can see that (B2) can be proven using $I_1 - e =: B_1$ and $I_2 = B_2$.

Proposition 3. *Let $E \neq \emptyset$ finite set and $\mathcal{B} \subseteq 2^E$ satisfying (B1) and (B2). Let $\mathcal{I} = \{X \subseteq E : \exists B \in \mathcal{B} \; : X \subseteq B\}$ then $\mathcal{M} = (E, \mathcal{I})$ is a matroid.*

Proof. (I1) and (I2) are trivial. For (I3) use the following lemma.

Lemma 2. *Let $\mathcal{B}$ be such that it satisfies (B1) and (B2). Then $\forall B_1, B_2 \in \mathcal{B} : |B_1| = |B_2|$.*

Proof. By contradiction suppose $|B_1| > |B_2|$ with minimal $|B_1 \setminus B_2|$. Then $e \in B_1 \setminus B_2 \Rightarrow \exists f \in B_2 \setminus B_1 : (B_1 - e) + f \in \mathcal{B}$ and also $|(B_1 - e) + f| = |B_1|$ which leads to $|((B_1 - e) + f) \setminus B_2| < |B_1 \setminus B_2|$ which is a contradiction with the minimality. □

□

1.3 Rank function

Definition 6. *For a matroid $\mathcal{M} = (E, \mathcal{I})$ define a **rank function** $r : 2^E \to \mathbb{Z}_0^+$, such that $r(X) = \max_{I \subseteq X, I \in \mathcal{I}} |I|$ and $r(\mathcal{M}) = r(E)$.*

Claim 4. *Rank function has the following properties:*

(R1) $X \subseteq E : 0 \leq r(X) \leq |X|$,

(R2) $X \subseteq Y \subseteq E \Rightarrow \mathsf{r}(X) \leq \mathsf{r}(Y)$ *and*

(R3) $X, Y \subseteq E : \mathsf{r}(X \cup Y) + \mathsf{r}(X \cap Y) \leq \mathsf{r}(X) + \mathsf{r}(Y)$ *(which is called* ***submodularity****).*

Proof of the properties. While (R1) and (R2) are obvious and now we will show that (R3) also holds. Let I_1 be the max independent in $X \cap Y$ and I_2 be an extension $I_2 \supseteq I_1$ and max independent in $X \cup Y$. Now $\mathsf{r}(X \cup Y) + \mathsf{r}(X \cap Y) = |I_2| + |I_1|$ and also $|I_2 \cap X| \leq \mathsf{r}(X)$ and $|I_2 \cap Y| \leq \mathsf{r}(Y)$. We apply simple rule $|A| + |B| = |A \cup B| + |A \cap B|$ and get

$$\mathsf{r}(X) + \mathsf{r}(Y) \geq |I_2 \cap X| + |I_2 \cap Y| = |I_2| + |I_1| = \mathsf{r}(X \cup Y) + \mathsf{r}(X \cap Y)$$

□

Theorem 5. *For $E \neq \emptyset$ finite set and $\mathsf{r} : 2^E \to \mathbb{Z}_0^+$ satisfying (R1), (R2) and (R3). Then $\mathcal{I} = \{X \subseteq E |\, \mathsf{r}(X) = |X|\}$ and $\mathcal{M} = (E, \mathcal{I})$ is a matroid.*

Lemma 3. *For $E \neq \emptyset$ finite set and $\mathsf{r} : 2^E \to \mathbb{Z}_0^+$ satisfying (R1), (R2) and (R3). It holds that if $X, Y \subseteq E$ $\forall y \in Y : \mathsf{r}(X) = \mathsf{r}(X + y)$ then $\mathsf{r}(X) = \mathsf{r}(X \cup Y)$.*

Proof of lemma 3. Let $Y \setminus X = \{y_1, y_2, \ldots, y_k\}$ and now we will prove it by induction on k. For $k = 1$ it obviously holds. For $k \geq 2$ we use the submodularity.

$$\begin{aligned}
\mathsf{r}(X) + \mathsf{r}(X) &= \mathsf{r}(X \cup \{y_1, y_2, \ldots, y_{k-1}\}) + \mathsf{r}(X + y_k) \quad \geq \mathsf{r}(X \cup \{y_1, y_2, \ldots, y_k\}) + \mathsf{r}(X) \\
&\quad \text{(by induction hypothesis)} \\
\mathsf{r}(X) &= \qquad \geq \mathsf{r}(X \cup \{y_1, y_2, \ldots, y_k\}) \\
\mathsf{r}(X) &\qquad \geq \mathsf{r}(X \cup Y)
\end{aligned}$$

For the other inequality we use (R2) and hence we obtain equality. □

Proof of theorem 5. *TODO: Finish the proof.*

□

TODO: Insert missing part.

1.4 Uniform matroids

Definition 7. *For $0 \leq r \leq n \neq 0, |E| = n$ and $\mathcal{I} = \{X \subseteq E : |X| \leq r\}$ is* ***Uniform matroid*** *$U_{r,n} = (E, \mathcal{I})$.*

All the properties should be formally proven, but one can already see that all (I1), (I2) and (I3) are really satisfied. Now we will show us some examples.

TODO: Add examples.

1.5 Visualization of matroids

TODO: Finish this part.

1.6 (Direct) Sum of matroids (also disjoint union)

Definition 8. *We have two matroids $\mathcal{M}_i = (E_i, \mathcal{I}_i)$ for $i = 1{,}2$, then the (direct) sum $\mathcal{M}_1 \bigoplus \mathcal{M}_2$ is defined as a matroid $\mathcal{M} = (E, \mathcal{I})$ where $E = E_1 \dot{\cup} E_2$ and $\mathcal{I} = \{X \subseteq E, X \cap E_i \in \mathcal{I}_i, i = 1{,}2\}$.*

Observation. *Lets see the basis and circuits:*

$$\mathcal{B}(\mathcal{M}_1 \bigoplus \mathcal{M}_2) = \{B_1 \cup B_2, B_i \in \mathcal{B}_i, i = 1{,}2\}$$
$$\mathcal{C}(\mathcal{M}_1 \bigoplus \mathcal{M}_2) = \mathcal{C}_1 \cup \mathcal{C}_2$$
$$X \subseteq E : \mathsf{r}(X) = \mathsf{r}_1(X \cap E_1) + \mathsf{r}_2(X \cap E_2)$$