



Correlations equalities and some upper bounds for the critical temperature for spin one systems

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ABSTRACT

Starting from correlation identities for the Blume–Capel spin 1 systems and using correlation inequalities, we obtain rigorous upper bounds for the critical temperature. The obtained results improve over effective field type results.

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1. Introduction

Correlation inequalities combined with exact identities are useful in obtaining rigorous results in statistical mechanics. Among the various questions that are resolved by them one is the decay of the correlation functions. The decay of the correlation functions give information about the critical couplings of statistical mechanics models. In this work, the method will be applied to study systems described by the spin one Blume–Capel model [1,2]. Firstly, we present the derivation of an exact relation for the two spin correlation function, valid in any dimension, which is an extension of Callen's identity for the spin 1/2 Ising model [3]. Starting from these identities we will then make use of the first and second Griffiths inequalities and Newman's inequalities to obtain the exponential decay of the two spin correlation functions. The coupling constant which are the upper bounds for the critical temperature are obtained for $d = 2$ and $d = 3$ dimensions. In this study the coupling parameters obtained improve effective field results. Upper bounds for the critical temperature T_c for Ising and multi-component spin systems have been obtained by showing (for $T > T_c$) the exponential decay of the two-point function [4–6]. Spin correlation inequalities and their iteration are used by Brydges et al. [6], Lieb [7] and Simon [5].

The aim of the present work is to obtain rigorous upper bounds for the critical couplings of the spin one Blume–Capel model in two- and three-dimensional lattices. The method we will employ is based on an exact two-point correlation function identity and rigorous inequalities for the correlation functions. Thus, our results are rigorous upper bounds for the critical temperature of the model. The results improve mean field ones, and are comparable to others obtained by effective field theory, with the advantage of representing rigorous limits for the true critical temperature.

The procedure to improve the bound for the critical temperature over the effective field result for the classical $S = 1$ model is as follows: starting from a two-point correlation function identity, a generalization of Callen's identity [3] for this model [8] and using Griffith's 1st and 2nd inequalities (Griffith I, II) (see Refs. [9–15]) and Newman's inequalities [11,16] we establish the inequality for the two-point function, $\langle S_0 S_l \rangle$, as

$$\langle S_0 S_l \rangle \leq \sum_j a_j \langle S_j S_l \rangle, \quad 0 \leq a_j \leq 1 \quad (1)$$

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which when iterated (see Ref. [5]) implies exponential decay for $T > T_c$. In Section 2 we present the derivation of the correlation identities for the Blume–Capel model [8]. In Section 3, we apply these identities to the $d = 2$ and $d = 3$ lattices. Next, in Section 4, we apply the correlation inequalities to obtain the upper bounds for T_c . Numerical results can be found in Section 5, and in Section 6 we present our concluding remarks.

We write the Hamiltonian for the classical spin one system, known as the Blume–Capel model, as

$$H = -J \sum_{i,j} S_i S_j - D \sum_i S_i^2, \quad (2)$$

where $J > 0$, D is the single ion anisotropy and the first sum is over the nearest neighbors spins on the lattice. We define the thermal average $\langle \dots \rangle$ by

$$\langle \dots \rangle = Z^{-1} \sum_{\{S_i\}} (\dots) e^{-\beta H}, \quad Z = \sum_{\{S_i\}} e^{-\beta H} \quad (3)$$

where each S_i is restricted by $S_i = -1, 0, +1$.

2. Correlation identity for the spin one model

We reproduce the generalization of Callen's identity for the spin 1 Blume–Capel model which has been obtained previously by Siqueira and Fittipaldi [8], derived in a manner analogous to the ones for the spin 1/2 Ising model [17], the transverse Ising model [18] and Z_2 gauge model [19]. Let

$$\langle F(S) S_i \rangle = \frac{\text{Tr}(F(S) S_i e^{-\beta H})}{\text{Tr}(e^{-\beta H})}, \quad (4)$$

where $F(S)$ is any function of S different from S_i . We can write $H = H_i + H'$, where

$$H_i = - \left(\sum_{|j|=1} J_{ij} S_j \right) S_i - D S_i^2, \quad (5)$$

is the Hamiltonian describing site i and its neighbors, and H' corresponds to the Hamiltonian of the rest of the lattice. Consequently $[H_i, H'] = 0$. From Eqs. (2) and (3), we get,

$$\langle F(S) S_i \rangle = \frac{\text{Tr} F(S) e^{-\beta(H_i+H')} S_i}{\text{Tr} e^{-\beta(H_i+H')}} = \frac{\text{Tr}' \text{Tr}_i F(S) e^{-\beta H_i} S_i e^{-\beta H'}}{\text{Tr}' \text{Tr}_i e^{-\beta H_i} e^{-\beta H'}} \quad (6)$$

or

$$\langle F(S) S_i \rangle = \frac{\text{Tr}' \text{Tr}_i F(S) e^{-\beta H_i} e^{-\beta H'} \frac{\text{Tr}_i e^{-\beta H_i} S_i}{\text{Tr}_i e^{-\beta H_i}}}{\text{Tr}' \text{Tr}_i e^{-\beta H_i} e^{-\beta H'}} \quad (7)$$

where $\text{Tr}' \text{Tr}_i = \text{Tr}$. Finally, we obtain,

$$\langle F(S) S_i \rangle = \left\langle F(S) \frac{\text{Tr}_i e^{-\beta H_i} S_i}{\text{Tr}_i e^{-\beta H_i}} \right\rangle. \quad (8)$$

Explicitly operating the trace Tr_i , we get,

$$\begin{aligned} \langle F(S) S_i \rangle &= \left\langle F(S) \frac{2e^{\beta D} \sinh(\sum_j \beta J_{ij} S_j)}{2e^{\beta D} \cosh(\sum_j \beta J_{ij} S_j) + 1} \right\rangle \\ &= \left\langle F(S) \prod_{|j|=1} e^{\beta J_{ij} S_j \nabla} \right\rangle f(x)|_{x=0}, \end{aligned} \quad (9)$$

with $\nabla \equiv \frac{\partial}{\partial x}$, such that $e^{\alpha \nabla} f(x) = f(x + \alpha)$, and

$$f(x) = \frac{2e^{\beta D} \sinh(x)}{2e^{\beta D} \cosh(x) + 1}. \quad (10)$$

As $S_j^{2n} = S_j^2$ and $S_j^{2n+1} = S_j$ for $n = 0, 1, 2, 3, \dots$, we obtain,

$$e^{S_j A} = S_j^2 \cosh(A) + S_j \sinh(A) + 1 - S_j^2, \quad (11)$$

and, applying Eqs. (10) and (11) in Eq. (9), we get

$$\langle F(S)S_i \rangle = \left\langle F(S) \prod_{j \neq i, |j|=1} (S_j^2 \cosh(\beta J_{ij} \nabla) + S_j \sinh(\beta J_{ij} \nabla) + 1 - S_j^2) \right\rangle f(x)|_{x=0}. \quad (12)$$

Similarly for the correlation function involving the square of the spin function S_i^2 , we obtain,

$$\langle G(S)S_i^2 \rangle = \left\langle G(S) \prod_{j \neq i, |j|=1} e^{\beta J_{ij} S_j \nabla} \right\rangle g(x)|_{x=0}, \quad (13)$$

with,

$$g(x) = \frac{2e^{\beta D} \cosh(x)}{2e^{\beta D} \cosh(x) + 1}, \quad (14)$$

resulting in,

$$\langle G(S)S_i^2 \rangle = \left\langle G(S) \prod_{j \neq i, |j|=1} (S_j^2 \cosh(\beta J_{ij} \nabla) + S_j \sinh(\beta J_{ij} \nabla) + 1 - S_j^2) \right\rangle g(x)|_{x=0}. \quad (15)$$

The function $G(S)$ is any function of S , except S_i^2 . Eqs. (12) and (15) are exact and generalize Callen's identity which was obtained for the $S = 1/2$ Ising model [3].

3. Exact correlation identities applied to the $d = 2$ and $d = 3$ lattices

Let us apply the previous results for $\langle F(S)S_i \rangle$ and $\langle G(S)S_i^2 \rangle$ given by Eqs. (12) and (15) for specific lattices in two- and three-dimensions. The two spin correlation functions, $\langle S_0 S_l \rangle$, are obtained from Eqs. (12) and (15) by defining $F(S) = S_l$.

3.1. For the $d = 2$ and $z = 3$, the honeycomb lattice

We obtain from Eq. (12)

$$\langle S_0 S_l \rangle = A_1 \sum_i \langle S_i S_l \rangle + A_2 \sum_{i < j} \langle S_i S_j^2 S_l \rangle + A_3 \sum_{i < j < k} \langle S_i S_j S_k S_l \rangle + A_4 \sum_{i < j < k} \langle S_i S_j^2 S_k^2 S_l \rangle, \quad (16)$$

where the A coefficients are given in Appendix A.1. We also obtain, from Eq. (15),

$$\langle S_0^2 S_l \rangle = B_0 + B_1 \sum_i \langle S_i^2 S_l \rangle + B_2 \sum_{i < j} \langle S_i S_j S_l \rangle + B_3 \sum_{i < j} \langle S_i^2 S_j^2 S_l \rangle + B_4 \sum_{i < j < k} \langle S_i S_j S_k^2 S_l \rangle + B_5 \sum_{i < j < k} \langle S_i^2 S_j^2 S_k^2 S_l \rangle, \quad (17)$$

where the B coefficients are given in Appendix A.1.

3.2. For $d = 2$ and $z = 4$, the square lattice

We obtain from Eq. (12) for the two spin correlation functions $\langle S_0 S_l \rangle$ the expression,

$$\begin{aligned} \langle S_0 S_l \rangle = & A_1 \sum_i \langle S_i S_l \rangle + A_2 \sum_{i < j} \langle S_i S_j^2 S_l \rangle + A_3 \sum_{i < j < k} \langle S_i S_j S_k S_l \rangle \\ & + A_4 \sum_{i < j < k} \langle S_i S_j^2 S_k^2 S_l \rangle + A_5 \sum_{i < j < k < m} \langle S_i S_j S_k S_m^2 S_l \rangle + A_6 \sum_{i < j < k < m} \langle S_i S_j^2 S_k^2 S_m^2 S_l \rangle, \end{aligned} \quad (18)$$

where the A coefficients are given in Appendix A.2. We also obtain, for the function $\langle S_0^2 S_l \rangle$,

$$\begin{aligned} \langle S_0^2 S_l \rangle = & B_0 + B_1 \sum_i \langle S_i^2 S_l \rangle + B_2 \sum_{i < j} \langle S_i S_j S_l \rangle + B_3 \sum_{i < j} \langle S_i^2 S_j^2 S_l \rangle + B_4 \sum_{i < j < k} \langle S_i S_j S_k^2 S_l \rangle \\ & + B_5 \sum_{i < j < k} \langle S_i^2 S_j^2 S_k^2 S_l \rangle + B_6 \sum_{i < j < k < m} \langle S_i S_i S_k S_m \rangle + B_7 \sum_{i < j < k < m} \langle S_i S_j S_k^2 S_m^2 \rangle + B_8 \sum_{i < j < k < m} \langle S_i^2 S_j^2 S_k^2 S_m^2 S_l \rangle, \end{aligned} \quad (19)$$

where the B coefficients are given in Appendix A.2.

3.3. For $d = 3$ and $z = 6$, the cubic lattice

We obtain from Eq. (12)

$$\begin{aligned} \langle S_0 S_l \rangle = & A_1 \sum_i \langle S_i S_l \rangle + A_2 \sum_{i < j} \langle S_i S_j^2 S_l \rangle + A_3 \sum_{i < j < k} \langle S_i S_j S_k S_l \rangle + A_4 \sum_{i < j < k} \langle S_i S_j^2 S_k^2 S_l \rangle + A_5 \sum_{i < j < k < m} \langle S_i S_j S_k S_m^2 S_l \rangle \\ & + A_6 \sum_{i < j < k < m} \langle S_i S_j^2 S_k^2 S_m^2 S_l \rangle + A_7 \sum_{i < j < k < m < n} \langle S_i S_j S_k S_m S_n S_l \rangle \\ & + A_8 \sum_{i < j < k < m < n} \langle S_i S_j S_k S_m^2 S_n^2 S_l \rangle + A_9 \sum_{i < j < k < m < n < p} \langle S_i S_j S_k S_m S_n S_p^2 S_l \rangle \\ & + A_{10} \sum_{i < j < k < m < n} \langle S_i S_j^2 S_k^2 S_m^2 S_n^2 S_l \rangle + A_{11} \sum_{i < j < k < m < n < p} \langle S_i S_j S_k S_m^2 S_n^2 S_p^2 S_l \rangle, \end{aligned} \quad (20)$$

where the A coefficients are given in [Appendix A.3](#). We also obtain, for the function $\langle S_0^2 S_l \rangle$,

$$\begin{aligned} \langle S_0^2 S_l \rangle = & B_0 + B_1 \sum_i \langle S_i^2 S_l \rangle + B_2 \sum_{i < j} \langle S_i S_j S_l \rangle + B_3 \sum_{i < j} \langle S_i^2 S_j^2 S_l \rangle \\ & + B_4 \sum_{i < j < k} \langle S_i S_j S_k^2 S_l \rangle + B_5 \sum_{i < j < k} \langle S_i^2 S_j^2 S_k^2 S_l \rangle + B_6 \sum_{i < j < k < m} \langle S_i S_j S_k S_m \rangle + B_7 \sum_{i < j < k < m} \langle S_i S_j S_k^2 S_m^2 \rangle \\ & + B_8 \sum_{i < j < k < m} \langle S_i^2 S_j^2 S_k^2 S_m^2 S_l \rangle + B_9 \sum_{i < j < k < m < n < p} \langle S_i S_j S_k S_m S_n S_p S_l \rangle \\ & + B_{10} \sum_{i < j < k < m < n} \langle S_i^2 S_j^2 S_k^2 S_m^2 S_n^2 S_l \rangle + B_{11} \sum_{i < j < k < m < n < p} \langle S_i^2 S_j^2 S_k^2 S_m^2 S_n^2 S_p^2 S_l \rangle \\ & + B_{12} \sum_{i < j < k < m < n} \langle S_i^2 S_j^2 S_k^2 S_m S_n S_l \rangle + B_{13} \sum_{i < j < k < m < n < p} \langle S_i^2 S_j^2 S_k^2 S_m^2 S_n S_p S_l \rangle \\ & + B_{14} \sum_{i < j < k < m < n} \langle S_i^2 S_j S_k S_m S_n S_l \rangle + B_{15} \sum_{i < j < k < m < n < p} \langle S_i^2 S_j^2 S_k S_m S_n S_p S_l \rangle, \end{aligned} \quad (21)$$

where the B coefficients are given in [Appendix A.3](#).

The sums over i, j, k, m, n and p are over the nearest neighbors of 0 to which we have given a numerical ordering. The proof of results (16) for the case Section 3.1, the honeycomb lattice, is presented in [Appendix B](#), as an example for the other cases.

4. Application of the correlation inequalities

In the following results we will make use of the following inequalities: $\langle S_A \rangle \geq 0$ (Griffiths I), $\langle S_A S_B \rangle - \langle S_A \rangle \langle S_B \rangle \geq 0$ (Griffiths II) (see Refs. [9,11,12,15]), $\langle S_i F \rangle \leq \sum_j \langle S_i S_j \rangle \langle dF/dS_j \rangle$ (Newman's) [11,16] and $\langle S_i^2 S_A \rangle \leq \langle S_A \rangle$ [13,14], where $\langle S_A \rangle = \prod_i S_i$, $\langle S_B \rangle = \prod_i S_i$ and F is a polynomial function of variables S .

From the equations for the two spin correlation functions obtained in Sections 3.1–3.3 and applying the Griffith's and Newman's inequalities we obtain an inequality of the form

$$\langle S_0 S_l \rangle \leq \sum_{|i|=1} a_i \langle S_i S_l \rangle, \quad (22)$$

where a_i is a sum of products of two-point functions.

4.1. Case $d = 2, z = 3$, honeycomb lattice

Using

$$\langle S_j^2 S_i S_l \rangle \leq \langle S_i S_l \rangle \quad (23)$$

in Eq. (19), in the A_2 term and Griffiths II, i.e.,

$$\langle S_i S_j S_k S_l \rangle \geq \langle S_i S_j \rangle \langle S_k S_l \rangle \quad (24)$$

in the A_3 term, and noticing that A_2 and A_3 are negative, we get for $d = 2, z = 3$,

$$\langle S_0 S_l \rangle \leq (A_1 - |A_2| - |A_3| \langle S_0 S_1 \rangle_{1D} + A_4) \sum_{|i|=1} \langle S_i S_l \rangle. \quad (25)$$

4.2. Case $d = 2, z = 4$, square lattice

Using inequality (23) in Eq. (20), in the A_2 term ($A_2 < 0$) term, Griffiths II in the A_3 term ($A_3 < 0$), the inequalities

$$\langle S_j^2 S_k^2 S_i S_l \rangle \leq \langle S_i S_l \rangle \quad (26)$$

in the A_4 term ($A_4 > 0$) and

$$\langle S_j^2 S_k^2 S_m^2 S_i S_l \rangle \leq \langle S_i S_l \rangle \quad (27)$$

in the A_6 term ($A_6 > 0$) and in the A_5 term using Griffiths II, we get for $d = 2, z = 4$,

$$\langle S_0 S_l \rangle \leq (A_1 - |A_2| - \langle S_1 S_2 \rangle_{1D} |A_3| + A_4 + \langle S_1 S_2 \rangle_{1D} A_5 + A_6) \sum_{|i|=1} \langle S_i S_l \rangle. \quad (28)$$

4.3. Case $d = 3, z = 6$, cubic lattice

As before, we use in Eq. (20), inequality (23) in the A_2 term ($A_2 < 0$) term, Griffiths II in the A_3 term ($A_3 < 0$), the inequalities (26) in the A_4 term ($A_4 > 0$), inequality (24) in the A_6 term ($A_6 > 0$), and Griffiths II in the A_5 term. For the term $A_7 (> 0)$ we use Newman's inequality and for the terms $A_8 (> 0)$, $A_9 (> 0)$, $A_{10} (> 0)$ and $A_{11} (> 0)$, we use inequality (22). Then, we get for $d = 3, z = 6$,

$$\langle S_0 S_l \rangle \leq (A_1 - |A_2| - \langle S_1 S_2 \rangle_{1D} |A_3| + A_4 + \langle S_1 S_2 \rangle_{1D} A_5 + A_6 + A_7 + A_8 + A_9 + A_{10} + A_{11}) \sum_{|i|=1} \langle S_i S_l \rangle. \quad (29)$$

The two-spin correlation function $\langle S_1 S_2 \rangle_{1D}$ is the one-dimension model two spin correlation separated by a distance of two lattice sites. By bounding the resulting two-point function occurring in the previous results from below with the two-point function of the one-dimensional infinite chain (see Appendix B), we get:

$$\langle S_0 S_l \rangle \leq \sum_{|i|=1} a_i \langle S_i S_l \rangle, \quad (30)$$

where, for $d = 2, z = 3$, honeycomb lattice,

$$a_j = A_1 - |A_2| - |A_3| \langle S_0 S_1 \rangle_{1D} + A_4; \quad (31)$$

for $d = 2, z = 4$, square lattice,

$$a_j = A_1 - |A_2| - \langle S_1 S_2 \rangle_{1D} |A_3| + A_4 + \langle S_1 S_2 \rangle_{1D} A_5 + A_6; \quad (32)$$

and for $d = 3, z = 6$, cubic lattice,

$$a_j = A_1 - |A_2| - \langle S_1 S_2 \rangle_{1D} |A_3| + A_4 + \langle S_1 S_2 \rangle_{1D} A_5 + A_6 + A_7 + A_8 + A_9 + A_{10} + A_{11}. \quad (33)$$

The one-dimensional correlation function is given by (see Appendix C):

$$\langle S_1 S_2 \rangle_{1D} = \frac{1 + \sqrt{1 - 2f(2\beta J)}}{f(2\beta J)} \quad (34)$$

and $f(2\beta J)$ is given by (10).

5. Numerical results

Evaluating numerically the value of T such that $\sum a_j \leq 1$, $a_j > 0$, we obtain, by sufficient condition, upper bounds for T_c , which are shown in Tables 1 and 2, for $D = 0$ and $D = \infty$, in comparison with results obtained by other methods. Although we have obtained the upper bounds for T_c for $D = 0$ and $D = \infty$, we can as well obtain the bounds for the critical temperature of the model as a function of the anisotropy D . The spin one model has been extensively studied by many authors applying different techniques, such as, molecular field approximation (MFA) [1,2], effective field approximation (EFT) [8,20], series expansion methods [21,22], renormalization group (RG) [23–27], Monte-Carlo methods (MC) [28–31], the Wang–Landau technique [32–34] and cluster variation methods (CVM) [35]. We will restrict our comparisons to some values of the critical temperatures obtained by these works mentioned above.

For the evaluation of the self-correlation terms ($\langle S_i^2 \rangle$) that emerge from the application of the Griffith's and Newman's inequalities, we use, for the $D = 0$ case, $\langle S_i^2 \rangle \leq 2/3$, correct for a spin 1 ferromagnetic system, and, for the $D = \infty$ case, $\langle S_i^2 \rangle = 1$, since in this limit the $S_i = 0$ spin value is suppressed. In other words, the $D = \infty$ case is the two state Ising model.

Table 1Estimates for kT_C/J for $D = 0$ in previous and in the present work.

	$d = 2, z = 3$	$d = 2, z = 4$	$d = 3, z = 6$
MFA	2	2.667	4
EFT [8]	1.518	2.188	3.516
EFT [20]	–	1.964	–
CVM [35]	–	–	2.886
Series [21,22]	–	1.688	3.192
RG [27]	–	2.128	3.474
Monte Carlo [29]	–	1.695	–
Monte Carlo [31]	–	1.681	–
Wang Landau [32]	–	1.714	–
Wang Landau [33,34]	–	1.693	–
Present work	1.591	2.322	3.678

Table 2Estimates for kT_C/J for $D = \infty$ in previous and in the present work.

	$d = 2, z = 3$	$d = 2, z = 4$	$d = 3, z = 6$
MFA	3	4	6
EFT [8]	2.103	3.088	5.076
CVM [35]	–	–	3.876
Series [21,22]	–	–	4.482
RG [27]	–	2.884	4.932
Monte Carlo [30]	–	–	4.504
Exact [36]	–	2.269	–
Present work	1.999	3.070	5.084

For the honeycomb lattice our result has to be compared with the mean field and the effective field calculations. Those results are not rigorous, as ours, and the numerical values we obtain improve those mean field type results and therefore represent the upper bounds. In particular, for $D = \infty$ our result is lower than all other estimative, so it is a better (more precise) result. For $D = 0$ the result we obtain here is of the same order (coincides in the second decimal figure) of the effective field one. For the square and cubic lattices besides the mean field calculation there are other results, better than mean field, obtained by series and renormalization group methods, which can be used as a comparison. As before, our results agree with the effective field ones (with a better estimative for the $D = \infty$ square lattice), with the advantage of being rigorous upper bounds, as already emphasized. In all the cases, our results are also close to those obtained by renormalization group methods.

The importance of the present numerical results lies in the fact that they were obtained using an identity and rigorous inequalities for the two-spin correlation function. For this reason they represent rigorous upper bounds for the critical temperature. In summary, the numerical results for the critical coupling obtained by the present method, which represent the rigorous bounds, are quite good when compared to other results obtained by approximated methods (MFA, CVM, EFT). Some of the other methods (MC, RG, Series) give better numerical results for the critical coupling, i.e. more precise results, although not based on rigorous procedures.

6. Final comments

We have presented the derivation of correlation identities for the Blume–Capel spin 1 model which are exact in all dimensions, and we have made use of correlation inequalities to obtain the upper bounds for the transition temperature. The coupling constants obtained for those bounds are calculated for $d = 2$ (honeycomb and square lattices) and $d = 3$ (cubic lattice). We obtain rigorous results that improve mean field type calculations. This was achieved by the use of an identity for the two-spin correlation function for the model, which is an exact result, and is derived explicitly in the paper, combined with correlation inequalities, which are rigorous and have been obtained, for the model, by various authors. This is the main advantage of the method—it is rigorous. The numerical results for the critical coupling, which represent the upper bounds, are quite good when compared to other results obtained by approximated methods. Other methods give better numerical results for the critical coupling (more precise results) although not based on rigorous procedures.

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Appendix A. Coefficients of the spin correlation identities for $d = 2$, $z = 3$ and $z = 4$

With $k = \beta J$ and $f(x)$ given by relation (10), we have for

A.1. $d = 2$, $z = 3$

$$A_1 = 3f(k) > 0, \quad (\text{A.1})$$

$$A_2 = (3f(2k) - 6f(k)) < 0, \quad (\text{A.2})$$

$$A_3 = \frac{1}{4}(f(3k) - 3f(k)) < 0 \quad (\text{A.3})$$

$$A_4 = \frac{3}{4}(5f(k) + f(3k) - 4f(2k)) > 0 \quad (\text{A.4})$$

and

$$B_0 = g(0), \quad (\text{A.5})$$

$$B_1 = 3(g(k) - g(0)), \quad (\text{A.6})$$

$$B_2 = \frac{3}{2}(g(2k) - g(0)), \quad (\text{A.7})$$

$$B_3 = \frac{3}{2}g(2k) + -6g(k) + \frac{9}{2}g(0), \quad (\text{A.8})$$

$$B_4 = \frac{3}{4}(g(3k) - g(k) - 2g(2k) + 2g(0)), \quad (\text{A.9})$$

$$B_5 = \frac{1}{4}g(3k) - \frac{3}{2}g(2k) + \frac{15}{4}g(k) - \frac{5}{2}g(0). \quad (\text{A.10})$$

A.2. $d = 2$, $z = 4$

$$A_1 = 4f(k) > 0, \quad (\text{A.11})$$

$$A_2 = 6f(2k) - 12f(k) < 0, \quad (\text{A.12})$$

$$A_3 = f(3k) - 3f(k) < 0, \quad (\text{A.13})$$

$$A_4 = 15f(k) - 12f(2k) + 3f(3k) > 0, \quad (\text{A.14})$$

$$A_5 = \frac{1}{2}f(4k) - f(3k) - f(2k) + 3f(k) > 0 \quad (\text{A.15})$$

$$A_6 = \frac{1}{2}f(4k) - 3f(3k) + 7f(2k) - 7f(k) < 0 \quad (\text{A.16})$$

and

$$B_0 = g(0), \quad (\text{A.17})$$

$$B_1 = 4(g(k) - g(0)), \quad (\text{A.18})$$

$$B_2 = 3(g(2k) - g(0)), \quad (\text{A.19})$$

$$B_3 = 3(g(2k) - 4g(k) + 3g(0)), \quad (\text{A.20})$$

$$B_4 = 3(g(3k) - 2g(2k) - g(k) + 2g(0)), \quad (\text{A.21})$$

$$B_5 = g(3k) - 6g(2k) + 15g(k) - 10g(0), \quad (\text{A.22})$$

$$B_6 = \frac{1}{8}(g(4k) - 4g(2k) + 3g(0)), \quad (\text{A.23})$$

$$B_7 = \frac{3}{4}g(4k) - 3g(3k) + 3g(2k) + 3g(k) - \frac{15}{4}g(0), \quad (\text{A.24})$$

$$B_8 = \frac{1}{8}g(4k) - g(3k) + \frac{7}{2}g(2k) - 7g(k) + \frac{35}{8}g(0). \quad (\text{A.25})$$

A.3. $d = 3$, $z = 6$

$$A_1 = 6f(k) > 0, \quad (\text{A.26})$$

$$A_2 = -30f(k) + 15f(2k) < 0, \quad (\text{A.27})$$

$$A_3 = 5f(3k) - 15f(k) < 0, \quad (\text{A.28})$$

$$A_4 = 75f(k) + 15f(3k) - 60f(2k) > 0, \quad (\text{A.29})$$

$$A_5 = -15f(3k) + 45f(k) + \frac{15}{2}f(4k) - 15f(2k) > 0, \quad (\text{A.30})$$

$$A_6 = -45f(3k) - 105(f(k) - f(2k)) + \frac{15}{2}f(4k) < 0, \quad (\text{A.31})$$

$$A_7 = \frac{3}{8}f(5k) - \frac{15}{8}f(3k) + \frac{15}{4}f(k) > 0, \quad (\text{A.32})$$

$$A_8 = \frac{45}{4}f(3k) - \frac{105}{2}f(k) + \frac{15}{4}f(5k) - 15f(4k) + 30f(2k) < 0, \quad (\text{A.33})$$

$$A_9 = -\frac{3}{8}f(5k) + \frac{15}{8}f(3k) - \frac{15}{4}f(k) + \frac{3}{16}f(6k) - \frac{3}{4}f(4k) + \frac{15}{16}f(2k) < 0, \quad (\text{A.34})$$

$$A_{10} = \frac{405}{8}f(3k) + \frac{315}{4}f(k) + \frac{15}{8}f(5k) - 15f(4k) - 90f(2k) > 0, \quad (\text{A.35})$$

$$A_{11} = -\frac{5}{4}f(3k) + \frac{45}{2}f(k) + \frac{15}{2}f(4k) - \frac{135}{8}f(2k) - \frac{15}{4}f(5k) + \frac{5}{8}f(6k) > 0 \quad (\text{A.36})$$

and

$$B_0 = g(0), \quad (\text{A.37})$$

$$B_1 = 6(g(k) - g(0)), \quad (\text{A.38})$$

$$B_2 = \frac{15}{2}(g(2k) - g(0)), \quad (\text{A.39})$$

$$B_3 = \frac{15}{2}(g(2k) - 4g(k) + 3g(0)), \quad (\text{A.40})$$

$$B_4 = 15(g(3k) - 2g(2k) - g(k) + 2g(0)), \quad (\text{A.41})$$

$$B_5 = 5(g(3k) - 6g(2k) + 15g(k) - 10g(0)), \quad (\text{A.42})$$

$$B_6 = \frac{15}{8}(g(4k) - 4g(2k) + 3g(0)), \quad (\text{A.43})$$

$$B_7 = 45\left(\frac{1}{4}g(4k) - g(3k) + g(2k) + g(k) - \frac{5}{4}g(0)\right), \quad (\text{A.44})$$

$$B_8 = 15\left(\frac{1}{8}g(4k) - g(3k) + \frac{7}{2}g(2k) - 7g(k) + \frac{35}{8}g(0)\right), \quad (\text{A.45})$$

$$B_9 = \frac{1}{32}(g(6k) - 6g(4k) + 15g(2k) - 10g(0)), \quad (\text{A.46})$$

$$B_{10} = \frac{3}{8}(-126g(0) + 45g(3k) + 210g(k) - 120g(2k) - 10g(4k) + g(5k)), \quad (\text{A.47})$$

$$B_{11} = \frac{3}{8}\left(-\frac{55}{3}g(3k) - 66g(k) + \frac{165}{4}g(2k) + \frac{77}{2}g(0) + \frac{1}{12}g(6k) + \frac{11}{2}g(4k) - g(5k)\right), \quad (\text{A.48})$$

$$B_{12} = \frac{15}{4}(-8g(2k) + 14g(0) + g(5k) - 14g(k) + 13g(3k) - 6g(4k)), \quad (\text{A.49})$$

$$B_{13} = \frac{15}{32}(-40g(3k) + 48g(k) + 15g(2k) - 42g(0) + 26g(4k) + g(6k) - 8g(5k)), \quad (\text{A.50})$$

$$B_{14} = \frac{15}{8}(-2g(4k) + 8g(2k) - 6g(0) + g(5k) - 3g(3k) + 2g(k)), \quad (\text{A.51})$$

$$B_{15} = \frac{15}{32}(2g(4k) - 17g(2k) + 14g(0) - 4g(5k) + 12g(3k) - 8g(k) + g(6k)). \quad (\text{A.52})$$

Appendix B. Proof of the correlation identity for the honeycomb lattice

From Eq. (12)

$$\langle F(S)S_i \rangle = \left\langle F(S) \prod_{j \neq i} (S_j^2 \cosh(\beta J_{ij} \nabla) + S_j \sinh(\beta J_{ij} \nabla) + 1 - S_j^2) \right\rangle f(x)|_{x=0} \quad (\text{B.1})$$

where,

$$f(x) = \frac{2e^{\beta D} \sinh(x)}{2e^{\beta D} \cosh(x) + 1}, \quad (\text{B.2})$$

we obtain $\langle S_0 S_l \rangle$, for the honeycomb lattice,

$$\begin{aligned} \langle S_0 S_l \rangle &= \langle S_l (1 + S_1 \sinh J \nabla + S_1^2 [\cosh J \nabla - 1]) \times (1 + S_2 \sinh J \nabla + S_2^2 [\cosh J \nabla - 1]) \\ &\quad \times (1 + S_3 \sinh J \nabla + S_3^2 [\cosh J \nabla - 1]) \rangle \end{aligned} \quad (\text{B.3})$$

where S_1, S_2 and S_3 are the neighbors of S_0 .

Or,

$$\langle S_0 S_l \rangle = 3a_1 \langle S_1 S_l \rangle + 6(a_2 - a_1) \langle S_1 S_2^2 \rangle + a_3 \langle S_1 S_2 S_3 \rangle + (a_1 - 2a_2 + a_4) \langle S_1 S_2^2 S_3^2 \rangle, \quad (\text{B.4})$$

where,

$$\begin{aligned} a_1 &= \sinh J \nabla \cdot f(x)|_{x=0} = f(\beta J) \\ a_2 &= \sinh J \nabla \cosh J \nabla \cdot f(x)|_{x=0} = 1/2f(2\beta J) \\ a_3 &= \sinh J^3 \nabla \cdot f(x)|_{x=0} = 1/4[f(3\beta J) - 3f(\beta J)] \\ a_4 &= \sinh J \nabla \cosh^2 J \nabla \cdot f(x)|_{x=0} = 1/4[f(3\beta J) + f(\beta J)]. \end{aligned} \quad (\text{B.5})$$

From those results we obtain Eqs. (16) and (17) of Section 3.1.

Appendix C. Spin correlation for the one-dimensional $S = 1$ Blume–Capel model

For the linear chain, we have,

$$\langle S_0 \rangle = \langle (1 + S_1 \sinh J \nabla + S_1^2 [\cosh J \nabla - 1])(1 + S_{-1} \sinh J \nabla + S_{-1}^2 [\cosh J \nabla - 1]) \rangle \cdot f(x)|_{x=0} \quad (\text{C.1})$$

with $f(x)$ given by expression (10) and S_1 and S_{-1} are neighbors of S_0 . We obtain for the two-spin correlation function

$$\langle S_0 S_R \rangle = \langle (S_1 S_R + S_{-1} S_R) \rangle f(k) + \langle (S_1 S_{-1} S_1 S_R + S_1 S_{-1} S_{-1} S_R) \rangle \left(\frac{1}{2} f(2k) - f(k) \right) \quad (\text{C.2})$$

where $k = \beta J$. Applying the inequalities [13,14]

$$\begin{aligned} \langle S_1^2 S_{-1} S_R \rangle &\leq \langle S_{-1} S_R \rangle \\ \langle S_{-1}^2 S_1 S_R \rangle &\leq \langle S_1 S_R \rangle \end{aligned} \quad (\text{C.3})$$

we get,

$$\langle S_0 S_R \rangle \leq (\langle S_1 S_R \rangle + \langle S_{-1} S_R \rangle) f(k) + (\langle S_{-1} S_R \rangle + \langle S_1 S_R \rangle) [1/2f(2k) - f(k)]. \quad (\text{C.4})$$

Defining $C(R) = \langle S_0 S_R \rangle$ we get

$$C(R) = A(k)(C(R+1) + C(R-1)), \quad (\text{C.5})$$

where $A(k) = f(2k)/2$.

If $\gamma(R) = C(R+1)/C(R)$ is inserted in the previous equation we get

$$1 = A(k)(\gamma(R) + \gamma(R)^{-1}). \quad (\text{C.6})$$

So, $C(R) = \gamma^R$ and

$$\gamma = \frac{1 + \sqrt{1 - 2f(2\beta J)}}{f(2\beta J)}. \quad (\text{C.7})$$

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