

# Numerical study of phase transitions in Potts models

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A model is defined that interpolates between integer dimension and integer  $q$ -state Potts models. Phase transitions in these models are studied, and in particular the critical value  $q_c$ , where the phase transition changes from second to first order, is determined for several dimensions. For  $(d, q_c)$  the values (2.0, 4.05), (2.5, 2.68), (3.0, 2.21), (3.5, 2.15), and (4.0, 2.07) are obtained.

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## I. INTRODUCTION

For the past few decades, Potts models have provided a continuous source of research [1,2]. This is mainly due to the fact that these models exhibit a rich critical behavior, although they have a simple definition. In particular, it is known that in the space  $\{(d, q)\}$  of  $d$ -dimensional  $q$ -state Potts models, a critical line  $(d, q_c(d))$  exists, separating the models with a second-order phase transition [ $q \leq q_c(d)$ ] from those with a first-order phase transition [ $q > q_c(d)$ ]. The only points that are known to be exactly on this line are  $(d, q) = (2, 4)$  [3] and for  $d > 4$   $(d, 2)$  [4], and there is convincing evidence that  $q_c(3) < 3$  [5–12]. Further progress is difficult because models with noninteger  $d$  and  $q$  are hard to treat numerically. The only results known to us for noninteger  $(d, q)$  are (1.58, 12.6), (2.4, 0.8), and (2.32, 2.85), which follow from a renormalization-group analysis [5]. Recently, the Ising model has been studied in noninteger dimensions [13,14], but these calculations have not been extended beyond  $q = 2$ .

In this paper we will define a family of models labeled by two parameters  $(d^*, q^*)$ , which we expect to be a reasonable approximation of the  $(d, q)$  Potts model, and which for integer  $(d^*, q^*)$  agrees with the  $(d, q)$  Potts model. The  $(d^*, q^*)$  is essentially an anisotropic Potts model equipped with a certain nonlocal cluster dynamics, analogous to the one introduced by Swendsen and Wang [15,16]. The latter has been quite successfully applied to a number of problems (for instance, to the Potts model, see, e.g., [17–19], or using the related Wolff algorithm [20], see, e.g., [21]). We will perform a numerical study of the  $(d^*, q^*)$  models. In particular, we will determine the critical temperature and try to measure the discontinuity in the internal energy that exists in the case of a first-order phase transition. Our results are in reasonably good agreement with the known theoretical results. A novel feature is that a considerable part of the simulations will be done at constant internal energy rather than constant temperature. This dramatically improves the statistics in the neighborhood of the critical point. For

more information about Monte Carlo simulations, see, e.g., [22].

This paper is organized as follows. In Sec. II we define the model and recall some exact results from  $d = 2$  for later use. In Sec. III we determine the critical temperature of these models using a Monte Carlo renormalization-group analysis [23,24]. This method is known to be highly insensitive to finite-size effects. In two dimensions we find that the critical temperature of the  $(2, q^*)$  models agrees with the theoretical value of the  $q^*$ -state Potts models at least up to 0.1%. In Sec. IV we describe our method to determine the discontinuity in the internal energy for first-order phase transitions, and how to use this to distinguish the different types of critical behavior. In Sec. V we present our results, which were obtained from extensive simulations of a large number of  $(d^*, q^*)$  models on a three-processor Apollo DN-10000 workstation. In particular, we determine  $q_c^*(d^*)$ , where  $d^*$  runs from 2 to 4 with steps of 0.5. We end with some conclusions in Sec. VI.

## II. MODEL

Consider the following Hamiltonian of an anisotropic Potts model in  $d$  dimensions,  $\sigma_i \in \{1, \dots, q\}$ :

$$\beta H = \sum_{r=1}^d J_r \sum_{\langle i, j \rangle_r} (1 - 2\delta_{\sigma_i \sigma_j}), \quad (1)$$

where  $\langle i, j \rangle_r$  means a pair of neighboring spins in the  $\mathbf{e}_r$  direction, and  $\beta = 1/k_B T$ . We assume the spins live on a  $d$ -dimensional lattice with  $L^d$  sites and periodic boundary conditions. We also define the following quantities  $s$  and  $s_r$ :

$$s_r = \frac{1}{L^d} \sum_{\langle i, j \rangle_r} (2\delta_{\sigma_i \sigma_j} - 1), \quad (2)$$

$$s = \frac{\sum_{r=1}^d J_r s_r}{\sum_{r=1}^d J_r}. \quad (3)$$

Clearly all  $s_r$  and hence also  $s$  are equal to 1 if all

nearest-neighbor spins are equal, whereas  $s = -1$  if all nearest-neighbor spins are different. With the partition function  $Z = \sum_{\langle i,j \rangle} \exp(-\beta H)$ , one has

$$\langle s_r \rangle = \frac{1}{L^d} \frac{\partial}{\partial J_r} \ln Z, \quad (4)$$

$$\langle s \rangle = \frac{\sum_{r=1}^d J_r \langle s_r \rangle}{\sum_{r=1}^d J_r}. \quad (5)$$

The internal energy per site is now equal to

$$\beta u = - \sum_r J_r \langle s_r \rangle = - \langle s \rangle \sum_r J_r. \quad (6)$$

In general, if  $J \rightarrow \infty$ ,  $\langle s \rangle \rightarrow 1$ , and if  $J \rightarrow 0$ ,  $\langle s \rangle \rightarrow -1 + (2/q)$ .

In two dimensions (see, e.g., [2]), phase transitions occur for pairs  $(J_1, J_2)$  satisfying

$$(e^{2J_1} - 1)(e^{2J_2} - 1) = \sqrt{q}. \quad (7)$$

If  $J_1 = J_2 = J$ , then at the critical coupling  $J = \ln[(1 + \sqrt{q})/2]$  one has

$$\langle s \rangle(J_{\text{crit}}) = \frac{1}{\sqrt{q}}, \quad (8)$$

and if  $q \geq 4$  the discontinuity in  $\langle s \rangle$  is given by

$$\langle s \rangle(J_{\text{crit}}^\pm) = \frac{1}{\sqrt{q}} \pm \left[ 1 + \frac{1}{\sqrt{q}} \right] \tanh \frac{\theta}{2} \prod_{n=1}^{\infty} (\tanh n \theta)^2, \quad (9)$$

where  $\cosh \theta = \frac{1}{2} \sqrt{q}$ .

The dynamics of this system that we used resemble those proposed in [15]. Given an arbitrary spin configuration, we make clusters by creating a bond between each pair of equal nearest-neighbor spins in the  $\mathbf{e}_r$  direction, with probability  $(1 - e^{-2J_r})$ . Then to all sites in each cluster a new state is assigned according to the following rule: the probability that all  $\sigma$ 's in a particular cluster obtain the value  $l \in \{1, \dots, q\}$  is  $p_l$ . Note that  $\sum_l p_l = 1$ . In the usual Swendsen-Wang algorithm  $p_l = 1/q$  for all  $l$ . We will, however, allow for the possibility that the  $p_l$ 's are different. In general, the resulting system will not correspond to a Potts model. We do not know whether it is possible to write down a Hamiltonian for which these dynamics give an equilibrium distribution, but if it exists we expect that it will necessarily be nonlocal.

We will now try to find coupling  $J_r$  and probabilities  $p_l$  such that this system describes as closely as possible a  $(d^*, q^*)$  Potts model with coupling  $J^*$ . To fix the  $p_l$  consider  $n$  clusters occurring in the Swendsen-Wang dynamics above. The probability that to all clusters the same spin will be assigned is equal to

$$P_n = \sum_{l=1}^q p_l^n. \quad (10)$$

For a  $q^*$ -state Potts model we would have  $P_n = (q^*)^{1-n}$ . Therefore, in order to have approximately the same multicluster interactions as in the  $q^*$ -state Potts model, we require that

$$\sum_{l=1}^q p_l^n = (q^*)^{1-n} \quad (11)$$

for as many values of  $n$  as possible. Alternatively, these conditions can be derived by looking at the high-temperature expansion of  $s$ . It turns out that for noninteger  $q^*$  these relations can only be realized for  $n=1,2$  but not for  $n=1,2,3$  simultaneously. If we now use standard variational techniques to minimize  $[\sum_l p_l^3 - (q^*)^{-2}]$ , subject to the conditions  $\sum_l p_l = 1$  and  $\sum_l p_l^2 = (q^*)^{-1}$ , we find that the best choice is to take  $q = [q^*] + 1$  different spin values, where  $[x]$  denotes the largest integer less than or equal to  $x$ , and the following values of  $p_l$ :

$$p_l = \frac{1}{q} \left\{ 1 + \left[ \left[ \frac{q}{q^*} - 1 \right] (q-1)^{-1} \right]^{1/2} \right\}, \quad 1 \leq l \leq q-1 \quad (12)$$

$$p_l = \frac{1}{q} \left\{ 1 - \left[ \left[ \frac{q}{q^*} - 1 \right] (q-1)^{-1} \right]^{1/2} \right\}, \quad l = q. \quad (13)$$

To fix the  $J_i$  is a much more difficult problem. The small  $J_i$  behavior of  $\beta u$  can be written as an expansion in  $\sum_i J_i^n$  with  $n \leq 1$ , so it would be tempting to try to realize

$$\sum_{i=1}^d J_i^n = d^* (J^*)^n \quad (14)$$

for as many values of  $n$  as possible. However, for large  $J_i$  there is an expansion in terms of  $\sum_i J_i \exp(-2nJ_i)$  and this suggests we should rather try to realize

$$\sum_{i=1}^d J_i e^{-2nJ_i} = d^* J^* e^{-2nJ^*}. \quad (15)$$

We have used a certain "hybrid" combination of (14) and (15) that includes these equations in the appropriate limits. We try to realize

$$\sum_{i=1}^d [J_i (1 - e^{-2J_i})]^{n/2} = d^* [J^* (1 - e^{-2J^*})]^{n/2} \quad (16)$$

for as many values of  $n$  as possible. This has two advantages: by putting  $z_i = (d^*)^{-1} [J_i (1 - e^{-2J_i}) / J^* (1 - e^{-2J^*})]^{1/2}$  we see that (16) is equivalent to  $\sum_{i=1}^d z_i^n = (d^*)^{1-n}$ , which can be solved in the same way as the  $p_l$  were determined, and the  $z_i$  uniquely determine the  $J_i$  because  $f(x) = x(1 - e^{-2x})$  is an invertible function for positive  $x$ . The solution is given by  $(d = [d^*] + 1)$ :

$$[J_i(1-e^{-2J_i})]^{1/2} = \begin{cases} [J_*(1-e^{-2J_*})]^{1/2} \frac{1}{d} \left\{ 1 + \left[ \left( \frac{d}{d^*} - 1 \right) (d-1)^{-1} \right]^{1/2} \right\}, & i < d \\ [J_*(1-e^{-2J_*})]^{1/2} \frac{1}{d} \left\{ 1 - \left[ \left( \frac{d}{d^*} - 1 \right) (d-1)^{-1} \right]^{1/2} \right\}, & i = d, \end{cases}$$

in which  $d = [d^*] + 1$ . Note that  $j_d \searrow 0$  as  $d^* \searrow (d-1)$ , so that one dimension effectively decouples from the theory as expected if we reduce the dimension by one. As a further consistency check, observe that all  $J_i \rightarrow 0$  if  $J^* \rightarrow 0$  and that all  $J_i \rightarrow \infty$  if  $J^* \rightarrow \infty$ . This completes the description of the model.

### III. LOCATION OF THE CRITICAL POINT

In order to determine the critical point we use the Monte Carlo renormalization-group method; for details see [23] and [24]. We start with a small lattice  $S$  with  $L = 2^{a-b}$  and a large lattice  $L$  with  $L = 2^a$ . Given a configuration of spins on a lattice with  $L = 2^a$  we define a renormalized configuration on a lattice with  $L = 2^{a-1}$ , by replacing blocks of  $2^d$  spins with one spin that is determined by the majority rule: Let  $r_1, \dots, r_h$  be the spins that occur most often in a block of  $2^d$  spins, then the resulting spin is picked from  $r_1, \dots, r_h$ , each with probability  $1/h$ . Let  $\langle s^{(l)} \rangle$  denote the average of  $s$  for an  $l$ -times renormalized lattice, then the critical point is the point where  $\langle s^{(l)} \rangle_S - \langle s^{(l+b)} \rangle_L$  changes sign. The advantage of this method is that it reduces finite-size effects very efficiently. The result can be further improved by increasing  $a$  or increasing  $l$ . We used  $b=1$  and  $a=4$ , and determined  $J_{\text{crit}}$  by finding the zero of a least-squares-fitted straight line through a set of points  $(J, \langle s^{(l)} \rangle_S - \langle s^{(l+b)} \rangle_L)$  in the direct vicinity of the phase transition. The statistical error was also obtained from these fits. After this, we took the average of the values for  $J_{\text{crit}}$  obtained for  $l=1$  and  $2$ . Comparing the results for different values of  $a$  showed that the statistical error is indeed larger than the finite-size error. The results for certain values of  $q^*$  in  $d=2$  are depicted in Table I. We took  $d=2$ , because in this case we know the theoretical value of  $J_{\text{crit}}$  (2.7), so that  $d=2$  can serve as a testing ground to see how good the fractional  $q^*$  model really is. As can be seen from Table I, the agreement is better than the statistical error, and no systematic errors are visible at this level of accuracy. This shows that the fractional  $q^*$  model is indeed a very reasonable approximation of the  $q$ -state Potts model.

The data were obtained from a Monte Carlo simulation as described previously. For every value of  $q^*$ , the final spin configuration obtained from the Monte Carlo simulation of the previous value of  $q^*$  was used as initialization. Every new data point was obtained from 5000 steps, of which the first 2000 steps were used for thermalization. The configuration for the first value of  $q^*$  was obtained by performing 5000 steps on a random configuration.

### IV. NATURE OF THE PHASE TRANSITION

In general, it is very difficult to determine numerically the order of a phase transition; see, e.g., [5–12]. The method we used essentially amounts to measuring the presence of a latent heat, which is only present for a first-order phase transition. Strictly speaking, the finite-sizes effect spoils the presence of a true latent heat, but one can nevertheless try to measure the corresponding gap in the  $(J, s)$  curve using the procedure below.

We performed simulations of Potts models for various values of  $(d^*, q^*)$ , and for each we determined numerically  $J$  as a function of  $s$ . This has the advantage that much more data are obtained that are in the direct vicinity of the critical point, as opposed to measuring  $s$  as a function of  $J$ . Because  $s$  is not a parameter of the model, we used a method that is well known in, e.g., constant-pressure simulations in molecular dynamics; namely, we build a “thermostat” into the simulation. More precisely, after every Monte Carlo sweep we determine  $s\{\sigma_r\}$ , and then adjust  $J$  according to the rule  $J' = J - \mu(s\{\sigma_r\} - s)$ , where  $\mu$  is a small multiplier. We used  $\mu=0.01$  during the 500 thermalization sweeps and  $\mu=0.002$  for the 1500 measurement sweeps performed for every point  $(s, J)$ . For every value of  $(d^*, q^*)$ , we obtained points  $(s, J)$  with  $s$  ranging from  $-1 + (q^*/2)$  to  $0.9$  with steps of  $0.02$ . As

TABLE I. Measurement of  $J_{\text{crit}}$  for various values of  $q^*$  and  $d=2$ .

$q^*$	$J_{\text{crit}}$ (expt)	$J_{\text{crit}}$ (theor)	$10^5 \Delta J$
2.0	0.440 60(65)	0.440 69	−9
2.2	0.455 07(82)	0.454 78	+29
2.4	0.467 64(76)	0.467 89	−25
2.6	0.479 73(52)	0.480 14	−42
2.8	0.492 41(70)	0.491 66	+75
3.0	0.502 53(66)	0.503 53	0
3.2	0.512 70(57)	0.512 82	−12
3.4	0.523 12(56)	0.522 59	+53
3.6	0.531 70(72)	0.531 90	−21
3.8	0.541 00(56)	0.540 79	+21
4.0	0.549 81(56)	0.549 31	+50
4.2	0.557 81(57)	0.557 47	+34
4.4	0.565 97(50)	0.565 32	+65
4.6	0.572 23(54)	0.572 87	−64
4.8	0.581 14(72)	0.580 15	+99
5.0	0.587 23(62)	0.587 18	+05
5.2	0.593 30(52)	0.593 98	−67
5.4	0.600 45(73)	0.600 55	−11
5.6	0.606 23(99)	0.606 93	−70
5.8	0.613 80(44)	0.613 11	+69

$s = -1 + (q^*/2)$  corresponds to an infinite-temperature configuration, we could take a purely random initial configuration and no initial thermalization is needed. We also performed a run with  $s$  running in the opposite direction to check for the presence of hysteresis. We could not detect any, indicating that the correlation time is sufficiently eliminated by the thermalization steps. Altogether it took about 100 000 sweeps to determine one curve  $(s, J)$ , each sweep taking about one second on our workstation.

From the values  $(s, J)$  thus obtained, we determined the point of minimal  $dJ/ds$  as an indication of the location of the critical point, and selected 12 points at both sides of this point. In the case of a large gap, extra points were added to compensate for the points in the gap. This set of points was used to perform a least-squares fit of a certain function  $J(s)$ . The function we have taken consists of two powerlike branches,  $J = c_l - a_l(x_l - s)^{e_l}$  for  $s < \min(x_l, (x_l + x_r)/2)$  and  $J = c_r + a_r(s - x_r)^{e_r}$  for  $s > \max(x_r, (x_l + x_r)/2)$ .  $c_l$  and  $c_r$  are chosen in such a way that both branches are glued together continuously. In the case where  $x_r > x_l$ ,  $J$  is given by  $J(s) = c_l = c_r$  for  $x_l < s < x_r$ . Clearly, this corresponds to the behavior of  $J$  in the scaling region of an ideal first-order phase transition. The latent heat we extract is  $x_r - x_l$ , and is negative if  $x_r < x_l$ . One might argue that, due to the finite-size effects, the function above is not a good description of the  $(s, J)$  curve for a finite lattice. However, we expect that the latent heat obtained from this fit will still be a reasonable approximation of the infinite lattice latent heat.

At this stage we have obtained the latent heat as function of  $q^*$ , which is negative in some cases. These negative values do not have any physical meaning, they are purely numerical artifacts. Finally, we have determined the value for  $q_c$  by defining it as the point where the latent heat changes sign. To find this point, we matched the latent heat with a function of  $q^*$  which is  $-a_l(q_c - q^*)^2$  for  $q^* < q_c$ , and which is  $a_r(q^* - q_c)^2$  for  $q^* > q_c$ , and optimized  $a_l$ ,  $a_r$ , and  $q_c$ . This is our final result for  $q_c$ .

## V. RESULTS

Because the correlation time increases as the lattice size increases [25], not only will the sweeps take more time for a larger lattice, but we also need more sweeps to overcome the correlation time. This strongly limits the lattice sizes that can be used in the simulations. We used a  $128 \times 128$  lattice for  $d^* = 2$ , a  $32 \times 32 \times 32$  lattice for  $d^* = 2.5, 3$  and a  $10 \times 10 \times 10 \times 10$  lattice for  $d^* = 3.5, 4$ . The results for the latent heat and the exponents  $e_l$  and  $e_r$  have been gathered in Table II. The final results for  $q_c$  are shown in Table III, together with the typical statistical errors for the gap sizes and exponents for each dimension. Here, the values of  $\Delta e^+$  and  $\Delta e^-$  indicate the typical positive and negative deviations of the values for the exponents given in Table II. A typical plot of some  $(s, J)$  curves is provided for  $d = 2$  and 3 in Figs. 1 and 2. The shaded areas indicate the fitted curve for the latent heat as a function of  $q^*$ . Finally, we have given a plot of  $q_c$

TABLE II. The latent heat and exponents for various values of  $d^*$  and  $q^*$ . For typical statistical errors see Table III.

$d^*$	$q^*$	$e_l$	$e_r$	Latent heat
2.0	5.0	1.97	2.00	0.094
	4.8	1.95	2.04	0.034
	4.6	2.14	1.92	0.005
	4.4	1.99	2.00	-0.004
	4.2	2.03	1.89	0.020
	4.0	1.97	1.97	0.019
	3.8	2.08	2.30	-0.058
	3.6	2.70	2.25	-0.172
2.5	2.9	1.73	1.82	0.076
	2.8	1.67	1.77	0.082
	2.7	1.77	1.96	0.059
	2.6	1.87	2.12	0.010
	2.5	3.29	2.00	-0.159
	2.4	3.26	2.29	-0.209
3.0	3.3	1.25	0.91	0.241
	3.1	1.15	1.55	0.233
	2.9	1.34	1.83	0.130
	2.7	1.71	1.67	0.061
	2.5	1.97	1.88	0.008
	2.3	2.01	2.11	-0.013
	2.1	1.77	2.29	0.007
	1.9	2.71	1.81	-0.111
	1.7	3.14	2.70	-0.208
3.5	2.5	1.08	1.15	0.192
	2.4	1.57	0.93	0.118
	2.3	1.37	1.17	0.097
	2.2	1.67	1.94	0.049
	2.1	1.80	2.31	-0.009
	2.0	2.04	2.15	-0.015
	1.9	3.85	2.04	-0.370
	1.8	3.27	1.88	-0.416
4.0	2.5	1.08	0.68	0.245
	2.4	1.05	0.67	0.223
	2.3	1.14	0.67	0.195
	2.2	1.14	0.91	0.180
	2.1	1.34	1.42	0.080
	2.0	1.26	1.77	0.096
	1.9	2.47	2.18	-0.083
	1.8	3.00	2.30	-0.354
	1.7	2.24	0.92	-0.428
	1.6	3.05	2.19	-0.456

TABLE III. Typical statistical errors in the exponents and latent heat as given in Table II;  $q_c^*$  is the value of  $q^*$  where the phase transition changes from second to first order.

$d^*$	$q_c^*$	$\Delta_{\text{gap}}$	$\Delta e^+$	$\Delta e^-$
2.0	4.05(16)	0.02	+40%	-15%
2.5	2.68(9)	0.05	+80%	-40%
3.0	2.21(5)	0.04	+50%	-30%
3.5	2.15(7)	0.03	+70%	-30%
4.0	2.07(9)	0.03	+80%	-25%

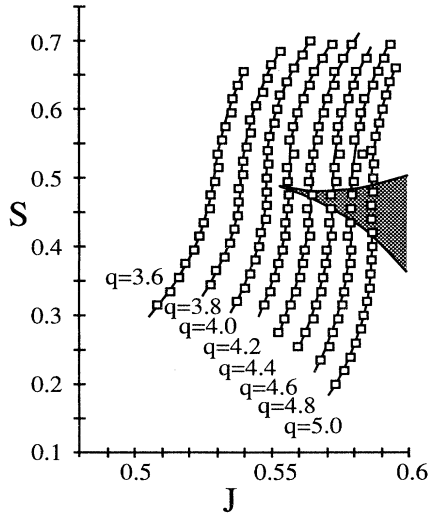


FIG. 1. The internal energy in two dimensions. The shaded area represents the latent heat.

versus  $d^*$  in Fig. 3.

If we compare the location and the size of the measured latent heat for  $d=2$  with the theoretical values that can be computed from (2.8) and (2.9), we see that the agreement for the location is good and for the size is reasonable. For  $(d,q)=(3,3)$  we can compare our result with the latent heat as obtained in [10,11], and again the agreement is reasonable. In comparing these values with the theoretical ones, one should bear in mind that the errors as indicated in Table III are purely statistical. We find the surprisingly good value of  $q_c^*=4.05$  for  $d=2$ . The value for  $d=4$  is also in good agreement with the theoretical value  $q_c=2$ . The result for  $d^*=2.5$  agrees

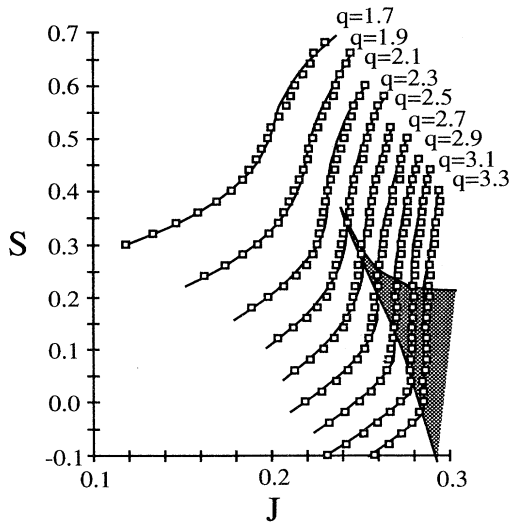


FIG. 2. The internal energy in three dimensions. The shaded area represents the latent heat.

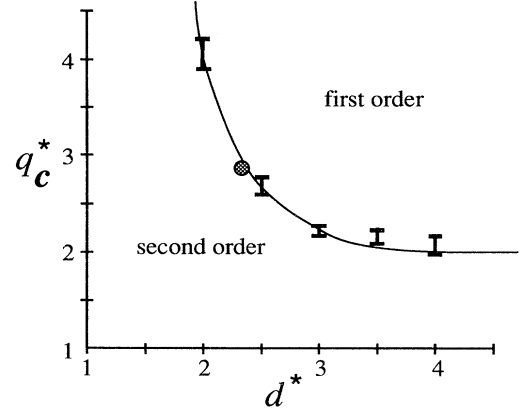


FIG. 3. Phase diagram of the  $(d^*, q^*)$  Potts model. ●: value taken from [5].

with the results of [5] (indicated by a circle in Fig. 3). What is certainly clear is that the phase transition for the three-dimensional three-state Potts model is of first order. On the other hand, our results seem to contradict the result of [4] that  $q_c(4-\epsilon)=2+\epsilon+O(\epsilon^2)$ . The behavior of  $q_c$  in Fig. 3 near the point (4,2) does not display any visible edge.

The exponents that are displayed in Table II are for  $q^*>q_c$  related to the standard exponent  $\alpha$  via  $\alpha=1-(1/e)$ . For  $q^*<q_c$  they have no direct physical meaning. Furthermore, the systematic error, due to the deviation from pure scaling in the region where we fitted  $J(s)$ , is quite large for the critical exponents, but is small for the latent heat. Therefore, one should not have too much confidence in the values for  $q^*>q_c$  either. If one's main interest is the determination of the values of the exponents, there are better methods to obtain these. One can, e.g., obtain  $e_l-1$  and  $e_r-1$  by fitting a powerlike curve through  $(s, dJ/ds)$ , where  $dJ/ds$  can be accurately determined as in [23]. Another method is to use the Monte Carlo renormalization-group method [23,24].

A final remark concerns the systematic errors involved in this method. As the derivative  $dJ/ds$  is typically  $\sim 0.02$  along the gaps in the  $(s, J)$  curves, we expect that the systematic errors due to the finite-size effect are certainly smaller than the statistical errors. The finite-size effect tends to increase the value of  $q_c$ . Comparing our value of  $q_c$  with the theoretical values for  $d=2,4$  suggests that the systematic error is altogether not much larger than 0.05.

## VI. CONCLUSIONS

In summary, we have simulated a large number of Potts models and have managed to confirm the suspected behavior of  $q_c$  versus  $d$ . Of course, our model is not equivalent to a real fractional  $(d,q)$  Potts model. From the definition, one sees that it does not interpolate analytically between the integer  $(d,q)$  models, but continuously. The fractional  $q$  behavior seems to be quite unambiguously defined, in contrast to the fractional  $d$  behavior, for

which other definitions are as plausible as ours. It would be interesting to try to “fractionalize” other systems that allow a description in terms of Swendsen-Wang dynamics.

The model can be improved in several ways. The definition of the  $J_i$  in terms of  $J^*$  can certainly be improved. One can also *define* the dimension of a Potts model by looking at the finite-size scaling of the correlation length [14], or at the finite-size scaling of the size of the largest cluster [18]. To determine the value of  $q_c$  one can also look at the finite-size scaling of the maximum

value of the specific heat, which is different for first- and second-order phase transitions [26,8]. Computation speed might be improved by using the algorithm of Wolff [20], or by using other methods to obtain the  $(J, s)$  curves [27,28]. Parallelizing or vectorizing the algorithm should also be possible. We leave these issues to future work.

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- [1] *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic, New York, 1972–1989), Vols. 1–13.
- [2] F. Y. Wu, *Rev. Mod. Phys.* **54**, 235 (1982).
- [3] R. J. Baxter, *J. Phys. C* **6**, L445 (1973).
- [4] A. Aharony and E. Pytte, *Phys. Rev. B* **23**, 362 (1981).
- [5] B. Nienhuis, E. K. Riedel, and M. Schick, *Phys. Rev. B* **23**, 6055 (1981).
- [6] S. J. Knak Jensen and O. G. Mouritsen, *Phys. Rev. Lett.* **43**, 1736 (1979).
- [7] H. W. J. Blöte and R. H. Swendsen, *Phys. Rev. Lett.* **43**, 799 (1979).
- [8] H. J. Herrmann, *Z. Phys. B* **35**, 171 (1979).
- [9] M. Fukugita and M. Okawa, *Phys. Rev. Lett.* **63**, 13 (1989).
- [10] W. G. Wilson and C. A. Vause, *Phys. Rev. B* **36**, 587 (1987).
- [11] O. F. de Alcantara Bonfim, *J. Stat. Phys.* **62**, 105 (1991).
- [12] R. V. Gavai, F. Karsch, and B. Petersson, *Nucl. Phys. B* **322**, 738 (1989).
- [13] Yu. Holovatch and N. Shpot (unpublished).
- [14] M. A. Novotny (unpublished).
- [15] R. H. Swendsen and J.-S. Wang, *Phys. Rev. Lett.* **58**, 86 (1987).
- [16] R. G. Edwards and A. D. Sokal, *Phys. Rev. D* **38**, 2009 (1988).
- [17] J.-S. Wang, *Physica A* **164**, 240 (1990).
- [18] M. D’Onorio De Meo, D. W. Heermann, and K. Binder, *J. Stat. Phys.* **60**, 585 (1990).
- [19] C.-K. Hu and K.-S. Mak, *Phys. Rev. B* **40**, 5007 (1989).
- [20] U. Wolff, *Phys. Rev. Lett.* **62**, 361 (1989).
- [21] C. F. Baillie and R. Gupta (unpublished).
- [22] *Applications of the Monte Carlo Method in Statistical Physics*, 2nd ed., edited by K. Binder, Topics in Current Physics Vol. 36 (Springer-Verlag, New York, 1987).
- [23] G. S. Pawley, R. H. Swendsen, D. J. Wallace, and K. G. Wilson, *Phys. Rev. B* **29**, 4030 (1984).
- [24] R. H. Swendsen, in *Real Space Renormalization*, edited by T. W. Burkhardt and J. M. J. van Leeuwen, Topics in Current Physics Vol. 30 (Springer-Verlag, New York, 1982).
- [25] D. W. Heermann and A. M. Burkitt, *Physica A* **162**, 210 (1990).
- [26] M. S. S. Challa, D. P. Landau, and K. Binder, *Phys. Rev. B* **34**, 1841 (1986).
- [27] A. M. Ferrenberg and R. H. Swendsen, *Phys. Rev. Lett.* **61**, 2635 (1988).
- [28] A. M. Ferrenberg and R. H. Swendsen, *Phys. Rev. Lett.* **63**, 1195 (1989).