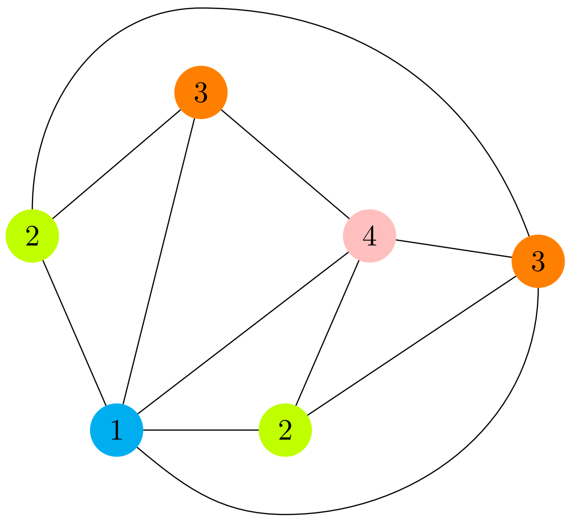
Barevnost grafů a kombinatorických struktur

Tomáš Turek



February 2, 2024

Information

Also there may be some mistakes. If you find some and want to update them, you may find all the sources on the GitHub.

Contents

1. Revision and introduction

One can already know some basics to the graph coloring and also some of the theorems. Therefore we will briefly introduce some basics and revisit some of the known theorems and perhaps some we will show throughout this document.

Definition 1 (Standard colorings)**.** *For a graph G with vertices $V(G)$ and edges $E(G)$ we define:*

- *Vertex coloring is an assignment $\varphi : V(G) \to \mathbb{N}$ such that no two adjacent vertices have the same color.*
- *Edge coloring is an assignment $\varphi : E(G) \to \mathbb{N}$ such that all edges incident to one vertex have different color.*
- *Graph is k-colorable (k-edge-colorable) if vertices (edges) can be colored by k colors.*
- *(Edge) chromatic number is min k such that G is k-(edge)-colorable.*

Notation. *First of all some basic notations.*

- *$\chi(G)$ A chromatic number of G.*
- *$\Delta(G)$ is the maximal degree of a graph G.*
- *$\delta(G)$ is the minimal degree of a graph G.*
- *$\chi_e(G)$ An edge chromatic number of G.*

Theorem 1 (Four colors)**.** *Every planar graph G has $\chi(G) \leq 4$.*

Theorem 2 (Brooks)**.** *If G is a connected graph and $G \neq K_n$ and $G \neq C_{2n+1}$ then $\chi(G) \leq \Delta(G)$.*

Theorem 3 (Vizing)**.** $\Delta(G) \leq \chi_e(G) \leq \Delta(G) + 1$

Theorem 4 (General Euler's formula)**.** *If G can be drawn on a surface of Euler genus g then $|E| \leq |V| + |F| - g$. Where $|F|$ is for the number of faces of the drawing.*

From Euler's formula one can see that if $|V| \geq 3$ then $|E| \leq 3|V| + 3g - 6$. Therefore the average degree is $\frac{2|E|}{|V|} = \frac{6(g-2)}{|V|}$.

Theorem 5 (Heawood's formula)**.** *If G can be drawn on a surface of Euler genus g then*

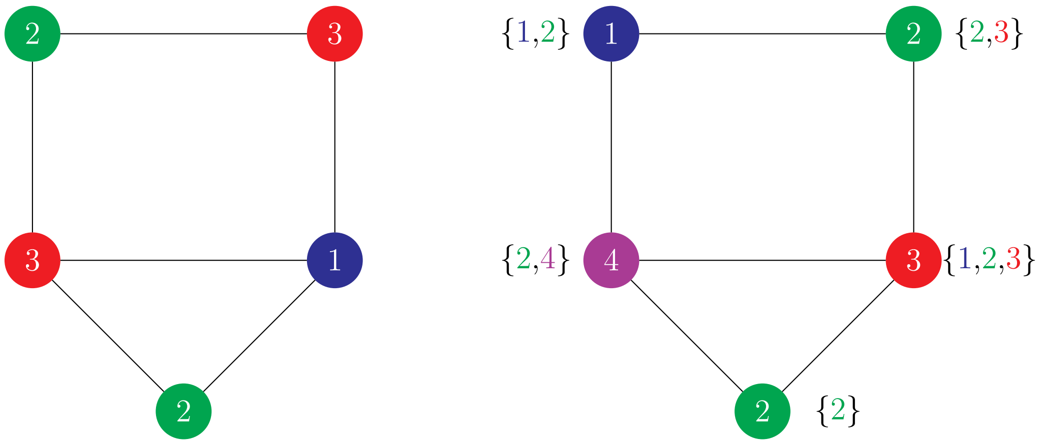
$$\chi(G) \leq \left\lfloor \frac{7 + \sqrt{24g + 1}}{2} \right\rfloor$$

Where this formula is tight for Klein's bottle. Then we will show us that deciding if a planar graph is 3-colorable is NP-hard problem. But on the other hand we have this theorem.

Theorem 6 (Grötsch)**.** *Every planar graph without triangle is 3-colorable.*

2. List coloring

Firstly we can see a graph G and its normal coloring which is depicted on a picture 2.1a. Whereas the list coloring shown on picture 2.1b is that each vertex has a assigned list for which we can choose colors. Otherwise the coloring is the same.



(a) Normal coloring of a graph G. (b) List coloring of a graph G.

Figure 2.1: Difference between basic and list coloring of a graph G.

Definition 2. *k-**list-assignment** is for an assignment for all vertices of size k.*

Definition 3. *The graph is k-**list-colorable** if G can be colored by every k-list-assignment.*

Definition 4. ***List chromatic number** of a graph G is denoted as $\chi_l(G)$. That is the min k s.t. G is k-list-colorable.*

Observation. $\chi(G) \leq \chi_l(G)$

Now one could prove that the Heawood's formula works pretty much the same, thus $\chi_l(G) \leq \left\lfloor \frac{7+\sqrt{24g+1}}{2} \right\rfloor$, but only if $g > 0$. On the other hand one can find a planar graph which has $\chi_l(G) > 4$, but it can be shown that for every planar graph G the $\chi_l(G) \leq 5$ which was proven by Thomassen. Vizing's theorem will be shown and the Brooks' theorem remains the same.

Definition 5. *L is a **degree-list-assignment** for graph G if $|L(v)| \geq \deg(v)\ \forall v \in V$. And G is a **degree-list-colorable** if G is colorable from every degree-list-assignment.*

Definition 6 (Gallai tree)**.** ***Gallai tree** is a graph where each 2-connected block is either a clique or an odd cycle.*

Observation. *Gallai trees are not degree-list-colorable.*

Theorem 7 (Brooks)**.** *If G is connected and G is not a gallai tree then G is degree-list-colorable.*

Proof.

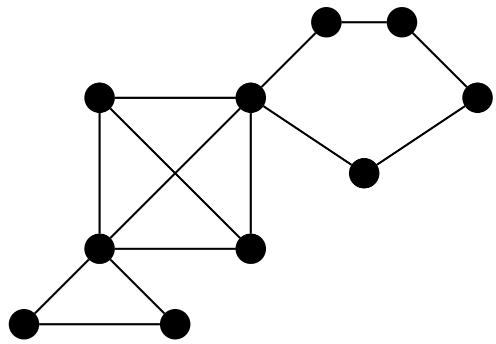


Figure 2.2: Example of a gallai tree.

Lemma 1. *G is connected, L is a degree-list-assignment* $(\exists v \in V(G))\ |L(v)| > \deg(v) \Rightarrow$ *G is L-colorable.*

Lemma 2. *G is connected, L is a degree-list-assignment, $uv \in E(G)$, u is not a cut vertex, G not L-colorable then $L(u) \subseteq L(v)$.*

Corollary. G connected, L is a degree-list-assignment, $uv \in E(G)$, u,v not cut vertices, G not L-colorable then $L(u) = L(v)$.

Corollary. G is 2-connected, L is degree-list-assignment, G is not L-colorable then $\forall u,v \in V(G) : L(u) = L(v)$ and G is a clique or an odd cycle.

□

TODO: From this part it is missing.

3. Critical graphs

Exercise. *If G is k-critical, then it is 2-connected.*

Solution. Lets take a graph G which has $\chi(G) = k$ but for all $H \subsetneq G$ $\chi(H) = k - 1$. If G is not connected then one component must have the largest $\chi(G)$, but if we remove vertex or edge from other component we don't achieve $\chi(H)$. That is contradiction.

Now assume it is 1-conencted. Therefore we have a vertex cut $M = \{v\}$ which splits the graph to two components G_1 and G_2. Lets fix graphs $G_1 \cup \{v\}$ and $G_2 \cup \{v\}$ where both of them have chromatic number $k - 1$ since they are subgraphs. Lets fix those colorings. Now for one coloring switch the colors so that v has the same color for both colorings. Therefore we have G colored by $k - 1$ colors which is a contradiction. □

Theorem 8. *If G is $(c + 1)$-critical for $c \geq 3$ and $G \neq K_{c+1}$ then the average degree is*

$$\bar{d} \geq c + \frac{c-2}{c^2 - 2c - 2} \approx c + \frac{1}{c} \quad \textit{(for large } c\textit{)}$$

Theorem 9. *If G is $(c + 1)$-critical, then*

$$\bar{d} \geq c + 1 - \frac{2}{c} - O\left(\frac{1}{|V(G)|}\right)$$

Theorem 10. *If G is 4-critical, then*

$$|E(G)| \geq \frac{5 \cdot |V(G)| - 2}{3} \Rightarrow \bar{d} \geq \frac{10}{3} - \frac{4}{3 \cdot |V(G)|}$$

Proof. Suppose H is 4-critical plane graph. Let f_i be the number of faces of length i in a drawing of H. Therefore by Euler's formula we get the following.

$$|V(G)| + |F(G)| = |E(G)| + 2$$
$$|V(G)| + \sum_{i \geq 3} f_i = |E(G)| + 2$$
$$\frac{5}{3}|V(G)| + \frac{5}{3}\sum_{i \geq 3} f_i = \frac{5}{3}|E(G)| + \frac{10}{3}$$

TODO: Finish this.

□

Theorem 11 (Grötsch)**.** *Every planar triangle-free graph is 3-colorable.*

TODO: Skipped.