Barevnost grafů a kombinatorických struktur

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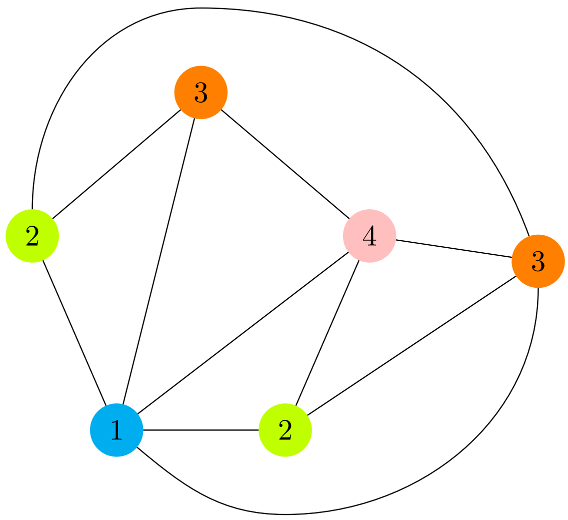

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Information

These are my notes from the course Graph coloring.

Keep in mind there may be some mistakes. You may visit GitHub.

Contents

1. Revision and introduction

One can already know some basics to the graph coloring and also some of the theorems. Therefore we will briefly introduce some basics and revisit some of the known theorems and perhaps some we will show throughout this document.

Definition 1 (Standard colorings). *For a graph G with vertices $V(G)$ and edges $E(G)$ we define:*

- *Vertex coloring is an assignment $\varphi : V(G) \to \mathbb{N}$ such that no two adjacent vertices have the same color.*
- *Edge coloring is an assignment $\varphi : E(G) \to \mathbb{N}$ such that all edges incident to one vertex have different color.*
- *Graph is k-colorable (k-edge-colorable) if vertices (edges) can be colored by k colors.*
- *(Edge) chromatic number is min k such that G is k-(edge)-colorable.*

Notation. *First of all some basic notations.*

- *$\chi(G)$ A chromatic number of G.*
- *$\Delta(G)$ is the maximal degree of a graph G.*
- *$\delta(G)$ is the minimal degree of a graph G.*
- *$\chi_e(G)$ An edge chromatic number of G.*

Theorem 1 (Four colors). *Every planar graph G has $\chi(G) \leq 4$.*

Theorem 2 (Brooks). *If G is a connected graph and $G \neq K_n$ and $G \neq C_{2n+1}$ then $\chi(G) \leq \Delta(G)$.*

Theorem 3 (Vizing). $\Delta(G) \leq \chi_e(G) \leq \Delta(G) + 1$

Theorem 4 (General Euler's formula). *If G can be drawn on a surface of Euler genus g then $|E| \leq |V| + |F| - g$. Where $|F|$ is for the number of faces of the drawing.*

From Euler's formula one can see that if $|V| \geq 3$ then $|E| \leq 3|V| + 3g - 6$. Therefore the average degree is $\frac{2|E|}{|V|} = 6 + \frac{6(g-2)}{|V|}$. Or in other words we have the following lemma.

Lemma 1. *If G is drawn on a surface with Euler genus g then*

$$\bar{d}(G) \leq 6 + \frac{6(g+2)}{|V(G)|}$$

Theorem 5 (Heawood's formula). *If G can be drawn on a surface of Euler genus g then*

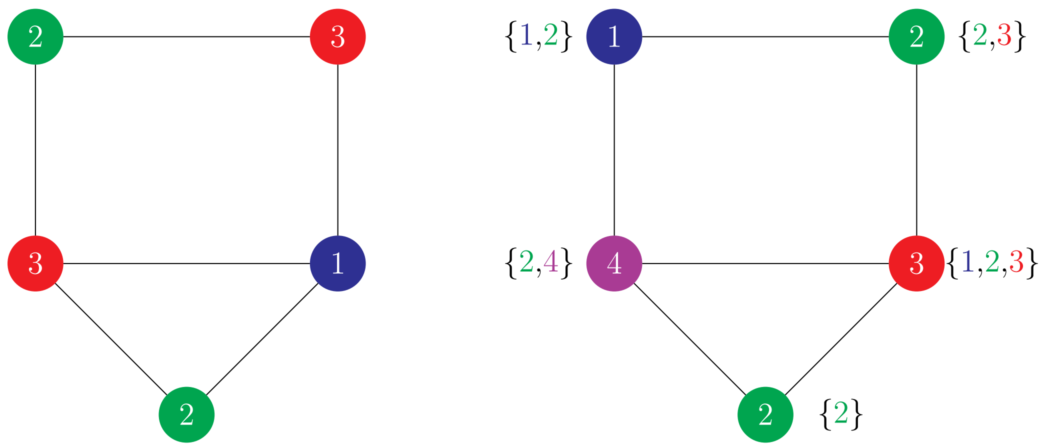
$$\chi(G) \leq \left\lfloor \frac{7 + \sqrt{24g+1}}{2} \right\rfloor$$

Where this formula is tight for Klein's bottle. Then we will show us that deciding if a planar graph is 3-colorable is NP-hard problem. But on the other hand we have this theorem.

Theorem 6 (Grötsch). *Every planar graph without triangle is 3-colorable.*

2. List coloring

Firstly we can see a graph G and its normal coloring which is depicted on a picture 2.1a. Whereas the list coloring shown on picture 2.1b is that each vertex has a assigned list for which we can choose colors. Otherwise the coloring is the same.



(a) Normal coloring of a graph G.   (b) List coloring of a graph G.

Figure 2.1: Difference between basic and list coloring of a graph G.

Definition 2. *k-**list-assignment** is for an assignment of lists for all vertices of size k.*

Definition 3. *The graph is k-**list-colorable** if G can be colored by every k-list-assignment.*

Definition 4. ***List chromatic number** of a graph G is denoted as $\chi_l(G)$. That is the min k s.t. G is k-list-colorable.*

Observation. $\chi(G) \leq \chi_l(G)$

Now one could prove that the Heawood's formula works pretty much the same, thus $\chi_l(G) \leq \left\lfloor \frac{7+\sqrt{24g+1}}{2} \right\rfloor$, but only if $g > 0$. On the other hand one can find a planar graph which has $\chi_l(G) > 4$, but it can be shown that for every planar graph G the $\chi_l(G) \leq 5$ which was proven by Thomassen. Vizing's theorem is an open problem and the Brooks' theorem will be shown.

Definition 5. *L is a **degree-list-assignment** for a graph G if $|L(v)| \geq \deg(v)$ $\forall v \in V$. And G is a **degree-list-colorable** if G is colorable from every degree-list-assignment.*

Definition 6 (Gallai tree)**.** ***Gallai tree** is a graph where each 2-connected block is either a clique or an odd cycle.*

Observation. *Gallai trees are not degree-list-colorable.*

Theorem 7 (Brooks)**.** *If G is connected and G is not a gallai tree then G is degree-list-colorable.*

Lemma 2. *G is connected, L is a degree-list-assignment $(\exists v \in V(G))$ $|L(v)| > \deg(v) \Rightarrow$ G is L-colorable.*

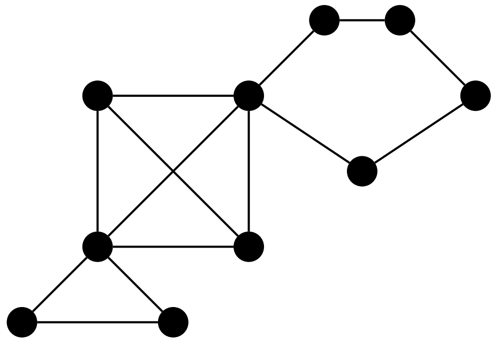


Figure 2.2: Example of a gallai tree.

Proof. By induction on $|V(G)|$. We firstly remove such vertex v from the graph, which will leave us with graph G' which has $|V(G')| + 1 = |V(G)|$ and also the neighbors have greater lists than their degrees. Also at least one neighbor exists since it is connected. $\square$

Lemma 3. *G is connected, L is a degree-list-assignment, $uv \in E(G)$, u is not a cut vertex, G not L-colorable then $L(u) \subseteq L(v)$.*

Proof. For a contradiction suppose $L(u) \nsubseteq L(v)$ therefore $\exists c \in L(u) \setminus L(v)$. We delete u and the color c will be assigned to u. Now we use the Lemma 2. $\square$

Corollary. G connected, L is a degree-list-assignment, $uv \in E(G)$, u,v not cut vertices, G not L-colorable then $L(u) = L(v)$.

Corollary. G is 2-connected, L is degree-list-assignment, G is not L-colorable then $\forall u,v \in V(G) : L(u) = L(v)$ and G is a clique or an odd cycle.

Proof of theorem 7. We will prove this by a contrapositive implication. We split the graph G by a cut $S \subseteq V(G)$. We denote the corresponding parts of graph as G_1 and G_2. Fix a as a number of edges between S and G_1 and b for S and G_2. Therefore $|S| \geq a + b$. Lets say that color c is **bad** if no L-coloring for G of given color c to vertex $v \in S$.

Observation. *$\leq a$ bad colors for G_1 and $\leq b$ bad colors for G_2.*

Observation. *Every color in S is bad for G_1 and G_2.*

Then $S = A \dot\cup B$ and $|A| \geq a$ and $|B| \geq b$. Now by induction G_1, G_2 are Gallai trees and therefore G is a Gallai tree as well. $\square$

3. Critical graphs

Definition 7. ***Girth*** *of a graph is the size of the shortest induced cycle.*

Definition 8. *A graph G is k-critical if $\chi(G) = c$ and for every proper subgraph $H \subsetneq G$ is $(c-1)$-colorable.*

TODO: Second lecture skipped for now.

Theorem 8. *If G is $(c+1)$-critical for $c \geq 3$ and $G \neq K_{c+1}$ then the average degree is*

$$\bar{d}(G) \geq c + \frac{c-2}{c^2 - 2c - 2} \approx c + \frac{1}{c} \quad (\textit{for large } c)$$

3.1 Coloring graph on a surface

# of colors	3	4	5	6	≥ 7
general	NP	?	P	P	P
triangle-free	P, $*$	P	P	P	P
girth > 4	P	P	P	P	P

Table 3.1: What is NP and P problem for coloring graph on surfaces. $*$ but infinitely many 4-critical triangle-free graphs.

Observation. *Now recall 1 therefore if $|V(G)| > 6(g-2)$ then $\delta(G) < 6$. Hence there exists an $O(n + f(g))$ algorithm to decide 7-colorability.*

Now for 6 colors we consider G a 7-critical graph, then $G = K_7$ or $\bar{d} \geq 6 + \frac{2}{23}$. If G is on surface of euler genus g then $6 + \frac{2}{23} \leq \bar{d}(G) \leq 6 + \frac{6(g+2)}{|V(G)|}$. Therefore we see that $|V(G)| \leq 69(g-2)$.

Corollary. For any surface Σ, there are finitely many 7-critical graphs that can be drawn on Σ.

Therefore we are able only to check finitely many number of graphs for the 6-colorability algorithm. Also there can be used **Epstein's** result which is $O(n)$ test for fixed F, Σ whether $F \subseteq G$.

3.1.1 Triangle-free G

Theorem 9 (Grötsch)**.** *Every planar triangle-free graph is 3-colorable.*

Lemma 4. *If G is drawn os surface of Euler genus g, G is triangle-free then*

$$\bar{d}(G) \leq 4 + \frac{4(g+2)}{|V(G)|}.$$

Observation. *If $|V(G)| > 4(g-2)$, then $\delta(G) \leq 4$. Thus the algorithm for 5-colorability.*

Lemma 5. *Suppose G is quadrangulation of the projective plane and G is not bipartite. For every proper coloring of G there exists a face of G whose vertices have four different colors. ($\Rightarrow \chi(G) \geq 4$)*

Proof. We will firstly split the graph on the projective plane by a five cycle, which can be seen on a picture **??**. So we obtain a 10-cycle A,B,C,D,E,A',B',C',D',E'. And then we will redraw the graph so that the 10-cycle is on the outer face. Now we will construct a flow between the faces. The orientation is that if we have an ab colored edge where $b < a$, then the orientation is shown on the picture **??**.

Observation. *Total flow through the outer face is $0 = 2(n_1 - n_2)$*

□

Observation. *If we took triangulation instead of quadrangulation then it is not 4-colorable.*

Lemma 6. *G planar triangulation graph where u,v vertices have odd degree and all other vertices have even degree then $\forall$ 4-coloring φ of G it holds that $\varphi(u) = \varphi(v)$.*

Proof. Now we will consider the following triangle counting. We will be talking about counter-clockwise order of vertices.

$$t_{a,b,c} = \#\text{ of } abc \text{ triangles } - \#\text{ of } acb \text{ triangles}$$

We may see that $t_{a,b,c} = t_{b,c,d}$ and that follows for others as well. Now compute the following.

$$t_{a,b,c} + t_{a,c,d} + t_{a,d,b} = 3t_{a,b,c} \equiv \sum_{v:\varphi(v)=a} \deg(v) \mod 2$$

Therefore if not sum includes u it must also include v because otherwise the sum will be odd and two even which is impossible. □

Exercise. *If G is k-critical, then it is 2-connected.*

Solution. Lets take a graph G which has $\chi(G) = k$ but for all $H \subsetneq G$ $\chi(H) = k - 1$. If G is not connected then one component must have the largest $\chi(G)$, but if we remove vertex or edge from other component we don't achieve $\chi(H)$. That is contradiction.

Now assume it is 1-conencted. Therefore we have a vertex cut $M = \{v\}$ which splits the graph to two components G_1 and G_2. Lets fix graphs $G_1 \cup \{v\}$ and $G_2 \cup \{v\}$ where both of them have chromatic number $k - 1$ since they are subgraphs. Lets fix those colorings. Now for one coloring switch the colors so that v has the same color for both colorings. Therefore we have G colored by $k - 1$ colors which is a contradiction. □

Observation. *If G is k-critical, then it is 3-connected.*

Theorem 10 (Korstochka, Tarcey)**.** *If G is $(c + 1)$-critical, then*

$$\bar{d} \geq c + 1 - \frac{2}{c} - O\left(\frac{1}{|V(G)|}\right)$$

Theorem 11. *If G is 4-critical, then*

$$|E(G)| \geq \frac{5 \cdot |V(G)| - 2}{3} \Rightarrow \bar{d} \geq \frac{10}{3} - \frac{4}{3 \cdot |V(G)|}$$

Now for an application. Suppose G is 4-critical plane graph. Let f_i be the number of faces of length i in a drawing of G. Therefore by Euler's formula we get the following.

$$\begin{aligned}
|V(G)| + |F(G)| &= |E(G)| + 2 \\
|V(G)| + \sum_{i \geq 3} f_i &= |E(G)| + 2 \\
\frac{5}{3}|V(G)| + \frac{5}{3}\sum_{i \geq 3} f_i &= \frac{5}{3}|E(G)| + \frac{10}{3} \\
\frac{1}{3}\sum_{i \geq 3} i f_i &= \frac{2}{3}|E(G)| \\
\frac{5}{3}|V(G)| + \frac{1}{3}\sum_{i \geq 3}(5 - i) f_i &= |E(G)| + \frac{10}{3} \\
|E(G)| &= \frac{5}{3}|V(G)| + \frac{1}{3}\sum_{i \geq 3}(5 - i) f_i - \frac{10}{3} \\
\frac{5}{3}|V(G)| - \frac{2}{3} &\leq \frac{5}{3}|V(G)| + \frac{1}{3}2f_3 + \frac{1}{3}f_4 - \frac{10}{3}
\end{aligned}$$

So if it is a 4-critical planar graph, then $2f_3 + f_4 \geq 8$.

Proof of theorem 9. Suppose there exists a triangle-free planar graph that is not 3-colorable. Choose one with $|V(H)| + |E(H)|$ minimal. Therefore H is 4-critical. By the last application it has at least 8 4-faces. $\square$