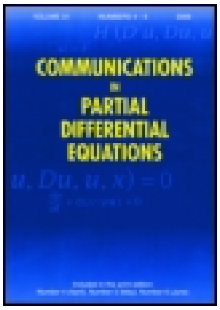

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The wave equation on the heisenberg group

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THE WAVE EQUATION ON THE HEISENBERG GROUP

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Introduction. *We consider the wave operator* $D_t^2 - \Box_b$ *with* $\Box_b$ *the* Laplacian associated to the $\bar{\partial}_b$ complex on the Heisenberg group $\mathbb{H}_n$. Written out in coordinates (cf. [3]) this amounts to studying, in $\mathbb{R}^{2n+2}$

$$(0.1) \quad \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{n} [(\frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial s})^2 + (\frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial s})^2] + 4i\gamma \frac{\partial}{\partial s}$$

where γ is one of the numbers $-n, -n+2, \ldots, n-2, n$. The Cauchy problem for this equation is well posed if and only if γ is real and $|\gamma| \leq n$ (cf. [6]). We will assume $|\gamma| < n$ (i.e. exclude the cases. when $\Box_b$ is not subelliptic).

We find a formula (cf. (3.8)) for the (forward) fundamental solution E of (0.1) using the group Fourier transform.

Working with certain fractional integrals G, H of E for which we have more tractable expressions we then find the "light cone" corresponding to our wave equation.

Let P denote the operator in (0.1), C its characteristic set and C_2 the subset where the principal symbol of P vanishes to precisely second order. It turns out that the light cone is generated by the bicharacteristics through the origin, projection of bicharacteristic strips lying in $C \setminus C_2$ (c.f §10). Once one realizes this one immediately conjectures that, as in the strictly hyperbolic case, for any distribution f, $[WF(f) \setminus WF(Pf)] \setminus C_2$ is invariant under the Hamilton flow while $[WF(f) \setminus WF(Pf)] \cap C_2$ does not change with time. This fact has recently been proved by Melrose [10] and Lascar [8] for a class of operators which includes $D_t^2 - \square_b$ on $\mathbb{H}_1$ (the proof in [10] appears to generalize to $\mathbb{H}_n$).

Next we prove that (at least for points $(t, x, y, s) \in \mathbb{R}^{2n+2}$ with $(x, y) \neq 0$) the fundamental solution is actually real analytic in the complement of the light cone. One extra difficulty here is that fractional integration in one of the variables is not a pseudolocal operator; however, we prove that it does preserve a certain part of the analytic wave front set of a distribution (§11.)

Finally, in Part III we compute the asymptotic behavior of the fundamental solution as we approach a generic point on the light cone.

The main results are summarized in §6, §14 and at the end of §11.

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I. Finding the Fundamental Solution

1. Notation and Preliminaries. The notation we will be using is, with minor modifications the one in Folland and Stein [3].

The Heisenberg group $\mathbb{H}_n$ is the Lie group with underlying manifold $\mathbb{C}^n \times \mathbb{R}$ and multiplication

$$(1.1) \qquad (z,s)\cdot(z',s') = (z+z', s+s'+2\, Im\, z\cdot z'), \quad z\cdot z' = \sum_{j=1}^{n} z_j \bar{z}'_j.$$

The vector fields

$$(1.2) \qquad Z_j = \frac{\partial}{\partial z_j} + i\,\bar{z}_j \frac{\partial}{\partial s}, \quad \bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} - iz_j \frac{\partial}{\partial s}, \quad 1 \le j \le n, \quad \text{and} \quad S = \frac{\partial}{\partial s}$$

form a basis for the left invariant vector fields. We will also use the real coordinate system (x, y, s) obtained from $z_j = x_j + iy_j$ and the basis for (real) left invariant vector fields

(1.3) $$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial s}\ , \ Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial s}\ , \ S = \frac{\partial}{\partial s}\ .$$

For $\gamma \in \mathbb{R}$, let $L_\gamma = 2 \sum_{j=1}^{n} (Z_j\overline{Z}_j + \overline{Z}_jZ_j) - 4i\gamma S$. We fix a (bi-invariant) Haar measure on $\mathbb{H}_n$ which in our coordinate system is just Lebesque measure on underlying $\mathbb{R}^{2n+1}$. For f, g functions on $\mathbb{H}_n$ define their convolution by

(1.4) $$f*g(w) = \int_{\mathbb{H}_n} f(wv^{-1})g(v)dv.$$

Powers of $-L_0$ can be used to define Sobolev-type spaces on $\mathbb{H}_n$ which take into account the different role played by the s-direction (see Folland [2]). We write S_α for S^2_α and $\|\ \|_\alpha$ for the corresponding norm; there is an analogue of Sobolev's theorem (proved in [2]):

(1.5) $$S_\alpha \subset C^k \quad \text{provided} \quad \alpha > 2k + n + 1.$$

We will need a few facts (see for example [9]) about Laguerre polynomials, defined by

(1.6) $$L_m^{(n)}(x) = \sum_{k=0}^{m} (-1)^k \binom{m+n}{m-k} \frac{x^k}{k!} \qquad x \geq 0\ .$$

They have a simple generating function:

$$\sum_{m=0}^{\infty} L_m^{(n)}(x)\ \varepsilon^n = (1-\varepsilon)^{-n-1} \exp[\ \frac{x\varepsilon}{\varepsilon-1}\ ]\ , \quad |\varepsilon| < 1$$

since this is true for x negative as well, and $|L_m^{(n)}(x)| \leq L_m^{(n)}(-x)$ for $x \geq 0$ we have the (rough) estimate

$$1.7)\qquad \sum_{m=0}^{\infty} |L_m^{(n)}(x)| \quad \varepsilon^m \le (1-\varepsilon)^{-n-1} \exp[\ \frac{x\varepsilon}{1-\varepsilon}\ ] \quad \text{for} \quad 0 < \varepsilon < 1,\ x \ge 0$$

Finally, the Laplace transform of $x^n L_m^{(n)}(x)$ is

$$(1.8)\qquad \int_0^\infty e^{-\rho x}\, x^n\, L_m^{(n)}(x)\, dx = \frac{(m+n)!}{m!} \quad \rho^{-m-n-1}(\rho-1)^m \qquad Re\ \rho > 0$$

2. <u>The Fourier Transform on $\mathbb{H}_n$</u> . We begin this section with some basic facts from Fourier Analysis on $\mathbb{H}_n$. (For further details and proofs see Geller [4] and [5]).

The irreducible unitary representations of $\mathbb{H}_n$ can be described as acting on Bargmann's spaces $\mathcal{H}_\lambda$; for $\lambda \neq 0$

$$\mathcal{H}_\lambda = \{F \text{ holomorphic on } \mathbb{C}^n : (\tfrac{2|\lambda|}{\pi})^n \int |F(\zeta)|^2 e^{-2|\lambda||\zeta|^2} d\zeta = \|F\|^2 < \infty\}$$

$\mathcal{H}_\lambda$ is a Hilbert space, and the monomials

$$F_{\alpha,\lambda}(\zeta) = \frac{(\sqrt{2|\lambda|}\zeta)^\alpha}{\sqrt{\alpha!}} \quad (\alpha \in (\mathbb{Z}^+)^n,\ \mathbb{Z}^+ = \{0, 1, 2, \ldots\})$$

form an orthonormal basis. Furthermore, $\mathcal{H}_\lambda$ has $K_\lambda(\zeta, \eta) = e^{2|\lambda|\zeta\cdot\eta}$ as reproducing kernel:

$$(2.1)\quad \text{if } F \in \mathcal{H}_\lambda,\ F(\zeta) = (\tfrac{2|\lambda|}{\pi})^n \int_{\mathbb{C}^n} F(\eta) K(\zeta,\eta) e^{-2|\lambda||\eta|^2} d\eta$$

For $(z,s) \in \mathbb{H}_n$ and $\lambda \neq 0$ define what turn out to be unitary operators $U^\lambda_{(z,s)}$ on $\mathcal{H}_\lambda$ by

$$(2.2)\qquad U^\lambda_{(z,s)}F(\zeta) = F(\zeta - \bar{z})e^{i\lambda s + 2\lambda(\zeta\cdot\bar{z} - |z|^2/2)} \qquad \text{if } \lambda > 0$$

and $$U^{\lambda}_{(z,s)}F(\zeta) = F(\zeta + z)e^{i\lambda s+2\lambda(\zeta\cdot z+|z|^2/2)} \quad \text{if} \quad \lambda < 0 .$$

The one parameter family $\{U^{\lambda}\}_{\lambda\neq 0}$ can be shown to give all irreducible representations of $\mathbb{H}_n$ (except those trivial on the center which can be ignored here) and for $f \in L^1(\mathbb{H}_n)$ the operator family

$$\hat{f}(\lambda) = \int_{\mathbb{H}_n} f(w)U^{\lambda}_w \, dw$$

is by definition the Fourier transform of f. We need:

$$(2.3) \qquad \left.\begin{aligned} (Z_jf)^{\wedge}(\lambda)F &= -2\lambda\hat{f}(\lambda)(\zeta_jF) \\ (\overline{Z}_jf)^{\wedge}(\lambda)F &= \hat{f}(\lambda)\left(\frac{\partial F}{\partial\zeta_j}\right) \end{aligned}\right\} \quad \text{when} \quad \lambda > 0$$

$$\left.\begin{aligned} (Z_jf)^{\wedge}(\lambda)F &= -\hat{f}(\lambda)\left(\frac{\partial F}{\partial\zeta_j}\right) \\ (\overline{Z}_jf)^{\wedge}(\lambda)F &= -2\lambda\hat{f}(\lambda)(\zeta_jF) \end{aligned}\right\} \quad \text{when} \quad \lambda < 0$$

$$\text{and} \quad (Sf)^{\wedge}(\lambda) = -i\lambda\hat{f}(\lambda)$$

in particular:

$$(2.4) \qquad \widehat{L_{\gamma}f}(\lambda)F_{\alpha,\lambda} = -4|\lambda|(2|\alpha| + n + \gamma \,\mathrm{sgn}\,\lambda)\,\hat{f}(\lambda)F_{\alpha,\lambda}$$

Plancherel's Theorem can be stated as follows:

Let A denote the Hilbert space of all one parameter families $\{A(\lambda)\}_{\lambda\neq 0}$ of operators on $\{H_{\lambda}\}$ which are Hilbert-Schmidt for a.e. $\lambda \in \mathbb{R}$, with $\|A(\lambda)\|_2$ measurable and with norm

$$\left(\frac{2^{n-1}}{\pi^{n+1}} \int_{-\infty}^{\infty} \|A(\lambda)\|_2^2 |\lambda|^n \, d\lambda\right)^{1/2} < \infty$$

The Fourier transform extends to an isometry from $L^2(\mathbb{H}_n)$ onto A.

We will also use the following form of the Inversion Theorem:

(2.5) i) If $\Sigma_{\alpha} \int_{-\infty}^{\infty} \|\hat{f}(\lambda)F_{\alpha,\lambda}\| |\lambda|^n \, d\lambda < \infty$ then

$$(2.6) \qquad \frac{2^{n-1}}{\pi^{n+1}} \int_{-\infty}^{\infty} \mathrm{tr}(U^{\lambda}_{w^{-1}}\hat{f}(\lambda))|\lambda|^n \, d\lambda = f(w) \quad \text{a.e.}$$

ii) The hypothesis above is satisfied if $f \in S^1_{2(n+2)}(\mathbb{H}_n)$:

(2.7) $$\Sigma_\alpha \int_{-\infty}^{\infty} \|\hat{f}(\lambda)F_{\alpha,\lambda}\| |\lambda|^n \, d\lambda \le C \sum_{k=0}^{n+2} \|L_0^k f\|_{L^1} .$$

For radial f, matters simplify.

2.8 Proposition

i) For $f \in L^1(\mathbb{H}_n)$ of the form $f(z, s) = g(|z|, s)$

$$\hat{f}(\lambda)F_{\alpha,\lambda} = R_{|\alpha|}(\lambda)F_{\alpha,\lambda}, \text{ where}$$

$$R_m(\lambda) = \frac{1}{\binom{m+n-1}{m}} \iint f(z, s)e^{i\lambda s} L_m^{(n-1)}(2|\lambda||z|^2)e^{-|\lambda||z|^2} \, dzds$$

(the $L_m^{(n-1)}$ are Laguerre polynomials -- see §1.)

ii) Conversely, if there are scalars $R_m(\lambda)$ with

$$\hat{f}(\lambda)F_{\alpha,\lambda} = R_{|\alpha|}(\lambda)F_{\alpha,\lambda} \text{ and}$$

$$\Sigma_m \binom{m+n-1}{m} \int_{-\infty}^{\infty} |R_m(\lambda)||\lambda|^n \, d\lambda < \infty \quad \text{then}$$

$$\frac{2^{n-1}}{\pi^{n+1}} \Sigma_m \int_{-\infty}^{\infty} e^{-i\lambda s} R_m(\lambda) L_m^{(n-1)}(2|\lambda||z|^2)e^{-|\lambda||z|^2} |\lambda|^n \, d\lambda = f(z,s) \text{ a.e. }.$$

What we will actually be using is the identity

(2.9) $$\Sigma_{|\alpha|=m}(U^\lambda_{(z,s)}F_{\alpha,\lambda}, F_{\alpha,\lambda}) = e^{i\lambda s} L_m^{(n-1)}(2|\lambda||z|^2)e^{-|\lambda||z|^2}$$

from which (2.8 ii) follows. It is easy to prove (2.8 i) and (2.9) using the Laplace transform, (1.8) and (2.1).

Given $f \in S(\mathbb{R}^{2n+1})$ let $\tilde{f}(z,\lambda)$ denote the Fourier transform of f in the s variable, $I_s^{1/2}f$ the fractional integral defined by

(2.10) $$(I_s^{1/2}f)^\sim(z,\lambda) = \frac{1}{|\lambda|^{1/2}} \tilde{f}(z,\lambda).$$

We will need the following technical result, the proof of which rests on the type of estimates used to prove (2.7). (see [4])

(2.11) Lemma If $f \in S(\mathbb{R}^{2n+1})$, $I_s^{1/2}f$ satisfies the hypothesis in the inversion theorem: in fact

$$\Sigma_\alpha \int_{-\infty}^{\infty} \|\widehat{I_s^{1/2}f}(\lambda)F_{\alpha,\lambda}\| |\lambda|^n \, d\lambda \le C(\|L_0^{n+1}f\|_{L^1} + \|f\|_{L^1}).$$

Proof: We use the inequalities: $\|\hat{f}(\lambda)F_{\alpha,\lambda}\| \le \|f\|_{L^1}$ and

$$((2|\alpha|+n)|\lambda|)^{n+1}\|\hat{f}(\lambda)F_{\alpha,\lambda}\| \le \|L_0^{n+1}f\|_{L^1}; \quad \text{pick} \quad 1 < \rho < 2$$

$$\sum_{|\alpha|>0} \int_{-\infty}^{\infty} \|\hat{f}(\lambda)F_{\alpha,\lambda}\| \; |\lambda|^{n-1/2}\, d\lambda$$

$$\le C \sum_{m=1}^{\infty} \binom{m+n-1}{m} \left(\int_0^{m^{-\rho}} \lambda^{n-1/2} d\lambda + \frac{1}{m^{n+1}} \int_{m^{-\rho}}^{\infty} \frac{\lambda^{n-1/2}}{\lambda^{n+1}} \, d\lambda \right)$$

$$< C \sum_{m=1}^{\infty} m^{n-1} (m^{-\rho(n+\frac{1}{2})} + m^{-n-1-\rho/2}) < \infty$$

The preceding results will be used in the following form.

2.12 Lemma Given measurable functions $a_\alpha(\lambda)$ with

$$|a_\alpha(\lambda)| < C \quad \text{for all} \quad \alpha, \lambda$$

i) there is a bounded operator $A : L^2(\mathbb{H}_n) \to L^2(\mathbb{H}_n)$ with $\widehat{Af}(\lambda)F_{\alpha,\lambda} = a_\alpha(\lambda)\, \hat{f}(\lambda)F_{\alpha,\lambda}$.

ii) There is a tempered distribution K (on $\mathbb{R}^{2n+1}$) with

$$Af = f*K \quad \text{for} \quad f \in S(\mathbb{R}^{2n+1})$$

($*$ denotes the group convolution defined in (1.4)).

iii) One can define the fractional integral $I_s^{1/2}K$ by duality from (2.11).

iv) If in addition $a_\alpha(\lambda) = a_{|\alpha|}(\lambda)$ and $\tilde{K}$ denotes the Fourier transform of K in the s variable

$$\tilde{K} = \frac{2^{n-1}}{\pi^{n+1}} \sum_{m=0}^{\infty} a_m(\lambda)|\lambda|^n L_m^{(n-1)}(2|\lambda||z|^2)e^{-|\lambda||z|^2}.$$

Write $K = \lim_{\varepsilon\to 1} K_\varepsilon$ where

$$\tilde{K}_\varepsilon = \frac{2^{n-1}}{\pi^{n+1}} \sum_{m=0}^{\infty} \varepsilon^m a_m(\lambda)|\lambda|^n L_m^{(n-1)}(2|\lambda||z|^2)e^{-|\lambda||z|^2} \quad ; \tag{2.13}$$

the *distributions* K_ε *have the following properties:*

v) K_ε is equal to a tempered distribution in s, $K_\varepsilon(z,\cdot)$ is analytic in ε and is real analytic in $z = (x,y)$,

iv) when $z \ne 0$ and $|\varepsilon| < \frac{1}{3}$ the distribution $K_\varepsilon(z,\cdot)$ is equal to a smooth function.

Proof: i) follows directly from Plancherel's theorem.

ii) From (2.4) we see that L_0 commutes with A whence $\|Af\|_\beta \le c\,\|f\|_\beta$ for any $\beta \ge 0$ so if $f \in S$, $Af \in C^\infty$; also

$$(2.14) \qquad \int \Sigma_\alpha \|\widehat{Af}(\lambda)F_{\alpha,\lambda}\|\,|\lambda|^n\,d\lambda \le C\,\|f\|_{S^1_{2(n+2)}} \qquad \text{by (2.7)}$$

so the inversion theorem applies to Af and we have $|Af(0)| \le C\|f\|_{S^1_{2(n+2)}}$ which shows that there is a tempered distribution K such that $Af(0) = K(f)$. From the inversion formula

$$Af(w) = Ag(0) \quad \text{where} \quad g(v) = f(wv^{-1})$$

thus

$$Af(w) = (f*K)(w).$$

iii) is clear from ii) and Lemma (2.11).

iv) It will be convenient to write $U^\lambda_{(z,s)} = e^{i\lambda s}\,W^\lambda_z$ (see the formula (2.2) for $U^\lambda_{(z,s)}$) with W^λ_z a unitary operator on H_λ

From the absolute convergence in (2.14) we know that

$$K(f) = \frac{2^{n-1}}{\pi^{n+1}}\ \Sigma_\alpha \iint \tilde{f}(z,-\lambda)a_\alpha(\lambda)|\lambda|^n\,(W^\lambda_z F_{\alpha,\lambda}, F_{\alpha,\lambda})\,dz d\lambda$$

this integral is absolutely convergent since

$$(2.15) \qquad |(W^\lambda_z F_{\alpha,\lambda}, F_{\alpha,\lambda})| \le 1$$

if $a_\alpha(\lambda) = a_{|\alpha|}(\lambda)$ iv) now follows from (2.9).

v) Note that (2.15) translates, via (2.9) into

$$|L^{(n-1)}_m(2x)e^{-x}| \le \binom{m+n-1}{m} \quad \text{for} \quad x \ge 0$$

whence

$$|\tilde{K}_\varepsilon(z,\lambda)| \le C\,|\lambda|^n \sum_{m=0}^{\infty}\binom{m+n-1}{m}|\varepsilon|^m = C\,\frac{|\lambda|^n}{(1-|\varepsilon|)^n}$$

vi) Here we use the estimate (1.7) to show

$$|\tilde{K}_\varepsilon(z,\lambda)| \le C\,|\lambda|^n\,\frac{1}{(1-|\varepsilon|)^n}\exp\Big(\frac{3|\varepsilon|-1}{1-|\varepsilon|}\,|\lambda|\,|z|^2\Big).$$

3. The Cauchy Problem. In this section we solve

$$(3.1)\qquad \left[\frac{\partial^2}{\partial t^2} - L_\gamma\right] u(t, w) = f(t, w) \qquad t \in \mathbb{R},\ w \in \mathbb{H}_n$$
$$u(0, w) = u_0(w)$$
$$\frac{\partial u}{\partial t}(0, w) = u_1(w)$$

(assuming $|\gamma| < n$).

For simplicity, we take $f \in S(\mathbb{R}^{2n+2})$, $u_0, u_1 \in S(\mathbb{R}^{2n+1})$. Write $g(t) = (\hat{u}(t, \lambda)F_{\alpha,\lambda}, F)$ (α, λ fixed, $F \in H_\lambda$) ; then g must satisfy

$$(3.2)\qquad g''(t) + 4|\lambda|(2|\alpha| + n + \gamma \operatorname{sgn} \lambda)g(t) = (\hat{f}(t, \lambda)F_{\alpha,\lambda}, F)$$
$$g(0) = (\hat{u}_0(\lambda)\ F_{\alpha,\lambda}, F)$$
$$g'(0) = (\hat{u}_1(\lambda)\ F_{\alpha,\lambda}, F)$$

The unique solution is:

$$g(t) = \cos(b_\alpha(\lambda)t)(\hat{u}_0(\lambda)F_{\alpha,\lambda}, F) + \frac{\sin(b_\alpha(\lambda)t)}{b_\alpha(\lambda)}(\hat{u}_1(\lambda)F_{\alpha,\lambda}, F)$$
$$+ \int_0^t \frac{\sin(b_\alpha(\lambda)(t-t'))}{b_\alpha(\lambda)}(\hat{f}(t', \lambda)F_{\alpha,\lambda}, F)dt'$$

where $b_\alpha(\lambda) = \sqrt{4|\lambda|(2|\alpha| + n + \gamma \operatorname{sgn} \lambda)}$.

Now, for each t, let A_t be the operator of Lemma (2.12) corresponding

$$\text{to}\quad a_\alpha(\lambda) = \frac{\sin(b_\alpha(\lambda)t)}{b_\alpha(\lambda)} .$$

Note that the operator corresponding to $a_\alpha'(\lambda) = \operatorname{cob}(b_\alpha(\lambda)t)$ is then $\frac{d}{dt} A_t$ and the unique solution to (3.1) is therefore

$$(3.3)\qquad u(t, \cdot) = \frac{dA_t}{dt}\ u_0 + A_t u_1 + \int_0^t A_{t-t'}\ f(t', \cdot)dt'$$

From (2.12) we already know that

$$\|A_t u_1\|_\beta \le C\ |t|\ \|u_1\|_\beta$$

and for any t

$$\left\| \frac{dA_t}{dt} u_0\right\|_\beta \le C\ \|u_0\|_\beta$$

but about A_t we can say more: since

$$\frac{\sqrt{4|\lambda|(2|\alpha|+n)}}{b_\alpha(\lambda)} \leq \sqrt{\frac{n}{n-\gamma}}$$

$$\|A_t u_1\|_1 \leq \max(|t|, 1)\ C_{n,\gamma}\|u_1\|_0 ,$$

it follows that

$$\text{(3.4)} \qquad \|u(t, \cdot)\|_{\beta+1} \leq C_{n,\gamma,\beta}\ [\|u_0\|_{\beta+1} + \max(1\ ,\ |t|)\ \|u_1\|_\beta + \\ + |\int_0^t \|f(t', \cdot)\|_\beta \max(1, |t-t'|) dt'|].$$

Similarly

$$\text{(3.5)} \qquad \|\frac{\partial u}{\partial t}(t, \cdot)\|_\beta \leq C\ [\|u_0\|_{\beta+1} + \|u_1\|_\beta + |\int_0^t \|f(t', \cdot)\|_\beta\ dt'|].$$

Let us return to (3.3) for a moment. The operator A_t appearing there is a convolution operator on the Heisenberg group, with kernel L_t. If $\mathbb{H}(t)$ denotes the Heaviside function then

$E(t, \cdot, \cdot) = H(t)L_t$ is the forward fundamental solution for our wave equation (thought of as an equation "with constant coefficients" on $\mathbb{H}_n$).

$E(t, \cdot, \cdot) = \lim_{\varepsilon\to 1} E_\varepsilon(t, \cdot, \cdot)$ where $E_\varepsilon(t, z, \cdot)$ is a tempered distribution in s with Fourier transform (in the s variable) $\tilde{E}_\varepsilon(t, z, \lambda)$ equal to

$$\text{(3.6)} \qquad C\sum_{m=0}^{\infty} \varepsilon^m \frac{H(t)\sin(2t\sqrt{|\lambda|(2m+n+n+\gamma \operatorname{sgn} \gamma)}}{\sqrt{|\lambda|(2m+n+\gamma \operatorname{sgn} \lambda)}}\ |\lambda|^n L_m^{(n-1)}(2|\lambda||z|^2)e^{-|\lambda||z|^2}.$$

From here on, the problem is principally to extract information from this formula.

4. <u>Integral Formulas for the Fundamental Solution</u>. To understand E, we try to write the series in (3.6) as an integral.

Let $E = E^+ + E^-$ where E^+ is such that

$$\tilde{E}^+(t, z, \lambda) = \chi_{(0,\infty)}(\lambda)\ \tilde{E}(t, z, \lambda).$$

To simplify matters, we use the invariance of our solution under dilations of $\mathbb{H}_n$ and rotations in the z-variable to obtain a distribution $K^+(t, s)$ which satisfies

$$E^+(t, z, s) = \frac{1}{|z|^{2n+1}} K^+\left(\frac{t^2}{|z|^2}, \frac{s}{|z|^2}\right) \quad \text{for} \quad t > 0, \ z \neq 0$$

(we have written this formally to avoid some cumbersome notation);

(4.1) K^+ is then $-i \frac{\partial}{\partial s} J$ where $J = \lim_{\varepsilon \to 1} J_\varepsilon$ with

$$\tilde{J}_\varepsilon(t, \lambda) = cH(t) \sum_{m=0}^{\infty} \varepsilon^m \chi_{(0,\infty)}(\lambda) \frac{\sin\sqrt{4t\lambda(2m+n+\gamma)}}{\sqrt{4(2m+n+\gamma)\lambda}} \lambda^{n-1} L_m^{(n-1)}(2\lambda) e^{-\lambda}.$$

In view of part (vi) of Lemma 2.12 we first assume $z \neq 0$ and $|\varepsilon| < \frac{1}{3}$ and see how far we can go in computing J_ε ; it will become clear on the way that the manipulations below are all justified.

Let $\tilde{j}_1(\lambda) = \chi_{(0,\infty)}(\lambda) \lambda^{n-1} L_m^{(n-1)}(2\lambda) e^{-\lambda/2}$

$$\tilde{j}_2(\lambda) = \chi_{(0,\infty)}(\lambda) \sqrt{t} \frac{\sin(2b\sqrt{\lambda})}{2b\sqrt{\lambda}} e^{-\lambda/2}, \quad b = \sqrt{(2m+n+\gamma)t}$$

$\tilde{j}_1$ and $\tilde{j}_2$ are integrable; we compute their (inverse) Fourier transform. From (1.8)

$$j_1(s) = \frac{(m+n-1)!}{m!} \left(\frac{1}{2} - is\right)^{-m-n} \left(-\frac{3}{2} - is\right)^m$$

For j_2 we start from $\int_0^\infty e^{-x^2} \sin 2bx\,dx = e^{-b^2} \int_0^b e^{\beta^2}\,d\beta$ change variables to obtain

$$j_2(s) = \frac{\sqrt{t}}{\frac{1}{2} - is} \int_0^1 \exp\left(b^2 \frac{\beta^2 - 1}{\frac{1}{2} - is}\right) d\beta.$$

$$\text{(4.2)} \qquad J_\varepsilon(t,s) = \sum_{m=0}^{\infty} \varepsilon^m \int_{-\infty}^{\infty} j_1(s-s') j_2(s')\,ds'$$

in the resulting integral make the change of variable

$u = \dfrac{1}{\frac{1}{2} - is'}$ to obtain, for $t > 0$

$$J_\varepsilon(t, s) = C\sqrt{t}\int_0^1\int_\Gamma e^{(n+\gamma)(\beta^2-1)tu}\,\frac{u^{n-1}}{(u(1-is)-1)^n}\sum\frac{(m+n-1)!}{m!}\times$$
$$\times\left[-\varepsilon e^{2(\beta^2-1)tu}\,\frac{u(1+is)+1}{u(1-is)-1}\right]^m du\,d\beta$$

where Γ is the circle (in the complex plane) of radius 1 centered at 1. It is shown below that for $u \in \Gamma$ (see (5.2)

$$\left|e^{-2tu}\,\frac{u(1+is)+1}{u(1-is)-1}\right| \le 3 \quad \text{if} \quad t \ge 0$$

therefore for $|\varepsilon| < \frac{1}{3}$ the series inside the integral coverges absolutely and we can write

$$(4.3) \qquad J_\varepsilon(t, s) = c\sqrt{t}\int_0^1 G_\varepsilon((1-\beta^2)t, s)\,d\beta, \quad \text{with}$$

$$(4.4) \qquad G_\varepsilon(t, s) = H(t)\int_\Gamma e^{-(n+\gamma)tu}\,u^{n-1}\,\frac{du}{[u(1-is)-1+\varepsilon e^{-2tu}(u(1+is)+1)]^n}$$

G is quite nice: we will eventually have detailed knowledge about it; note that a change of variable in (4.3) shows that

$$(4.5) \qquad J_\varepsilon(t, s) = c\int_0^t \frac{G_\varepsilon(t', s)}{\sqrt{t-t'}}\,dt'$$

so J_ε is a fractional integral of G_ε in the t variable. (If τ is the dual variable to t, on the Fourier transform side

$$(4.6) \qquad \tilde{J}_\varepsilon(\tau, s) = c|\tau|^{-1/2}(1-i\ \mathrm{sgn}\ \tau)\ \tilde{G}_\varepsilon(t, s)).$$

We extend the definition of G_ε to all $|\varepsilon| < 1$ by requiring (4.5) to hold; in other words, G_ε is an appropriate fractional derivative of J_ε (which makes sense: it is the convolution in t of two distributions which vanish for $t < 0$).

The G_ε then converge to G, the fractional derivative of J. Along with G, and with negligible extra effort, we will also study a fractional integral of E in the s variable; this is to circumvent certain later difficulties which arise when trying to obtain information about E from G.

We know from part iii) of (2.12) that we can talk about $I_s^{1/2}E^+$; again, using the dilation and rotation invariance we can find a distribution $H(t, s)$ which is such that

$$H(t, s) = \frac{1}{\sqrt{t}} I_s^{1/2} K(t, s) \quad \text{for} \quad t > 0 \tag{4.7}$$

(the $\frac{1}{\sqrt{t}}$ in this relation is inserted for cosmetic reasons to become clear later). H will then be the limit of H_ε with

$$\tilde{H}_\varepsilon(t,\lambda) = c\, \chi_{(0,\infty)}(t)\chi_{(0,\infty)}(\lambda) \sum_{m=0}^{\infty} \varepsilon^m \frac{\sin\sqrt{4t\lambda(2m+n+\gamma)}}{\sqrt{4t(2m+n+\gamma)}} \lambda^{n-1} L_m^{(n-1)}(2\lambda)e^{-\lambda} \tag{4.8}$$

Computations very similar to the ones done for J yield, for $t > 0$ and $|\varepsilon| < \frac{1}{3}$

$$H_\varepsilon(t, s) = c \int_\Gamma e^{-(n+\gamma)tu}\, u^{n-\frac{1}{2}} \frac{du}{[u(1-is)-1+\varepsilon e^{-2tu}(u(1+is)+1)]^n}. \tag{4.9}$$

5. <u>The Zeros of a Certain Analytic Function</u>. Looking at (4.4) and (4.9) we are faced with the question: how many zeros u does

$$f(u, t, s, \varepsilon) = u(1-is) -1 + \varepsilon e^{-2tu}(u(1 + is) + 1) \tag{5.1}$$

have in Γ ? Answer: for any $t > 0$, $s \in \mathbb{R}$ and $|\varepsilon| \le 1$ there is exactly one (simple) zero $u(t, s, \varepsilon)$ in the (open) right half plane; for $|\varepsilon| < \frac{1}{3}$ it lies inside Γ. To see this, for a, b real let $\Gamma(a, b)$ be the circle centered on the real axis passing through a and b. We shall use Rouché's Theorem: consider $f(u) = f(u, t, s, \varepsilon)$ and $g(u) = u(1-is)-1$ as well as the circle $\Gamma(0, M)$ where $M > \frac{1+|\varepsilon|}{1-|\varepsilon|}$.
The image of $\Gamma(0, M)$ under the map $u \to \frac{u(1+is)+1}{u(1-is)-1}$ is $\Gamma\left(-1, \frac{M+1}{M-1}\right)$.
Therefore, if $u \in \Gamma(0, M)$ we have

$$|f(u) - g(u)| = |\varepsilon e^{-2tu}(u(1+is)+1)| \le |\varepsilon| \frac{M+1}{M-1} |u(1-is)-1| < |g(u)|. \tag{5.2}$$

We conclude that f has exactly one zero inside $\Gamma(0, M)$. Since $\frac{\partial f}{\partial u}(u(t, s, \varepsilon), t, s, \varepsilon) \neq 0$ it follows from the implicit function theorem that $u(t, s, \varepsilon)$ is analytic in ε for $|\varepsilon| < 1$, real analytic in t, s.

For $|\varepsilon| < \frac{1}{3}$ we now know that

$$(5.3) \quad G_\varepsilon(t, s) = c \text{ Residue of } \frac{u^{n-1} e^{-(n+\gamma)tu}}{[f(u,t,s,\varepsilon)]^n} \text{ at } u(t, s, \varepsilon).$$

It is a simple matter to show that (5.3) remains true for all $|\varepsilon| < 1$.

6. Summary of Part I. It may be worthwhile to restate briefly what we have found out so far about the fundamental solution E, if only to set down the notation we want to keep.

The function

$$(6.1) \qquad f(u, t, s, \varepsilon) = u(1 - is) - 1 + \varepsilon e^{-2tu}(u(1 + is) + 1)$$

has a unique zero $u(t, s, \varepsilon)$ in the right half plane $\mathcal{Re}\ u > 0$. For $t > 0$, $s \in \mathbb{R}$, let

$$(6.2) \quad G_\varepsilon(t, s) = \text{Residue of } \frac{u^{n-1} e^{-(n+\gamma)tu}}{[f(u, t, s, \varepsilon)]^n} \text{ at } u(t, s, \varepsilon)$$

$G_\varepsilon^-(t, s)$ = same formula, with γ, s replaced by $-\gamma$, $-s$

$$H_\varepsilon(t, s) = \text{Residue of } \frac{u^{n-\frac{1}{2}} e^{-(n+\gamma)tu}}{[f(u,t,s,\varepsilon)]^n} \text{ at } u(t, s, \varepsilon)$$

$H_\varepsilon^-(t, s)$ = same as H_ε, with γ, s replaced by $-\gamma$, $-s$.

(6.3) Proposition: Let E be the forward fundamental solution described in §3.

$E = \lim_{\varepsilon \to 1} E_\varepsilon$ as distribution, where

for $z \neq 0$, $t > 0$ and any $|\varepsilon| < 1$ $E_\varepsilon(t, z, s)$ is a function, (6.4) $$E_\varepsilon(t, z, s) = \frac{1}{|z|^{2n+1}} K\left(\frac{t^2}{|z|^2}, \frac{s}{|z|^2}\right)$$

and for K_ε we have two formulas:

(6.5) $$K_\varepsilon(t, s) = C \frac{\partial}{\partial s} \int_0^t \frac{G_\varepsilon(t',s)-G_\varepsilon^-(t',s)}{\sqrt{t-t'}} dt'$$

(6.6) $$K_\varepsilon(t, s) = c\sqrt{t}\; D_s^{1/2}[H_\varepsilon(t, s) + H_\varepsilon^-(t, s)]$$

($D_s^{1/2}$ is multiplication on the Fourier transform side by $|\lambda|^{1/2}$, λ the dual variable to s; G_ε, H_ε, etc are fefined in (6.2)). Furthermore, as $\varepsilon \to 1$, G_ε, H_ε, K_ε converge to tempered distributions G, H, K satisfying (6.4), (6.5), (6.6) in the appropriate sense.

II The Singular Set

7. <u>What the Residue Formulas Tell Us</u>. To find the singular supports of the distributions G and H we first study the pointwise limits

$$\lim_{\varepsilon\to 1} G_\varepsilon(t, s) \quad\text{and}\quad \lim_{\varepsilon\to 1} H_\varepsilon(t, s).$$

Throughout most of this section t and s will be fixed and we will write $f_\varepsilon(u)$, u_ε, G_ε instead of $f(u, t, s, \varepsilon)$, etc; also let

(7.1) $$f(u) = f(u, t, s, 1),\quad g(u) = u^{n-1}e^{-(n+\gamma)tu},\quad h(u) = u^{n-\frac{1}{2}} e^{-(n+\gamma)tu}$$

Near u_ε, $\frac{u-u_\varepsilon}{f_\varepsilon(u)}$ is analytic and G_ε is the coefficient of $(u-u_\varepsilon)^{n-1}$ in the expansion of $g(u)\left(\frac{u-u_\varepsilon}{f_\varepsilon(u)}\right)^n$

If $\left(\frac{u-u_\varepsilon}{f_\varepsilon(u)}\right)^n = \sum_{k=0}^{\infty} a_k(\varepsilon)\,(u-u_\varepsilon)^k$ then

(7.2) $$G_\varepsilon = \sum_{k=0}^{n-1} a_{n-k-1}(\varepsilon) \frac{g^{(k)}(u_\varepsilon)}{k!}$$

(7.3) $$H_\varepsilon = \sum_{k=0}^{n-1} a_{n-k-1}(\varepsilon) \frac{h^{(k)}(u_\varepsilon)}{k!}.$$

In section 8, we show that $\lim_{\varepsilon \to 1} u_\varepsilon = u_1$ exists, with $u_1 = u_1(t, s)$ a (finite) point in the closed right half plane satisfying $f(u_1) = 0$; the derivatives $g^{(k)}(u_\varepsilon)$, $h^{(k)}(u_\varepsilon)$ of order $k \le n - 1$ will all tend to $g^{(k)}(u_1)$, $h^{(k)}(u_1)$ (even if $u_1 = 0$). What about $a_k(\varepsilon)$? For later purposes, we give the formulas in greater detail than needed for this argument.

$$(7.4)\quad \frac{u-u_\varepsilon}{f_\varepsilon(u)} = \frac{1}{f'(u_\varepsilon)}\left[1 + \sum_{m=1}^{\infty} b_m(\varepsilon)(u-u_\varepsilon)^m\right]^{-1}, \quad b_m(\varepsilon) = \frac{f_\varepsilon^{(m+1)}(u_\varepsilon)}{(m+1)!\,f_\varepsilon'(u_\varepsilon)}$$

$$(7.5)\qquad = \frac{1}{f'(u_\varepsilon)}\left[1 + \sum_{m=1}^{\infty} c_m(\varepsilon)(u-u_\varepsilon)^m\right] \quad \text{where}$$

$$c_1 = -b_1,\ c_2 = -b_2 + b_1^2,\ -c_m = b_m + b_{m-1}c_1 + \ldots + b_1 c_{m-1}$$

Now

$$\left(\frac{u-u_\varepsilon}{f_\varepsilon(u)}\right)^n = \frac{1}{(f_\varepsilon'(u_\varepsilon))^n}\,[1 + c_1(\varepsilon)(u-u_\varepsilon) + \ldots + c_{n-1}(u-u_\varepsilon)^{n-1}]^n + O(u-u_\varepsilon)^n$$

$$= \frac{n!}{(f'(u_\varepsilon))^n} \sum_{j_0+j_1+\ldots+j_{n-1}=n} \left[\frac{1}{j_0!\,j_1!\ldots j_{n-1}!}\, c_1^{j_1} \ldots c_{n-1}^{j_{n-1}} \times \times (u-u_\varepsilon)^{j_1+2j_2+\ldots+(n-1)j_{n-1}}\right] + \ldots$$

Whence

$$(7.6)\quad a_k(\varepsilon) = \frac{n!}{(f_\varepsilon'(u_\varepsilon))^n} \sum_{\substack{j_0+j_1+\ldots+j_{n-1}=n \\ j_1+2j_2+\ldots+(n-1)j_{n-1}=k}} \frac{1}{j_0!\,j_1!\ldots j_{n-1}!}\, c_1^{j_1}(\varepsilon)\, c_2^{j_2}(\varepsilon)\ldots c_{n-1}^{j_{n-1}}(\varepsilon)$$

On the other hand,

$$(7.7)\quad f_\varepsilon(u) = u(1 - is) - 1 + \varepsilon e^{-2tu}(u(1 + is) + 1)$$

$$(7.8)\quad f_\varepsilon'(u) = 1 - is + \varepsilon e^{-2tu}[-2t(u(1 + is) + 1) + 1 + is]$$

and for $m \ge 2$

$$(7.9)\quad f_\varepsilon^{(m)}(u) = (-2t)^{m-1}\,\varepsilon e^{-2tu}\,[-2t(u(1 + is) + 1) + m(1 + is)]$$

From (7.7) and (7.8)

$$(7.10)\qquad f'_\varepsilon(u_\varepsilon) = 2\ \frac{1-t[1+2u_\varepsilon is-u_\varepsilon^2(1+s^2)]}{u_\varepsilon(1+is)+1}$$

So $\lim\limits_{\varepsilon\to 1} f'_\varepsilon(u_\varepsilon) = f'(u_1)$ and similarly

$$(7.11)\qquad \lim_{\varepsilon\to 1} f_\varepsilon^{(m)}(u_\varepsilon) = f_\varepsilon^{(m)}(u_1) \quad \text{for any} \quad m \geq 0.$$

This, together with (7.4)-(7.6) proves that as $\varepsilon \to 1$ the coefficients $a_k(\varepsilon)$ tend to a finite limit provided u_1 is not a zero of f' ; in that case

$$(7.12)\qquad \lim_{\varepsilon\to 1} G_\varepsilon = \sum_{k=0}^{n-1} a_{n-1-k}(1)\,\frac{g^{(k)}(u_1)}{k!}$$

and similarly for H.

The problem now is to find the set of points (t, s) for which the function $u \mapsto f(u, t, s)$ admits multiple zeros and also to find a way of distinguishing among these multiple zeros the ones arising as $\lim\limits_{\varepsilon\to 1} u(t, s, \varepsilon)$.

8. <u>More on the Zeros of our Analytic Function</u>. Fix t, s ; let $\varepsilon(u) = \frac{1-u(1-is)}{1+u(1+is)}\, e^{2tu}$ *and let us try to find the set* B of points u in the right half plane with $|\varepsilon(u)| < 1$. Write $a + ib = u$; then $|\varepsilon(u)|^2 < 1$ is the same as:

$$(8.1)\qquad \left(b - \frac{s}{1+s^2}\right)^2 < \frac{4a}{(1+s^2)(e^{4ta}-1)} - \left(a - \frac{1}{1+s^2}\right)^2$$

B turns out to be a bounded simply connected region such that:

(8.4) if $t > 1 + s^2$ B is away from the b-axis;

if $t \leqq 1 + s^2$ the boundary of B and the b-axis have in common the segment of points $(0, b)$ with

$$(8.5)\qquad \left(b - \frac{s}{1+s^2}\right)^2 \leqq \frac{1}{1+s^2}\left(\frac{1}{t} - \frac{1}{1+s^2}\right)$$

The function $u \mapsto \varepsilon(u)$ defines a comformal mapping of B onto the unit disc D: its inverse is $\varepsilon \mapsto u_\varepsilon$. Since the boundary of B

is well behaved, u_ε extends to a homeomorphism of $\overline{D}$ onto $\overline{B}$. In particular $u_1(t, s) = \lim_{\varepsilon\to 1} u(t, s, \varepsilon)$ exists, is a point on the boundary of B and is such that $\varepsilon(u_1) = 1$ i.e. $f(u_1) = 0$.

Let us start to put all this to work:

since $u_1(t, s)$ is unique and since $f(0) = 0$, $u_1(t, s) = 0$ whenever t and s are such that $0 \in$ boundary of B ; from (8.5) we conclude

(8.6) $$u_1(t, s) = 0 \text{ if and only if } t \le 1$$

From this, it is quite easy to obtain some information about the fundamental solution E (the result will be improved upon in §10); looking at (7.8) we see that $f'(0) = 0$ if and only if $t = 1$; from the formula (7.12) for H we conclude

(8.7) $$\text{if } t < 1 \ \lim_{\varepsilon\to 1} H_\varepsilon(t, s) = 0; \text{ hence (see §9.)}$$

$$\text{if } t < 1 \ H(t, s) = 0.$$

The proposition (6.3) now yields:

(8.8) E is zero if $t < |z|$.

9. Finding the Singular Support of G and H. We are trying to find the set of points (t, s) with

(9.1) $$t > 0 \text{ and } f'(u_1(t, s), t, s) = 0.$$

We already know that all points (t, s) with $t = 1$ are in this set and all those with $t < 1$ are not.

Let Σ denote the set of (t, s) satisfying (9.1) with $t \ne 1$. Formula (7.10) shows that $f'(u_1) = 0$ iff

(9.2) $$t = \frac{1}{1 + 2isu_1 - u_1^2(1+s^2)}$$

If $u_1 = a + ib$, $a \geq 0$ the denominator is real provided

i) $a = 0$ or ii) $b = \frac{s}{1+s^2}$

case ii) is easily taken care of: for then

$$t = \frac{1+s^2}{1-a^2(1+s^2)^2} \quad \text{hence} \quad a < \frac{1}{1+s^2} \quad \text{and} \quad t \leq 1 + s^2 ;$$

graphing B, we find that the only possible

$u_1 = a + i \frac{s}{1+s^2}$ with $0 \leq a < \frac{1}{1+s^2}$ is $u_1 = i \frac{s}{1+s^2}$. Thus:

(9.3) if u_1 is a multiple zero of f, it must be imaginary.

To find all such u_1 we rewrite f as follows

$$f(u) = u(1 - is) - 1 + e^{-2tu}(u(1 + is) + 1)$$

$$= 2e^{-tu}\, u \sinh(tu) \,[\coth tu - \frac{1}{u} - is]$$

we assume $u_1 \neq 0$ (see 8.6) and note that if $tu = ik\pi$ $f(u) \neq 0$ so every nonzero zero of f is a zero of

$$\coth tu - \frac{1}{u} - is$$

if in addition $f'(u_1) = 0$ we must have

$$- \frac{t}{\sin h^2(tu_1)} + \frac{1}{u_1^2} = 0$$

this, together with (9.3) and (8.1) proves:

(9.4) $(t, s) \in \Sigma$ if and only if the equations:

$$s = \frac{1}{b} - \cot bt, \quad t = \frac{\sin^2 bt}{b^2}$$

admit a real solution b satisfying (8.5).

Suppose such a b exists and let $p = tb$ (we know $p \neq k\pi$); then $u_1(t, s) = i \sin^2 p/p$, $t = t(p)$ and $s = s(p)$ where

$$(9.5) \qquad t(p) = \frac{p^2}{\sin^2 p} \quad , \quad s(p) = \frac{p - \sin p \cos p}{\sin^2 p} \quad .$$

Therefore Σ consists precisely of the curve(s) $(t(p), s(p))$, $p \in \mathbb{R}$, $p \neq k\pi$. To see this let p be arbitrary real, $p \neq k\pi$ and let $t = t(p)$, $s = s(p)$; then since $i \sin^2 p/p$ is a multiple zero of f and satisfies (8.5) it must equal $u_1(t, s)$ and thus $(t, s) \in \Sigma$.

(9.6) Proposition: The distributions G, H, G^-, H^- (related to the forward fundamental solution E as described in §6) are all real analytic in the complement of the set

$$\Sigma \cup \{t=1\} = \{(t, s) : t = 1 \text{ or } (t, s) = (t(p), s(p)),\ p \text{ real} \neq k\pi\} .$$

Proof: Note that the set above is invariant under $(t, s) \to (t, -s)$ and does not depend on γ, so consider only G and H.

If $(t_0, s_0) \notin \Sigma \cup \{t = 1\}$ the function $f(u, t, s, \varepsilon)$ (analytic in all the variables) has, at $t = t_0$, $s = s_0$, $\varepsilon = 1$ the simple zero $u_1(t_0, s_0)$; it follows from the implicit function theorem that $u_1(t, s)$ is real analytic in a neighborhood of (t_0, s_0). If the neighborhood is small enough, (7.12) is valid throughout, which shows that $\lim_{\varepsilon\to 1} G_\varepsilon(t, s)$ (pointwise limit) is real analytic near (t_0, s_0). To prove the same for $\lim_{\varepsilon\to 1} H_\varepsilon(t, s)$ we first look at $t_0 < 1$, in which case the limit is identically zero near (t_0, s_0); if $t_0 > 1$ then $u_1(t_0, s_0) \neq 0$ and, away from $u = 0$ h is as nice as g (see (7.1)).

On the other hand, the implicit function theorem also implies that $u(t, s, \varepsilon)$ is continuous in all the variables in a neighborhood of $(t_o, s_0, 1)$; we may assume $\frac{\partial f}{\partial u}(u(t, s, \varepsilon),\ t, s, \varepsilon) \neq 0$ there; then G_ε is continuous in (t, s, ε) near $(t_0, s_0, 1)$. Therefore, if φ is a test function supported in a small neighborhood of (t_0, s_0) $\lim_{\varepsilon\to 1} \int G_\varepsilon \varphi = \int \lim G_\varepsilon \varphi$. We conclude that, for (t, s) near (t_0, s_0), the distribution G is equal to the real analytic function $\lim_{\varepsilon\to 1} G_\varepsilon(t, s)$.

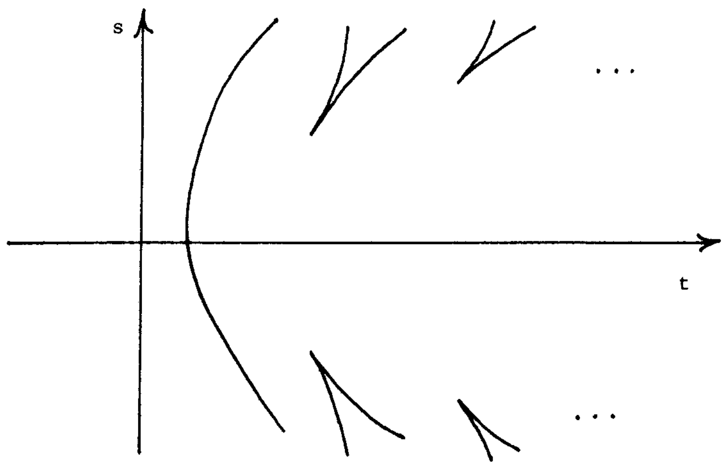

Figure 1

The set Σ is pictured in Figue 1. If we denote by $p_1, p_2, \ldots$ the numbers $p > 0$ for which $p = \tan p$, the cusp points correspond to $p = \pm p_k$, $k = 1, 2, 3, \ldots$ and lie on the parabola $t = s^2 + 1$; therefore any horizontal line intersects only finitely many of the curves making up Σ ; the same is true of any vertical line.

The line $t = 1$ is actually a red herring: even though it turns out that G and H are singular on it, this will no longer be true of K (defined in §6): it is here that the non-pseudolocal property of fractional integration in one of the variables becomes significant. This also means that we cannot immediately conclude that K is real analytic away from Σ. We will, however prove this in §10 and §11. Furthermore, it will follow from our computation of asymptotics in Part III that the singular support of K is actually equal to the set Σ, not just contained in it. In terms of the initial variables (t, z, s) on $\mathbb{R} \times \mathbb{H}_n$ this will show that for $z \neq 0$ the fundamental solution E is real analytic in the complement of the set

$$\{(t, z, s) : |z|^2 = t^2 \sin^2 p/p^2, \quad s = t^2(p - \sin p \cos p)/p^2\}$$

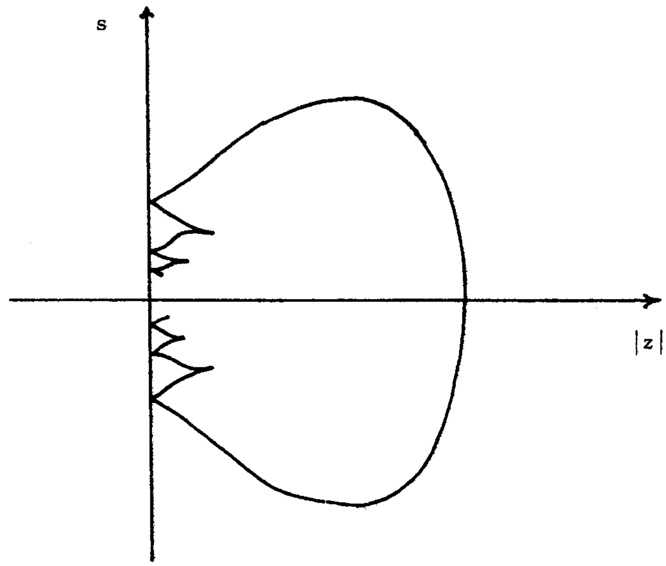

Figure 2

We show in Figure 2 a section of this cone for t constant.

In the next section we show how the singular set we have found can be recovered from the geometry of bicharacteristics.

10. Bicharacteristics. Domain of Dependence. The Hamilton-Jacobi equations are quite easy to solve explicitly in our case. The principal symbol is

$$-\tau^2 + \sum_{j=1}^{n} [(\xi_j + 2y_j\sigma)^2 + (\eta_j - 2x_j\sigma)^2] \tag{10.1}$$

So we have to solve

$$\begin{aligned} \dot{t} &= -2\tau & \dot{\tau} &= 0 \\ \dot{x}_j &= 2\xi_j + 4y_j\sigma & \dot{\xi}_j &= 4\eta_j\sigma - 8x_j\sigma^2 \\ \dot{y}_j &= 2\eta_j - 4x_j\sigma & \dot{\eta}_j &= -4\xi_j\sigma - 8y_j\sigma^2 \\ \dot{s} &= \Sigma[4(y_j\xi_j - x_j\eta_j) + 8(x_j^2 + y_j^2)\sigma] & \dot{\sigma} &= 0 \end{aligned} \tag{10.2}$$

with $(x(0), y(0), s(0)) = 0$ and, if the curve is to be in the characteristic set $\tau^2 = |\xi(0)|^2 + |\eta(0)^2|$.

If $\sigma = 0$ then ξ and η are constant and the bicharacteristics are straight lines in the plane $s = 0$:

$$\text{(10.3)} \qquad t = -2\mu\tau,\ x = 2\mu\xi,\ y = 2\mu\eta$$

$$\text{with } |\xi|^2 + |\eta|^2 = \tau^2, \quad \text{hence} \quad t^2 = |z|^2$$

If $\sigma \neq 0$ solving first for x_j, y_j, ξ_j, η_j we obtain

$$\text{(10.4)} \qquad t = -2\tau\mu$$

$$x_j = \frac{\sin(4\sigma\mu)}{2\sigma}[\xi_j(0)\cos(4\sigma\mu) + \eta_j(0)\sin(4\sigma\mu)]$$

$$y_j = \frac{\sin(4\sigma\mu)}{2\sigma}[-\xi_j(0)\sin(4\sigma\mu) + \eta_j(0)\cos(4\sigma\mu)]$$

$$\xi_j = \cos(4\sigma\mu)\ [\xi_j(0)\cos(4\sigma\mu) + \eta_j(0)\sin(4\sigma\mu)$$

$$\eta_j = \cos(4\sigma\mu)\ [-\xi_j(0)\sin(4\sigma\mu) + \eta_j(0)\cos(4\sigma\mu)]$$

$$s = \frac{\tau^2}{8\sigma^2}[8\sigma\mu - \sin(8\sigma\mu)]$$

Note that if $\tau = 0$ (hence also $\xi(0) = \eta(0) = 0$) that is, if the bicharacteristic strip starts off in the degenerate part of the characteristic set then it does indeed reduce to a point. If not, with $c = \dfrac{\tau^2}{4\sigma^2} > 0$ and the new variable $p = 4\sigma\mu$ we have

$$\text{(10.5)} \qquad t^2 = cp^2,\ |z|^2 = c\sin^2 p,\ s = c(p - \sin p\cos p)$$

Conversely, any point on the (conical) surface (10.5) is the projection of a point in the cotangent space on a bicharacteristic strip lying in the nondegenerate part of the characteristic set.

The points with $z \neq 0$ in (10.5) and (10.3) are exactly those satisfying

$$\frac{t^2}{|z|^2} = t(p), \quad \frac{s}{|z|^2} = s(p)$$

Thus, once we prove that the singular support of K is equal to Σ

we will have shown (except for points with $z = 0$) that the singular support of the fundamental solution E is equal to the cone generated by the bicharacteristics passing through the origin.

Since our equation has analytic coefficients we can now expect to be able to show using Holmgren's uniqueness theorem that the fundamental solution is zero outside the cone described above (a section of which is depicted in Figue 2). For the sake of completeness we give below the proof of this fact using the explicit formulas obtained thus far.

Let $p(s) : R \to (-\pi, \pi)$ be the inverse function to

$$s(p) : (-\pi, \pi) \to \mathbb{R}, \; s(p) = \frac{p - \sin p \cos p}{\sin^2 p} \quad \text{and let} \tag{10.6}$$

$$q(r, s) = \begin{cases} r^2 t(p(s/r^2)) & \text{if} \quad r \neq 0 \\ \pi s & \text{if} \quad r = 0 \end{cases} \tag{10.7}$$

(the function $t(p)$ is defined in (9.5)).

The set where $q(|z|, s) = t^2$ is precisely the "outer sheet" of our characteristic cone; in this proof q will play the role of the square of the Euclidean norm in the classical case.

10.8. Proposition The fundamental solution E is zero whenever $t < q(|z|, s)^{1/2}$.

First, some properties of q:

10.9. Lemma

i) q is real analytic away from $r = 0$ and continuous everywhere.

ii) $\left(\frac{\partial q}{\partial r}\right)^2 + 4r^2\left(\frac{\partial q}{\partial s}\right)^2 = 4q$ away from $r = 0$.

The proof is straightforward.

We will work with a smoothed out version of q:

$$q_\varepsilon(r, s) = q\left(\sqrt{r^2 + \varepsilon^2}, s\right)$$

q_ε is analytic, $\geq \varepsilon^2$ and

$$\left(\frac{\partial q_\varepsilon}{\partial r}\right)^2 + 4r^2\left(\frac{\partial q_\varepsilon}{\partial s}\right)^2 \le 4q_\varepsilon \tag{10.10}$$

By abuse of notation we also regard q_ε as a function on $\mathbb{H}_n$.

<u>Proof of Proposition (10.8)</u>: Let (t_0, z_0, s_0) be any point with

$$t_0 < q^{1/2}(|z_0|, s_0).$$

Choose first t_1 and then (in view of the continuity of q) $\varepsilon > 0$ so that

$$t_0 < t_1 < q^{\frac{1}{2}}\left(\sqrt{|z_0|^2 + \varepsilon^2}, s_0\right), \quad 10\varepsilon < t_1 - t_0 ;$$

then the "disc" $q_\varepsilon((z_0, s_0)^{-1} \cdot (z, s)) \le t_1^2$ in the plane $t = 0$ i.e. $\{(0, z, s) = (0, x, y, s) : q_\varepsilon(|z-z_0|, s-s_0 + 2x_0y - 2y_0x) \le t_1^2\}$ stays away from the origin. We want to construct non-characteristic deformations of this disc which sweep out a cone containing the point (t_0, z_0, s_0). Let

$$\psi(t, z, s) = - t + \ell\, [q_\varepsilon^{1/2}((z_0, s_0)^{-1} \cdot (z, s))]$$

with ℓ a nonnegative, smooth, decreasing function defined on $[0, \infty)$ with $\ell(\rho) = 0$ for $\rho \ge t_1$.

To see when the surface $\psi = 0$ is non-characteristic we compute

$$\begin{aligned} &- 1 + \sum_{j=1}^{n} [(\frac{\partial\psi}{\partial x_j} + 2y_j \frac{\partial\psi}{\partial s})^2 + (\frac{\partial\psi}{\partial y_j} - 2x_j \frac{\partial\psi}{\partial s})^2] \\ &= - 1 + \frac{1}{4q_\varepsilon} (\ell'(q_\varepsilon^{1/2}))^2[(\frac{\partial q_\varepsilon}{\partial r})^2 + 4 |z-z_0|^2 (\frac{\partial q_\varepsilon}{\partial s})^2] \\ &\le - 1 + (\ell'(q_\varepsilon^{1/2}))^2 \end{aligned}$$

So to make ψ non-characteristic it is enough to add the requirement $|\ell'| < 1$.

We can now start with $\ell \equiv 0$ and adhering to this requirement continuously deform ℓ to arrive at a function with $\ell(0)$ arbitrarily close to t_1,

$\ell(\varepsilon)$ arbitrarily close to $t_1 - \varepsilon$. (Note that

$$\psi(t, z_0, s_0) = - t + \ell(\varepsilon))$$

Holmgren's uniqueness theorem and a simple compactness argument now imply that E is 0 in a neighborhood of (t_0, z_0, s_0).

11. In Lieu of Pseudolocality. In this section we prove that the distribution K is real analytic at all points where G and H are both real analytic.

The operator which transforms G to K corresponds to the multiplier

$$p(\tau, \lambda) = i\lambda|\tau|^{-\frac{1}{2}}(1 - i \operatorname{sgn} \tau)$$

If we restrict our attention to any cone Γ in (τ, λ) space which stays away from the line $\tau = 0$ then in Γ p is real analytic. We prove below that p then preserves the part lying in $\mathbb{R}^n \times \Gamma$ of the (analytic) wave front set of a distribution. A similar argument holds for $p_2(\tau, \lambda) = |\lambda|^{1/2}$ which takes H to K. Using both G and H we thus gain control on the smoothness of K in all cotangent directions.

The proof is somewhat complicated by the fact that G and H are not of compact support.

We are using the notion of analytic wave front set $WF_A u$ of a distribution as defined by Hormander [7]: $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n \setminus 0$ is not in $WF_A u$ if and only if there is an open neighborhood U of x_0 and a bounded sequence u_N of distributions of compact support such that

$$u_N = u \text{ in } U \text{ and } |\hat{u}_N(\xi)| < C\,(CN/|\xi|)^N$$

for all ξ in a conic neighborhood of ξ_0.

With the cut-off functions introduced in [7] it is then easy to prove

11.1 Lemma

a tempered distribution with $\hat{u}$ equal to 0 in

$$\Gamma' \cap \{\xi : |\xi| \geq C\}$$

then any (x, ξ) with ξ in Γ is not in $WF_A u$.

We can now state the main result of this section (only in the generality needed here).

11.2 Proposition Let Γ be an open cone in $\mathbb{R}^n$ and $p(\xi)$ a homogeneous function which extends to a holomorphic function $p(\xi+i\eta)$ for ξ in a conic neighborhood of $\overline{\Gamma}$ and away from 0, and $|\eta| < \varepsilon|\xi|$.

Let P denote the operator corresponding to the multiplier p and assume Pu is defined, u and Pu tempered distributions.

Then if u is analytic near x and ξ is in Γ it follows that (x, ξ) is not in $WF_A Pu$.

The proof consists in splitting p into two pieces, one of which is equal to 0 in Γ, the other one equal to the Fourier transform of a distribution real analytic away from 0.

The existence of the cut-off function we need is guaranteed by the Andersson-Hörmander Lemma in [1] (or rather by its proof):

given $\Gamma_1 \subset \Gamma_2$ cones in $\mathbb{R}^n \setminus 0$, Γ_1 closed, Γ_2 open there exists $\varphi \in C^\infty(\mathbb{R}^n)$ supported in Γ_2 such that

(11.3) $$\varphi(\xi) \equiv 1 \quad \text{for} \quad \xi \in \Gamma_1, \quad |\xi| \geq 1$$

(11.4) $$|D^{\alpha+\beta}\varphi(\xi)| \leq C_\beta |\xi|(1+|\xi|)^{-|\beta|}(C2^{-\nu})^{|\alpha|} \quad \text{whenever} \quad 4^\nu|\alpha| \leq |\xi|$$

Furthermore, it is proved in [1] that if φ satisfying (11.4) with $\beta = 0$ is extended to $\mathbb{C}^n$ by

$$\varphi(\xi + i\eta) = \Sigma_\alpha \varphi^{(\alpha)}(\xi)(i\eta)^\alpha \chi(|\xi| - (|\alpha| + 1))/\alpha!$$

where $\chi \in C^\infty(\mathbb{R})$ is supported in $(0, \infty)$ and equal to 1 in $[1, \infty)$ then there are constants C_1, C_2 such that

(11.5) $\quad |\bar{\partial}\varphi(\xi + i\eta)| \le C_1|\xi|e^{-|\xi|} \qquad$ if $\quad C_1(|\eta| + 1) \le |\xi|$

(11.6) $\quad |\varphi(\xi + i\eta)| \le C_1|\xi|^2 \exp(C_2 2^{-\nu}|\eta|) \quad$ if $\quad C_1 2^{\nu}(|\eta| + 2^{\nu}) \le |\xi|$

also $\quad \varphi(\xi + i\eta) = 0 \quad$ if $\quad \xi \notin \Gamma_2$

Proof of Proposition 11.2:

With Γ_2 the conic neighborhood of $\overline{\Gamma}$ in the hypothesis let Γ_1 be a closed cone with $\overline{\Gamma} \subset \overset{o}{\Gamma}_1 \subset \Gamma_1 \subset \Gamma_2$ and construct φ as above.

Note that φp is a slowly increasing C^{∞} function so its (inverse) Fourier transform L is in the space $\mathcal{O}_c'$ of Schwartz and $L*u$ is well defined.

By virture of condition (11.3) and Lemma 11.1 the proposition will be proved if we show

(11.7) $\qquad \text{sing supp}_A\, L*u \subset \text{sing supp}_A u.$

We first prove that L is equal to an analytic function for $x \neq 0$. (Similar proofs, in more difficult contexts can be found in Treves [13] and Andersson [1].)

(11.8) $\quad$ Fix $x_0 \neq 0$, choose ν so that $C_2 2^{-\nu} < \frac{|x_0|}{4}$

(11.9) $\quad$ then ε, M so that $\varepsilon\, C_1 2^{\nu} < 1$ and $M \ge C_1 4^{\nu}$

L is, modulo an entire function, the limit (in the snese of distributions) of the functions

$$(11.10) \qquad \int_{|\xi| \ge M} e^{ix\cdot\xi}\, \varphi(\xi)p(\xi)e^{-\delta|\xi|^2}\, d\xi$$

Assume x stays within distance at most $\frac{|x_0|}{4}$ from x_0 and let $v = \frac{x_0}{|x_0|}$. We want to move the domain of integration in (11.10) to

$$\xi' = \xi + i\varepsilon(|\xi| - M)v \qquad |\xi| \ge M$$

Stokes' theorem applied to the n-form $f(\zeta)d\zeta = f(\zeta)d\zeta_1 \wedge d\zeta_2 \wedge \ldots \wedge d\zeta_n$

yields the formula

$$(11.11)\quad \int_{M\le|\xi|\le R} f(\xi)\,d\xi = \int_{M\le|\xi|\le R} f(\xi')\,d\xi' - iR^{n-1}(R-M)\int_0^{\varepsilon}\int_{|\omega|=1} f(R\omega+it(R-M)v)\,v\cdot\omega\,d\omega\,dt$$

$$+ 2i\int_0^{\varepsilon}\int_{M\le|\xi|\le R} (|\xi|-M)\sum_{j=1}^{n} v_j \frac{\partial f}{\partial\bar{\zeta}_j}(\xi + i\,t(|\xi|-M)v)\,d\xi\,dt$$

$$= I_1 + I_2 + I_3$$

We take $f(\zeta) = e^{ix\cdot\zeta}\varphi(\zeta)p(\zeta)e^{-\delta\zeta^2}$ (here $\zeta^2 = \zeta_1^2+\zeta_2^2+\dots\zeta_n^2$) ; the convergence factor $e^{-\delta\zeta^2}$ allows us to let R tend to infinity in (11.11) and I_2 drops out. More interestingly I_1 and I_3 are convergent even for $\delta = 0$:

note that $\eta = \varepsilon(|\xi|-M)v$ satisfies the condition in (11.6), so

$$f(\xi') \le C|\xi|^{m+2}\exp(-x\cdot\eta + C_2 2^{-\nu}|\eta|)$$
$$\le C|\xi|^{m+2}\exp[-\varepsilon(|\xi| - M)|x_0|/2]$$

As for I_3, since in the region of integration

$$\bar{\partial} f = e^{ix\cdot\zeta}\,p(\zeta)e^{-\delta\zeta^2}\,\bar{\partial}\varphi$$

the integrand is $\le C(|\xi| - M)|\xi|^{m+1}e^{-|\xi|}$

Therefore we can let δ tend to zero and obtain $L(x)$ as a sum of convergent integrals. Moreover, we can replace x by $x + iy$ in these integrals (with x in the same neighborhood of x_0 as before) and they will remain convergent if

$$|y| < \min(\varepsilon|x_0|/4,\ 1/2)$$

L thus extends to a holomorphic function in a complex neighborhood of x_0. We can say a little more about L. Note that ε and M in (11.9) can be chosen independent of x_0 for x_0 large and $\varepsilon = O(|x_0|)$, $M = O(|x_0|^{-2})$ for x_0 small.

Our estimates now show that if

$$|x_0| \geq d, \; |x-x_0| < |x_0|/4, \quad |y| < \min(c|x_0|^2, 1/2)$$

then $L(x + iy)$ is bounded by a constant depending only on d. Therefore

$$|D^\alpha L(x)| \leq C_d^{|\alpha|+1} \alpha! \quad \text{for} \quad |x| \geq d.$$

It is clear from (11.4) that for any β, $D^\beta\varphi$ is just as good a cut-off function as φ; *the same proof as above will give, for any* β,γ

$$|D^\alpha x^\beta D^\gamma L(x)| \leq C_{\beta,\gamma,d}^{|\alpha|+1} \alpha! \quad \text{for} \quad |x| \geq d$$

hence also

$$|x^\beta D^\gamma D^\alpha L(x)| \leq C_{\beta,\gamma,d}^{|\alpha|+1} \alpha! \quad \text{for} \quad |x| \geq d. \tag{11.12}$$

It is now easy to prove that if u is any tempered distirbution

$$\text{sing supp}_A \, L*u \subset \text{supp } u. \tag{11.13}$$

To refine this, assume $x_0 \notin \text{sing supp}_A u$, let $\psi \equiv 1$ near x_0 be supported in a sufficiently small neighborhood of x_0; then ψu is C^∞ of compact support and analytic around x_0, L is analytic in $\mathbb{R}^n \setminus 0$ therefore (by a standard argument) $L*\psi u$ is analytic around x_0. By (11.13) so is $L*(1-\psi)u$. This proves (11.7) and the proposition.

Now let (t, s) be a point where G is analytic. By the proposition

$$\text{if} \quad \tau \neq 0, \quad (t, s, \tau, \lambda) \notin WF_A K$$

If H is also analytic around (t, s)

$$\lambda \neq 0 \Rightarrow (t, s, \tau, \lambda) \notin WF_A K$$

This completes the proof of the following.

<u>11.14 Theorem</u> Let E be the forward fundamental solution under study and K the distribution satisfying (formally)

$$E(t, z, s) = \frac{1}{|z|^{2n+1}} K\left(\frac{t^2}{|z|^2}, \frac{s}{|z|^2}\right) \quad \text{for } z > 0$$

Then sing $\text{supp}_A K \subset \left\{ \left(\frac{p^2}{\sin^2 p}, \frac{p - \sin p \cos p}{\sin^2 p} \right) : p \in \mathbb{R}, p \neq k\pi \right\}$

The reverse inclusion will follow from our results in Part III.

III Asymptotics of the Fundamental Solution

12. Asymptotics of H. In this section we want to compute $H(t, s)$ asymptotically for points (t, s) approaching $(t_0, s_0) \in \Sigma$. (The calculations for G are very similar, but not needed in what follows). We also want our formula to be well prepared for the fractional differentiation in s to come.

The solution to this problem depends on the order of the multiple zero $u_1(t_0, s_0)$ of f. Fortunately in our case this order is at most equal to 3 (easily seen from (7.9)). At most singular points it is actually equal to 2. The exceptional ones (order 3) turn out to be the cusp points in Figue 1 and the point $(1, 0)$. We will not study these exceptional points here.

Let $\varphi(t, s) = \frac{\partial f}{\partial u}(u_1(t, s), t, s)$ and let us go back to the formulas in section 7 giving $H(t, s)$ when (t, s) is non-singular: ε is replaced by 1 throughout.

Starting with (7.4) we see that

$$(12.1) \quad b_1 = \frac{f''(u_1)}{2\varphi}, \ldots, b_m = O(\varphi^{-1}) \quad \text{for all } m \geq 1 \text{ as } (t, s) \to (t_0, s_0)$$

then

$$(12.2) \quad c_1 = -\frac{f''(u_1)}{2\varphi} \qquad c_2 = \left(\frac{f''(u_1)}{2\varphi}\right)^2 + O(\varphi^{-1}), \ldots$$
$$c_m = \left(-\frac{f''(u_1)}{2}\right)^m + O(\varphi^{-m+1})$$

and substituting in (7.6) we obtain

$$(12.3) \quad a_k = d_{n,k}\, \varphi^{-n-k} (f''(u_1))^k + O(\varphi^{-n-k+1})$$

with

$$(12.4) \qquad d_{n,k} = \frac{n!}{(-2)^k} \sum_{\substack{j_0+\dots+j_{n-1}=n \\ j_1+\dots+(n-1)j_{n-1}=k}} \frac{1}{j_0!j_1!\dots j_{n-1}!}$$

Therefore (see (7.3))

$$(12.5) \qquad H = d_{n,n-1} \frac{1}{\varphi^{2n-1}} (f''(u_1))^{n-1} h(u_1) + 0(\varphi^{-2n+2})$$

(note that for the points (t_0, s_0) considered here, $u_1(t_0, s_0) \neq 0$).

To compute φ asymptotically, we first look at $u_1(t, s)$. We have seen that it satisfies

$$\frac{i}{u} - i \coth tu = s$$

hence $v(t, s) = - it\, u_1(t, s)$ is a solution of

$$(12.6) \qquad s = \frac{t}{v} - \cot v$$

We consider the following problem: given $F(v, t)$ analytic and v_0, t_0, s_0 such that

$$s_0 = F(v_0, t_0)$$
$$F_v(v_0, t_0) = 0$$
$$F_{vv}(v_0, t_0) \neq 0$$

compute $v(t, s)$ satisfying $F(v(t, s), t) = s$ for (t, s) near (t_0, s_0). By the implicit function theorem, there is in a neighborhood of t_0 an analytic function $b(t)$ such that

$$F_v(b(t), t) = 0, \quad b(t_0) = v_0$$

then $F(v, t) - F(b(t), t) = (v - b(t))^2 B(v, t)$
where B is analytic and non-zero in a neighborhood U of (v_0, t_0); we have to solve

$$(12.7) \qquad s - F(b(t), t) = (v-b(t))^2 B(v, t)$$

From here on the argument is classical: in U there is an analytic $B_1(v, t)$ with $B_1^2(v, t) = B(v, t)$

(12.8) in particular $B_1^2(b(t), t) = \frac{F_{vv}(b(t),t)}{2}$

If w is a new variable, the equation

$$w = (v-b(t))B_1(v, t), \quad B_1 \neq 0 \text{ in } U$$

uniquely defines $v = D_1(w, t)$ with D_1 analytic in a neighborhood of $(0, t_0)$ and $\frac{\partial D_1}{\partial w}$ non-zero there. We can write

$$v = b(t) + \frac{w}{B_1(b(t),t)}(1 + D_2(w, t))$$

with D_2 analytic and $D_2(0, t) = 0$

Now if we write a sufficiently small neighborhood of (t_0, s_0) as the union of the surface $s = F(b(t), t)$ and two pieces where $\sqrt{s-F(b(t),t)}$ can be defined, then in each of those pieces

$$(12.9) \qquad v = b(t) \pm \frac{\sqrt{s-F(b(t),t)}}{B_1(b(t),t)}\left(1 + D_2(\pm\sqrt{s-F(b(t),t)},t)\right)$$

There is a nice interpretation for the quantity $s - F(b(t), t)$: it is equal to zero precisely for those (t, s) which admit a double root v to $F(v, t) = s$ therefore it represents the (vertical) distance to our characteristic curve.

We will need an expansion for $[F_v(v(t, s), t)]^{-1}$

$F_v(v, t) = (v-b(t))F_{vv}(b(t), t)\,[1 + (v - b(t))D_3(v, t)]$.

Substituting $v = v(t, s)$ given by (12.9):

$$(12.10) \qquad \frac{1}{F_v(v(t,s),t)} = \frac{1}{\sqrt{s-F(b(t),t)}}\,\frac{1}{\sqrt{2F_{vv}(b(t),t)}}\left[1+D_4(\pm\sqrt{s-F(b(t),t)},t)\right]$$

with D_4 analytic, $D_4(0, t) = 0$

Now let $F(v, t) = \frac{t}{v} - \cot v$, $(t_0, s_0) = (t(p_0), s(p_0))$, $v = p_0$ where $p_0 \neq k\pi$ and $p_0 \neq$ the "cusp values" $\pm p_1, \pm p_2, \ldots$

The equation for $b(t)$ becomes $t = \frac{(b(t))^2}{\sin^2(b(t))}$; $t(p)$ has an inverse $b(t)$ in each of the intervals $0 < p < \pi$, $k\pi < p < p_k$, $p_k < p < (k+1)\pi$ and similarly for $p < 0$. $F(b(t), t) = \frac{b(t)}{\sin^2 b(t)} - \cot b(t) = s(b(t))$; as noted, $s - s(b(t))$ is the vertical distance to Σ.

Also note that, with the results of §8, it is easy to determine which of the two square roots in (12.9) corresponds to the solution $u_1 = \lim_{\varepsilon\to 1} u_\varepsilon$.

We summarize our calculations in the following.

(12.11) Lemma Under the assumptions and with the notation given above for (t, s) in a neighborhood of (t_0, s_0), $s \neq s(b(t))$

$$(12.12) \quad H(t, s) = \frac{\psi_1(t)}{[s-s(b(t))]^{\frac{2n-1}{2}}} \left[1 + 0\left(\sqrt{s-s(b(t))}\right)\right]$$

ψ_1 is real analytic and non-zero near t_0, and is computed below.

Proof:

$$f(u, t, s) = 2i\, e^{-tu}\, u \sinh(tu)\, [F(-itu, t) - s]$$

$$\varphi(t, s) = \frac{\partial f}{\partial u}(u_1(t, s), t, s) = -2e^{-iv(t,s)} v(t, s) \sin(v(t, s)) F_v(t, s), t)$$

Using (12.9) and (12.10) we find

$$(12.13) \quad \frac{1}{\varphi} = -\frac{1}{2}\, \frac{e^{ib(t)}}{b(t)\sin(b(t))}\, \frac{1}{\sqrt{2F_{vv}(b(t), t)}}\, \frac{1}{\pm\sqrt{s-s(b(t))}}\, [1+D_5(\pm\sqrt{s-s(b(t))}, t)]$$

with D_5 analytic, $D_5(0, t) = 0$

Substituting this in (12.5) along with corresponding expansions for $f''(u_1)$ and $h(u_1)$ yields (12.12) with

$$(12.14) \quad \psi_1(t) = \pm \text{ const. } \frac{\exp(-i\gamma b(t))}{[tb(t)F_{vv}(b(t), t)]^{1/2}(\sin b(t))^n}.$$

$H^-(t, s)$ has, of course, a similar expansion, but we need to check that the main terms do not cancel. Recall that $H^-(t, s)$ is $H(t, -s)$ with γ replaced by $-\gamma$; if (t, s) is near (t_0, s_0), $(t, -s)$ is near

$(t_0, -s_0)$ and the "new" $b(t) = -$ "old" $b(t)$; since as a function of v, $F_{vv}(v, t)$ is odd and so is $s(p)$ we have:

$$(12.15) \quad H(t, s) + H^{-}(t, s) = \frac{(1\pm i)\psi_1(t)}{[s-s(b(t))]^{\frac{2n-1}{2}}} + \text{error terms}$$

with $\psi_1(t)$ given by (12.14).

As mentioned before, the sign in (12.14) and (12.15) can be determined: we look at v, solution of (12.6). If v is complex we choose the square root to make $\mathcal{Im}\, v < 0$. (For we must have $\mathcal{Re}\, u_1 \geq 0$). If v is purely real then (8.5), rewritten using (12.6), shows that it must satisfy

$$(12.16) \qquad t \geq \frac{v^2}{\sin^2 v} \text{ in addition to (12.6);}$$

if p_0 is in any of the intervals

$$(12.17) \qquad \ldots (-p_2, -2\pi), (-p_1, -\pi), (0, \pi), (p_1, 2\pi), (p_2, 3\pi), \ldots$$

then we choose the square root to make $v < p_0$. If p_0 is in the remaining intervals

$$(12.18) \qquad \ldots (-\pi, 0), (\pi, p_1), (2\pi, p_2), \text{ etc}$$

we must choose v so that $v > p_0$.

(12.19) Lemma: for (t, s) in a neighborhood of $(t(p_0), s(p_0))$, $p_0 > 0$, $s \neq s(b(t))$

$$(12.20) \quad H(t, s) + H^{-}(t, s) = \frac{[\operatorname{sgn}(-s+s(b(t)))]^{n-\epsilon(p_0)}}{|s-s(b(t))|^{n-\frac{1}{2}}} \psi_2(t)(1+O(\sqrt{s-s(b(t))}))$$

here $\epsilon(p_0) = 1$ if $p_0 > 0$ is in one of the intervals (12.17)

0 if $p_0 > 0$ is in one of the intervals (12.18)

$$(12.21) \quad \psi_2(t) = \text{pos. real const.} \times \frac{\exp(-i\gamma b(t))}{|t\, b(t) F_{vv}(b(t), t)|^{1/2} (\sin b(t))^n} .$$

One more remark is in order. In the next section we will need information about the distribution H in a whole neighborhood of (t_0, s_0), not just where it is equal to a function. It is not hard to obtain such information with the methods we already have: recall that as distributions $H = \lim_{\varepsilon\to 1} H_\varepsilon$ and that H_ε is equal to a function everywhere; this function $H_\varepsilon(t, s)$ we can compute for (t, s, ε) near $(t_0, s_0, 1)$. We indicate briefly how the previous calculations are to be modified.

Write, with $\varepsilon = e^{-2\delta}$

$$f(u, t, s, \varepsilon) = 2i\ e^{-\delta-tu}\ u \sinh(\delta + tu)[-i \coth(\delta + tu) + \frac{i}{u} - s]$$

As before, let $tu = iv$ and

$$F(v, t, \delta) = \frac{t}{v} - \cot(v - i\delta)$$

If $b(t, \delta)$, analytic in both variables near $(t_0, 0)$, is such that

$$F_v(b(t, \delta), t, \delta) = 0$$

and if we set

$$s(t, \delta) = F(b(t, \delta), t, \delta)$$

we have

$$H = \lim \frac{\psi(t,\delta)}{[s-s(t,\delta)]^{\frac{2n-1}{2}}} \left[1 + D_6(\sqrt{s-s(t,\delta)}, t,\delta)\right]$$

with D_6 analytic, $D_6(0, t, \delta) = 0$ and

$$\psi(t,\delta) = \frac{\exp[-i\gamma b(t,\delta)+n\delta]}{\sqrt{t\ b(t,\delta)F_{vv}(b(t,\delta),t,\delta)}\ [\sin(b(t,\delta)-i\delta)]^n}$$

In terms of functions computed earlier, one finds

$$b(t, \delta) = \dot{b}(t) + i\frac{b(t)}{b(t)-\tan b(t)}\ \delta + 0(\delta^2)$$

$$s(t, \delta) = s(b(t)) - i\ \frac{\delta}{\sin^2 b(t)} + 0(\delta^2)$$

We can now conclude that in a neighborhood of (t_0, s_0) the distribution H = distribution $\dfrac{\psi_1(t)}{[s-s(b(t))]^{\frac{2n-1}{2}}}$ + error terms.

13. Asymptotics of K and E. In this section we want to compute the fractional derivative $D_s^{1/2}$ of the formula (12.20).

Fix a point (t_0, s_0). We assume that the line $t = t_0$ contains none of the exceptional points

$$(t(p), s(p)) : p = 0, \pm p_1, \pm p_2, \ldots$$

Let $(t_0, s_0), (t_0, s_1), (t_0, s_2), \ldots, (t_0, s_k)$ for some finite $k \geq 0$ be the points where the line $t = t_0$ intersects Σ ; let $b_j(t)$ be the corresponding local inverses of $t(p)$ and let $s_j(t) = s(b_j(t))$.

One can find a neighborhood U of t_0 and $\varepsilon > 0$ so that the tubular neighborhoods $V_j = \{(t, s) : t \in U, |s-s_j(t)| < \varepsilon\}$ are disjoint. Let η , φ be cut-off functions with

η supported in U, $\eta \equiv 1$ near t_0

φ supported in $(-\varepsilon, \varepsilon)$, $\eta \equiv 1$ near 0

then

$\eta(t)[H + H^- - \sum_{j=0}^{k} \varphi(s-s_j(t))(H + H^-)]$ is equal to a smooth function and the problem is reduced to the computation of the fractional derivative of

$$\eta(t)\, \varphi(s - s_j(t))(H + H^-). \tag{13.1}$$

For each of these distributions we have a formula like (12.20). In view of the translation invariance of $D_s^{1/2}$ it remains to compute

$$D_s^{1/2}\{(-\operatorname{sgn} s)^{n-\varepsilon(p_0)} \varphi\,(s)|s|^{-n+\frac{1}{2}}\}$$

$$= D_s^{1/2}\{(-\operatorname{sgn} s)^{n-\varepsilon(p_0)} |s|^{-n+\frac{1}{2}}\} + C^\infty \text{ function}$$

If $\varepsilon(p_0) = 1$ then the function above is even when n is odd, odd when n is even; its Fourier transform is known explicitly and we obtain, with c_n a constant depending on n which is not hard to determine,

$$D_s^{1/2}\,[(-\operatorname{sgn} s)^{n-1}|s|^{-n+\frac{1}{2}}] = c_n\,\delta^{(n-1)}(s),$$

the appropriate derivative of the delta function.

If $\varepsilon(p_0)= 0$ the function is even when n is even, etc, and

$$D_s^{1/2}\,[(-\operatorname{sgn} s)^{n}|s|^{-n+\frac{1}{2}}] = \text{const.}\ /s^n.$$

Using these formulas for each of the distributions in (13.1) gives the following (see (6.6)).

(13.2) Proposition Suppose $(t_0, s_0) = (t(p_0), s(p_0))$, $p_0 > 0$, is a point in the singular set Σ such that

$$t_0 \neq t(p),\ p = p_1, p_2, \ldots$$

Then in a neighborhood of (t_0, s_0) (with the notation of §12)

if $\varepsilon(p_0) = 1$, $K = \text{const.} \times \psi(t)\delta^{(n-1)}(s-s(b(t)) + O((s-s(b(t))^{-n+\frac{1}{2}})$

if $\varepsilon(p_0) = 0$, $K = \text{const.} \times \psi(t)(s-s(b(t))^{-n} + O((s-s(b(t))^{-n+\frac{1}{2}})$.

Here $\psi(t)$ is real analytic near t_0 and given explicitly by

(13.3) $$\psi(t) = |b(t)F_{vv}(b(t),t)|^{-\frac{1}{2}}(\sin(b(t))^{-n}\exp(-i\gamma b(t)).$$

Since the "outer sheet" of our characteristic cone (for $s > 0$) corresponds to $p_0 \in (0, \pi)$, (hence $\varepsilon(p_0) = 1$) these calculations are consistent with the result proved earlier: $E = 0$ outside the cone; they also show that E is not identically zero in any other of the domains in the complement of the cone.

14. Summary of Part III

We first recall the pertinent notation:

let $p_1, p_2, \dots$ be the positive real solutions to $p = \tan p$.

Let $\varepsilon(p) = 1$ if p is in any of the intervals $(0, \pi)$, $(p_k, (k+1)\pi), k \geq 1$

$\varepsilon(p) = 0$ if $p \in (k\pi, p_k)$ for some $k \geq 1$

Let $t(p) = \dfrac{p^2}{\sin^2 p}$, $s(p) = \dfrac{p - \sin p \cos p}{\sin^2 p}$

Let (t_0, z_0, s_0) be a singular point of the forward fundamental solution E, with $z \neq 0$,

$$\text{i.e.} \quad \left(\frac{t_0}{|z_0|^2}, \frac{s_0}{|z_0|^2}\right) = (t(p_0), s(p_0)) \quad \text{for some } p_0 \neq k\pi$$

Assume $\dfrac{t_0}{|z_0|^2} \neq t(p)$ $p = 0, p_1, p_2, \dots$

and, to fix notation $s_0 > 0$.

Finally, let $b(t)$ be the local inverse of $t(p)$ near p_0 and

$$\rho(t, z, s) = s - |z|^2 \times s\left(b\left(\frac{t}{|z|^2}\right)\right)$$

which near (t_0, z_0, s_0) is a distance to the singular cone.

<u>(14.1) Theorem</u> Under the assumptions and with the notation given above, in a neighborhood of (t_0, z_0, s_0)

$$E = \text{const.} \frac{1}{|z|} \psi\left(\frac{t}{|z|^2}\right) \delta^{(n-1)}(\rho(t, z, s)) + 0(\rho^{-n+\frac{1}{2}}) \quad \text{if } \varepsilon(p_0) = 1$$

$$E = \text{const.} \frac{1}{|z|} \psi\left(\frac{t}{|z|^2}\right) (\rho(t, z, s))^{-n} + 0(\rho^{-n+\frac{1}{2}}) \quad \text{if } \varepsilon(p_0) = 0$$

with ψ real analytic near $\dfrac{|t_0|}{|z_0|^2}$ and explicit in formula (13.3).

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