# Undergraduate Research: Metric Distances Between Cayley Tables of Finite Groups

Mark Lewers

May 3, 2012

## 1 Basics of $d(\mathbb{Z}_n, \mathbb{Z}_m)$ (Assume $n \geq m$)

This paper deals with finding ways of comparing the Caley tables of groups and as a result, a way of comparing the actual structure of groups themselves. Most of this paper deals with examples and conjectures about patterns observed. The motivation of this work is to develop an analogue of the space of marked groups as studied by Grigorchuk, Gromov and others, using Cayley tables instead of Cayley graphs. Is it possible to put a metric on the set of all groups which will yield an interesting metric space? In this paper we only consider finite groups.

### 1.1 What is $d$?

Definition: The distance represented by $d(\mathbb{Z}_n, \mathbb{Z}_m)$ is the size of the union of the two Cayley tables of each group minus the intersection of the Cayley tables. This can be described as the best - fit interesction of the two Cayley tables (so the fit where the most cells of the table intersect).

Example: Consider $d(\mathbb{Z}_4, \mathbb{Z}_2)$:

- Cayley table For $\mathbb{Z}_2$:

| + | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

- Cayley table For $\mathbb{Z}_4$:

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 0 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 2 |

So for our superposition, the 0 in $\mathbb{Z}_2$ can be mapped to the 0 in $\mathbb{Z}_4$ and the 1 in $\mathbb{Z}_2$ can be mapped to the 2 in $\mathbb{Z}_4$. Note that we can re-label our elements in order to be able to do a symmetric mapping.

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 4 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 2 |

The blue cells represents the superposition of the $\mathbb{Z}_2$ elements and the green marks the intersection of the two Cayley tables. Notice that with these two groups, there are no additional union of cells created. Therefore the distance $\boldsymbol{d}$ here will be the **union of elements** minus their **intersection**. Here, the union is all the elements of $\mathbb{Z}_4$ (16) and the intersection is 4. So:

$$d(\mathbb{Z}_2, \mathbb{Z}_4) = 16 - 4 = 12.$$

Also note that the order in which these groups are written doesn't matter:

$$d(\mathbb{Z}_n, \mathbb{Z}_m) = d(\mathbb{Z}_m, \mathbb{Z}_n)$$

Here is another example for when the two groups **don't** divide each other. Consider $d(\mathbb{Z}_2, \mathbb{Z}_3)$. So we have that

- Cayley table For $\mathbb{Z}_2$:

| + | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

- Cayley table For $\mathbb{Z}_3$:

| + | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 2 | 0 |
| 2 | 2 | 0 | 1 |

To achieve the best mapping, we are going to have $0(\mathbb{Z}_2) \to 0(\mathbb{Z}_3)$. Thus we have two mappings to consider.

$\mathbf{1(\mathbb{Z}_2) \to 1(\mathbb{Z}_3)}$

| + | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 2 | 0 |
| 2 | 2 | 0 | 1 |

$\mathbf{1(\mathbb{Z}_2) \to 2(\mathbb{Z}_3)}$

| + | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 2 | 0 |
| 2 | 2 | 0 | 1 |

Just like the previous example, green represents the elements that do intersect but red represents the cells where they don't intersect.

In both cases we have mappings where all map except one element. In order for that one element to intersect, it needs to be mapped to the identity element (0) but there is no identity element in that spot. Thus the element from $\mathbb{Z}_2$ is considered a "stacked" element.

So the best mapping here is the elements of $\mathbb{Z}_3$ Cayley table ($3^2$) plus one stacked element from the Cayley table of $\mathbb{Z}_2$ minus the remaining interesecting elements from the Cayley table of $\mathbb{Z}_2$. So

$$d(\mathbb{Z}_2, \mathbb{Z}_3) = \Big((3^2 + 1) - 3\Big) = 3^2 - 2 = 7.$$

The obvious question to ask if there is a pattern of behavior when comparing the distances. Below are patterns that were observed and, in one case, proved.

- $n|m$:

$$d(\mathbb{Z}_n, \mathbb{Z}_m) = m^2 - n^2 \tag{1}$$

*Proof.* Since $n|m$, and we are talking about cyclic moduli groups, then $\mathbb{Z}_n$ is isomorphic to a subgroup of $\mathbb{Z}_m$ so all the elements in $\mathbb{Z}_n$ have a one-to-one relationship to a subgroup of $\mathbb{Z}_m$ and so to certain elemenets of $\mathbb{Z}_m$. Therefore when you map to $\mathbb{Z}_m$'s Cayley table, the Cayley table of $\mathbb{Z}_m$ is superimposed with no elements left over. By our definition of distance, then it is the elements of $\mathbb{Z}_m$ ($m^2$) plus the non-intersecting elements but we have none here so that number is zero, minus all the superimposed elements of $\mathbb{Z}_n$ which here is all the elements of the Cayley table of $\mathbb{Z}_n$ ($n^2$). Therefore, the distance is $m^2 - n^2$. □

  – if $n = m$:

$$d(\mathbb{Z}_n, \mathbb{Z}_n) = 0 \tag{2}$$

- **Conjecture**: $n \nmid m$:

$$^*d(\mathbb{Z}_n, \mathbb{Z}_m) \leq \begin{cases} n^2 - \left(\dfrac{m^2}{2}\right) & \text{if } m \text{ is even} \\ n^2 - \left(\dfrac{m^2+1}{2}\right) & \text{if } m \text{ is odd} \end{cases} \tag{3}$$

**This is a relationship that I have observed calculating the distances between $\mathbb{Z}$ groups from orders 2-11. The reason it is "less than or equal to" is that since that the statement is not proven. So there could be a distance mapping "smaller" than this. All this cojecture says is that it cannot be larger than the two cases. However, I will assume it is true (for this paper) and use it to prove observations.*

| $d$ | $\mathbb{Z}_2$ | $\mathbb{Z}_3$ | $\mathbb{Z}_4$ | $\mathbb{Z}_5$ | $\mathbb{Z}_6$ | $\mathbb{Z}_7$ | $\mathbb{Z}_8$ | $\mathbb{Z}_9$ | $\mathbb{Z}_{10}$ | $\mathbb{Z}_{11}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbb{Z}_2$ | 0 | $3^2-2$ | $4^2-2^2$ | $5^2-2$ | $6^2-2^2$ | $7^2-2$ | $8^2-2^2$ | $9^2-2$ | $10^2-2^2$ | $11^2-2$ |
| $\mathbb{Z}_3$ | | 0 | $4^2-5$ | $5^2-5$ | $6^2-3^2$ | $7^2-5$ | | $9^2-3^2$ | | |
| $\mathbb{Z}_4$ | | | 0 | $5^2-8$ | $6^2-8$ | $7^2-8$ | $8^2-4^2$ | | | |
| $\mathbb{Z}_5$ | | | | 0 | $6^2-13$ | $7^2-13$ | | | | |
| $\mathbb{Z}_6$ | | | | | 0 | $7^2-18$ | | | | |
| $\mathbb{Z}_7$ | | | | | | 0 | $8^2-25$ | | | |
| $\mathbb{Z}_8$ | | | | | | | 0 | $9^2-32$ | | |
| $\mathbb{Z}_9$ | | | | | | | | 0 | $10^2-41$ | |
| $\mathbb{Z}_{10}$ | | | | | | | | | 0 | $11^2-50$ |
| $\mathbb{Z}_{11}$ | | | | | | | | | | 0 |

This table show the known **lowest** values of $d$ distance between cyclic groups ranging from $\mathbb{Z}_2$ to $\mathbb{Z}_{11}$.

## 1.2 Is $d$ metric?

In order to consider this a **metric distance**, then we need to confirm that

1. $d(\mathbb{Z}_m, \mathbb{Z}_n) \geq 0$
2. $d(\mathbb{Z}_m, \mathbb{Z}_n) = 0$ if and only if $\mathbb{Z}_n = \mathbb{Z}_m$
3. $d(\mathbb{Z}_m, \mathbb{Z}_n) = d(\mathbb{Z}_n, \mathbb{Z}_m)$
4. $d(\mathbb{Z}_m, \mathbb{Z}_n) + d(\mathbb{Z}_n, \mathbb{Z}_p) \geq d(\mathbb{Z}_p, \mathbb{Z}_m)$ for $m, n, p \in \mathbb{Z}$

Well since we are subtracting the smaller ordered group from the larger ordered group, then the distance is never less than 0 and it is only 0 if the two groups are the same group (if the two groups are isomorphic to each other, we will consider them to be the same group).

Earlier it was stated that it doesn't matter what order the groups are written in so condition 3 is met. Therefore we need to prove that Triangle Inequality holds for some $m \leq n \leq p$.

$$d(\mathbb{Z}_m, \mathbb{Z}_n) + d(\mathbb{Z}_n, \mathbb{Z}_p) \geq d(\mathbb{Z}_p, \mathbb{Z}_m) \tag{4}$$

*Proof.* Let us define $f(n) = n^2, \frac{n^2}{2}$, or $\frac{n^2+1}{2}$ and $f(m) = m^2, \frac{m^2}{2}$, or $\frac{m^2+1}{2}$. Each of the cases corresponds to if it divides the order of the higher group, doesn't divide it and its own order is even, or if it doesn't divide the order and it's own order is odd, respectively.
In general, we have two main cases:

- Suppose $m|p$: First we note that since $m|p$, then $d(\mathbb{Z}_m, \mathbb{Z}_p) = p^2 - m^2$. So

$$\begin{aligned} d(\mathbb{Z}_n, \mathbb{Z}_m) + d(\mathbb{Z}_p, \mathbb{Z}_n) &= \Big(n^2 - f(m)\Big) + \Big(p^2 - f(n)\Big) \\ &= p^2 + \Big(n^2 - f(n)\Big) - f(m) \\ &\geq p^2 - f(m) \end{aligned}$$

  Since the greatest element of $f(m)$ is $m^2$, then

$$p^2 - f(m) \geq p^2 - m^2$$

- Suppose $m \not| p$:
  - $m \not| n$: Note that $d(\mathbb{Z}_p, \mathbb{Z}_m) = p^2 - f'(m)$ and $d(\mathbb{Z}_m, \mathbb{Z}_n) = n^2 - f'(m)$ where $f'(m)$ is $f(m)$ excluding $m^2$. Since $m$ doesn't divide either $n$ or $p$, then the value of $f'(m)$ in both $d(\mathbb{Z}_m, \mathbb{Z}_n)$ and $d(\mathbb{Z}_m, \mathbb{Z}_p)$ are the same. Therefore

$$\begin{aligned} d(\mathbb{Z}_n, \mathbb{Z}_m) + d(\mathbb{Z}_p, \mathbb{Z}_n) &= \Big(n^2 - f'(m)\Big) + \Big(p^2 - f(n)\Big) \\ &= p^2 + \Big(n^2 - f(n)\Big) - f'(m) \\ &\geq p^2 - f'(m) \end{aligned}$$

- $m|n$, $n \not| p$, $m$ is odd, and $n$ is odd: $d(\mathbb{Z}_n, \mathbb{Z}_m) + d(\mathbb{Z}_p, \mathbb{Z}_n) = (n^2 - m^2) + (p^2 - \frac{n^2+1}{2})$. Rearrange terms to get

$$\begin{aligned} p^2 + \left(n^2 - \left(\frac{n^2+1}{2}\right)\right) - m^2 &\geq p^2 + \left(m^2 - \left(\frac{m^2+1}{2}\right)\right) - m^2 \\ &= p^2 - \left(\frac{m^2+1}{2}\right) = d(\mathbb{Z}_p, \mathbb{Z}_m). \end{aligned}$$

- $m|n$, $n \not| p$, $m$ is odd, and $n$ is even: $d(\mathbb{Z}_n, \mathbb{Z}_m) + d(\mathbb{Z}_p, \mathbb{Z}_n) = (n^2 - m^2) + (p^2 - \frac{n^2}{2})$. Rearrange terms to get

$$p^2 + \left(\frac{n^2}{2}\right) - m^2 > p^2 + \left(\frac{(m+1)^2}{2}\right) - m^2 = p^2 + \left(\frac{m^2+2m+1}{2}\right) - m^2$$

You can further manipulate it by using $m^2 + 2m + 1 > m^2 - 1$

$$\begin{aligned} p^2 + \left(\frac{m^2+2m+1}{2}\right) - m^2 &> p^2 + \left(\frac{m^2-1}{2}\right) - m^2 \\ &= p^2 - \left(\frac{m^2+1}{2}\right) = d(\mathbb{Z}_p, \mathbb{Z}_m). \end{aligned}$$

- $m|n$, $n \not| p$, $m$ is even, and $n$ is odd: $d(\mathbb{Z}_n, \mathbb{Z}_m) + d(\mathbb{Z}_p, \mathbb{Z}_n) = (n^2 - m^2) + (p^2 - \frac{n^2+1}{2})$. Rearrange terms to get

$$\begin{aligned} p^2 + \left(n^2 - \left(\frac{n^2+1}{2}\right)\right) - m^2 &= p^2 + \frac{n^2-1}{2} - m^2 \\ &> p^2 + \frac{m^2}{2} - m^2 \\ &= p^2 - \frac{m^2}{2} = d(\mathbb{Z}_p, \mathbb{Z}_m) \end{aligned}$$

- $m|n$, $n \not| p$, $m$ is even, and $n$ is even: $d(\mathbb{Z}_n, \mathbb{Z}_m) + d(\mathbb{Z}_p, \mathbb{Z}_n) = (n^2 - m^2) + (p^2 - \frac{n^2}{2})$. Rearrange terms to get

$$\begin{aligned} p^2 + \left(\frac{n^2}{2}\right) - m^2 &\geq p^2 + \left(\frac{m^2}{2}\right) - m^2 \\ &= p^2 - \frac{m^2}{2} = d(\mathbb{Z}_p, \mathbb{Z}_m) \end{aligned}$$

We have exhausted all possibilities. Therefore, the triangle inequality holds. $\square$

## 2 Modifications of $d(\mathbb{Z}_n, \mathbb{Z}_m)$

Now that we found a basic notion of metric distance, the next question is whether this is the only metric distance we can come up with? Are there other versions of distance that are variations of this?

### 2.1 $d'(\mathbb{Z}_n, \mathbb{Z}_m)$

Definition: The distance notation $d'(\mathbb{Z}_n, \mathbb{Z}_m)$ means that

$$d'(\mathbb{Z}_n, \mathbb{Z}_m) = \frac{d(\mathbb{Z}_n, \mathbb{Z}_m)}{|\mathbb{Z}_n| + |\mathbb{Z}_m|} \tag{5}$$

This has not been given much focus and research on however.

### 2.2 $d''(\mathbb{Z}_n, \mathbb{Z}_m)$

Definition: The distance notation $d''(\mathbb{Z}_n, \mathbb{Z}_m)$ means that

$$d''(\mathbb{Z}_n, \mathbb{Z}_m) = \frac{d(\mathbb{Z}_n, \mathbb{Z}_m)}{|\mathbb{Z}_n|^2 + |\mathbb{Z}_m|^2} \tag{6}$$

One property observed by this definition of distance is that

$$\frac{1}{4} < d''(\mathbb{Z}_n, \mathbb{Z}_m) < 1.$$

More specifically, when $m|n$, $m \neq n$,

$$\frac{3}{5} \leq d''(\mathbb{Z}_n, \mathbb{Z}_m) < 1.$$

*Proof.* We have three cases to consider.

- Consider $m|n$. Then we can write

  $$d''(\mathbb{Z}_n, \mathbb{Z}_m) = \frac{d(\mathbb{Z}_n, \mathbb{Z}_m)}{n^2 + m^2} = \frac{n^2 - m^2}{n^2 + m^2} = \frac{n^2}{n^2 + m^2} - \frac{m^2}{n^2 + m^2}.$$

  It is obvious by looking at these fractions the number that you get is going to be always less than 1 for $m \neq 0$. Now to show that the lower bound is $\frac{3}{5}$.

  Since $m|n$ and they are not equal, then we can say that

  $$n \geq 2m.$$

  Since $m, n$ are integers, then their square holds the same inequality

  $$\begin{aligned} n^2 &\geq 4m^2 \\ n^2 - 4m^2 &\geq 0 \\ 2n^2 - 8m^2 &\geq 0 \end{aligned}$$

If we add $3n^2 + 3m^2$ to both sides we get

$$\begin{aligned} 5n^2 - 5m^2 &\geq 3n^2 + 3m^2 \\ 5(n^2 - m^2) &\geq 3(n^2 + m^2) \\ \frac{n^2 - m^2}{n^2 + m^2} &\geq \frac{3}{5} \end{aligned}$$

Therefore, the lower bound for $d''$ when $m|n$ is $\frac{3}{5}$.

- <u>Consider $m \not| n$ and $m$ is even.</u> Then we can write

$$d''(\mathbb{Z}_n, \mathbb{Z}_m) = \frac{d(\mathbb{Z}_n, \mathbb{Z}_m)}{n^2 + m^2} = \frac{n^2 - \frac{m^2}{2}}{n^2 + m^2} = \frac{2n^2}{2n^2 + 2m^2} - \frac{m^2}{2n^2 + 2m^2}.$$

Likewise, this fraction will always be less than 1. Now to show that the fraction is bounded below by $\frac{1}{4}$.

We are assuming that $n > m$. Since integers, then their square would preseve the inequality

$$\begin{aligned} n^2 &> m^2 \\ 3n^2 &> 3m^2 \\ 3n^2 - 3m^2 &> 0 \end{aligned}$$

Add $n^2 + m^2$ to both sides to get

$$\begin{aligned} 4n^2 - 2m^2 &> n^2 + m^2 \\ 8n^2 - 4m^2 &> 2n^2 + 2m^2 \\ 4(2n^2 - m^2) &> 2n^2 + 2m^2 \end{aligned}$$

Divide both sides by $2n^2 + 2m^2$ and 4 to get

$$\begin{aligned} \frac{2n^2 - m^2}{2n^2 + 2m^2} &> \frac{1}{4} \\ \frac{n^2 - \frac{m^2}{2}}{n^2 + m^2} &> \frac{1}{4} \end{aligned}$$

Therefore when $m \not| n$ and $m$ is even, $d''$ will always be between $\frac{1}{4}$ and 1.

- <u>Consider $m \not| n$ and $m$ is odd.</u> Then we can write

$$d''(\mathbb{Z}_n, \mathbb{Z}_m) = \frac{d(\mathbb{Z}_n, \mathbb{Z}_m)}{n^2 + m^2} = \frac{n^2 - \frac{m^2+1}{2}}{n^2 + m^2} = \frac{2n^2}{2n^2 + 2m^2} - \frac{m^2 + 1}{2n^2 + 2m^2}.$$

This is also bounded from above by 1. Now to show that it is bounded below by $\frac{1}{4}$.

Since $m \not| n$ and $m$ is odd, then $n - m$ is at least 1. And $n^2 - m^2 > 1$ so we also state that

$$n^2 - m^2 > \frac{2}{3}$$
$$3n^2 - 3m^2 > 2$$

Add $n^2 + m^2$ to both sides and move the 2 over to get

$$4n^2 - 2m^2 - 2 > n^2 + m^2$$
$$4(2n^2 - m^2 - 1) > 2n^2 + 2m^2$$

Divide by $2n^2 + 2m^2$ and 4 to get

$$\frac{2n^2 - m^2 - 1}{2n^2 + 2m^2} > \frac{1}{4}$$
$$\frac{n^2 - \frac{m^2+1}{2}}{n^2 + m^2} > \frac{1}{4}$$

Thus this distance is bounded by $\frac{1}{4}$ and 1.

We have exhausted all cases. This completes the proof. □

But is this new notion of distance useful? One thing that would help us in determining a use for if it is something that can consider metric. So like what we did for $d$ distance, does this $d''$ distance satisfy the triangle inequality? We will quickly show that at least for $n \leq m \leq p$ (all positive integers) where $n|m$ and $m|p$ that in fact it does hold.

*Proof.* So here

$$d''(\mathbb{Z}_n, \mathbb{Z}_m) + d''(\mathbb{Z}_p, \mathbb{Z}_n) = \frac{n^2 - m^2}{n^2 + m^2} + \frac{p^2 - n^2}{p^2 + n^2} = \frac{(n^2 - m^2)(p^2 + n^2) + (n^2 + m^2)(p^2 - n^2)}{(n^2 + m^2)(p^2 + n^2)}$$

$$= \frac{(p^2n^2 + n^4 - p^2m^2 - n^2m^2) + (p^2n^2 - n^4 + p^2m^2 - n^2m^2)}{p^2n^2 + n^4 + p^2m^2 + n^2m^2}$$

$$= \frac{n^2(2p^2 - 2m^2)}{n^2\left(n^2 + p^2 + m^2 + \frac{p^2m^2}{n^2}\right)} = \frac{2p^2 - 2m^2}{p^2 + p^2\left(\frac{m^2}{n^2}\right) + n^2 + m^2}$$

For the next part, keep in mind that the quantity $2p^2 + 2m^2 = p^2 + p^2\left(\frac{m^2}{m^2}\right) + m^2 + m^2 \geq p^2 + p^2\left(\frac{m^2}{n^2}\right) + n^2 + m^2$.

$$\geq \frac{2p^2 - 2m^2}{p^2 + p^2\left(\frac{m^2}{m^2}\right) + m^2 + m^2} = \frac{2(p^2 - m^2)}{2(p^2 + m^2)}$$

$$= \frac{p^2 - m^2}{p^2 + m^2} = d''(\mathbb{Z}_p, \mathbb{Z}_m)$$

□

But this inequality is more that $\geq$. It is a strict $>$ inequality. Since each $d''$ is between $\frac{3}{5}$ and 1, then the sum of two of them will always be greater than 1 which means it is allways be greater than any $d''$ it is being compared to.

### 2.3 $\delta(\mathbb{Z}_n, \mathbb{Z}_m)$

Definition: The distance notation $\delta(\mathbb{Z}_n, \mathbb{Z}_m)$, is:

$$\delta(\mathbb{Z}_n, \mathbb{Z}_m) = d(\mathbb{Z}_n, \mathbb{Z}_m) - (|\mathbb{Z}_m|^2 - |\mathbb{Z}_n|^2) \tag{7}$$

For this definition of distance, the triangular inquality always works.

## 3 Distance Of Direct Product Groups: $d(\mathbb{Z}_m \times \mathbb{Z}_n, \mathbb{Z}_r \times \mathbb{Z}_s)$

After anaylzing general cyclic groups and ways of coming up with metric distance between them, the next type of groups to think about are the direct products of cyclic groups ($\mathbb{Z}_m \times \mathbb{Z}_n$). If we attempt to take the distance $d$ between two direct product groups say $\mathbb{Z}_m \times \mathbb{Z}_n$ and $\mathbb{Z}_r \times \mathbb{Z}_s$, will be come up with a similar relationship and will the triangle inequality hold. Note, for these first two sections, we are dealing with $\mathbb{Z}_n \times \mathbb{Z}_n$ and these groups may NOT be cyclic.

### 3.1 $d(\mathbb{Z}_m \times \mathbb{Z}_m, \mathbb{Z}_{m^2})$

We will first look at distance between a direct product of group $\mathbb{Z}_m$ and itself squared. So what is the distance, $d$, here?

Let us start by looking at an example: $\boldsymbol{d(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4)}$:

So the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ has elements $\{(0,0),(1,0),(0,1),(1,1)\}$ while $\mathbb{Z}_4$ has $\{0,1,2,3\}$. Note that both groups have the same number of elements so it is arbitrary which Cayley table is chosen to be mapped onto. Earlier we have shown that

- Cayley Table For $\mathbb{Z}_4$:

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 0 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 2 |

  Also we can see that

- Cayley Table For $\mathbb{Z}_2 \times \mathbb{Z}_2$:

| + | (0,0) | (1,0) | (0,1) | (1,1) |
|---|---|---|---|---|
| (0,0) | (0,0) | (1,0) | (0,1) | (1,1) |
| (1,0) | (1,0) | (0,0) | (1,1) | (0,1) |
| (0,1) | (0,1) | (1,1) | (0,0) | (1,0) |
| (1,1) | (1,1) | (0,1) | (1,0) | (0,0) |

Since order is same in both groups, let us map the $\mathbb{Z}_4$ table onto the $\mathbb{Z}_2 \times \mathbb{Z}_2$ table. To determine the best fit mapping, it is imperative that the identity in $\mathbb{Z}_4$ (0) be mapped to the identity of $\mathbb{Z}_2 \times \mathbb{Z}_2$ (0,0). If that is done, then all different mapping labels yield the same table (here let $\mathbf{1} \to \mathbf{(1,0)}$, $\mathbf{2} \to \mathbf{(0,1)}$, and $\mathbf{3} \to \mathbf{(1,1)}$).

| + | (0,0) | (1,0) | (0,1) | (1,1) |
|---|---|---|---|---|
| (0,0) | (0,0) | (1,0) | (0,1) | (1,1) |
| (1,0) | (1,0) | (0,0) | (1,1) | (0,1) |
| (0,1) | (0,1) | (1,1) | (0,0) | (1,0) |
| (1,1) | (1,1) | (0,1) | (1,0) | (0,0) |

where the green cells are the intersecting elements and the red cells are the stacked elements. So

$$d(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4) = (4^2 + 4) - 12 = 4^2 - 8 = 2^4 - 2^3 = 8.$$

The question is that does this one example lead anywhere? Notice that one way the answer is expressed is as

$$d(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_{2^2}) = 2^4 - 2^3.$$

Upon inspection, one can look at $d(\mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_{3^2})$ and the best mapping obtained (thus far) is such that

$$d(\mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_{3^2}) \leq 3^4 - 3^3 = 54$$

Similarly for if $n = 4$, then the lowest mapping so far obtained is that such

$$d(\mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_{4^2}) \leq 4^4 - 4^3 = 192$$

So it seems that there is a relationship developing.

***Conjecture***: For $n \in \mathbb{Z}^+$:

$$d(\mathbb{Z}_n \times \mathbb{Z}_n, \mathbb{Z}_{n^2}) \leq n^4 - n^3 \tag{8}$$

*For all we know, this could be the equation for the distance. But it isn't proven and while this is the best fit relationship obtained, there could be an even better mapping not found.*

## 3.2 $d(\mathbb{Z}_m \times \mathbb{Z}_m, \mathbb{Z}_n \times \mathbb{Z}_n)$

Here we will take a look at when finding distance between two direct product groups of different order. But the ones here are such so that each group is a direct product with itself (i.e. $\mathbb{Z}_n \times \mathbb{Z}_n$). This section only contains examples that were found and hopefully someone can find further relationships/proofs for these. Here are some solutions found when $\mathbb{Z}_2 \times \mathbb{Z}_2$ is the lower ordered group.

- $d(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_3) = (3^2)^2 - 2 = 9^2 - 2$
- $d(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4 \times \mathbb{Z}_4) = (4^2)^2 - (2^2)^2 = 16^2 - 4^2$

- $d(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_5 \times \mathbb{Z}_5) = (5^2)^2 - 2 = 25^2 - 2$

So again we have that if the order of $(\mathbb{Z}_n \times \mathbb{Z}_n)$ divides the order of $(\mathbb{Z}_m \times \mathbb{Z}_m)$, then it looks like we again have the relationship that the distance is the order of the higher groups squared minus the order of the lower group squared. The same arguement holds true in fact where you have the lowered order group is isomorphic to a subgroup of the higher ordered group. Hence it is a one-to-one mapping so when mapping a Cayley table, all the elements intersect.

***Conjecture***: if $2 \not| n$ and $n > 2$, then

$$d(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_n \times \mathbb{Z}_n) \leq (n^2)^2 - 2.$$

Other solutions discovered include:

- $d(\mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_4 \times \mathbb{Z}_4) \leq (4^2)^2 - 17$
- $d(\mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_5 \times \mathbb{Z}_5) \leq (5^2)^2 - 17$

It is too early to make a conjecture that the distance is the higher ordered group squared minus 17. More examples need to be done to determine if this relationship holds for larger groups.

# 4 Future Research

The recent findings with metric distances between groups have not come close to scratching the surface of what we could know. Further research includes analyzing the mappings between Cayley tables that are the best fit by trying to find more patterns to what elements are being mapped to and what order these elements are. Further research into the modifications of $d$ proposed in the paper is also in order.

One project that would help tremendously with finding the closest distance between cyclic/noncyclic groups is to write a computer program that would find the total number of possible mappings and, after doing the mapping, returns the best fit map.