Geometry of Nilpotent and Solvable Groups

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Lecture 1: Basic definitions and overview of the course

The objects that we study are infinite groups G that are finitely generated.

Question 1. *How to endow such groups with a geometry ?*

There are several possible ways of doing that, we list three below.

(Answer 1) By defining a *Cayley graph.*

Let S be a finite generating set for G, such that $S^{-1} = \{s^{-1} \mid s \in S\} = S$ and $1 \notin S$.

NB From now on we always assume that generating sets of the group that we consider satisfy the above.

The Cayley graph $\mathrm{Cayley}(G, S)$ of G with respect to the generating set S is a non-oriented graph defined as follows:

- its set of vertices is G;
- every pair of elements $g_1, g_2 \in G$ such that $g_1 = g_2 s$, with $s \in S$, is joined by an edge.

We suppose that every edge has length 1 and we endow $\mathrm{Cayley}(G, S)$ with the shortest path metric: $\mathrm{d}_S(u, v)$ is the length of the shortest path joining u and v. Note that since S generates G this graph is connected.

The restriction of d_S to its set of vertices, i.e. G, is called the word metric on G associated to S.

Notation 2. *We denote by $|g|_S$ the distance $\mathrm{d}_S(1, g)$, for every $g \in G$.*

Remark 3. A Cayley graph can be constructed also for an infinite set of generators. In this case the graph has infinite valency in each point.

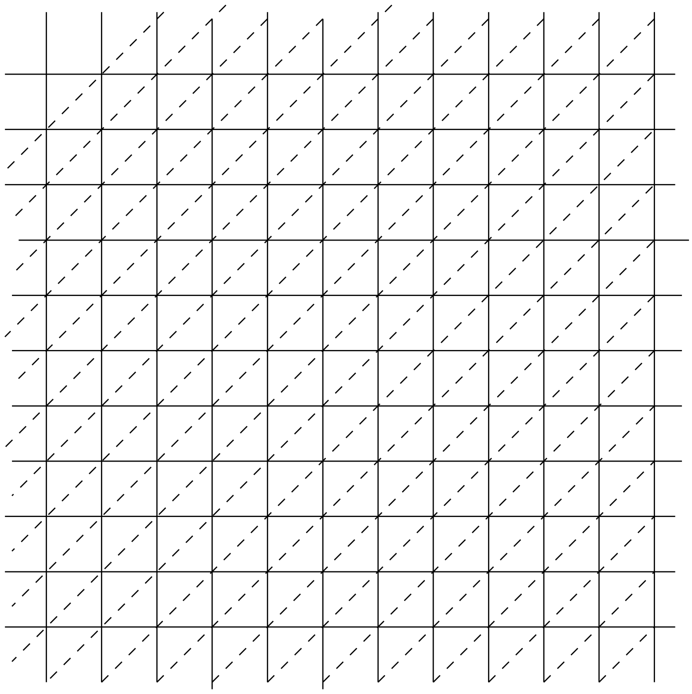

Figure 1: Cayley graph of $\mathbb{Z}^2$.

Example 1. In Figure 1 are represented the Cayley graph of $\mathbb{Z}^2$ with set of generators $\{(\pm 1, 0), (0, \pm 1)\}$ and the Cayley graph of $\mathbb{Z}^2$ with set of generators $\{(\pm 1, 0), \pm(1, 1)\}$.

If S and $\bar{S}$ are two finite generating sets of G then d_S and $\mathrm{d}_{\bar{S}}$ are bi-Lipschitz equivalent, that is for every g, h in G

$$\frac{1}{L}\mathrm{d}_S(g,h) \leq \mathrm{d}_{\bar{S}}(g,h) \leq L\mathrm{d}_S(g,h)\,,$$

where L is the maximum of the following two numbers: $\max_{\bar{s}\in\bar{S}} |\bar{s}|_S$ and $\max_{s\in S} |s|_{\bar{S}}$.

Example 2. Consider the free group of rank 2, $\mathbf{F}_2$. Let $\{a, b, a^{-1}, b^{-1}\}$ be the set of generators. Recall that $\mathbf{F}_2$ is composed of reduced words in a, b, that is words in the alphabet $\{a, b, a^{-1}, b^{-1}\}$ not containing any sub-word of the form $aa^{-1}, bb^{-1}, b^{-1}b, a^{-1}a$.

The group operation on $\mathbf{F}_2$ is the concatenation of two words followed by reduction, i.e. deletion of any sub-word of the form $aa^{-1}, bb^{-1}, b^{-1}b, a^{-1}a$.

See Figure 2 for the Cayley graph of the free group of rank two $\mathbf{F}_2 = \langle a, b\rangle$.

The action of a group on itself to the left is an action by isometries, that is for every $g \in G$ the left-translation map $x \mapsto gx$ is an isometry. This is because every such map sends an edge to an edge.

(Answer 2) Sometimes the group has a nice action on a nice metric space.

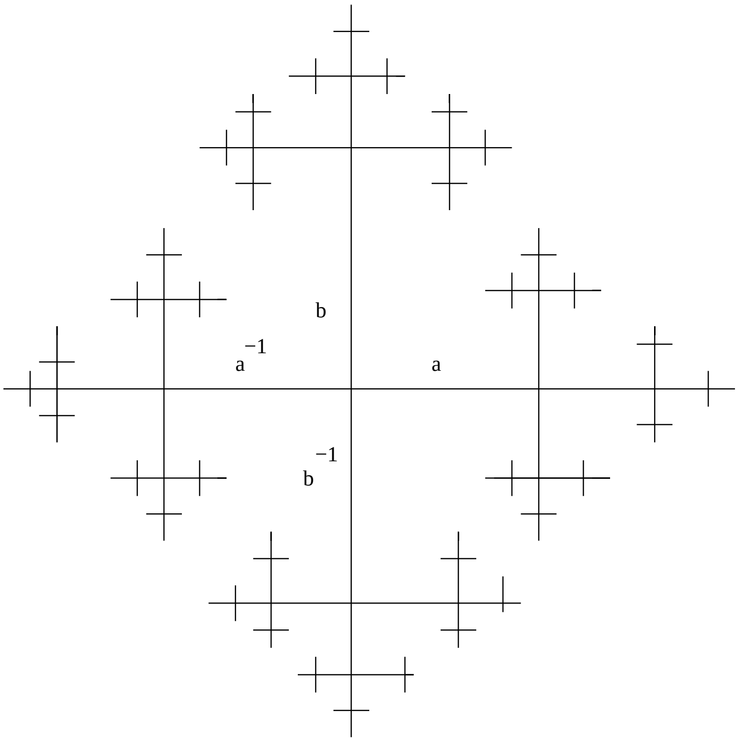

Figure 2: Cayley graph of $\mathbf{F}_2$.

When a group G acts on a metric space we can always endow G with a pseudo-metric: fix $x \in X$ and define $\mathrm{d}_G(g,h) = \mathrm{d}(gx,hx)$. It is a pseudo-metric because distinct elements in G might be at distance zero.

When is a metric space (X, d) considered "nice"?

- When it is geodesic, i.e. such that for any two points $x, y \in X$ there exists a map
$$\rho : [0, \mathrm{d}(x,y)] \to X, \rho(0) = x, \rho(\mathrm{d}(x,y)) = y\,,$$
moreover ρ is an isometry, that is $\mathrm{d}(\rho(t), \rho(s)) = |t-s|$.

  Such a map ρ is called a geodesic joining x, y. Sometimes we use the same terminology for its image.

- When it is proper, that is all closed balls $\bar{B}(x,r) = \{y \in X \mid \mathrm{d}(x,y) \le r\}$ (where x is an arbitrary point in X and r is an arbitrary positive number) are compact sets.

When is the action of a group on a metric space (X, d) considered "nice"?

- When it is by isometries, that is we have a homomorphism $\phi : G \to \mathrm{Isom}(X)$.
- When the action is properly discontinuous, that is given any compact subset K of X, the set
$$\{g \in G \mid gK \cap K \neq \emptyset\}$$
is finite.

When the action is too nice, the metric introduced by it might be essentially the same as a word metric. More precisely, we have the following:

Theorem 4 (Milnor–Schwarz). *Let (X, d) be a proper geodesic metric space, and let G be a group acting properly discontinuously by isometries on X, with compact quotient $G \backslash X$. Then:*

1. *the group G is finitely generated;*
2. *for any word metric d_w on G and any point $x \in X$, the map $G \to X$ given by $g \mapsto gx$ is a quasi-isometry.*

A proof of this can be found in the book of Pierre de la Harpe - Topics in Geometric Group Theory, Theorem IV.B.23.

Recall that a quasi-isometry between two metric spaces (X, d_X) and (Y, d_Y) is a map $\mathfrak{q} : X \to Y$ such that:

- for every $x_1, x_2 \in X$,
$$\frac{1}{L}\mathrm{d}_X(x_1, x_2) - C \leq \mathrm{d}_Y(\mathfrak{q}(x_1), \mathfrak{q}(x_2)) \leq L\mathrm{d}_X(x_1, x_2) + C\,, \tag{1}$$
for some constants $L \geq 1$ and $C \geq 0$.
- Y is contained in the C–tubular neighborhood of $\mathfrak{q}(X)$.

Notation 5. *We call the R–tubular neighborhood of a set A the open set*
$$\mathcal{N}_R(A) = \{z \mid \mathrm{d}(z, A) < R\}\,.$$

(Answer 3) The group might be represented as a linear group (if we are lucky).

That is, we could have an injective homomorphism $\phi : G \to SL(n, \mathbb{R})$.

The group $SL(n, \mathbb{R})$ is essentially the group of isometries of a symmetric space of non-positive curvature, and as such it has a natural metric, invariant by left multiplication. The distance from the identity matrix I_n to an arbitrary matrix $M \in SL(n, \mathbb{R})$ is approximately the same as $\ln(1 + \|M\|)$.

So when a group is linear, i.e. an injective homomorphism $\phi : G \to SL(n,\mathbb{R})$ exists, an essential question is to compare a word distance on G, $\mathrm{d}_S(g,g')$, to the metric induced by ϕ, i.e. $\mathrm{d}_{SL(n,\mathbb{R})}(\phi(g),\phi(g'))$.

This is what the distortion of linear groups will be about.

Answers 1 and 2 provide two good reasons to study the features of a group with a word metric that are preserved by quasi-isometry:

- we want to find features intrinsic to the group, i.e. independent of the generating set involved, hence invariant by bi-Lipschitz equivalence;

- we want to be able to exchange a group G with a metric space X, when the hypotheses of Theorem 4 are satisfied.

There are two types of problems we'll be interested in:

1. What algebraic properties are quasi-isometrically rigid ?

   That is: if a group G has a certain algebraic property (Π) and H is another group quasi-isometric to it, does H have the property (Π) ?

   As such, this cannot work for almost any algebraic property, because one can take $H = G \times F$, with F a finite group without property (Π). So reformulate:

   If a group G has a certain algebraic property (Π) and H is another group quasi-isometric to it, does H virtually have the property (Π) ?

   We say a group H virtually has property (Π) if it contains a finite index subgroup with property (Π).

2. Classify the groups in a given class up to quasi-isometry.

   Example: Consider the set of all abelian groups. Every such group is isomorphic (hence quasi-isometric) to a product $\mathbb{Z}^n \times F$, with F a finite abelian group. So it suffices to classify the set of groups $\mathbb{Z}^n$ with $n \geq 0$, up to quasi-isometry. This can be done for instance using the growth function.

Let G be a finitely generated group with finite generating set S and word metric d_S. The growth function is the function $\mathfrak{G}_S^G : \mathbb{R}_+ \to \mathbb{R}$ defined as

$$\mathfrak{G}_S^G(R) := \mathrm{card}\ \bar{B}(1,R),$$

the cardinality of the closed R-ball centred at 1 in (G,d_S).

Since G acts transitively on itself, this definition does not depend on the choice of the centre, i.e. if instead of 1 we consider some other element $g_0 \in G$.

The growth function obviously depends on the choice of generating set S but only up to the following equivalence relation.

Notation 6. *We introduce the following order relation between two functions $f, g : X \to \mathbb{R}$ with $X \subset \mathbb{R}$: we write $f \preceq g$ if there exist $C > 0$ such that $f(x) \leq Cg(Cx + C) + Cx + C$ for every $x \in X$.*

If $f \preceq g$ and $g \preceq f$ then we write $f \asymp g$ and we say that f and g are asymptotically equal.

Lemma 7. 1. *If S and S' are two generating sets for G then $\mathfrak{G}_S^G \asymp \mathfrak{G}_{S'}^G$.*

2. *If G and G' are quasi-isometric then $\mathfrak{G}^G \asymp \mathfrak{G}^{G'}$.*

It is straightforward that when $G = \mathbb{Z}^k$, $\mathfrak{G}_G \asymp x^k$. Hence $\mathbb{Z}^k$ and $\mathbb{Z}^m$ with $k \neq m$ cannot be quasi-isometric.

This course aims to explain the proofs of the following two theorems:

Theorem 8 (the Bass-Gromov equivalence)**.** *A finitely generated group G is virtually nilpotent if and only if $\mathfrak{G}_G \preceq x^d$ for some $d > 0$ (in which case $\mathfrak{G}_G \asymp x^m$ for some integer m).*

The main ingredient necessary to prove the converse part of this equivalence is the following.

Theorem 9 (Jacques Tits' alternative theorem)**.** *A subgroup of $SL(n, \mathbb{R})$ is either virtually solvable or it contains a copy of the free group of rank 2, $\mathbf{F}_2$.*

An immediate consequence of the Bass-Gromov equivalence is the quasi-isometry rigidity for virtually nilpotent groups: a group quasi-isometric to a virtually nilpotent group is itself virtually nilpotent.

This is no longer true for the larger class of solvable groups: Anna Dioubina-Erschler provided examples of pairs of quasi-isometric groups G and H, with G solvable and H not virtually solvable.

We recall the notions of semi-direct product and of short exact sequence.

Definition 10. 1. (with the ambient group as given data) A group G is a semidirect product of two subgroups N and H, which is sometimes denoted by $G = N \rtimes H$, if and only if

- N is a normal subgroup of G;
- H is a subgroup of G;
- $G = NH = \{nh \,;\, n \in N, h \in H\}$ and $N \cap H = \{1\}$.

Let $\mathrm{Aut}\,(N)$ denote the group of all automorphisms of N. The map $\varphi : H \to \mathrm{Aut}\,(N)$ defined by $\varphi(h)(n) = hnh^{-1}$, is a group homomorphism.

2. (with the factor groups as given data) Given any two groups N and H (not necessarily subgroups of the same group) and a group homomorphism $\varphi : H \to \mathrm{Aut}\,(N)$, one can define a new group $G = N \rtimes_{\varphi} H$ which is a semidirect product of a copy of N and a copy of H in the above sense, defined as follows. As a set, $N \rtimes_{\varphi} H$ is defined as the cartesian product $N \times H$. The binary operation $*$ on G is defined by

$$(n_1, h_1) * (n_2, h_2) = (n_1\varphi(h_1)(n_2), h_1h_2)\,,\ \forall n_1, n_2 \in N \text{ and } h_1, h_2 \in H\,.$$

The group $G = N \rtimes_{\varphi} H$ is called the semidirect product of N and H with respect to φ.

An exact sequence is a sequence of group homomorphisms

$$...G_{n-1} \xrightarrow{\varphi_{n-1}} G_n \quad \xrightarrow{\varphi_n} G_{n+1}....$$

such that $\mathrm{Im}\,\varphi_{n-1} = \ker\varphi_n$.

A short exact sequence is an exact sequence as follows:

$$\{1\} \longrightarrow N \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow \{1\}\,. \tag{2}$$

In other words N is isomorphic to a normal subgroup N' in G, and $G/N' \simeq H$.

If N is a group of type (A) and Q a group of type B we say that G is a group of type A-by-B.

For instance we will talk of abelian-by-finite groups, of free-by-cyclic groups, and so on.

If G decomposes as $N \rtimes H$, this defines a short exact sequence

$$1 \to N \xrightarrow{i} G \xrightarrow{\pi} H \to 1\,,$$

where i is the inclusion map of N in G, and $\pi : G \to H$ is defined by $\pi(nh) = h$ for every $n \in N$ and $h \in H$ (i.e. π is the 'projection' onto the 'factor' H).

The converse is not always true: the existence of a short exact sequence as in (2) does not imply that G is isomorphic to a semidirect product $N \rtimes H$.

Example 11. The short exact sequence

$$1 \longrightarrow 2\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 1$$

does not correspond to a decomposition of $\mathbb{Z}$ into a semidirect product.

Lemma 12. *A short exact sequence* $1 \xrightarrow{i} N \to G \xrightarrow{\pi} H\mathbf{F}_n \to 1$, *where* $\mathbf{F}_n$ *is a free group of rank* n, *corresponds to a decomposition of* G *as a semidirect product* $G = N \rtimes H$, *where* H *is isomorphic to* $\mathbf{F}_n$.

Proof. Indeed let $\{a_1^{\pm 1}, ..., a_n^{\pm 1}\}$ be a generating set for F_n. For each a_i consider an element $t_i \in G$ projecting onto a_i, and take $H = \langle t_1, ...t_n \rangle$. Clearly $t_1, ...t_n$ generate a free group (any non-trivial relation in $t_1, ...t_n$ would project to a non-trivial relation in $a_1, ...a_n$), hence $H \simeq \mathbf{F}_n$.

Any non-empty word $w(t_1, ...t_n)$ in H that is also contained in N has the property that its projection onto $\mathbf{F}_n$ must be 1, a contradiction.

For every g in G, $\pi(g)$ equals a word $w(a_1, ..., a_n)$, hence $g \in w(t_1, ..., t_n)N$. Thus $G = HN$. □

Nilpotent groups

The commutator of two elements h, k is

$$[h, k] = hkh^{-1}k^{-1} .$$

Let H, K be two subgroups of G. We denote by $[H, K]$ the subgroup generated by all commutators $[h, k]$ with $h \in H$, $k \in K$.

We define by induction a decreasing sequence of subgroups in G:

$$C^1 G = G , \ C^{n+1}G = [G, C^n G] .$$

The decreasing sequence $G \trianglerighteq C^2 G \trianglerighteq \cdots \trianglerighteq C^n G \trianglerighteq C^{n+1} G \trianglerighteq \dots$ is called the lower central series of the group G.

Recall that a subgroup K in a group G is called characteristic if for every automorphism $\phi : G \to G$, $\phi(K) = K$.

All subgroups $C^k G$ are characteristic subgroups of G, because automorphisms preserve commutators.

Definition 13. A group G is called (k-step) nilpotent if there exists k such that $C^{k+1}G = \{1\}$. The minimal such k is the class of G.

Examples 14.

1. An abelian group is nilpotent of class 1.

2. The group $\mathcal{U}_n$ of upper triangular $n \times n$ matrices with 1 on the diagonal is nilpotent of class $n - 1$.

   Indeed, let $\mathcal{U}_{n,k}$ be the subset of $\mathcal{U}_n$ composed of matrices (a_{ij}) such that $a_{ij} = \delta_{ij}$ for $j < i + k$. Note that $\mathcal{U}_{n,1} = \mathcal{U}_n$.

   For every $k \geq 1$ the map

$$\begin{array}{rcc} \varphi_k : \mathcal{U}_{n,k} & \to & (\mathbb{R}^{n-k} , +) \\ A = (a_{ij}) & \mapsto & (a_{1k+1}, a_{2k+2}, \dots , a_{n-kn}) \end{array}$$

   is a homomorphism. It follows that $(\mathcal{U}_{n,k})' \subset \mathcal{U}_{n,k+1}$ and that $\mathcal{U}_{n,k+1} \lhd \mathcal{U}_{n,k}$ for every $k \geq 1$.

   Moreover $C^k \mathcal{U}_n \leq \mathcal{U}_{n,k+1}$ for every $k \geq 0$. Hence $\mathcal{U}_n$ is $(n-1)$-step nilpotent.

3. The discrete Heisenberg group

$$H_{2n+1} = \left\{ \begin{pmatrix} 1 & x_1 & \dots & x_n & z \\ 0 & 1 & 0 & \dots & y_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & y_1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} ; \; x_1, ..., x_n, y_1, ..., y_n, z \in \mathbb{Z} \right\}$$

is nilpotent of class 2.

Indeed $C^2 H_{2n+1}$ is the subgroup with all $x_i = 0$ and $y_i = 0$.

Basic properties of nilpotent groups (to be found in any book on the subject):

1. A subgroup H of a nilpotent group G is nilpotent.

   This is because $C^k H \leq C^k G$.

2. A quotient G/N of a nilpotent group G (where N is a normal subgroup) is nilpotent.

   Because, given $\pi : G \to G/N$ the canonical projection, $\pi(C^k G) = C^k(G/N)$.

3. $[C^k G, C^n G] \leq C^{k+n} G$.