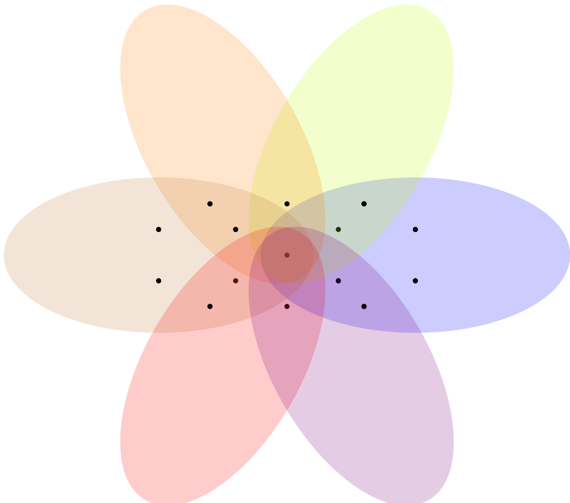
Selected chapters from combinatorics

Tomáš Turek[1]



October 31, 2024

[1]These are my nots from the lecture selected chapters from combinatorics in the year 2024-2025. Keep in mind there may be some mistakes. You may visit GitHub.

Contents

Chapter 1

Introduction to Steiner systems

1.1 Simple hypergraphs

Definition 1. *Hypergraph is a tuple* $(X, \mathcal{M})$ *where* $\mathcal{M} \subseteq \mathcal{P}(X)$. *Or generally just a set system.*

Definition 2. *A simple hypergraph (linear, k-graph) is* $(X, \mathcal{M})$ *if* $M_1 \neq M_2 \in \mathcal{M} \Rightarrow |M_1 \cap M_2| \leq 1$.

Example. Lets see some of the easier examples of simple hypergraphs.

- Graphs themselves are simple hypergraphs. Where $(X, \mathcal{M})$ and $\mathcal{M} \subseteq \binom{X}{2}$.
- Or generally k-graphs, where $\mathcal{M} \subseteq \binom{X}{k}$.
- A well known Fano plane, see picture 2.1.
- Lets have a set of points A and define $X = \binom{A}{2}$; which are edges in A and $\mathcal{M} = \{\binom{T}{2} | |T| = 3, T \subseteq A\}$; which are triangles in A. This is also simple.

Lets also define a chromatic number of such hypergraphs as:

$$\chi(X, \mathcal{M}) := \min\{k | \exists \bigcup_{i=1}^{k} X_i = X \text{ and no } X_i \text{ contains } M \in \mathcal{M}\}.$$

In other words: At least two "colors" for each $M \in \mathcal{M}$. And by Ramsey theory we may state that $\forall k \; \exists X : \chi(X, \mathcal{M}) > k$.

Example. For fixed $k \in \mathbb{N}$ we have k committees, each of them has k members and they are meeting in a room with k seats. Any two committees are disjoint. Can someone sit at the same place? And how many of them? – This was stated by Erdos, Faber and Lovász in 1972.

Theorem 1 (Kuhn, Osthus, Kang, Kelly, Methuku, 2023)**.** *Showed that the previous example is true for large* k.

And some different formulation is by using simple hypergraphs. Lets have simple hypergraf $(X, \mathcal{M})$ where $|\mathcal{M}| = k$ and line chromatic number $\leq k$. That is coloring the edges instead. If they meet they have to be distinct.

Proposition 2. $\chi_l(K_{2k}) = 2k - 1$ *for* $k \in \mathbb{N}$.

Sketch of proof. Lets draw the graph, so the vertices are on a circle. Then take the edges across in the same direction and one from the inside of the circle to the boundary and color them. Then rotate and color once again, until colored. □

1.2 Dual hypergraphs

Lets now define a dual hypergraphs, which may not be so intuitive at a first glance. Lets see a picture 1.1 showing the incidence graph for $(X, \mathcal{M})$. Then the dual is obtained by switching the parts of $(X, \mathcal{M})$ and $(\mathcal{M}', X')$. Lets denote the dual of $(X, \mathcal{M})$ as $d(X, \mathcal{M})$.

Lemma 1. $(X, \mathcal{M})$ *is simple if and only if* $d(X, \mathcal{M})$ *is simple.*

Proof by picture. For the proof see the picture 1.2. This C_4 like structure happens if it is not simple and hence when we flip the diagram, obtaining the dual, the diagram does not change. □

Now lets denote $A(X, \mathcal{M})$ as an incidence matrix of a given hypergraph, then the dual has incidence matrix $A(d(X, \mathcal{M})) = A^T(X, \mathcal{M})$.

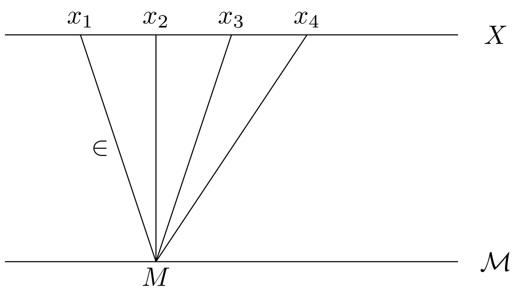


Figure 1.1: Diagram for the dual hypergraph.

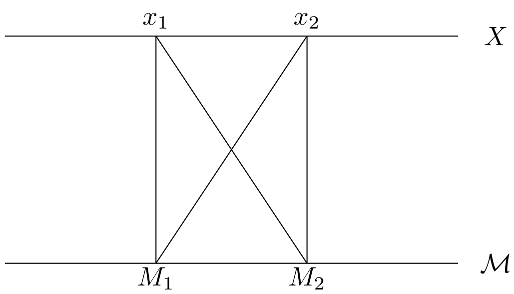


Figure 1.2: Simple dual graphs proof.

Chapter 2

Projective planes

Some may already encountered the projective planes or at least their finite versions. In this chapter we will recall the definition and also some properties they have.

2.1 Basics

Definition 3. *Projective plane is hypergraf* $(X, \mathcal{M})$ *such that*

1. *every two (different) edges (or lines) intersect in exactly 1 point,*
2. *for every 2 (distinct) points there exist exactly one edge containing them,*
3. *there are four points so no 3 of them lies on same edge.*

One of the most well known finite projective plane is Fano plane, which can be seen on a picture 2.1.

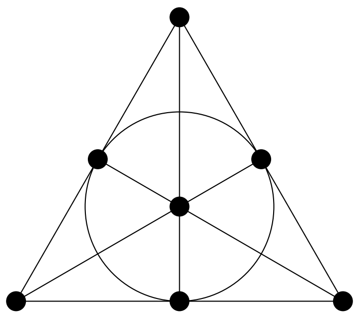


Figure 2.1: Fano plane.

Example. Now what about euclidean space $\mathbb{R}^2$? We can obviously see that part 3 is satisfied, and also the property 2. Only for the very first one 1 we may encounter two lines which are parallel, hence they do not share any point. But we may establish an infinite point for which all such parallel lines in this direction go to. Therefore we must create a lot of infinite points, for every possible direction. But with such augmentation we have broken the property 2 and so we need to add a line which goes through all infinite points.

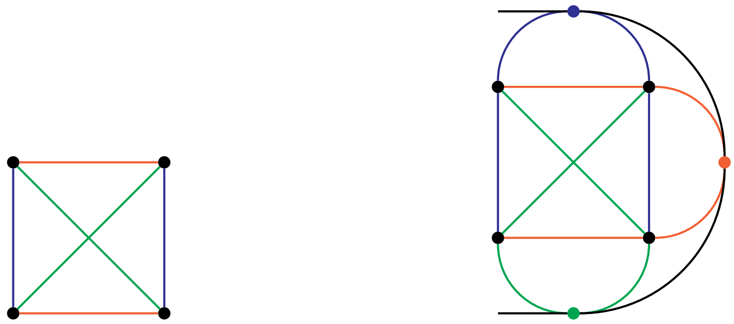
2.2 Construction of projective planes

As in the example 2.1 shown before we will furthermore establish general technique to create a projective plane. Firstly in a geometric way and later on also in algebraic way.

Lets firstly start by taking 4 points, so we are trying to create a smallest possible finite projective plane. To fulfil all properties lets add few lines and end up with a box having two diagonals, see picture 2.2a. Now we encounter the same problem as it was before, so we also add infinite points and extend the lines to them and also creating a line going through all of such infinite points. Note that *parallel lines* now are those lines which don't cross each other in a point. With this procedure we get the following picture 2.2b and we may see that it is indeed isomorphic to the well known Fano plane.

We can also apply to this to other starting points. We may see the result of applying to 3×3 grid of points and resulting in a projective plane depicted on picture 2.3.

But now one question may arise. In all cases we set few parallel lines and mainly decided which lines are so called *diagonal.* Lets now generate such planes by using algebraic methods.



(a) Starting box with 6 lines and 4 points. (b) Ending with 7 lines and 7 points.

Figure 2.2: Generating smallest projective plane.

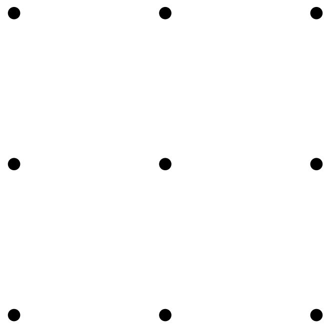


Figure 2.3: Creating a projective plane from 3×3 grid.

2.2.1 Construction by algebraic methods

Lets have $\mathbb{F}$ as a finite field. For such field we would like to create a projective plane. There are few approaches. We will show two of them.

1. Take a vector space $\mathbb{F}^2$; that is tuples of elements from $\mathbb{F}$. The main lines are obviously those which are of a type (c, x) and (x, c) where c is some element from $\mathbb{F}$ and x is increasing elements from the same field. And the diagonals are such lines which has the same difference between the two points, or in other words the same *slope*.

2. Lets now take a vector space $\mathbb{F}^3$. Now the points are (sub)spaces of dimension 1; and the lines are (sub)spaces of dimension 2. Therefore points are lines and lines are planes. Therefore all properties 1, 2 and 3 are satisfied from the perspective of linear algebra.

2.3 Further definitions and observations

Lets talk about some other propositions and definitions of projective planes.

Definition 4. *Order of projective plane is the number of points on every line* -1.

For this definition it is crucial to show that each line has the same number of points. For this see the next lemma.

Lemma 2. *Every projective plane has all lines of same size.*

Proof. When we have two different lines p, q and a point x not lying on any of those, then we set a bijection of points from p to points from q by the lines derived from x and the point of p. Since all such lines intersect in a common point x then they cannot intersect in any point of q.

Note that the existence of such x is not obtained by default. Either it exists from the property 3. If all points from this property are on p or q it must happen that exactly two of them are on p and the rest on q thus seeing a lines going through these four points we get a common meeting point, which will be our desired x. $\square$

Now one can already see that if we take projective planes of order 2 we have $2 \cdot 2 + 2 + 1$ points and for order 3 we have $3 \cdot 3 + 3 + 1$. So that the next proposition is true.

Proposition 3. *Projective plane of order n has $n^2 + n + 1$ points.*

Proof. Lets take a line p and a point x, for every point on p (where there is $n+1$ of them) we see the line going through x and such point. On all of these lines there is another $n-1$ points. Therefore in total we have $n+1+(n+1)\cdot(n-1)+1 = n^2+n+1$.

Also note that we haven't missed any of the points. Otherwise there is a path going through x and such point and this line must intersect p in one point, therefore it was already considered. $\square$

Lastly see the table of known results.

2	3	4	5	6	7	8	9	10	11	12
Fano plane	Shown		No plane in 1900.					Computer search.		OPEN

2.4 Latin squares

Lets now jump to another topic which is related to the projective planes. Latin squares are pretty much generalized sudoku.

Definition 5. *Latin square of order n is a table A of size $n \times n$, where every entry of A is from a collection of n items (we will assume it is n numbers). Then there are two constraints:*

1. *Every column has distinct entries.*
2. *Every row has distinct entries.*

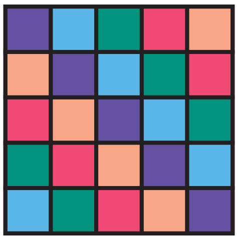


Figure 2.4: Example of a latin square.

Note that the table may represent how the multiplication in an inverse grupoid is defined. Now lets establish the connection to projective planes. If we have a projective plane of order q, then by taking one line we say that taking element from such line will be i and for another line we will take it as j. Then the element for which connect is k and hence $a_{ij} = k$. But note that there is more choices of the other lines, therefore we have much more latin squares.

Definition 6. *Two latin squares L, L' are said to be orthogonal; $L \perp L'$ if $\forall k, k' \exists! i, j$ such that $a_{ij} = k, a'_{ij} = k'$. (Or in other words pairs are unique.)*

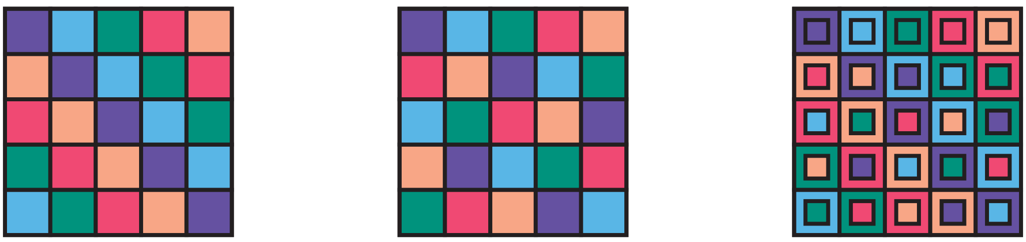


Figure 2.5: Example of a two orthogonal latin square.

Therefore we may see that the existence of projective plane or order q lead to $q-1$ pairwise orthogonal latin squares.

Proposition 4. *$q-1$ pairwise orthogonal latin squares exists if and only if projective plane $PG(1)$ exists.*

We may now ask ourselves if this is the most number of pairwise orthogonal latin squares of order q. See that this can be viewed only from the perspective of permutations.

Proposition 5. *There can be at most $q-1$ pairwise orthogonal latin squares of order q.*

Proof. Lets assume we have t pairwise orthogonal latin squares $L^1, L^2, \dots, L^t$. Lets denote $L^r = (a^r_{ij})$. Now we will change the labeling. For all squares set $a^r_{1j} = j$ for all $r \in [t]$, so first row is identity. Now check that a^r_{21} has to differ from 1, and also due to the orthogonality we have from the first row all pairs (i, i), therefore a^{r+1}_{21} has to differ from a^k_{21} for $1 \le k \le r$. Hence $t \le q - 1$. $\square$

2.5 Another application of projective planes

Lets now take a graph $G = (V, E)$ and suppose that we forbid K_3 being a subgraph of G. Then by either Turán's result we get that $|E| \le \frac{n^2}{4}$ when $|V| = n$ or we look at graph $K_{n/2,n/2}$ which is also sufficient. Similarly look at the example if we forbid K_4 being a subgraph of G, then $|E| \le O(n^3)$. Which can also be seen by Turán's result or looking at a graph which has two parts V_1, V_2, V_3 and all three has close to $n/3$ vertices and edges are only going between these parts. On the other hand if we forbid C_4 being a subgraph of G then we obtain much smaller bound, which is $|E| \le O(n^{3/2})$. Which is somewhat not expected and was proved by Erdös in 1940.

Proposition 6. *For a graph $G = (V, E)$ and $n = |V|$ if $C_4 \not\subseteq G$ then $|E| \le c \cdot n^{3/2}$ for some constant $c \in \mathbb{R}$.*

Proof. We will be counting the pairs $(v, \{v_1, v_2\})$ which are sometimes called forks. When counting from the tuple side we get that we have $\binom{n}{2}$ such pairs and for each such pair there can be at most 1. Otherwise we will have C_4.

When we count from the other side we et that $\sum_{v \in V} \binom{\deg(v)}{2}$. Therefore when combining it we obtain the following inequality

$$\sum_{v \in V} (\deg(v) - 1)^2 \le \sum_{v \in V} \binom{\deg(v)}{2} \le \binom{n}{2} \le n^2.$$

Now lets use Cauchy-Schwarz: $\sum x_i y_i = \sqrt{\sum x_i^2}\sqrt{\sum y_i^2}$. And substitute $x_i = \deg(v) - 1$ and $y_i = 1$.

$$2 \cdot |E| - n = \sum \deg(v) - 1 \le \sqrt{n^2}\sqrt{n} = n^{3/2}$$

And therefore $|E| \le \frac{n^{3/2}+n}{2}$. $\square$

Proposition 7. *The previous upper bound is tight.*

Proof. Lets take a projective plane of order q. So we have a hypergraf $(X, \mathcal{M})$ where $|X| = q^2 + q + 1 = |\mathcal{M}|$ where $\mathcal{M} \subseteq \binom{X}{k}$ for $k = q + 1$. Draw a diagram, where one line is for elements from $\mathcal{M}$ and the other are from X. See the picture 2.6. And we can see that the red drawing cannot happen since it will induce C_4.

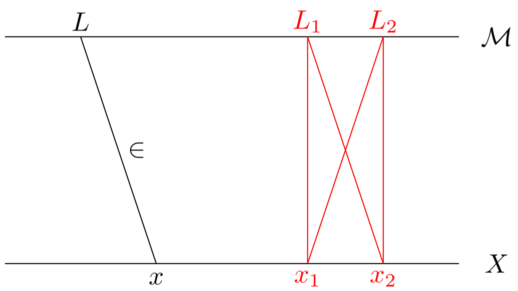


Figure 2.6: Diagram for the proof.

Therefore we have that $|E| = |\mathcal{M}| \cdot (q - 1) = q^3 + \dots$ and $|V| \sim 2(q^2 + q + 1)$ therefore it is tight. $\square$

2.6 Existence of BIBD

Now lets get back to BIBD and if one exists with parameters (n, k, λ, t). Recall that we have steiner triple systems $(n, 3, 1, 2)$ and we can further generalize it to **Steiner system** which is for $\lambda = 1$; i.e. $(n, k, 1, 2)$. It is known that $(n, 4, 1, 2)$ and $(n, 5, 1, 2)$ exist non-trivial Steiner system, but for $k > 6$ it is not known.

Theorem 8 (P. Keevash)**.** *$\forall k, \lambda, t$ $\exists n_0$ s.t. BIBD (n, k, λ, t) where $n > n_0$ exists if and only if Integrality conditions hold. Integrality conditions are the following.*

$$\lambda \frac{\binom{n-i}{k-i}}{\binom{k-i}{t-i}} \textit{ for all } i \in [t-1] \textit{ has to be integers.}$$

Proving that integrality conditions are necessary. Lets again draw a simple diagram, which can be seen on picture 2.7. Then each such T is in λ M's and M is in $\binom{k}{t}$ number of T's. Now lets fix a point x and set $\mathcal{M}' = \{M \in \mathcal{M}; |M \cap X| > 0\}$ and also in the same way $T' = \{T \in \binom{X}{t}; |T \cap X| > 0\}$. And vice versa for all numbers, not just zero.

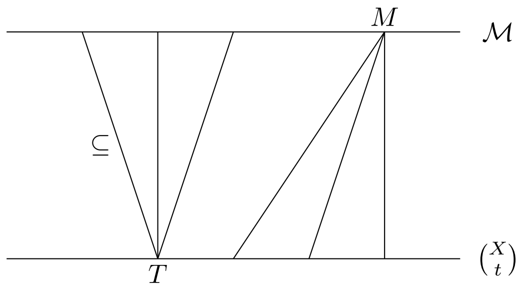


Figure 2.7: Diagram for the proof.

□

Note that n_0 is dependent on all k, λ, t. So for answering the question if $(k^2 + k + 1, k + 1, 1, 2)$ exists we cannot do much.

Now consider this following problem. How many blocks (or hyperedges) can you find so that every tuple is in **at most** λ sets? See that we exchanged equality for an inequality.

Definition 7. *Lets define a function $m :=$ maximum number of such blocks.*

Theorem 9 (Erdös, Hanani)**.** $\forall \epsilon\ \exists n_0\ \forall n \geq n_0$ *the following holds*

$$m(n, k, \lambda, t) \geq \lambda \frac{\binom{n}{k}}{\binom{k}{t}}(1 - \epsilon).$$

Erdös and Hanani stated this problem and in 1985 V. Rödl solved this problem and proved, that it really holds. He proved it by a method which later on was called Rödl **nibbling**, which is also essential in Keevash.

Example. We have 15 schoolgirls, 7 days in a week and we want to form a groups of 3. Moreover we want that every pair will be together in a group in exactly one day. We may only compute the value

$$\frac{\binom{15}{2}}{\binom{3}{2}} = \frac{105}{3} = 35$$

and so it is solvable.

Generally we would like to check if for a hypergraf $(X, \mathcal{M})$ there exists $\mathcal{M} = \bigcup_i M_i$ where M_i is exactly matching of size $2l + 1$.