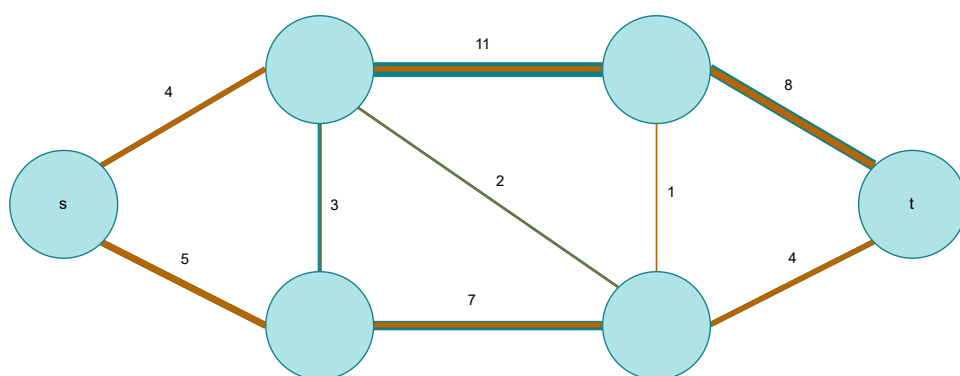


# Flows, paths and cuts

Tomáš Turek



November 10, 2023

# Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
1.1	Network flow . . . . .	2
1.2	Min $s,t$ -cut . . . . .	2
<b>2</b>	<b>Multi-commodity problem</b>	<b>3</b>
2.1	Multi cut problem . . . . .	3
2.2	Example . . . . .	3
2.3	Preparation for algorithm . . . . .	4
2.4	Pipe cut algorithm . . . . .	6
2.5	How to solve LP . . . . .	6
2.6	Is there any better approximation? . . . . .	7
<b>3</b>	<b>The Sparsest cut problem</b>	<b>8</b>
3.1	Concurrent multicommodity flow . . . . .	8
3.2	Metric spaces . . . . .	14
3.3	Applications . . . . .	16

# 1. Introduction

Firstly we remind some basics from flows and cuts.

## 1.1 Network flow

**Flow** is defined on a **network**. Network is on a oriented graph  $G = (V, E)$  and it has two special vertices  $s, t \in V$  called source and target. Also we have a capacity, which is a mapping  $c : E \rightarrow \mathbb{R}_0^+$ . Flow then is a mapping  $f : E \rightarrow \mathbb{R}_0^+$  which has two properties.

1.  $\forall e \in E : f(e) \leq c(e)$
2. Kirchoff's law:  $\forall v \neq s, t \in V : \sum_{uv \in E} f(uv) - \sum_{vu \in E} f(vu) = 0$

## 1.2 Min $s, t$ -cut

Now we also remind ourselves another term which is a  $s, t$ -cut. Which is a  $M \subset E$  such that no  $s, t$ -path exists in  $G' = (V, E \setminus M)$ .

These basic terms can be generalized to a **multi-commodity flow** problem and **multi-cut** problem.

## 2. Multi-commodity problem

On a graph  $G = (V, E)$  and  $k$  tuples  $(s_1, t_1), (s_2, t_2), \dots, (s_k, t_k)$  in  $V^2$  known as **commodities** we can also define a network and a flow. Same as before we have a capacity  $c : E \rightarrow \mathbb{R}_0^+$ . These commodities are shared on the same resources.

We may define  $\mathcal{P}_i$  as all paths between  $s_i$  and  $t_i$ . And also  $\mathcal{P} := \bigcup_{i=1}^k \mathcal{P}_i$ .

For a single commodity flow problem we could define an *linear program* (LP):

$$\begin{aligned} \forall e \in E \quad & \max \sum_{p \in \mathcal{P}_{st}} f_p \\ & \sum_{p: e \in p \in \mathcal{P}_{st}} f_p \leq c(e) \\ & f \geq 0 \end{aligned}$$

From this we may define an linear program denoted as **LP1** for multi commodity problem.

$$\begin{aligned} \forall e \in E \quad & \max \sum_{p \in \mathcal{P}} f_p = \sum_{i=1}^k \sum_{p \in \mathcal{P}_i} f_p \\ & \sum_{j=1}^k \sum_{p: e \in p \in \mathcal{P}_j} f_p \leq c(e) \\ & f \geq 0 \end{aligned}$$

Alternatively we could define it by a single variables for flows on each edge. For multi commodity problem we would have  $k$  flows which adds up to one single flow. Thus we can see that the problem is in  $P$  (polynomial).

### 2.1 Multi cut problem

As a cuts for single flow we may define somewhat similiar definition. Such that between every tuple of  $s_i$  and  $t_i$  there is no such path. We will also look at a linear program which will be denoted as **LP2**.

$$\begin{aligned} \forall e : x_e &= \begin{cases} 1 & e \text{ in cut} \\ 0 & \text{otherwise} \end{cases} \\ \Phi &:= \min \sum_{e \in E} x(e)c(e) \\ \forall i \in [k] \forall p \in \mathcal{P}_i & \sum_{e \in p \in \mathcal{P}_i} x(e) \geq 1 \\ \text{(ILP)} & \quad x(e) \in \{0, 1\} \\ \text{(LP)} & \quad x(e) \geq 0 \end{aligned}$$

First ILP is integer linear program which is generally NP hard. Thus we will look on the relaxation of LP program. We may observe that we don't need to specify that  $x(e) \geq 1$ .

Now we may see that indeed **LP1** and **LP2** are dual programs. Thus resulting in knowing that the maximum flow is the same as minimum fractional cut.

### 2.2 Example

Before we continue we will take a look at a simple example (2.1). The graph is as follows. All capacities are equal to 1.

We may observe that the max flow is  $|f| = \frac{3}{2}$  because we can put  $\frac{1}{2}$  on every edge. Then multi-cut is 2 since we need to remove always at least two edges. But minimum

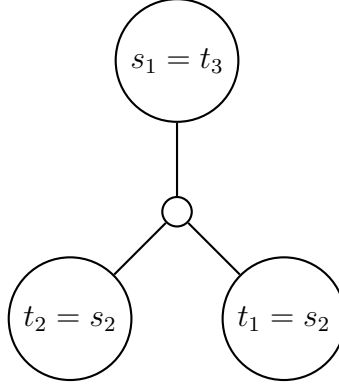


Figure 2.1: Example graph.

fractional multi-cut is only  $\frac{3}{2}$  because we only have to "cut" half of each edge. Thus it is the exact same result as max flow.

## 2.3 Preparation for algorithm

We will show an algorithm which will give an approximate result of a multi-cut problem. We may look at it like we would cut off some parts of the graph which are close to the sources and continue.

Given  $\bar{G} = (\bar{V}, \bar{E})$ ,  $(s_i, t_i)$ ,  $k, c : E \rightarrow \mathbb{R}_0^+$  and solution  $x$  of **LP2**. We define:

$$B_x(s_i, r) := \{u \in \bar{V} \mid d_x(s_i, u) \leq r\}$$

As a ball around  $s_i$  with diameter w.r.t  $x$ .  $d_x(s_i, u) :=$  the length of the shortest  $s_i u$  path in  $\bar{G}$  w.r.t. the edge length  $x(e)$ .

$$\delta(B_x(s_i, r)) = \{\{u, v\} \in \bar{E} : |\{u, v\} \cap B_x(s_i, r)| = 1\}$$

$$V_x(s_i, r) = \frac{\Phi}{k} + \sum_{\{u, v\} \in \bar{E}, u, v \in B_x(s_i, r)} c(e)x(e) + \sum_{\{u, v\} \in \delta(B_x(s_i, r)) : u \in B_x(s_i, r)} c(e)(r - d_x(s_i, u))$$

This we call a volume of a ball. We will denote as a function of  $r$   $f(r)$ . The last sum is of all edges which are only partly inside the ball. Next we also define the following.

$$C_x(s_i, r) = \sum_{\{u, v\} \in \delta(B_x(s_i, r))} c(u, v)$$

This will be denoted as a function  $g(r)$ . We may see some really nice properties these functions have. For instance  $f(r)$  is a growing function which is increasing linearly and then it jumps to another point. On the other hand  $g(r)$  is constant at some parts and then it jumps to a certain point, these jumps are for both function in the exact same spots. Next we can see that for some nice points it holds that  $f'(r) = g(r)$ .

**Lemma 1.** For each  $i \in [n]$  s.t.  $s_i, t_i \in \bar{V}$  there exist  $r \in (0, 1/2)$  s.t.

$$\frac{C_x(s_i, r)}{V_x(s_i, r)} \leq 2 \ln 2k.$$

*Proof.* By contradiction. For fixed  $i \forall r \in (0, 1/2)$ ,  $\frac{f'(r)}{f(r)} > 2 \ln 2k$ . We will have  $r_0 = 0 < r_1 < r_2 < \dots < r_{l-1} < 1/2 = r_l$  which are the values where  $f(r)$  is not continuous (there are these "jumps"). First we consider  $(r_j, r_{j+1})$  for some  $j$ .

$$\frac{f'(r)}{f(r)} = (\ln f(r))'$$

So  $\forall r \in (r_j, r_{j+1})$ ,  $(\ln f(r))' > 2 \ln 2k$ . We will compute the integral over all of these values. We may see that the right side is just a constant so we get

$$\int_{r_j}^{r_{j+1}} (\ln f(r))' > (r_{j+1} - r_j) 2 \ln 2k.$$

In  $r_{j+1}$  there may be jump so we instead take the  $\lim_{r \rightarrow r_{j+1}^-} f(r) = f^-(r_{j+1})$ . Note that  $f^-(r_{j+1}) \leq f(r_{j+1})$ .

$$\ln f(r_{j+1}) - \ln f(r_j) \geq \ln f^-(r_{j+1}) - \ln f(r_j) = \int_{r_j}^{r_{j+1}} (\ln f(r))' > (r_{j+1} - r_j) 2 \ln 2k.$$

Now we sum our inequality over all intervals  $(r_j, r_{j+1})$ ,  $j = 0, \dots, l$ . We will only have the very ends because the rest will be once added and once removed.

$$\sum_{j=0}^{l-1} (\ln f(r_{j+1}) - \ln f(r_j)) > \sum_{j=0}^{l-1} (r_{j+1} - r_j) 2 \ln 2k$$

$$\ln f(r_l) - \ln f(r_0) > (r_l - r_0) 2 \ln 2k$$

$$\ln f(1/2) - \ln(0) > \ln 2k$$

$$\ln \frac{f(1/2)}{\frac{\Phi}{k}} > \ln 2k$$

$$\frac{f(1/2)}{\frac{\Phi}{k}} > 2k$$

$$f(1/2) > 2\Phi$$

As the volume of the entire pipe system is at most  $2\Phi$  it means that we have a contradiction. □

Note that choosing  $1/2$  is not necessary for the proof, but for the algorithm to work. Because if we choose  $1/2$  it means that no  $s_j, t_j$  will both be in a ball for the index  $i$ . That is because the length w.r.t  $x$  of paths from  $s_j$  to  $t_j$  need to be at least 1.

---

**Algorithm 1** Pipe cut algorithm

---

**Require:**  $\bar{G} = (\bar{V}, \bar{E})$ .

**Ensure:**  $F$  multi cut.

```
1:  $F \leftarrow \emptyset$ 
2: for  $i = 1 \dots k$  do
3:   if  $s_i - t_i$  are still connected in  $(\bar{V}, \bar{E} \setminus F)$  then
4:     Choose  $r \in (0, 1/2)$  by Lemma.
5:      $F \leftarrow F \cup \delta(B_x(s_i, r))$ 
6:     Remove edges inside  $B_x(s_i, r)$  and  $\delta(B_x(s_i, r))$ .
7:   end if
8: end for
9: return  $F$ 
```

---

## 2.4 Pipe cut algorithm

There are few things to talk about. To get  $r$  we will check all "almost ends" of all intervals. The time complexity is polynomial since everything that is inside the code is polynomial. Correctness of the algorithm is easily observable since no pair  $p_i, t_i$  is inside some other ball and all balls will separate pairs  $s_j, t_j$ . Other thing to consider is what is the approximation ratio?

**Theorem 1.** *Approximation ratio of the Pipe cut algorithm is  $O(\log k)$ .*

*Proof.* Lets define  $C_i$  as the cost of the cut of the ball from iteration  $i$  and  $V_i$  as the volume of it. We know that  $C_i \leq 2 \ln 2k \cdot V_i$ .

$$\sum_i C_i \leq 2 \ln 2k \sum_i V_i \leq 2 \ln 2k \cdot 2\Phi = 4 \ln 2k \cdot \Phi = O(\log k)\Phi$$

□

For a single commodity we know that max flow = min cut. Where  $\leq$  is trivial and  $\geq$  is a little harder. This is a case of **exact duality**. On the other hand we already shown that this doesn't hold for multi-commodity, but what if we can define **approximate duality**.

*Corollary.* Max flow  $\leq$  min cut  $\leq O(\log k)$  max flow. For multi-commodity.

*Proof.* Because of the duality of LP1 and LP2 we know that max flow is the same as min fractional multi-cut. And because of the algorithm we know that the fractional multi-cut is in  $O(\log k)$ .

□

## 2.5 How to solve LP

There is still a problem with our LP which can have up to exponential many of constraints. But this can be solved fast by using **Ellipsoid algorithm** on  $Ax \leq b$ . Only think it needs is an so called **ORACLE** which is that for given  $\bar{x}$ , check whether  $A\bar{x} \leq b$  and if not return a violated constraint.

In our case **ORACLE** is for each  $i$  find the shortest  $s_i - t_i$  path w.r.t  $\bar{x}$ . This can be either 1 and we are happy or  $< 1$  then this constraint is violated.

## 2.6 Is there any better approximation?

We will show that indeed this approximation is the best we can get. Firstly we will define a new property of graphs.

A graph  $G = (V, E)$  is an  $\alpha$ -**expander** if  $\forall S \subseteq V, |S| \leq \frac{n}{2}, \delta(S) \leq \alpha|S|$ .

We take as granted that it holds: 3-regular  $\alpha$ -expanders exist for  $\alpha > 0$ . Now lets observe (2.2) that at most  $1 + 3 \cdot 2^{l-1}$  vertices are reachable by a path of length  $\leq l$ .

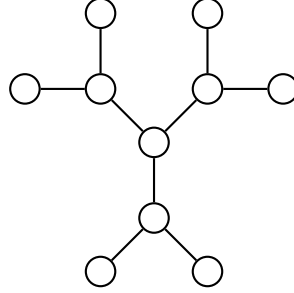


Figure 2.2: How to get the upper bound.

If we take  $l = \log_2 \frac{n-2}{6} + 1$  so with the upper bound we get  $1 + 3 \frac{n-2}{6} = 1 + \frac{n-2}{2} = \frac{n}{2}$ . And also we define an instance of multi-commodity problem:  $T = \{\{u, v\} | \delta(u, v) > l\}$ . A unit of flow consumes  $\geq l$  units of volume of the entire system. Thus  $|E| = O(n)$ . Therefore  $\max \text{ flow} \leq O(\frac{n}{l}) = O(\frac{n}{\log n})$ . But for min cut we take the optimum  $F \subseteq E$ . Every path in  $G = (V, E \setminus F)$  is  $\leq \frac{n}{2}$  so min cut is  $\Theta(n)$ . Thus it is indeed tight.



### 3. The Sparsest cut problem

Same as before we have an undirected graph  $G = (V, E)$  and  $k$ -pairs of sources and targets  $(s_1, t_1), (s_2, t_2), \dots, (s_k, t_k) \in V^2$ . But we will introduce a new parameters  $d_1, d_2, \dots, d_k \in \mathbb{R}^+$  called **demands**.

Firstly we will take a look at linear program for solving this problem for  $k = 1$ .

$$\begin{aligned} & \max f \\ & \sum_{p \in \mathcal{P}_{st}} x_p \geq f \cdot d_1 \\ \forall e \in E \quad & \sum_{e \in \mathcal{P}_{st}} x_p \leq c(e) \\ & x \geq 0 \end{aligned}$$

Where  $\mathcal{P}_{st}$  are all paths between  $s$  and  $t$ . We may see that the optimum of the max flow is the same as this optimum just divided by  $d_1$ . We will denote  $\mathcal{P}_i = \mathcal{P}_{s_i, t_i}$ .

#### 3.1 Concurrent multicommodity flow

Thus we are getting this LP for all  $k$  commodities and  $k$  demands.

$$\begin{aligned} & \max f \\ \forall i \in [k] \quad & \sum_{p \in \mathcal{P}_i} x_p \geq f \cdot d_i \\ \forall e \in E \quad & \sum_{i=1}^k \sum_{e \in \mathcal{P}_i} x_p \leq c(e) \\ & x \geq 0 \end{aligned}$$

We will take a look at the matrix of this LP and after that find a dual program. But firstly we modify  $\sum_{p \in \mathcal{P}_i} x_p \geq f \cdot d_i$  to  $f \cdot d_i - \sum_{p \in \mathcal{P}_i} x_p \leq 0$ . Then the matrix is as follows:

	$f$		$\mathcal{P}_1$		$\mathcal{P}_2$		$\dots$	
1	$d_1$	-1	-1	$\dots$	0	$\dots$	0	$\dots$
2	$d_2$	0	0	$\dots$	-1	-1	$\dots$	0
$\vdots$								
$k$	$d_k$	0	0	$\dots$	0	0	0	$\dots$
$e_1$	0	1	0		1	0	0	$\dots$
$\vdots$								
$e_{ E }$	0	0	1		1	0	1	$\dots$

Where for the first  $k$  lines are  $\leq 0$  and for edges it is  $\leq c(e)$ . We visualized the matrix and thus we can make the dual. We will have variables  $x_e$  for edges and  $y_i$  for  $i \in [k]$ . Thus the dual is:

$$\begin{aligned} & \min \sum_{e \in E} x_e c(e) \\ & \sum_{i=1}^k y_i d_i \geq 1 \\ \forall i \in [k] \forall p \in \mathcal{P}_i \quad & \sum_{e \in p} x_e - y_i \geq 0 \\ & x, y \geq 0 \end{aligned}$$

**Definition 1.** For  $S \subseteq V$  we define  $\delta(S) = \{\{u, v\} \in E : |\{u, v\} \cap S| = 1\}$  and then  $I(S) = \{i \in [k] : |\{s_i, t_i\} \cap S| = 1\}$ . Then the **sparcity** of  $S$  is

$$\rho(S) = \frac{\sum_{e \in \delta(S)} c(e)}{\sum_{i \in I(S)} d_i}$$

*Example.* We will have a simple example where all capacities are 1 and all demands are 1. So we have the graph ??.

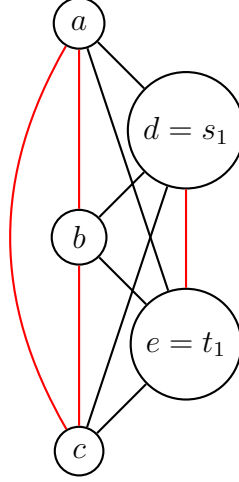


Figure 3.1: Sparse cut example.

The demands are the red edges. We may see that if we choose  $S = \{c, e\}$  then  $\sum_{e \in \delta(S)} c(e) = 3$  and  $\sum_{i \in I(S)} d_i = 3$  therefore  $\rho(S) = 1$ .

We may see that each pair  $s_i t_i$  consumes at least 2 units of a flow of the network for a single unit of the flow. Then we set  $f$  as a max flow and see what we get. For example for paths  $\mathcal{P}_1 = \{(d, a, e) = p_1, (d, b, e) = p_2, (d, c, e) = p_3\}$ .  $x_{p_1} + x_{p_2} + x_{p_3} \geq f \cdot d - 1 = f$ . Thus the total volume consumed by a flow with objective value  $f$  is  $\geq k2f = 8f$ . Total volume of  $G$  is 6. Therefore  $f \leq \frac{6}{8} = \frac{3}{4}$ .

Maybe we can ask if there exist such a flow with this volume. We can obtain it by pushing  $\frac{1}{4}$  from  $d$  to  $c$  on each path and  $\frac{3}{8}$  between all other pairs on all paths. All edges are not over their capacities and we get  $\frac{3}{4}$  for all demands.

**Definition 2.** Now  $F \subseteq E : I(F) = \{i \in [k] : s_i, t_i \text{ are in different components in } (V, E \setminus F)\}$ . And **sparsity** of  $F$  be

$$\rho(F) = \frac{\sum_{e \in F} c(e)}{\sum_{i \in I(F)} d_i}$$

**Lemma 2.**  $\min_{S \subseteq V} \rho(S) = \min_{F \subseteq E} \rho(F)$ .

*Proof.* The  $\geq$  inequality can be easily seen if we set  $F$  to be  $\delta(S)$ . Now we need to show the other inequality. For given  $F \subseteq E$ , let  $s_1, \dots, s_l$  be the components of connectivity of  $(V, E \setminus F)$ . For that we will proof that  $\min_{i \in [l]} \rho(s_i) \leq \rho(F)$ . This will be shown by a contradiction. Assume  $\forall i$ :

$$\begin{aligned} \frac{\sum_{e \in \delta(s_i)} c(e)}{\sum_{j \in I(s_i)} d_j} &> \frac{\sum_{e \in F} c(e)}{\sum_{j \in I(F)} d_j} \\ \sum_{e \in \delta(s_i)} c(e) &> \rho(F) \cdot \sum_{j \in I(s_i)} d_j \end{aligned}$$

Now we sum all  $i$  inequalities.

$$\sum_{i=1}^l \sum_{e \in \delta(s_i)} c(e) > \rho(F) \cdot \sum_{i=1}^l \sum_{j \in I(s_i)} d_j$$

We can see that  $\sum_{i=1}^l \sum_{e \in \delta(s_i)} c(e) = 2 \sum_{e \in F} c(e)$  because all edges are counted twice and similarly  $\sum_{i=1}^l \sum_{j \in I(s_i)} d_j = 2 \sum_{j \in I(F)} d_j$ . So we get:

$$\sum_{e \in F} c(e) > \rho(F) \cdot \sum_{j \in I(F)} d_j$$

Which is a contradiction. So for each  $F$  we can find  $s_i$  that satisfies the inequality.  $\square$

Now we can use this for integer program and then use relaxation. The program will look like this:

$$\begin{aligned} \min & \frac{\sum_{e \in E} c(e)x_e}{\sum_{i=1}^k d_i y_i} \\ \forall i \in [k] \forall p \in \mathcal{P}_i & \quad \sum_{e \in p} x_e \geq y_i \\ & \quad \sum_{i=1}^k d_i y_i \geq 1 \\ \forall e \in E & \quad x_e \in \{0,1\} \\ \forall i \in [k] & \quad y_i \in \{0,1\} \end{aligned}$$

At least one edge has to be removed from each path. Plus we assume that  $d_i \geq 1$ . Now we could just put  $x_e \geq 0$  and  $y_i \geq 0$ . But the thing is that we don't have a linear function in the objective function. What if we have a vector  $(x,y) \rightarrow (\alpha x, \alpha y)$  for  $\alpha > 0$ . You can see that the feasible solution don't change and also the objective is the same. So we could put  $\alpha = \frac{1}{\sum_{i=1}^k d_i y_i}$  and we know that the  $\sum_{i=1}^k d_i y_i = 1$ . Thus the linear program will be:

$$\begin{aligned} \min & \sum_{e \in E} c(e)x_e \\ \forall i \in [k] \forall p \in \mathcal{P}_i & \quad \sum_{e \in p} x_e \geq y_i \\ & \quad \sum_{i=1}^k d_i y_i = 1 \\ \forall e \in E & \quad x_e \geq 0 \\ \forall i \in [k] & \quad y_i \geq 0 \end{aligned}$$

Before we continue we remind ourselves the Manhattan distance  $\|z\|_1 = \sum_{j=1}^k |z_j|$ . This is indeed a metric, which means that it is non-negative, symmetric and triangular inequality holds.

**Lemma 3.** *Let  $f$  be a mapping  $f : V \rightarrow \mathbb{R}^d$  for some  $d > 0$  and let*

$$\begin{aligned} \forall \{u,v\} \in E : \quad \hat{x}(\{u,v\}) &= \|f(u) - f(v)\|_1 \\ \forall i \in [k] : \quad \hat{y}(i) &= \|f(s_i) - f(t_i)\|_1 \\ \beta &= \sum_{i=1}^k d(i)\hat{y}(i). \end{aligned}$$

*Then  $(\frac{\hat{x}}{\beta}, \frac{\hat{y}}{\beta})$  is feasible solution. Also this is called **solution induced by  $f$** . And we will denote  $(\frac{\hat{x}}{\beta}, \frac{\hat{y}}{\beta}) = (x', y')$ .*

*Proof.* We need to show that all conditions of LP are satisfied. Easily the non-negativity still holds. Also

$$\sum_{i=1}^k y'(i)d(i) = \sum_{i=1}^k \frac{\hat{y}(i)}{\beta} d(i) = 1$$

where the last equality holds by the definition of  $\beta$ . Lastly we need to check that  $\sum_{e \in E} x(e) \geq y(i)$  of our LP still holds. This can be easily proven by the fact that  $\|\cdot\|_1$  is metric so in particular triangular inequality is satisfied and by induction on the length of the path we would prove it. Also keep in mind that scaling by  $\beta$  doesn't change anything for the whole inequality since it is on both sides.  $\square$

**Lemma 4 (A).** *Let  $(x', y')$  be a solution induced by  $f : V \rightarrow \mathbb{R}^d$ . Then one can find in polynomial time cut  $S \subseteq V$  of sparsity  $\rho(S) \leq \sum_{e \in E} x'(e)c(e)$ .*

**Lemma 5 (B).** *Given any feasible solution  $(x, y)$  of LP, one can construct a mapping  $f : V \rightarrow \mathbb{R}^d$  (by random algorithm with high probability) which induces a solution  $(\bar{x}, \bar{y})$  s.t.*

$$\sum_{e \in E} c(e)\bar{x}(e) = O(\log k) \sum_{e \in E} c(e)x(e)$$

**Theorem 2.** *There exist a randomized polynomial-time algorithm for the sparsest cut problem that is  $O(\log k)$ -approximation.*

*Proof.* By Lemma B (5) we generate  $(\bar{x}, \bar{y})$  and then by Lemma A (4) we construct the cut.  $\square$

*Proof of Lemma A 4.* Given  $f : V \rightarrow \mathbb{R}^d$  let

$$\forall u, v \in V \quad \mu(u, v) = \|f(u) - f(v)\|_1$$

For  $S \subseteq V$  we define  $\forall u, v \in V$

$$\mu_S(u, v) = \begin{cases} 1 & \text{iff } |\{u, v\} \cap S| = 1 \\ 0 & \text{otherwise} \end{cases}$$

This will be called **cut mapping** and we can easily see that it is non-negative, symmetric and triangular inequality is satisfied thus it is metric. Before we continue we will use another lemma.

**Lemma 6 (lemma).**  $\forall S \subseteq V \exists \lambda_S \geq 0$  s.t.  $\forall u, v \in V : \mu(u, v) = \sum_{S \subseteq V} \lambda_S \mu_S(u, v)$ . Moreover  $|\{S | \lambda_S > 0\}| \leq n \cdot d$ .

*Proof of lemma 6.* Consider the contribution of the first coordinate to  $\mu(u, v)$ : order the vertices according to  $f_1$  where  $f = (f_1, f_2, \dots, f_d)$ , s.t.  $f_1(v_1) \leq f_1(v_2) \leq \dots \leq f_1(v_n)$ . Now let  $S(l) = \{v_1, \dots, v_l\}$  for  $l \in [n]$ . Consider any two vertices  $v_i, v_j$  s.t.  $i > j$ .

$$f_1(v_i) - f_1(v_j) = \sum_{l=j}^{i-1} (f_1(v_{l+1}) - f_1(v_l)) = \sum_{l=1}^{n-1} (f_1(v_{l+1}) - f_1(v_l)) \mu_{S(l)}(v_i, v_j)$$

Where  $(f_1(v_{l+1}) - f_1(v_l)) = \lambda_{S(l)}$ . This can be used to prove this for all dimensions  $f_2, \dots, f_d$  thus it is true for  $f$ .  $\square$

**Observation.** *For any non-negative numbers  $a_1, \dots, a_n$  and positive numbers  $b_1, \dots, b_n$  holds:*

$$\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i} \geq \min_{i \in [n]} \frac{a_i}{b_i}$$

*Proof of observation.* By a contradicton assume  $\forall j :$

$$\begin{aligned}\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i} &< \frac{a_j}{b_j} \\ b_j \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i} &< a_j \\ \sum_j b_j \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i} &< \sum_j a_j\end{aligned}$$

Where the last line is summing all the inequalities together. We get a contradiction. *Note that geometricaly that can be represent as vectors and values of the tan function and it would state that there is a tan smaller of one of the vectors than the sum of them.*  $\square$

Now we continue to proof the Lemma A.

$$\sum_{e \in E} c(e)x'(e) = \frac{\sum_{e \in E} c(e)x'(e)}{\sum_{i=1}^k y'(i)d(i)}$$

Which is just a division by 1 from the conditions in LP. Then by lemma:

$$\begin{aligned}&= \frac{\sum_{e \in E} c(e) \sum_{S \subseteq V} \lambda_S \mu_S(e)}{\sum_{i=1}^k d(i) \sum_{S \subseteq V} \lambda_S \mu_S(s_i, t_i)} = \\&= \frac{\sum_{S \subseteq V} \lambda_S \sum_{e \in E} c(e) \mu_S(e)}{\sum_{S \subseteq V} \lambda_S \sum_{i=1}^k d(i) \mu_S(s_i, t_i)} \geq \min_{S \subseteq V} \rho(S)\end{aligned}$$

The last part is due to the previous observation and the fact that  $\sum_{e \in E} c(e) \mu_S(e) = a_S$  and  $\sum_{i=1}^k d(i) \mu_S(s_i, t_i) = b_S$  taken means  $\frac{a_S}{b_S} = \rho(S)$ .  $\square$

Now we will be proving the Lemma A 5. For that we denote  $T = \{s_i | i \in [k]\} \cap \{t_i | i \in [k]\}$  and without loss of generality assume that  $|T| = 2^\tau$  (we can add arbitrary sources and targets that are essentially the same). Let us denote  $d_x(u, v)$  the length of the  $x$ -shortest  $u - v$  path. Where  $x$  is the result of our LP. For  $A \subseteq V : d_x(A, u) = \min_{v \in A} d_x(v, u)$ .

Also we put  $L = q \log(k)$  where  $q$  is some constant to be decided later on and  $k$  is for number of commodities. Also  $d = L \cdot \tau = O(\log^2(k))$ . For  $t = 1, \dots, \tau$  and  $l = 1, \dots, L$ : let  $A_{tl}$  be a set that is constructed by  $2^{\tau-t}$ -times selecting uniformly at random  $v \in V$ .

**Definition 3.**  $\forall v \in V : f_{tl}(v) = d_x(v, A_{tl})$ .

*Note that both  $A_{tl}$  and  $f_{tl}$  are not dependent on  $l$ . One can say that  $l$  is for repeating the selection.*

**Lemma 7.**  $\forall \{u, v\} \in E : \|f(u) - f(v)\|_1 \leq d \cdot x(u, v)$ .

*Proof.* We proceed by the definition and some algebra.

$$\|f(u) - f(v)\|_1 = \sum_{t=1}^{\tau} \sum_{l=1}^L |f_{tl}(u) - f_{tl}(v)| = \sum_{t=1}^{\tau} \sum_{l=1}^L |d_x(u, A_{tl}) - d_x(v, A_{tl})|$$

Now lets take a look at these inequalities which follows from the triangle inequalities.

$$\begin{aligned}
d_x(u, A_{tl}) &\leq x(u, v) + d_x(v, A_{tl}) \\
d_x(v, A_{tl}) &\leq x(u, v) + d_x(u, A_{tl}) \\
&\Downarrow \\
d_x(u, A_{tl}) - d_x(v, A_{tl}) &\leq x(u, v) \\
d_x(v, A_{tl}) - d_x(u, A_{tl}) &\leq x(v, u)
\end{aligned}$$

Which leads to  $|d_x(u, A_{tl}) - d_x(v, A_{tl})| \leq x(u, v)$  and thus getting the last inequality:

$$\sum_{t=1}^{\tau} \sum_{l=1}^L |d_x(u, A_{tl}) - d_x(v, A_{tl})| \leq \tau \cdot L \cdot x(u, v) = d \cdot x(u, v)$$

□

**Lemma 8.** *With probability  $\geq 1/2$ :  $\forall i \in [k]$  holds that*

$$\|f(s_i) - f(t_i)\|_1 \geq \frac{L}{88} y_i.$$

Before proving this lemma we will take a look, how useful it is.  $\beta = \sum_{i=1}^k d(i) \cdot \|f(s_i) - f(t_i)\|_1 = \Omega(\log k) \cdot \sum_{i=1}^k d(i) y_i = \Omega(\log k)$  where  $y_i$  is from our LP and thus  $\sum_{i=1}^k d(i) y_i$  is equal to 1. The second equality is from the lemma before. And now from the lemma even before that we get  $\leq \sum_{e \in E} c(e) \cdot d \cdot x(e) = d \sum_{e \in E} c(e) x(e)$  which is the objective function result of our LP. Thus  $= O(\log^2(k)) \sum_{e \in E} x(e) c(e)$ . But that is scaled by  $\beta$  thus the objective value of the solution induced by  $f$  is  $\leq O(\frac{\log^2(k)}{\log(k)} \sum_{e \in E} c(e) x(e) = O(\log k) \sum_{e \in E} x(e) c(e)$  and so we have  $O(\log k)$ -approximation. So this proves the Lemma B 5.

*Proof.* To prove the lemma we will prove a simple version that for fixed  $i \in [k]$  with probability  $\geq 1 - 1/2k$  it holds that

$$\|f(s_i) - f(t_i)\|_1 \geq \frac{L}{88} y_i.$$

We can easily see that for doing this for all  $i \in [k]$  the lemma will follow. To prove this we will define few more things.

$$\forall v \in \{s_i, t_i\} : B_x(v, r) = \{w \in T | d_x(v, w) \leq r\}$$

$$\forall v \in \{s_i, t_i\} : B_x^\circ(v, r) = \{w \in T | d_x(v, w) < r\}$$

Now we will look at this sequence of radii.  $r_0 = 0$ ,

$$r_t = \min\{r > 0 : |B_x(s_i, r)| \geq 2^t \wedge |B_x(t_i, r)| \geq 2^t\}$$

$$\hat{t} = \min \left\{ t | r_t \geq \frac{y(i)}{4} \right\}$$

and also redefine  $r_{\hat{t}} = \frac{y(i)}{4}$ . This is define with respect to LP. And also it means that  $B_x(s_i, r_{\hat{t}}) \cap B_x(t_i, r_{\hat{t}}) = \emptyset$ .

Now we observe that for  $A_{tl} \subseteq V : A_{tl} \cap B_x^\circ(s_i, r_t) = \emptyset \Leftrightarrow d_x(s_i, A_{tl}) \geq r_t$ . And also  $A_{tl} \cap B_x(t_i, r_{t-1}) \neq \emptyset \Leftrightarrow d_x(t_i, A_{tl}) \leq r_{t-1}$ . Let  $E_{tl}$  be the event such that  $A \cap B = \emptyset$  and  $A \cap G \neq \emptyset$  where  $B = B_x(s_i, r_t)$  and  $G = B_x(t_i, r_{t-1})$ .

We may observe that if  $E_{tl}$  happens then  $|f_{tl}(s_i) - f_{tl}(t_i)| = |d_x(s_i, A_{tl}) - d_x(t_i, A_{tl})| \geq r_t - r_{t-1}$ . We will look at the probability of happening this.

$$\begin{aligned} \Pr[E_{tl}] &= \Pr[A_{tl} \cap G \neq \emptyset | A_{tl} \cap B = \emptyset] \Pr[A_{tl} \cap B = \emptyset] \\ &\geq \Pr[A_{tl} \cap G \neq \emptyset] \Pr[A_{tl} \cap B = \emptyset] \end{aligned}$$

Let us assume wlog  $s_i$  defines  $r_t$ .

$$\Pr[A_{tl} \cap B = \emptyset] = \left(1 - \frac{|B|}{|V|}\right)^{2^{\tau-t}} \geq \left(1 - \frac{2^t}{2^\tau}\right)^{\frac{2^\tau}{2^t}} \geq \frac{1}{e} \geq \frac{1}{4}$$

$$\begin{aligned} \Pr[A_{tl} \cap G \neq \emptyset] &= (1 - \Pr[A_{tl} \cap G = \emptyset]) = 1 - \left(1 - \frac{|G|}{|V|}\right)^{2^{\tau-t}} \geq \\ &\geq 1 - \left(1 - \frac{2^{t-1}}{2^\tau}\right)^{\frac{2^\tau}{2^{t-1}} \cdot \frac{1}{2}} \geq 1 - \left(\frac{1}{e}\right)^{\frac{1}{2}} \geq \frac{4}{11} \end{aligned}$$

Thus the  $\Pr[E_{tl}] \geq \frac{1}{11}$ . Now we fix  $t = \{1, \dots, \tau\}$  and define:

$$X_{tl} = \begin{cases} 1 & \text{iff } E_{tl} \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

For  $l = 1, \dots, L$  let  $\mu = \mathbb{E} \left[ \sum_{l=1}^L X_{tl} \right]$ . We may observe that  $\mu \geq \frac{L}{11}$  by linearity of  $\mathbb{E}$ . We now may use the Chernoff bound.

$$\Pr \left[ \sum_{l=1}^L X_{tl} \leq \frac{\mu}{2} \right] \leq e^{-\frac{\mu}{8}} \leq e^{-\frac{q \log k}{88}} \leq e^{-\log 2k - \log \log 2k} = \frac{1}{2k \log 2k}$$

Where there is hidden analysis to proper choice of  $q$ . If  $\sum_{l=1}^L X_{tl} \geq \frac{\mu}{2}$  then

$$\sum_{l=1}^L |f_{tl}(s_i) - f_{tl}(t_i)| \geq \sum_{l=1}^L X_{tl}(r_t - r_{t-1}) \geq \frac{L}{22}(r_t - r_{t-1})$$

Therefore with probability  $\geq 1 - \frac{\tau}{2k \log 2k} \geq 1 - \frac{1}{2k} \forall t \in [\hat{t}]$  the previous statement holds. Thus

$$\|f(s_i) - f(t_i)\|_1 \geq \frac{L}{88} y_i = \frac{L}{88} 4 \sum_{t=1}^{\tau} (y_t - y_{t-1})$$

□

## 3.2 Metric spaces

Some of basic definitions are for metric spaces which reader may already know, but we will remind it once again.

**Definition 4.** *Metric space*  $(M, d)$  when  $d : M \times M \rightarrow \mathbb{R}^+$  and

- (i)  $\forall x, y \in M : d(x, y) \geq 0$  and  $d(x, y) = 0 \Leftrightarrow x = y$ ,
- (ii)  $\forall x, y \in M : d(x, y) = d(y, x)$ ,

(iii)  $\forall x, y, z \in M : d(x, z) \leq d(x, y) + d(y, z)$ .

We may already know some examples. One is for  $G = (V, E)$  and for  $x : E \rightarrow \mathbb{R}^+$  the metric is  $d(z, y) = \min_{z-y \text{ paths}} \sum_{e \in P} x(e)$ . This means that  $(V, d)$  is a metric system.

**Definition 5.** Let  $(X, d)$  and  $(Y, \bar{d})$  be metric spaces. An injective function  $f : X \rightarrow Y$  is  $D$ -embedding for some  $D \geq 1$ , if  $\exists r > 0$  such that  $\forall x, y \in X$  the following holds

$$r \cdot d(x, y) \leq \bar{d}(f(x), f(y)) \leq D \cdot r \cdot d(x, y).$$

**Definition 6.** The inf of  $D$  values satisfying the above property is called **distortion** of  $f$ .

**Theorem 3** (Bourgain, 1985). Every  $n$ -point metric space  $(V, d)$  can be embedded in  $(\mathbb{R}^p, l_1)$  with distortion  $O(\log n)$  with  $p = O(\log^2 n)$ . Where  $l_1 = \|x\|_1 = \sum_i x_i$ .

We remind ourselves what we did. We constructed  $f : V \rightarrow \mathbb{R}^+$  and  $p = O(\log^2 k)$  such that

- (i)  $\forall u, v \in V : \|f(u) - f(v)\|_1 \leq p \cdot d_x(u, v)$ ,
- (ii)  $\forall i \in [k] : \|f(s_i) - f(t_i)\|_1 \geq \Omega(\log k) d_x(s_i, t_i)$ .

So if we take  $T = V$ , think about all pair of vertices as commodities. Everything in the proof still works. So we technically proved this theorem before.

Now the question one can ask is: *Is the analysis tight?* For the answer we may recall that in 3-regular graphs, there are  $\Omega(n^2)$  pairs of vertices at distance  $\Omega(\log k)$ . That was for the multi-commodity system. Lets consider 3-regular  $\beta$ -expanders (i.e.  $\delta(S) \geq \beta|S|, \forall S \subseteq V, |S| \leq |V|/2$ ).

With that consider the instance: Commodity for each pair of vertices and set all  $d = 1$ . This will lead to

$$\min_{S \subseteq V} \frac{E(S, V \setminus S)}{I(S)} \geq \min_{S \subseteq V} \frac{\beta|S|}{|S||V \setminus S|} \geq \frac{\beta}{|V|} = \Omega(1/n)$$

then the max concurrent flow is at most the available capacity  $O(n)$  divided by what unit of flow consumes. Thus get

$$\frac{O(n)}{\Omega(n^2 \log n)} = O\left(\frac{1}{n \log n}\right)$$

which means that it is tight. Alternatively integrality gap of our LP is  $\Omega(\log n)$ . Also it implies that the asymptotic optimality of the theorem is tight. Otherwise if there was better version we could use that for better approximation of our LP which is a contradiction. Also there exists  $O(\sqrt{\log n})$ -approximation for sparsest cut using positive semidefinite programming.

*Corollary.* Max flow  $\leq$  min cut  $\leq O(\log k)$  max flow. For sparsest cut problem.



### 3.3 Applications

**Definition 7.** Cut  $(S, V \setminus S)$  is  $b$ -balanced (for some  $b \leq 1/2$ ) if

$$bn \leq |S| \leq (1 - b)n$$

where  $G = (V, E)$  and  $|V| = n$ .

$1/2$ -balanced is called **bisection**. Also there is a problem for finding a  $b$ -balanced cut minimizing the number of edge between  $E(S, V \setminus S)$  (cost of cut). This problem is generally NP-hard.

**Theorem 4.** If there is a  $b$ -balanced cut  $T$  in  $G = (V, E)$ , then for any  $b' < \min\{1/3, b\}$  one can find in polynomial time  $b'$ -balanced cut of cost  $O\left(\frac{E(T, V \setminus T) \log n}{b - b'}\right)$ .

*Proof.* First we define the algorithm.

---

**Algorithm 2** Find  $b'$ -balanced cut.

---

**Require:** Graph  $G$ .

**Ensure:**  $b'$ -balanced cut.

- 1:  $i := 0, G_i = G, S = \emptyset$
  - 2: **while**  $|V(G_i)| > (1 - b')|V|$  **do**
  - 3:     find an approximation of the sparsest cut in  $G_i$  and denote it as  $S_i \subseteq V(G_i)$
  - 4:     let  $G_{i+1} = G_i[V(G_i) \setminus S_i], S = S \cup S_i, i = i + 1$
  - 5: **end while** **return**  $S$
- 

where in the sparsest cut problem is on the network where all vertices are terminals and demands are 1.

Correctness of the algorithm: Before the last iteration it is true that  $|S| < b'n$ . In the last iteration at most  $|V(G_i)|/2$  are added to  $S$ . Therefore at the end

$$\leq |S| + \frac{n - |S|}{2} = \frac{n + |S|}{2} < \frac{(1 + b')n}{2} \leq (1 - b')n$$

Where the last inequality is due the value  $b' \leq 1/3$  and the fact that  $1 + b' \leq 2 - 2b'$ . Also because it ended  $|S| \geq b'n$  so it is indeed a  $b'$ -balanced cut.

Approximation of the cost: Consider an optimal  $b$ -balanced cut  $(T, V \setminus T)$ . In each iteration  $|T \setminus S| \geq (b - b')n$ . What is the sparsity of the cut  $T \setminus S$ ? Lets denote  $\text{opt} = E(T, V \setminus V)$ . The sparsity is

$$\leq \frac{\text{opt}}{b - b')n(1 - b)n} \leq \frac{2\text{opt}}{(b - b')n^2}$$

so the sparsity of the  $O(\log n)$ -approximation  $S_i$  found by the algorithm is

$$\frac{E(S_i, V_i \setminus S_i)}{|S_i||V_i \setminus S_i|} \leq O(\log n) \frac{\text{opt}}{(b - b')n^2}$$

which means that  $E(S_i, V_i \setminus S_i) \leq O(\log n) \frac{\text{opt}}{(b - b')n^2} |S_i|$ . Now we sum it up.

$$E(S, V \setminus S) \leq \sum_i E(s_i, V_i \setminus S_i) \leq O(\log n) \frac{\text{opt}}{(b - b')n^2} \sum_i |S_i| = O(\log n) \frac{\text{opt}}{b - b'}$$

□