Kombinatorika a grafy

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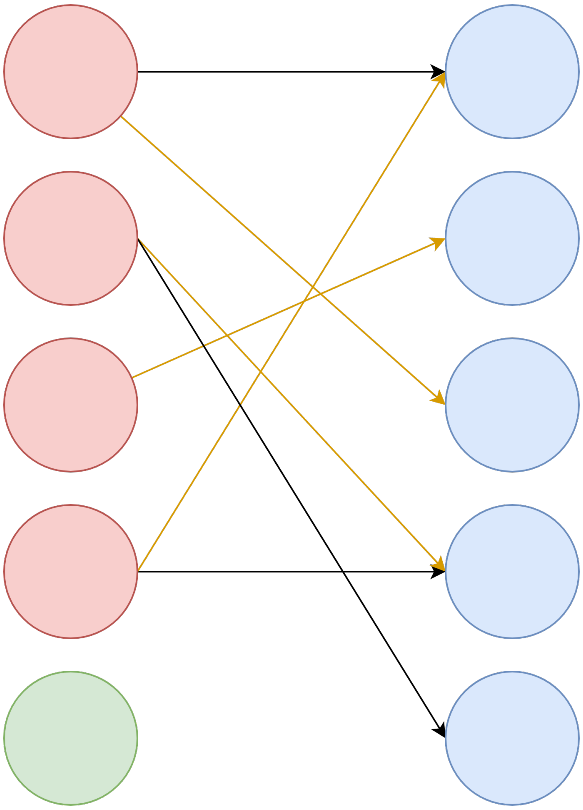

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1. Structural graph theory

Definition 1. *$H \leq_t G$ means that subdivision of H is a subgraph of G, also known as* ***topological minor****.*

Definition 2. *$H \leq_m G$ means that H is a* ***minor*** *of G.*

Definition 3. *$H \subseteq G$ means that H is a* ***subgraph*** *of G.*

Definition 4. *$H \sqsubseteq G$ means that H is a* ***induced subgraph*** *of G.*

Theorem 1 (Kuratowski)**.**

$$K_5, K_{3,3} \not\leq_t G \Leftrightarrow G \text{ planar}$$

$$K_5, K_{3,3} \not\leq_m G \Leftrightarrow G \text{ planar}$$

Definition 5. *$\chi(G)$ means that G has a coloring of size $\chi(G)$.*

Observation. *$C_3, C_5, C_7, \dots \nsubseteq G \Leftrightarrow \chi(G) \leq 2$ which holds also for $\sqsubseteq$.*

Observation. *$C_3 \not\leq_m G \Leftrightarrow G$ is a forest also holds for $\leq_t$.*

Definition 6. *$Forb_{leq}(\mathcal{F}) = \{G | (\forall F \in \mathcal{F}) F \not\leq G\}$*

We will try to show $\mathcal{G} = Forb_{\leq_m}(\mathcal{F})$. If $G \in \mathcal{G}$ then all minors of G belong to $\mathcal{G}$.

Observation. *If $\mathcal{G} = Forb_{\leq}(\mathcal{F})$ then $\mathcal{G}$ is $\leq$-closed. Which means that $\forall G,G'$ if $G \in \mathcal{G}$ and $G' \leq G$ then $G' \in \mathcal{G}$.*

Lemma 1. *Let $\leq$ be a partial ordering of graphs. If a class $\mathcal{G}$ of graphs is $\leq$-closed, then there exist $\mathcal{F}$ s.t. $\mathcal{G} = Forb_{\leq}(\mathcal{F})$.*

Proof. $\mathcal{F} = \{F : F \not\leq G\}$. □

Definition 7. *F is* ***minimal $\leq$-obstruction*** *for $\mathcal{G}$ if $F \notin \mathcal{G}$ but for every $F' \lneq F$ and $F' \in \mathcal{G}$.*

Lemma 2. *Let $\leq$ be an ordering og graphs* ***without infinite decreasing chains****. If $\mathcal{F}$ is $\leq$-closed, then $\mathcal{G} = Forb_{\leq}(\{F : F$ is a minimal $\leq$ -obstruction for $\mathcal{G}\})$.*

Proof. $G \notin \mathcal{G}$ is min $\leq$-obstruction or $\exists G' \lneq G : G \notin \mathcal{G} \Rightarrow G'$ is obstruction or we continue and because we don't have **without infinite decreasing chains** we will eventually end. □

If $\mathcal{G}$ is $\leq_m$-closed, then there exists a **finite** $\mathcal{F}$ such that $\mathcal{G} = Forb_{\leq_m}(\mathcal{F})$.

Theorem 2 (Robertson-Seymor)**.** *For every F there exists an algorithm that for input graph G decides whether $F \leq_m G$ in time $O_F(|G|^3)$.*

Definition 8. *For graph $G = (V,E)$ we define $|G| = |V|$ and $||G|| = |E|$. Also for some $U \subseteq V$ $G[U]$ is a induced subgraph of G that has only vertices from U. Then $N_G(v)$ stands for the neighborhood of vertex v in graph G.*

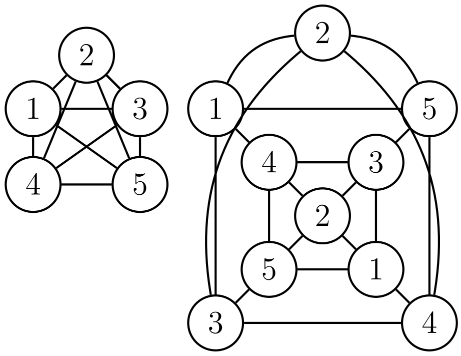


Figure 1.1: Example of G and G' as covers.

Definition 9. *G' is a **cover** of G if $(\exists f : V(G') \to V(G)) \forall v \in V(G')$ for $N_{G'}(v)$ is a bijection with $N_G(f(v))$.*

Example. We may see an example 1.1:

$$\{G"\exists \text{ planar } G' \text{ cover of } G\}$$
$$\Updownarrow$$
$$F_1, \ldots, F_n \not\leq_m G$$

Contrary we take $\mathcal{G} = \{G : (\forall uv \in V(G) U \neq v, \deg(u) \geq 5, \deg(v) \geq 5)(\exists X \subseteq E(G) : |X| \leq 1) u$ and v are in different component of $G - X\}$ which is $\leq_t$-closed. But take these graphs:

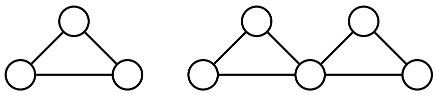


Figure 1.2: Obstructions.

Where each one of them is an obstruction. And we could create much more of them.

Now we take a look at some nice properties of graphs if we forbid some graphs as a minors.

$$\begin{array}{lcl}
K_1 \not\leq_m G & \Leftrightarrow & V(G) = \emptyset \\
K_2 \not\leq_m G & \Leftrightarrow & E(G) = \emptyset \\
K_3 \not\leq_m G & \Leftrightarrow & G \text{ is a forest} \\
 & & G \text{ is obtained from } K_1, K_2 \text{ by clique sums} \\
K_4 \not\leq_m G & \Leftrightarrow & G \text{ is obtained from } K_1, K_2, K_3 \text{ by clique sums}
\end{array}$$

Definition 10. *Graph G can be obtained from G_1 and G_2 by **clique-sum** if the intersection that these graphs have in G form a clique. In other way it is that we bind together two graphs by identifying their vertices and edges in the same size clique.*

Observation. *If G is obtained from G_1 and G_2 by a clique-sum then:*

$$K_m \leq_m G \Leftrightarrow K_m \leq_m G_1 \vee K_m \leq_m G_2$$

Lemma 3. *If $K_k \leq_m G$ and G is the clique-sum of G_1 and G_2 then $K_k \leq_m G_1 \vee K_k \leq_m G_2$.*

Lemma 4. *If G is not 3-connected then there exist $G_1, G_2 \lneq_m G$ s.t. G is a clique-sum of G_1 and G_2.*

Proof. If G is not connected then it is done since it is a clique sum on K_0. If G is connected, but not 2-connected then it is a clique-sum on K_1 since there exist a articulation. If G is 2-connected then there must be two vertices which splits the graph. And these two vertices form a K_2 as a minor. That is because we split G to two parts where we leave the major one side and add a edge to these two vertices, which we can do because they need to have a path between them so we contract all the edges alongside the path. □

Definition 11. *$\delta(G)$ is a minimum degree of a graph G.*

Theorem 3. *If G is K_4-minor-free then G is obtained from $K_{\leq 3}$'s by clique-sums.*

Proof. By induction on $|V(G)|$.

(a) If G is not 3-connected. G is a clique-sum of G_1,$G_2 \lneq_m G$. Since $K_4 \not\leq_m G_1$ and $K_4 \not\leq_m G_2$ we use induction hypothesis and we are done.

(b) If G is 3-connected. If $|V(G)| \leq 3$, then $G = K_{\leq 3}$, wlog $|V(G)| \geq 4$. $\delta(G) > 1 \Rightarrow G$ contains a cycle. Let C be a shortest cycle in G. C is induced in G 3-connected $\Rightarrow G \neq C$ so $\exists v \in V(G) \setminus V(C)$. By Merger's theorem there exists three paths from v to C intersecting only in v. That gives us K_4 as a minor of the graph. Which is contradiction.

□

$$K_5 \not\leq_m G \quad \Leftrightarrow \quad G \text{ is obtained from planar graphs and } W_8 \text{ by clique sums}$$

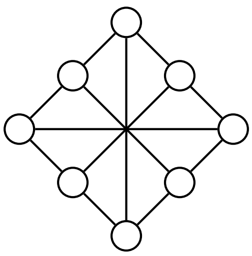


Figure 1.3: W_8 graph.

Observation. *If G is a clique-sum of G_1 and G_2 then*

$$\chi(G) \leq \max(\chi(G_1), \chi(G_2))$$

Proof. We just need to match the coloring of the cliques. Other than that we don't have any problem. □

1.1 Hadwiger's conjecture

K_t-minor-free graphs are $(t-1)$ colorable.

$$\begin{array}{lll} K_1 \not\leq_m G & \chi \leq 1 & \delta \leq 0 \\ K_2 \not\leq_m G & \chi \leq 2 & \delta \leq 1 \\ K_3 \not\leq_m G & \chi \leq 3 & \delta \leq 2 \\ K_4 \not\leq_m G & \chi \leq 4 & \delta \leq 5 \\ K_5 \not\leq_m G & \chi \leq 5 & \end{array}$$

Theorem 4. *$\exists f$ every K_t-minor-free graph G has $\delta(G) \leq f(t)$.*

The function is somewhere near $f(t) = (1{,}6 \cdots + O(1))t\sqrt{\log t}$. But we won't show this result. Instead we will show $f(t) = O(t^2)$. Before we continue it is better to remind ourselves **chordal graph** and **elimination ordering** (known as PES).

Definition 12 (Chordal decomposition of G)**.** *$V(G) = \mathcal{P}_1 \dot\cup \mathcal{P}_2 \dot\cup \ldots \dot\cup \mathcal{P}_n \dot\cup$ and*

1. *$(\forall i) G[\mathcal{P}_i]$ is connected.*
2. *"$\mathcal{P}_i$'s form elimination ordering" Precisely: $(\forall i \in [n])(forall j_1, j_2 < i)$ if G has an edge between $\mathcal{P}_i$ and $\mathcal{P}_{j_1}$ and also between $\mathcal{P}_i$ and $\mathcal{P}_{j_2}$ then it also has an edge between $\mathcal{P}_{j_1}$ and $\mathcal{P}_{j_2}$.*

Definition 13. *Chordal partition is **geodesic** if $(\forall i)(\exists v_i \in \mathcal{P}_i)$ s.t. if $v_1, \ldots, v_t < i$ are the indices s.t. G has an edge between $\mathcal{P}_i$ and $\mathcal{P}_{j_1}, \mathcal{P}_{j_2}, \ldots, \mathcal{P}_{j_t}$ then $v_1, \ldots, v_t \in \mathcal{P}_i$ s.t. v_i has a neighbor in $\mathcal{P}_{j_1}, \mathcal{P}_{j_2}, \ldots, \mathcal{P}_{j_t}$ and $G - \bigcup_{j<i} \mathcal{P}_j$ contains shortest paths from v_i to $v_1, \ldots, v_t$ which cover all vertices in $\mathcal{P}_i$.*

Theorem 5. *Every graph has a geodesic chordal partition.*

Before we show us a proof we will take a look at a simple application. If G is K_k-minor-free last part has neighbours in $t \leq k-2$ parts (otherwise it will have K_k as a minor). Then we may take a look at a $\deg(v) \leq (k-2) + (k-2)(k-2)3 \leq 3k^2$. Thus getting the upper bound $\delta(G) \leq 3k^2$.

Definition 14. *Part is called **terminal** if there is no edge from any vertex in that part going to some vertex in one of the parts on the right.*

Proof. Let $\mathcal{P}$ be a chordal decomposition of G into parts satisfying both properties of definition of chordal decomposition (i) abd (ii) and geodesity (iii) for all non-terminal parts.

This can be easily done by creating parts based on the components of connectivity. For them all properties hold, since they are all connected and "chordal" property is also satisfied since there are no edges. Also all of them are terminal (iii) doesn't have to be satisfied.

Now we proof by that by choosing $\mathcal{P}$ with largest number of parts. Lets say that there is a part that does not satisfy (iii). This means that it is terminal part. Lets take vertex from the part and find the shortest paths to the vertices that are connected to some of the parts to the left. Now we put vertices to separate components and these components will make a new parts. We will also remove all these vertices from the origin part. Note that all properties are satisfied. (i) is trivial. (ii) If there are any vertices from the new parts to other parts then they are to the ones which are already connected to the origin part, which satisfied (ii) before so it is fine. Also (iii) is satisfied.

The thing is that we created $\mathcal{P}$ with larger number of parts which is contradiction. $\square$

Observation. $H \leq_t G \Rightarrow H \leq_m G$

Observation. $\Delta(H) \leq 3 : H \leq_m G \Rightarrow H \leq_t G$

Lets remind ourselves a table and add some new thinks.

$$\begin{array}{lcl}
K_1 \not\leq_t G \Leftrightarrow K_1 \not\leq_m G & \Leftrightarrow & V(G) = \emptyset \\
K_2 \not\leq_t G \Leftrightarrow K_2 \not\leq_m G & \Leftrightarrow & E(G) = \emptyset \\
K_3 \not\leq_t G \Leftrightarrow K_3 \not\leq_m G & \Leftrightarrow & G \text{ is a forest} \\
 & & G \text{ is obtained from } K_1, K_2 \text{ by clique sums} \\
K_4 \not\leq_t G \Leftrightarrow K_4 \not\leq_m G & \Leftrightarrow & G \text{ is obtained from } K_1, K_2, K_3 \text{ by clique sums} \\
K_5 \not\leq_t G \not\Leftrightarrow K_5 \not\leq_m G & \Leftrightarrow & G \text{ is obtained from planar graphs and } W_8 \text{ by clique sums}
\end{array}$$

Well technically $K_5 \not\leq_t G \Rightarrow K_5 \not\leq_m G$ but the other way around is what doesn't work $K_5 \not\leq_m G \not\Rightarrow K_5 \not\leq_t G$. For that we can see an example 1.4. We may see that $\mathcal{G} = \{G : G \text{ has } \leq 4 \text{ vertices of degree } \geq 4\}$ these graphs are so that $K_5 \not\leq_t G$.

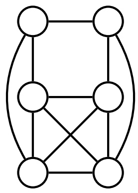


Figure 1.4: A counter example.

1.2 Hájos conjecture

If we remember Headwiger's conjecture then Hájos conjecture is the same only with topological minors. Thus it is that $K_t \leq_t G \Rightarrow \chi(G) \leq t - 1$. This is actually true for $t < 4$ but it is false for $t \geq 7$ and 5,6 are open questions.

Theorem 6. $\exists f_m(k) = O(k\sqrt{\log k})$ *Every K_k-minor-free graph G satisfies $\delta(k) \leq f_m(k)$.*

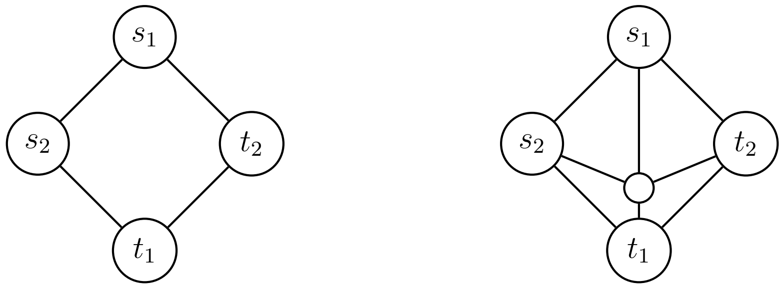
We won't proof this, but we will proof something similiar, that is for topological minors.

Theorem 7. $\exists f_t(k) = O(k^2)$ *Every G s.t. $K_k \not\leq_t G$ satisfies $\delta(G) \leq f_t(k)$.*

The corollary to this is that $\chi)G_{\leq} f_t(k) + 1$. We will proof this theorem, but to do that we need to do some steps beforehand.

Firstly imagine that the enemy gives you a graph and you need to prove that. But the enemy is kind enough to give you a graph H with connectivity $>> k^2$. We could apply Merger's theorem. Though this will only give certain number of vertex disjoint paths from one vertex to another. We would more likely have this many paths between more pairs of sources and targets.

Definition 15. *Graph G is k-**linked** if $|V(G)| \geq 2k$ and $\forall s_1, s_2, \ldots, s_k, t_1, t_2, t_k$ distinct vertices of G. G contains pairwise vertex-disjoint paths $P_1, P_2, \ldots, P_k$. When P_i has ends s_i and t_i.*



(a) 2-connectivity (b) 3-connectivity

Figure 1.5: A counter example to 2-linked graphs.

We may see that there exist a graph that is 2-connected and yet not 2-linked. You may see this on the picture 1.5a. Also not even 3-connected graph has to be 2-linked. Which is also on the picture 1.5b (though we can change the vertex inside for any planar graph). We could continue and end up with that not even 5-connectivity forces 2-linked.

Observation. *Every k-linked graph is $(2k-1)$ connected.*

Proof. That is simply because we put all the s_i, t_i for $i \in [k-1]$ to the edge cut and then choose s_k in the left part and t_k in the right part then we can see that it is indeed $(2k-1)$-connected. □

Theorem 8. *If G is $2k$-connected, $K_{4k} \leq_m G$ then G is k-linked.*

Proof. Next week... □

Corollary. If G is $\max(2k, f_m(4k)+1)$-connected then G is K-linked.

Proof. We use the theorem to get that $\delta > f_m(4k)$ thus $K_{4k} \leq_m G$. □

Also we can say $\exists f_l(k) = O(k\sqrt{\log k})$. If G is $f_l(k)$-connected then G is k-linked.

Corollary. If G is $f_l\left(\frac{k(k-1)}{2}\right)$-connected then $K_k \leq_t G$.

Proof. To see this we choose k vertices and for every one of them $k-1$ neighbors. Then we give s_i and t_i to every single one of these vertex so that every neighborhood has pair with all others. Then we find such paths between them. □

Lemma 5. *If $\bar{d}(G) \geq 4d$ then G contains a $(d+1)$-connected subgraph H of minimum degree $2d+1$.*

Proof. Let H be a minimal subgraph of G s.t. $|V(H)| \geq 2d$ and $|E(H)| > 2d(|V(H)|-d)$. We may see that $|V(H)| > 2d$ that is if it has $2d$ vertices then

$$\frac{2d^2-d}{2} = \binom{2d}{2} > |E(H)| > 2d^2$$

which is a contradiction.

Then we also have that $\delta(H) \geq 2d+1$. If we have $\delta(H) \leq 2d$ we may remove the certain vertex. But we need to show that given properties still hold. We will split the graph to two parts $|A|, |B| \geq 2d+2 > 2d$. Then

$$\begin{aligned}
|E(G)| &\leq |E(A)| + |E(B)| \\
(1) &\leq 2d(|V(A)| - d) + 2d(|V(A)| - d) \\
&= 2d(|V(A)| + |V(B)| - 2d) \\
&= 2d(|V(H)| - |V(A \cap B)| - 2d) \\
|E(G)| &> 2d(|V(H)| - d)
\end{aligned}$$

Where (1) is due to the minimality of H. The thing is with the last two lines we get that $|A \cap B| > d$. □

Proof. This actually is enough for the theorem to be proven since the enemy doesn't have to be kind anymore. □