Selected chapters from combinatorics

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[1]These are my nots from the lecture selected chapters from combinatorics in the year 2024-2025. Keep in mind there may be some mistakes. You may visit GitHub.

Contents

Chapter 1

Introduction to Steiner systems

1.1 Simple hypergraphs

Definition 1. *Hypergraph is a tuple* $(X, \mathcal{M})$ *where* $\mathcal{M} \subseteq \mathcal{P}(X)$*. Or generally just a set system.*

Definition 2. *A simple hypergraph (linear, k-graph) is* $(X, \mathcal{M})$ *if* $M_1 \neq M_2 \in \mathcal{M} \Rightarrow |M_1 \cap M_2| \leq 1$*.*

Example. Lets see some of the easier examples of simple hypergraphs.

- Graphs themselves are simple hypergraphs. Where $(X, \mathcal{M})$ and $\mathcal{M} \subseteq \binom{X}{2}$.
- Or generally k-graphs, where $\mathcal{M} \subseteq \binom{X}{k}$.
- A well known Fano plane, see picture 2.1.
- Lets have a set of points A and define $X = \binom{A}{2}$; which are edges in A and $\mathcal{M} = \{\binom{T}{2} \| |T| = 3, T \subseteq A\}$; which are triangles in A. This is also simple.

Lets also define a chromatic number of such hypergraphs as:

$$\chi(X, \mathcal{M}) := \min\{k | \exists \bigcup_{i=1}^{k} X_i = X \text{ and no } X_i \text{ contains } M \in \mathcal{M}\}.$$

In other words: At least two "colors" for each $M \in \mathcal{M}$. And by Ramsey theory we may state that $\forall k \; \exists X : \chi(X, \mathcal{M}) > k$.

Example. For fixed $k \in \mathbb{N}$ we have k committees, each of them has k members and they are meeting in a room with k seats. Any two committees are disjoint. Can someone sit at the same place? And how many of them? – This was stated by Erdos, Faber and Lovász in 1972.

Theorem 1 (Kuhn, Osthus, Kang, Kelly, Methuku, 2023)**.** *Showed that the previous example is true for large* k*.*

And some different formulation is by using simple hypergraphs. Lets have simple hypergraf $(X, \mathcal{M})$ where $|\mathcal{M}| = k$ and line chromatic number $\leq k$. That is coloring the edges instead. If they meet they have to be distinct.

Proposition 2. $\chi_l(K_{2k}) = 2k - 1$ *for* $k \in \mathbb{N}$*.*

Sketch of proof. Lets draw the graph, so the vertices are on a circle. Then take the edges across in the same direction and one from the inside of the circle to the boundary and color them. Then rotate and color once again, until colored. □

1.2 Dual hypergraphs

Lets now define a dual hypergraphs, which may not be so intuitive at a first glance. Lets see a picture 1.1 showing the incidence graph for $(X, \mathcal{M})$. Then the dual is obtained by switching the parts of $(X, \mathcal{M})$ and $(\mathcal{M}', X')$. Lets denote the dual of $(X, \mathcal{M})$ as $d(X, \mathcal{M})$.

Lemma 1. $(X, \mathcal{M})$ *is simple if and only if* $d(X, \mathcal{M})$ *is simple.*

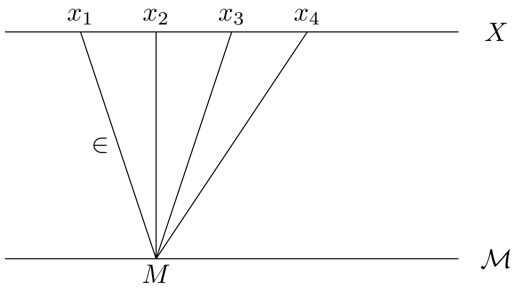


Figure 1.1: Diagram for the dual hypergraph.

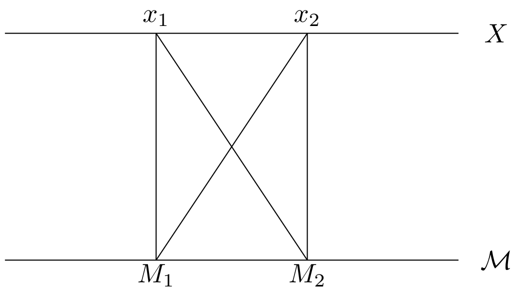


Figure 1.2: Simple dual graphs proof.

Proof by picture. For the proof see the picture 1.2. This C_4 like structure happens if it is not simple and hence when we flip the diagram, obtaining the dual, the diagram does not change.

$\square$

Now lets denote $A(X, \mathcal{M})$ as an incidence matrix of a given hypergraph, then the dual has incidence matrix $A(d(X, \mathcal{M})) = A^T(X, \mathcal{M})$. Lets consider $(X, \mathcal{M})$ a k-uniform hypergraph. Can we somehow boudn the size of $|\mathcal{M}|$? We may establish trivial bounds as $0 \leq |\mathcal{M}| \leq \binom{X}{k}$. We will further make better bound. lets see the picture 1.3. We will be using double counting, for which we notice that for some $\binom{X}{2}$ we have at least one $M \in \mathcal{M}$.

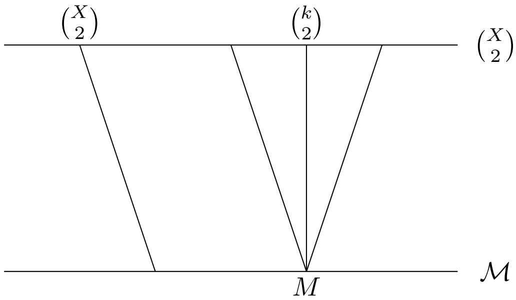


Figure 1.3: Providing better bound.

Hence we may compute the following.

$$|\mathcal{M}| \cdot \binom{k}{2} = \sum_{M \in \mathcal{M}} \binom{|M|}{2} \leq \binom{|X|}{2}$$

Therefore we can obtain the bound.

$$|\mathcal{M}| \leq \frac{\binom{|X|}{2}}{\binom{k}{2}}$$

See that this bound is actually tight. For $k = 2$ we can consider a graph K_n and for $k = 3$ we may look at Fano plane. If the equality hold we call it *Steiner system.* Or in other words it is true if $\forall x \neq y \in X \ \exists! M \in \mathcal{M}$ such that $\{x, y\} \subseteq M$. For $k = 3$ we call this *Steiner triple system* or STS for short (one can be seen as Fano plane and the other as another seen on picture 1.4). This is particularly used in experiments and mainly in agriculture. Usually this is then denoted as *BIBD* or balanced incomplete block design.

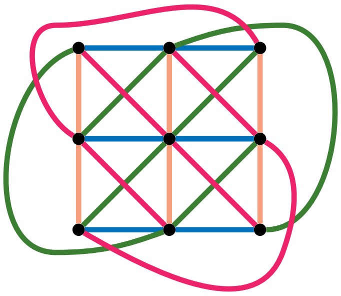


Figure 1.4: Another Steiner triple system.

We may say that for STS to exists it must hold that both $n-1$ and $\frac{\binom{n}{2}}{\binom{3}{2}}$ must be integers. So it only exists if n is either $6k+1$ or $6k+3$.

Theorem 3. *Steiner triple system exists if and only if n is either $6k+1$ or $6k+3$.*

Proof. We will be showing how it can be generated. That is from two STS we create a new one. □

1.3 Introducing BIBD

Definition 3 (BIBD)**.** *Hypergraph $(X, \mathcal{M})$ is BIBD (n, k, λ, t) if $|X| = n$ $\mathcal{M} \subseteq \binom{X}{k}$ and $\forall x_1, \ldots x_t \in \binom{X}{t}$ we have that $|\{M \in \mathcal{M} | \{x_1, \ldots, x_t\} \subseteq M\}| = \lambda$.*

With the similar arguments we may see that $|\mathcal{M}| = \frac{\binom{|X|}{t}}{\binom{k}{t}} \cdot \lambda$. Now the question is whether there actually exists BIBD with given values (n, k, λ, t)? There is pretty simple observation that if we take $A \subseteq X$ of size $|A| = a$ it must be true that $\frac{\binom{n-a}{t-a}}{\binom{k-a}{t-a}} \cdot \lambda$ must be integer for the number of such M's containing A. From these properties we establish the integer constraints for the existence of BIBD.

Proposition 4 (BIBD necessary constraints)**.** *If we have BIBD (n, k, λ, t) everything has to hold. Firstly the size of $|\mathcal{M}$*

$$|\mathcal{M}| = \frac{\binom{|X|}{t}}{\binom{k}{t}} \cdot \lambda$$

and also the integrality of the following fractions.

$$\frac{\binom{n-a}{t-a}}{\binom{k-a}{t-a}} \cdot \lambda \text{ for } a = 1, \ldots, t.$$

Now recall that we have shown how two STS can create a new STS. Now we will show a similar proposition.

Proposition 5. *If there exists BIBD $(n, 3, 2, 1)$ then also BIBD $(2v+1, 3, 2, 1)$ exists.*

Proof. The proof is by a picture. Firstly lets have STS and duplicate it. Then also add new vertex. We will create new M's so that the properties of STS are still satisfied. Which also includes the newly created vertex.

Figure 1.5: Newly created larger STS.

□

Theorem 6 (R. Wilson, R. Chatouri)**.** $\forall k, \lambda, t = 2$ *for every large n the integrality conditions are sufficient for the existence of BIBD.*

Chapter 2

Projective planes

Some may already encountered the projective planes or at least their finite versions. In this chapter we will recall the definition and also some properties they have.

2.1 Basics

Definition 4. *Projective plane is hypergraf* $(X, \mathcal{M})$ *such that*

1. *every two (different) edges (or lines) intersect in exactly 1 point,*
2. *for every 2 (distinct) points there exist exactly one edge containing them,*
3. *there are four points so no 3 of them lies on same edge.*

One of the most well known finite projective plane is Fano plane, which can be seen on a picture 2.1.

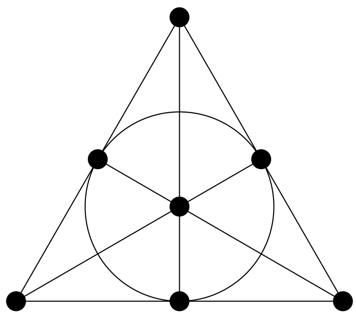


Figure 2.1: Fano plane.

Example. Now what about euclidean space $\mathbb{R}^2$? We can obviously see that part 3 is satisfied, and also the property 2. Only for the very first one 1 we may encounter two lines which are parallel, hence they do not share any point. But we may establish an infinite point for which all such parallel lines in this direction go to. Therefore we must create a lot of infinite points, for every possible direction. But with such augmentation we have broken the property 2 and so we need to add a line which goes through all infinite points.

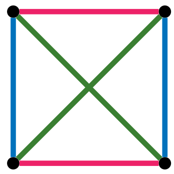
2.2 Construction of projective planes

As in the example 2.1 shown before we will furthermore establish general technique to create a projective plane. Firstly in a geometric way and later on also in algebraic way.

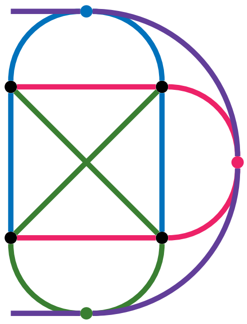
Lets firstly start by taking 4 points, so we are trying to create a smallest possible finite projective plane. To fulfil all properties lets add few lines and end up with a box having two diagonals, see picture 2.2a. Now we encounter the same problem as it was before, so we also add infinite points and extend the lines to them and also creating a line going through all of such infinite points. Note that *parallel lines* now are those lines which don't cross each other in a point. With this procedure we get the following picture 2.2b and we may see that it is indeed isomorphic to the well known Fano plane.

We can also apply to this to other starting points. We may see the result of applying to 3×3 grid of points and resulting in a projective plane depicted on picture 2.3.

But now one question may arise. In all cases we set few parallel lines and mainly decided which lines are so called *diagonal.* Lets now generate such planes by using algebraic methods.

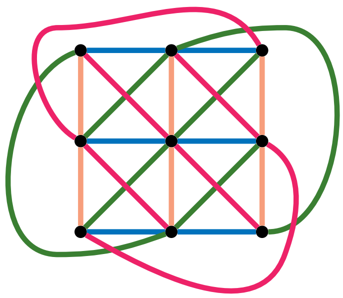


(a) Starting box with 6 lines and 4 points.

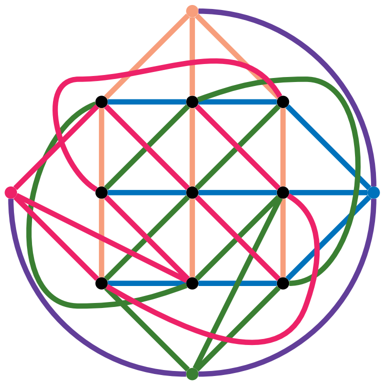


(b) Ending with 7 lines and 7 points.

Figure 2.2: Generating smallest projective plane.



(a) Starting 3×3 grid with lines and diagonals.



(b) Extension with infinite points and line.

Figure 2.3: Creating a projective plane from 3×3 grid.

2.2.1 Construction by algebraic methods

Lets have $\mathbb{F}$ as a finite field. For such field we would like to create a projective plane. There are few approaches. We will show two of them.

1. Take a vector space $\mathbb{F}^2$; that is tuples of elements from $\mathbb{F}$. The main lines are obviously those which are of a type (c, x) and (x, c) where c is some element from $\mathbb{F}$ and x is increasing elements from the same field. And the diagonals are such lines which has the same difference between the two points, or in other words the same *slope*.

2. Lets now take a vector space $\mathbb{F}^3$. Now the points are (sub)spaces of dimension 1; and the lines are (sub)spaces of dimension 2. Therefore points are lines and lines are planes. Therefore all properties 1, 2 and 3 are satisfied from the perspective of linear algebra.

2.3 Further definitions and observations

Lets talk about some other propositions and definitions of projective planes.

Definition 5. *Order of projective plane is the number of points on every line* -1.

For this definition it is crucial to show that each line has the same number of points. For this see the next lemma.

Lemma 2. *Every projective plane has all lines of same size.*

Proof. When we have two different lines p, q and a point x not lying on any of those, then we set a bijection of points from p to points from q by the lines derived from x and the point of p. Since all such lines intersect in a common point x then they cannot intersect in any point of q.

Note that the existence of such x is not obtained by default. Either it exists from the property 3. If all points from this property are on p or q it must happen that exactly two of them are on p and the rest on q thus seeing a lines going through these four points we get a common meeting point, which will be our desired x. $\square$

Now one can already see that if we take projective planes of order 2 we have $2 \cdot 2 + 2 + 1$ points and for order 3 we have $3 \cdot 3 + 3 + 1$. So that the next proposition is true.

Proposition 7. *Projective plane of order n has $n^2 + n + 1$ points.*

Proof. Lets take a line p and a point x, for every point on p (where there is $n + 1$ of them) we see the line going through x and such point. On all of these lines there is another $n - 1$ points. Therefore in total we have $n + 1 + (n + 1) \cdot (n - 1) + 1 = n^2 + n + 1$.

Also note that we haven't missed any of the points. Otherwise there is a path going through x and such point and this line must intersect p in one point, therefore it was already considered. □

Lastly see the table of known results.

2	3	4	5	6	7	8	9	10	11	12
Fano plane	Shown		No plane in 1900.					Computer search.		OPEN

2.4 Latin squares

Lets now jump to another topic which is related to the projective planes. Latin squares are pretty much generalized sudoku.

Definition 6. *Latin square of order n is a table A of size $n \times n$, where every entry of A is from a collection of n items (we will assume it is n numbers). Then there are two constraints:*

1. *Every column has distinct entries.*
2. *Every row has distinct entries.*

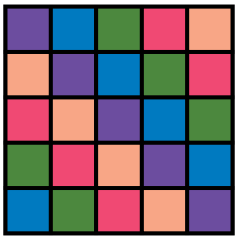


Figure 2.4: Example of a latin square.

Note that the table may represent how the multiplication in an inverse grupoid is defined. Now lets establish the connection to projective planes. If we have a projective plane of order q, then by taking one line we say that taking element from such line will be i and for another line we will take it as j. Then the element for which connect is k and hence $a_{ij} = k$. But note that there is more choices of the other lines, therefore we have much more latin squares.

Definition 7. *Two latin squares L, L' are said to be orthogonal; $L \perp L'$ if $\forall k, k' \exists! i, j$ such that $a_{ij} = k, a'_{ij} = k'$. (Or in other words pairs are unique.)*

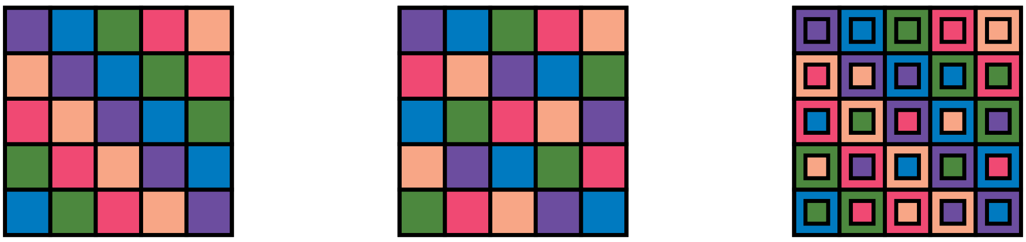


Figure 2.5: Example of a two orthogonal latin square.

Therefore we may see that the existence of projective plane or order q lead to $q - 1$ pairwise orthogonal latin squares.

Proposition 8. *$q - 1$ pairwise orthogonal latin squares exists if and only if projective plane $PG(1)$ exists.*

We may now ask ourselves if this is the most number of pairwise orthogonal latin squares of order q. See that this can be viewed only from the perspective of permutations.

Proposition 9. *There can be at most $q-1$ pairwise orthogonal latin squares of order q.*

Proof. Lets assume we have t pairwise orthogonal latin squares $L^1, L^2, \dots, L^t$. Lets denote $L^r = (a^r_{ij})$. Now we will change the labeling. For all squares set $a^r_{1j} = j$ for all $r \in [t]$, so first row is identity. Now check that a^r_{21} has to differ from 1, and also due to the orthogonality we have from the first row all pairs (i,i), therefore a^{r+1}_{21} has to differ from a^k_{21} for $1 \le k \le r$. Hence $t \le q-1$. □

2.5 Another application of projective planes

Lets now take a graph $G = (V, E)$ and suppose that we forbid K_3 being a subgraph of G. Then by either Turán's result we get that $|E| \le \frac{n^2}{4}$ when $|V| = n$ or we look at graph $K_{n/2,n/2}$ which is also sufficient. Similarly look at the example if we forbid K_4 being a subgraph of G, then $|E| \le O(n^3)$. Which can also be seen by Turán's result or looking at a graph which has two parts V_1, V_2, V_3 and all three has close to $n/3$ vertices and edges are only going between these parts. On the other hand if we forbid C_4 being a subgraph of G then we obtain much smaller bound, which is $|E| \le O(n^{3/2})$. Which is somewhat not expected and was proved by Erdős in 1940.

Proposition 10. *For a graph $G = (V, E)$ and $n = |V|$ if $C_4 \not\subseteq G$ then $|E| \le c \cdot n^{3/2}$ for some constant $c \in \mathbb{R}$.*

Proof. We will be counting the pairs $(v, \{v_1, v_2\})$ which are sometimes called forks. When counting from the tuple side we get that we have $\binom{n}{2}$ such pairs and for each such pair there can be at most 1. Otherwise we will have C_4.

When we count from the other side we et that $\sum_{v \in V} \binom{\deg(v)}{2}$. Therefore when combining it we obtain the following inequality

$$\sum_{v \in V} (\deg(v) - 1)^2 \le \sum_{v \in V} \binom{\deg(v)}{2} \le \binom{n}{2} \le n^2.$$

Now lets use Cauchy-Schwarz: $\sum x_i y_i = \sqrt{\sum x_i^2}\sqrt{\sum y_i^2}$. And substitute $x_i = \deg(v) - 1$ and $y_i = 1$.

$$2 \cdot |E| - n = \sum \deg(v) - 1 \le \sqrt{n^2}\sqrt{n} = n^{3/2}$$

And therefore $|E| \le \frac{n^{3/2}+n}{2}$. □

Proposition 11. *The previous upper bound is tight.*

Proof. Lets take a projective plane of order q. So we have a hypergraf $(X, \mathcal{M})$ where $|X| = q^2 + q + 1 = |\mathcal{M}|$ where $\mathcal{M} \subseteq \binom{X}{k}$ for $k = q + 1$. Draw a diagram, where one line is for elements from $\mathcal{M}$ and the other are from X. See the picture 2.6. And we can see that the red drawing cannot happen since it will induce C_4.

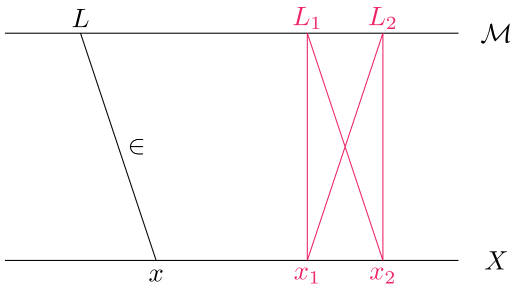


Figure 2.6: Diagram for the proof.

Therefore we have that $|E| = |\mathcal{M}| \cdot (q-1) = q^3 + \dots$ and $|V| \sim 2(q^2 + q + 1)$ therefore it is tight. □

2.6 Existence of BIBD

Now lets get back to BIBD and if one exists with parameters (n, k, λ, t). Recall that we have steiner triple systems $(n, 3, 1, 2)$ and we can further generalize it to **Steiner system** which is for $\lambda = 1$; i.e. $(n, k, 1, 2)$. It is known that $(n, 4, 1, 2)$ and $(n, 5, 1, 2)$ exist non-trivial Steiner system, but for $k > 6$ it is not known.

Theorem 12 (P. Keevash)**.** *$\forall k, \lambda, t\ \exists n_0$ s.t. BIBD (n, k, λ, t) where $n > n_0$ exists if and only if Integrality conditions hold. Integrality conditions are the following.*

$$\lambda \frac{\binom{n-i}{k-i}}{\binom{k-i}{t-i}} \text{ for all } i \in [t-1] \text{ has to be integers.}$$

Proving that integrality conditions are necessary. Lets again draw a simple diagram, which can be seen on picture 2.7. Then each such T is in λ M's and M is in $\binom{k}{t}$ number of T's. Now lets fix a point x and set $\mathcal{M}' = \{M \in \mathcal{M}; |M \cap X| > 0\}$ and also in the same way $T' = \{T \in \binom{X}{t}; |T \cap X| > 0\}$. And vice versa for all numbers, not just zero.

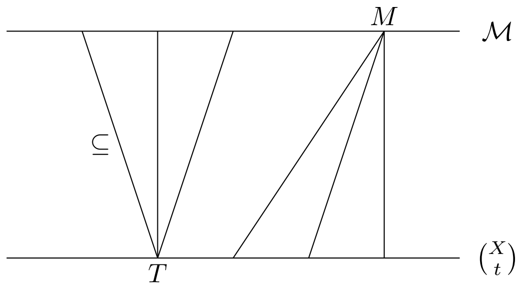


Figure 2.7: Diagram for the proof.

$\square$

Note that n_0 is dependent on all k, λ, t. So for answering the question if $(k^2 + k + 1, k + 1, 1, 2)$ exists we cannot do much.

Now consider this following problem. How many blocks (or hyperedges) can you find so that every tuple is in **at most** λ sets? See that we exchanged equality for an inequality.

Definition 8. *Lets define a function $m :=$ maximum number of such blocks.*

Theorem 13 (Erdős, Hanani)**.** *$\forall \epsilon\ \exists n_0\ \forall n \geq n_0$ the following holds*

$$m(n, k, \lambda, t) \geq \lambda \frac{\binom{n}{k}}{\binom{k}{t}} (1 - \epsilon).$$

Erdős and Hanani stated this problem and in 1985 V. Rödl solved this problem and proved, that it really holds. He proved it by a method which later on was called Rödl **nibbling**, which is also essential in Keewash.

Example. We have 15 schoolgirls, 7 days in a week and we want to form a groups of 3. Moreover we want that every pair will be together in a group in exactly one day. We may only compute the value

$$\frac{\binom{15}{2}}{\binom{3}{2}} = \frac{105}{3} = 35$$

and so it is solvable.

Generally we would like to check if for a hypergraf $(X, \mathcal{M})$ there exists $\mathcal{M} = \bigcup_i M_i$ where M_i is exactly matching of size $2l + 1$.

2.7 Conjectures and theorems

As it was stated before we may furthermore extend the list of conjectures and sometimes even proven theorems.

Existence of partial BIBD. As it was stated in theorem 13 for large enough set X there exists such BIBD $(X, \mathcal{M})$.

Existence of BIBD. Furthermore Keewash showed stronger theorem which is about an exsitence of BIBD. The theorem 14 is stated below.

Theorem 14 (Keewash, Osthus-Kuhn)**.** *$\exists (X, \mathcal{M})$ BIBD (v, k, λ, t) for every k, λ, t and $v \geq v_0(k, \lambda, t)$ with integrality conditions.*

Ringel tree packing problem. We may recall steiner tripple systems in which when we take K_n we want to find edge-disjoint triangles in K_n such that every edge is in one triangle. This problem (and all other similar sounding ones) are typically called *packing problems.* In our special case we know that $|E(K_n)| = \binom{n}{2} = \frac{n}{2}(n-1)$ and in K_{2n+1} we take an arbitrary tree T with $n+1$ vertices. Now the question is if K_{2n+1} can be packed by $2n+1$ copies of T.

This was proved as true by Sudakar and Keewash. Also observe that if $|T| = n$ and we would have K_n then having a tree having one vertex with degree $n-1$ it is not possible to pack such tree.

Rosa conjecture. Suppose we have a tree $T = (V, E)$ where $|V| = n$/ Does there exist labeling $l : V \to \{1, 2, \ldots, n\}$ such that $\{|l(v) - l(u)|; \{u, v\} \in E\} = \{1, 2, \ldots, n-1\}$, i.e. the differences are distinct. This is an open problem.

We may not see the relation to the previous topics at a first glance, but imagine having a circle with numbers around it. And we would connect the numbers by edges, which will be corresponding to the labeling, then we would be looking at packing of such model.

Gyarfas. Most people will already know that $\binom{n}{2} = \frac{n}{2}(n-1) = 1 + 2 + \cdots + (n-1)$ which is well known fact already shown by Gauss. So lets use this to state another packing problem.

Given trees T_i for all $i = 2, 3, \ldots, n$ where T_i tree has $i-1$ edges (or i vertices). The question is whether such trees packs K_n?

- When all trees are starts (T_i have one "middle" vertex with degree $i-1$) then it is true. This can be somewhat easily seen.
- On the other hand if we have trees which are either of a type star or path it is also true.
- But in general it is not known and it is open.

Graph dimension. Before we state any theorem we must firstly define what is a product of graphs and also dimension of graph.

Definition 9 (Graph product)**.** *For graphs $G = (V, E)$ and $G' = (V', E')$ their product $G \times G'$ is defined as a new graph $H = (V \times V', E'')$ where $\{(x, x'), (y, y')\} \in E'' \iff \{x, y\} \in E$ and $\{x', y'\} \in E'$.*

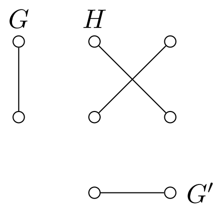


Figure 2.8: Simple example of graph product, where both G and G' are P_1.

Definition 10 (Graph dimension)**.** *For a graph $G = (V, E)$ we define dimension* $\dim(G) = \min(|I|)$ *and G is induced subgraph of $\prod_{i \in I} K_{n(i)}$.*

Theorem 15. *For every G there exists $n(i); i \in I$ such that G is induced subgraph of $\prod_{i \in I} K_{n(i)}$.*

Example. Lets see some examples of dimensions of graphs, where some may be easier and some harder. Firstly trivially $\dim(K_n) = 1$. Secondly for graph G consisting of a isolated vertex and K_n we have that $\dim(G) = n$. Lastly for a graph H of a matching we get that $\dim(H) = \log n$ which was proved by Lovász, Pultr and Nešetřil.

When we take a graph $G_{V,1/2}$ which is a random graph with V vertices and every edge has probability 1/2 that it exists. The lower bound $\dim(G_{V,1/2}) \geq \frac{n}{\log n}$ can be seen by a probabilistic argument and the fact that $\dim(G)$ corresponds to the covering of edges of complement of G by equivalences. Moreover Guo and Warke showed that for constants c, c' we have the following.

$$c\frac{n}{\log n} \leq \dim(G_{V,1/2}) \leq c'\frac{n}{\log n}$$

Back to STS. For STS $(v, 3, 1, 2)$ simple hypergraph $(X, \mathcal{M})$ where $\mathcal{M}$ contains on k points at least $k-3$ triples; that is for $k \geq 4$. Erdős stated if there exists STS $(X, \mathcal{M})$ such that on any l points there are $\leq l-3$ edges. This was proven as being true.

But what about $|\mathcal{M}| \geq n^{1+\epsilon}$? This will be shown as being actually hard.

For simple k-graphs $(X, \mathcal{M})$ we have shown that χ may be unbounded, due to Ramsey and Hales Jewett, but $PG(g)$ has $\chi = 2$ for $g > 2$. Also $\mathcal{M} \sim \binom{|X|}{2}$ due to Erdš and Hanani. Now lets see if $|\mathcal{M}| \geq |X|^{1+\epsilon}$ can be true. Before showing the proof lets some direct results from such statement.

Theorem 16 (Ordering property)**.** *Assume $|\mathcal{M}| \geq |X|^{1+\epsilon}$ fo given k, ϵ and large X. $\forall G = (V, E)$ and $\forall$ linear ordering of V there exists graph $H = (W, F)$ such that for every linear ordering $\preceq$ of W there exist monotone embedding $f : (G, \leq) \to (H, \preceq)$. In particular $f : V \to W$, $\{x, y\} \in E \iff \{f(x), f(y)\} \in F$ and $x \leq y \iff f(x) \preceq f(y)$.*

Proof. Let $G = (V, E)$ be given and $|V| = k$. Let $(X, \mathcal{M})$ be simple k-graph with n vertices. Let $\mathcal{G}$ be the set of all graphs $H = (X, F)$ with property that for any $M \in \mathcal{M}$ the graph $H|_M \simeq G$ and every edge of F is a subset of an $M \in \mathcal{M}$. Then the number of placements of different orderings of G is $a = \frac{k!}{\mathtt{Aut}(G)} = a$ where $\mathtt{Aut}(G)$ is the automorfism group of graph G.

Then $|\mathcal{G}| = a^{|\mathcal{M}|} \geq a^{n^{1+\epsilon}}$. How many graphs H in $\mathcal{G}$ contain embedding $(G, \leq) \to (H, \preceq)$ for some $\preceq$? That is $\leq n!(a-1)^{n^{1+\epsilon}}$ which is $\ll a^{n^{1+\epsilon}}$, which can be seen by taking logarithms.

$$n^{1+\epsilon} \log a > n \log n + n^{1+\epsilon} \log(a-1)$$

Hence there exists $H \; in \mathcal{G}$ such that it suffices the property. $\square$

Other remark is that there exists H where all orderings of $(G, \leq)$ appear almost equally likely, which was shown by Angel, Kechris and Lyons. The technique is called *random placement method* used in 1991 by Nešetřil, Rodl and Ramsey.

Now we will show a stronger version of our stated property.

Theorem 17. *$\forall k \forall l \exists \epsilon$ There exist $(X, \mathcal{M})$ k-graph such that*

1. *$|\mathcal{M}| \geq |X|^{1+\epsilon}$ and*
2. *$(X, \mathcal{M})$ has no cycles of length $\leq l$.*

Note that we can have cycles of length 2 by just having $M \neq M' \in \mathcal{M}$ where $|M \cap M'| = 2$. This can be seen by drawing the incidence bipartite graph, where we have cycles of length $2l$ in comparison to the circles inside a k-graph. Hence for $l = 3$ we obtain the original statement.

Proof. Proof with stronger theorem. Let $G = (V, E)$ be given and $|V| = k$. Let $(X, \mathcal{M})$ be simple k-graph with n vertices. Let $\mathcal{G}$ be the set of all graphs $H = (X, F)$ with property that for any $M \in \mathcal{M}$ the graph $H|_M \simeq G$ and every edge of F is a subset of an $M \in \mathcal{M}$. Also H contains cycles of length $\leq l$ only in copies of G. $\square$

Theorem 18. *$\forall k \; \forall l \; \exists G_{k,l} = G$:*

1. *$\chi(G) \geq k$ and*
2. *G contains no cycles $C_3, C_4, \ldots, C_{l-1}$.*

Proof. Put $K = k + 1$ and apply stronger theorem 17 for K, l, then we get $(X, \mathcal{M})$ K-graph and $|X| = n$ and $|\mathcal{M}| \geq |X|^{1+\epsilon}$. We take $M \in \mathcal{M}$ and we put there one edge. $\mathcal{G} = \{(X, E); (X, E)|_M \simeq \text{only one edge}\}$.

Now $a = \binom{K}{2}$ and any $H \in \mathcal{G}$ does not contain short cycles. How many graphs H in $\mathcal{G}$ have a $\chi(H) \leq k$?

So $\exists f : X \to \{1, 2, \ldots, k\}$ on every $M \in \mathcal{M}$ $\exists x \neq y$ such that $f(x) \neq f(y)$. $|\mathcal{G}| = a^{n^{1+\epsilon}} \gg |\{G | \chi(G) \leq k, G \in \mathcal{G}\}| = k^n \cdot (a-1)^{n^{1+\epsilon}}$. $\square$

Now suppose we have a poset $P = (X, \leq)$ and we create Hase diagram. Question: Which graphs are diagrams? For planar graphs it holds that G is diagram $\iff K_3 \nsubseteq G \Rightarrow \chi(G) \leq 3$. Problem is that whether there exists graph without C of length $[3, l]$ which fails to be a diagram? Then there is a theorem which states that indeed it is true for all l. G is not a diagram $\iff$ $\forall$ ordering $\leq$ of $V(G)$ there exists a cycle of length t for some t.

Proof. Proof of theorem 17 Set $\epsilon = \frac{1}{l}$ and put $m = 2 \cdot n^{1+\epsilon}$. Consider all k-graphs with m edges and n vertices. There is exactly $\binom{\binom{n}{k}}{m}$ such k-graphs. Observe that if $(X, \mathcal{M})$ has no cycles then $|\mathcal{M}|(k-1) + 1 \leq |X|$. How many of these k-graphs contain a cycle of length $l' \leq l$? By this observation it must be violated so it must be

$$\leq c(k, l') n^{(k-1)l'} \binom{\binom{n}{k} - l'}{m - l'}$$

where

$$c(k,l') = \binom{\binom{l'(l-1)}{k}}{l'}.$$

Lets divide it be $\binom{\binom{n}{k}}{m}$ which leads to upper bound for average number of cycles of length l'. So we proceed by summing it over all $l' \leq l$. Obtaining the following.

$$\sum_{l' \leq l} \frac{c(k,l') n^{(k-1)l'} \binom{\binom{n}{k} - l'}{m - l'}}{\binom{\binom{n}{k}}{m}}$$

We can simplify the binomials and just obtain

$$\frac{m \cdot (m-1) \cdots 2 \cdot 1}{\binom{n}{k} \cdot (\binom{n}{k} - 1) \cdots (\binom{n}{k} - l' + 1)} \cdot n^{kl' - l'}$$

and we are considering only with going to ∞, that is the leading elements of such polynomials. Hence we have

$$\leq c \cdot n^{kl'-l'} \cdot \frac{m^{l'}}{n^{kl'}} = c \cdot n^{-l'} \cdot n^{(1+\epsilon)l'} = c \cdot n^{\epsilon l'} \leq n \cdot c.$$

Therefore there has to be k-graph with such small number of short cycles. For that we will delete at most cn edges to get rid of every short cycle. □

Note that for this theorem there does not exist a constructive proof.

Corollary. $\forall p, k \geq 2, l \geq 2\ \exists (Y, \mathcal{N})$ such that

1. $\mathcal{N} \subseteq \binom{Y}{k}$,
2. $(Y, \mathcal{N})$ has girth $> l$ and
3. $\chi(Y, \mathcal{N}) > p$.

Proof. Put $K = p(k-1)+1$ and let $(X, \mathcal{M})$ be simple hypergraph such that $\mathcal{M} \subseteq \binom{X}{K}$ and $(X, \mathcal{M})$ has girth $> l$ and $|\mathcal{M}| \geq |X|^{1+\epsilon}$ (by using the theorem 17). Let $\mathcal{H}$ be the set of all graphs $(X, \mathcal{N})$, where $\mathcal{N} \subseteq \binom{X}{k}$ such that

1. for every $M \in \mathcal{M}$ there exists ≤ 1 $N \in \mathcal{N}$ where $N \subseteq M$ and
2. there are no other edges.

Any $(X, \mathcal{N}) \in \mathcal{H}$ has girth $> l$. How many k-graphs from $\mathcal{H}$ have a chromatic number $< p$? That would be

$$\leq p^{|X|} \cdot \left(\binom{K}{k} - 1 \right)^{|X|^{1+\epsilon}}$$

since it is the number of colorings multiplied by the number of k-graphs with given coloring (respectively). This is

$$\ll |\mathcal{H}| = \left(\binom{K}{k} \right)^{|X|^{1+\epsilon}}.$$

□

We will also present a constructive proof in this case. But before we do so, we state some remarks. Usually the *chromatic number* of a graphs is something like the complexity, or in other words it cannot be split into small pieces. On the other hand *girth* is something like local simplicity. So the theorem states that even a graph which locally seems simple it is still complex.

Definition 11. *Graph F is χ-unavoidable if $\exists n_0(F)$ for every graph G with $\chi(G) \geq n_0(F)$ contains F as a subgraph (not necessarily induced).*

Corollary. Forests are only χ-unavoidable graphs.

Proof. If F is not a forest then F has a cycle of length l which by using the theorem 2.7 can be still made arbitrarily colourful.

Suppose $\chi(G) \gg n_0$ and F is a tree. This implies that $\Delta(G) > n_0$, where $\Delta(G)$ denotes the maximum degree.

Lets have vertices with small degrees and then delete them. Now we may end up with other vertices which now have small degree so we proceed in the same way until we can. After that we take a vertex with the maximum degree this has to have other vertices which also have a high degree and so on, this is how the tree can be constructed. □

Corollary. $\forall k\ \forall F$ K-graph tree $\exists n_0(k, F)$ such that any $\chi(X, \mathcal{M}) \geq n_0$ contains F.

Lemma 3. *Any k-graph with high chromatic number contains large degree. (Sometimes this is called a sunflower or Δ system).*

Proof. Suppose $\chi(X, \mathcal{M}) \geq (k-1) \cdot t = n_0$. Then we split vertices into a groups $X_1, X_2, \ldots, X_{n_0}$ by their colours. We will take X_1 and enlarge it to X_1' as much as possible without violating the colouring. Then for X_2 we take $X_2' = X_2 \setminus X_1'$ and again enlarge it. We will continue with all groups. All of them have to be non-empty, otherwise the colouring was not optimal. Lets have $x \in X_{n_0}'$. There has to be an edge to $X_1', X_2', \ldots,$ because otherwise we would add x to one of them. This is how we got our sunflower. □

Conjecture 1. *Any high chromatic graphs contains either large K_n or given induced tree T. Alternatively: Fix T tree, K_n. Let G be a graph $K_n \nsubseteq G$ and $T \nsubseteq G$ then $\chi(G) \leq n_0(T, K_n)$.*

Conjecture 2. *Same statement is not true for k-graphs.*

Example. Lets consider $k = 2$ so graphs and $l = 4$, therefore triangle-free graphs. Try to create $\forall n$ a graph such that $C_3 \nsubseteq G$ and $\chi(G) = n$. For $n = 3$ we take an odd cycle of length 5. Now we can proceed by Mycielski and construct this graph 2.9

(a) Five cycle.

(b) Adding brothers and common point.

Figure 2.9: Mycielski creation of a graph.

This can be actually generalized. Lets have G_n, for every vertex of such graph create a *"brother"* which will share the same neighbours. Then also create one common vertex and connect all *brothers* to him. See that this has chromatic number at least $n+1$ if G_n had chromatic at least n.

Example. Another construction is so called *Shift graph.* We will create G_n by taking V to be $E(K_n)$ and $E(G_n)$ being two consecutive edges in K_n by an ordering $\leq$. This graph also does not have triangles and $\chi(G_n) = \log n$. Suppose that $\chi(G_n) \leq t$ then $E = E_1 \dot{\cup} E_2 \dot{\cup} \ldots \dot{\cup} E_t = E(K_n)$ easily seen that $\chi([n], E_i) \leq 2$ since no beginnings and ends can overlap. Therefore $\chi(K_n) \leq 2^t$, where the logarithm follows.

Chapter 3

Hales-Jewett theorem

We will now jump to another topic and mainly the Hales-Jewett theorem. This was actually motivated by famous Van der Waerden theorem 19.

Theorem 19. *For every $n > 0$ and every finite coloring of integers there exists monochromatic arithmetic progression of length n, i.e. $a, a+b, a+2b, \ldots, a+(n-1)b$ for $b > 0$.*

But the main issue is that this theorem is rather more algebraic and also used in combinatorics. For example we could create an auxiliary hypergraph $H = (\mathbb{N}, \text{all pregressions of length } n)$ so that we are looking at edges.

Multidimensional Tic-Tac-Toe. Lets consider Σ finite alphabet (set). Then Σ^n is set of functions $[n] \to \Sigma$ (usually called words). Lets define a hypergraph by the following set. $L \subseteq \Sigma^n$ is combinatorial line if $M \subseteq \{1, 2, \ldots, n\}$ nonempty and $f : [n] \to \Sigma$ such that

$$L = \left\{ g | \exists c \in \Sigma \; g(i) = \left\{ \begin{array}{ll} c & \text{if } i \in M \\ f(i) & \text{if } i \notin M \end{array} \right\} \right\}.$$

Example. Lets have $w = (\{\lambda\} \cup \Sigma)^n$ then lets have $A\lambda\lambda B$ which can either be $AAAB$ or $ABBB$. So formally $M = \{2, 3\}$ and $f = AAAB$.

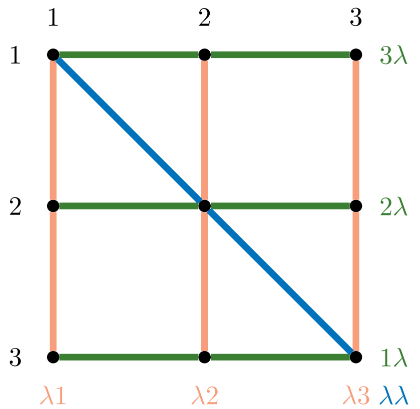


Figure 3.1: Combinatorial representation of combinatorial lines for $\Sigma = \{1, 2, 3\}$ and $n = 2$. Which we can see we obtain a hypergraph $H = (\Sigma^n, \text{Lines})$.

Theorem 20. *Hales-Jewett, 1964 For every finite Σ and finite $n > 0$ $\exists N > 0$ denoted as $\mathtt{HJ}(|\Sigma|, r)$ such that the chromatic number of $H(\Sigma^n, Lines)$ is at least n, or alternatively if Σ^n is r-colored then there exist monochromatic combinatorial line (which is a subspace of dimension 1).*

Theorem 21. *Van der Warden, 1927 For r number of colors and l length of progression is known then there is finite version $N = \mathtt{VW}(r, k)$. And N is not primitive recursion.*

Actually in the terms of Van der Warden theorem Shelah showed that $\mathtt{VW}$ has upper bound via a primitive recursion which is of a kind

$$2^{2^{r^{2^{2^{k+q}}}}}.$$

Proof of Hales Jewett theorem.

Definition 12. *Parameter spaces are for Σ finite alphabet and its n-dimensional parameter Σ^n of functions $[r] \to \Sigma$ space.*

So the d-dimensional subspace is a subset of Σ^n described by a word in $\Sigma \cup \{\lambda_1, \lambda_2, \ldots, \lambda_d\}$ where every λ_i appears at least once and is a set $S = \{w(u) | u \in \Sigma^d\}$.

Example. $\Sigma = \{a, b\}$ let $W = a\lambda_1\lambda_2 b\lambda_1$, which by (aa) is replaced to $aaaba$.

Now we will proceed with improvement of the Hales Jewett theorem.

Theorem 22. *Hales Jewett improved $\forall$ finite Σ $\forall r > 0, d > 0$ there exists $N = \mathtt{HJ}(|\Sigma, r, d)$ such that if Σ^N is r-colored then there exist monochromatic d-dimensional subspace.*

Observation. *$\forall s, d :$ $\mathtt{HJ}(s, 1, d) = d$ and also $\forall r, d :$ $\mathtt{HJ}(1, r, d) = d$. Lets see that $\mathtt{HJ}(2, 2, 1) = 2$, or generally that $\mathtt{HJ}(2, r, 1) = r$ which can be viewed by pigeonhole principle, so that there is k zeroes followed by $r - k$ ones, then every pair makes a line. In Σ^r are $r + 1$ step functions.*

Lemma 4. $\forall r > 0$ $\mathtt{HJ}(s, r, 1) \Rightarrow \forall r, d > 0$ $\mathtt{HJ}(s, r, d)$

Lemma 5. $\forall r, d > 0$ $\mathtt{HJ}(s, r, d) \Rightarrow \forall r > 0$ $\mathtt{HJ}(s + 1, r, 1)$

Proof of theorem 22. Simply put both lemmas 4 and 5 together and we are able to recreate HJ for all s, r, d. $\square$

Proof of lemma 4. We will show that if $\forall r, d > 0$ $\mathtt{HJ}(s, r, 1)$ is finite then $\forall r > 0$ $\mathtt{HJ}(s+, r, 2)$ is finite. Fix s, r and assume $\mathtt{HJ}(s, r', 1)$ is finite. Put $n_1 = \mathtt{HJ}(s, r, 1)$ and $n_2 = \mathtt{HJ}(s, r^{s^{n-1}}, 1)$ and then $\mathtt{HJ}(s, r, 2) \leq n_2 + n_1$. We may see this as that we have n_2 initial segment and n_1 suffix.

Let $\Sigma^{n_1+n_2}$ be r-colored, then define $r \cdot s^{n_1}$ coloring of Σ^{n_2}. By the choice od n_2 then Σ^{n_2} contains a monochromatic line. $\chi : \Sigma^{n_2+n_1} \to r$ and χ' is a function from Σ^{n_2} to the set of all functions $\Sigma^{n_1} \to r$.

Color of this line is a r-coloring of Σ^{n_1}. By the choice of n_1 it contains a monochromatic line. For higher d replace $n_1 = \mathtt{HJ}(s, r^{s^{n_1}}, d)$. $\square$

Proof of lemma 5. Define S_a as a set of specials words. Suppose we know it for Σ and we want to add new letter $a \notin \Sigma$ and find $\mathtt{HJ}(|\Sigma \cup \{a\}|, r, 1)$. Suppose for $S_d \subseteq \Sigma^d$ we have k-times a and then the rest, that is some word $u \in \Sigma^{d-k}$.

Claim 23. *$\forall r$ if S_r is r-colored in a way that color depends only on length of the initial segment (the number of a's) then S_r contains monochromatic line.*

The claim is basically for the step function as was seen earlier.

Claim 24. *$\forall r, d, l$ $\exists N$ such that if $(\Sigma \cup \{a\})^N$ is r-colored then there exists d-dimensional subspace such that color of words with at most l number of a's followed by $u \in \Sigma^{d-l}$ depends only on number of a's.*

Proof. For $l = 0$ use $N = \mathtt{HJ}(|\Sigma|, r, d)$. Let $l = 1$ then set $n_1 = \mathtt{HJ}(|\Sigma|, r, d - 1)$ and $n_2 = \mathtt{HJ}(|\Sigma|, r, n_1 + 1)$. Then $N = n_2$. $\square$

$\square$

Van der Warden from Hales Jewett. We may see a simple example 3.1 for which we would create a progression of numbers $(11, 12, 13, 21, 22, 23, 31, 32, 33)$ for which the progression are in a bijection to the example above. This is somehow the principle of doing it in general. We encode the combinatorial lines into a progressions.