Kombinatorika a grafy

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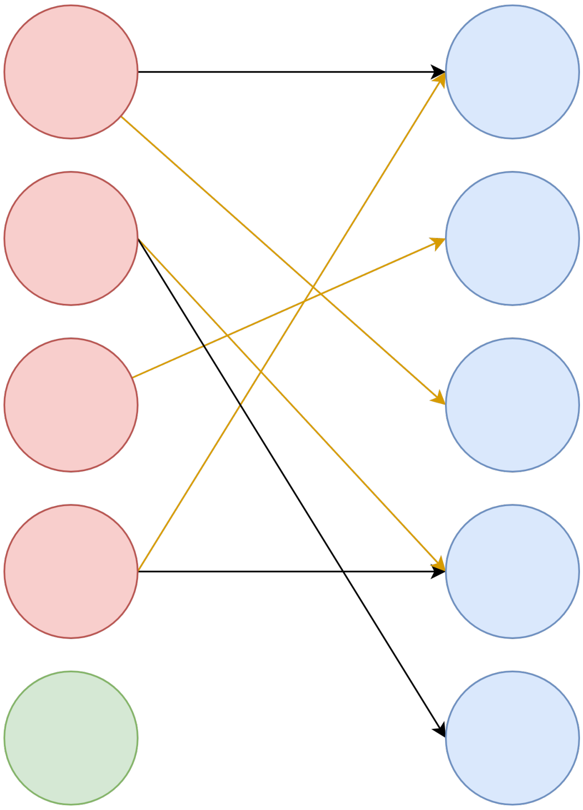

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1. Structural graph theory

Definition 1. *$H \leq_t G$ means that subdivision of H is a subgraph of G, also known as **topological minor**.*

Definition 2. *$H \leq_m G$ means that H is a **minor** of G.*

Definition 3. *$H \subseteq G$ means that H is a **subgraph** of G.*

Definition 4. *$H \sqsubseteq G$ means that H is a **induced subgraph** of G.*

Theorem 1 (Kuratowski)**.**

$$K_5, K_{3,3} \nleq_t G \Leftrightarrow G \text{ planar}$$

$$K_5, K_{3,3} \nleq_m G \Leftrightarrow G \text{ planar}$$

Definition 5. *$\chi(G)$ means that G has a coloring of size $\chi(G)$.*

Observation. *$C_3, C_5, C_7, \dots \nsubseteq G \Leftrightarrow \chi(G) \leq 2$ which holds also for $\sqsubseteq$.*

Observation. *$C_3 \nleq_m G \Leftrightarrow G$ is a forest also holds for $\leq_t$.*

Definition 6. *$Forb_{leq}(\mathcal{F}) = \{G | (\forall F \in \mathcal{F}) F \nleq G\}$*

We will try to show $\mathcal{G} = Forb_{\leq_m}(\mathcal{F})$. If $G \in \mathcal{G}$ then all minors of G belong to $\mathcal{G}$.

Observation. *If $\mathcal{G} = Forb_{\leq}(\mathcal{F})$ then $\mathcal{G}$ is $\leq$-closed. Which means that $\forall G,G'$ if $G \in \mathcal{G}$ and $G' \leq G$ then $G' \in \mathcal{G}$.*

Lemma 1. *Let $\leq$ be a partial ordering of graphs. If a class $\mathcal{G}$ of graphs is $\leq$-closed, then there exist $\mathcal{F}$ s.t. $\mathcal{G} = Forb_{\leq}(\mathcal{F})$.*

Proof. $\mathcal{F} = \{F : F \nleq G\}$. ☐

Definition 7. *F is **minimal $\leq$-obstruction** for $\mathcal{G}$ if $F \notin \mathcal{G}$ but for every $F' \lneq F$ and $F' \in \mathcal{G}$.*

Lemma 2. *Let $\leq$ be an ordering og graphs **without infinite decreasing chains**. If $\mathcal{F}$ is $\leq$-closed, then $\mathcal{G} = Forb_{\leq}(\{F : F$ is a minimal $\leq$ -obstruction for $\mathcal{G}\})$.*

Proof. $G \notin \mathcal{G}$ is min $\leq$-obstruction or $\exists G' \lneq G : G \notin \mathcal{G} \Rightarrow G'$ is obstruction or we continue and because we don't have **without infinite decreasing chains** we will eventually end. ☐

If $\mathcal{G}$ is $\leq_m$-closed, then there exists a **finite** $\mathcal{F}$ such that $\mathcal{G} = Forb_{\leq_m}(\mathcal{F})$.

Theorem 2 (Robertson-Seymor)**.** *For every F there exists an algorithm that for input graph G decides whether $F \leq_m G$ in time $O_F(|G|^3)$.*

Definition 8. *For graph $G = (V,E)$ we define $|G| = |V|$ and $||G|| = |E|$. Also for some $U \subseteq V$ $G[U]$ is a induced subgraph of G that has only vertices from U. Then $N_G(v)$ stands for the neighborhood of vertex v in graph G.*

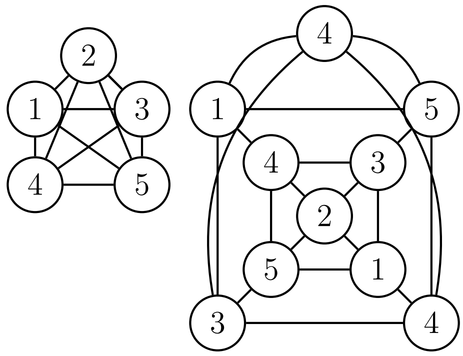

Figure 1.1: Example of G and G' as covers.

Definition 9. *G' is a **cover** of G if $(\exists f : V(G') \to V(G))\forall v \in V(G')$ for $N_{G'}(v)$ is a bijection with $N_G(f(v))$.*

Example. We may see an example 1.1:

$$\{G\text{"}\exists \text{ planar } G' \text{ cover of } G\}$$
$$\Updownarrow$$
$$F_1, \ldots, F_n \not\leq_m G$$

Contrary we take $\mathcal{G} = \{G : (\forall uv \in V(G)U \neq v, \deg(u) \geq 5, \deg(v) \geq 5)(\exists X \subseteq E(G) : |X| \leq 1)u$ and v are in different component of $G - X\}$ which is $\leq_t$-closed. But take these graphs:

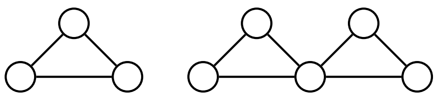

Figure 1.2: Obstructions.

Where each one of them is an obstruction. And we could create much more of them.

Now we take a look at some nice properties of graphs if we forbid some graphs as a minors.

$$\begin{array}{lcl} K_1 \not\leq_m G & \Leftrightarrow & V(G) = \emptyset \\ K_2 \not\leq_m G & \Leftrightarrow & E(G) = \emptyset \\ K_3 \not\leq_m G & \Leftrightarrow & G \text{ is a forest} \\ & & G \text{ is obtained from } K_1, K_2 \text{ by clique sums} \\ K_4 \not\leq_m G & \Leftrightarrow & G \text{ is obtained from } K_1, K_2, K_3 \text{ by clique sums} \end{array}$$

Definition 10. *Graph G can be obtained from G_1 and G_2 by **clique-sum** if the intersection that these graphs have in G form a clique. In other way it is that we bind together two graphs by identifying their vertices and edges in the same size clique.*

Observation. *If G is obtained from G_1 and G_2 by a clique-sum then:*

$$K_m \leq_m G \Leftrightarrow K_m \leq_m G_1 \vee K_m \leq_m G_2$$

Lemma 3. *If $K_k \leq_m G$ and G is the clique-sum of G_1 and G_2 then $K_k \leq_m G_1 \vee K_k \leq_m G_2$.*

Lemma 4. *If G is not 3-connected then there exist $G_1, G_2 \lneq_m G$ s.t. G is a clique-sum of G_1 and G_2.*

Proof. If G is not connected then it is done since it is a clique sum on K_0. If G is connected, but not 2-connected then it is a clique-sum on K_1 since there exist a articulation. If G is 2-connected then there must be two vertices which splits the graph. And these two vertices form a K_2 as a minor. That is because we split G to two parts where we leave the major one side and add a edge to these two vertices, which we can do because they need to have a path between them so we contract all the edges alongside the path. □

Definition 11. *$\delta(G)$ is a minimum degree of a graph G.*

Theorem 3. *If G is K_4-minor-free then G is obtained from $K_{\leq 3}$'s by clique-sums.*

Proof. By induction on $|V(G)|$.

(a) If G is not 3-connected. G is a clique-sum of G_1,$G_2 \lneq_m G$. Since $K_4 \nleq_m G_1$ and $K_4 \nleq_m G_2$ we use induction hypothesis and we are done.

(b) If G is 3-connected. If $|V(G)| \leq 3$, then $G = K_{\leq 3}$, wlog $|V(G)| \geq 4$. $\delta(G) > 1 \Rightarrow G$ contains a cycle. Let C be a shortest cycle in G. C is induced in G 3-connected $\Rightarrow G \neq C$ so $\exists v \in V(G) \setminus V(C)$. By Merger's theorem there exists three paths from v to C intersecting only in v. That gives us K_4 as a minor of the graph. Which is contradiction.

□

$$K_5 \nleq_m G \quad \Leftrightarrow \quad G \text{ is obtained from planar graphs and } W_8 \text{ by clique sums}$$

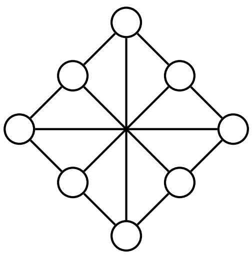

Figure 1.3: W_8 graph.

Observation. *If G is a clique-sum of G_1 and G_2 then*

$$\chi(G) \leq \max(\chi(G_1), \chi(G_2))$$

Proof. We just need to match the coloring of the cliques. Other than that we don't have any problem. □

1.1 Hadwiger's conjecture

K_t-minor-free graphs are $(t-1)$ colorable.

$$\begin{array}{lll} K_1 \not\leq_m G & \chi \leq 1 & \delta \leq 0 \\ K_2 \not\leq_m G & \chi \leq 2 & \delta \leq 1 \\ K_3 \not\leq_m G & \chi \leq 3 & \delta \leq 2 \\ K_4 \not\leq_m G & \chi \leq 4 & \delta \leq 5 \\ K_5 \not\leq_m G & \chi \leq 5 & \end{array}$$

Theorem 4. *$\exists f$ every K_t-minor-free graph G has $\delta(G) \leq f(t)$.*

The function is somewhere near $f(t) = (1{,}6 \cdots + O(1))t\sqrt{\log t}$. But we won't show this result. Instead we will show $f(t) = O(t^2)$. Before we continue it is better to remind ourselves **chordal graph** and **elimination ordering** (known as PES).

Definition 12 (Chordal decomposition of G)**.** *$V(G) = \mathcal{P}_1 \dot\cup \mathcal{P}_2 \dot\cup \ldots \dot\cup \mathcal{P}_n \dot\cup$ and*

1. *$(\forall i) G[\mathcal{P}_i]$ is connected.*
2. *"$\mathcal{P}_i$'s form elimination ordering" Precisely: $(\forall i \in [n])(forall j_1, j_2 < i)$ if G has an edge between $\mathcal{P}_i$ and $\mathcal{P}_{j_1}$ and also between $\mathcal{P}_i$ and $\mathcal{P}_{j_2}$ then it also has an edge between $\mathcal{P}_{j_1}$ and $\mathcal{P}_{j_2}$.*

Definition 13. *Chordal partition is **geodesic** if $(\forall i)(\exists v_i \in \mathcal{P}_i)$ s.t. if $v_1, \ldots, v_t < i$ are the indices s.t. G has an edge between $\mathcal{P}_i$ and $\mathcal{P}_{j_1}, \mathcal{P}_{j_2}, \ldots, \mathcal{P}_{j_t}$ then $v_1, \ldots, v_t \in \mathcal{P}_i$ s.t. v_i has a neighbor in $\mathcal{P}_{j_1}, \mathcal{P}_{j_2}, \ldots, \mathcal{P}_{j_t}$ and $G - \bigcup_{j<i} \mathcal{P}_j$ contains shortest paths from v_i to $v_1, \ldots, v_t$ which cover all vertices in $\mathcal{P}_i$.*

Theorem 5. *Every graph has a geodesic chordal partition.*

Before we show us a proof we will take a look at a simple application. If G is K_k-minor-free last part has neighbours in $t \leq k-2$ parts (otherwise it will have K_k as a minor). Then we may take a look at a $\deg(v) \leq (k-2) + (k-2)(k-2)3 \leq 3k^2$. Thus getting the upper bound $\delta(G) \leq 3k^2$.