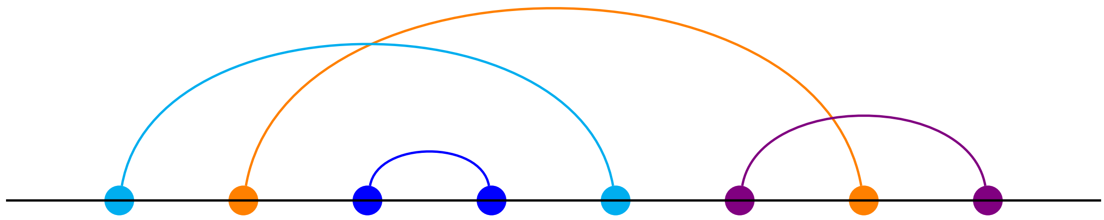
Geometrická reprezentace grafů

Tomáš Turek



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Information

Many parts of text are taken from the handouts made by Jan Kratochvíl. Also there may be some mistakes. If you find some and want to update them, you may find all the sources on the GitHub.

Contents

Part I

Geometrická reprezentace grafů I

1. Introducing

Firstly we will start by the introduction to the main characters – intersection defined graph classes; characterization of chordal graphs.

1.1 Intersection defined graph classes

Definition 1. *The intersection graph of a set family $\mathcal{A}$ is the graph*

$$IG(\mathcal{A}) = (\mathcal{A}, \{ab : a \neq b, a \cap b \neq \emptyset, a, b \in \mathcal{A}\}).$$

Definition 2. *Let $\mathcal{M}$ be a family of sets. We say that a graph G is an intersection graph of (members of) $\mathcal{M}$ if G is isomorphic to the graph $IG(\mathcal{A})$ for some family $\mathcal{A}$ whose all elements belong to $\mathcal{M}$. We write*

$$IG(\mathcal{M}) = \{\mathcal{IG}(\mathcal{A}) : \mathcal{A} \subseteq \mathcal{M}\}.$$

Observation. *For every graph G and every set family $\mathcal{M}, G \in \mathcal{IG}(\mathcal{M})$ if and only if there is a mapping $f : V(G) \to \mathcal{M}$ such that $uv \in E(G)$ iff $f(u) \cap f(v) \neq \emptyset$ holds true for all pairs of distinct vertices u, v of G.*

Observation. *For every family $\mathcal{M}$ (containing at least one nonempty set), it holds that $\mathcal{IG}(\mathcal{M}$ contains all complete graphs and is hereditary (i.e., every induced subgraph of every graph from $\mathcal{IG}(\mathcal{M})$ also belongs to $\mathcal{IG}(\mathcal{M})$).*

1.1.1 Examples

In many cases, the members of $\mathcal{M}$ are defined by their geometric shape. And in most of these cases, the members of $\mathcal{M}$ are arc-connected sets in the plane.

- **Interval graphs** INT $= \mathcal{IG}(\{$intervals on a line$\})$
- **Circle graphs** CIR $= \mathcal{IG}(\{$chords of a circle$\})$
- **Circular arc graphs** CA $= \mathcal{IG}(\{$arcs on a circle$\})$
- **Permutation graphs** PER $= \mathcal{IG}(\{$segments connecting two parallel lines$\})$
- **Function graphs** FUN $= \mathcal{IG}(\{$curves connecting two parallel lines$\})$
- **Polygon circle graphs** PC $= \mathcal{IG}(\{$polygons inscribed in a circle$\})$
- **Segment graphs** SEG $= \mathcal{IG}(\{$straight-line segments in the plane$\})$
- **Convex graphs** CONV $= \mathcal{IG}(\{$convex sets in the plane$\})$
- **String graphs** STRING $= \mathcal{IG}(\{$curves in the plane$\})$

1.2 Chordal graphs

Definition 3. *A graph is **chordal** if it does not contain any cycle of length greater than three as an induced subgraph.*

Definition 4. *A vertex u of a graph G is **simplicial** if $G[N_G(u)]$ is a clique.*

Definition 5 (PES)**.** *A **perfect elimination scheme** for a graph G is a linear ordering $u_1, u_2, \ldots, u_n$ of its vertices such that for every i, u_i is simplicial in the induced subgraph $G[\{u_1, u_2, \ldots, u_i\}]$.*

Lemma 1. *Every minimal vertex cut in a chordal graph induces a clique.*

Proof. Let $A \subset V(G)$ be a minimal vertex cut, and suppose u, v be two vertices of A. These vertices are connected by a path in each component of $G \setminus A$. If u and v were not adjacent, a pair of shortest such paths would give rise to an induced cycle of length greater than 3 in G. □

Lemma 2. *Every chordal graph, which is not a complete graph, contains two nonadjacent simplicial vertices.*

Proof. By induction. If G is a complete graph, the claim of the lemma is fulfilled. If G is not complete, it has a vertex cut, say A. Let B be a connected component of $G \setminus A$, and set $G_1 = G[B \cup A]$ and $G_2 = G \setminus B$. By induction hypothesis, each of G_1, G_2 is either complete or has two nonadjacent simplicial vertices. Thus each of them has a simplicial vertex which does not belong to A. Each of these is then also simplicial in entire G, and they are clearly nonadjacent. □

Corollary. Every nonempty chordal graph contains a simplicial vertex.

Definition 6 (Clique-tree decomposition)**.** *A **clique-tree decomposition** of a graph G is a tree $T = (\mathcal{Q}, F)$, with $\mathcal{Q}$ being the set of all maximal cliques of G, such that for every vertex $u \in V(G)$, the subgraph $T[\{Q : u \in Q \in \mathcal{Q}\}]$ is connected.*

Warning!! The vertex set of a clique-tree decomposition of a graph G is uniquely defined, but not necessarily the edge set!!

Theorem 1. *For any graph G, the following statements are equivalent:*

1. *G is chordal,*
2. *G allows a PES.*
3. *G has a clique-tree decomposition, and*
4. *G is an intersection graph of subtrees of a tree.*

Proof. "1. $\Rightarrow$ 2." By induction on the number of vertices, using Lemma 2.

"2. $\Rightarrow$ 3." By induction on the number of vertices again. Suppose $G + 0 = G \setminus v_n$ has a clique-tree $T = (\mathcal{Q}', F')$. If $Q' = N_G(v_n) \in \mathcal{Q}'$, then $Q = N_G[v_n]$ is a maximal clique in $G, \mathcal{Q} = (\mathcal{Q}' \setminus \{Q'\}) \cup \{Q\}$ and $T = (\mathcal{Q}, F)$ is a clique-tree for G, where $F = (F' \setminus \{Q'P : P \in \mathcal{Q}'\}) \cup \{QP : Q'P \in F'\}$. If, on the other hand, $Q' = N_G(v_n) \notin \mathcal{Q}'$, then $\mathcal{Q} = \mathcal{Q}' \cap N_G[v_n]$ and $(\mathcal{Q}, F' \cup N_G[v_n]P)$ is a clique-tree for G for any $P \in \mathcal{Q}'$ such that $N_G(v_n) \subset P$.

"3. $\Rightarrow$ 4." Given a clique-tree decomposition $T = (\mathcal{Q}, F)$, define $T_u = T[\{Q : u \in Q \in \mathcal{Q}\}]$ for $u \in V(G)$. Clearly $V(T_u) \cap V(T_v) \neq \emptyset$ iff u and v belong to the same maximal clique of G, which happens if and only u and v are adjacent in G.

"4. $\Rightarrow$ 1." Let G be the intersection graph of a collection $\{T_u\}_{u \in V(G)}$ of subtrees of a tree T. Suppose $v_1, v_2, \ldots, v_k$ be an induced cycle in G, with $k > 3$. Then the subtrees T_{v_1} and T_{v_3} are vertex disjoint, and hence there is an edge $e \in E(T)$ which lies on every path connecting T_{v_1} and T_{v_3} in T . This edge separates T into T_1 and T_2 such that T_{v_1} and T_{v_3} belong to different components of $T \setminus e$, say, $T_{v_1} \subseteq T_1$ and $T_{v_3} \subseteq T_2$. One can show by induction on i that for every $i \geq 3$, $T_{v_i} \subseteq T_2$. But then T_{v_k} and T_{v_1} must be disjoint, contradicting the assumption that $v_1 v_k \in E(G)$. $\square$

Corollary. Chordal graphs are perfect (i.e., $\chi(H) = \omega(H)$ for every induced subgraph H of G).

Proof. Consider a PES $u_1, u_2, \ldots, u_n$ for G and color the vertices from u_1 to u_n by the First Fit Method. $\square$