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Discrete Cocompact Subgroups of the Four-Dimensional Nilpotent Connected Lie Group and Their Group C^*-Algebras[1]

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Let G_4 be the unique, connected, simply connected, four-dimensional, nilpotent Lie group. In this paper, the discrete cocompact subgroups H of G_4 are classified and shown to be in 1–1 correspondence with triples $(p_1, p_2, p_3) \in \mathbb{Z}^3$ that satisfy $p_2, p_3 > 0$ and a certain restriction on p_1. The K-groups of the group C^*-algebra $C^*(\mathrm{H})$ are computed and shown to involve all three parameters. Furthermore, for each such subgroup H, the set of faithful simple quotients (i.e., those generated by a faithful representation of H) of the group C^*-algebra $C^*(\mathrm{H})$ is shown to be independent of p_1 and p_3 and to be in 1–1 correspondence with the irrational θ's in $[0, 1/2)$. The other infinite-dimensional simple quotients of $C^*(\mathrm{H})$ (those generated by a representation of H that is not faithful) are shown to be isomorphic to matrix algebras over irrational rotation algebras. © 2001 Academic Press

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1. INTRODUCTION

In three dimensions there is a unique (up to isomorphism) connected, simply connected, nilpotent Lie group, which we call G_3 (following Nielsen [8]); G_3 ($= \mathbb{R}^3$ as a set) is the Heisenberg group with multiplication

$$(k, m, n)(k', m', n') = (k + k' + nm', m + m', n + n').$$

It is not hard to show that every cocompact discrete subgroup H of G_3 is isomorphic to a group $\mathrm{H}_3(p)$ ($= \mathbb{Z}^3$ as a set) with multiplication

$$(k, m, n)(k', m', n') = (k + k' + pnm', m + m', n + n'),$$

for some $p \in \mathbb{N}$, and that different p's give non-isomorphic groups. The K-groups of the group C^*-algebras $C^*(\mathrm{H}_3(p))$ were calculated in [5, Remark 5], and so different p's were shown to give non-isomorphic $C^*(\mathrm{H}_3(p))$'s, even though the set of infinite-dimensional simple quotients is the same for each $C^*(\mathrm{H}_3(p))$, just the set of irrational rotation algebras $\{A_\theta \mid \theta \in (0, 1/2),\ \ \theta \notin \mathbb{Q}\}$ which is usually thought of as the set of infinite-dimensional simple quotients of the group C^*-algebra of the integer Heisenberg group $\mathrm{H}_3 = \mathrm{H}_3(1)$, the lattice subgroup of G_3.

The analogous (unique) four-dimensional Lie group G_4 is studied in the present paper. G_4, as a set, is equal to $\mathbb{R}^4$ and has multiplication given by

$$\begin{aligned}(w, x, y, z)(w', x', y', z') = (&w + w' + zx' + y'z(z - 1)/2,\\ &x + x' + zy', y + y', z + z');\end{aligned}$$

see [8] and also [5], where we looked at the natural discrete cocompact lattice subgroup $\mathrm{H}_4 = \mathbb{Z}^4 \subset \mathrm{G}_4$ and determined all the simple infinite-dimensional quotients of its associated group C^*-algebra $C^*(\mathrm{H}_4)$. In this paper we obtain all discrete cocompact subgroups H of G_4 (up to isomorphism) and determine the K-group structure of $C^*(\mathrm{H})$, as well as classify all of the simple infinite-dimensional quotients of $C^*(\mathrm{H})$. We will show that the non-isomorphic groups H are differentiated by three integral parameters (p_1, p_2, p_3) that satisfy $p_2, p_3 > 0$ and $0 \leq p_1 \leq \gcd\{p_2, p_3\}/2$ (see Theorem 1).

These integer parameters determine the K-groups $K_j(C^*(\mathrm{H}))$ of the group C^*-algebra, and in effecting this, use is made of the Connes Chern character on K-theory (Theorem 7). The simple quotients of $C^*(\mathrm{H})$ that arise from a faithful representation are also determined and classified (Theorem 3), and (as in [5]) the remaining infinite-dimensional quotients are identified as matrix algebras over an irrational rotation algebra (Theorem 5).

We take this opportunity to thank the referee for his careful reading of the paper and for his helpful suggestions; see, in particular, Remark 4 at the end of the paper.

2. DETERMINATION OF THE DISCRETE COCOMPACT SUBGROUPS

1. THEOREM. *Every discrete cocompact subgroup* H *of* G_4 *has the following form: there are integers* p_1, p_2, *and* p_3 *satisfying* $p_2, p_3 > 0$ *and*

$$0 \leq p_1 \leq \gcd\{p_2, p_3\}/2,$$

yielding $H \cong H_4(p_1, p_2, p_3) = \mathbb{Z}^4$ *with multiplication*

$$\begin{aligned}(j, k, m, n)&(j', k', m', n') \\ &= (j + j' + p_3 nk' + p_1 nm' + p_3 p_2 m'n(n-1)/2, \\ &\quad k + k' + p_2 nm', m + m', n + n').\end{aligned} \tag{m}$$

Furthermore, different choices for the p's give non-isomorphic groups, each of which is, in fact, isomorphic to a subgroup of H_4, *the lattice subgroup of* G_4.

Proof. It is possible that abelian methods, as in Chap. 2 of [7], could be adapted to prove this result. However, the proof given here is complete and more self-contained.

Suppose H is a discrete cocompact subgroup of G_4, and consider elements

$$s_i = (w_i, x_i, y_i, z_i) \in H, \quad 1 \leq i \leq 3.$$

Then

$$s_1 s_2 = (z_1 x_2 - z_2 x_1 + y_2 z_1(z_1 - 1)/2 - y_1 z_2(z_2 - 1)/2, z_1 y_2 - z_2 y_1, 0, 0) \cdot s_2 s_1$$

so

$$\begin{aligned}[s_1, s_2] = (z_1 x_2 - z_2 x_1 &+ y_2 z_1(z_1 - 1)/2 \\ &- y_1 z_2(z_2 - 1)/2, z_1 y_2 - z_2 y_1, 0, 0) \in H,\end{aligned}$$

hence $[s_3, [s_1, s_2]] = (z_3(z_1 y_2 - z_2 y_1), 0, 0, 0) \in H$. The discreteness and cocompactness now tell us that there is

(1) a minimal non-zero $|z_1 y_2 - z_2 y_1|$ for all $s_1, s_2 \in H$, and also

(2) a minimal non-zero $|z_3|$ for all $s_3 \in H$.

Hence there is a $J > 0$ such that

$$\{(w, x, y, z) \in H \mid x = y = z = 0\} = (J\mathbb{Z}, 0, 0, 0); \tag{3}$$

J divides $|z_3(z_1 y_2 - z_2 y_1|$ for all $s_1, s_2, s_3 \in H$.

Specific members s_i, $1 \leq i \leq 4$, will now be identified. Let $s_4 = (w_4, x_4, y_4, z_4)$ have the minimal $z_4 > 0$. Then z_4 must divide z for all

$s = (w, x, y, z) \in \mathrm{H}$; otherwise, for some integer r one has $0 < z + rz_4 < z_4$ and hence ss_4^r has a last coordinate $z + rz_4$. So, for each $s \in \mathrm{H}$, there is a unique $r \in \mathbb{Z}$ giving a unique element ss_4^r in the (abelian) subgroup

$$\mathrm{H}' = \{(w, x, y, z) \in \mathrm{H} \mid z = 0\} \subset \{(w, x, y, z) \in \mathrm{G}_4 \mid z = 0\} \cong \mathbb{R}^3.$$

(Note that H' is a discrete cocompact subgroup of $\mathbb{R}^3$.)

Let $s_3 = (w_3, x_3, y_3, 0) \in \mathrm{H}'$ have minimal $y_3 > 0$. We can now assume that $0 \leq y_4 < y_3$ (by replacing s_4 with $s_3^r s_4$ as appropriate). Next consider

$$\mathrm{H}'' = \{(w, x, y, z) \in \mathrm{H} \mid y = z = 0\};$$

there is an $s_2 = (w_2, x_2, 0, 0) \in \mathrm{H}''$ with minimal $x_2 > 0$ (because if there was a sequence of s_2's with x_2's converging to 0 we could use (3) to modify the sequence and have first coordinates also converging). Again, we may now assume that $0 \leq x_3, x_4 < x_2$. Finally, we set $s_1 = (w_1, 0, 0, 0) = (J, 0, 0, 0)$ and may assume that $0 \leq w_4, w_3, w_2 < w_1$. Thus a bijection $\Phi\colon \mathbb{Z}^4 \to \mathrm{H}$ is given by

$$\Phi\colon (j, k, m, n) \mapsto s_1^j s_2^k s_3^m s_4^n.$$

Note that s_1, s_2, and s_3 commute and that Φ is onto by the minimality choices made above.

We want the product for $\mathbb{Z}^4$ that makes Φ a homomorphism, and use commutators to determine it. We have

$$[s_4, s_3] = (z_4 x_3 + y_3 z_4(z_4 - 1)/2,\ z_4 y_3, 0, 0) = s_1^{p_1} s_2^{p_2}$$

and

$$[s_4, s_2] = (z_4 x_2, 0, 0, 0) = s_1^{p_3}$$

for suitable $p_3, p_2, p_1 \in \mathbb{Z}$ with $p_3, p_2 \neq 0$. Now use the commutators to collect terms and get

$$\begin{aligned}(s_1^j s_2^k s_3^m s_4^n)&(s_1^{j'} s_2^{k'} s_3^{m'} s_4^{n'}) \\ &= (s_1^{j+j'+p_3nk'+p_1nm'+p_3p_2m'n(n-1)/2} s_2^{k+k'+p_2nm'} s_3^{m+m'} s_4^{n+n'});\end{aligned}$$

the exponents give the desired product formula

$$\begin{aligned}(j, k, m, n)&(j', k', m', n') \\ &= (j + j' + p_3nk' + p_1nm' + p_3p_2m'n(n-1)/2, \\ &\quad k + k' + p_2nm', m + m', n + n')\end{aligned} \tag{m}$$

for $\mathrm{H}_4(p_1, p_2, p_3)$ ($= \mathbb{Z}^4$ as a set).

To get the restrictions on the p's, note the three isomorphisms

$$(j,k,m,n)\mapsto(-j,k,m,n),\quad \mathrm{H}_4(p_1,p_2,p_3)\to\mathrm{H}_4(-p_1,p_2,-p_3),$$
$$(j,k,m,n)\mapsto(j,-k,m,n),\quad \mathrm{H}_4(p_1,p_2,p_3)\to\mathrm{H}_4(p_1,-p_2,-p_3),$$

and

$$(j,k,m,n)\mapsto(j,k,-m,n),\quad \mathrm{H}_4(p_1,p_2,p_3)\to\mathrm{H}_4(-p_1,-p_2,p_3),$$

which show that we may assume $p_2, p_3 > 0$. To get $0 \le p_1 < \gcd\{p_2, p_3\}$, use the isomorphism

$$(j,k,m,n)\mapsto(j+qk,k-rm,m,n),$$
$$\mathrm{H}_4(p_1,p_2,p_3)\to\mathrm{H}_4(p_1+qp_2+rp_3,p_2,p_3), \tag{†}$$

so that appropriate choices for r and q yield the conclusion. Then, if we have arrived at a p_1' with $0 \le p_1' < \gcd\{p_2, p_3\}$ and this p_1' also satisfies

$$0 \le p_1' \le \gcd\{p_2,p_3\}/2 \tag{$*$}$$

we are done. If p_1' does not satisfy $(*)$, use one more isomorphism

$$(j,k,m,n)\mapsto(j,-k,m,-n),$$
$$\mathrm{H}_4(p_1',p_2,p_3)\to\mathrm{H}_4(-p_1'+p_2p_3,p_2,p_3); \tag{†′}$$

one can then apply (†) again and get $p_1'' = \gcd\{p_2, p_3\} - p_1'$, which does satisfy $(*)$.

Now, note that $\mathbb{Z}^4/Z$ ($Z = (\mathbb{Z}, 0, 0, 0)$ is the center of $\mathbb{Z}^4$) is just $\mathrm{H}_3(p_2)$; also, the second commutator subgroup is $p_3p_2\mathbb{Z} \subset Z$ and $Z/(p_3p_2\mathbb{Z}) = \mathbb{Z}_{p_3p_2}$ (the cyclic group of order p_3p_2). So, different (positive) p_2's or p_3's give non-isomorphic groups. It must still be shown that the manipulations at (†) and (†′) above give the only way of changing just p_1 in $\mathrm{H}_4(p_1, p_2, p_3)$.

Suppose that $\varphi\colon \mathrm{H}_4(p_1, p_2, p_3) \to \mathrm{H}_4(p_1', p_2, p_3)$ is an isomorphism. Then

$$\varphi\colon Z = K_1 = (\mathbb{Z},0,0,0)\to(\mathbb{Z},0,0,0) = K_1' = Z',$$
$$K_2 = (\mathbb{Z},\mathbb{Z},0,0)\to(\mathbb{Z},\mathbb{Z},0,0) = K_2',$$
$$K_3 = (\mathbb{Z},\mathbb{Z},\mathbb{Z},0)\to(\mathbb{Z},\mathbb{Z},\mathbb{Z},0) = K_3',$$

since the Z's are the centers, the K_2's contain the commutator subgroups, and the K_3's are the centralizers of the commutator subgroups. (For example, one verifies readily that the commutator subgroup of $\mathrm{H}_4(p_1, p_2, p_3)$ is

$$\mathbb{Z}(p_3,0,0,0)+\mathbb{Z}(p_1,p_2,0,0)$$

and hence concludes that $\varphi(0, p_2, 0, 0)$ and $\varphi(0, 1, 0, 0)$ are in K_2'.) So then we must have

$$\varphi(0,0,0,1) = (*,*,*,\epsilon) = s \qquad \text{with } \epsilon = \pm 1,$$
$$\varphi(0,0,1,0) = (*,*,\delta,0) = t \qquad \text{with } \delta = \pm 1,$$
$$\varphi(p_1, p_2, 0, 0) = (*, p_2\epsilon\delta, 0, 0),$$

and

$$\varphi(p_2p_3, 0,0,0) = (p_2p_3\epsilon^2\delta, 0,0,0) = p_2p_3v$$
$$\text{with } v = (\epsilon^2\delta, 0,0,0) = \varphi(1,0,0,0)$$

(using commutators). Using the last two equations, we get

$$\varphi(0,1,0,0) = (q, \epsilon\delta, 0, 0) = u$$

(say). Also, set $t = (*, r, \delta, 0)$.

Now, in $\mathrm{H}_4(p_1, p_2, p_3)$ the multiplication formula is

$$\begin{aligned}(j,k,m,n)&(j',k',m',n')\\ &= (j + j' + p_3nk' + p_1nm' + p_3p_2m'n(n-1)/2,\\ &\quad k + k' + p_2nm', m+m', n+n'),\end{aligned}$$

and in $\mathrm{H}_4(p_1', p_2, p_3)$ it is

$$\begin{aligned}(j,k,m,n)&\cdot(j',k',m',n')\\ &= (j + j' + p_3nk' + p_1'nm' + p_3p_2m'n(n-1)/2,\\ &\quad k + k' + p_2nm', m+m', n+n'),\end{aligned}$$

so

$$\begin{aligned}\varphi(j,k,m,n) &= \varphi((j,0,0,0)(0,k,0,0)(0,0,m,0)(0,0,0,n))\\ &= (jv)\cdot(ku)\cdot(mt)\cdot s^n = jv + ku + mt + s^n \in \mathrm{H}'.\end{aligned}$$

Note that $s^n \neq ns$, but $s^n = (*,*,*,n\epsilon)$, and that the (ku) term puts a kq in the first entry of $\varphi(j,k,m,n)$ and so also puts $(k + k' + p_2nm')q$ in the first entry of $\varphi((j,k,m,n)(j',k',m',n'))$. Thus, equating the coefficients of the nm' terms in the first entry of

$$\varphi((j,k,m,n)(j',k',m',n')) \qquad \text{and} \qquad \varphi(j,k,m,n)\cdot\varphi(j',k',m',n')$$

gives

$$p_1\epsilon^2\delta + qp_2 - p_3p_2\epsilon^2\delta/2 = rp_3\epsilon + p_1'\epsilon\delta - p_3p_2\epsilon\delta/2,$$

which shows that the manipulations at (†) and (†′) above give the only way of changing just p_1 in $\mathrm{H}_4(p_1, p_2, p_3)$.

An isomorphism of $\mathrm{H}_4(p_1, p_2, p_3)$ into $\mathrm{H}_4 \subset \mathrm{G}_4$ is given by

$$(j, k, m, n) \mapsto (j, p_3 k + p_1 m, p_2 p_3 m, n).$$

This completes the proof. ∎

Remark. The previous paragraph gives an isomorphism of $\mathrm{H}_4(p_1, p_2, p_3)$ into H_4; conversely, there is always an isomorphism of H_4 into $\mathrm{H}_4(p_1, p_2, p_3)$, namely

$$(j, k, m, n) \mapsto (p_3^2 p_2 j, p_3 p_2 k - p_1 m, p_3 m, n), \quad \mathrm{H}_4 \to \mathrm{H}_4(p_1, p_2, p_3).$$

So, here we have an infinite class of non-isomorphic groups, each of which is isomorphic to a subgroup of any other one. (Of course, this was also the case for the three-dimensional groups $\mathrm{H}_3(p)$, $p \in \mathbb{N}$.)

3. INFINITE DIMENSIONAL SIMPLE QUOTIENTS OF $C^*(\mathrm{H}_4(p_1, p_2, p_3))$ AND THEIR CLASSIFICATION

Let $\lambda = e^{2\pi i\theta}$ for an irrational θ, and consider the flow $\mathscr{F} = (\mathbb{Z}, \mathbb{T}^2)$ generated by the homeomorphism

$$\psi\colon (w, v) \mapsto (\lambda^{p_3} w, \lambda^{p_1} w^{p_2} v),$$
$$n\colon (w, v) \mapsto \psi^n(w, v) = (\lambda^{p_3 n} w, \lambda^{p_1 n + p_2 p_3 n(n-1)/2} w^{p_2 n} v).$$

Let v and w denote (as well as members of $\mathbb{T}$) the functions in $\mathscr{C}(\mathbb{T}^2)$ defined by

$$(w, v) \mapsto v \text{ and } w,$$

respectively. Define unitaries U, V, and W on $L^2(\mathbb{T}^2)$ by

$$U\colon f \mapsto f \circ \psi, \qquad V\colon f \mapsto vf \text{ and } W\colon f \mapsto wf.$$

These unitaries satisfy

$$UV = \lambda^{p_1} W^{p_2} VU, \qquad UW = \lambda^{p_3} WU, \qquad VW = WV \tag{CR}$$

equations which ensure that

$$\pi\colon (j, k, m, n) \mapsto \lambda^j W^k V^m U^n$$

is a representation of $\mathrm{H}_4(p_1, p_2, p_3)$. Denote by $A_\theta^4(p_1, p_2, p_3)$ the C^*-subalgebra of $B(L^2(\mathbb{T}^2))$ generated by π, i.e., by U, V and W. Another construction in this situation where there is a homeomorphism of a compact space X generating an action of $\mathbb{Z}$ on X and so also on $\mathscr{C}(X)$ leads to the C^*-crossed product algebra $C^*(\mathscr{C}(X), \mathbb{Z})$, which is the enveloping C^*-algebra of the ℓ_1-algebra $\ell_1(\mathbb{Z}, \mathscr{C}(\mathbb{T}^2))$; see [12] or [6] for more details.

Since $A^4_\theta(p_1, p_2, p_3)$ is generated by a representation of $\mathrm{H}_4(p_1, p_2, p_3)$, it is a quotient of the group C^*-algebra $C^*(\mathrm{H}_4(p_1, p_2, p_3))$. There are a number of methods to prove that quotients of group C^*-algebras are simple; see pp. 318–319 in [6]. Of these, we will use the minimal flow method involving Corollary 5.16 in [3]. The minimal flow situation is appealing because of its connection with geometry and topology. Following the same arguments as in [6] one obtains the following result.

2. THEOREM. *Let $\lambda = e^{2\pi i\theta}$ for an irrational θ.*

(a) *There is a unique* (*up to isomorphism*) *simple C^*-algebra $A^4_\theta(p_1, p_2, p_3)$ generated by unitaries U, V, and W satisfying*

$$UV = \lambda^{p_1} W^{p_2} VU, \qquad UW = \lambda^{p_3} WU, \qquad VW = WV. \tag{CR}$$

Let the flow $\mathscr{F} = (\mathbb{Z}, \mathscr{C}(\mathbb{T}^2))$ be as above; *then*

$$A^4_\theta(p_1, p_2, p_3) \cong C^*(\mathscr{C}(\mathbb{T}^2), \mathbb{Z}).$$

(b) *Let π' be a representation of $\mathrm{H}'_4 = \mathrm{H}_4(p_1, p_2, p_3)$ such that $\pi = \pi'$* (*as scalars*) *on the center* $(\mathbb{Z}, 0, 0, 0)$ *of* H'_4, *and let A be the C^*-algebra generated by π'. Then $A \cong A^4_\theta(p_1, p_2, p_3) = A'^4_\theta$* (*say*) *via a unique isomorphism ω such that the following diagram commutes*:

$$\begin{array}{ccc} \mathrm{H}'_4 & \xrightarrow{\pi} & A'^4_\theta \\ \pi' \searrow & & \swarrow \omega \\ & A & \end{array}$$

(c) *The C^*-algebra A'^4_θ has a unique tracial state.*

Next, let us turn our attention to the classification of the algebras $A^4_\theta(p_1, p_2, p_3)$. Let $\lambda = e^{2\pi i\theta}$ for an irrational θ. The operator equations (CR) for $A^4_\theta(p_1, p_2, p_3)$ can be simplified by changing two of the variables, i.e., by substituting $W_0 = e^{2\pi i\theta p_1/p_2}W$ and putting $\lambda_0 = \lambda^{p_3}$. The equations (CR) become

$$UV = W_0^{p_2} VU,\ \ UW_0 = \lambda_0 W_0 U,\ \ VW_0 = W_0 V, \tag{CR$_0$}$$

which are the equations for $A^4_{p_3\theta}(0, p_2, 1)$, so

$$A^4_\theta(p_1, p_2, p_3) \cong A^4_{p_3\theta}(0, p_2, 1).$$

It is not hard to see from the Pimsner–Voiculescu six-term exact sequence [10] that one has

$$K_0(A^4_\theta(p_1, p_2, p_3)) = \mathbb{Z}^3, \qquad K_1(A^4_\theta(p_1, p_2, p_3)) = \mathbb{Z}^3 \oplus \mathbb{Z}_{p_2}$$

(where $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$). Further, by Pimsner's result on the range of the trace [9], one easily shows that $\tau_* K_0(A^4_\theta(0, p_2, 1)) = \mathbb{Z} + \mathbb{Z}\theta$, where τ is the unique tracial state of $A^4_\theta(0, p_2, 1)$. Therefore, $\tau_* K_0(A^4_\theta(p_1, p_2, p_3)) = \mathbb{Z} + \mathbb{Z}p_3\theta$. From these one immediately obtains the isomorphism classification for the algebras $A^4_\theta(p_1, p_2, p_3)$.

3. THEOREM. *A C^*-algebra A is isomorphic to a faithful simple quotient of the group C^*-algebra of a discrete cocompact subgroup of* G_4 *if and only if A is isomorphic to an $A^4_\theta(0, p_2, 1)$ (generated by $(w, v) \mapsto (\lambda w, w^{p_2} v)$ with $\lambda = e^{2\pi i\theta}$) for some $p_2 \in \mathbb{N}$ and irrational $\theta \in (0, 1/2)$. Also, for such θ's and p_2's, $A^4_\theta(0, p_2, 1) \cong A^4_{\theta'}(0, p_2', 1)$ iff $\theta = \theta'$ and $p_2 = p_2'$. More generally, $A^4_\theta(p_1, p_2, p_3) \cong A^4_{\theta'}(p_1', p_2', p_3')$ iff $p_2' = p_2$ and $p_3'\theta' = n \pm p_3\theta$ for some integer n.*

4. OTHER SIMPLE QUOTIENTS OF $C^*(\mathrm{H}_4(p_1, p_2, p_3))$

Now assume that λ is a primitive q'th root of unity and that U, V, and W are generating unitaries for a simple quotient A of $C^*(\mathrm{H}_4(p_1, p_2, p_3))$, i.e., they satisfy (CR). We may assume that A is irreducibly represented. Let q be the smallest positive integer with $\lambda^{p_3 q} = 1$, i.e., $q = q'/(\gcd\{q', p_3\})$. Then W^q commutes with U and V and so by irreducibility equals μ', a multiple of the identity. Put $W = \mu W_1$ for $\mu^q = \mu'$, so that $W_1^q = 1$, and substitute $W = \mu W_1$ in (CR) to get

$$UV = \lambda^{p_1}\mu^{p_2} W_1^{p_2} VU,\ UW_1 = \lambda^{p_3} W_1 U,\ VW_1 = W_1 V,\ W_1^q = 1. \quad (\mathrm{CR}_1)$$

It is possible here to attempt to reduce to the case $p_1 = 0$ and $p_3 = 1$, as above, for the faithful simple quotients, but the process introduces some other complications (e.g., the substitution $W_0 = e^{2\pi i\theta p_1/p_2} W_1$ could give a unipotent W_0 not satisfying $W_0^q = 1$), so this approach seems not worth pursuing, in view of the conclusion of Theorem 5 below: all these "other" simple quotients are just matrix algebras over irrational rotation algebras.

1. If μ is not a root of unity, let $\mathbb{Z}_q \subset \mathbb{T}$ denote the cyclic group of order q. We can modify the flow $\mathcal{F} = (\mathbb{Z}, \mathbb{T}^2)$ used above to generate $A^4_\theta(p_1, p_2, p_3)$ and get $\mathcal{F}' = (\mathbb{Z}, \mathbb{Z}_q \times \mathbb{T})$ generated by the homeomorphism

$$\varphi\colon (w, v) \mapsto (\lambda^{p_3} w, \lambda^{p_1}\mu^{p_2} w^{p_2} v) \text{ of } \mathbb{Z}_q \times \mathbb{T},$$

$$n\colon (w, v) \mapsto \varphi^n(w, v) = (\lambda^{p_3 n} w, \lambda^{p_1 n + p_3 p_2 n(n-1)/2} (\mu^{p_2} w^{p_2})^n v).$$

$\mathcal{F}'$ is minimal (by [4, 3.3.12] or an argument as on p. 331 of [6]) and effective, so the C^*-crossed product $C^*(\mathcal{C}(\mathbb{Z}_q \times \mathbb{T}), \mathbb{Z})$ is simple and isomorphic to A, with V, W, and U corresponding to v_0, w_0, and δ_1, respectively, in

$$\ell_1(\mathbb{Z}, \mathcal{C}(\mathbb{Z}_q \times \mathbb{T})) \subset C^*(\mathcal{C}(\mathbb{Z}_q \times \mathbb{T}), \mathbb{Z}).$$

(Here, for $a \in A$, $a_0 \in \ell_1(\mathbb{Z}, \mathscr{C}(\mathbb{Z}_q \times \mathbb{T}))$ denotes the delta function equal to a at 0, and to 0 elsewhere; δ_1 is the delta function equal to the identity of A at 1 and equal to 0 elsewhere.)

2. If μ is also a root of unity, then (CR_1) (along with irreducibility) shows that U and V, as well as W', are unipotent, so A is finite-dimensional.

The preceeding comments are summarized in the next theorem.

4. THEOREM. *A C^*-algebra A is isomorphic to a simple infinite-dimensional quotient of $C^*(\mathrm{H}_4(p_1, p_2, p_3))$ if and only if A is isomorphic to $A^4_\theta(0, p_2, 1)$ for an irrational $\theta \in (0, 1/2)$ or to a $C^*(\mathscr{C}(\mathbb{Z}_q \times \mathbb{T}), \mathbb{Z})$, as in Statement* 1 *above*.

Experience indicates that simple quotients involving unipotent operators can often be represented as matrix algebras over algebras of lower dimension. (For example, see [5] and [6].)

5. THEOREM. *Let λ be a primitive q'th root of unity and suppose that*

$$\mu, \quad q \text{ and } C^*(\mathscr{C}(\mathbb{Z}_q \times \mathbb{T}), \mathbb{Z})$$

are as in Statement 1 *above. Then $C^*(\mathscr{C}(\mathbb{Z}_q \times \mathbb{T}), \mathbb{Z})$ is isomorphic to $M_q(A_\gamma)$ (a matrix algebra over an irrational rotation algebra) for suitable γ.*

Proof. Let unitaries U_0 and V_0 generate A_γ, i.e.,

$$U_0V_0 = e^{2\pi i\gamma} V_0U_0, \qquad \text{with} \qquad e^{2\pi i\gamma} = \lambda^{p_1q+p_2p_3q(q-1)/2}\mu^{p_2q}.$$

Then define unitaries in $M_q(A_\gamma)$ as follows. Let U' have U_0 in the upper right-hand corner and 1's on the subdiagonal, i.e.,

$$U' = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & U_0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

let V' have

$$V_0,\ (\overline{\mu^{p_2}\lambda^{p_1}})\lambda^{p_2p_3}V_0,\ (\overline{\mu^{p_2}\lambda^{p_1}})^2\lambda^{3p_2p_3}V_0,$$
$$(\overline{\mu^{p_2}\lambda^{p_1}})^3\lambda^{6p_2p_3}V_0, \ldots, (\overline{\mu^{p_2}\lambda^{p_1}})^{q-1}\lambda^{p_2p_3q(q-1)/2}V_0$$

on the diagonal, and let W' have

$$1,\ \overline{\lambda}^{p_3},\ \overline{\lambda}^{2p_3},\ \overline{\lambda}^{3p_3}, \ldots, \overline{\lambda}^{(q-1)p_3}$$

on the diagonal. Then U', V', and W' satisfy (CR_1) and generate A_γ. ∎

5. THE K-GROUPS OF THE GROUP C^*-ALGEBRA OF $\mathrm{H}_4(p_1, p_2, p_3)$

The following fairly detailed analysis of some quotients of $\mathbb{Z}^2$ is required for the proof of the main result in this section.

6. LEMMA. *For any integers k, ℓ, m, where k, m are non-zero, one has*

$$\frac{\mathbb{Z}^2}{\mathbb{Z}(k,0)+\mathbb{Z}(\ell,m)} \cong \mathbb{Z}_\delta \oplus \mathbb{Z}_{mk/\delta}$$

where $\delta = \gcd\{k, \ell, m\}$.

Proof. Let $d = \gcd\{k, \ell\}$, $r = mk/d$, and $G = \mathbb{Z}(k,0)+\mathbb{Z}(\ell,m)$. Then the map

$$\psi\colon \mathbb{Z}_k \oplus \mathbb{Z}_r \to \mathbb{Z}^2/G, \quad ([a]_k,\ [b]_r) \mapsto a[(1,0)] + b[(0,1)] = [(a,b)]$$

is a well-defined surjection. (Here, $[(a,b)], [a]_k$, and $[b]_r$ are classes in $\mathbb{Z}^2/G$, $\mathbb{Z}_k$, and $\mathbb{Z}_r$, respectively.) In fact, it is clear that $[(1,0)]$ has order k. To see that $[(0,1)]$ has order r in $\mathbb{Z}^2/G$, note that $r[(0,1)] = 0$ and that if $n[(0,1)] = 0$ in $\mathbb{Z}^2/G$, then $(0,n) = a(k,0) + b(\ell,m)$ for some integers a, b. Hence $n = bm = (bd/k)r$ and bd/k is an integer since k/d divides b, which follows from $a(k/d) = -b(\ell/d)$ since k/d and ℓ/d are relatively prime.

Now the kernel of ψ is easily seen to be the cyclic subgroup $\langle \xi \rangle$ generated by the element $\xi := ([\ell]_k, [m]_r)$, which has order k/d in $\mathbb{Z}_k \oplus \mathbb{Z}_r$ (since each of $[\ell]_k, [m]_r$ has order k/d in $\mathbb{Z}_k$ and $\mathbb{Z}_r$, respectively). Thus $\mathbb{Z}^2/G$ is isomorphic to $(\mathbb{Z}_k \oplus \mathbb{Z}_r)/\langle \xi \rangle$.

After observing that $[\ell/d]_k$ generates $\mathbb{Z}_k$ and $[1]_r$ generates $\mathbb{Z}_r$, so that $([\ell/d]_k, 0)$, $(0, [1]_r)$ is a basis for $\mathbb{Z}_k \oplus \mathbb{Z}_r$, the next step is to make a basis change for $\mathbb{Z}_k \oplus \mathbb{Z}_r$. With

$$\delta = \gcd\{k, \ell, m\} = \gcd\{d, m\},$$

this basis change is achieved with a matrix

$$\begin{pmatrix} d/\delta & m/\delta \\ b & a \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

Such a matrix exists because d/δ and m/δ are relatively prime. So the members for the new basis are

$$\xi' = ((d/\delta)[\ell/d]_k, (m/\delta)[1]_r) = ([\ell/\delta]_k, [m/\delta]_r) \text{ and } \eta' = (b[\ell/d]_k, a[1]_r).$$

Of course, $\mathbb{Z}_k \oplus \mathbb{Z}_r = \langle \xi' \rangle \oplus \langle \eta' \rangle$. Since $\xi = \delta\xi'$, the order of ξ' is $\delta k/d$, and the order χ of η' must satisfy $kr = (\delta k/d)\chi$, hence $\chi = dr/\delta = mk/\delta$. Finally, we have

$$\mathbb{Z}^2/G \cong \frac{\mathbb{Z}_k \oplus \mathbb{Z}_r}{\langle \xi \rangle} = \frac{\langle \xi' \rangle \oplus \langle \eta' \rangle}{\langle \delta\xi' \rangle} \cong \frac{\langle \xi' \rangle}{\langle \delta\xi' \rangle} \oplus \langle \eta' \rangle \cong \mathbb{Z}_\delta \oplus \mathbb{Z}_{mk/\delta},$$

which ends the proof. ∎

7. THEOREM. *For integers* p_1, p_2, p_3 *as in Theorem* 1, *and with* $\mathrm{H} = \mathrm{H}_4(p_1, p_2, p_3)$, *the K-groups of the group C^*-algebra* $C^*(\mathrm{H})$ *are*

$$K_0(C^*(\mathrm{H})) \cong K_1(C^*(\mathrm{H})) \cong \mathbb{Z}^4 \oplus \mathbb{Z}_\delta \oplus \mathbb{Z}_{p_2 p_3/\delta},$$

where $\delta = \gcd\{p_1, p_2, p_3\}$.

Proof. The group $\mathrm{H} = \mathrm{H}_4(p_1, p_2, p_3)$ can be viewed as the universal group generated by elements e_1, e_2, e_3, e_4 satisfying the relations

$$e_4 e_1 = e_1 e_4, \qquad e_4 e_2 = e_1^{p_3} e_2 e_4, \qquad e_4 e_3 = e_1^{p_1} e_2^{p_2} e_3 e_4,$$

with $\{e_1, e_2, e_3\}$ pairwise commuting. The group H is a semidirect product $\mathbb{Z}^3 \times \mathbb{Z}$, and accordingly its group C^*-algebra $C^*(\mathrm{H})$ is isomorphic to the C^*-crossed product $C^*(C(\mathbb{T}^3), \mathbb{Z})$, where the action of $\mathbb{Z}$ on $C(\mathbb{T}^3)$ is generated by the automorphism σ of $\mathscr{C}(\mathbb{T}^3)$, which acts on the free commuting unitary generators X, Y, Z (corresponding to e_1, e_2, e_3, respectively) of $\mathscr{C}(\mathbb{T}^3)$ by

$$\sigma(X) = X, \qquad \sigma(Y) = X^{p_3} Y, \qquad \sigma(Z) = X^{p_1} Y^{p_2} Z.$$

In order to apply the Pimsner–Voiculsecu exact sequence [10] corresponding to the above C^*-crossed product,

$$\begin{tikzcd}
K_0(C(\mathbb{T}^3)) \arrow[r, "id_* - \sigma_*"] & K_0(C(\mathbb{T}^3)) \arrow[r, "i_*"] & K_0(C^*(\mathrm{H})) \arrow[d, "\delta_0"] \\
K_1(C^*(\mathrm{H})) \arrow[u, "\delta_1"] & K_1(C(\mathbb{T}^3)) \arrow[l, "i_*"'] & K_1(C(\mathbb{T}^3)), \arrow[l, "id_* - \sigma_*"']
\end{tikzcd}$$

one needs to know the action of σ_* on the K-groups of $C(\mathbb{T}^3)$. To do this, recall that a basis for $K_1(C(\mathbb{T}^3)) = \mathbb{Z}^4$ is $\{[X], [Y], [Z], [\xi]\}$ where $\xi = I_2 + (Z-1) \otimes P_{XY} \in M_2(C(\mathbb{T}^3))$ and $P_{X,Y}$ is a Bott projection in the X, Y variables. Note that since $\sigma(P_{X,Y}) = P_{X, X^{p_3} Y}$ is unitarily equivalent to $P_{X,Y}$ (since they have the same K_0 class and $C(\mathbb{T}^2)$ has cancellation), one gets

$$\begin{aligned}
\sigma_*[\xi] &= [I_2 + (\sigma(Z) - 1) \otimes P_{X, X^{p_3} Y}] \\
&= [I_2 + (\sigma(Z) - 1) \otimes P_{X,Y}] \\
&= [I_2 + (X^{p_1} Y^{p_2} Z - 1) \otimes P_{X,Y}] \\
&= [I_2 + (X^{p_1} - 1) \otimes P_{X,Y}] \\
&\qquad + [I_2 + (Y^{p_2} - 1) \otimes P_{X,Y}] + [I_2 + (Z - 1) \otimes P_{X,Y}] \\
&= p_1[X] + p_2[Y] + [\xi].
\end{aligned}$$

Clearly, one has

$$(\sigma_* - id_*)[X] = 0, \qquad (\sigma_* - id_*)[Y] = p_3[X],$$
$$(\sigma_* - id_*)[Z] = p_1[X] + p_2[Y],$$
$$(\sigma_* - id_*)[\xi] = p_1[X] + p_2[Y],$$

hence

$$\ker(\sigma_* - id_*) \text{ on } K_1(C(\mathbb{T}^3)) = \mathbb{Z}[X] + \mathbb{Z}([\xi] - [Z]), \tag{4}$$
$$\text{Image}(\sigma_* - id_*) \text{ on } K_1(C(\mathbb{T}^3)) = \mathbb{Z}p_3[X] + \mathbb{Z}(p_1[X] + p_2[Y]). \tag{5}$$

Let ϕ denote the fundamental cyclic cocycle on $\mathbb{T}^2$,

$$\phi(f^0, f^1, f^2) = \frac{1}{2\pi i}\int\int f^0[f_x^1 f_y^2 - f_y^1 f_x^2]\, dxdy,$$

where $f_x = \partial f/\partial x$. Let B denote the Bott projection in $M_2(C(\mathbb{T}^2))$ written out as

$$B = \begin{bmatrix} 1-f & g \\ \bar{g} & f \end{bmatrix}$$

where $f, g \in C(\mathbb{T}^2)$ are smooth functions satisfying

$$(\phi\#\ \mathrm{Tr})(B, B, B) = -6\phi(f, g, \bar{g}) = -\frac{6}{2\pi i}\int\int f[g_x\bar{g}_y - g_y\bar{g}_x]\, dxdy = 1,$$

where $\#$ is the cup product. (This is the Connes pairing of $[B]$ with $[\phi]$ as in [2]—also called the "twist" of B.) For $1 \le i < j \le 3$, let P_{ij} denote the Bott projection in $M_2(C(\mathbb{T}^3))$ in the variables i, j. More specifically,

$$P_{12}(r, s, t) = B(r, s), \qquad P_{13}(r, s, t) = B(r, t), \qquad P_{23}(r, s, t) = B(s, t).$$

Putting $b_{ij} = [P_{ij}] - [1]$ (the Bott elements), it is not hard to check that $\{[1], b_{12}, b_{13}, b_{23}\}$ is a basis for $K_0(C(\mathbb{T}^3)) \cong \mathbb{Z}^4$.

The (numerical) Connes Chern character ch_0 is the group homomorphism

$$\mathrm{ch}_0\colon \mathrm{K}_0(\mathrm{C}(\mathbb{T}^3)) \to \mathbb{Z}^4$$

given by

$$\mathrm{ch}_0(x) = (\tau(x), \langle x, \phi_{12}\rangle, \langle x, \phi_{13}\rangle, \langle x, \phi_{23}\rangle)$$

where

$$\phi_{ij}(f^0, f^1, f^2) = \frac{1}{2\pi i}\int f^0[f_i^1 f_j^2 - f_j^1 f_i^2]\, dx_1 dx_2 dx_3$$

is a cyclic 2-cocycle on $C(\mathbb{T}^3)$ and $f_k := \partial f/\partial x_k$. (Henceforth, we shall write all integrals over the 3-torus simply with one integral.) From the above one gets

$$\langle [P_{ij}], [\phi_{k\ell}] \rangle = \delta_{i,k}\delta_{j,\ell}$$

which yields

$$\begin{aligned} \mathrm{ch}_0[1] &= (1,0,0,0), \ \ \mathrm{ch}_0(b_{12}) = (0,1,0,0), \\ \mathrm{ch}_0(b_{13}) &= (0,0,1,0), \ \ \mathrm{ch}_0(b_{23}) = (0,0,0,1), \end{aligned}$$

so that ch_0 is injective on $K_0(C(\mathbb{T}^3))$.

We now need a lemma.

8. LEMMA. *The action of σ_* on $K_0(C(\mathbb{T}^3))$ is given by*

$$\begin{aligned} \sigma_*[1] &= [1], \ \ \sigma_*(b_{12}) = b_{12}, \ \ \sigma_*(b_{13}) = p_2 b_{12} + b_{13}, \\ \sigma_*(b_{23}) &= (p_2 p_3 - p_1) b_{12} + p_3 b_{13} + b_{23}. \end{aligned}$$

Proof. For simplicity consider the change of variables $(u, v, w) = (x_1, p_3x_1 + x_2, p_1x_1 + p_2x_2 + x_3)$, and note that from the chain rule

$$\frac{\partial}{\partial x_k} F(u,v,w) = F_1(u,v,w)\frac{\partial u}{\partial x_k} + F_2(u,v,w)\frac{\partial v}{\partial x_k} + F_3(u,v,w)\frac{\partial w}{\partial x_k}$$

one has

$$\begin{aligned} \frac{\partial}{\partial x_1} h(u,v,w) &= h_1(u,v,w) + p_3 h_2(u,v,w) + p_1 h_3(u,v,w), \\ \frac{\partial}{\partial x_2} h(u,v,w) &= h_2(u,v,w) + p_2 h_3(u,v,w), \\ \frac{\partial}{\partial x_3} h(u,v,w) &= h_3(u,v,w). \end{aligned}$$

Writing

$$P_{ij} = \begin{bmatrix} 1-f & g \\ \bar{g} & f \end{bmatrix}$$

where f, g depend only on the i, j coordinates ($i < j$), so that

$$\sigma(P_{ij}) = \begin{bmatrix} 1 - f(u,v,w) & g(u,v,w) \\ \bar{g}(u,v,w) & f(u,v,w) \end{bmatrix},$$

one has

$$\begin{aligned}\langle[\sigma(P_{ij})],[\phi_{k\ell}]\rangle &= (\phi_{k\ell}\#\mathrm{Tr})(\sigma(P_{ij}),\sigma(P_{ij}),\sigma(P_{ij}))\\ &= -6\phi_{k\ell}(f(u,v,w),g(u,v,w),\bar{g}(u,v,w))\\ &= -\frac{6}{2\pi i}\int f(u,v,w)\bigg[\frac{\partial}{\partial x_k}g(u,v,w)\frac{\partial}{\partial x_\ell}\bar{g}(u,v,w)\\ &\qquad -\frac{\partial}{\partial x_\ell}g(u,v,w)\frac{\partial}{\partial x_k}\bar{g}(u,v,w)\bigg]\,dx_1dx_2dx_3.\end{aligned}$$

Let us first calculate the pairings with P_{13}. Since g for P_{13} depends only on x_1 and x_3, $g_2 = 0$, and one gets

$$\begin{aligned}&\langle[\sigma(P_{13})],[\phi_{12}]\rangle\\ &\quad= -\frac{6}{2\pi i}\int f(u,v,w)\bigg[\frac{\partial}{\partial x_1}g(u,v,w)\frac{\partial}{\partial x_2}\bar{g}(u,v,w)\\ &\qquad -\frac{\partial}{\partial x_2}g(u,v,w)\frac{\partial}{\partial x_1}\bar{g}(u,v,w)\bigg]\,dx_1dx_2dx_3\\ &\quad= -\frac{6}{2\pi i}\int f(u,v,w)\bigg[(g_1+p_3g_2+p_1g_3)(\bar{g}_2+p_2\bar{g}_3)\\ &\qquad -(g_2+p_2g_3)(\bar{g}_1+p_3\bar{g}_2+p_1\bar{g}_3)\bigg]\,dx_1dx_2dx_3\\ &\quad= -p_2\cdot\frac{6}{2\pi i}\int f(u,v,w)[g_1\bar{g}_3-g_3\bar{g}_1]\,dx_1dx_2dx_3\\ &\quad= p_2\langle[P_{13}],[\phi_{13}]\rangle = p_2,\end{aligned}$$

where we have simply written $g_j = g_j(u,v,w)$ and have used the change of variables formula for integrals since the transformation $(u,v,w) = (x_1, p_3x_1+x_2, p_1x_1+p_2x_2+x_3)$ has Jacobian determinant 1. Similarly,

$$\begin{aligned}&\langle[\sigma(P_{13})],[\phi_{13}]\rangle\\ &\quad= -\frac{6}{2\pi i}\int f(u,v,w)\bigg[\frac{\partial}{\partial x_1}g(u,v,w)\frac{\partial}{\partial x_3}\bar{g}(u,v,w)\\ &\qquad -\frac{\partial}{\partial x_3}g(u,v,w)\frac{\partial}{\partial x_1}\bar{g}(u,v,w)\bigg]\,dx_1dx_2dx_3\\ &\quad= -\frac{6}{2\pi i}\int f(u,v,w)\bigg[(g_1+p_3g_2+p_1g_3)\,\bar{g}_3\\ &\qquad -g_3\,(\bar{g}_1+p_3\bar{g}_2+p_1\bar{g}_3)\bigg]\,dx_1dx_2dx_3\\ &\quad= -\frac{6}{2\pi i}\int f[g_1\bar{g}_3-g_3\bar{g}_1]\,dx_1dx_2dx_3 = \langle[P_{13}],[\phi_{13}]\rangle = 1,\end{aligned}$$

$$\begin{aligned}
&\langle[\sigma(P_{13})],[\phi_{23}]\rangle\\
&\quad= -\frac{6}{2\pi i}\int f(u,v,w)\Big[\frac{\partial}{\partial x_2}g(u,v,w)\frac{\partial}{\partial x_3}\bar{g}(u,v,w)\\
&\qquad -\frac{\partial}{\partial x_3}g(u,v,w)\frac{\partial}{\partial x_2}\bar{g}(u,v,w)\Big]\,dx_1dx_2dx_3\\
&\quad= -\frac{6}{2\pi i}\int f(u,v,w)\Big[(g_2+p_2g_3)\,\bar{g}_3-g_3\,(\bar{g}_2+p_2\bar{g}_3)\Big]\,dx_1dx_2dx_3\\
&\quad= -\frac{6}{2\pi i}\int f[(g_2+p_2g_3)\bar{g}_3-g_3(\bar{g}_2+p_2\bar{g}_3)]\,dx_1dx_2dx_3=0.
\end{aligned}$$

Hence $\mathrm{ch}_0(b_{13})=(0,p_2,1,0)$ which yields $\sigma_*(b_{13})=p_2b_{12}+b_{13}$. The pairings with P_{23} are (where now $g_1=0$)

$$\begin{aligned}
&\langle[\sigma(P_{23})],[\phi_{12}]\rangle\\
&\quad= -\frac{6}{2\pi i}\int f(u,v,w)\Big[\frac{\partial}{\partial x_1}g(u,v,w)\frac{\partial}{\partial x_2}\bar{g}(u,v,w)\\
&\qquad -\frac{\partial}{\partial x_2}g(u,v,w)\frac{\partial}{\partial x_1}\bar{g}(u,v,w)\Big]\,dx_1dx_2dx_3\\
&\quad= -\frac{6}{2\pi i}\int f(u,v,w)\Big[(g_1+p_3g_2+p_1g_3)(\bar{g}_2+p_2\bar{g}_3)\\
&\qquad -(g_2+p_2g_3)(\bar{g}_1+p_3\bar{g}_2+p_1\bar{g}_3)\Big]\,dx_1dx_2dx_3\\
&\quad= -(p_2p_3-p_1)\cdot\frac{6}{2\pi i}\int f(u,v,w)[g_2\bar{g}_3-g_3\bar{g}_2]\,dx_1dx_2dx_3\\
&\quad= (p_2p_3-p_1)\langle[P_{23}],[\phi_{23}]\rangle = p_2p_3-p_1,
\end{aligned}$$

$$\begin{aligned}
&\langle[\sigma(P_{23})],[\phi_{13}]\rangle\\
&\quad= -\frac{6}{2\pi i}\int f(u,v,w)\Big[\frac{\partial}{\partial x_1}g(u,v,w)\frac{\partial}{\partial x_3}\bar{g}(u,v,w)\\
&\qquad -\frac{\partial}{\partial x_3}g(u,v,w)\frac{\partial}{\partial x_1}\bar{g}(u,v,w)\Big]\,dx_1dx_2dx_3\\
&\quad= -\frac{6}{2\pi i}\int f(u,v,w)\Big[(g_1+p_3g_2+p_1g_3)\,\bar{g}_3\\
&\qquad -g_3\,(\bar{g}_1+p_3\bar{g}_2+p_1\bar{g}_3)\Big]\,dx_1dx_2dx_3\\
&\quad= -p_3\cdot\frac{6}{2\pi i}\int f[g_2\bar{g}_3-g_3\bar{g}_2]\,dx_1dx_2dx_3 = p_3\,\langle[P_{23}],[\phi_{23}]\rangle = p_3,
\end{aligned}$$

and

$$
\begin{aligned}
&\langle[\sigma(P_{23})],[\phi_{23}]\rangle\\
&\quad= -\frac{6}{2\pi i}\int f(u,v,w)\biggl[\frac{\partial}{\partial x_2}g(u,v,w)\frac{\partial}{\partial x_3}\bar{g}(u,v,w)\\
&\qquad -\frac{\partial}{\partial x_3}g(u,v,w)\frac{\partial}{\partial x_2}\bar{g}(u,v,w)\biggr]\,dx_1dx_2dx_3\\
&\quad= -\frac{6}{2\pi i}\int f(u,v,w)\Bigl[(g_2+p_2g_3)\,\bar{g}_3-g_3\,(\bar{g}_2+p_2\bar{g}_3)\Bigr]\,dx_1dx_2dx_3\\
&\quad= -\frac{6}{2\pi i}\int f[g_2\bar{g}_3-g_3\bar{g}_2]\,dx_1dx_2dx_3=\langle[P_{23}],[\phi_{23}]\rangle=1.
\end{aligned}
$$

Hence $\mathrm{ch}_0(b_{23})=(0,p_2p_3-p_1,p_3,1)$ which gives $\sigma_*(b_{23})=(p_2p_3-p_1)b_{12}+p_3b_{13}+b_{23}$. In a similar manner, and more easily, one can check that $\sigma_*(b_{12})=b_{12}$. ∎

From this lemma one gets

$$
\begin{aligned}
\ker(\sigma_*-id_*)\text{ on }K_0(C(\mathbb{T}^3)) &= \mathbb{Z}[1]+\mathbb{Z}b_{12},\\
\mathrm{Image}(\sigma_*-id_*)\text{ on }K_0(C(\mathbb{T}^3)) &= \mathbb{Z}p_2b_{12}+\mathbb{Z}\bigl((p_2p_3-p_1)b_{12}+p_3b_{13}\bigr)\\
&= \mathbb{Z}p_2b_{12}+\mathbb{Z}\bigl(-p_1b_{12}+p_3b_{13}\bigr).
\end{aligned}
$$

Together with the Pimsner–Voiculescu exact sequence these yield

$$
\begin{aligned}
K_0(C^*(\mathrm{H})) &\cong \mathbb{Z}^2\oplus\frac{\mathbb{Z}^4}{\mathbb{Z}p_2b_{12}+\mathbb{Z}\bigl(-p_1b_{12}+p_3b_{13}\bigr)}\\
&\cong \mathbb{Z}^2\oplus\frac{\mathbb{Z}^4}{\mathbb{Z}(p_2,0,0,0)+\mathbb{Z}(-p_1,p_3,0,0)}\\
&\cong \mathbb{Z}^4\oplus\frac{\mathbb{Z}^2}{\mathbb{Z}(p_2,0)+\mathbb{Z}(-p_1,p_3)}
\end{aligned}
$$

which, by Lemma 6, becomes

$$K_0(C^*(\mathrm{H}))\cong\mathbb{Z}^4\oplus\mathbb{Z}_\delta\oplus\mathbb{Z}_{p_2p_3/\delta}$$

where $\delta=\gcd\{p_1,p_2,p_3\}$. Similarly, from Eqs. (4) and (5),

$$K_1(C^*(\mathrm{H}))\cong\mathbb{Z}^4\oplus\frac{\mathbb{Z}^2}{\mathbb{Z}(p_3,0)+\mathbb{Z}(p_1,p_2)}\cong\mathbb{Z}^4\oplus\mathbb{Z}_\delta\oplus\mathbb{Z}_{p_2p_3/\delta}.$$

This completes the proof of Theorem 7. ∎

Concluding Remarks. 1. Of course, isomorphic groups $H_4(p_1, p_2, p_3)$ must be assigned isomorphic K-groups by Theorem 7; for example, if $H = H_4(p_1, p_2, p_3)$ and $H' = H_4(p_1', p_2', p_3')$ are isomorphic via one of the isomorphisms in the proof of Theorem 1, then the K-groups given for H and H′ by Theorem 7 are isomorphic.

2. The K-groups in Theorem 7 sometimes do not always distinguish the group C^*-algebras of non-isomorphic groups. For example, $H_4(p_1, p_2, p_3) \not\cong H_4(p_1, p_3, p_2)$ if $p_2 \neq p_3$, but the group C^*-algebras of these groups have the same K-groups (given by Theorem 7). Also, for prime $p > 2$, $C^*(H_4(p_1, p, p))$ has K-groups isomorphic to $\mathbb{Z}^4 \oplus \mathbb{Z}_{p^2}$ for $0 < p_1 < p/2$, but $C^*(H_4(0, p, p))$ has its own group $\mathbb{Z}^4 \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$.

QUESTION. *Is it possible that*

$$C^*(H_4(p_1, p_2, p_3)) \cong C^*(H_4(p_1', p_2', p_3'))$$

if $H_4(p_1, p_2, p_3) \not\cong H_4(p_1', p_2', p_3')$?

3. One can attempt to classify the discrete cocompact subgroups of the five-dimensional nilpotent connected groups using the methods of Theorem 1, an enterprise that is already more involved because there are six such connected groups (see [7] or [6] for details and notation about these connected groups). Preliminary calculations indicate that it may be possible to achieve a complete classification of the discrete cocompact subgroups for $G_{5,3}$ (one of the five-dimensinal connected groups); but, as is not surprising, the problem of such a classification seems very challenging (and perhaps intractable) for most higher dimensional groups.

4. It was pointed out by the referee that a simpler proof of Lemma 8 could be given by using the basic properties of the Bott element $B(u, v) \in K_0(A)$, where u, v are any two commuting unitaries in a unital C^*-algebra A—these properties being

$$B(u, u) = 0, \qquad B(u, vw) = B(u, v) + B(u, w), \qquad B(u, v) = -B(v, u),$$

where u, v, w are any commuting unitaries. However, we prefer the proof given here based on cyclic cocycles in view of its more universal connection with Connes' non-commutative geometry and the Connes Chern character. Our method of proof is applicable in situations where the Bott element is unavailable; for instance, our method of proof is used in [11, Sect. 4] for the C^*-algebra $C(\mathbb{T}) \otimes A_\theta$ and an automorphism that mixes the two tensor factors—here, the cyclic cocycles approach seems to be the only known tool one could use to determine K-group elements.

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