Università degli Studi di Trento
Facoltà di Scienze
Matematiche, Fisiche e Naturali

Corso di Laurea Magistrale
in Matematica

Eberhard Karls
Universität Tübingen

Studiengang Mathematik

Master Thesis

The Isoperimetric Problem in the Sub-Riemannian Heisenberg Group

Supervisors:
Prof. Frank Loose
Prof. Francesco Serra Cassano

Student:
Elisa Paoli

Academic Year 2010-2011

Contents

Introduction

In this survey we will discuss the *isoperimetric problem* in the sub-Riemannian Heisenberg group. This problem has its origin in the more ancient isoperimetric problem in the Euclidean plane, which has absorbed mathematicians of any time. There the question is to *find the figure with maximal area, bounded by a fixed perimeter*.

This is a practical problem known for so long time that it is documented even in the legend of the foundation of Carthage. According to the story, the Phoenician princess Dido, because of dynastic fight, was forced to take refuge on the coast of North Africa. There she managed to obtain from the local potentate a small bit of land for temporary refuge, as wide as could be encompassed by the hide of a bull. That's not a lot, but Dido found a very ingenious solution: she instructed to cut the hide into a series of long thin strips, to obtain a long string, by which she could mark out a vast region, forming the eventual line of the walls of ancient Carthage.

The problem that Dido had to solve was a variant of the classical isoperimetric problem: she had to find the maximal land that can be enclosed with a fixed perimeter. An equivalent formulation is: given a fixed area, find the region with this area and minimal perimeter. The solution is very intuitive and, as Dido correctly guessed, is the disk with length of the circumference equal to the given perimeter.

The formal solution of this problem has requested over the centuries the efforts of many scientists, starting from the great Greek mathematicians. A first partial solution was found by the Swiss *Jacob Steiner* (1796-1863), who introduced its celebrated method of simmetrization, even if he didn't proved that an isoperimetric profile exists. The question in the Euclidean space was solved in 1958 by *Ennio De Giorgi* (1928-1996), when he presented the formalization of the problem and proved the isoperimetric property of the circle in the plane, the sphere in the space (and more generally of the hypersphere in a space of arbitrary dimension) among a very broad class of sets called sets of finite perimeter.

Here we are interested in the isoperimetric problem in spaces with a sub-

Riemannian metric structure, also known as Carnot-Carathéodory spaces. This metric structure may be viewed as a constrained geometry, where motion is only possible along a given set of directions, changing from point to point. They play a central role in the general problem of analysis and geometry in metric spaces, with applications ranging from differential geometry to non holomorphic mechanics, getting through partial differential equations and control theory. One of the most significant and simplest example of such a space is the so-called (first) Heisenberg group.

In 1982 the French mathematician *Pierre Pansu* studied the isoperimetric problem in the Heisenberg group and proposed a conjecture for its solution. From then on several developments have been made and lots of instances support Pansu's conjecture, but the problem is still an open question. The goal of this survey is to introduce the notions for the comprehension of the conjecture and the presentation of the recent results in this topic.

After a first chapter, in which we recall the main concepts of analysis and geometry necessary for the comprehension of this work, Chapter 2 is devoted to the study of the isoperimetric problem in the Euclidean space, with the purpose to give an idea of the notions that are involved.

The next chapters treat the generalization of the problem to the Heisenberg groups, in particular to the first Heisenberg group. These are special Lie groups, whose Lie algebra is nilpotent and can be expressed as a direct sum $V = V_1 \oplus V_2$ of two vector spaces such that the commutator $[V_1, V_1] = V_2$. We will see that this property allows us to introduce a distance, provided we give a Riemannian metric on V_1 and we extend it to a sub-Riemannian metric on the whole space. This construction will be done in Chapter 3, in which we introduce the Heisenberg group, its structure and its Riemannian approximants.

In Chapter 4 and Chapter 5 we deal with the main geometric quantities in the sub-Riemannian geometry of the Heisenberg group as the perimeter measure and its first variation.

Finally, the last chapter is the most interesting from the point of view of the research. We will describe how Pansu generalized the isoperimetric problem to the Heisenberg group and his conjecture. In particular we will present the proof of the existence of an isoperimetric set among sets of finite perimeter, which is inspired by the Euclidean technique due to De Giorgi.

At this point I would like to thank professor Francesco Serra Cassano, whose experience and competence have been an example for me and a great help for my study in these months of work. I'd would like to thank him also for the time he has given me, but even more for the humane words of support.

Finally I would like to thank professor Frank Loose for his helpfulness and kindness. I have appreciated his enthusiasm for mathematics which he communicates to the students and the readiness with which he accepted to supervise me.

Chapter 1

Recalls on Riemannian geometry and Geometric Measure Theory

In this section we recall that definitions and theorems on Riemannian geometry and geometric measure theory, which will be used in the survey. We will sometimes state these classical results without proof.

1.1 Manifolds

We assume here that the concepts of C^∞ manifold, differentiable map, tangent space, differential of a function and tangent bundle are known (for istance see [20] or [5]). We restrict ourself to recall those notions on manifold and Riemannian geometry which will be heavily used in this survey.

1.1.1 Vector fields and Frobenius' Theorem

Let M be a m-dimensional manifold.

Definition 1.1. A *vector field* X on an open set U in M is a map $X : U \to TM$ such that $X(p)$ (often denoted X_p) is a vector in T_pM, the tangent space to M at p. The vector field is smooth if it is smooth as a map from U to TM, the tangent bundle of M.

We denote by $\mathfrak{X}(M)$ the vector space of all smooth vector fields on M.

If (U, φ) is a neighborhood of $p \in M$, the vectors

$$\left(\frac{\partial}{\partial x_1}\right)_p := (d\varphi_p)^{-1}\left(\frac{\partial}{\partial u_1}\right), \dots, \left(\frac{\partial}{\partial x_m}\right)_p := (d\varphi_p)^{-1}\left(\frac{\partial}{\partial u_m}\right)$$

form a basis for the tangent space T_pM. Here $u_1, \ldots, u_m$ denote the canonical coordinates in $\mathbb{R}^m$. Then the vector field X is smooth if and only if its local expression

$$X = \sum_{i=1}^{m} a_i \frac{\partial}{\partial x_i}$$

has smooth coordinates functions $a_1, \ldots, a_m : U \to \mathbb{R}$.

Definition 1.2. If X and Y are smooth vector fields on M, we define a vector field $[X, Y]$ called the *Lie bracket* of X and Y by setting

$$[X, Y]_p(f) = X_p(Yf) - Y_p(Xf).$$

The following proposition underlines some properties of the Lie bracket, whose proofs are an easy computation.

Proposition 1.3. *For each $X, Y, Z \in \mathfrak{X}(M)$ hold:*

1. $[X, Y]$ *is indeed a smooth vector field on M.*
2. *If $f, g \in C^\infty(M)$, then $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$.*
3. $[X, Y] = -[Y, X]$.
4. (Jacobi identity)

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

A vector space with a bilinear operation satisfying (3) and (4) is called a *Lie algebra.*

Definition 1.4. Let X be a smooth vector field on M. A C^∞ curve γ in M is an *integral curve of* X if $\dot{\gamma}(t) = X(\gamma(t))$ for each t in the domain of γ.

Since a differentiable manifold is locally diffeomorphic to $\mathbb{R}^n$, the fundamental theorem on existence, uniqueness and dependence on initial conditions of ordinary differential equations extends naturally to differentiable manifolds. By this theorem, to each $p \in M$ there exists a neighborhood $U \subset M$ of p, an intervall $(-\delta, \delta)$, with $\delta > 0$, and a differentiable mapping $\varphi : (-\delta, \delta) \times U \to M$ such that the curve $t \to \varphi(t, q)$, with $t \in (-\delta, \delta)$ and $q \in U$ is the unique integral curve of X with $\varphi(0, q) = q$. The map $\varphi_t : U \to M$ such that $\varphi_t(q) = \varphi(t, q)$ is called the *local flow* of X.

Let us generalize these concepts to greater dimensional objects.

Definition 1.5. Let M be a m-dimensional manifold and let $n \leq m$ be an integer. A n-dimensional *distribution* Δ on M is a choice for each point $p \in M$ of an n-dimensional subspace Δ_p of the tangent space T_pM. Δ is *smooth* if for each $p \in M$ there is a neighborhood U of p and there are n linearly independent C^∞-vector fields $X_1, \ldots, X_n$ on U which form a basis of Δ_q for every $q \in U$.

We say that a vector field X on M lies in the distribution Δ, $X \in \Delta$, if $X_p \in \Delta_p$ for each $p \in M$. A distribution Δ is called *involutive* if it is closed under the operation of Lie brackets, that is if $[X, Y] \in \Delta$ for each $X, Y \in \Delta$.

Let Δ be a C^∞ distribution on M and N a connected C^∞ manifold. Consider a function $F : N \to M$. If F is a one-to-one immersion such that for each $q \in N$ we have $dF_q(T_qN) \subset \Delta_{F(q)}$, then we say that the immersed submanifold $F(N) \subset M$ is an *integral manifold* of Δ. Notice that an integral manifold may be of lower dimension than Δ.

A more restrictive definition characterizes a *completely integrable* distribution. Let Δ be a distribution on M. If each point $p \in M$ admits a coordinate neighborhood (U, φ) such that the first n vectors

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$$

of the local basis of T_pM are a local basis for Δ on U, we say that the distribution Δ is *completely integrable.* In other words, for a completely integrable distribution there exists an n-dimensional integral manifold N with Δ as tangent space. In fact, if $(q_1, \ldots, q_m)$ denote the local coordinates of q, then the integral manifold through q is the set of all points whose coordinates satisfy $u_{n+1} = q_{n+1}, \ldots, u_m = q_m$, that is $N = \varphi^{-1}(\{u \in \varphi(U) : u_j = q_j, j = n+1, \ldots, m\})$.

Notice that if Δ is completely integrable, then it is also involutive, in fact

$$\left[\left(\frac{\partial}{\partial x_i}\right)_p, \left(\frac{\partial}{\partial x_j}\right)_p\right] = d\varphi^{-1}\left[\left(\frac{\partial}{\partial u_i}\right)_p, \left(\frac{\partial}{\partial u_j}\right)_p\right] = 0,$$

lies in Δ_p for each $i, j = 1, \ldots, n$.

Frobenius' theorem states that also the inverse is true.

Theorem 1.6 (Frobenius)**.** *A distribution Δ on a manifold M is completely integrable if and only if it is involutive.*

To conclude this section we give a classical geometric interpretation of the Lie bracket.

Proposition 1.7. *Let X, Y be differentiable vector fields on a differentiable manifold M, let $p \in M$, and let φ_t be the local flow of X in a neighborhood U of p. Then*

$$[X,Y](p) = \lim_{t\to 0} \frac{1}{t}[Y - d\varphi_t Y](\varphi_t(p)).$$

For the proof, we need the following lemma from calculus.

Lemma 1.8. *Let $\delta > 0$ and $h : (-\delta,\delta)\times U \to \mathbb{R}$ be a differentiable function with $h(0,q) = 0$ for all $q \in U$. Then there exists a differentiable mapping $g : (-\delta,\delta)\times U \to \mathbb{R}$ with $h(t,q) = tg(t,q)$, in particular,*

$$g(0,q) = \left.\frac{\partial h(t,q)}{\partial t}\right|_{t=0}.$$

Proof of Lemma 1.8. It sufficies to define for fixed t

$$g(t,q) = \int_0^1 \frac{\partial h(ts,q)}{\partial ts} ds,$$

indeed, after a change of variable, we have

$$tg(t,q) = \int_0^t \frac{\partial h(r,q)}{\partial r} dr = h(t,q).$$

□

Proof of Proposition 1.7. Let f be a differentiable function in a neighborhood of p. Define

$$h(t,q) = f(\varphi_t(q)) - f(q)$$

and apply the lemma to h to obtain a differentiable function $g(t,q)$ such that

$$f \circ \varphi_t(q) = f(q) + tg(t,q)$$

and $g(0,q) = Xf(q)$. Then we obtain

$$((d\varphi_t Y)f)(\varphi_t(p)) = (Y(f\circ\varphi_t))(p) = Yf(p) + t(Yg(t,p))$$

and we can conclude that

$$\begin{aligned}
\lim_{t\to 0}\frac{1}{t}[Y - d\varphi_t Y]f(\varphi_t(p)) &= \lim_{t\to 0}\frac{(Yf)(\varphi_t(p)) - Yf(p)}{t} - (Yg(0,p)) \\
&= (X(Yf))(p) - (Y(Xf))(p) \\
&= ((XY - YX)f)(p) = [X,Y]f(p).
\end{aligned}$$

□

1.1.2 Lie groups and Lie algebras

Definition 1.9. A *Lie group*, $(\mathbb{G}, \cdot)$, is a group with a structure of differential manifold with respect to which the operation of the group and the inverse operation are smooth functions, that is

$$(x, y) \longmapsto x^{-1}y \quad \forall x, y \in \mathbb{G}$$

is a smooth function between $\mathbb{G} \times \mathbb{G}$ and $\mathbb{G}$.

An example of Lie group that will be usefull later is the *general linear group*, $GL(n, \mathbb{R})$, the set of all $n \times n$ invertible matrices. The operation is the usual product between matrices and the differentiable structure is induced by $\mathbb{R}^{n^2}$.

On a Lie group $\mathbb{G}$ are naturally defined the *left translation*, L_x, and the *right translation*, R_x, that are given respectively by

$$L_x : \mathbb{G} \longrightarrow \mathbb{G}, \qquad L_x(y) = xy \tag{1.1}$$

$$R_x : \mathbb{G} \longrightarrow \mathbb{G}, \qquad R_x(y) = yx. \tag{1.2}$$

They are smooth functions, so we may consider their differential dL_x and dR_x. A smooth vector field, X, on the Lie group $\mathbb{G}$, is said to be *left invariant* if $dL_xX = X$ for all $x \in \mathbb{G}$, in other words, for each $x, y \in \mathbb{G}$ it holds

$$dL_x(y)(X(y)) = X(xy).$$

In the same way, we say that X is *right invariant* if $dR_xX = X$ for each $x \in \mathbb{G}$.

The vector space of all left invariant vector fields on $\mathbb{G}$ is a Lie algebra. In fact, let X, Y be two left invariant smooth vector fields on $\mathbb{G}$, then

$$\begin{aligned} dL_x[X, Y]f &= [X, Y](f \circ L_x) \\ &= X(dL_xY)f - Y(dL_xX)f \\ &= (XY - YX)f \\ &= [X, Y]f, \end{aligned}$$

for each element $x \in \mathbb{G}$ and each differentiable function f on $\mathbb{G}$. Then the Lie product $[X, Y]$ of two left invariant vector fields is still left invariant.

Notice that a left invariant smooth vector field is completely determined by its value at a unique point, for example at the neutral element e of the group. In fact $X(x) = dL_x(X(e))$ for each $x \in \mathbb{G}$. This observation allows us to give to $T_e\mathbb{G}$ a structure of Lie algebra induced from that of the left invariant vector fields: let us identify each vector $X_e \in T_e\mathbb{G}$ with the left invariant vector field defined by

$$X_x = dL_x X_e \quad \forall x \in \mathbb{G}$$

and let us define the Lie product of two vectors $X_e, Y_e \in T_e\mathbb{G}$ as

$$[X_e, Y_e] = [X, Y]_e.$$

Thus $T_e\mathbb{G}$ equipped with this operation is a Lie algebra.

The Lie algebra of $\mathbb{G}$ is denoted by $\mathfrak{g}$ and may be equivalently defined as the tangent space at the identity, $T_e\mathbb{G}$, or as the set of all left invariant tangent vector fields.

1.2 Riemannian geometry

In this section we recall the notion of curvature and exponential map on a Riemannian surface. We start by the notions of curvature for a space curve and for a plane curve. Then we introduce the affine connection and in particular the Riemannian connection, in order to give at the end the definitions of exponential map, curvature and second fundamental form of a Riemannian manifold.

Let I denote an open interval in $\mathbb{R}$. A C^2 space curve $\gamma : I \to \mathbb{R}^3$ is called regular if $\gamma'(t) \neq 0$ for each t. Given an interval $[a, b] \subset I$, we recall that the length of γ between a and b is defined as

$$\text{Length}(\gamma) := \sup\{\sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})| : a = t_0 < t_1 < \cdots < t_n = b\}$$

where the supremum is taken over all finite parametrizations of $[a, b]$. A classical result of geometry states that for a regular space curve,

$$\text{Length}(\gamma) = \int_a^b |\gamma'(t)| dt.$$

This computation for the length of a curve, suggests us to take a preferred parameter for γ. Given a point $\gamma(t_0)$ the arclength parameter is defined by

$$s(t) := \int_{t_0}^{t} |\gamma'(h)| dh.$$

In particular, if γ is parametrized by arclength, the vector γ' has constant unitary norm.

Let $\gamma : I \to \mathbb{R}^3$ be parametrized by arclength. The *Frenet orthonormal frame* of γ is given by

$$\vec{t}(s) := \gamma'(s) \qquad \vec{n}(s) := \frac{\vec{t}\,'(s)}{|\vec{t}\,'(s)|} \quad \text{and} \quad \vec{b}(s) := \vec{t}(s) \wedge \vec{n}(s).$$

Notice that, since $\vec{t}$ has constant unitary norm, $\frac{d}{dt}\left\langle \vec{t}, \vec{t} \right\rangle = 2\left\langle \vec{t}\,', \vec{t} \right\rangle = 0$ and $\vec{n}$ is orthogonal to $\vec{t}$. Hence $\vec{t}, \vec{n}, \vec{b}$ form indeed an orthonormal frame. The vector

$$\vec{k} := \vec{t}\,' = \gamma''$$

is called *curvature vector* of γ and represents the acceleration of the curve. Therefore it is a normal vector to γ that points towards the inside of the curve. The quantity $k(s)$ such that

$$\vec{t}\,'(s) = k(s)\vec{n}(s)$$

is called *curvature* of γ. In particular $k = |\gamma''|$ is always positive.

Some special comment is required for the case of curves lying on a plane.

Let $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ define a regular C^2 plane curve in $\mathbb{R}^2$. The length of the curve and the arclength parameter are defined as for a space curve. Let γ be parametrized by arclength. As before, the tangent vector to γ is defined by

$$\vec{t}(s) := \gamma'(s) = (\gamma_1', \gamma_2')(s).$$

The normal unit vector is now choosen so that $\vec{t}$ and $\vec{n}$ form an orthonormal frame. So $\vec{n} := (-\gamma_2', \gamma_1')$. The curvature vector still represents the acceleration of γ and is given by $\vec{k} := \gamma''$; so it is the vector normal to γ that points towards the inside of the curve. Therefore it can go in the same direction of $\vec{n}$ or in the opposite direction and the curvature $k(s)$ such that $\vec{t}\,'(s) = k(s)\vec{n}(s)$ may be positive, zero or negative.

Let us explicite the last formulas in the identification of $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Let $\gamma : I \to \mathbb{C}$ be a regular C^2 curve, parametrized by arclength. Then

$$\vec{t} = (\gamma_1', \gamma_2') \quad \vec{n} = \mathbf{i}(\gamma_1', \gamma_2') \quad \vec{k} = (\gamma_1'', \gamma_2'') = k(s)\vec{n} = k(s)\mathbf{i}(\gamma_1', \gamma_2'). \tag{1.3}$$

1.2.1 Affine connection and Riemannian connection

We now introduce the concept of affine connection on a smooth manifold. The underlying idea is the following: given a smooth manifold M immersed in $\mathbb{R}^n$ and a tangent vector field X on M, the Euclidean derivative of X along a curve lying in M will not in general be tangent to the manifold.

Hence, the concept of differentiating a vector field is not an intrinsic geometric notion on M. We will consider therefore the orthogonal projection of the derivative of X on the tangent space to M and we will call this projection the *covariant derivative* of X along the curve. The idea of the affine connection is a generalization of the covariant derivative.

Let us indicate with $\mathfrak{X}(M)$ the set of all smooth vector fields on M.

Definition 1.10. An *affine connection* D on a differentiable manifold M is a mapping

$$D : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) : (X, Y) \mapsto D_X Y$$

which satisfies the following properties:

(i) $D_{fX+gY}Z = fD_X Z + gD_Y Z$;

(ii) $D_X(Y + Z) = D_X Y + D_X Z$;

(iii) $D_X(fY) = fD_X Y + X(f)Y$

for each $X, Y, Z \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$.

We remark that an affine connection is frequently denoted by ∇. Here we have adopted this convention for coherence with the notation used in next chapters and to avoid confusion with the gradient operator.

The following proposition will clarify the relation between affine connection and covariant derivative.

Proposition 1.11. *Let M be a differentiable manifold with an affine connection D. There exists a unique correspondence which associates to a vector field X along a differentiable curve $\gamma : (-\epsilon, \epsilon) \to M$ another vector field $\frac{DX}{dt}$ along γ, called the* covariant derivative of X along γ, *such that:*

1. $\frac{D}{dt}(X + Y) = \frac{D}{dt}X + \frac{D}{dt}Y$;
2. $\frac{D}{dt}(fX) = \frac{df}{dt}X + f\frac{DX}{dt}$ *for each smooth vector field Y along γ and differentiable function f on $(-\epsilon, \epsilon)$;*
3. *if X is the restriction to γ of a vector field $\tilde{X} \in \mathfrak{X}(M)$, then* $\frac{DX}{dt} = D_{d\gamma/dt}\tilde{X}$.

Proof. See for example [10]. □

Let M be endowed with a Riemannian metric $\langle\cdot,\cdot\rangle_M$. We say that the connection D is *compatible* with the metric $\langle\cdot,\cdot\rangle_M$, if for each vector field $X, Y, Z \in \mathfrak{X}(M)$ we have

$$X\langle Y, Z\rangle_M = \langle D_X Y, Z\rangle_M + \langle Y, D_X Z\rangle_M. \tag{1.4}$$

We say that D is *symmetric* if

$$D_X Y - D_Y X = [X, Y]$$

for each $X, Y \in \mathfrak{X}(M)$.

Finally we state the main result of this section.

Theorem 1.12 (Levi-Civita)**.** *Let $(M, \langle\cdot,\cdot\rangle_M)$ be a Riemannian manifold. There exists a unique affine connection D on M that is symmetric and compatible with the Riemannian metric.*

Proof. We first prove the uniqueness. Let us suppose that such a connection exists. Since D is compatible with the metric we have

$$\begin{aligned} X\langle Y, Z\rangle_M &= \langle D_X Y, Z\rangle_M + \langle Y, D_X Z\rangle_M & (1.5)\\ Y\langle Z, X\rangle_M &= \langle D_Y Z, X\rangle_M + \langle Z, D_Y X\rangle_M & (1.6)\\ Z\langle X, Y\rangle_M &= \langle D_Z X, Y\rangle_M + \langle X, D_Z Y\rangle_M. & (1.7) \end{aligned}$$

Adding (1.5) and (1.6) and subtracting (1.7), and using the symmetry of D, we obtain

$$\begin{aligned} &X\langle Y, Z\rangle_M + Y\langle Z, X\rangle_M - Z\langle X, Y\rangle_M = \\ &\qquad = \langle [X, Z], Y\rangle_M + \langle [Y, Z], X\rangle_M + \langle [X, Y], Z\rangle_M + 2\langle Z, D_Y X\rangle_M. \end{aligned}$$

Therefore

$$\begin{aligned} \langle Z, D_Y X\rangle_M = \frac{1}{2}\{&X\langle Y, Z\rangle_M + Y\langle Z, X\rangle_M - Z\langle X, Y\rangle_M \\ &- \langle [X, Z], Y\rangle_M - \langle [Y, Z], X\rangle_M - \langle [X, Y], Z\rangle_M\}. \end{aligned} \tag{1.8}$$

This expression is usually referred to as the *Kozul identity.* It shows in particular that D is uniquely determined by the metric $\langle\cdot,\cdot\rangle_M$, hence if it exists, it will be unique.

To prove existence, let us define D as in (1.8). It is easy to verify that D is well-defined and that it satisfies the desired conditions. □

Definition 1.13. Given a Riemannian manifold M, the unique symmetric connection D compatible with the metric is called the *Levi-Civita* (or *Riemannian*) *connection* on M.

1.2.2 The exponential map

The definition of affine connection allows us to give a notion of acceleration of a smooth vector field on a manifold. Consequently we may define the geodesic curves, that are smooth curves which have constant velocity with respect to the manifold.

Let M be a Riemannian manifold endowed with the respective Riemannian connection.

Definition 1.14. A parametrized curve $\gamma : (-\epsilon, \epsilon) \to M$ is called *geodesic at* $t_0 \in (-\epsilon, \epsilon)$ if $\frac{D}{dt}(\gamma') = 0$ at t_0. If γ is a geodesic at all points $t \in (-\epsilon, \epsilon)$ we say that γ is a *geodesic.*

Notice that, if $\gamma : (-\epsilon, \epsilon) \to M$ is a geodesic, then using the fact that the Riemannian connection is compatible with the metric, we have

$$\frac{d}{dt}\left\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right\rangle = 2\left\langle \frac{D}{dt}\frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right\rangle = 0$$

and the velocity vector $\frac{d\gamma}{dt}$ is constant as desired.

With respect to the local coordinates, it may be seen that the equation describing a geodesic satisfyies a second order differential equation.

It may be stated a theorem of existence and uniqueness of the solution of differential equation on a manifold M analogous to the classical one in $\mathbb{R}^n$. For the proof and more details, see [10].

Proposition 1.15. *Let M be a Riemannian manifold, endowed with its Riemannian connection. Given a point $p \in M$, there exist a neighborhood V of p in M and a number $\epsilon > 0$, such that if we denote by $\mathcal{U} = \{(q, w) \in TM; q \in V, w \in T_qM, |w| < \epsilon\}$, there exists a C^∞ mapping $\gamma : (-2, 2) \times \mathcal{U} \to M$ such that $t \to \gamma(t, q, w)$, $t \in (-2, 2)$, is the unique geodesic of M which, at the instant $t = 0$, passes through q with velocity w, for every $q \in V$ and for every $w \in T_qM$, with $|w| < \epsilon$.*

Definition 1.16. Let $p \in M$ and let $\mathcal{U} \subset TM$ be an open set as in Proposition 1.15. The *exponential map* on $\mathcal{U}$ is defined as the map $\exp : \mathcal{U} \to M$ such that

$$\exp(q, v) = \gamma(1, q, v) = \gamma(|v|, q, \frac{v}{|v|}), \quad (q, v) \in \mathcal{U}.$$

We will frequently utilize the restriction of exp to an open subset of the tangent space T_qM, that is, we define

$$\exp_q : B_\epsilon(0) \subset T_qM \to M$$

by $\exp_q(v) = \exp(q, v)$. Here $B_\epsilon(0)$ denotes the open ball with center at the origin of T_qM and of radius ϵ.

Geometrically, $\exp_q(v)$ is a point of M obtained by going out the length equal to $|v|$, starting from q, along a geodesic which passes through q with velocity equal to $\frac{v}{|v|}$.

1.2.3 Curvature of a Riemannian manifold

Let M denote a Riemannian manifold with Riemannian connection D.

Definition 1.17. *The* curvature R *of a Riemannian manifold* M *is a correspondence that associates to every pair* $X, Y \in \mathfrak{X}(M)$ *a mapping* $R(X,Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ *given by*

$$R(X,Y)Z = D_Y D_X Z - D_X D_Y Z + D_{[X,Y]}Z$$

for each $Z \in \mathfrak{X}(M)$.

If for example $M = \mathbb{R}^n$, then $R(X,Y)Z = 0$ for all $X, Y, Z \in \mathfrak{X}(\mathbb{R}^n)$. For this reason we can think of R as a way of measuring how much M deviates from being Euclidean.

Another way of viewing the curvature is to consider a local system of coordinates $(x_1, \ldots, x_n)$. Since $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$, we obtain

$$R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = \left(D_{\frac{\partial}{\partial x_j}} D_{\frac{\partial}{\partial x_i}} - D_{\frac{\partial}{\partial x_i}} D_{\frac{\partial}{\partial x_j}}\right)\frac{\partial}{\partial x_k},$$

that is, the curvature measures the non-commutativity of the covariant derivative.

The following proposition is a consequence of the definition and can be proved by direct computations.

Proposition 1.18. *The curvature* R *has the following properties:*

1. R *is bilinear in* $\mathfrak{X}(M) \times \mathfrak{X}(M)$, *that is,*

$$\begin{aligned} R(fX_1 + gX_2, Y_1) &= fR(X_1, Y_1) + gR(X_2, Y_1) \\ R(X_1, fY_1 + gY_2) &= fR(X_1, Y_1) + gR(X_1, Y_2) \end{aligned}$$

for each $f, g \in C^\infty(M)$ *and* $X_1, X_2, Y_1, Y_2 \in \mathfrak{X}(M)$.

2. *for any* $X, Y \in \mathfrak{X}(M)$, *the curvature operator is linear* $R(X,Y)$, *that is,*

$$R(X,Y)(fZ + W) = fR(X,Y)Z + R(X,Y)W,$$

for each $f \in C^\infty(M)$, $Z, W \in \mathfrak{X}(M)$.

We recall also the celebrated *Bianchi identity*:

Proposition 1.19 (Bianchi identity)**.**

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0,$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Moreover for the curvature tensor the following properties hold:

Proposition 1.20. *For each $X, Y, Z, T \in \mathfrak{X}(M)$ we have*

1. $\langle R(X,Y)Z,T\rangle + \langle R(Y,Z)X,T\rangle + \langle R(Z,X)Y,T\rangle = 0$;
2. $\langle R(X,Y)Z,T\rangle = -\langle R(Y,X)Z,T\rangle$;
3. $\langle R(X,Y)Z,T\rangle = -\langle R(X,Y)T,Z\rangle$;
4. $\langle R(X,Y)Z,T\rangle = \langle R(Z,T)X,Y\rangle$.

Closely related to the curvature operator is the sectional curvature. Let M be an n-dimensional Riemannian manifold.

Definition 1.21. Given a point $p \in M$ and a two-dimensional subspace $\pi \subset T_pM$, the real number

$$K(X,Y) = \frac{\langle R(X,Y)X,Y\rangle}{|X \wedge Y|} \tag{1.9}$$

does not depend on the choice of a basis X, Y of π and is called the *sectional curvature* of π at p.

To see that $K(X,Y)$ is invariant from the choice of the basis, observe that any other basis can be obtained by X, Y by iterating the following linear transformations $\{X,Y\} \to \{Y,X\}$, $\{X,Y\} \to \{\lambda X, Y\}$ and $\{X,Y\} \to \{X + \lambda Y, Y\}$, which do not modify the sectional curvature of π.

Definition 1.22. Given a point $p \in M$ and a vector $X = X_n$ in T_pM, take an orthonormal basis $X_1, \ldots, X_{n-1}$ of the hyperplane orthogonal to X. Then

$$\mathrm{Ric}_p(X) = \sum_{i=1}^{n-1} \langle R(X,X_i)X,X_i)\rangle = \sum_{i=1}^{n-1} K(X,X_i) \tag{1.10}$$

and

$$K(p) = \sum_{j=1}^{n} \mathrm{Ric}_p(X_j) \tag{1.11}$$

are independent of the choice of the corresponding orthonormal basis and are called respectively the *Ricci curvature* in the direction X and the *scalar curvature* at p.

1.2.4 Second fundamental form and shape operator

We introduce now the second fundamental form and the mean curvature of a surface. They are very important notions, which describe the geometry of the manifold. For sake of simplicity and since it will be the only case in which we consider them, we restrict our study to the surfaces embedded in $\mathbb{R}^3$.

Let M be a surface embedded in $\mathbb{R}^3$, that is M is a manifold of dimension 2 and there exists an embedding $F : M \to F(M) \subset \mathbb{R}^3$. Let us denote by M also the manifold $F(M)$ in $\mathbb{R}^3$. Let $p \in M$ be a point and take a coordinate neighborhood (U, φ) of p, with $W = \varphi(U)$ a connected open subset of $\mathbb{R}^2$. Let (u, v) be the coordinates of $\mathbb{R}^2$. The function $X : W \to U \subset M$ is a parametrization of U and induces the vector frame $X_u = \partial X/\partial u$ and $X_v = \partial X/\partial v$ which is a basis for the tangent space to U. The matrix of the metric g induced on M by $\mathbb{R}^3$ with respect to the basis X_u, X_v is given by

$$G = \begin{pmatrix} \langle X_u, X_u\rangle & \langle X_u, X_v\rangle \\ \langle X_v, X_u\rangle & \langle X_v, X_v\rangle \end{pmatrix}$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product in $\mathbb{R}^3$. The matrix G is called the *first fundamental form* on M.

Suppose M is orientable, then it is well defined a unit normal vector field N on M with respect to the Riemannian metric g on M. Let us choose the vector field N in such a way so that X_u, X_v and N form a frame with the same orientation as the standard orthonormal frame $(\partial/\partial x_1)$, $(\partial/\partial x_2)$, $(\partial/\partial x_3)$ of $\mathbb{R}^3$. We will study the structure of M by means of the derivative of N with respect to the various vectors in T_pM.

Let v be a tangent vector in T_pM and choose a C^∞ curve $\gamma : (-\epsilon, \epsilon) \to U$ such that $\gamma(0) = p$ and $\dot\gamma(0) = v$. Restricting N to γ we may differentiate $N(\gamma(t))$ as a vector field along a curve in $\mathbb{R}^3$. Since N has constant unitary norm, we have

$$0 = \frac{d}{dt}\langle N(t), N(t)\rangle_g\bigg|_{t=0} = 2\left\langle \frac{dN(t)}{dt}\bigg|_{t=0}, N_p\right\rangle_g ,$$

where $\langle\cdot,\cdot\rangle_g$ denotes the scalar product on M induced from the Riemannian metric g. This proves that $(dN/dt)_{t=0}$ is orthogonal to N and therefore lies in T_pM. The following theorem shows some more properties of this derivative.

Theorem 1.23. *The vector $(dN/dt)_{t=0}$ depends only on the vector X_p and not on the chosen curve γ. The map*

$$S : T_pM \to T_pM \quad \textit{with} \quad S(v_p) := -\frac{dN(\gamma)}{dt}\bigg|_{t=0} \tag{1.12}$$

is well defined and is a linear map.

Proof. Let $v_p = aX_u + bX_v$ be expressed in coordinates with respect to X_u, X_v and let

$$\gamma(t) = (X^1(u(t), v(t)), X^2(u(t), v(t)), X^3(u(t), v(t)))$$

be a differentiable curve such that $\gamma(0) = p$ and $\dot\gamma(0) = v_p$. Computing $\dot\gamma(0)$ we have

$$\dot\gamma(0) = \dot u(0)X_u + \dot v(0)X_v.$$

By the uniqueness of the coordinates, we then have $a = \dot u(0)$ and $b = \dot v(0)$.

Let us express the vector field N with respect to the standard coordinate system of $\mathbb{R}^3$ as follows

$$N = n^1(u,v)\frac{\partial}{\partial x^1} + n^2(u,v)\frac{\partial}{\partial x^2} + n^3(u,v)\frac{\partial}{\partial x^3}.$$

Then its derivative along the curve is given by

$$\begin{aligned}
\left.\frac{dN(\gamma)}{dt}\right|_{t=0} &= \sum_{i=1}^{3}\left(\left.\frac{\partial n^i}{\partial u}\right|_{\varphi(0)}\dot u(0) + \left.\frac{\partial n^i}{\partial v}\right|_{\varphi(0)}\dot v(0)\right)\frac{\partial}{\partial x^i} \\
&= a\sum_{i=1}^{3}\left.\frac{\partial n^i}{\partial u}\right|_{\varphi(0)}\frac{\partial}{\partial x^i} + b\sum_{i=1}^{3}\left.\frac{\partial n^i}{\partial v}\right|_{\varphi(0)}\frac{\partial}{\partial x^i} \\
&= -S(v_p).
\end{aligned}$$

This shows that $S(v_p)$ depends only on the coordinates $(u(0), v(0))$ of p and on the components $(\dot u(0), \dot v(0))$ of v_p, and not on the chosen curve. The latter formula shows also that $S(v_p)$ depends linearly on the coordinates of v_p, i.e. S is a linear map on T_pM. □

Let D be the Levi-Civita connection on $\mathbb{R}^3$ induced from the metric and note that

$$S(v) = -\frac{dN(\gamma(t))}{dt} = -D_vN, \tag{1.13}$$

where γ is a C^∞ curve with $\gamma(0) = p$ and $\dot\gamma = v$. It can be seen, by observing that dN/dt is symmetric and compatible with the metric and using the uniqueness of the Levi-Civita connection. Then the operator S is intrinsic for M and does not depend on the particular coordinate neighborhood.

Definition 1.24. The map S defined in (1.12) is called the *shape operator* or *Weingarten operator* of M.

Proposition 1.25. *The shape operator S is self-adjoint, i.e.*

$$\langle S(v), w\rangle_g = \langle v, S(w)\rangle_g \quad \textit{for each } v, w \in T_pM.$$

Proof. We use the coordinate neighborhood (U, φ) of p and the induced frame X_u, X_v introduced before. Then

$$X_u = \sum_{i=1}^{3} X_u^i \frac{\partial}{\partial x^i}, \quad X_v = \sum_{i=1}^{3} X_v^i \frac{\partial}{\partial x^i}, \text{ and } N = \sum_{i=1}^{3} n^i \frac{\partial}{\partial x^i}.$$

Since N is normal to the surface we have $\langle X_u, N \rangle_g = 0 = \langle X_v, N \rangle_g$. Differentiating them we obtain

$$\begin{aligned} -\left\langle \frac{\partial N}{\partial u}, X_u \right\rangle_g &= \langle N, X_{uu} \rangle_g = \sum_{i=1}^{3} n^i X_{uu}^i \\ -\left\langle \frac{\partial N}{\partial v}, X_u \right\rangle_g &= \langle N, X_{uv} \rangle_g = \sum_{i=1}^{3} n^i X_{vu}^i = \langle N, X_{vu} \rangle_g = -\left\langle \frac{\partial N}{\partial u}, X_v \right\rangle_g \\ -\left\langle \frac{\partial N}{\partial v}, X_v \right\rangle_g &= \langle N, X_{vv} \rangle_g = \sum_{i=1}^{3} n^i X_{vv}^i. \end{aligned}$$

These equations show that the components of the shape operator are C^∞ and the second relation in particular proves that S is symetric. □

Definition 1.26. The symmetric bilinear form on T_pM defined by

$$II(v, w) := \langle S(v), w \rangle_g \tag{1.14}$$

for each $v, w \in T_pM$ is called the *second fundamental form* on M.

Let II_p denote the matrix of the second fundamental form in p with respect to X_u and X_v and S the matrix of the shape operator with respect to the same basis. Then for all vectors $v, w \in T_pM$ we have

$$v^t II_p w = II(v, w) = \langle S(v), w \rangle_g = \langle v, S(w) \rangle_g = \langle v, GS(w) \rangle = v^t GSw.$$

Hence we have found the relation:

$$II_p = GS. \tag{1.15}$$

Making use of equation (1.13) we can write explicitly the matrix of the second fundamental form with respect to a basis v, w as follows:

$$II = \begin{pmatrix} \langle \tilde{D}_v N, v \rangle & \langle \tilde{D}_v N, w \rangle \\ \langle \tilde{D}_w N, v \rangle & \langle \tilde{D}_w N, w \rangle \end{pmatrix}. \tag{1.16}$$

The shape operator of a surface M describes the geometry of the manifold. A clarifying result is the following theorem.

Theorem 1.27. *Let M be a surface embedded in $\mathbb{R}^3$. At each point $p \in M$ the eigenvalues of the linear transformation S are real numbers k_1 and k_2, $k_1 \geq k_2$. If $k_1 \neq k_2$, then the eigenvectors belonging to them are orthogonal; if $k_1 = k_2 = k$ at p, them $S(v_p) = kv_p$ for every vector X_p in T_pM. The numbers k_1 and k_2 are the maximum and minimum values of $II(v_p, v_p) = (S(v_p), v_p)$ over all unit vectors $v_p \in T_pM$.*

Proof. The statements to prove are consequences of the theory of linear algebra for symmetric operators.

Fix $p \in M$ and consider S defined on the tangent space T_pM. Since S is a symmetric operator, it is diagonalizable and there exist real eigenvalues k_1, k_2.

If $k_1 = k_2 = k$ the eigenvalues of k span the tangent space T_pM. Choose any base v_1, v_2. Each vector $v \in T_pM$ can be written as $v = av_1 + bv_2$ and

$$S(v) = aS(v_1) + bS(v_2) = k(av_1 + bv_2) = kv.$$

The proof is complete in this case.

If $k_1 \neq k_2$, let v_1, v_2 be respective eigenvectors with unt norm. Then

$$k_1\langle v_1, v_2\rangle_g = \langle S(v_1), v_2\rangle_g = \langle v_1, S(v_2)\rangle_g = k_2\langle v_1, v_2\rangle_g,$$

implies that v_1 and v_2 are orthogonal since $k_1 \neq k_2$. Replacing v_2 with $-v_2$ if necessary, we may suppose that v_1, v_2 have the same orientation of T_pM. Then any vector $v \in T_pM$ may be written as $v = \cos\theta v_1 + \sin\theta v_2$. Let $k(\theta) := (S(v), v) = II(v, v)$. Then

$$k(\theta) = k_1 \cos^2\theta + k_2 \sin^2\theta,$$

because v_1, v_2 are orthonotmal. (This equation is also known as *Euler's formula*). Differentiating it we have

$$\frac{dk}{d\theta} = 2(k_2 - k_1)\sin\theta\cos\theta = (k_2 - k_1)\sin(2\theta).$$

Hence the extrema of $k(\theta)$ occur when $\theta = 0, \pi/2, \pi, 3\pi/2$, that correspond to $v = \pm v_1$, $v = \pm v_2$ as desidered. $\square$

The trace of the symmetric operator S and its determinant are the coefficients of the characteristic polynomial of S. They are invariants that describe the geometry of the surface and take special names.

Definition 1.28. The *Gaussian curvature* of a surface is the determinant of any representative matrix of S and is equal to

$$K = k_1 k_2$$

where k_1 and k_2 are the eigenvalues of S.

The *mean curvature*of a surface is the trace of S and is equal to

$$\mathcal{H} = k_1 + k_2.$$

1.3 Recalls on geometric measure theory

We take now into account the analytic aspects that will be used in the survey. To this end we explicite some notions given in the previous paragraph in the particular case of a C^k submanifold of $\mathbb{R}^n$, with $k < \infty$ in general. We then define the Hausdorff measure in a metric space, some fundamental theorems of geometric measure theory and their application to the calculus of the first variation. Finally we define a doubling measure and the Hardy-Littlewood maximal operator.

1.3.1 The divergence theorem for manifolds embedded in $\mathbb{R}^n$

Let us consider an n-dimensional C^k submanifold M of $\mathbb{R}^{n+r}$, with $k \geq 1$ and $r \geq 0$. As said in the previous paragraph, this means in particular that M is a subset of $\mathbb{R}^{n+r}$ such that for each $y \in M$ there exist open sets $U, V \subset \mathbb{R}^{n+r}$ with $y \in U$, $0 \in V$ and a C^k diffeomorphism $\varphi : U \to V$ such that $\varphi(0) = y$ and $\varphi(M \cap U) = W = V \cap \mathbb{R}^n$. By $\mathbb{R}^n \subset \mathbb{R}^{n+r}$ we mean the subspace of $\mathbb{R}^{n+r}$ consisting of all points $x = (x_1, \ldots, x_{n+r})$ such that $x_{n+1} = \ldots = x_{n+r} = 0$.

Let $\psi := \varphi^{-1}|_W$. Then $\psi : W \to \mathbb{R}^{n+r}$ is a diffeomorphism on the image $\psi(W) = M \cap V$ and $\psi(0) = y$, that is ψ is a local representation for M at y, and the vectors

$$\frac{\partial \psi}{\partial x_1}(0), \ldots, \frac{\partial \psi}{\partial x_n}(0)$$

are linearly independent in $\mathbb{R}^{n+r}$. In particular, this is a basis for the tangent space T_yM to M at y.

We want to give the definition of divergence on M. To this end, let us recall that a function $f : M \to \mathbb{R}^m$ is C^l on a manifold $M \subset \mathbb{R}^{n+r}$, with $l \leq k$, if there exist an open set $U \subset \mathbb{R}^{n+r}$ such that $M \subset U$, and a function $\tilde{f} : U \to \mathbb{R}^m$, such that $\tilde{f}\Big|_M = f$.

In case $f : M \to \mathbb{R}$ is a C^1 real-valued function, that is $m = 1$, we define the *gradient* $\nabla_M f$ of f at $y \in M$ by

$$\nabla_M f(y) := \sum_{j=1}^{n} (D_{v_j} f) v_j \tag{1.17}$$

where $v_1, \dots, v_n$ is any orthonormal base for the tangent space to M at y. Here $D_v f$ denotes the directional derivative of f with respect to the vector $v \in T_yM$. More explicitly, given a C^1 curve $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = y$ and $\dot\gamma(0) = v$, the directional derivative $D_v f \in \mathbb{R}^{n+r}$ of f is

$$D_v f := \frac{d}{dt}(f \circ \gamma(t))|_{t=0}$$

and depends only on the vector v and the function f and not on the particular curve γ that we have choosen. (Here $\epsilon > 0$ is choosen so that γ is well-defined on M.)

Let $e_1, \dots, e_{n+r}$ be the standard basis of $\mathbb{R}^{n+r}$ and

$$\nabla^j_M f := \langle \nabla_M f, e_j \rangle_{\mathbb{R}^{n+r}}$$

be the component of $\nabla_M f$ with respect to e_j, for each $j = 1, \dots, n+r$. Here $\langle \cdot, \cdot \rangle_{\mathbb{R}^{n+r}}$ denotes the standard scalar product in $\mathbb{R}^{n+r}$. Then the expression of $\nabla_M f$ with respect to the standard basis is

$$\nabla_M f(y) = \sum_{j=1}^{n+r} \nabla^j_M f(y) e_j.$$

Note that if f is the restriction to M of a C^1 function $\tilde f$, defined on an open subset $U \subset \mathbb{R}^{n+r}$ containing M, then from elementary calculus follows that

$$D_{v_i} f = \left\langle \mathrm{grad} \tilde f, v_i \right\rangle,$$

where $\mathrm{grad}\tilde f$ is the usual gradient $(D_1 \tilde f, \dots, D_{n+r}\tilde f)$ on $U \subset \mathbb{R}^{n+r}$, so that we can write equation (1.17) in the form

$$\nabla_M f(y) = \sum_{i=1}^{n} \left\langle \mathrm{grad}\tilde f(y), v_i \right\rangle v_i = \pi_M(\mathrm{grad}\tilde f(y))$$

where $\pi_M : \mathbb{R}^{n+r} \to T_yM$ is the orthogonal projection of $\mathbb{R}^{n+r}$ onto T_yM. In other words, the gradient of a function f with respect to M is simply the orthogonal projection of the classical gradient on the tangent space to M.

Let $X = (X_1, \dots, X_{n+r}) : M \to \mathbb{R}^{n+r}$ be a vector function with $X_j \in C^1(M)$, $j = 1, \dots, n+r$. We define the *divergence* of X with respect to M as

$$\mathrm{div}_M X := \sum_{j=1}^{n+r} \nabla^j_M X_j.$$

Notice that X can be more general than a vector field, since we do not require $X_y \in T_y M$. If we write more explicitely the expression of $\nabla_M^j X_j$ we have

$$\begin{aligned}
\mathrm{div}_M X &= \sum_{j=1}^{n+r} \langle \nabla_M X_j, e_j \rangle = \sum_{j=1}^{n+r} \left\langle \sum_{i=1}^{n} (D_{v_i} X_j) v_i, e_j \right\rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n+r} \langle (D_{v_i} X_j) v_i, e_j \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n+r} \langle v_i, (D_{v_i} X_j) e_j \rangle \qquad (1.18) \\
&= \sum_{i=1}^{n} \langle D_{v_i} X, v_i \rangle , \qquad (1.19)
\end{aligned}$$

for any orthonormal basis $v_1, \ldots, v_n$ of $T_x M$. The last equality follows from the expression $X = \sum_{j=1}^{n+r} X_j e_j$.

We are now able to state the *divergence theorem* for manifolds.

Theorem 1.29 (Divergence theorem). *Let M be a n-dimensional smooth compact manifold. Let us denote by $\bar{M}$ the closure of M and by $\partial M = \bar{M} \setminus M$ its boundary. Let $X = (X_1, \ldots, X_{n+r}) : M \to \mathbb{R}^{n+r}$ be a vector function such that $X_y \in T_y M$ for each $y \in M$. Then*

$$\int_M \mathrm{div}_M X d\mathcal{H}^n = - \int_{\partial M} \langle X, \nu \rangle \, d\mathcal{H}^{n-1},$$

where ν is the inward pointing unit normal of ∂M, that is $|\nu| = 1$, ν is normal to ∂M, tangent to M, and points into M at each point of ∂M.

From this theorem follows immediately that

$$\int_M \mathrm{div}_M X d\mathcal{H}^n = 0$$

whenever X has compact support in M.

To conclude this section, we give a Leibniz formula for the divergence. Let $X = (X_1, \ldots, X_{n+r}) : M \to \mathbb{R}^{n+r}$ be a vector function with $X_j \in C^1(M)$, $j = 1, \ldots, n+r$ and $f : M \to \mathbb{R}$ be a C^1 function. From equation (1.18) applied to the product fX we obtain

$$\begin{aligned}
\mathrm{div}_M (fX) &= \sum_{i=1}^{n} \langle D_{v_i}(fX), v_i \rangle = \sum_{i=1}^{n} (\langle D_{v_i}(f) X, v_i \rangle + \langle f D_{v_i} X, v_i \rangle) \\
&= \sum_{i=1}^{n} \langle X, D_{v_i}(f) v_i \rangle + f \mathrm{div}_M X \\
&= X_{\mathrm{tang}}(f) + f \mathrm{div}_M X, \qquad (1.20)
\end{aligned}$$

where in the last equality, X_{tang} is the C^1 vector field on M defined as the tangential component of X with respect to the tangent space to M. $X_{\text{tang}}(f)$ denotes the derivative of f with respect to this vector field.

1.3.2 Hausdorff measures

Definition 1.30. Let (X, d) be a metric space and $A \subset X$. For any $\alpha \geq 0$, let

$$\omega_\alpha := \frac{\pi^{\frac{\alpha}{2}}}{\Gamma(1 + \frac{\alpha}{2})},$$

where $\Gamma(t) := \int_o^\infty x^{t-1} e^{-x} dx$ denotes the Euler function. For $\delta > 0$ define

$$\mathcal{H}^\alpha_\delta(A) := \frac{\omega_\alpha}{2^\alpha} \inf \left\{ \sum_{i \in \mathbb{N}} (\text{diam}(A_i))^\alpha \text{ s.t. } \text{diam}(A_i) < \delta, A \subset \bigcup_{i \in \mathbb{N}} A_i \right\}. \quad (1.21)$$

The *α-dimensional Hausdorff measure* of A is defined as

$$\mathcal{H}^\alpha(A) := \sup_{\delta > 0} \mathcal{H}^\alpha_\delta(A). \quad (1.22)$$

Notice that the Hausdorff measure is a metric concept and depends only on the given distance.

Remark 1.31. If $X = \mathbb{R}^n$ endowed with the usual euclidean distance and $\alpha = k$ is a positive integer, then

$$\omega_k = \mathcal{L}^k(B(0, 1)),$$

where $B(0, 1)$ is the unit ball in $\mathbb{R}^n$ centered in 0 and $\mathcal{L}^k$ is the k-dimensional Lebesgue measure in $\mathbb{R}^n$.

Remark 1.32. Observe that the quantity in (1.21) is non-decreasing as $\delta \to 0$. Then the supremum in (1.22) is equivalent to

$$\mathcal{H}^\alpha(A) = \lim_{\delta > 0} \mathcal{H}^\alpha_\delta(A).$$

Definition 1.33. If in Definition 1.30 we require that the covering sets A_i are balls, we obtain the *spherical Hausdorff measure* denoted by $\mathcal{S}^\alpha(A)$.

Notice that for each $A \subset X$,

$$2^{-\alpha} \mathcal{S}^\alpha(A) \leq \mathcal{H}^\alpha(A) \leq \mathcal{S}^\alpha(A).$$

We recall the following basic properties of the Hausdorff measure.

Proposition 1.34. *Let* $0 \leq \alpha < \alpha' < \infty$ *and* $A \subset X$*, then*

1. $\mathcal{H}^{\alpha}(A) < \infty$ *implies* $\mathcal{H}^{\alpha'}(A) = 0$*;*

2. $\mathcal{H}^{\alpha'}(A) > \infty$ *implies* $\mathcal{H}^{\alpha}(A) = +\infty$*.*

Proof. To prove 1, choose $\delta > 0$ and let $A \subset \bigcup_i A_i$, with $\mathrm{diam}(A_i) < \delta$ and $(\omega_\alpha/2^\alpha)\sum_i(\mathrm{diam}(A_i))^\alpha \leq \mathcal{H}^\alpha_\delta(A) + 1$. Then

$$\mathcal{H}^{\alpha'}_\delta(A) \leq C\sum_i(\mathrm{diam}(A_i))^{\alpha'} \leq C\delta^{\alpha'-\alpha}\sum_i(\mathrm{diam}(A_i))^{\alpha} \leq C\delta^{\alpha'-\alpha}(\mathcal{H}^\alpha_\delta(A)+1),$$

where C is a suitable constant. As δ goes to zero, this estimate proves 1.

Statement 2 is just the contrapositive of 1. □

These properties of the Hausdorff measure allow us to give the following definition.

Definition 1.35. Let $A \subset X$, we define the *Hausdorff dimension of* A as

$$\begin{aligned} \mathcal{H} - \dim(A) &:= \inf\{\alpha \geq 0 | \mathcal{H}^\alpha(A) = 0\} \\ &:= \sup\{\alpha \geq 0 | \mathcal{H}^\alpha(A) = +\infty\}. \end{aligned}$$

The Hausdorff dimension of a metric space (X, d) is defined as the Hausdorff dimension of its unit ball centered in 0.

Proposition 1.34 implies that if the dimension of A is α then $\mathcal{H}^{\alpha'}(A) = +\infty$ for each $\alpha' < \alpha$ and $\mathcal{H}^{\alpha'}(A) = 0$ for each $\alpha' > \alpha$, but we cannot deduce any information about the value of $\mathcal{H}^\alpha(A)$. However if for a set $A \subset X$ there exists $\alpha \geq 0$ such that $0 < \mathcal{H}^\alpha(A) < +\infty$ then α is the Hausdorff dimension of A.

1.3.3 Rademacher's Theorem and Area and Coarea formulas

Definition 1.36. Let $A \subset \mathbb{R}^n$. A function $f : A \to \mathbb{R}^m$ is called *Lipschitz* if there exists a constant $L \geq 0$ such that

$$|f(x) - f(y)| \leq L|x - y| \text{ for all } x, y \in \mathbb{R}^n. \tag{1.23}$$

The smallest constant L such that (1.23) holds is called the *Lipschitz constant* of f and is denoted by

$$\mathrm{Lip}(f) := \sup\left\{\frac{|f(x) - f(y)|}{|x - y|} : x, y \in A, x \neq y\right\}.$$

We recall the well-known extension theorem for Lipschitz functions.

Theorem 1.37. *Let $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ be Lipschitz. Then there exists a Lipschitz function $\bar{f} : \mathbb{R}^n \to \mathbb{R}^m$ such that $\bar{f}$ is an extension of f, that is $\bar{f} = f$ on A, and $\mathrm{Lip}(\bar{f}) \leq \sqrt{m}\mathrm{Lip}(f)$.*

We state without proof the following important theorem:

Theorem 1.38 (Rademacher's Theorem)**.** *Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz function. Then f is differentiable $\mathcal{L}^n$-a.e. in A, that is, there exists a linear mapping $T : A \to \mathbb{R}^m$ such that*

$$\lim_{y \to x} \frac{|f(y) - f(x) - T(x-y)|}{|x-y|} = 0 \text{ for } \mathcal{L}^n\text{-a.e.} x \in A.$$

To state the Area and Coarea Formulas we need the notion of Jacobian of a function. Let $k \leq n$ and $v_1, \dots, v_k$ linearly independent vectors in $\mathbb{R}^n$. A k-dimensional parallelepiped P in $\mathbb{R}^n$ is the set defined by

$$P = \left\{ \sum_{i=1}^{k} \lambda_i v_i : 0 \leq \lambda_i \leq 1, \text{ for } i = 1, \dots, k \right\}.$$

Proposition 1.39. *Let $v_1, \dots, v_k$ be linearly independent vectors in $\mathbb{R}^n$ with $k \leq n$, then the parallelepiped determined by those vectors has k-dimensional area $\sqrt{\det(V^t V)}$, where V is the $n \times k$ matrix having $v_1, \dots, v_k$ as its columns.*

Definition 1.40. Let $A \subset \mathbb{R}^m$ and $f : A \to \mathbb{R}^n$ be differentiable at x. Let $k \leq m$. We define the *k-dimensional Jacobian of f at x*, denoted by $J_k f(x)$, by setting

$$J_k f(x) = \sup \left\{ \frac{\mathcal{H}^k(Df(x)(P))}{\mathcal{H}^k(P)} \right\}.$$

where the supremum is taken over all k-dimensional parallelepipeds P contained in $\mathbb{R}^m$. Here $Df(x)$ denotes the differential of f in x.

The following lemma simplifies the computation of the k-dimensional Jacobian in particular cases.

Lemma 1.41. *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be differentiable at $x \in \mathbb{R}^m$:*

1. *if $m = n$, then $J_m f(x) = J_n f(x) = |\det(Df(x))|$;*
2. *if $m \leq n$, then $J_m f(x) = \sqrt{\det\left((Df(x))^t (Df(x))\right)}$;*
3. *if $m \geq n$, then $J_n f(x) = \sqrt{\det\left((Df(x))(Df(x))^t\right)}$.*

We can now state the following important theorems of geometric measure theory.

Theorem 1.42 (Area Formula). *If $f : \mathbb{R}^m \to \mathbb{R}^n$ is a Lipschitz function and $m \le n$, then*

$$\int_A J_m f(x) d\mathcal{L}^m(x) = \int_{\mathbb{R}^n} \mathcal{H}^0(A \cap f^{-1}(y)) d\mathcal{H}^m(y)$$

for each Lebesgue measurable subset $A \subset \mathbb{R}^m$.

The following is a generalization of the area formula, which can be proved by approximations by simple functions.

Corollary 1.43. *If $f : \mathbb{R}^m \to \mathbb{R}^n$ is a Lipschitz function and $m \le n$, then*

$$\int_A g(x) J_m f(x) d\mathcal{L}^m(x) = \int_{\mathbb{R}^n} \sum_{x \in A \cap f^{-1}(y)} g(x) d\mathcal{H}^m(y)$$

for each Lebesgue measurable subset $A \subset \mathbb{R}^m$ and each nonnegative $\mathcal{L}^m$-measurable function $g : A \to \mathbb{R}$.

Theorem 1.44 (Coarea Formula). *If $f : \mathbb{R}^m \to \mathbb{R}^n$ is a Lipschitz function and $m \ge n$, then*

$$\int_A J_n f(x) d\mathcal{L}^m(x) = \int_{\mathbb{R}^n} \mathcal{H}^{m-n}(A \cap f^{-1}(y)) d\mathcal{H}^m(y)$$

for each Lebesgue measurable subset $A \subset \mathbb{R}^m$.

In case $m = n$, the n-dimensional Jacobian agrees with the usual Jacobian $|\det(Df)|$, and the area and coarea formulas coincide. In case $m > n$, and $f : \mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^n$ is the orthogonal projection onto the first factor, then the coarea formula simplifies to Fubini's theorem. Hence one can think of the coarea formula as a generalization of Fubini's theorem to functions more complicated than orthogonal projection.

We give a generalization of the co-area formula similar to that in Corollary 1.43.

Corollary 1.45. *If $f : \mathbb{R}^m \to \mathbb{R}^n$ is a Lipschitz function and $m \ge n$, then*

$$\int_A g(x) J_n f(x) d\mathcal{L}^m(x) = \int_{\mathbb{R}^n} \int_{A \cap f^{-1}(y)} g \, d\mathcal{H}^{m-n} d\mathcal{L}^n(y)$$

for each Lebesgue measurable subset $A \subset \mathbb{R}^m$ and each nonnegative $\mathcal{L}^m$-measurable function $g : A \to \mathbb{R}$.

1.3.4 First variation formula

We apply the just defined notions, to compute the first variation of a 1-parameter family of diffeomerphisms.

Let M be a n-dimensional C^1 submanifold in $\mathbb{R}^{n+r}$ and V be an open subset of $\mathbb{R}^{n+r}$ sucht that $M \cap U \neq \emptyset$ and $\mathcal{H}^n(K \cap M) < \infty$ for each compact $K \subset V$.

A *1-parameter family of diffeomorphisms* of V is a family $\{\varphi_t\}_{t\in\mathbb{R}}$ of diffeomorphisms $\varphi_t : V \to V$, such that

$$\varphi_{t+s}(x) = \varphi_t(\varphi_s(x))$$

for each $t, s \in \mathbb{R}$ and $x \in V$.

Let $\{\varphi_t\}$ be such a family of diffeomorphisms on V (it is enough to require $t \in (-1, 1)$) such that $\varphi(t, x) := \varphi_t(x)$ is a C^1 map from $(-1, 1) \times V$ to V, φ_0 is the identity map and φ_t is the identity map out of a compact $K \subset V$ for each $t \in (-1, 1)$.

Since $\varphi(t, x)$ is a C^1 map, for each x it determines a C^1 curve in V with starting point $x = \varphi(0, x)$. Let us define the vector field U of initial velocities:

$$U_x = \left.\frac{\partial\varphi(t, x)}{\partial t}\right|_{t=0} \qquad \text{for each } x \in V.$$

Then U has compact support contained in K and

$$\varphi_t(x) = x + tU_x + o(t^2).$$

Let us denote by M_t the image of $M \cap K$ under φ_t. In particular $M_0 = M \cap K$. The computation of the first variation of the measure $\mathcal{H}^n(M_t)$ may be simplified by an application of the area formula. Indeed, let $\psi_t = \varphi_t|_{M\cap V}$. Then by the area formula 1.42 we have

$$\left.\frac{d}{dt}\mathcal{H}^n(M_t)\right|_{t=0} = \left.\frac{d}{dt}\int_{M_t} d\mathcal{H}^n\right|_{t=0} = \left.\frac{d}{dt}\int_{M\cap K} J\psi_t d\mathcal{H}^n\right|_{t=0}.$$

Here we have omitted the indication of the dimension n in the Jacobian, for the sake of simplicity. Differentiating under the integral, we may reduce to the calculation of $\frac{\partial}{\partial t}(J\psi_t)$.

Fix $t \in (-1, 1)$. In order to obtain the Jacobian, we have find the differential of ψ_t. For each $v \in T_xM$ we have

$$d\psi_t|_x(v) = D_v\psi_t(x) = D_v(x + tU(x) + o(t^2)) = v + tD_vU(x) + o(t^2).$$

So the matrix of $d\psi_t$ relative to an orthonormal basis $v_1, \ldots, v_n$ of T_xM and the standard basis $e_1, \ldots, e_{n+r}$ of $\mathbb{R}^{n+r}$ is given by

$$[D\psi_t(x)]_{ki} = v_i^{(k)} + tD_{v_i}U_k(x) + o(t^2)$$

for $i = 1, \ldots, n$ and $k = 1 \ldots, n + r$. Here $v_i^{(k)}$ is the k-th component of v_i. Then $(D\psi_t(x))^t \cdot (D\psi_t(x))$ is the matrix with entries

$$b_{ij} = \delta_{ij} + t\left(\langle v_i, D_{v_j}U(x)\rangle + \langle v_j, D_{v_i}U(x)\rangle\right) + o(t^2).$$

Using the expansion formula $\det(I + tA) = 1 + t\,\mathrm{tr}(A) + o(t^2)$, where A is any square matrix and I is the identity matrix, we have

$$\begin{aligned}(J\psi_t)^2 &= 1 + t\,\mathrm{tr}\left(\langle v_i, D_{v_j}U\rangle + \langle v_j, D_{v_i}U\rangle\right)_{ij} + o(t^2)\\ &= 1 + 2t\left(\sum_{i=1}^{n}\langle v_i, D_{v_i}U\rangle\right) + o(t^2)\\ &= 1 + 2t\,\mathrm{div}_M U + o(t^2)\end{aligned}$$

where the last equality follows from the definition of the divergence (1.18). Finally by the Taylor expansion $\sqrt{1+x} = 1 + \frac{1}{2}x + o(x^2)$ we have $J\psi_t = 1 + t\mathrm{div}_M U + o(t^2)$. Then the derivative of the Jacobian in $t = 0$ is

$$\left.\frac{d}{dt}J\psi_t\right|_{t=0} = \mathrm{div}_M U. \tag{1.24}$$

We will make use of this formula in the computation of the first variation of the perimeter in Section 5.3.2.

1.3.5 Doubling measure and the Hardy-Littlewood maximal operator

We introduce here the doubling property of a measure and the related Hardy-Littlewood maximal operator. Its weak-type estimate will be fundamental to state a Sobolev-Gagliardo-Nierenberg inequality in the Heisenberg group (see Section 5.2).

In this section let (X, d) be a metric space and let $\mathcal{B}(X)$ denote the σ-algebra of its Borel sets.

Definition 1.46 (Doubling measures)**.** A measure $\mu : \mathcal{B}(X) \to [0, +\infty]$ is said to be *doubling* if μ is finite on bounded sets and there exists a constant C such that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) \quad \textit{for each } x \in X, \textit{ for each } r > 0. \tag{1.25}$$

The best constant C in the above inequality is called *the doubling constant* of μ.

The following theorem provides a lower bound for a doubling measure on balls and gives a characterization of the doubling property.

Theorem 1.47. *A measure $\mu : \mathcal{B}(X) \to [0, +\infty]$ finite on bounded sets, is doubling if and only if there exist constants $s, C' > 0$ such that*

$$\frac{\mu(B(x,r))}{\mu(B(y,R))} \geq C' \left(\frac{r}{R}\right)^s \tag{1.26}$$

for all $x, y \in X$ and all $R \geq r > 0$ such that $x \in B(y, R)$.

Proof. On one hand, if (1.26) holds, we have in particular with $R = 2r$ and $y = x$ that

$$\mu(B(x, 2r)) \leq 2^s (C')^{-1} \mu(B(x,r))$$

which proves the doubling property (1.25) with $C := 2^s (C')^{-1}$.

For the other implication, suppose that μ is doubling and define for $i \in \mathbb{N}$

$$R_i := 2^i r \quad \text{ and } \quad j := \min\{i : B(x, R_i) \supseteq B(y, R)\}.$$

Iterating (1.25) j times yields

$$\mu(B(y,R)) \leq \mu(B(x, R_i)) \leq C^j \mu(B(x,r))$$

which implies that

$$\frac{\mu(B(x,r))}{\mu(B(y,R))} \geq C^{-j}.$$

On the other hand, $B(y, R)$ is not contained in $B(x, R_{j-1})$: hence $R_{j-1} \leq 2R$ (since $x \in B(y, R)$, by assumption) and $j \leq \log_2(4R/r)$. Then the proof is completed by letting $C' := C^{-2}$ and $s := -\log C / \log 2$. □

Definition 1.48 (Hardy-Littlewood maximal operator)**.** Given two non-negative measures μ and ν on the metric space X, define the *Hardy-Littlewood maximal operator* of the measure ν (with respect to the measure μ) as

$$M_\nu(x) := \sup_{r>0} \frac{\nu(B(x,r))}{\mu(B(x,r))} \quad \text{for each } x \in X. \tag{1.27}$$

Let $X = \mathbb{R}^n$ endowed with the ususal Euclidean distance. A Borel function f in $L^1(\mathbb{R}^n)$ defines a non-negative finite Borel measure

$$\nu_f(A) = \int_A |f(y)| dy$$

for each Borel set $A \subset \mathbb{R}^n$.

The Hardy-Littlewood maximal operator of ν_f with respect to the measure μ is then given by

$$Mf(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|dy. \tag{1.28}$$

Theorem 1.49 (Weak-type $(1,1)$ estimate)**.** *Suppose that μ is doubling and that the support of μ coincides with the whole of X. Then there exists $C > 0$ (depending only on the doubling constant of μ) such that*

$$\mu(\{x \in X : M_\nu(x) > \lambda\}) \leq \frac{C}{\lambda}\nu(X) \tag{1.29}$$

for all $\lambda > 0$.

Proof. If $\nu(X) = +\infty$, there is nothing to prove. Otherwise, choose $\lambda > 0$, $R > 0$, and define for all $x \in X$

$$\begin{aligned} M_{\nu,R}(x) &:= \sup_{0<r<R} \frac{\nu(B(x,r))}{\mu(B(x,r))} \\ A_R &:= \{x \in X : M_{\nu,R}(x) > \lambda\}. \end{aligned} \tag{1.30}$$

Let $\mathcal{F}$ be the family of all open balls $B(x,r)$ such that $x \in A_R$, $0 < r < R$, and $\nu(B(x,r)) > \lambda\mu(B(x,r))$. $\mathcal{F}$ is a covering of A_R, because from the definitions in (1.30), for all $x \in A_R$ there exists $0 < r_x < R$ so that $B(x,r_x) \in \mathcal{F}$. By a Vitali-type covering theorem, there exists an at most countable disjoint subfamily $\mathcal{F}' \subseteq \mathcal{F}$ such that $A_R \subseteq \bigcup_{B\in\mathcal{F}'} \hat{B}$, where $\hat{B}$ is the ball with the same center as B and a radius five times that of B. Therefore, since μ is doubling and $5 < 2^3$, we have

$$\begin{aligned} \mu(A_R) &\leq \sum_{B\in\mathcal{F}'} \mu(\hat{B}) \leq C^3 \sum_{B\in\mathcal{F}'} \mu(B) \\ &\leq \frac{C^3}{\lambda} \sum_{B\in\mathcal{F}'} \nu(B) \leq \frac{C^3}{\lambda}\nu(X). \end{aligned}$$

Finally, since R was arbitrary, we obtain

$$\mu(\{x \in X : M_\nu(x) > \lambda\}) = \lim_{R\to+\infty} \mu(A_R) \leq \frac{C^3}{\lambda}\nu(X).$$

□

Chapter 2

The Isoperimetric problem in Euclidean Space

In this chapter we describe the *classical isoperimetric problem.* Being very ancient, it was approached many times and there are several proofs for the planar Euclidean case; we will analyse some of them to give an idea of the various technics that may be involved in this problem. The last section is devoted to the problem in the n-dimensional Euclidean space. In particular we will introduce the generalized notion of perimeter given by De Giorgi, to deal with the largest class of sets.

2.1 The isoperimetric inequality in the plane

The isoperimetric problem in the plane has substantially three equivalent formulations: consider all bounded isoperimetric domains in $\mathbb{R}^2$, i.e. all connected open sets of $\mathbb{R}^2$ with fixed given perimeter, and find that one which contains the greatest area. The answer, as suggested by the intuition, will be the disk; equivalently, one may fix the area of all bounded domains and require that the perimeter is minimized; finally, we may express the problem as an analytic inequality: we ask which is the greatest constant C such that the inequality $L^2 \geq CA$ holds for every domain, where A denotes the area of the domain under consideration and L deontes the length of its boundary. We would like to show that the best constant is $C = 4\pi$ and that the equality is achieved exactly when the domain is a disk.

Here we will recall some proofs for relatively compact domains with $\mathbb{C}^1$ boundary consisting of a single component, namely we will prove the following theorem.

Theorem 2.1 (Isoperimetric inequality in $\mathbb{R}^2$)**.** *Let Ω be a relatively compact* *domain, with boundary $\partial\Omega \in C^1$ consisting of one component. Then*

$$L^2(\partial\Omega) \geq 4\pi A(\Omega). \tag{2.1}$$

First proof of (2.1)*: using complex variables.* We identify $\mathbb{R}^2$ with the complex plane and denote any element with $z = x + \mathbf{i}y$. The area measure is then given by

$$dA = dx \wedge dy = \frac{\mathbf{i}}{2} dz \wedge d\bar{z}.$$

Then we obtain

$$4\pi A(\Omega) = \int_\Omega 4\pi dA = \int_\Omega 2\pi \mathbf{i} dz \wedge d\bar{z}$$

By the residue theorem $2\pi\mathbf{i} = \int_{\partial\Omega} \frac{1}{\omega - z} d\omega$ for each $z \in \Omega$. Then from this observation, together with Fubini's theorem it follows that

$$4\pi A(\Omega) = \int_\Omega \int_{\partial\Omega} \frac{1}{\omega - z} d\omega dz \wedge d\bar{z} = \int_{\partial\Omega} \int_\Omega \frac{1}{\omega - z} dz \wedge d\bar{z} d\omega. \tag{2.2}$$

Notice that $d(\frac{\bar{\omega}-\bar{z}}{\omega - z} dz) = (\omega - z)^{-1} dz \wedge d\bar{z}$, then applying Green's theorem to the inner integral of (2.2) we have

$$4\pi A(\Omega) = \int_{\partial\Omega} \int_{\partial\Omega} \frac{\bar{\omega} - \bar{z}}{\omega - z} dz d\omega.$$

Since $\left|\frac{\bar{\omega}-\bar{z}}{\omega-z}\right| = 1$ a.e. on $\partial\Omega$, the latter integral is less than or equal to $L(\partial\Omega)^2$ and we get the desired conclusion.

In particular, if $\Omega = B(0, \rho)$ is a disk with center in 0 and radius ρ, take the substitutions $\omega = \rho e^{i\varphi}$ and $z = \rho e^{i\theta}$ in the last step, then we have

$$\int_{\partial\Omega} \int_{\partial\Omega} \frac{\bar{\omega} - \bar{z}}{\omega - z} dz d\omega = \int_0^{2\pi} \int_0^{2\pi} \rho^2 d\theta d\varphi = 4\pi\rho^2 = L(\partial B(0, \rho))^2.$$

This proves that 4π is indeed the smallest suitable constant in (2.1). □

The second proof makes use of Fourier series. First of all we need the following

Theorem 2.2 (Wirtinger's Inequality)**.** *If f is a C^1, L-periodic function on $\mathbb{R}$ such that $\int_0^L f(t)dt = 0$, then*

$$\int_0^L |f'(t)|^2 dt \geq \frac{4\pi^2}{L^2} \int_0^L |f(t)|^2 dt,$$

with equality if and only if there exist constants a_{-1} and a_1 such that $f(t) = a_{-1} e^{-2\pi \mathbf{i} t/L} + a_1 e^{2\pi \mathbf{i} t/L}$.

Proof. This result follows from the theory of Fourier series: a function g, that is L-periodic and in $L^2(0, L)$, admits a Fourier expansion

$$g(t) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi \mathbf{i} kt/L},$$

where

$$c_k = \frac{1}{L} \int_0^L g(t) e^{-2\pi \mathbf{i} kt/L} dt$$

are the Fourier coefficients of g.

Since $f(t)$ is a C^1, L-periodic function, it has a Fourier expansion together with its derivative. Let

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi \mathbf{i} kt/L} \quad \text{with} \quad a_k = \frac{1}{L} \int_0^L f(t) e^{-2\pi \mathbf{i} kt/L} dt,$$

$$f'(t) = \sum_{k=-\infty}^{\infty} b_k e^{2\pi \mathbf{i} kt/L} \quad \text{with} \quad b_k = \frac{1}{L} \int_0^L f'(t) e^{-2\pi \mathbf{i} kt/L} dt.$$

By hypothesis $a_0 = 0$. Moreover, if we integrate by parts the formula for b_0, the continuity of f implies $b_0 = 0$. For each $k \neq 0$, an analogous integration by parts implies $b_k = 2\pi \mathbf{i} k a_k / L$. This computation together with Parseval's identity yields

$$\int_0^L |f'(t)|^2 dt = L \sum_{k \neq 0} |b_k|^2 \geq L \frac{4\pi^2}{L^2} \sum_{k \neq 0} |a_k|^2 = \frac{4\pi^2}{L^2} \int_0^L |f(t)|^2 dt,$$

which implies the desired inequality. The equality holds if and only if $a_k = 0$ for all $|k| \neq 1$. $\square$

We use this last result to give another proof of (2.1) which characterizes the sets that realize the equality in (2.1).

Second proof of (2.1)*: Method of Fourier series.* Without loss of generality let us assume $\int_{\partial\Omega} x ds = 0$, where $x = (x_1, x_2) \in \mathbb{R}^2$ and ds is the length measure on $\partial\Omega$ (if necessary, translate Ω). Let us consider the vector field $\vec{x} = x_1 e_1 + x_2 e_2$. Note that $\mathrm{div}\vec{x} = 2$, then by the divergence theorem we have

$$2A(\Omega) = \int_\Omega \mathrm{div}\vec{x} dA = -\int_{\partial\Omega} \langle \vec{x}, \vec{n} \rangle \leq \int_{\partial\Omega} |\vec{x}| ds, \tag{2.3}$$

where $\vec{n}$ is the inward pointing unit normal to $\partial\Omega$. By Hölder's inequality, we can majorize this last quantity with

$$2A(\Omega) \leq \left(\int_{\partial\Omega} |\vec{x}|^2 ds\right)^{\frac{1}{2}} \left(\int_{\partial\Omega} 1^2 ds\right)^{\frac{1}{2}} = L(\partial\Omega)^{\frac{1}{2}} \left(\int_{\partial\Omega} |x_1|^2 + |x_2|^2 ds\right)^{\frac{1}{2}}.$$

Let us parametrize $\partial\Omega$ by arc length. By the hypothesis $\int_{\partial\Omega} x ds = 0$, we can apply Wirtinger's inequality to each coordinate function x_1 and x_2, hence

$$2A(\Omega) \leq L(\partial\Omega)^{\frac{1}{2}} \left(\frac{L(\partial\Omega)^2}{4\pi} \int_{\partial\Omega} |x'|^2 ds\right)^{\frac{1}{2}} = \frac{L(\partial\Omega)^2}{2\pi}$$

as desired.

Let us characterize the sets which achieve the equality in (2.1). To have an equality in (2.3), we must require $\vec{x}$ in the same direction of $\vec{n}$ a.e. Since the boundary of Ω is C^1, this holds for every $\vec{x}$, hence $\vec{n}$ is radial. This implies that $\partial\Omega$ is a circumference and we have equality everywhere. $\square$

The last proof that we will present uses geometric arguments. It is based on symmetry and convexity methods. However, it will give only a weak result, in the sense that it will find the nature of the minimizer, provided we assume its existence. On the other hand, it is interesting because it generalizes to only piecewise C^1 boundaries.

Third proof of (2.1)*: Geometric methods.* Let Ω be a minimizer such that its boundary is piecewise C^1. First notice that each line π which divides Ω in two pieces of equal area must separate also the boundary in two pieces of equal length. In fact, let Ω_1 and Ω_2, be the two parts in which Ω is divided by π, with $A(\Omega_1) = A(\Omega_2)$. If, for example, $L(\partial\Omega_1) < L(\partial\Omega_2)$, we could obtain a set with less perimeter than Ω by taking the union of Ω_1 and its reflection over the line π. This contradicts the hypothesis that Ω is an isoperimetric minimizer.

By carrying out this argument succesively, we may assume that there exists a coordinate system in $\mathbb{R}^2$ such that Ω is symmetric with respect to the two coordinate axes. Then every line through the origin cuts Ω into two open sets of equal area and hence equal bounding length.

Notice also that we may assume the closure of Ω, $\bar{\Omega}$, to be convex. Indeed, if it weren't, we may replace it by its convex hull, which increases the area and decreases the bounding length.

Since the origin is in Ω, for each $y \in \partial\Omega$ such that there exists the tangent line to Ω at y, consider the line l that passes through the origin and y. We claim that l is perpendicular to the tangent line at y. If not, by reflecting over l one of the two parts in which Ω is divided by l, we obtain a non-convex domain, with the same area and the same perimeter of Ω. Its convex hull has however greater area and smaller perimeter, which contradicts the hypothesis on $\partial\Omega$ to be minimal.

Therefore, Ω is convex with every point of differentiability of $\partial\Omega$ having tangent line orthogonal to the position vector $\vec{x}$. Consider the unit circle $\mathbb{S}$, denote by $\vec{n}$ its unitary position vector and by θ the local coordinate on $\mathbb{S}$. For what we have just said we have $\vec{x} = \tau\vec{n}$, at all points of differentiability of $\partial\Omega$, so that

$$\frac{\partial\vec{x}}{\partial\theta} = \frac{\partial\tau}{\partial\theta}\vec{n} + \tau\frac{\partial\vec{n}}{\partial\theta}.$$

Since $\vec{n}$ is perpendicular to the tangent space to $\partial\Omega$ a.e., we have

$$0 = \langle\frac{\partial\vec{x}}{\partial\theta}, \vec{n}\rangle = \langle\frac{\partial\tau}{\partial\theta}\vec{n}, \vec{n}\rangle + \langle\tau\frac{\partial\vec{n}}{\partial\theta}, \vec{n}\rangle = \frac{\partial\tau}{\partial\theta}$$

and we conclude that τ is differentiable with $\partial\tau/\partial\theta = 0$ at all points of differentiability of $\partial\Omega$; then τ is locally constant on the C^1 arcs of $\partial\Omega$. Hence $\partial\Omega$ consists of a finite number of circular arcs connected by segments of radial lines. The convexity of Ω implies $\partial\Omega$ is a circle. $\square$

Remark 2.3. One could be tempted to carry out this argument also in higher dimensions, but we have to be careful about the convexity. For example, passing to the convex hull of a domain does not in general decrease its isoperimetric quotient. In fact, if we consider a standard imbedded solid torus in $\mathbb{R}^3$, where the circle in the xy-plane has radius 1, and the rotated circle perpendicular to the $xy-plane$ has radius $\epsilon << 1$. Then its isoperimetric quotient is asymptotic to a constant multiple of $\epsilon^{-1/3}$ as $\epsilon \to 0$. On the other hand, the convex hull has isopermetric quotient asymptotic to a constant multiple of $\epsilon^{-2/3}$ as $\epsilon \to 0$.

2.2 The classical isoperimetric problem

The classical formulation of the isoperimetric problem in the n-dimensional Euclidean space, with $n > 2$, reads as follows:

Theorem 2.4 (Isoperimetric inequality)**.** *Let $E \subset \mathbb{R}^n$ be a Borel set with finite perimeter $P(E)$ and area denoted by $|E|$. Then*

$$\min\{|E|^{(n-1)/n}, |\mathbb{R}^n \setminus E|^{(n-1)/n}\} \leq C_{\text{iso}}(\mathbb{R}^n)P(E), \tag{2.4}$$

where

$$C_{\text{iso}}(\mathbb{R}^n) = (n^{1-1/n}\omega_{n-1}^{1/n})^{-1}. \tag{2.5}$$

Moreover the equality in (2.4) *holds if and only if* $E = B(x, R)$ *for some* $x \in \mathbb{R}^n$ *and* $R > 0$*.*

The proof was given by De Giorgi in 1958 and can be found in [9] pagg.185-197. There are also several works that prove the theorem in some restricted classes of sets. Some solutions of different kinds can be found for istance in [1], [7] or [14]. In this survey we will follow that one of [11].

2.2.1 BV functions and perimeter

Here we analyze the generalized concept of perimeter involved in this setting and study some properties that will be usefull also for the sub-Riemannian case.

Let E be any measurable subset of $\mathbb{R}^n$. If E were sufficiently smooth, that is if the boundary ∂E were C^1, then one has the surface measure $d\sigma$ of ∂E, and

$$P(E) = \int_{\partial E} d\sigma.$$

For rougher domains we have to define the notion of perimeter introduced by De Giorgi.

Definition 2.5. Let $\Omega \subset \mathbb{R}^n$ be an open set. A function $u \in L^1(\Omega)$ has *bounded variation* in Ω if

$$\mathrm{Var}(u; \Omega) := \sup\left\{\int_\Omega u \;\mathrm{div} g \; dx \middle| \; g \in C_c^1(\Omega, \mathbb{R}^n), |g| \leq 1\right\} < +\infty. \tag{2.6}$$

We denote the vector space of functions of bounded variation in Ω with $BV(\Omega)$.

A function $u \in L^1_{\mathrm{loc}}(U)$ is said to have *locally bounded variation* in Ω if for each open set V, with compact closure in U,

$$\sup\left\{\int_V u \;\mathrm{div} g \; dx \middle| \; g \in C_c^1(V, \mathbb{R}^n), |g| \leq 1\right\} < +\infty.$$

The space of such functions will be denoted by $BV_{\mathrm{loc}}(\Omega)$.

Definition 2.6. Let $E \subset \mathbb{R}^n$ be a measurable set and $\Omega \subset \mathbb{R}^n$ open. We say that E has *finite perimeter* (or is a *Caccipoli set*) in Ω if

$$P(E, \Omega) := \mathrm{Var}(\chi_E; \Omega) < +\infty, \tag{2.7}$$

where χ_E is the characteristic function of E.

E has *locally finite perimeter* in Ω if $\chi_E \in BV_{\mathrm{loc}}(\Omega)$.

Remark 2.7. Note that if E is a bounded set with C^1 boundary in Ω, we can apply the divergence theorem to each $g \in C_c^1(\Omega, \mathbb{R}^n)$. So if ν is the outer normal to ∂E and σ_{n-1} the surface measure, then

$$\int_E \mathrm{div} g \, dx = \int_{\partial E} \langle g, \nu \rangle d\sigma_{n-1}.$$

Taking the supremum we obtain the usual formula $P(E, \Omega) = \sigma_{n-1}(\Omega \cap \partial E)$.

Theorem 2.8 (Lower semicontinuity of the variation functional)**.** *The variation functional is lower semicontinuous with respect to the $L^1_{\mathrm{loc}}(\Omega)$ convergence.*

Proof. Fix $g \in C_c^1(\Omega; \mathbb{R}^n)$, let $V \subset \Omega$ open with compact closure in Ω and let $\{u_k\}_k$ be a sequence of $BV(\Omega)$ functions which converge to $u \in BV(\Omega)$ in the $L^1_{\mathrm{loc}}(\Omega)$ convergence, then

$$\left| \int_V (u - u_k) \mathrm{div} g \, dx \right| \leq ||\mathrm{div} g||_\infty \int_V |u - u_k| dx \to 0$$

as k tends to ∞. (Notice that $||\mathrm{div} g||_\infty$ is bounded because the derivative of g is continuous and with compact support). This means that the functional $u \mapsto \int_\Omega u \mathrm{div} g \, dx$ is continuous for each g. Now the variation functional is lower semicontinuous because it is the supremum of these continuous functionals. □

Theorem 2.9. *Let $u \in L^1(\Omega)$. Then u is in $BV(\Omega)$ if and only if there exists a Radon measure $Du = (D_1 u, \ldots, D_n u)$ in Ω such that* $\mathrm{Var}(u; \Omega) = |Du|(\Omega) \leq M < +\infty$ *and which represents the derivative of u in the sense of the distributions, i.e.*

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} dx = - \int_\Omega \varphi d D_i u \qquad \textit{for each } \varphi \in C_c^1(\Omega).$$

Here $|\mu|(B)$ denotes the variation of a scalar or vector measure μ on a Borel set B and is equal to

$$|\mu|(B) := \sup \left\{ \sum_{i=0}^{\infty} |\mu(B_i)| : B = \bigcup_{i=0}^{\infty} B_i \right\}$$

where the supremum is taken over all partitions $\{B_i\}_i$ of B in Borel sets.

Proof. Let $u \in L^1(\Omega)$. If the measure Du exists, then for each $g \in C_c^1(\Omega, \mathbb{R}^n)$ with $|g| \leq 1$ we have

$$\int_\Omega u \operatorname{div} g \, dx = -\int_\Omega \sum_{i=1}^n g_i dD_i u$$

then $\operatorname{Var}(u; \Omega) = |Du|(\Omega) \leq M$.

For the other implication, let $u \in BV(\Omega)$. From equation (2.6) we obtain

$$\left| \int_\Omega u \operatorname{div} g \, dx \right| \leq \operatorname{Var}(u; \Omega) ||g||_\infty \quad \text{for each } g \in C_c^1(\Omega, \mathbb{R}^n).$$

Let us extend by density the functional $g \mapsto \left(\int_\Omega u \operatorname{div} g \, dx \right)$ to the whole $C_c(\Omega, \mathbb{R}^n)$, so that we obtain a linear bounded functional $L(g)$ with norm less or equal to $\operatorname{Var}(u; \Omega)$. Then by Riesz theorem there exists a vector measure $\mu = (\mu_1, \dots, \mu_n)$ such that

$$L(g) = -\sum_{i=1}^n \int_\Omega g_i(x) \, d\mu_i(x) \quad \text{for each } g \in C_c(\Omega, \mathbb{R}^n)$$

with $|\mu|(\Omega) \leq \operatorname{Var}(u; \Omega)$. Hence the measure μ represents the derivative of u in the sense of the distributions. $\square$

We give for later purpose an useful approximation result.

Theorem 2.10 (Local approximation by smooth functions). *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in BV(\Omega)$. There exist functions $u_k \in BV(\Omega) \cap C^\infty(\Omega)$, $k \in \mathbb{N}$, such that $u_k \to u$ in $L^1(\Omega)$ as $k \to \infty$ and*

$$\lim_{k \to \infty} \operatorname{Var}(u_k, \Omega) = \operatorname{Var}(u, \Omega).$$

Theorem 2.11 (Product with Lipschitz functions). *Let $\Omega \subset \mathbb{R}^n$ be an open set, $u \in BV(\Omega)$ and ψ a Lipschitz real function on Ω which gradient has compact support where it exists. Then the product $u\psi$ has bounded variation on Ω and the associated measure is given by*

$$D(u\psi) = \psi Du + u \nabla \psi \mathcal{L}^n. \tag{2.8}$$

Proof. By Rademacher's theorem ψ is differentiable a.e., therefore for every $g \in C_c^1(\Omega, \mathbb{R}^n)$ the product law gives $\psi \partial g_i / \partial x_i = \partial \psi g_i / \partial x_i - g_i \partial \psi / \partial x_i$ where ψ is differentiable. Then since $u \in BV(\Omega)$, there exists a scalar measure $Du = (D_1 u, \dots, D_n u)$ such that

$$\int_\Omega u\psi \frac{\partial g_i}{\partial x_i} dx = -\int_\Omega g_i \psi d(D_i u) - \int_\Omega g_i u \frac{\partial \psi}{\partial x_i} dx. \tag{2.9}$$

Hence the measure $D_i(u\psi) := \psi D_i u + u\partial\psi/\partial x_i \mathcal{L}^n$ represent the i-derivative of $u\psi$ in the sense of the distributions. Since the gradient of ψ has compact support, the measure $D_i(u\psi)$ has bounded variation. So $u\psi \in BV(\Omega)$ and

$$D(u\psi) = \psi Du + u\nabla\psi\mathcal{L}^n.$$

□

For later purpose we derive here a theorem of localization for the perimeter measure. We will apply it in Section 6.3 in the proof of the existence of minimizers for the Heisenberg isoperimeter problem.

Theorem 2.12. *Let E be a set of finite perimeter in Ω, $x_0 \in \Omega$ and $\delta =$ $\operatorname{dist}(x_0, \partial\Omega)$. Then*

$$P(E \cap B_\rho(x_0), \mathbb{R}^n) \leq P(E, \overline{B_\rho(x_0)}) + m'_+(\rho)$$

holds for each $\rho \in (0, \delta)$, where $m(\rho) = |E \cap B_\rho(x_0)|$ and m'_+ is the lower right derivative of m.

Proof. By translation, it is not restrictive to assume $x_0 = 0$. Let $u(x) = \chi_E(x)$. We will prove the following more general inequality:

$$\operatorname{Var}(u_\rho, \mathbb{R}^n) \leq \operatorname{Var}(u, \overline{B_\rho(0)}) + \left(\int_{B_\rho(0)} |u(x)| dx \right)'_+ \tag{2.10}$$

for each $\rho \in (0, \delta)$ and for any $u \in BV(\Omega)$, where $u_\rho = u\chi_{B_\rho(0)}$. Given any $\sigma \in (0, \delta - \rho)$ we construct the family of functions $u_\sigma \in BV(\mathbb{R}^n)$ supported in $\overline{B_{\rho+\sigma}}(0)$ given by $u_\sigma(x) = u(x)\gamma_\sigma(|x|)$, where

$$\gamma_\sigma(t) := \begin{cases} 1 & \text{if } t \leq \rho \\ 1 + \frac{\rho - t}{\sigma} & \text{if } \rho \leq t \leq \rho + \sigma \\ 0 & \text{if } t \geq \rho + \sigma. \end{cases}$$

Since the functions γ_σ are Lipschitz functions, by theorem 2.11 the u_σ are still of bounded variation in Ω and the respective variation measure is

$$Du_\sigma = \gamma_\sigma(|x|)Du + u(x)\gamma'_\sigma(|x|)\frac{x}{|x|}\mathcal{L}^n.$$

Since $\gamma'_\sigma(t) = -\sigma^{-1}\, \chi_{B_{\rho+\sigma(0)} \setminus B_\rho(0)}$ we have the estimate

$$|Du_\sigma|(\mathbb{R}^n) \leq |Du|(B_{\rho+\sigma}) + \sigma^{-1} \int_{B_{\rho+\sigma(0)} \setminus B_\rho(0)} |u(x)| dx.$$

Finally, since $u\sigma$ convergies to u_ρ in $L^1(\mathbb{R}^n)$ as σ goes to 0, equation (2.10) follows from the semicontinuity of the variation. □

2.2.2 Sobolev's and Poincaré's Inequalities for C_c^1 functions

The proof of the isoperimetric inequality (2.4) is a consequence of Sobolev's and Poincaré's inequalities for BV functions. In this section we will therefore prove such inequalities and at the end we will show how the isoperimetric inequality follows from them.

Let us consider first a function $f \in C_c^1(\mathbb{R}^n)$. For each $1 \leq p < n$ we define the *Sobolev conjugate* of p as

$$p^* := \frac{np}{n-p}.$$

Notice that the Sobolev conjugate of p is such that $1/p - 1/p^* = 1/n$.

Theorem 2.13 (Gagliardo-Nirenberg-Sobolev Inequality). *Let $1 \leq p < n$. There exists a constant C_1, depending only on p and n, such that*

$$\left(\int_{\mathbb{R}^n} |f|^{p*} dx \right)^{1/p*} \leq C_1 \left(\int_{\mathbb{R}^n} |Df|^p dx \right)^{1/p} \tag{2.11}$$

for all $f \in C_c^1(\mathbb{R}^n)$.

To prove the Gagliardo-Nirenberg-Sobolev inequality, we need the following theorem.

Theorem 2.14 (General Hölder inequality). *Let $U \subset \mathbb{R}^n$ be an open set and let $1 \leq p_1 \leq \ldots \leq p_n \leq \infty$ be such that $1/p_1 + \ldots + 1/p_n = 1$. Given functions $u_1, \ldots, u_n$ such that $u_k \in L^{p_k}(U)$ for each $k = 1, \ldots, n$ the following inequality holds*

$$\int_U |u_1 \cdots u_n| dx \leq \prod_{i=1}^n ||u_k||_{L^{p_k}(U)}. \tag{2.12}$$

Proof. The proof is an application of the mathematical induction and the classical Hölder inequality.

For $n = 1$, there's nothing to prove.

Let $n > 1$. If $p_n = \infty$, then $1/p_1 + \ldots + 1/p_{n-1} = 1$ and

$$\int_U |u_1 \cdots u_n| dx \leq ||u_n||_{L^\infty(U)} \int_U |u_1 \cdots u_{n-1}| dx \leq \prod_{k=1}^n ||u_k||_{L^{p_k}(U)}$$

where the latter inequality follows from the inductive hypothesis.

In $p_n \neq \infty$, let $q = p_n$ and

$$\frac{1}{p} := 1 - \frac{1}{p_n} = \frac{1}{p_1} + \ldots + \frac{1}{p_{n-1}}.$$

Then $1/p + 1/q = 1$ and we may apply the classical Hölder inequality to the functions $u_1 \cdots u_{n-1}$ and u_n:

$$\begin{aligned} \int_U |u_1 \cdots u_n| dx &\leq \left(\int_U |u_1 \cdots u_{n-1}|^p dx \right)^{1/p} \left(\int_U |u_n|^q dx \right)^{1/q} && (2.13) \\ &= \left(\int_U ||u_1|^p \cdots |u_{n-1}|^p|\, dx \right)^{1/p} ||u_n||_{L^{p_n}(U)}. && (2.14) \end{aligned}$$

Let $\tilde{p}_k = p_k/p$ for $k = 1, \ldots, n-1$. Then $1/\tilde{p}_1 + \ldots + 1/\tilde{p}_{n-1} = p(1/p_1 + \ldots + 1/p_{n-1}) = 1$. Moreover $|u_k|^p \in L^{\tilde{p}_k}(U)$, because

$$\int_U |u_k|^{p\tilde{p}_k} dx = \int_U |u_k|^{p_k} dx < \infty.$$

Then we can apply the inductive hypothesis to the functions $|u_1|^p, \ldots, |u_{n-1}|^p$ with the new coefficients $\tilde{p}_1, \ldots, \tilde{p}_{n-1}$ and we obtain

$$\begin{aligned} \left(\int_U ||u_1|^p \cdots |u_{n-1}|^p|\, dx \right)^{\frac{1}{p}} &\leq \left(\prod_{k=1}^{n-1} \left(\int_U |u_k|^{p\tilde{p}_k} dx \right)^{\frac{1}{\tilde{p}_k}} \right)^{\frac{1}{p}} \\ &= \left(\int_U |u_1|^{p_1} dx \right)^{\frac{1}{p_1}} \cdots \left(\int_U |u_{n-1}|^{p_{n-1}} dx \right)^{\frac{1}{p_{n-1}}}. \end{aligned}$$

This inequality, together with (2.14) proves inequality (2.12). □

Proof of Theorem 2.13. For each $1 \leq i \leq n$, we have

$$f(x_1, \ldots, x_i, \ldots, x_n) = \int_{-\infty}^{x_i} \frac{\partial f}{\partial x_i}(x_1, \ldots, t_i, \ldots, x_n) dt_i,$$

hence

$$\begin{aligned} |f(x_1, \ldots, x_i, \ldots, x_n)| &\leq \int_{-\infty}^{+\infty} \left| \frac{\partial f}{\partial x_i}(x_1, \ldots, t_i, \ldots, x_n) \right| dt_i \\ &\leq \int_{-\infty}^{+\infty} |Df(x_1, \ldots, t_i, \ldots, x_n)|\, dt_i. \end{aligned}$$

Multiplying for $i = 1, \ldots, n$ we obtain

$$|f(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{-\infty}^{+\infty} |Df(x_1, \ldots, t_i, \ldots, x_n)|\, dt_i \right)^{\frac{1}{n-1}}. \qquad (2.15)$$

Notice that $\frac{n}{n-1} = 1^*$. Let us integrate (2.15) with respect to x_1:

$$\begin{aligned}\int_{-\infty}^{+\infty} |f|^{1^*} dx_1 &\leq \int_{-\infty}^{+\infty} \prod_{i=1}^{n} \left(\int_{-\infty}^{+\infty} |Df(x_1, \dots, t_i, \dots, x_n)|\, dt_i \right)^{\frac{1}{n-1}} dx_1 \\ &= \left(\int_{-\infty}^{+\infty} |Df|\, dt_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{+\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{+\infty} |Df|\, dt_i \right)^{\frac{1}{n-1}} dx_1 \\ &\leq \left(\int_{-\infty}^{+\infty} |Df|\, dt_1 \right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |Df|\, dx_1 dt_i \right)^{\frac{1}{n-1}}\end{aligned}$$

where the latter inequality follows from (2.12) with

$$u_k(x_1) = \left(\int_{-\infty}^{+\infty} |Df(x_1, \dots, t_k, \dots, x_n)|\, dt_k \right)^{\frac{1}{n-1}}$$

and $p_k = n-1$ for $k = 1, \dots, n-1$.

Next integrate with respect to x_2, so that it results

$$\begin{aligned}&\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f|^{1^*} dx_1 dx_2 \\ &\quad\leq \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |Df|\, dx_1 dt_2 \right)^{\frac{1}{n-1}} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} |Df|\, dt_1 \right)^{\frac{1}{n-1}} dx_2 \\ &\qquad\cdot \int_{-\infty}^{+\infty} \prod_{i=3}^{n} \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |Df|\, dx_1 dt_i \right)^{\frac{1}{n-1}} dx_2 \\ &\quad\leq \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |Df|\, dx_1 dt_2 \right)^{\frac{1}{n-1}} \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |Df|\, dt_1 dx_2 \right)^{\frac{1}{n-1}} \\ &\qquad\cdot \prod_{i=3}^{n} \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |Df|\, dx_1 dx_2 dt_i \right)^{\frac{1}{n-1}}\end{aligned}$$

Continue till x_n and we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} |f|^{1^*} dx_1 &\leq \prod_{i=1}^{n} \left(\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} |Df|\, dx_1 \dots dt_i \dots dx_n \right)^{\frac{1}{n-1}} \\ &= \left(\int_{\mathbb{R}^n} |Df| dx \right)^{\frac{n}{n-1}}.\end{aligned}$$

So we have proven the inequality for $p = 1$:

$$\left(\int_{\mathbb{R}^n} |f|^{1^*} dx \right)^{1/1^*} \leq \int_{\mathbb{R}^n} |Df| dx. \tag{2.16}$$

If $1 < p < n$, let $g = |f|^\gamma$ with $\gamma > 0$ to be chosen later. Applying (2.16) to g, we get

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |f|^{\frac{\gamma n}{n-1}} dx\right)^{\frac{n-1}{n}} &\leq \gamma \int_{\mathbb{R}^n} |f|^{\gamma-1}|Df|dx \\ &\leq \gamma \left(\int_{\mathbb{R}^n} |f|^{(\gamma-1)\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |Df|^p dx\right)^{\frac{1}{p}}\end{aligned} \tag{2.17}$$

where the latter inequality follows from the classical Hölder inequality. Now we choose γ so that $\frac{\gamma n}{n-1} = (\gamma - 1)\frac{p}{p-1}$, i.e.

$$\gamma = \frac{p(n-1)}{n-p} \geq 1.$$

In this way we have $\frac{\gamma n}{n-1} = (\gamma - 1)\frac{p}{p-1} = \frac{np}{n-p} = p^*$. Then (2.17) becomes

$$\left(\int_{\mathbb{R}^n} |f|^{p^*} dx\right)^{\frac{n-1}{n}} \leq \gamma \left(\int_{\mathbb{R}^n} |f|^{p^*} dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |Df|^p dx\right)^{\frac{1}{p}},$$

that is

$$\left(\int_{\mathbb{R}^n} |f|^{p^*} dx\right)^{\frac{n-1}{n} - \frac{p-1}{p}} \leq \gamma \left(\int_{\mathbb{R}^n} |Df|^p dx\right)^{\frac{1}{p}},$$

which is equivalent to (2.11). □

We state now the Poincaré's inequality for C^1_c functions on the balls. To this end, we fix here the notation. We will denote by $B(x,r)$ the open ball in $\mathbb{R}^n$ with center in x and radius $r > 0$. For an integrable function f on $\mathbb{R}^n$ the average of f on a ball $B(x,r)$ is given by

$$(f)_{x,r} = \fint_{B(x,r)} f(y)dy = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy.$$

Theorem 2.15 (Poincaré's inequality)**.** *For each $1 \leq p < n$ there exists a constant C_2, depending only on p and n, such that*

$$\left(\fint_{B(x,r)} |f - (f)_{x,r}|^{p*} dx\right)^{1/p*} \leq C_2 r \left(\fint_{B(x,r)} |Df|^p dx\right)^{1/p}$$

for all $B(x,r) \subset \mathbb{R}^n$ and $f \in C^1_c(\mathbb{R}^n)$.

The proof of this theorem is a consequence of the next lemma.

Lemma 2.16. *For each $1 \leq p < \infty$ there exists a constant C, depending only on n and p, such that*

$$\int_{B(x,r)} |f(y) - f(z)|^p dy \leq C r^{n+p-1} \int_{B(x,r)} |Df(y)|^p |y - z|^{1-n} dy$$

for all $B(x,r) \subset \mathbb{R}^n$, $f \in C^1(B(x,r))$, and $z \in B(x,r)$.

Proof. If $y, z \in B(x,r)$, then

$$f(y) - f(z) = \int_0^1 \frac{d}{dt} f(z + t(y-z))dt = \int_0^1 Df(z + t(y-z)) \cdot (y-z)dt,$$

and so

$$|f(y) - f(z)|^p \leq |y - z|^p \int_0^1 |Df(z + t(y-z))|^p dt.$$

Thus, for $s > 0$,

$$\begin{aligned}
&\int_{B(x,r)\cap\partial B(z,s)} |f(y) - f(z)|^p d\mathcal{H}^{n-1}(y) \\
&\qquad \leq s^p \int_0^1 \int_{B(x,r)\cap\partial B(z,s)} |Df(z + t(y-z))|^p d\mathcal{H}^{n-1}(y)dt \\
&\qquad \leq s^p \int_0^1 \frac{1}{t^{n-1}} \int_{B(x,r)\cap\partial B(z,ts)} |Df(\omega)|^p d\mathcal{H}^{n-1}(\omega)dt
\end{aligned}$$

where we have used the change of variable $\omega = z + t(y-z) \in \partial B(z, ts)$. Since $|\omega - z| = st$ we further get

$$\begin{aligned}
&\leq s^{n+p-1} \int_0^1 \int_{B(x,r)\cap\partial B(z,ts)} |Df(\omega)|^p |\omega - z|^{1-n} d\mathcal{H}^{n-1}(\omega)dt \\
&= s^{n+p-2} \int_{B(x,r)\cap B(z,s)} |Df(\omega)|^p |\omega - z|^{1-n} d\omega.
\end{aligned}$$

Hence the co-area formula implies

$$\int_{B(x,r)} |f(y) - f(z)|^p dy \leq C r^{n+p-1} \int_{B(x,r)} |Df(\omega)|^p |\omega - z|^{1-n} d\omega.$$

□

Proof of Theorem 2.15. Using the previous lemma, we may compute

$$\begin{aligned}
\fint_{B(x,r)} |f-(f)_{x,r}|^p dy &= \fint_{B(x,r)} \left| \fint_{B(x,r)} f(y)-f(z)dz \right|^p dy \\
&\leq \fint_{B(x,r)} \fint_{B(x,r)} |f(y)-f(z)|^p dz dy \\
&\leq C \fint_{B(x,r)} r^{p-1} \int_{B(x,r)} |Df(z)|^p |y-z|^{1-n} dz dy \\
&\leq C r^p \fint_{B(x,r)} |Df|^p dz.
\end{aligned} \tag{2.18}$$

Moreover it holds a general inequality: there exists a constant $C = C(n,p)$ such that for each $g \in C^1_c(B(x,r))$

$$\left(\fint_{B(x,r)} |g|^{p^*} dy \right)^{\frac{1}{p^*}} \leq C \left(r^p \fint_{B(x,r)} |Dg|^p dy + \fint_{B(x,r)} |g|^p dy \right)^{\frac{1}{p}}. \tag{2.19}$$

Indeed, we may suppose $x = 0$. Let R such that $spt(g) \subset B(0,R)$, then, by replacing $g(y)$ by $(R/r)g((r/R)y)$ if necessary, we may assume $g = 0$ out of $B(0,1)$ and $r = 1$. Then the Gagliardo-Nirenberg-Sobolev inequality (2.11) implies

$$\begin{aligned}
\left(\int_{B(0,1)} |g|^{p^*} dy \right)^{\frac{1}{p^*}} &\leq \left(\int_{\mathbb{R}^n} |g|^{p^*} dy \right)^{\frac{1}{p^*}} \\
&\leq C_1 \left(\int_{\mathbb{R}^n} |Dg|^p dy \right)^{\frac{1}{p}} \\
&\leq C \left(\int_{\mathbb{R}^n} |Dg|^p + |g|^p dy \right)^{\frac{1}{p}}.
\end{aligned}$$

Now we may conclude, indeed (2.18) and (2.19) with $g = f - (f)_{x,r}$ give the desired inequality. □

2.2.3 Isoperimetric Inequalities

The relations proved in the latter section have done all the heavy job towards the isoperimetric inequality. From Theorem 2.13 and Theorem 2.15 follows the next theorem, which proves an equivalent of the latter relations for BV functions.

Theorem 2.17 (Gagliardo-Nirenberg-Sobolev inequality for BV functions)**.** *There exists a constant C_1 such that*

$$||u||_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq C_1 \mathrm{Var}(u, \mathbb{R}^n) \tag{2.20}$$

for all $u \in BV(\mathbb{R}^n)$.

Proof. By the approximation theorem for BV functions, we may choose $u_k \in C_c^\infty(\mathbb{R}^n)$, with $k = 1, 2, \ldots$, such that u_k converges to u in $L^1(\mathbb{R}^n)$ and

$$\lim_{k\to\infty} \mathrm{Var}(u_k, \mathbb{R}^n) = \mathrm{Var}(u, \mathbb{R}^n).$$

Passing to a subsequence we may further assume that u_k converges to u a.e.

Observe that the functions u_k are also BV functions and their variation $\mathrm{Var}(u, \mathbb{R}^n) = |Du|(\mathbb{R}^n)$ is equal to

$$\begin{aligned}
\mathrm{Var}(u; \mathbb{R}^n) &= \sup\left\{\int_{\mathbb{R}^n} u \ \mathrm{div} g \ dx | \ g \in C_c^1(\mathbb{R}^n, \mathbb{R}^n), |g| \leq 1\right\} \\
&= \sup\left\{-\int_{\mathbb{R}^n} Du \ \cdot g \ dx | \ g \in C_c^1(\mathbb{R}^n, \mathbb{R}^n), |g| \leq 1\right\} \\
&= ||Du||_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

Then, keeping in mind this observation, we may apply Fatou's lemma and Theorem 2.13 to obtain

$$\begin{aligned}
||u||_{L^{n/n-1}(\mathbb{R}^n)} &\leq \liminf_{k\to\infty} ||u_k||_{L^{n/n-1}(\mathbb{R}^n)} \\
&\leq \lim_{k\to\infty} C_1 \mathrm{Var}(u_k, \mathbb{R}^n) \\
&= C_1 \mathrm{Var}(u, \mathbb{R}^n).
\end{aligned}$$

□

Theorem 2.18 (Poincaré inequality for BV functions)**.** *There exists a constant C_2 such that*

$$||u - (u)_{x,r}||_{L^{\frac{n}{n-1}}(B(x,r))} \leq C_2 \mathrm{Var}(u, B(x, r)) \tag{2.21}$$

for all $B(x, r) \subset \mathbb{R}^n$, $u \in BV_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. The proof follows similarly to the proof of Theorem 2.17, using the Poincaré inequality 2.15. Indeed, take the sequnce of functions $u_k \in C_c^\infty(\mathbb{R}^n)$

as in the proof of Theorem 2.17 and apply Fatou's lemma and Theorem 2.15 to the functions $u_k - (u_k)_{x,r}$ to obtain

$$\begin{aligned}
||u-(u)_{x,r}||_{L^{n/n-1}(B(x,r))} &\le \liminf_{k\to\infty} ||u_k-(u_k)_{x,r}||_{L^{n/n-1}(B(x,r))} \\
&= \liminf_{k\to\infty}\left(\int_{B(x,r)} |u_k-(u_k)_{x,r}|^{n/n-1}dy\right)^{\frac{n-1}{n}} \\
&= \liminf_{k\to\infty}\left(r^n ⨍_{B(x,r)} |u_k-(u_k)_{x,r}|^{n/n-1}dy\right)^{\frac{n-1}{n}} \\
&\le \lim_{k\to\infty} C_2 r^n ⨍_{B(x,r)} |Du_k|dy \\
&= \lim_{k\to\infty} C_2 \mathrm{Var}(u_k, B(x,r)) \\
&= C_2\mathrm{Var}(u, B(x,r)).
\end{aligned}$$

□

Now we are able to give a proof of the first part of Theorem 2.4 and we can also prove something more, namely:

Theorem 2.19. *Let $E \subset \mathbb{R}^n$ be a bounded Borel set of finite perimeter $P(E)$ and area $|E|$. Then*

$$|E|^{\frac{n-1}{n}} \le C_1 P(E). \tag{2.22}$$

This relation is called isoperimetric inequality*.*

Moreover, for each ball $B(x,r) \subset \mathbb{R}^n$

$$\min\{|B(x,r)\cap E|, |B(x,r)\setminus E|\}^{\frac{n-1}{n}} \le 2C_2 P(E, B(x,r)) \tag{2.23}$$

is called the relative isoperimetric inequality*.*

Proof. Equation (2.22) is the statement of Theorem 2.17 with $u = \chi_E$.

For equation (2.23), apply Theorem 2.18 with $u = \chi_{B(x,r)\cap E}$. So $(u)_{x,r} = |B(x,r)\cap E|/|B(x,r)|$ and the integrand $|u-(u)_{x,r}|$ assumes two different values, depending on whether we integrate on $B(x,r)\cap E$ or on $B(x,r)\setminus E$. Thus

$$\begin{aligned}
\int_{B(x,r)} |u-(u)_{x,r}|^{\frac{n}{n-1}}dy = &\left(\frac{|B(x,r)\setminus E|}{|B(x,r)|}\right)^{\frac{n}{n-1}} |B(x,r)\cap E| \\
&+ \left(\frac{|B(x,r)\cap E|}{|B(x,r)|}\right)^{\frac{n}{n-1}} |B(x,r)\setminus E|.
\end{aligned}$$

Now, if for example, $|B(x,r) \cap E| \geq |B(x,r) \setminus E|$, then $\frac{|B(x,r) \setminus E|}{|B(x,r)|} \geq \frac{1}{2}$ and applying Theorem 2.18 we have

$$\begin{aligned} C_2 P(E, B(x,r)) &\geq \left(\int_{B(x,r)} |u - (u)_{x,r}|^{\frac{n}{n-1}} dy \right)^{\frac{n-1}{n}} \\ &\geq \left(\frac{|B(x,r) \setminus E|}{|B(x,r)|} \right) |B(x,r) \cap E|^{\frac{n-1}{n}} \\ &\geq \frac{1}{2} \min\{|B(x,r) \cap E|, |B(x,r) \setminus E|\}^{\frac{n-1}{n}}. \end{aligned}$$

The other case is similar. □

Untill now we have analysed the isoperimetric problem in the Euclidean space. Since the next chapter we will start the study of its extension to the Heisenberg group.

Chapter 3

The Heisenberg Group and Sub-Riemannian Geometry

In this chapter we introduce the first Heisenberg group, the simplest example of space with a sub-Riemannian metric structure. We will start with a detailed description of its algebraic structure and differentiable structure, defining the horizontal subbundle. Next we present the Carnot-Carathéodory metric as the least time required to travel between two given points at unit speed along horizontal paths. Subsequently we introduce the notion of sub-Riemannian metric and show how it may be approximated by Riemannian metrics. Finally we compute in these Riemannian approximants some of the standard differential geometric tools that will be useful in later chapters.

3.1 The first Heisenberg group $\mathbb{H}$

The first Heisenberg group admits several representations. The first that we consider is given by a matrix model and the group with such a representation is called the *polarized Heisenberg group*: it is the following subgroup of the group of three by three upper triangular matrices equipped with the usual matrix product:

$$\mathbb{H} = \left\{ \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(3, \mathbb{R}) : x_1, x_2, x_3 \in \mathbb{R} \right\}. \tag{3.1}$$

This is a three dimensional subgroup of the general linear group $GL(3\mathbb{R})$ and it receives from it the structure of Lie group. Its Lie algebra has dimension 3 and $\mathfrak{h}$ can be considered as the tangent space at the identity, $T_I\mathbb{H}$, or equivalently as the set of all left invariant tangent vectors, as seen in Section 1.1.2.

Let us give a base for the tangent space of $\mathbb{H}$ in the point

$$\begin{pmatrix} 1 & u_1 & u_3 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

We will indicate this point with (u_1, u_2, u_3). To this end we translate it by three independent elements, to obtain curves in $\mathbb{H}$. For example, we can choose the curves $\gamma_1, \gamma_2, \gamma_3 : [-1, 1] \to \mathbb{H}$ defined by

$$\gamma_1(t) = \begin{pmatrix} 1 & u_1 & u_3 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 + t & u_3 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\gamma_2(t) = \begin{pmatrix} 1 & u_1 & u_3 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 & tu_1 + u_3 \\ 0 & 1 & t + u_2 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\gamma_3(t) = \begin{pmatrix} 1 & u_1 & u_3 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 & t + u_3 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

If we derive these curves we obtain the tangent vectors

$$U_1 = \left.\frac{d\gamma_1(t)}{dt}\right|_{t=0} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$U_2 = \left.\frac{d\gamma_2(t)}{dt}\right|_{t=0} = \begin{pmatrix} 0 & 0 & u_1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$U_3 = \left.\frac{d\gamma_3(t)}{dt}\right|_{t=0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By construction, U_1, U_2 e U_3 are left invariant vector fields and they span the tangent space to $\mathbb{H}$ at (u_1, u_2, u_3). Computing the Lie brackets of these vector fields, we observe that

$$[U_1, U_2] = U_1 U_2 - U_2 U_1 = U_3$$

while all other brackets are zero.
Using the basis U_1, U_2, U_3 we may introduce a system of coordinates generally

known as *polarized coordinates* or *canonical coordinates of the second kind.* We identify the point (u_1, u_2, u_3) with the matrix product

$$\exp(u_3 U_3|_I)\exp(u_2 U_2|_I)\exp(u_1 U_1|_I),$$

where we have denoted by $\exp(U) = I + U + \frac{1}{2}U^2 + \ldots$ and by $U|_I$ the vector field U evaluated at the identity. The group law in polarized coordinates writes explicitly

$$(u_1, u_2, u_3)(v_1, v_2, v_3) = (u_1 + v_1, u_2 + v_2, u_3 + v_3 + u_1 v_2).$$

The Heisenberg group is a nilpotent step two Lie group that means that the central series

$$\{e\} = G_0 \leq G_1 \leq G_2 \leq \ldots \leq G_n = G$$

is finite and stops for $n = 2$. Here G_i are such that $[G, G_{i+1}] \leq G_i$ and $[G, H]$ represents the commutative subgroup of G generated by $g^{-1}h^{-1}gh$ for $g \in G$ and $h \in H$. In $\mathbb{H}$ we can compute explicitly the series: $G_0 = \{(0, 0, 0)\}$;

$$\begin{aligned}
G_1 &= \{c = (c_1, c_2, c_3) \in \mathbb{H} \text{ s.t. } x^{-1}c^{-1}xc = (0, 0, 0)\ \forall x \in \mathbb{H}\} \\
&= \{(c_1, c_2, c_3) \in \mathbb{H} \text{ s.t. } (u_1 + c_1, u_2 + c_2, u_3 + c_3 + u_1 c_2) = \\
&\qquad = (c_1 + u_1, c_2 + u_2, c_3 + u_3 + c_1 u_2)\ \forall (u_1, u_2, u_3) \in \mathbb{H}\} \\
&= \{(c_1, c_2, c_3) \in \mathbb{H} \text{ s.t. } u_1 c_2 = c_1 u_2\ \forall u_1, u_2 \in \mathbb{R}\} \\
&= \{(0, 0, c) \in \mathbb{H} : c \in \mathbb{R}\}; \\
G_2 &= \{d = (d_1, d_2, d_3) \in \mathbb{H} \text{ s.t. } x^{-1}d^{-1}xd = (0, 0, c)\ \forall x \in \mathbb{H} \text{ for a } c \in \mathbb{R}\} \\
&= \{(d_1, d_2, d_3) \in \mathbb{H} \text{ s.t. } (u_1 + d_1, u_2 + d_2, u_3 + d_3 + u_1 d_2) = \\
&\qquad = (d_1 + u_1, d_2 + u_2, d_3 + u_3 + d_1 u_2 + c)\ \forall\, x \in \mathbb{H} \text{ for a } c \in \mathbb{R}\} \\
&= \mathbb{H}.
\end{aligned}$$

In conclusion, the Heisenberg group $\mathbb{H}$ is the unique analytic nilpotent step two Lie group whose background manifold is $\mathbb{R}^3$ and whose Lie algebra $\mathfrak{h}$ has the following properties:

- $\mathfrak{h} = V_1 \oplus V_2$, where V_1 has dimension 2 and V_2 has dimension 1
- $[V_1, V_1] = V_2$, $[V_1, V_2] = 0$ and $[V_2, V_2] = 0$.

We are now ready to introduce a more intrinsic representation of $\mathbb{H}$, using the structure of its Lie algebra, $\mathfrak{h}$. The key point of this presentation is that, via the exponential map and the Baker-Campell-Hausdorff formula, we can derive the Lie group from its Lie algebra.

First, note that since $\mathfrak{h}$ is nilpotent the exponential map $\exp : \mathfrak{h} \to \mathbb{H}$ is a diffeomorphism. Fix an arbitrary basis X_1, X_2 of V_1 and let $X_3 = [X_1, X_2] \in V_2$. The Baker-Campell-Hausdorff formula states that

$$\begin{aligned}
&\exp^{-1}(\exp(X)\exp(Y)) = \\
&\quad = \sum_{n>0} \frac{(-1)^{n-1}}{n} \sum_{\substack{r_i+s_i>0\\ 1\le i\le n}} \frac{\left(\sum_{i=1}^n r_i + s_i\right)^{-1}}{r_1!s_1!\cdots r_n!s_n!} \left[X^{r_1}Y^{s_1}X^{r_2}Y^{s_2}\cdots X^{r_n}Y^{s_n}\right] \\
&\quad = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \dots \qquad (3.2)
\end{aligned}$$

where

$$\begin{aligned}
&\left[X^{r_1}Y^{s_1}X^{r_2}Y^{s_2}\cdots X^{r_n}Y^{s_n}\right] = \\
&\quad = \underbrace{[X,[X,\dots[X,}_{r_1}\underbrace{[Y,[Y,\dots[Y,}_{s_1}\dots\underbrace{[X,\dots[X,}_{r_n}\underbrace{[Y,[Y,\dots,Y]]}_{s_n}\dots] .
\end{aligned}$$

It is a powerful formula that allows us to obtain all the information about the group product by knowing the Lie algebra.

In this case, since the Lie algebra $\mathfrak{h}$ is step two, the higher terms in (3.2) are zero, so equation (3.2) takes the simpler form:

$$\exp^{-1}(\exp(x)\exp(y)) = x + y + \frac{1}{2}[x,y]. \qquad (3.3)$$

where we have denoted by $x = x_1X_1 + x_2X_2 + x_3X_3 = (x_1, x_2, x_3)$ a generic vector in $\mathfrak{h}$. Using the commutation relation $X_3 = [X_1, X_2]$ and $[X_i, X_3] = 0$ we obtain

$$\begin{aligned}
x + y + \frac{1}{2}[x,y] &= (x_1+y_1)X_1 + (x_2+y_2)X_2 + (x_3+y_3)X_3 \\
&\quad + \frac{1}{2}\left\{x_1y_1\,[X_1,X_1] + (x_1y_2 - x_2y_1)\,[X_1,X_2] + x_2y_2\,[X_2,X_2]\right\} \\
&= \left(x_1+y_1, x_2+y_2, x_3+y_3+\frac{1}{2}(x_1y_2 - x_2y_1)\right).
\end{aligned}$$

Then formula (3.3) reads more explicitly as

$$\exp^{-1}(\exp(x)\exp(y)) = \left(x_1+y_1, x_2+y_2, x_3+y_3+\frac{1}{2}(x_1y_2 - x_2y_1)\right). \qquad (3.4)$$

We will identify $\mathbb{H}$ with $\mathbb{C}\times\mathbb{R}$ by associating $\exp(x_1X_1 + x_2X_2 + x_3X_3) \in \mathbb{H}$ with the point $(z, x_3) \in \mathbb{C}\times\mathbb{R}$, where $z = x_1 + \mathbf{i}x_2 \in \mathbb{C}$ and $x_3 \in \mathbb{R}$. The coordinates $x = (x_1, x_2, x_3) = (z, x_3) \in \mathbb{H}$ are called *canonical coordinates of the first kind* or simply *exponential coordinates*. Using these coordinates, by (3.4), the group law reads

$$(z, x_3)(w, y_3) = (z + w, x_3 + y_3 - \frac{1}{2}\mathrm{Im}(z\bar{w})). \tag{3.5}$$

From this equation we get immediately that the group identity is $0 = (0, 0, 0)$, while $x^{-1} = (-x_1, -x_2, -x_3)$.

An isomorphism between this latter model of $\mathbb{H}$ and the polarized Heisenberg group defined in (3.1) is obtained by mapping the element $\exp(x_1X_1 + x_2X_2 + x_3X_3)$ to the matrix

$$\begin{pmatrix} 1 & x_1 & x_3 + \frac{1}{2}x_1x_2 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

For use in next paragraphs, it is useful to write explicitly the vector fields X_1, X_2, X_3 with respect to the standard derivations $\partial_{x_1}, \partial_{x_2}, \partial_{x_3}$.

To this end let us choose for the tangent space at the identity the canonical base $X_1(0) = \partial_{x_1}$, $X_2(0) = \partial_{x_2}$ and $X_3(0) = \partial_{x_3}$. The left invariant vector fields X_1, X_2, X_3 generated by this base are

$$X_i(x) = dL_x(0)(X_i(0)) = dL_x(0)(\partial_{x_i}).$$

Now, since the operation of left translation, $L_x(y) = xy$, has differential with matrix

$$dL_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2}x_2 & \frac{1}{2}x_1 & 1 \end{pmatrix}, \tag{3.6}$$

the explicit form of the vector fields X_1, X_2, X_3 is

$$X_1 = \partial_{x_1} - \frac{1}{2}x_2\partial_{x_3}, \quad X_2 = \partial_{x_2} + \frac{1}{2}x_1\partial_{x_3} \quad \text{e} \quad X_3 = \partial_{x_3}. \tag{3.7}$$

Notice that the desired commutation relations hold.

We want to give to $\mathbb{H}$ a structure of homogeneous group. Let us consider the following family of non-isotropic *dilations*: namely, for each $s \in \mathbb{R}^+$ let us define

$$\delta_s(x) = (sx_1, sx_2, s^2x_3).$$

These maps are bijections on $\mathbb{H}$ that preserve the group structure, indeed it is easy to see that

$$\delta_s(xy) = \delta_s(x)\delta_s(y).$$

Moreover

$$\delta_s \circ \delta_{s'} = \delta_{ss'} \quad \forall s, s' \in \mathbb{R}^+$$

and δ_s is the identity transformation exactly when $s = 1$.

Let us observe that ordinary Euclidean volumes are preserved by L_x, that means that the volume of a measurable set E is the same as the volume of its image $L_x(E)$ under L_x. This statement follows because the Jacobian of the mapping L_x, which is the determinant of the differential of L_x, is equal to 1 everywhere on $\mathbb{H}$, as we can see from (3.6). In short, ordinary volume measure is invariant under left translations in the Heisenberg group, and in the same way it can be seen that it is invariant also under right translation. In conclusion, the Lebesgue measure on $\mathbb{R}^3$ is the Haar measure of $\mathbb{H}$.

If we consider the dilations, we observe that the volume of $\delta_s(E)$ is equal to s^4 times the volume of E for all $s > 0$ and all measurable sets $E \subset \mathbb{H}$. The number $4 = \dim V_1 + 2 \cdot \dim V_2$ is called the *homogeneous dimension* of $\mathbb{H}$ and is one dimension greater than the topological dimension.

The homogeneous dimension takes an important role in the isoperimetric inequality. The reason why this dimension has not been introduced in the classical isoperimetric problem is that in the Euclidean case, $\mathbb{R}^3$, there is no difference between the topological and the homogeneous dimensions, since they are both equal to 3. In the Euclidean case, it is therefore insignificant which dimension we take. We will see that this is not the case for the Heisenberg group.

3.1.1 The horizontal distribution and the horizontal linear maps on $\mathbb{H}$

The horizontal fibration $H(x)$ of $\mathbb{H}$ is a distribution of planes on $\mathbb{H}$. It is defined as the subbundle of the tangent bundle spanned at every point by the left invariant frame X_1, X_2. With an easy computation we can see that $H(x) = \text{Ker}[dx_3 - \frac{1}{2}(x_1 dx_2 - x_2 dx_1)]$, indeed if we define

$$\omega = dx_3 - \frac{1}{2}(x_1 dx_2 - x_2 dx_1) \tag{3.8}$$

then $\omega(aX_1 + bX_2 + cX_3) = c$. Note that ω is a contact form in $\mathbb{R}^3$, i.e., $\omega \wedge d\omega = -dx_1 \wedge dx_2 \wedge dx_3 \neq 0$.

We can define on the horizontal fibration a new gradient, which considers only the derivations with respect to the vector fields X_1 and X_2 that lie on the horizontal fibration. Namely, for any C^1 function ϕ defined in an open set of $\mathbb{H}$, its *horizontal gradient* is

$$\nabla_0 \phi = X_1 \phi X_1 + X_2 \phi X_2.$$

To conclude this section, we define the notion of a linear map on $\mathbb{H}$.

Definition 3.1. Consider the Heisenberg group bbH with dilation δ_s. A map $L : \mathbb{H} \to \mathbb{H}$ is a *horizontal linear map* if L is a group homomorphism which respects the dilations, i.e.

$$L(\delta_s x) = \delta_s L(x).$$

More explicitly

$$L(x \cdot y) = L(x) \cdot L(y) \quad \forall x, y \in \mathbb{H} \tag{3.9}$$

and

$$\begin{aligned} L_i(sx_1, sx_2, s^2x_3) &= sL_i(x_1, x_2, x_3) \qquad \text{for } i = 1, 2 \\ L_3(sx_1, sx_2, s^2x_3) &= s^2L_3(x_1, x_2, x_3) \end{aligned} \tag{3.10}$$

where $L = (L_1, L_2, L_3)$.

The next Proposition gives a characterization of the linear maps.

Proposition 3.2. *Each horizontal linear map $L : \mathbb{H} \to \mathbb{H}$ takes the form $L(x) = Ax$, where the matrix A is equal to*

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix}$$

for some $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$.

Proof. The proof follows by some remarks about the definition and by some explicite computations. Equations (3.9) and (3.10) are equivalent to require

$$L_i(x \cdot y) = L_i(x) + L_i(y) \quad \text{and} \quad L_i(\delta_s(x)) = sL_i(x) \tag{3.11}$$

for $i = 1, 2$, and

$$\begin{aligned} L_3(x \cdot y) &= L_3(x) + L_3(y) + \tfrac{1}{2}(L_1(x)L_2(y) - L_2(x)L_1(y)) \\ L_3(\delta_s(x)) &= s^2L_3(x) \end{aligned} \tag{3.12}$$

From (3.11) follows that L_i is linear for $i = 1, 2$, hence we can write $L_i(x_1, x_2, x_3) = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3$. Rewrite equations (3.11) with $x = y = (0, 0, c)$, so that

$$2L_i(0, 0, c) = L_i((0, 0, c) \cdot (0, 0, c)) = L_i(\delta_{\sqrt{2}}(0, 0, c) = \sqrt{2}L_i(0, 0, c).$$

Then we have found the desired conclusion about the first two components of L, that is:

$$L_i(0,0,c) = 0 \quad \text{for } i = 1,2 \quad \text{and} \quad a_{13} = a_{23} = 0. \tag{3.13}$$

To reach the conclusion also for the third component, let $x = y = (x_1, x_2, 0)$ in (3.12), then

$$2L_3(x_1,x_2,0) = L_3((x_1,x_2,0)\cdot(x_1,x_2,0)) = L_3(\delta_2(x_1,x_2,0)) = 4L_3(x_1,x_2,0).$$

Then

$$L_3(x_1,x_2,0) = 0 \quad \forall x_1, x_2 \in \mathbb{R}. \tag{3.14}$$

Observe that for the dilation properties

$$L_3(0,0,c) = \begin{cases} cL_3(0,0,1) & \text{if } c \geq 0 \\ -cL_3(0,0,-1) & \text{if } c < 0, \end{cases}$$

and since $L(0,0,0) = (0,0,0)$, it holds $L_3(0,0,1) = -L_3(0,0,-1) := a_{33}$. Then for every $c \in \mathbb{R}$

$$L_3(0,0,c) = cL_3(0,0,1) = ca_{33}. \tag{3.15}$$

Now, let $x = (x_1,x_2,x_3) = (x_1,x_2,0)\cdot(0,0,x_3)$, then the first equation of (3.12), together with (3.13), (3.14) and (3.15), yields

$$L_3(x) = a_{33}x_3 \quad \text{and} \quad a_{31} = a_{32} = 0. \tag{3.16}$$

Summing up equations (3.13) and (3.16), it follows that

$$L(x_1,x_2,x_3) = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

The last step is to prove that $a_{33} = a_{11}a_{22} - a_{12}a_{21}$, but this follows easily using the latter result to calculate properties (3.12) with $x = (x_1,x_2,x_3)$ and $y = (y_1,y_2,y_3)$. □

3.2 Carnot-Carathéodory distance

3.2.1 Horizontal paths

We want to endow $\mathbb{H}$ with a notion of distance. We are particularly interested in two metrics, the Carnot-Carathéodory distance and the Korányi metric.

The *Carnot-Carathéodory distance* between two points is defined as the shortest time required to go from one point to the other, travelling at unit speed along an admissible path, an horizontal path.

Let x and y be points in $\mathbb{H}$. For $\delta > 0$ we define the class $C(\delta)$ of absolutely continuous paths $\gamma : [0,1] \to \mathbb{R}^3$ with endpoints $\gamma(0) = x$ and $\gamma(1) = y$, so that

$$\gamma'(t) = a(t)X_1|_{\gamma(t)} + b(t)X_2|_{\gamma(t)} \tag{3.17}$$

and

$$a(t)^2 + b(t)^2 \le \delta^2 \tag{3.18}$$

for a.e. $t \in [0,1]$. Paths satisfying (3.17) are called *horizontal* or *Legendrian* paths. In other words, Legendrian paths are paths for which the component of the speed vector γ' in direction X_3 vanishes. More explicitly from expression (3.7) we obtain

$$\partial_{x_1} = X_1 + \frac{1}{2}x_2X_3 \quad \partial_{x_2} = X_2 - \frac{1}{2}x_1X_3 \quad \partial_{x_3} = X_3,$$

then, given a curve $\gamma = (\gamma_1, \gamma_2, \gamma_3)$,

$$\begin{aligned}\gamma'(t) &= \gamma_1'(t)\partial_{x_1} + \gamma_2'(t)\partial_{x_2} + \gamma_3'(t)\partial_{x_3} \\ &= \gamma_1'X_1 + \gamma_2'X_2 + (\gamma_3' - \frac{1}{2}(\gamma_1\gamma_2' - \gamma_2\gamma_1'))X_3. \end{aligned} \tag{3.19}$$

So γ is a Legendrian path if and only if

$$\omega(\gamma') = \gamma_3' - \frac{1}{2}(\gamma_1\gamma_2' - \gamma_2\gamma_1') = 0 \tag{3.20}$$

a.e., where ω is the contact form on $\mathbb{R}^3$ given in (3.8).

Not all the paths are Legendrian, but, as we will see in a moment, it is possible to join every pair of points in $\mathbb{H}$ with such paths.

Let $\pi : \mathbb{H} \to \mathbb{C}$ denote the projection $\pi(x) = x_1 + \mathbf{i}x_2$. Given any absolutely continuous planar curve $\alpha : [0,1] \to \mathbb{C}$ and a point $x = (\alpha(0), h) \in \mathbb{H}$ it is possible to *lift* α to a Legendrian path $\gamma : [0,1] \to \mathbb{H}$ starting at x and satisfying $\pi(\gamma) = \alpha$. To accomplish this we let $\gamma_1(t) = \alpha_1(t)$, $\gamma_2(t) = \alpha_2(t)$ and

$$\gamma_3(t) = h + \frac{1}{2}\int_0^t (\gamma_1\gamma_2' - \gamma_2\gamma_1')(s)ds.$$

Given $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, let us lift the curves $\alpha, \beta : [0,1] \to \mathbb{C}$ that join respectively $x_1 + \mathbf{i}x_2$ and $y_1 + \mathbf{i}y_2$ to the origin. The problem of joining x and y with a Legendrian curve is then equivalent to show how to connect $(0,0,h)$ to $(0,0,k)$ with a horizontal path, for every choice of $h, k \in \mathbb{R}$, or also to connect $(0,0,0)$ with $(0,0,c)$ for each $c \in \mathbb{R}$.

Let us show a particular case, that is to go from the origin to the point $(0,0,1)$. First, we travel in the X_1 direction; as we begin at the origin, this is simply travel along the x_1 axis. From the point $(1,0,0)$, we travel in the X_2 direction to the point $(1,1,\frac{1}{2})$. We then travel from this point in the $-X_1$ direction to the point $(0,1,1)$. Finally, we travel in the $-X_2$ direction, arriving at the terminus $(0,0,1)$.

To travel from the origin to a generic point $(0,0,c)$, we just have to take the latter path multiplied by $\sqrt{c}$ if $c > 0$, by $-\sqrt{-c}$ if $c < 0$.

From this procedure it is clear that for any choice of x and y, the set $C(\delta)$ is nonempty for sufficiently large δ. So we can define the Carnot-Carathéodory (CC) metric

$$d(x,y) = \inf\{\delta \text{ such that } C(\delta) \neq \emptyset\}.$$

A dual formulation is

$$d(x,y) = \inf\left\{T : \begin{array}{c} \exists\gamma : [0,T] \to \mathbb{R}^3, \gamma(0) = x, \gamma(T) = y, \\ \text{and } \gamma' = a\, X_1|_\gamma + b\, X_2|_\gamma \text{ with } a^2 + b^2 \leq 1 \text{ a.e.} \end{array}\right\},$$

in other words, $d(x,y)$ is the shortest time required to travel from x to y, travelling at unit speed along horizontal paths.

Since the vector fields X_1 and X_2 are left invariant, left translates of horizontal curves are still horizontal. In particular, given a horizontal curve γ,

$$(L_x(\gamma))'(t) = \sum_{i=1}^{2} \gamma_i'(t)\, X_i|_{L_x\gamma(t)},$$

therefore also the distance is preserved by left translation and $d(x,y) = d(y^{-1}x, 0)$.

Note that if γ is a horizontal curve, then so is its dilation $\delta_s\gamma$. In fact, the expression of $(\delta_s\gamma)'$ with respect to the basis X_1, X_2, X_3 is

$$(\delta_s\gamma)' = \left(s\gamma_1', s\gamma_2', \frac{1}{2}s^2(\gamma_1\gamma_2' - \gamma_2\gamma_1')\right) = \sum_{i=1}^{2} s\gamma_i'\, X_i|_{\delta_s\gamma}.$$

Moreover, if $\gamma \in C(\delta)$ then $\delta_s\gamma \in C(s\delta)$ (observe that the endpoints must be dilated as well). Consequently

$$d(\delta_s(x), \delta_s(y)) = sd(x,y).$$

The map $x \mapsto d(x,0)$ is continuous, in fact from the definition of the CC-distance and the composition of paths, it follows that

$$d(x,0) \leq d(x,y) + d(y,0) \quad \text{and} \quad d(y,0) \leq d(x,y) + d(x,0),$$

then $|d(x, 0) - d(y, 0)| \le d(x, y)$, that implies the continuity of the CC-distance.

The properties of the CC metric transfer directly to the Hausdorff measure $\mathcal{H}^\alpha$ of $(\mathbb{H}, d)$, with $\alpha > 0$. Recall the definition of Hausdorff measure given in Definition 1.30. The left invariance and scaling properties of the CC metric imply that

$$\mathcal{H}^\alpha(L_y E) = \mathcal{H}^\alpha(E) \quad \text{and} \quad \mathcal{H}^\alpha(\delta_s E) = s^\alpha \mathcal{H}^\alpha(E),$$

for all $s, \alpha > 0$, $y \in \mathbb{H}$, and $E \subset \mathbb{H}$. In particular, for each α there exists $c(\alpha) = \mathcal{H}^\alpha(B(x, 1)) \in [0, \infty]$ so that

$$\mathcal{H}^\alpha(B(x, r)) = \mathcal{H}^\alpha(\delta_r B(0, 1)) = c(\alpha) r^\alpha \tag{3.21}$$

for all $x \in \mathbb{H}$ and $r > 0$, where $B(x, r)$ denotes the metric ball with center x and radius r in $(\mathbb{H}, d)$.

On the other hand, the Hausdorff measure of a ball is given by

$$\mathcal{H}^\alpha(B(x, r)) = \int_{B(x,r)} d\mathcal{H}^\alpha = r^4 \int_{B(0,1)} d\mathcal{H}^\alpha = c(\alpha) r^4, \tag{3.22}$$

where r^4 is the Jacobian of the dilations.

By comparing (3.21) and (3.22), we must have $c(\alpha) = -\infty$ when $0 < \alpha < 4$, while for $\alpha > 4$ we have $c(\alpha) = 0$. In case $\alpha = 4$,

$$c(4) = \mathcal{H}^4(B(0, 1)) \in (0, +\infty).$$

Thus the Hausdorff dimension of $(\mathbb{H}, d)$ is 4, equal to the homogeneous dimension.

3.2.2 The Korányi gauge and metric

An equivalent distance on $\mathbb{H}$ is defined by the so-called *Korányi metric*: for each $x, y \in \mathbb{H}$ define

$$d_{\mathbb{H}}(x, y) = \left\| y^{-1} x \right\|_{\mathbb{H}},$$

where

$$\|x\|_{\mathbb{H}}^4 = (x_1^2 + x_2^2)^2 + 16 x_3^2 = |x_1 + \mathbf{i} x_2|^4 + 16 x_3^2$$

is the *Korányi gauge*. To verify that $d_{\mathbb{H}}$ is a metric, one needs to prove the triangle inequality:

$$d_{\mathbb{H}}(x, y) \le d_{\mathbb{H}}(x, z) + d_{\mathbb{H}}(z, y),$$

or equivalently

$$\left\|y^{-1}x\right\|_{\mathbb{H}} \leq \left\|z^{-1}x\right\|_{\mathbb{H}} + \left\|y^{-1}z\right\|_{\mathbb{H}}.$$

This can be done by direct computation as we now recall. By replacing $z^{-1}x$ with y and $y^{-1}z$ with x, it sufficies to prove that

$$\|xy\|_{\mathbb{H}} \leq \|x\|_{\mathbb{H}} + \|y\|_{\mathbb{H}}.$$

Writing $x = (z, x_3)$ and $y = (w, y_3)$ and using the group law (3.5), we find

$$\begin{aligned}
\|xy\|_{\mathbb{H}}^4 &= |z+w|^4 + 16(x_3 + y_3 - \frac{1}{2}\mathrm{Im}(z\overline{w}))^2 \\
&= \left| |z+w|^2 + 4\mathbf{i}(x_3 + y_3 - \frac{1}{2}\mathrm{Im}(z\overline{w})) \right|^2 \\
&= \left| |z|^2 + 4\mathbf{i}x_3 + 2\bar{z}w + |w|^2 + 4\mathbf{i}y_3 \right|^2 \\
&\leq \left(\left| |z|^2 + 4\mathbf{i}x_3 \right| + 2|\bar{z}||w| + \left| |w|^2 + 4\mathbf{i}y_3 \right| \right)^2 \\
&= \left(\|x\|_{\mathbb{H}}^2 + 2|\bar{z}||w| + \|y\|_{\mathbb{H}}^2 \right)^2 \\
&\leq (\|x\|_{\mathbb{H}} + \|y\|_{\mathbb{H}})^4 .
\end{aligned}$$

From the definition of $d_{\mathbb{H}}$ follows easily that it is homogeneous of order 1 with respect to the dilations (δ_s), i.e.

$$\|\delta_s x\|_{\mathbb{H}} = s\,\|x\|_{\mathbb{H}}.$$

Notice that both distances $d_{\mathbb{H}}$ and the CC-metric d are anisotropic. In particular, both behave like the Euclidean distance in horizontal directions (X_1 and X_2), and behave like the square root of the Euclidean distance in the missing direction (X_3).

Indeed the two described distances are equivalent, i.e., there exist constants $C_1, C_2 > 0$ so that

$$C_1\,\|x\|_{\mathbb{H}} \leq d(x,0) \leq C_2\,\|x\|_{\mathbb{H}} \tag{3.23}$$

for any $x \in \mathbb{H}$. To show this statement, recall that the Korányi unit sphere $\{x \in \mathbb{H} : \|x\|_{\mathbb{H}} = 1\}$ is compact for the Riesz theorem. From the continuity of the function

$$\begin{aligned}
(\mathbb{H}, d) &\to \mathbb{R} \\
x &\mapsto d(x,0)
\end{aligned}$$

there exist

$$C_1 := \min_{x \in S_{\mathbb{H}}} d(x,0) \quad \text{and} \quad C_2 := \max_{x \in S_{\mathbb{H}}} d(x,0).$$

So (3.23) holds for each $x \in S_{\mathbb{H}}$ and obviously for $x = 0$. Let $y \in \mathbb{H} \setminus \{0\}$, there exist $x \in S_{\mathbb{H}}$ and $s \in \mathbb{R}^+$ such that $y = \delta_s(x)$, then

$$C_1 \|y\|_{\mathbb{H}} = sC_1 \|x\|_{\mathbb{H}} \leq sd(x,0) = d(y,0) \leq sC_2 \|x\|_{\mathbb{H}} \leq C_2 \|y\|_{\mathbb{H}},$$

as desired.

3.2.3 Sub-Riemannian structure

Our next purpose is to endow $\mathbb{H}$ with a sub-Riemannian structure. This can be done by choosing an inner product on the horizontal subbundle of the Lie algebra $\mathfrak{h}$ and extending it to the whole $\mathfrak{h}$. Using the sub-Riemannian structure one may define the length of horizontal curves with respect to the sub-Riemannian product and equip $\mathbb{H}$ with the structure of a metric *length space*. A metric space (X,d) is called *length* if the distance between any two points x and y is realized by the infimum of the lengths of rectifiable paths joining x to y. We will see that this metric turns out to agree with the Carnot-Carathéodory metric.

Let us give now a sub-Riemannian metric on $\mathbb{H}$: since we have already made an arbitrary choice of coordinates to present $\mathbb{H}$, we may define the inner product at each point x so that X_1 and X_2 form an orthonormal basis of the horizontal space $H(x)$. We extend this inner product to an inner product defined on the full tangent space, i.e., a Riemannian metric, by requiring that the two layers in the stratification of the Lie algebra are orthogonal and that X_1, X_2 and X_3 form an orthonormal system. We denote this extended inner product by g_1 or $\langle \cdot, \cdot \rangle_1$ as dictation by specific situations.

Accordingly, we define the *horizontal length* of a curve γ to be

$$\mathrm{Length}_{\mathbb{H},CC}(\gamma) = \int_0^1 \sqrt{\left\langle \gamma'(t),\, X_1|_{\gamma(t)} \right\rangle_1^2 + \left\langle \gamma'(t),\, X_2|_{\gamma(t)} \right\rangle_1^2}\, dt, \tag{3.24}$$

and we claim that for the CC distance holds

$$d(x,y) = \inf_{\gamma} \mathrm{Length}_{\mathbb{H},CC}(\gamma), \tag{3.25}$$

where the infimum is taken over all horizontal curves joining x and y. By proving (3.25), we will show that $\mathbb{H}$ with the CC distance has the structure of a metric length space.

Note that if $\pi : \mathbb{H} \to \mathbb{C}$ denotes the projection $\pi(x) = x_1 + \mathbf{i}x_2$, $d\pi : \mathfrak{h} \to \mathbb{C}$ denotes its differential, and $\mathrm{Length}_{\mathbb{C},\mathrm{Eucl}}(\cdot)$ denotes Euclidean length in the plane, then

$$\mathrm{Length}_{\mathbb{H},CC}(\gamma) = \mathrm{Length}_{\mathbb{C},\mathrm{Eucl}}(\pi(\gamma)), \tag{3.26}$$

indeed since $\gamma' = \gamma_1' X_1 + \gamma_2' X_2 + (\gamma_3' - \frac{1}{2}(\gamma_1\gamma_2' - \gamma_2\gamma_1')X_3$, both the lengths that we are considering are equal to

$$\int_0^1 \sqrt{|\gamma_1'(t)|^2 + |\gamma_2'(t)|^2} dt.$$

To prove (3.25) we fix $x, y \in \mathbb{H}$ and let $\bar{d} = \inf_\gamma \mathrm{Length}_{\mathbb{H},CC}(\gamma)$. For any $\delta > d(x,y)$ we consider a curve $\gamma \in C(\delta)$ and note that

$$\bar{d} \leq \mathrm{Length}_{\mathbb{C},\mathrm{Eucl}}(\pi(\gamma)) \leq \int_0^1 \sqrt{a^2+b^2} dt \leq \delta,$$

by (3.18). Thus $\bar{d} \leq d(x,y)$.

For the opposite inequality let $\epsilon > 0$ and let $\gamma : [0,1] \to \mathbb{H}$ be a curve connecting x and y so that $\mathrm{Length}_{\mathbb{C},\mathrm{Eucl}}(\pi(\gamma)) = \bar{d} + \epsilon$. If we reparametrize γ to have constant velocity $|d\pi(\gamma')| = \bar{d} + \epsilon$, then $\gamma \in C(\bar{d} + \epsilon)$. Hence $\bar{d} + \epsilon \geq d(x,y)$. Since $\epsilon > 0$ was arbitrary, $\bar{d} = d(x,y)$.

The next lemma shows that the Korányi and CC metrics generate the same infinitesimal structure. This fact follows from the definition of the Korányi metric. Indeed this behaves similarly to the CC metric in the X_1 and X_2 directions, while in the missing direction it behaves as the square root of the distance. An usefull exercise to understand the behaviour of the length of a curve with respect to the square root of the norm is the following:

Example 1. *Consider the real plane $\mathbb{R}^2$ with metric $\sqrt{\|\cdot\|}$, i.e. the square root of the Euclidean metric in $\mathbb{R}^2$, and let $\gamma : [a,b] \to \mathbb{R}^2$ be a regular non constant curve. Then*

$$\mathrm{Length}_{\sqrt{\|\cdot\|}}(\gamma) = \infty.$$

Proof. The length of a curve in the Euclidean plane is given by:

$$\mathrm{Length}_{\sqrt{\|\cdot\|}}(\gamma) = \sup \left\{ \sum_{i=1}^n \sqrt{\|\gamma(x_i) - \gamma(x_{i-1})\|} : a = x_0 < \ldots < x_n = b \right\}.$$

Let $L := \mathrm{Length}_{\sqrt{\|\cdot\|}}(\gamma)$ and suppose $L < +\infty$. For each $\epsilon > 0$ there exists a partition $a = x_0 < x_1 < \ldots < x_n = b$ such that

$$\sum_{i=1}^n \sqrt{\|\gamma(x_i) - \gamma(x_{i-1})\|} > L - \epsilon. \tag{3.27}$$

For each $N \in \mathbb{N}$ big enough, refining the partition if necessary, we can suppose that

$$\frac{1}{N} < \|\gamma(x_i) - \gamma(x_{i-1})\| < \frac{5}{N} \tag{3.28}$$

for each $i = 1, \ldots, n$. Note that by refining the partition, property (3.27) still holds. From (3.27) and (3.28) it follows that $n > \frac{N}{5}(L - \epsilon)$. Hence

$$\mathrm{Length}_{\sqrt{|\cdot||}}(\gamma) \geq \sum_{i=1}^{n} \sqrt{\|\gamma(x_i) - \gamma(x_{i-1})\|} > n\frac{1}{\sqrt{N}} > \frac{N(L-\epsilon)}{5\sqrt{N}}.$$

If we send N to infinity (and consequently choose an appropriate n), then $\mathrm{Length}_{\sqrt{|\cdot||}}(\gamma)$ tends to infinity and we reach a contraddiction. So $L = \infty$. □

Now we may state the Lemma about the Korányi and CC metric.

Lemma 3.3. *If $\gamma : [0,1] \to \mathbb{R}$ is a C^1 curve and $t_i = \frac{i}{n}$, for $i = 1, \ldots, n$, is a partition of $[0,1]$, then*

$$\limsup_{n\to\infty} \sum_{i=1}^{n} d_{\mathbb{H}}(\gamma(t_i), \gamma(t_{i-1})) = \begin{cases} \mathrm{Length}_{\mathbb{H},CC}(\gamma) & \textit{if } \gamma \textit{ is horizontal} \\ \infty & \textit{otherwise.} \end{cases}$$

Proof. Let us denote $\gamma(t) = (\gamma^1(t), \gamma^2(t), \gamma^3(t))$, $\gamma_i = \gamma(t_i) = (\gamma_i^1, \gamma_i^2, \gamma_i^3)$, and $\dot{\gamma}_i^j = (\frac{d\gamma^j}{dt})(t_i)$. Then

$$d_{\mathbb{H}}(\gamma(t_i), \gamma(t_{i-1})) = ||\gamma_{i-1}^{-1}\gamma_i||_{\mathbb{H}} = |a_{in}^2 + 16b_{i,n}^2|^{\frac{1}{4}}$$

where $a_{i,n} = (\gamma_i^1 - \gamma_{i-1}^1)^2 + (\gamma_i^2 - \gamma_{i-1}^2)^2$ and $b_{i,n} = \gamma_i^3 - \gamma_{i-1}^3 - \frac{1}{2}[\gamma_i^1(\gamma_i^2 - \gamma_{i-1}^2) - \gamma_i^2(\gamma_i^1 - \gamma_{i-1}^1)]$. Applying simple inequalities and then summing up for $i = 1, \ldots, n$ we obtain

$$\sum_{i=1}^{n} |a_{i,n}|^{\frac{1}{2}} \leq \sum_{i=1}^{n} d_{\mathbb{H}}(\gamma(t_i), \gamma(t_{i-1})) \leq \sum_{i=1}^{n} |a_{i,n}|^{\frac{1}{2}} + 2\sum_{i=1}^{n} |b_{i,n}|^{\frac{1}{2}} \quad (3.29)$$

$$\sum_{i=1}^{n} |b_{i,n}|^{\frac{1}{2}} \leq \sum_{i=1}^{n} d_{\mathbb{H}}(\gamma(t_i), \gamma(t_{i-1})). \quad (3.30)$$

Notice that $\lim_{n\to\infty} \sum_{i=1}^{n} |a_{i,n}|^{\frac{1}{2}} = \mathrm{Length}_{\mathbb{C},\mathrm{Eucl}}(\pi(\gamma))$, as follows from the classical properties of the Euclidean length. Hence it is enough to prove that

$$\lim_{n\to\infty} \sum_{i=1}^{n} |b_{i,n}|^{\frac{1}{2}} = \begin{cases} 0 & \text{if } \gamma \text{ is horizontal,} \\ +\infty & \text{otherwise.} \end{cases} \quad (3.31)$$

To show (3.31) let us rewrite $|b_{i,n}|$ as

$$
\begin{aligned}
|b_{i,n}| &= \left|\int_{t_{i-1}}^{t_i} \dot\gamma^3(s)ds - \frac{1}{2}\left[\gamma_i^1 \int_{t_{i-1}}^{t_i} \dot\gamma^2(s)ds - \gamma_i^2 \int_{t_{i-1}}^{t_i} \dot\gamma^1(s)ds\right]\right| \\
&= \left|\int_{t_{i-1}}^{t_i} (\dot\gamma^3 - \frac{1}{2}(\gamma^1\dot\gamma^2 - \gamma^2\dot\gamma^1))(s)ds\right. \\
&\quad \left. -\frac{1}{2}\left[\int_{t_{i-1}}^{t_i} (\gamma_i^1 - \gamma^1)\dot\gamma^2 ds - (\gamma_i^2 - \gamma^2)\dot\gamma^1 ds\right]\right| \\
&= \left|\int_{t_{i-1}}^{t_i} \omega(\dot\gamma)(s)ds\right. \\
&\quad \left. -\frac{1}{2}\int_{t_{i-1}}^{t_i}\int_s^{t_i} \left(\dot\gamma^1(\sigma)\dot\gamma^2(s) - \dot\gamma^2(\sigma)\dot\gamma^1(s)\right) d\sigma ds\right|. \qquad (3.32)
\end{aligned}
$$

Let us consider the integral in (3.32). Since $\dot\gamma$ is continuous in $[0,1]$, it is also bounded. Let $M > 0$ be such that $|\dot\gamma^j(s)| < M$ for each $s \in [0,1]$ and $j = 1, 2$. Since $[0,1]$ is compact, $\dot\gamma$ is uniformly continuous and for each $\epsilon > 0$ there exists $\delta > 0$ such that $|\dot\gamma^j(t) - \dot\gamma^j(s)| < \epsilon$ for each $|t - s| < \delta$ in $[0,1]$ and for each $j = 1, 2$. Let us take n so that $\frac{1}{n} < \delta$. Hence the second integral in (3.32) is controlled by

$$
\begin{aligned}
&|\int_{t_{i-1}}^{t_i}\int_s^{t_i} \left(\dot\gamma^1(\sigma)\dot\gamma^2(s) - \dot\gamma^2(\sigma)\dot\gamma^1(s)\right) d\sigma ds| \\
&\qquad \le \int_{t_{i-1}}^{t_i}\int_s^{t_i} |(\dot\gamma^1(\sigma) - \dot\gamma^1(s))\dot\gamma^2(s) - (\dot\gamma^2(\sigma) - \dot\gamma^2(s))\dot\gamma^1(s)| d\sigma ds \\
&\qquad \le \int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t_i} 2M\epsilon \, d\sigma ds = \frac{2M\epsilon}{n^2}.
\end{aligned}
$$

Now, if γ is horizontal, then $\omega(\dot\gamma) = 0$ and

$$
\lim_{n\to\infty} \sum_{i=1}^n |b_{i,n}|^{\frac{1}{2}} \le \lim_{n\to\infty} \sum_{i=1}^n \sqrt{\frac{M\epsilon}{n^2}} = \sqrt{M\epsilon}.
$$

Since ϵ was arbitrary, the claim is proved in case γ horizontal.

If γ is not horizontal, we have to prove that $\lim_{n\to\infty} \sum_{i=1}^n |b_{i,n}|^{\frac{1}{2}} = +\infty$. Notice that

$$
\sqrt{|b_{i,n}|} = \sqrt{\left|\int_{t_{i-1}}^{t_i} \omega(\dot\gamma)(s)ds - c_{i,n}\right|} \ge \sqrt{\left|\int_{t_{i-1}}^{t_i} \omega(\dot\gamma)(s)ds\right|} - \sqrt{|c_{i,n}|}
$$

where
$$c_{i,n} = \frac{1}{2}\int_{t_{i-1}}^{t_i}\int_{s}^{t_i}\big(\dot\gamma^1(\sigma)\dot\gamma^2(s) - \dot\gamma^2(\sigma)\dot\gamma^1(s)\big)\,d\sigma ds$$
is as in (3.32). Since we have just shown that $\lim_{n\to\infty}\sum_{i=1}^n \sqrt{|c_{i,n}|} = 0$, let us prove that
$$\lim_{n\to\infty}\sum_{i=1}^n \sqrt{\left|\int_{t_{i-1}}^{t_i}\omega(\dot\gamma)(s)ds\right|} = +\infty.$$
Since γ is not horizontal, there exists $t \in (0,1)$ such that $\omega(\dot\gamma)(t) \neq 0$, for example $\omega(\dot\gamma)(t) > 0$. Then for continuity, there esists a neighborhood of t where $\omega(\dot\gamma)$ is positive. Let us suppose for simplicity that $\omega(\dot\gamma) > 0$ on the whole $[0,1]$. Since $\omega(\dot\gamma)$ is continuous, there exists $c > 0$ such that $\omega(\dot\gamma) \geq c$. Then
$$\lim_{n\to\infty}\sum_{i=1}^n \sqrt{\left|\int_{t_{i-1}}^{t_i}\omega(\dot\gamma)ds\right|} \geq \lim_{n\to\infty}\sum_{i=1}^n\sqrt{\frac{c}{n}} = +\infty$$
and the proof is complete. □

3.3 Geodesic and bubble sets

In this section we describe the length minimizing curves joining pairs of points in the Heisenberg group and define the so-called *bubble sets* which appear in Pansu's conjecture on the isoperimetric profile of $\mathbb{H}$.

Without loss of generality, and thanks to left-invariance of the CC metric, we may assume that the two points are the origin $0 = (0,0,0)$ and $x = (x_1, x_2, x_3)$. Let $\gamma : [0,1] \to \mathbb{H}$ denote a Legendrian curve joining 0 and x. Let S be the region in the (x_1,x_2)-plane bounded by $\pi(\gamma)$ and by the segment joining $\pi(x)$ to the origin, and let $\tilde\gamma$ be the closed curve obtained by closing $\pi(\gamma)$ with the segment. By Stoke's theorem we have
$$\begin{aligned} x_3 &= \int_0^1 \gamma_3'(t)dt = \frac{1}{2}\int_0^1(\gamma_1\gamma_2' - \gamma_2\gamma_1')(t)dt \\ &= \frac{1}{2}\int_{\tilde\gamma} x_1dx_2 - x_2dx_1 = \int_S dx_1\wedge dx_2 = \mathrm{Area}(S), \end{aligned} \tag{3.33}$$
where the equality from the first to the second row holds because the integral on the segment joining 0 to $\pi(x)$ is zero. In view of (3.33), since the horizontal length of a Legendrian curve γ depends only on the length of $\pi(\gamma)$, we can rephrase the problem of finding the Legendrian curve from 0 to x with minimal length with the following problem: Find the plane curve from the

origin to (x_1, x_2) with minimum length, subject to the constraint that the region S delimited by the curve and the segment joining $(0,0)$ to (x_1, x_2) has fixed area. This is one formulation of *Dido's problem*, closely related to the isoperimetric problem, and is solved by choosing the plane curve to be an arc of a circle.

In conclusion, a length minimizing curve between 0 and x is the lift of a circular arc joining the origin in $\mathbb{C}$ with (x_1, x_2), whose convex hull has area x_3. The family of such curves emanating from 0 is parametrized by $e^{i\phi} \in \mathbb{S}^1$ and $c \in \mathbb{R}$ and is given explicitly in the form

$$\gamma_{c,\phi}(s) = \left(e^{i\phi} \frac{1 - e^{-ics}}{c}, \frac{cs - \sin(cs)}{2c^2} \right); \tag{3.34}$$

it is length minimizing over any interval of length $\frac{2\pi}{|c|}$. These curves are all the Legendrian curves that have as projection, $\pi(\gamma)$, the arc of a circle with center in $\frac{e^{i\phi}}{c}$ and radius $\frac{1}{c}$, which passes through the origin and is clockwise parametrized if $c > 0$, anticlockwise otherwise. If $c = 0$, $\gamma_{c,\phi}$ is a straight line through 0 in the xy-plane. We call c the *curvature* of the geodesic arc $\gamma_{c,\phi}$.

If $x = (0, 0, x_3)$ lies on the x_3-axis, then the projection of the geodesic in $\mathbb{H}$ joining 0 to x is a circle of area x_3 passing through the origin. Clearly there are infinitely many such circles, so geodesics are not unique in this case. Choosing $t = cs$, $R = 1/c$ and $\phi = 0$ in (3.34) gives the following representation:

$$\gamma(t) = (R(1 - e^{-it}), \frac{1}{2} R^2 (t - \sin t)), \tag{3.35}$$

with $0 \leq t \leq 2\pi$. The request that the area is equal to x_3, gives $R^2 = x_3/\pi$.

Rotating such a geodesic around the x_3-axis produces a surface of revolution Σ whose profile curve is given parametrically as

$$t \mapsto \left(2R \sin(\frac{t}{2}), \frac{1}{2} R^2 (t - \sin t) \right), \tag{3.36}$$

that is determined by taking $(||\pi(\gamma_\phi(t))||, \gamma_{3,\phi}(t))$.

For each $z \in \mathbb{C}$ with $|z| < 2R$ there are two distinct points $(z, x_{3,1})$ and $(z, x_{3,2})$ on Σ, determined respectively by

$$\frac{t_1}{2} = \arcsin\left(\frac{|z|}{2R}\right) = \frac{\pi}{2} - \arccos\left(\frac{|z|}{2R}\right)$$

and

$$\frac{t_2}{2} = \pi - \arcsin\left(\frac{|z|}{2R}\right) = \frac{\pi}{2} + \arccos\left(\frac{|z|}{2R}\right).$$

So Σ may be seen as the boundary of the open set Ω consisting of all the points $(z, x_3) \in \mathbb{H}$ such that

$$R^2\left(\frac{\pi}{2} - \arccos\left(\frac{|z|}{2R}\right)\right) - \frac{|z|}{2}\sqrt{R^2 - \frac{|z|^2}{4}} < x_3 < R^2\left(\frac{\pi}{2} + \arccos\left(\frac{|z|}{2R}\right)\right) + \frac{|z|}{2}\sqrt{R^2 - \frac{|z|^2}{4}}. \quad (3.37)$$

To simplify the notation we dilate the ball by a factor of 2 and translate vertically by $-\pi R^2/2$. The resulting domains

$$\mathcal{B}(0, R) := \{(z, x_3) \in \mathbb{H} \, : |x_3| < f_R(|z|)\}, \quad (3.38)$$

$$f_R(r) = \frac{1}{4}\left(R^2 \arccos(\frac{r}{R}) + r\sqrt{R^2 - r^2}\right),$$

are the conjectures extremals for the sub-Riemannian isoperimetric problem in $\mathbb{H}$ and are often called *bubble sets.*

We conclude this section with the following result on the regularity of the boundary of the bubble sets. Unlike the classical case, the conjectured solution for the sub-Riemannian isoperimetric problem in $\mathbb{H}$ is not C^∞.

Theorem 3.4. *The boundary of the set $\mathcal{B}(0, R)$ is C^2 but not C^3. More precisely, $\partial\mathcal{B}(0, R)$ is not C^3 in $(0, 0, \pm f_R(0))$ and is C^∞ away from these points.*

Remark 3.5. Notice that $(0, 0, \pm f_R(0))$ are the unique points on $\partial\mathcal{B}(0, R)$ such that the tangent space to the surface coincides with the horizontal plane. Points with this property take a special name. We will call them *characteristic points* (see Definition 4.2) and we will discuss them more deeply in the next chapters.

Proof. By smooth dilating the bubble set, we can restrict our proof to the unitary bubble set $\mathcal{B}(0, 1)$ and the points $(0, 0, \pm\frac{\pi}{8})$.

Let us denote $f_1(r)$ by $f(r)$ in what follows. The regularity of $\partial\mathcal{B}(0, 1)$ at $(0, 0, \pm\frac{\pi}{8})$ is equivalent to verifying up to what order of derivatives n we have

$$\lim_{r\to 0^+} f^{(n)}(r) = \lim_{r\to 0^-} f^{(n)}(-r). \quad (3.39)$$

Then let us compute the derivatives:

$$\begin{aligned}
f'(r) &= -\frac{r^2}{2\sqrt{1-r^2}} & f'(-r) &= -f'(r);\\
f''(r) &= \frac{r^3-2r}{2\sqrt{1-r^2}(1-r^2)} & f''(-r) &= (f'(-r))' = -f''(r);\\
f'''(r) &= -\frac{r^2+2}{2\sqrt{1-r^2}(1-r^2)} & f'''(-r) &= (f''(-r))' = -f'''(r).
\end{aligned}$$

Hence, equation (3.39) holds for $n = 0, 1, 2$, whereas for $n = 3$ we have different limits. This shows that the boundary is C^2 but not C^3 near $(0, 0, \frac{\pi}{8})$.

The boundary is C^∞ away from these points because, after a dilation and a translation, it is obtained by revolution around the x_3-axis of the C^∞ curve given by (3.36), that is a C^∞ surface away from $(0, 0, \pm\frac{\pi}{8})$. □

3.4 Riemannian approximants to $\mathbb{H}$

In this section we will study the connection between the sub-Riemannian geometry of the Heisenberg group and its Riemannian approximants. For this purpose we will introduce the approximant Riemannian metrics and a definition of convergence between metric spaces, the Gromov-Hausdorff convergence. The main result is that the Heisenberg group equipped with the CC metric may be realized as the Gromov-Hausdorff limit of a sequence of Riemannian manifolds $(\mathbb{R}^3, g_L)$, as $L \to \infty$. This *Riemannian approximant scheme* plays a central role in the development of sub-Riemannian submanifold geometry of Chapter 4.

3.4.1 The g_L metrics

To describe the Riemannian approximants to $\mathbb{H}$, let $L > 0$ and define a metric g_L on $\mathbb{R}^3$ so that the left invariant basis $X_1, X_2, X_3/\sqrt{L}$ of $\mathfrak{h}$ is orthonormal. This family of metrics is essentially obtained as an anisotropic blow-up of the Riemannian metric g_1 defined in section 3.2.3.The left invariant property for the scalar product, requires that

$$\langle X_i(x), X_j(x)\rangle_x = \langle (dL_x)^{-1}X_i(x), (dL_x)^{-1}X_j(x)\rangle_e = \langle X_{i,e}, X_{j,e}\rangle_e,$$

for each $x \in \mathbb{R}^3$, where $X_i(x)$ and $X_{i,e}$, $\langle\cdot,\cdot\rangle_x$ and $\langle\cdot,\cdot\rangle_e$ denote the canonical vector fields and the scalar products evaluated in x and the neutral element respectively. This leads to the conclusion that the matrix of the scalar product $g_L(x)$ must be equal to

$$g_L(x) = C^T I_L C,$$

where

$$C = (dL_x)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2}x_2 & -\frac{1}{2}x_1 & 1 \end{pmatrix}$$

is the inverse of the matrix defining left translation (see (3.6)) and

$$I_L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & L \end{pmatrix}$$

is the matrix of the g_L metric in the neutral element. We may represent g_L explicitly in exponential coordinates $x = (x_1, x_2, x_3)$ via the positive definite matrix

$$g_L(x) = \begin{pmatrix} 1 + \frac{1}{4}x_2^2 L & -\frac{1}{4}x_1 x_2 L & \frac{1}{2}x_2 L \\ -\frac{1}{4}x_1 x_2 L & 1 + \frac{1}{4}x_1^2 L & -\frac{1}{2}x_1 L \\ \frac{1}{2}x_2 L & -\frac{1}{2}x_1 L & L \end{pmatrix}. \tag{3.40}$$

Observe that the Riemannian volume element in $(\mathbb{H}, g_L)$ is

$$\sqrt{\det g_L} dx_1 \wedge dx_2 \wedge dx_3 = \sqrt{L} dx_1 \wedge dx_2 \wedge dx_3. \tag{3.41}$$

When calculating with the Riemannian metric g_L, we will sometimes use $\langle \cdot, \cdot \rangle_L$ to denote the inner product on vectors. In other words, for $\vec{a} = \sum_{i=1}^3 a_i X_i$ and $\vec{b} = \sum_{i=1}^3 b_i X_i$ in $T\mathbb{H}$,

$$\langle \vec{a}, \vec{b} \rangle_L = a_1 b_1 + a_2 b_2 + L a_3 b_3. \tag{3.42}$$

As usual, the length of a vector is given as

$$|\vec{a}|_L = \langle \vec{a}, \vec{a} \rangle_L^{1/2}.$$

The metric $\langle \cdot, \cdot \rangle_L$ restricted to the horizontal directions gives the sub-Riemannian inner product on $\mathbb{H}$. Moreover, suppose $\gamma : [0,1] \to \mathbb{H}$ is a C^1 curve and that $\gamma'(t) = a_1(t)X_1 + a_2(t)X_2 + a_3(t)X_3$. Then

$$\begin{aligned} \text{Length}_L(\gamma) &= \int_0^1 \langle \gamma'(t), \gamma'(t) \rangle_L^{\frac{1}{2}} dt & (3.43) \\ &= \int_0^1 (a_1(t)^2 + a_2(t)^2 + L a_3(t)^2)^{\frac{1}{2}} dt, & (3.44) \end{aligned}$$

where we have written $\text{Length}_L = \text{Length}_{d_L}$ for simplicity. Note that γ has finite length in the limit as $L \to \infty$ if and only if $a_3 = 0$, i.e., γ is a horizontal curve. We define d_L to be the standard path distance associated to g_L.

3.4.2 Levi-Civita connection and curvature in the Riemannian approximants

In this section, we compute the sectional, Ricci and scalar curvatures of the Heisenberg group with respect to g_L. This is a quite computational section, but we will see that such explicite results are necessary to carry out the proof of the isoperimetric problem. In particular we will use these formulae in the computation of the generalized first variation of the perimeter (see Proposition 5.24).

Let us denote by D the Levi-Civita connection on $(\mathbb{H}, g_L)$. Given vector fields U, V, W on a Riemannian manifold (M, g), the *Kozul identity* defined in (1.8) is

$$\begin{aligned} \langle \nabla_U V, W\rangle_L &= \frac{1}{2}\{U\langle V, W\rangle_L + V\langle W, U\rangle_L - W\langle U, V\rangle_L \\ &\quad -\langle W, [V, U]\rangle_L - \langle [V, W], U\rangle_L - \langle V, [U, W]\rangle_L\} \quad (3.45) \end{aligned}$$

where $\langle\cdot,\cdot\rangle_L$ is the inner product associated to g_L. To simplify the notation let us introduce the functions

$$\alpha_{ijk} = \langle \tilde{X}_i, [\tilde{X}_j, \tilde{X}_k]\rangle_L$$

where $\tilde{X}_i = X_i$ for $i = 1, 2$ and $\tilde{X}_3 = L^{-1/2}X_3$. Using the Kozul identity and considering that $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3$ is an orthonormal basis with respect to g_L, we have

$$\langle D_{\tilde{X}_i}\tilde{X}_j, \tilde{X}_k\rangle_L = -\frac{1}{2}(\alpha_{kji} + \alpha_{ijk} + \alpha_{jik}).$$

Since the only non-trivial Lie bracket is $[X_1, X_2] = X_3 = \sqrt{L}\tilde{X}_3$, we have $\alpha_{312} = \sqrt{L}$, $\alpha_{321} = -\sqrt{L}$, and $\alpha_{ijk} = 0$ for all other triples (i, j, k). Then

$$\begin{array}{ll} D_{\tilde{X}_1}\tilde{X}_2 = \frac{1}{2}X_3 & D_{\tilde{X}_2}\tilde{X}_1 = -\frac{1}{2}X_3 \\ D_{\tilde{X}_1}\tilde{X}_3 = D_{\tilde{X}_3}\tilde{X}_1 = -\frac{\sqrt{L}}{2}X_2 & D_{\tilde{X}_2}\tilde{X}_3 = D_{\tilde{X}_3}\tilde{X}_2 = \frac{\sqrt{L}}{2}X_1 \\ D_{\tilde{X}_i}\tilde{X}_j = 0 \text{ for all other } (i, j). & \end{array} \quad (3.46)$$

Using these results we are now able to compute the curvatures introduced in Subsection 1.2.3. The curvature tensor for a Riemannian manifold (M, g) and Riemannian connection D as defined in Definition 1.17 is

$$R(U, V)W = D_U D_V W - D_V D_U W - D_{[U,V]}W.$$

Then the sectional curvatures of the two-planes spanned by the basis vectors

$\tilde{X}_i$ and $\tilde{X}_j$ are (see equation (1.9)):

$$\begin{aligned} K_{12} &= \langle R(X_1, X_2)X_1, X_2\rangle_L = \langle D_{X_1}D_{X_2}X_1 - D_{X_2}D_{X_1}X_1 - D_{X_3}X_1, X_2\rangle_L \\ &= \left\langle D_{X_1}\left(-\frac{1}{2}X_3\right) - D_{X_2}(0) + \frac{L}{2}X_2, X_2\right\rangle_L \\ &= \frac{L}{4} + \frac{L}{2} = \frac{3}{4}L, \end{aligned} \tag{3.47}$$

$$\begin{aligned} K_{13} &= \langle R(X_1, \tilde{X}_3)X_1, \tilde{X}_3\rangle_L = \langle D_{X_1}D_{\tilde{X}_3}X_1 - D_{\tilde{X}_3}D_{X_1}X_1, \tilde{X}_3\rangle_L \\ &= \left\langle D_{X_1}\left(-\frac{\sqrt{L}}{2}X_2\right) - D_{\tilde{X}_3}(0), \tilde{X}_3\right\rangle_L \\ &= \left\langle -\frac{\sqrt{L}}{4}X_3, \tilde{X}_3\right\rangle_L = -\frac{1}{4}L, \end{aligned} \tag{3.48}$$

$$\begin{aligned} K_{23} &= \langle R(X_2, \tilde{X}_3)X_2, \tilde{X}_3\rangle_L = \langle D_{X_2}D_{\tilde{X}_3}X_2 - D_{\tilde{X}_3}D_{X_2}X_2, \tilde{X}_3\rangle_L \\ &= \left\langle D_{X_2}\frac{\sqrt{L}}{2}X_1 - D_{\tilde{X}_3}(0), \tilde{X}_3\right\rangle_L \\ &= \left\langle -\frac{\sqrt{L}}{4}X_3, \tilde{X}_3\right\rangle_L = -\frac{1}{4}L. \end{aligned} \tag{3.49}$$

More generally, denote by $R_{ijkl} = \langle R(\tilde{X}_i, \tilde{X}_j)\tilde{X}_k, \tilde{X}_l\rangle$ the full Riemannian curvature tensor, then

$$R_{ijkl} = \begin{cases} \frac{3}{4}L & \text{if } (ijkl) = (1212) \text{ or } (2121), \\ -\frac{3}{4}L & \text{if } (ijkl) = (1221) \text{ or } (2112), \\ -\frac{1}{4}L & \text{if } (ijkl) = (1313), (3131), (2323) \text{ or } (3232), \\ \frac{1}{4}L & \text{if } (ijkl) = (1331), (3113), (2332) \text{ or } (3223), \\ 0 & \text{otherwise.} \end{cases}$$

Finally, the Ricci curvatures $\mathrm{Ric}_i = K_{i1} + K_{i2} + K_{i3}$ are

$$\mathrm{Ric}_1 = \mathrm{Ric}_2 = \frac{1}{2}L \quad \text{and} \quad \mathrm{Ric}_3 = -\frac{1}{2}L,$$

and the scalar curvature $\sigma = \mathrm{Ric}_1 + \mathrm{Ric}_2 + \mathrm{Ric}_3$ is

$$\sigma = \frac{1}{2}L.$$

3.4.3 Gromov-Hausdorff convergence

A way to value the distance between metric spaces is the Gromov-Hausdorff distance. In this section we will see its definition and the corresponding notion of convergence.

First we give a preliminary definition.

Definition 3.6. Let (Z, d) be a metric space. The *Hausdorff distance* between E_1 and E_2 is given by

$$\mathrm{Haus}_Z(E_1, E_2) = \inf\{\epsilon > 0 | E_1 \subset (E_2)_\epsilon, E_2 \subset (E_1)_\epsilon\}$$

where $(E_i)_\epsilon = \{z \in \mathbb{Z} : \mathrm{dist}(z, E_i) < \epsilon\}$ is the ϵ-neighborhood of E_i in (Z, d).

Definition 3.7. Let (X, d_X) and (Y, d_Y) be metric spaces. The *Gromov-Hausdorff distance* between X and Y is given by

$$d_{GH}(X, Y) = \inf_{f,g,Z} \mathrm{Haus}_Z(f(X), g(Y)),$$

where the infimum is taken over all metric spaces Z and isometric embeddings f and g of X and Y respectively into Z.

Using this metric, we have a corresponding notion of convergence:

Definition 3.8. A sequence of compact metric spaces (X_n) *Gromov-Hausdorff converges* to a compact metric space X if $d_{GH}(X_n, X) \to 0$ as $n \to \infty$.

We would like to apply this notion of convergence to the Riemannian approximants $(\mathbb{R}^3, g_L)$ and the limit $(\mathbb{H}, d)$, but this definition is unnatural for noncompact spaces and limit. Indeed, consider the dilated spheres $(\mathbb{S}^n, \lambda d_{\mathbb{S}^n})$, where $d_{\mathbb{S}^n}$ is the geodesic distance on $\mathbb{S}^n$. We would like to state that they converge to $\mathbb{R}^n$ as $\lambda \to \infty$. However, according to Definition 3.7, $d_{GH}((\mathbb{S}^n, \lambda d_{\mathbb{S}^n}), \mathbb{R}^n) = +\infty$ for all $\lambda > 0$.

To generalize this notion of convergence to non-compact metric spaces we have to restrict ourselves to proper pointed metric length spaces. A metric space (X, d) is called *proper* if all closed balls in X are compact, while a *pointed* metric space (X, d, x) is a metric space (X, d) equipped with a fixed basepoint $x \in X$. The generalized definition is the following:

Definition 3.9. A sequence of proper pointed length spaces (X_n, d_n, x_n) *Gromov-Hausdorff converges* to a proper pointed length space (X, d, x) if the sequence of closed balls $\overline{B_{X_n}(x_n, r)}$ Gromov-Hausdorff converges (in the sense of definition 3.8) to $\overline{B_X(x, r)}$, uniformly in r.

The following proposition gives a sufficient condition for the Gromov-Hausdorff convergence, which will guaranty that the Riemannian approximants scheme Gromov-Hausdorff convergies to the sub-Riemannian Heisenberg group.

Proposition 3.10. *Let X be a set equipped with a family of metrics $(d_t)_{t\geq 0}$ generating a common topology. For K compact in X, let:*

$$\omega_K(\epsilon) := \sup_{x,y\in K, t\geq 0} d_t(x,y) - d_{t+\epsilon}(x,y).$$

Assume that:

(i) For each $t \geq 0$, (X, d_t) is a proper length space.

(ii) For fixed $x, y \in X$, the function $t \mapsto d_t(x,y)$ is non-increasing.

(iii) For each compact set K in X, $\omega_K(\epsilon) \to 0$ as $\epsilon \to 0$.

Then (X, d_t) converges, in the sense of pointed Gromov-Hausdorff convergence, to (X, d_0).

Proof. By hypothesis (ii) and the definition of ω_K we easily verify the following additional facts for each compact set K:

(iv) the map $\epsilon \mapsto \omega_K(\epsilon)$ is non-decreasing in ϵ: indeed, let $\epsilon_1 < \epsilon_2$. By hypothesis (ii) $d_{t+\epsilon_1}(x,y) \geq d_{t+\epsilon_2}(x,y)$ for each $t \geq 0, x, y \in K$, then

$$\begin{aligned}\omega_K(\epsilon_2) &= \sup_{x,y\in K, t\geq 0} d_t(x,y) - d_{t+\epsilon_2}(x,y) \\ &\geq \sup_{x,y\in K, t\geq 0} d_t(x,y) - d_{t+\epsilon_1}(x,y) = \omega_K(\epsilon_1);\end{aligned}$$

(v) ω_K is sublinear, i.e. $\omega_K(a+b) \leq \omega_K(a) + \omega_K(b)$ for all $a, b \geq 0$;

(vi) if we denote by $B_t(x_0, R)$ the closed metric ball with center x_0 and radius R in the metric (X, d_t), $t \geq 0$, then

$$B_0(x_0, R) \subset B_t(x_0, R) \subset B_0(x_0, R + \omega_K(t))$$

for any $x_0, R > 0$ and $t \geq 0$ so that $B_t(x_0, R) \subset K$: the first inclusion holds because $d_0(x_0, x) \geq d_t(x_0, x)$ for each $t \geq 0$, $x \in K$. The second inclusion follows because $d_0(x_0, x) = d_0(x_0, x) - d_t(x_0, x) + d_t(x_0, x) \leq d_t(x_0, x) + \omega_K(t)$, then $d_t(x_0, x) \geq d_0(x_0, x) - \omega_K(t)$.

From (vi) and (i) we further conclude

(vii) For any $x_0, R > 0$ and $t \geq 0$, to each $y \in B_t(x_0, R)$ there corresponds a point $x \in B_0(x_0, R)$ with $d_t(x, y) \leq \omega_K(t)$: in fact, if $y \in B_0(x_0, R)$ it is trivial. Let $y \in B_t(x_0, R) \setminus B_0(x_0, R)$, then $y \in B_0(x_0, R + \omega_K(t))$ and for each $\epsilon > 0$, by (i), there exists $\gamma : [0, 1] \to (X, d_0)$ such that $\gamma(0) = x_0$, $\gamma(1) = y$ and $\text{Length}_0(\gamma) \leq d_0(x_0, y) + \epsilon \leq R + \omega_K(t) + \epsilon$. Let $t_0 = \max\{t : \gamma(t) \in B_0(x_0, R)\}$ (it is indeed a maximum because $B_0(x_0, R)$ is closed by (i)) and let $x := \gamma(t_0) \in B_0(x_0, R)$. We claim that $\text{Length}_0(\gamma|_{[0,t_0]}) \geq R$: if it weren't, there would exist $t > t_0$ such that $\text{Length}_0(\gamma|_{[0,t]}) \leq R$ and $\gamma(t) \in B_0(x_0, R)$, which contradicts the maximality of t_0. Hence

$$\begin{aligned} R + \omega_K(t) + \epsilon &\geq \text{Length}_0(\gamma) \\ &\geq \text{Length}_0(\gamma|_{[0,t_0]}) + \text{Length}_0(\gamma|_{[t_0,1]}) \\ &\geq R + d_0(x, y). \end{aligned}$$

Since ϵ was arbitrary $d_t(x, y) \leq d_0(x, y) \leq \omega_K(t)$ and statement (vii) follows.

We now establish the desired conclusion. Fix a basepoint $x_0 \in X$. It sufficies to prove that the compact metric balls $B_t(x_0, R)$ converge (in the sense of Definition 3.8) to $B_0(x_0, R)$, for each $R > 0$. We restrict our attention to $t \in [0, 1]$, let $K = B_1(x_0, R)$, and equip the space

$$Z = K \times [0, 1]$$

with the metric

$$d_Z((x, t), (x', t')) = d_{max\{t,t'\}}(x, x') + \omega_K(|t - t'|) + |t - t'|.$$

We claim that d_Z is a metric. If $d_Z((x, t), (x', t')) = 0$ then $t = t'$ and $d_t(x, x') = 0$, hence also $x = x'$ (since d_t is a metric).

Next we verify the triangle inequality. Let $(x, t), (x', t'), (x'', t'') \in Z$. If $\max\{t, t', t''\} \neq t''$, then

$$\begin{aligned} d_Z((x, t), (x', t')) &= d_{max\{t,t',t''\}}(x, x') + \omega_K(|t - t'|) + |t - t'| \\ &\leq d_{max\{t,t''\}}(x, x'') + d_{max\{t'',t'\}}(x'', x') + \omega_K(|t - t'|) \\ &\quad + \omega_K(|t'' - t'|) + |t - t''| + |t'' - t'| \\ &= d_Z((x, t), (x'', t'')) + d_Z((x'', t''), (x', t')) \end{aligned}$$

where we used (ii) and (v) in the middle step.

If $\max\{t, t', t''\} = t''$, then

$$\begin{aligned} d_Z((x, t), (x', t')) &= d_{max\{t,t'\}}(x, x') + \omega_K(|t - t'|) + |t - t'| \\ &\leq d_{t''}(x, x') + \omega_K(t'' - \max\{t, t'\}) + \omega_K(|t - t'|) + |t - t'| \end{aligned}$$

by the definition of ω_K, and by (iv) follows

$$\begin{aligned} &\leq d_{t''}(x,x'') + d_{t''}(x'',x') + \omega_K(t''-t) + \omega_K(t''-t') + |t-t''| + |t''-t'| \\ &= d_Z((x,t),(x'',t'')) + d_Z((x'',t''),(x',t')). \end{aligned}$$

It is clear that the map $x \mapsto (x,t)$ is an isometric embedding of $B_t(x_0,R)$ into Z. To complete the proof, it sufficies to verify that the Hausdorff distance (in Z) between $B_t(x_0,R) \times \{t\}$ and $B_0(x_0,R) \times \{0\}$ tends to zero as $t \to 0$. In fact, we claim that

$$\mathrm{Haus}_Z(B_t(x_0,R) \times \{t\}, B_0(x_0,R) \times \{0\}) \leq 2\omega_K(t) + t; \tag{3.50}$$

the result then follows from assumption (iii).

To see that (3.50) is true: if $x \in B_0(x_0,R)$, then $x \in B_t(x_0,R)$ and

$$d_Z((x,t),(x,0)) = \omega_K(t) + t \leq 2\omega_K(t) + t,$$

while if $y \in B_t(x_0,R)$ we choose x as in (vii) and conclude

$$d_Z((y,t),(x,0)) = d_t(x,y) + \omega_K(t) + t \leq 2\omega_K(t) + t.$$

□

We now state the main result of this section.

Corollary 3.11. *The sequence of metric spaces* $(\mathbb{R}^3, d_L)$ *converges to* $(\mathbb{H}, d)$ *in the pointed Gromov-Hausdorff sense as* $L \to \infty$.

The proof follows from Proposition 3.10 where we choose $X = \mathbb{R}^3$, d_0 equal to the Carnot-Carathéodory metric d on X for the usual Heisenberg structure, and d_t the distance function associated to the Riemannian metric g_L, where $L = 1/t$. The third hypothesis remains to be proved and it is the content of the next section.

3.4.4 Carnot-Carathéodory geodesics and Gromov-Hausdorff convergence

The CC geodesics in the Heisenberg group can be obtained also through the approximation scheme, using the geodesics in the Riemannian manifolds $(\mathbb{R}^3, g_L)$.

Let $\gamma : [0,1] \to \mathbb{H}$ be a Lipschitz curve and let $\omega = dx_3 - \frac{1}{2}(x_1 dx_2 - x_2 dx_1)$ denote the contact form defined in (3.8). The g_L-length of γ is given by

$$\mathrm{Length}_L(\gamma) = \int_0^1 \sqrt{|\gamma_1'(t)|^2 + |\gamma_2'(t)|^2 + L \left|\omega(\gamma'(t))|_{\gamma(t)}\right|^2}\, dt. \tag{3.51}$$

In our computation, however, we will consider the "penalized" energy of γ:

$$E_L(\gamma) = \int_0^1 |\gamma_1'(t)|^2 + |\gamma_2'(t)|^2 + L \left| \omega(\gamma'(t))|_{\gamma(t)} \right|^2 dt. \tag{3.52}$$

Since minimizing curves of (3.52) are also minimizing curve of (3.51) and viceversa, we will find the solutions of the Euler-Lagrange equations of the energy rather than of the length.

Let us substitute for γ a one-parameter family of compactly supported perturbations $\{\gamma^\lambda\}$, where

$$\gamma_i^\lambda(s) = \gamma_i(s) + \lambda f_i(s),$$

with $f_i \in C_0^\infty([0,1])$ for $i = 1,2,3$, and compute the derivative of $E_L(\gamma^\lambda)$ with respect to λ at $\lambda = 0$ to obtain the Euler-Lagrange equations for the critical points of the penalized energy. The equations are given by

$$\frac{d}{dt}\left[\frac{d}{d\gamma_i'} g(t,\gamma,\gamma')\right] = \frac{d}{d\gamma_i} g(t,\gamma,\gamma'), \tag{3.53}$$

with $i = 1,2,3$, where $g(t,\gamma,\gamma') = |\gamma_1'(t)|^2 + |\gamma_2'(t)|^2 + L \left| \omega(\gamma'(t))|_{\gamma(t)} \right|^2$. Computing explicitely equations (3.53) we obtain the following three equations for the critical points of the penalized energy:

$$\gamma_1'' = -L\omega\gamma_2', \quad \gamma_2'' = L\omega\gamma_1', \quad (\omega(\gamma')|_\gamma)' = 0, \tag{3.54}$$

where to obtain the first two equations, we have to use the third one. The right-hand side of the first two equations contains L and hence could potentially blow up as $L \to \infty$. However, note that $\omega(\gamma')|_\gamma$ equals a constant depending only on the initial data $\gamma(0)$ and $\gamma'(0)$ by the third equation. If we choose $\gamma(0) = 0$ and $\gamma'(0) = (h_1, h_2, a_L/L)$, with $h_1, h_2, a_L \in \mathbb{R}$, then $\omega(\gamma'(t))|_{\gamma(t)} = a_L/L$ for all t and the (3.54) yield

$$(\gamma_1 + \mathbf{i}\gamma_2)'' = \mathbf{i}a_L(\gamma_1 + \mathbf{i}\gamma_2)', \tag{3.55}$$

with $(\gamma_1, \gamma_2)(0) = (0,0)$ and $(\gamma_1', \gamma_2')(0) = (h_1, h_2)$. Comparing (3.34) with (3.55) indicates that solutions to (3.55) corresponding to $h_1^2 + h_2^2 \neq 0$ (which are arcs of circles) are projections of length minimizing arcs emanating from the origin with initial velocity $h_1X_1 + h_2X_2$, parametrized by $a_L \in \mathbb{R}$. In particular, g_L-geodesic arcs with horizontal initial velocity ($a_L = 0$) are also length minimizing arcs in $(\mathbb{H}, d)$, the horizontal segments. In general, if $a_L \neq 0$, the solutions of (3.54) may be not horizontal. However, choosing a sequence $a_L \to a_\infty \in \mathbb{R}$ as $L \to \infty$, one can easily prove, using energy estimates for the ODE (3.55), that the corresponding solutions γ_L of (3.54) converge uniformly to a length minimizing arc in $(\mathbb{H}, d)$ as described in (3.34).

As a result of the above observations, we can state the following corollary, which completes the proof of Corollary 3.11, showing that we can indeed apply Proposition 3.10.

Proposition 3.12. *Given $x \in \mathbb{H}$, any length minimizing horizontal curve γ joining x to the origin $0 \in \mathbb{H}$ is the uniform limit as $L \to \infty$ of geodesic arcs from 0 to x in the Riemannian spaces $(\mathbb{R}^3, g_L)$. Moreover, the convergence is uniform both in x and in the parameter L, in the sense that*

$$\lim_{\epsilon \to 0} \sup_{0 < L \leq \infty, x \in K} d_L(0, x) - d_{L-\epsilon}(0, x) = 0$$

for any compact $K \subset \mathbb{R}^3$.

Chapter 4

Horizontal geometry of hypersurfaces in $\mathbb{H}$

We now focus our attention to the sub-Riemannian geometry of hypersurfaces in $\mathbb{H}$, introducing the main geometric notions, that are necessary to study the isoperimetric problem. We introduce them as a limit of the correspondent Riemannnian objects.

4.1 The second fundamental form in $(\mathbb{R}^3, g_L)$

Let us consider a C^2 regular surface S in the Heisenberg group, given as the locus of the zeros of a function $u \in C^2(\mathbb{R}^3)$ with nonvanishing gradient along S:

$$S = \left\{ x \in \mathbb{R}^3 : u(x) = 0 \right\}. \tag{4.1}$$

Let us introduce some additional useful notation. Let X_1, X_2 be as in (3.7) and $\tilde{X}_3 = \frac{X_3}{\sqrt{L}}$. Set $p = X_1 u$, $q = X_2 u$ and $r = \tilde{X}_3 u$. To simplify the upcoming formulas, we set

$$l = \sqrt{p^2 + q^2}, \quad l_L = \sqrt{p^2 + q^2 + r^2},$$

and $\bar{p} = p/l$, $\bar{q} = q/l$, $\bar{p}_L = p/l_L$, $\bar{q}_L = q/l_L$, $\bar{r}_L = r/l_L$. Observe that the Riemannian normal to S is

$$\nu_L = \bar{p}_L X_1 + \bar{q}_L X_2 + \bar{r}_L \tilde{X}_3,$$

while the tangent space to S is spanned by the orthonormal basis

$$e_1 = \bar{q} X_1 - \bar{p} X_2, \quad \text{and} \quad e_2 = \bar{r}_L \bar{p} X_1 + \bar{r}_L \bar{q} X_2 - \frac{l}{l_L} \tilde{X}_3. \tag{4.2}$$

As seen in Section 1.2.4, in particular in equation (1.16), the second fundamental form of S with respect to this orthonormal frame is

$$II^L = \begin{pmatrix} \langle -D_{e_1}\nu_L, e_1\rangle_L & \langle -D_{e_1}\nu_L, e_2\rangle_L \\ \langle -D_{e_2}\nu_L, e_1\rangle_L & \langle -D_{e_2}\nu_L, e_2\rangle_L \end{pmatrix},$$

where D denotes the Levi-Civita connection associated to g_L. Since $\{e_1, e_2\}$ form an orthonormal system, the matrix G of the metric with respect to this basis is the identity matrix. Then the relation between the matricies of the shape operator and of the second fundamental form (equation (1.15)) implies that the two matricies are equal and we can compute the Gaussian and mean crvature of S with respect to the second fundamental form.

We explicite in the next theorem the geometric quantities of $S \subset (\mathbb{R}^3, g_L)$, but, since the proof implies only long computations, we refer to [6] (Theorem 4.3) for the whole calculus.

Theorem 4.1. *Let $S \subset (\mathbb{R}^3, g_L)$ be a regular C^2 surface defined as in* (4.1). *Relative to the orthonormal frame $\{e_1, e_2\}$ defined in* (4.2), *the second fundamental form of S is*

$$II^L = \begin{pmatrix} -\frac{l}{l_L}(X_1\bar{p} + X_2\bar{q}) & \frac{l_L}{l}\langle e_1, \nabla_0 \bar{r}_L\rangle_L + \frac{\sqrt{L}}{2} \\ \frac{l_L}{l}\langle e_1, \nabla_0 \bar{r}_L\rangle_L + \frac{\sqrt{L}}{2} & \frac{l^2}{l_L^2}\langle e_2, \nabla_0(\frac{r}{l})\rangle_L - \tilde{X}_3\bar{r}_L \end{pmatrix}. \tag{4.3}$$

In particular, the main curvature $\mathcal{H}_L = \mathrm{tr} II^L$ of S is

$$-\frac{l}{l_L}(X_1\bar{p} + X_2\bar{q}) + \frac{l^2}{l_L^2}\langle e_2, \nabla_0(\frac{r}{l})\rangle_L - \tilde{X}_3\bar{r}_L, \tag{4.4}$$

while the Gauss curvature $\mathcal{K}_L = \det II^L$ is

$$\frac{l}{l_L}(X_1\bar{p} + X_2\bar{q})\left(-\frac{l^2}{l_L^2}\langle e_2, \nabla_0(\frac{r}{l})\rangle_L + \tilde{X}_3\bar{r}_L\right) - \left(\frac{l_L}{l}\langle e_1, \nabla_0\bar{r}_L\rangle_L + \frac{\sqrt{L}}{2}\right)^2. \tag{4.5}$$

4.2 Horizontal geometry of hypersurfaces

We want to study the behaviour of these classical differential geometric objects as $L \to \infty$. We will see that the horizontal components have well-defined limits, which are natural candidates for sub-Riemannian analogs. On the other hand, the vertical components are unbounded.

Before initiating this analysis we introduce a fundamental notion in the study of sub-Riemannian submanifold geometry.

Definition 4.2. Let $S \subset \mathbb{H}$ be a C^1 surface defined as in (4.1). The *characteristic set* of S is the closed set

$$\Sigma(S) = \{x \in S : \nabla_0 u(x) = 0\}. \tag{4.6}$$

In other words, $\Sigma(S)$ is the set of points $x \in S$ where the tangent space coincides with the horizontal plane $H(x)$.

Note that $r, \bar{r}_L$ and $\tilde{X}_3$ all converge to zero as $L \to \infty$ at rates of the order of $L^{-1/2}$. On the other hand, $\bar{q}_L \to \bar{q}$, $\bar{p}_L \to \bar{p}$, $l_L \to l$, and $e_2 \to 0$. Hence the Riemannian normal ν_L converges to the so-called *horizontal normal*

$$\nu_H = \frac{X_1 u X_1 + X_2 u X_2}{|\nabla_0 u|} = \bar{p} X_1 + \bar{q} X_2 \in L^\infty(S \setminus \Sigma(S)) \tag{4.7}$$

in the complement of the characteristic set. Note that ν_H is simply the projection of ν_L onto the horizontal subbundle rescaled to have unitary norm. The vectors ν_H and e_1 form an orthonormal frame of the horizontal subbundle.

A direct computation shows that the Gauss curvature $\mathcal{K}_L$ in (4.5) diverges as $L \to \infty$ (similarly to the behaviour of the sectional, Ricci and scalar curvatures discussed in Section 3.4.2). Indeed

$$\lim_{L \to \infty} \frac{\mathcal{K}_L}{L} = -\frac{1}{4}.$$

This reflects the fact that $\nu_L \to \bar{p} X_1 + \bar{q} X_2 = \nu_H$ as $L \to \infty$, i.e. the tangent plane to S tends, as $L \to \infty$, towards a plane, that is orthogonal to the horizontal plane. As (3.48) and (3.49) show, the curvature of such planes computed with respect to g_L equals $-L/4$.

Surprisingly, while the Gauss curvature does not have a limit as $L \to \infty$, the mean curvature presents a rather different behaviour. The following lemma is an immediate consequence of (4.4).

Lemma 4.3. *Let S be a C^2 regular surface defined as in* (4.1). *Then*

$$\lim_{L \to \infty} \mathrm{tr} II^L = \lim_{L \to \infty} \mathcal{H}_L = -(X_1 \bar{p} + X_2 \bar{q}) \tag{4.8}$$

at noncharacteristic points.

This behaviour of the mean curvature suggests us the definition of its sub-Riemannian analogous.

Definition 4.4. Let $S \subset \mathbb{H}$ be a C^2 regular surface, given as a level set of a function u. The *horizontal mean curvature* of S at a noncharacteristic point is

$$\mathcal{H}_0 = -X_1 \bar{p} - X_2 \bar{q}.$$

We can rewrite the horizontal mean curvature $\mathcal{H}_0$ in several ways. First, its explicite expression is:

$$\mathcal{H}_0 = -\sum_{i=1}^{2} X_i \left(\frac{X_i u}{|\nabla_0 u|} \right). \tag{4.9}$$

Next, a direct computation shows that

$$\mathcal{H}_0 = \frac{X_1 u X_1(|\nabla_0 u|) + X_2 u X_2(|\nabla_0 u|)}{|\nabla_0 u|^2} - \frac{(X_1^2 u + X_2^2 u)|\nabla_0 u|}{|\nabla_0 u|^2}$$

with $X_i(|\nabla_0 u|) = (|\nabla_0 u|)^{-1}(X_1 u X_i X_1 u + X_2 u X_i X_2 u)$. Then, if we set $\mathcal{L}u = X_1^2 u + X_2^2 u$ for the *Heisenberg Laplacian* of u and

$$\mathcal{L}_\infty u := \sum_{i,j=1}^{2} X_i u X_j u X_i X_j u \tag{4.10}$$

for the Heisenberg *infinite Laplacian* of $u : \mathbb{H} \to \mathbb{R}$, the horizontal mean curvature can also be expressed via the identity

$$\mathcal{H}_0 |\nabla_0 u| = \frac{\mathcal{L}_\infty u}{|\nabla_0 u|^2} - \mathcal{L}u. \tag{4.11}$$

Finally, if $u(x) = x_3 - f(|z|)$ and we let $r = |z|$, then direct computations show:

$$\mathcal{H}_0 = \frac{\frac{1}{4} r^2 f'' + \frac{(f')^3}{r}}{((f')^2 + \frac{1}{4} r^2)^{\frac{3}{2}}}. \tag{4.12}$$

The horizontal mean curvature turn out to have a fundamental role in the isoperimetric problem, indeed we will see in section 5.3 that the *first variation* of the perimeter functional among all horizontal perturbations is determined by the horizontal mean curvature. We emphasize this relation with the following definition.

Definition 4.5. A C^2 regular surface $S \subset \mathbb{H}$ is called a *horizontal minimal surface* if it has vanishing horizontal mean curvature along its noncharacteristic locus.

We conclude by giving a kind of extension of the horizontal mean curvature to the characteristic locus. Rearranging equations (4.4) and (4.9) using the relation between the coefficients $\bar{r}_L$, l and l_L, we can express the mean curvature $\mathcal{H}_L$ as

$$\mathcal{H}_L = -\mathrm{div}_{g_L}(\nu_L) = -\sum_{i=1}^{3} \tilde{X}_i \left(\frac{\tilde{X}_i u}{|\nabla_L u|} \right). \tag{4.13}$$

Direct computations as those done to obtain (4.11) show that

$$\mathcal{H}_L|\nabla_L u| = -\sum_{i,j=1}^{2}\left(\delta_{ij} - \frac{\tilde{X}_i u \tilde{X}_j u}{|\nabla_L u|^2}\right)\tilde{X}_i\tilde{X}_j u.$$

Since $\tilde{X}_i u = 0$ for $i = 1, 2$ at the characteristic locus, both sides of this equation converge at every point, characteristic or not, as $L \to \infty$, giving

$$\lim_{L\to\infty} \mathcal{H}_L|\nabla_L u| = \begin{cases} \mathcal{H}_0|\nabla_0 u| = \frac{\mathcal{L}_\infty u}{|\nabla_0 u|^2} - \mathcal{L}u & \text{on } S \setminus \Sigma(S), \\ -\mathcal{L}u & \text{on } \Sigma(S), \end{cases} \tag{4.14}$$

where $\mathcal{L}u = X_1^2 u + X_2^2 u$ and $\mathcal{L}_\infty$ as in (4.10) denote the sub-Laplacian and the infinite Laplacian in $\mathbb{H}$, respectively.

Let π_H denote the orthogonal projection of Lie algebra vectors onto the horizontal bundle. In view of equation (4.14), we can extend the function $\mathcal{H}_0|\pi_H(\nu_1)|$ from $S \setminus \Sigma(S)$ to all of S as a continuous function.

4.2.1 The Legendrian foliation

Let $S = \{x \in \mathbb{H} : u(x) = 0\}$ be a C^2 regular surface with characteristic set $\Sigma(S)$.

For every point $x \in S \setminus \Sigma(S)$ consider the intersection of the horizontal plane $H(x)$ with the tangent space $T_x S$. Since x is noncharacteristic, this interesection is a line $H_x S$, that is horizontal and tangent; we denote by HS the distribution of these lines and call it the *horizontal line subbundle.*

The unit vectors e_1 defined in (4.2) form a tangent vector field in HS, that at each point is orthonormal to ν_H with respect to the g_1-metric. For each $x \in S\setminus\Sigma(S)$ define the curve $\gamma_x = (\gamma_x^1, \gamma_x^2, \gamma_x^3)$ determined by the intersection $S \cap H(x)$, with $\gamma_x(0) = x$ and $\gamma_x'(0) = e_1(x)$. We call the family of all such curves the *Legendrian foliation* of S.

As we are going to see, the Legendrian foliation describes well the horizontal mean curvature of the relative surface. Let us investigate more deeply one of its curve.

The curve γ_x has tangent vector $\dot\gamma_x = (\dot\gamma_x^1, \dot\gamma_x^2, \dot\gamma_x^3)$ which in general is not in the horizontal distribution (notice indeed that γ_x is not in general horizontal away from 0). The components of $\dot\gamma_x$ with respect to the horizontal base X_1, X_2 are $(\dot\gamma_x^1, \dot\gamma_x^2) = \pi(\dot\gamma_x)$, as computed in (3.19), where π denotes the projection of a vector $v = v_1\partial_{x_1} + v_2\partial_{x_2} + v_3\partial_{x_3}$ on the first two components.

Let us parametrize γ_x so that $(\dot\gamma_x^1, \dot\gamma_x^2)$ is a unitary vector. Then (γ_1, γ_2) defines a planar curve on $\mathbb{C}$ parametrized by arc length.

The horizontal normal unitary vector to S at x was defined as

$$\nu_H = \frac{X_1uX_1 + X_2uX_2}{|\nabla_0 u|} = \frac{X_1u\partial_{x_1} + X_2u\partial_{x_2} + \frac{1}{2}(x_1X_2u - x_2X_1u)\partial_{x_3}}{|\nabla_0 u|},$$

then the projection $\vec{n}(t) := \pi(\nu_H(\gamma(t))) \in \mathbb{C}$ is a unitary vector normal to the planar curve (γ_x^1, γ_x^2) in 0.

The vector $\pi(e_1) = (\dot\gamma_x^1, \dot\gamma_x^2)(0)$ is tangent to the curve (γ_x^1, γ_x^2) in 0 and is orthogonal to $\vec{n}$. More precisely, with respect to the defining function u we have

$$\pi(e_1) = (\bar q, -\bar p) \circ \gamma = \frac{(X_2u, -X_1u)}{|\nabla_0 u|} \circ \gamma = -\mathbf{i}\vec{n}.$$

Recall from equation (1.3) that the curvature vector of the planar curve (γ_1, γ_2) is defined as

$$\vec{k} = (\gamma_1'', \gamma_2'') = ki(\gamma_1', \gamma_2') = k\vec{n}.$$

We are now going to see that it is deep related with the horizontal mean curvature of the surface. Indeed, let us compute it with respect to u:

$$\begin{aligned}
\vec{k} &= (\gamma_1'', \gamma_2'') = \left(\left(\frac{X_2u}{|\nabla_0 u|}, \frac{-X_1u}{|\nabla_0 u|}\right)\circ\gamma\right)' \\
&= \left(\left(\frac{d\gamma_1}{dt}X_1 + \frac{d\gamma_2}{dt}X_2\right)\left(\frac{X_2u}{|\nabla_0 u|}\right), \left(\frac{d\gamma_1}{dt}X_1 + \frac{d\gamma_2}{dt}X_2\right)\left(-\frac{X_1u}{|\nabla_0 u|}\right)\right)\circ\gamma
\end{aligned}$$

This is equal to

$$\frac{1}{|\nabla_0 u|}\left((X_2uX_1 - X_1uX_2)\left(\frac{X_2u}{|\nabla_0 u|}\right), (X_2uX_1 - X_1uX_2)\left(-\frac{X_1u}{|\nabla_0 u|}\right)\right)\circ\gamma$$

and by a direct computation we obtain

$$\frac{-(X_2u)^2X_1X_1u + X_1uX_2u(X_1X_2u + X_2X_1u) - (X_1u)^2X_2X_2u}{|\nabla_0 u|^4}(X_1u, X_2u)\circ\gamma.$$

So

$$\begin{aligned}
\vec{k} &= \left(\frac{\mathcal{L}_\infty u}{|\nabla_0 u|^2} - \mathcal{L}u\right)\frac{(X_1u, X_2u)}{|\nabla_0 u|^2}\circ\gamma \\
&= \mathcal{H}_0\pi(\nu_H)\circ\gamma = \mathcal{H}_0\vec{n}.
\end{aligned}$$

Since $\vec{k} = k\vec{n}$ we have proved the following proposition.

Proposition 4.6. *Let S be a C^2 regular surface in $\mathbb{H}$, and let $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ be a curve in the Legendrian foliation of $S \setminus \Sigma(S)$. Then the curvature k of γ at $(\gamma_1(t), \gamma_2(t))$ equals the horizontal mean curvature $\mathcal{H}_0$ of S at $(\gamma_1(t), \gamma_2(t), \gamma_3(t))$.*

Chapter 5

Sobolev and BV Spaces and first variation of the perimeter in $\mathbb{H}$

In this chapter we introduce the notions of Sobolev spaces, BV functions and perimeter of a set with respect to the sub-Riemannian structure of $\mathbb{H}$. These are crucial notions in the sub-Riemannian geometric measure theory. At the end, we will conclude by giving an explicite formula for the first variation of the perimeter.

5.1 Sobolev spaces, perimeter measure and total variation

We begin by introducing the sub-Riemannian analog of the classical first-order Sobolev spaces. Let $\Omega \subset \mathbb{H}$ be an open set and $p \geq 1$. We define the Sobolev space $S^{1,p}(\Omega)$ as the set of functions $f \in L^p(\Omega)$ such that $\nabla_0 f$ exists in the sense of distributions with $|\nabla_0 f| \in L^p(\Omega)$. We endow it with the following norm

$$||f||_{S^{1,p}(\Omega)} = ||f||_{L^p(\Omega)} + ||\nabla_0 f||_{L^p(\Omega)},$$

where

$$||f||_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}} < \infty & \text{if } p < \infty \\ \inf\{M : |f(x)| \leq M < \infty \text{ a.e. } x \in \Omega\} & \text{if } p = \infty \end{cases}$$

and if $\nabla_0 f = (y_1, y_2)$, $||\nabla_0 f||_{L^p(\Omega)} = ||\sqrt{y_1^2 + y_2^2}||_{L^p(\Omega}$. The underlying measure in use here to compute these norms is the Haar measure in $\mathbb{H}$. As seen in Sections 3.1 and 3.2.1, it agrees with the exponential of the Lebesgue measure on $\mathfrak{h}$ or equivalently with the Hausdorff 4-measure associated with the Carnot-Carathéodory metric.

Using group convolution

$$f * g(x) = \int_{\mathbb{H}} f(xy^{-1})g(y)dy$$

and following the out-line of the Euclidean argument one can easily verify that $S^{1,p}(\mathbb{H})$ is the closure of $C_0^\infty(\mathbb{H})$ in the norm $||\cdot||_{S^{1,p}}$.

We may also define the local Sobolev space $S^{1,p}_{loc}(\Omega)$ by replacing L^p with L^p_{loc} in the above definition.

Let us now introduce the sub-Riemannian analogs of the classical notions of variation of a function and perimeter of a set.

Let $\Omega \subset \mathbb{H}$. We will denote by $\mathcal{F}(\Omega)$ the class of $\mathbb{R}^2$-valued functions $\phi = (\phi_1, \phi_2) : \Omega \to \mathbb{R}^2$ such that $\phi \in C_0^1(\Omega, \mathbb{R}^2)$ and $|\phi| \leq 1$.

Definition 5.1. Let $\Omega \subset \mathbb{H}$ be open and $f \in L^1_{loc}(\Omega)$. We define the *variation* of f in Ω as

$$\mathrm{Var}_{\mathbb{H}}(f, \Omega) = \sup_{\phi \in \mathcal{F}(\Omega)} \int_\Omega f(x)(X_1\phi_1 + X_2\phi_2)(x)dx.$$

Notice that if f is C^1, we can integrate by parts the above formula, so that

$$\mathrm{Var}_{\mathbb{H}}(f, \Omega) = \sup_{\phi \in \mathcal{F}(\Omega)} \int_\Omega (X_1 f(x))\,\phi_1(x) + (X_2 f(x))\,\phi_2(x)dx = \int_\Omega |\nabla_0 f|.$$

Remark 5.2. Let us denote by A the 2×3 smooth matrix whose rows are the coefficients of the vector fiels $X_j = \sum_{i=1}^3 a_{ji}\partial_{x_i}$:

$$A = \begin{pmatrix} 1 & 0 & -\frac{1}{2}x_2 \\ 0 & 1 & \frac{1}{2}x_1 \end{pmatrix}. \tag{5.1}$$

Then

$$A^T\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \frac{1}{2}(x_1\phi_2 - x_2\phi_1) \end{pmatrix},$$

so that $\mathrm{div}(A^T\phi) = \partial_{x_1}\phi_1 + \partial_{x_2}\phi_2 - \frac{1}{2}x_2\partial_{x_3}\phi_1 + \frac{1}{2}x_1\partial_{x_3}\phi_2 = X_1\phi_1 + X_2\phi_2$ for all $\phi \in \mathcal{F}(\Omega)$. Then the above definition is equivalent to

$$\begin{aligned} \mathrm{Var}_{\mathbb{H}}(f, \Omega) &= \sup_{\phi \in \mathcal{F}(\Omega)} \int_\Omega f(x)(X_1\phi_1 + X_2\phi_2)(x)dx \\ &= \sup_{\phi \in \mathcal{F}(\Omega)} \int_\Omega f(x)\mathrm{div}(A^T\phi)dx. \end{aligned}$$

Definition 5.3. The space $BV(\Omega)$ of functions with *bounded variation* in Ω is the space of all functions $f \in L^1(\Omega)$ such that

$$||f||_{BV(\Omega)} := ||f||_{L^1(\Omega)} + \mathrm{Var}_{\mathbb{H}}(f,\Omega) < \infty.$$

Clearly if $f \in S^{1,1}(\Omega)$, then $f \in L^1(\Omega)$ and there exists $\nabla_0 f$ in the sense of distributions, such that

$$\int_\Omega f(x)(X_1\phi_1 + X_2\phi_2)(x)dx = \int_\Omega \langle \nabla_0 f, \phi\rangle (x)dx,$$

which is bounded because $|\nabla_0 f| \in L^1(\Omega)$. Therefore $S^{1,1}(\Omega) \subset BV(\Omega)$.

In the next chapters we will need the following useful approximation results.

Lemma 5.4. *Let $\Omega \subset \mathbb{H}$ be an open and bounded set and let $p \geq 1$. For any $u \in S^{1,p}(\Omega)$ there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subset C^\infty(\Omega)$ such that $u_k \to u$ in $L^1(\Omega)$ as $k \to \infty$ and*

$$\lim_{k\to\infty} \int_\Omega |\nabla_0 u_k - \nabla_0 u|^p = 0.$$

Lemma 5.5. *Let $\Omega \subset \mathbb{H}$ be an open set. For any $u \in BV(\Omega)$ there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subset C^\infty(\Omega)$ such that $u_k \to u$ in $L^1(\Omega)$ as $k \to \infty$ and*

$$\lim_{k\to\infty} \mathrm{Var}_{\mathbb{H}}(u_k,\Omega) = \mathrm{Var}_{\mathbb{H}}(u,\Omega).$$

Definition 5.6. Let $E \subset \mathbb{H}$ be a measurable set and $\Omega \subset \mathbb{H}$ be an open set. The *horizontal perimeter* of E in Ω is given by

$$P_{\mathbb{H}}(E,\Omega) = \mathrm{Var}_{\mathbb{H}}(\chi_E,\Omega),$$

where χ_E denotes the characteristic function of E. A set with finite perimeter is called *Cacciopoli set.* For $\Omega = \mathbb{H}$ we let $P_{\mathbb{H}}(E,\Omega) = P_{\mathbb{H}}(E)$.

Next lemma is a consequence of the properties of the vector fields X_i.

Lemma 5.7. *For any $y \in \mathbb{H}$, $s > 0$, $\Omega \subset \mathbb{H}$ open and $E \subset \mathbb{H}$ Cacciopoli,*

$$P_{\mathbb{H}}(\delta_s E, \delta_s\Omega) = s^3 P_{\mathbb{H}}(E,\Omega) \quad \textit{and} \quad P_{\mathbb{H}}(L_y(E), L_y(\Omega)) = P_{\mathbb{H}}(E,\Omega),$$

where $L_y(x) = yx$ denotes the operation of left translation by y. In other words, the perimeter is homogeneous under dilations and invariant under left translation.

Proof. To prove the homogeneity property, recall that $\delta_s(x) = \delta_s(x_1, x_2, x_3) = (sx_1, sx_2, s^2x_3)$. Then $d\delta_s X_1 = sX_1$, $d\delta_s X_2 = sX_2$ and $|J(d\delta_s)| = s^4$. Therefore, with the change of variable $y = \delta_s(x)$, we obtain

$$\begin{aligned} P_{\mathbb{H}}(\delta_s E, \delta_s \Omega) &= \sup_{\phi \in \mathcal{F}(\delta_s \Omega)} \int_{\delta_s(E\cap\Omega)} (X_1\phi_1 + X_2\phi_2)(y)dy \\ &= \sup_{\phi \in \mathcal{F}(\delta_s \Omega)} s^3 \int_{E\cap\Omega} s(X_1\phi_1 + X_2\phi_2)(\delta_s x)dx \\ &= \sup_{\phi \in \mathcal{F}(\Omega)} s^3 \int_{E\cap\Omega} (X_1\phi_1 + X_2\phi_2)(x)dx \\ &= s^3 P_{\mathbb{H}}(E, \Omega). \end{aligned}$$

For the other equation, since the vector fields X_i are left invariant, then $dL_y X_i = X_i$, hence, using the change of variable $z = L_y x$, we compute

$$\begin{aligned} P_{\mathbb{H}}(L_y(E), L_y(\Omega)) &= \sup_{\phi \in \mathcal{F}(L_y(\Omega))} \int_{L_y(E\cap\Omega)} (X_1\phi_1 + X_2\phi_2)(z)dz \\ &= \sup_{\phi \in \mathcal{F}(L_y(\Omega))} \int_{E\cap\Omega} (dL_y X_1\phi_1 + dL_y X_2\phi_2)(x)dx \\ &= \sup_{\phi \in \mathcal{F}(\Omega)} \int_{E\cap\Omega} (X_1\phi_1 + X_2\phi_2)(x)dx = P_{\mathbb{H}}(E, \Omega). \end{aligned}$$

□

The perimeter of smooth sets has a very explicit integral representation in terms of the underlying Euclidean geometry.

Proposition 5.8. *Let E be a C^1 set, $d\sigma$ the surface measure on ∂E, and $\vec{n}$ the outer unit normal. Then*

$$P_{\mathbb{H}}(E, \Omega) = \int_{\partial E\cap\Omega} \left(\langle X_1, \vec{n}\rangle^2 + \langle X_2, \vec{n}\rangle^2\right)^{1/2} d\sigma(x).$$

Proof. In view of Definition 5.6 and Remark 5.2 and by an application of the divergence theorem 1.29, we have

$$\begin{aligned} P_{\mathbb{H}}(E, \Omega) &= \sup_{\phi \in \mathcal{F}(\Omega)} \int_{\Omega\cap E} \operatorname{div}(A^T\phi)dx = \sup_{\phi \in \mathcal{F}(\Omega)} \int_{\Omega\cap\partial E} \langle A^T\phi, \vec{n}\rangle d\sigma \\ &= \sup_{\phi \in \mathcal{F}(\Omega)} \int_{\Omega\cap\partial E} \langle \phi, A\vec{n}\rangle d\sigma = \int_{\Omega\cap\partial E} |A\vec{n}| d\sigma. \end{aligned}$$

Since $A\vec{n} = (\langle X_1, \vec{n}\rangle, \langle X_2, \vec{n}\rangle)^T$ we conclude that

$$P_{\mathbb{H}}(E,\Omega) = \int_{\Omega\cap\partial E} |A\vec{n}| d\sigma = \int_{\Omega\cap\partial E} \left(\sum_{i=1}^{2} \langle X_i, \vec{n}\rangle^2\right)^{1/2} d\sigma.$$

□

Corollary 5.9. *If $S = \{u = 0\}$ is a C^1 hypersurface in $\mathbb{H}$ which bounds an open set $E = \{u < 0\}$, then*

$$P_{\mathbb{H}}(E,\Omega) = \int_{S\cap\Omega} d\mu \tag{5.2}$$

for every domain $\Omega \subset \mathbb{H}$, where

$$d\mu = \frac{|\nabla_0 u|}{|\nabla u|} d\sigma = |\pi_H(\nu_1)| d\sigma = |A\vec{n}| d\sigma.$$

Proof. From the previous Lemma, $d\mu = |A\vec{n}|d\sigma$. As computed in Section 4.2 the Riemannian unitary normal to S is

$$\vec{n} = \frac{X_1 u X_1 + X_2 u X_2 + X_3 u X_3}{|X_1 u X_1 + X_2 u X_2 + X_3 u X_3|} = \nu_1,$$

thus $|\pi_H(\nu_1)| = |X_1 u X_1 + X_2 u X_2| / |X_1 u X_1 + X_2 u X_2 + X_3 u X_3| = |A\vec{n}|$, which is equal to $|\pi_H(\nu_1)| = |\nabla_0 u|/|\nabla u|$. □

Note that, since the value of $|\nabla_0 u|/|\nabla u|$ does not change if we choose a different parametrization of the surface, $S = \{v = 0\}$, then the perimeter will not change under a reparametrization of S.

5.1.1 Riemannian perimeter approximation

We may also obtain the representation formula (5.2) via the Riemannian approximation scheme introduced in Subsection 3.4.1, with the Heisenberg perimeter being the limit of the Riemannian perimeters, provided we rescale them.

Let us consider a C^1 parametrized surface $S = f(D)$ in $\mathbb{R}^3$, where $f = (f_1, f_2, f_3) : D \subset \mathbb{R}^2 \to \mathbb{R}^3$. Recall from elementary calculus that if $g : S \to \mathbb{R}$ is a continuous function then

$$\int_S g d\sigma = \int_D g \circ f(u,v) |\vec{N}(u,v)| du dv \tag{5.3}$$

where $\vec{N}(u,v) = f_u \times f_v$ is the Euclidean normal vector to S determined by the parameterization. Consider the sequence of Riemannian approximating spaces $(\mathbb{R}^3, g_L)$ introduced in Section 3.4.1. Recall that the vector fields X_1, X_2, and $\tilde{X}_3$ form an orthonormal frame with respect to the metrics g_L. Let us call this frame $\mathcal{F} = \{X_1, X_2, \tilde{X}_3\}$. In these coordinates the standard derivations in $\mathbb{R}^3$ are expressed by

$$\partial_{x_1} = X_1 + \frac{\sqrt{L}}{2} x_2 \tilde{X}_3 \qquad \partial_{x_2} = X_2 - \frac{\sqrt{L}}{2} x_1 \tilde{X}_3 \qquad \partial_{x_3} = \sqrt{L}\tilde{X}_3$$

so that the basis of the tangent bundle of S with respect to the frame $\mathcal{F}$ is

$$[\partial_u f]_{\mathcal{F}} = \left(\partial_u f_1, \partial_u f_2, \sqrt{L}\left[\partial_u f_3 - \frac{(\partial_u f_2 x_1 - \partial_u f_1 x_2}{2} \right] \right),$$
$$[\partial_v f]_{\mathcal{F}} = \left(\partial_v f_1, \partial_v f_2, \sqrt{L}\left[\partial_v f_3 - \frac{(\partial_v f_2 x_1 - \partial_v f_1 x_2}{2} \right] \right).$$

Let ν_L denote the Riemannian normal to S in $(\mathbb{R}^3, g_L)$, given by $[\nu_L]_{\mathcal{F}} = [\partial_u f]_{\mathcal{F}} \times [\partial_v f]_{\mathcal{F}}$, and let us call with C_L^T the 3×3 matrix whose rows are the coefficients of the vector fields X_1, X_2, and $\tilde{X}_3$:

$$C_L^T = \begin{pmatrix} 1 & 0 & -\frac{1}{2}x_2 \\ 0 & 1 & \frac{1}{2}x_1 \\ 0 & 0 & L^{-1/2} \end{pmatrix}.$$

Simple computations show that $\nu_L = \sqrt{L} C_L^T \vec{N}$. On the other hand, recall that for any vector $w_1, w_2 \in \mathbb{R}^3$ hold

$$||w_1 \wedge w_2||^2 = \det(W_{ij}),$$

where the entries of the matrix W are the scalar products $W_{ij} = w_i \cdot w_j$. In particular, we can express the norm of ν_L as

$$||\nu_L||_L^2 = \det \mathcal{G},$$

where $\mathcal{G} = (g_{ij})$ is the 2×2 matrix of the first fundamental form with entries $g_{ij} = \langle \partial_i f, \partial_j f \rangle_L$. Thus $\sqrt{\det \mathcal{G}} = ||\nu_L||_L = \sqrt{L}|C_L^T \vec{N}|$ and the change of variable formula (5.3) on $(\mathbb{R}^3, g_L)$ yields

$$\int_S d\sigma_L = \int_D ||\nu_L||_L du dv = \int_D \sqrt{L}|C_L^T \vec{N}| du dv. \tag{5.4}$$

Here we have denoted by $d\sigma_L$ the Riemannian surface area element on S induced by the metric g_L on $\mathbb{H}$.

Note that if we send L to ∞, the matrix C_L^T becomes the matrix A defined in (5.1) with a row of zeros added, thus

$$\begin{aligned}\lim_{L\to\infty}\frac{1}{\sqrt{L}}\int_S d\sigma_L &= \lim_{L\to\infty}\int_D |C_L^T\vec{N}|dudv \\ &= \int_D |A\vec{N}|dudv = \int_S \frac{|A\vec{N}|}{|\vec{N}|}d\sigma,\end{aligned}\tag{5.5}$$

where for the last equation we have used the change of variable formula (5.3).

In particular, if $S=\{u=0\}$ is given as in Corollary 5.9 as a level set, we have $|A\vec{N}|=|\nabla_0 u|$ and $|\vec{N}|=|\nabla u|$, thus

$$\lim_{L\to\infty}\frac{1}{\sqrt{L}}\int_S d\sigma_L = \int_S d\mu = \int_S \frac{|\nabla_0 u|}{|\nabla u|}d\sigma.\tag{5.6}$$

We conclude that the perimeter measure in $\mathbb{H}$ can be obtained as the limit as $L\to\infty$ of the Riemannian surface measures in S, computed with respect to g_L and rescaled by the factor $1/\sqrt{L}$.

5.2 Embedding theorems for the Sobolev and BV spaces

As seen in Chapter 2, the Sobolev inequalities are strong related to the isoperimetric one. For this reason, we want to study here an analog relation for $\mathbb{H}$.

The Sobolev conjugate of a number $p\geq 1$ is no more determined by the topological dimension of the space, but it turns out that the right dimension is the homogeneous one. Then the general Sobolev embedding theorem in $\mathbb{H}$ takes the following form.

Theorem 5.10 (Sobolev embedding theorem in the Heisenberg group)**.** *Let $p\geq 1$. Then*

$$S^{1,p}(\mathbb{H}) \hookrightarrow L^{\frac{4p}{4-p}}(\mathbb{H}) \text{ for } 1\leq p<4$$

and

$$S^{1,p}(\mathbb{H}) \hookrightarrow C^{0,1-\frac{4}{p}}(\mathbb{H}) \text{ for } p>4.$$

More precisely, there exist constants $C_p(\mathbb{H})<\infty$ for each $p\neq 4$ so that

$$||f||_{4p/(4-p)} \leq C_p(\mathbb{H})||\nabla_0 f||_p\tag{5.7}$$

for all $f \in S^{1,p}(\mathbb{H})$ if $1 \leq p < 4$, while if $p > 4$ and $f \in S^{1,p}(\mathbb{H})$, then there exists a representative $\tilde{f}$ of f satisfying the Hölder condition

$$|\tilde{f}(x) - \tilde{f}(y)| \leq C_p(\mathbb{H}) d(x,y)^{1-4/p} ||\nabla_0 f||_p$$

for all $x, y \in \mathbb{H}$.

As seen in Chapter 2, the case $p = 1$ is the most interesting from the point of view of the isoperimetric inequality. Consequently, we will restrict our attention to this case.

5.2.1 The geometric case (Sobolev-Gagliardo-Nirenberg inequality)

We prove Theorem 5.10 in the geometric case $p = 1$. To obtain it we will first prove a weak-type estimate.

Let $B(0,r) \subset \mathbb{H}$ be a metric ball[1], and let $q > 1$. The weak L^q space $L^{q,*}(B(0,r))$ consists of all measurable functions $f : B(0,r) \to \mathbb{R}$ for which the quantity

$$||f||^q_{L^{q,*}(B(0,r))} = \sup_{\lambda>0} \lambda^q |\{x \in B(0,r) | \ |f(x)| > \lambda\}|$$

is finite. (We have denoted by |A| the Haar measure of $A \subset \mathbb{H}$).

Recall the Hardy-Littlewood maximal operator defined in Section 1.3.5 with respect to Carnot-Carathéodory metric balls is

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy,$$

where $f \in L^1(\mathbb{H})$. By equation (3.21), $|B(x,2r)| = 2^4 |B(x,r)|$. This proves that the Carnot-Carathéodory metric d is doubling and we can apply a weak-type $(1,1)$ estimate (Theorem 1.49): there exists a constant $C > 0$ such that

$$|\{x \in \mathbb{H} : |Mf(x)| > \lambda\}| \leq \frac{C}{\lambda} ||f||_{L^1(\mathbb{H})} \tag{5.8}$$

for all $f \in L^1(\mathbb{H})$ and $\lambda > 0$.

The weak-type Sobolev-Gagliardo-Nirenberg inequality reads as follows.

[1] By homogeneity we could simply assume $r = 1$ and then rescale the resulting estimates. Similarly, by translation invariance the same estimates hold for balls centered at any point in $\mathbb{H}$.

Proposition 5.11. *There exists a constant $C > 0$ such that*

$$|\{x \in B(0,r) : |f(x)| > \lambda\}| \leq C\lambda^{-\frac{4}{3}} ||\nabla_0 f||^{4/3}_{L^1(B(0,r))}$$

for all Lipschitz functions f compactly supported in $B(0,r)$ and for all $\lambda > 0$.

Before giving the proof of this inequality, let us present a singular integral representation formula for an $f \in C_0^\infty(\mathbb{H})$. Such formula derives from the *sub-Riemannian Green's formula*, that we are going to introduce.

Let D be a bounded C^1 domain in $\mathbb{R}^3$ equipped with the approximant metric g_L. Let us recall from equation (3.41), that the respective Riemannian volume element is $\sqrt{L} dx_1 \wedge dx_2 \wedge dx_3$. Then Green's formula in $(\mathbb{R}^3, g_L)$ takes the form

$$\int_D (f\Delta_L g - g\Delta_L f)\sqrt{L} dx_1 \wedge dx_2 \wedge dx_3 = \int_{\partial D} (f\langle \nabla g, \nu_L\rangle_L - g\langle \nabla f, \nu_L\rangle_L) d\sigma_L$$

for each $f, g \in C^2(\bar{D})$, where $\Delta_L = X_1^2 + X_2^2 + \tilde{X}_3^2 = X_1^2 + X_2^2 + L^{-1}X_3^2$ denotes the Laplacian in $(\mathbb{R}^3, g_L)$ and ν_L is the outer unit normal to ∂D. Dividing both sides of the equation by $\sqrt{L}$ and using (5.6), we obtain the sub-Riemannian Green's formula

$$\int_D (f\Delta g - g\Delta f) dx = \int_{\partial D} (f\langle \nabla_0 g, \nu_H\rangle_1 - g\langle \nabla_0 f, \nu_H\rangle_1) d\mu \tag{5.9}$$

where $\Delta = X_1^2 + X_2^2$ is the Heisenberg laplacian introduced in Section 4.2.

Let us consider the case when ∂D is the boundary S_ϵ of the Heisenberg ball $B_\epsilon := \{x \in \mathbb{H} : ||x||_{\mathbb{H}} = \epsilon\}$. We may parameterize S_ϵ as $S_\epsilon = f(D)$, where $f = (f_1 + \mathbf{i} f_2, f_3)$ is given by

$$(f_1 + \mathbf{i} f_2)(\varphi, \theta) = \epsilon\sqrt{\cos\varphi}\exp(\mathbf{i}\theta)$$

and

$$f_3(\varphi, \theta) = \frac{1}{4}\epsilon^2 \sin\varphi,$$

with $D = \{(\varphi, \theta) \in \mathbb{R}^2 : -\pi/2 < \varphi < \pi/2, 0 \leq \theta < 2\pi\}$. Let us compute the perimeter measure $d\mu$ on S_ϵ as given in equations (5.5) and (5.6). By explicite computations we obtain

$$f_\varphi \times f_\theta = \left(-\frac{\epsilon^3}{4}\cos^{3/2}\varphi\cos\theta, -\frac{\epsilon^3}{4}\cos^{3/2}\varphi\sin\theta, -\frac{\epsilon^2}{2}\sin\varphi\right),$$

thus

$$|A\vec{n}| = |A(f_\varphi \times f_\theta)| = \frac{1}{4}\epsilon^3\sqrt{\cos\varphi}.$$

We therefore may conclude that

$$\begin{aligned} P_{\mathbb{H}}(B_\epsilon, \Omega) &= \int_{S_\epsilon \cap \Omega} d\mu \\ &= \int_{\{(\varphi,\theta)\in D:(\epsilon\sqrt{\cos\varphi}\exp(\mathbf{i}\theta),\frac{1}{4}\epsilon^2\sin\varphi)\in\Omega\}} \frac{1}{4}\epsilon^3\sqrt{\cos\varphi}\,d\varphi d\theta. \quad (5.10)\end{aligned}$$

We want to explicite (5.9) with $f \in C_0^\infty(\mathbb{H})$, $g = ||x||_{\mathbb{H}}^{-2}$ and $D = D_\epsilon = B_\epsilon^c \cap \mathrm{spt}(f)$. To simplify the notation in what follows we write $N(x) = ||x||_{\mathbb{H}}$ for the Heisenberg norm. Then $|\nabla_0 N(x)| = |z|/N(x)$, where $x = (z, x_3)$. By (4.7), $\nu_H = \nabla_0 u/|\nabla_0 u| = -\nabla_0 N/|\nabla_0 N|$, then

$$\langle \nabla_0 g, \nu_H\rangle_1 = \left\langle \nabla_0 g, -\frac{\nabla_0 N}{|\nabla_0 N|}\right\rangle_1 = 2N^{-3}|\nabla_0 N|. \quad (5.11)$$

Moreover, by a long but easy computation, one can show that

$$\Delta g = 0 \text{ in } \mathbb{H}\setminus\{0\}. \quad (5.12)$$

Thus, by equations (5.11) and (5.12), (5.9) becomes

$$-\int_{D_\epsilon} N^{-2}\Delta f = \int_{S_\epsilon} 2fN^{-3}|\nabla_0 N| + N^{-2}\left\langle \nabla_0 f, \frac{\nabla_0 N}{|\nabla_0 N|}\right\rangle_1 d\mu. \quad (5.13)$$

Lemma 5.12.

$$\int_{S_\epsilon} N^{-2}\left\langle \nabla_0 f, \frac{\nabla_0 N}{|\nabla_0 N|}\right\rangle_1 d\mu = o(\epsilon).$$

Proof. Using (5.10) in the desired integral gives

$$\int_{-\pi/2}^{\pi/2}\int_0^{2\pi} \epsilon^2 \left\langle \nabla_0 f, \frac{\nabla_0 N}{|\nabla_0 N|}\right\rangle_1 \Bigg|_{x=(\epsilon\sqrt{\cos\varphi}\exp(\mathbf{i}\theta),\frac{1}{4}\epsilon^2\sin\varphi)} \frac{\epsilon^3}{4}\sqrt{\cos\varphi}\,d\theta d\varphi.$$

□

Another use of (5.10) gives

$$\int_{S_\epsilon} 2N^{-3}|\nabla_0 N| d\mu = 8\pi.$$

In the first term on the right-hand side of (5.13), we add and substract $f(0)$ and observe that the term

$$\int_{S_\epsilon} 2(f - f(0))N^{-3}|\nabla_0 N| = o(1)$$

by uniform continuity of f in $\overline{B_\epsilon}$. In conclusion, passing to the limit as $\epsilon \to 0$ in (5.13), we deduce the identity

$$-\int_{\mathbb{H}} N^{-2}\Delta f = 8\pi f(0),$$

valid for all $f \in C_0^\infty(\mathbb{H})$. An application of the horizontal integration by parts formulas $\int_{\mathbb{H}}(X_i f)g = -\int_{\mathbb{H}} f(X_i g)$, for $i = 1, 2$, converts this into the following singular integral representation formula:

$$\begin{aligned} f(0) &= \frac{1}{8\pi}\int_{\mathbb{H}} \langle \nabla_0 f(y), \nabla_0(N^{-2})(y)\rangle_1 dy \\ &= -\frac{1}{4\pi}\int_{\mathbb{H}} \langle \nabla_0 f(y), \nabla_0 N(y)\rangle_1 N(y)^{-3} dy. \end{aligned} \tag{5.14}$$

Another form for (5.14) can be obtained by inserting the test function $f_x(y) = f(xy^{-1})$ and transforming the integrand via the isometry $y \mapsto xy^{-1}$:

$$\begin{aligned} f(x) &= \frac{1}{8\pi}\int_{\mathbb{H}} \langle \nabla_0 f(y), \nabla_0(N^{-2})(y^{-1}x)\rangle_1 dy \\ &= -\frac{1}{4\pi}\int_{\mathbb{H}} \langle \nabla_0 f(y), \nabla_0 N(y^{-1}x)\rangle_1 N(y^{-1}x)^{-3} dy. \end{aligned} \tag{5.15}$$

Now we are ready to prove the weak-type Sobolev-Gagliardo-Nirenberg inequality.

Proof of Proposition 5.11. Our starting point is the representation formula (5.15). The estimate $|\nabla_0 N| \leq 1$ yields

$$\begin{aligned} |f(x)| &\leq \frac{1}{4\pi}\int_{\mathbb{H}} |\nabla_0 f(y)|(d_{\mathbb{H}}(x,y))^{-3} dy \\ &\leq C\int_{\mathbb{H}} |\nabla_0 f(y)| d(x,y)^{-3} dy. \end{aligned} \tag{5.16}$$

Here the last inequality holds by the equivalence of the Carnot-Carathéodory metric d and $d_{\mathbb{H}}$ (see equation (3.23)). In view of (5.16), we see that in order to show the theorem we need only to prove that the fractional integration operator

$$g \to I_1 g(x) = \int_{\mathbb{H}} |g(y)| d(x,y)^{-3} dy,$$

satisfies a weak-type $(1, 4/3)$ estimate, i.e., there exists a positive constant C such that

$$|\{x \in B(0,r) : |I_1 g(x)| > \lambda\}| \leq C\lambda^{-\frac{4}{3}} ||g||_{L^1(\mathbb{H})}^{\frac{4}{3}} \tag{5.17}$$

for all $g \in L^1(\mathbb{H})$ with compact support in $B(0,r)$. Notice that $I_1 g = 0$ outside $B(0,r)$. Let $\epsilon > 0$ be a sufficiently small constant (to be chosen later) which depends on λ and $||g||_{L^1(\mathbb{H})}$ and write

$$I_1 g = I_1^1 g + I_1^2 g,$$

where

$$I_1^1 g(x) = \int_{B(x,\epsilon)} |g(y)| d(x,y)^{-3} dy$$

and

$$I_1^2 g(x) = \int_{B(0,r)\setminus B(x,\epsilon)} |g(y)| d(x,y)^{-3} dy.$$

By a dyadic decomposition argument and by the definition of the Hardy-Littlewood maximal operator, we get the existence of a constant $C_1 > 0$ such that

$$\begin{aligned} I_1^1(g) &\le \sum_{k=0}^{\infty} (2^{-k-1}\epsilon)^{-3} \int_{B(x,2^{-k}\epsilon)\setminus B(x,2^{-k-1}\epsilon)} |g(y)| dy \\ &\le C_1 \sum_{k=0}^{\infty} 2^{-k}\epsilon \left(\frac{1}{|B(x,2^{-k}\epsilon)|} \int_{B(x,2^{-k}\epsilon)} |g(y)| dy \right) \\ &\le 2C_1 \epsilon M g(x). \end{aligned} \tag{5.18}$$

We also have the trivial estimate

$$I_1^2 g(x) \le \epsilon^{-3} ||g||_{L^1(\mathbb{H})}. \tag{5.19}$$

Next, using (5.8) and (5.18), we have

$$\begin{aligned} |\{I_1 g \ge \lambda\}| &\le \left| \left\{ I_1^1 g \ge \frac{\lambda}{2} \right\} \right| + \left| \left\{ I_1^2 g \ge \frac{\lambda}{2} \right\} \right| \\ &\le \left| \left\{ M g > \frac{\lambda}{4C_1\epsilon} \right\} \right| + \left| \left\{ I_1^2 g \ge \frac{\lambda}{2} \right\} \right| \\ &\le \frac{4C_1\epsilon}{\lambda} ||g||_{L^1(\mathbb{H})} + \left| \left\{ I_1^2 g \ge \frac{\lambda}{2} \right\} \right|. \end{aligned} \tag{5.20}$$

Now, if $\lambda \le 2r^{-3}||g||_{L^1(\mathbb{H})}$, then $\lambda^{-4/3}||g||^{4/3}_{L^1(\mathbb{H})} \ge 2^{-4/3} r^4$, that is

$$|B(0,r)| \le C\lambda^{-4/3} ||g||^{4/3}_{L^1(\mathbb{H})}$$

and we obtain the desired estimate (5.17):

$$|\{x \in B(0,r) : |I_1 g(x)| > \lambda\}| \le |B(0,r)| \le C\lambda^{-4/3} ||g||^{4/3}_{L^1(\mathbb{H})}.$$

Hence, to estimate the second term on the right-hand side of (5.20) we may assume

$$\lambda > 2r^{-3}||g||_{L^1(\mathbb{H})}. \tag{5.21}$$

Choose $\epsilon = (\lambda^{-1}2||g||_{L^1(\mathbb{H})})^{1/3} < r$. By (5.19), $I_1^2 g(x) < \lambda/2$ whence $|\{I_1^2 > \lambda/2\}| = 0$ and

$$|\{I_1 g \geq \lambda\}| \leq \frac{4C_1}{\lambda}||g||_{L^1(\mathbb{H})}(\lambda^{-1}2||g||_{L^1(\mathbb{H})})^{1/3} \leq C\lambda^{-4/3}||g||^{4/3}_{L^1(\mathbb{H})}$$

which concludes the proof. □

Making use of the weak-type inequality, we prove the corresponding strong-type inequality.

Proposition 5.13. *There exists a constant $C_1(\mathbb{H}) < \infty$ so that*

$$||f||_{4/3} \leq C_1(\mathbb{H})||\nabla_0 f||_1$$

for all $f \in S^{1,1}(\mathbb{H})$.

Proof. By the approximation lemma 5.4, it suffices to prove the estimate for smooth, compactly supported functions f on $\mathbb{H}$. Moreover, if we prove the proposition for all nonnegative functions, we easily conclude the proof for every f. Indeed, $f = f_+ - f_-$, where f_+ and f_- denote the positive and negative part of f respectively. Then

$$\begin{aligned} ||f||_{\frac{4}{3}} &\leq ||f_+||_{\frac{4}{3}} + ||f_-||_{\frac{4}{3}} \leq C_1(\mathbb{H})(||\nabla_0 f_+||_1 + ||\nabla_0 f_-||_1) \\ &= C_1(\mathbb{H})\int_{\mathbb{H}} |\nabla_0 f_+| + |\nabla_0 f_-| dy = C_1(\mathbb{H})||\nabla_0 f||_1. \end{aligned}$$

Hence, let f be a nonnegative, smooth, compactly supported function, choose $R > 0$ so that $B(0,R)$ contains the support of f, and write

$$A_j = \{x \in B(0,R) : 2^j < f(x) \leq 2^{j+1}\}, \quad j \in \mathbb{Z}.$$

Let $f_j = \max\{0, \min\{f - 2^j, 2^j\}\}$, i.e.

$$f_j(x) = \begin{cases} 2^j & \text{if } x \in \bigcup_{i>j} A_i \\ f - 2^j & \text{if } x \in A_j \\ 0 & \text{otherwise.} \end{cases}$$

We observe that $\nabla_0 f_j$ is supported on A_j and that if $x \in A_{j+1}$ then $f_j(x) = 2^j > 2^{j-1}$. Since f_j is Lipschitz we may apply the weak-type estimate in

Proposition 5.11 and

$$
\begin{aligned}
|A_{j+1}| &\leq |\{f_j > 2^{j-1}\}| \\
&\leq |\{I_1(|\nabla_0 f_j|) > C^{-1}2^{j-1}\}| \\
&\leq C\left(2^{-j}\int_{A_j} |\nabla_0 f_j|\right)^{4/3} = C\left(2^{-j}\int_{A_j} |\nabla_0 f|\right)^{4/3}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\int_{\mathbb{H}} |f|^{4/3} &= \sum_{j\in\mathbb{Z}} \int_{A_j} |f|^{4/3} \leq \sum_{j\in\mathbb{Z}} (2^{j+1})^{4/3}|A_j| \\
&\leq C\sum_{j\in\mathbb{Z}}\left(\int_{A_j} |\nabla_0 f|\right)^{4/3} \\
&\leq C\left(\sum_{j\in\mathbb{Z}}\int_{A_j} |\nabla_0 f|\right)^{4/3} = C\left(\int_{\mathbb{H}} |\nabla_0 f|\right)^{4/3},
\end{aligned}
$$

which completes the proof. □

The just proved Sobolev embedding theorem 5.13 holds for functions defined on all of $\mathbb{H}$. However, for future applications, we are interested in a similar relation which can be applied to functions defined only on a ball $B \subset \mathbb{H}$. It is easy to see that the desidered estimate cannot be exactly that of theorem 5.13; infact, consider for example a constant function f different from zero, then the inequality $||f||_{4/3,B} \leq C||\nabla_0 f||_{1,B}$ doesn't hold for any constant C. An appropriate analog to theorem 5.13 is the following *Poincaré inequality*

$$
\left(\frac{1}{|C^{-1}B|}\int_{C^{-1}B} |f(x) - f_B|^{\frac{4}{3}}dx\right)^{\frac{3}{4}} \leq Cr\left(\frac{1}{|B|}\int_B |\nabla_0 f(x)|dx\right) \tag{5.22}
$$

which holds for functions $f \in S^{1,1}(B)$ defined on a ball B with radius r, where C is independent of both f and B. Here $f_B = \frac{1}{|B|}\int_B f$ denotes the average of f on B, and $C^{-1}B$ is the ball concentric with B whose radius is C^{-1} times that of B.

Applying Hölder's inequality to both sides of (5.22), we obtain the following family of Poincaré-type inequalities:

$$\begin{aligned}
\frac{1}{|C^{-1}B|}\int_{C^{-1}B}|f(x)-f_B|dx &\leq \left(\frac{1}{|C^{-1}B|}\int_{C^{-1}B}|f(x)-f_B|^{4/3}dx\right)^{3/4} \\
&\leq Cr\left(\frac{1}{|B|}\int_B|\nabla_0 f(x)|dx\right) \\
&\leq Cr\frac{1}{|B|}|B|^{1-1/p}\left(\int_B|\nabla_0 f(x)|^p dx\right)^{1/p} \\
&= Cr\left(\frac{1}{|B|}\int_B|\nabla_0 f(x)|^p dx\right)^{1/p},
\end{aligned}$$

for each $1 \leq p < \infty$. Summing up, we obtain the so-called $(1,p)$-*Poincaré inequality*:

$$\frac{1}{|C^{-1}B|}\int_{C^{-1}B}|f(x)-f_B|dx \leq Cr\left(\frac{1}{|B|}\int_B|\nabla_0 f(x)|^p dx\right)^{1/p}. \tag{5.23}$$

5.2.2 Compactness of the embedding $BV \subset L^1$ on John domains

In this section we focus our attention to functions with bounded variation defined on a John domain.

Definition 5.14. A bounded, connected open set Ω in a metric space (X,d) is a *John domain* if there exists a point $x_0 \in \Omega$ (the *center* of the domain) and a constant $\delta > 0$ such that for any point $x \in \Omega$ there exists an arc length parameterized rectifiable path $\gamma : [0,L] \to \Omega$ so that $\gamma(0) = x$, $\gamma(L) = x_0$ and $d(\gamma(t), X \setminus \Omega) \geq \delta t$.

We are interested in John domains because we can state some theorems on them, which are crucial for proving the existence of minimizers for the isoperimetric inequality in $\mathbb{H}$. In Euclidean spaces every Lipschitz domain is a John domain. In the Heisenberg group it is more difficult to verify the John property.

Lemma 5.15. *For each $x \in \mathbb{H}$ and $R > 0$, both Carnot-Carathéodory ball $B(x,R)$ and gauge ball $B_{\mathbb{H}}(y,R) = \{x \in \mathbb{H} : d_{\mathbb{H}}(x,y) < R\}$ are John domains.*

Theorem 5.16 (Garofalo-Nhieu)**.** *Let $\Omega \subset \mathbb{H}$ be a John domain. The space $BV(\Omega)$ is compactly embedded in $L^1(\Omega)$.*

To prove Theorem 5.16 we need the following two lemmas.

Lemma 5.17. *Let $\Omega \subset \mathbb{H}$ be a John domain. Then there exists a constant $C_{1,\Omega} > 0$ so that*

$$\left(\frac{1}{|\Omega|}\int_\Omega |u-u_\Omega|^{4/3}\right)^{3/4} \leq C_{1,\Omega}\frac{\mathrm{diam}(\Omega)}{|\Omega|}\mathrm{Var}_{\mathbb{H}}(u,\Omega)$$

for all $u \in BV(\Omega)$. Here we have denoted by u_Ω the average of the function u over the domain Ω.

The second lemma is a relative isoperimetric inequality for John domains (similar to that one given in the euclidean case). We state it only in the case of metric balls.

Lemma 5.18. *Let $B \subset \mathbb{H}$ be a metric ball. Then there exists a constant $C > 0$ so that*

$$\min(|A|, |B \setminus A|)^{3/4} \leq CP_{\mathbb{H}}(A, B)$$

for any measurable set $A \subset B$.

Proof of Theorem 5.16. We have to show that $BV(\Omega) \subset L^1(\Omega)$ is compact for each John domain Ω. By the characterization of compact sets in a metric space, a set in a metric space is compact if and only if it is closed and totally bounded. Therefore, in view of the approximation Lemma 5.5 it is sufficient to prove that the set $S = \{u \in BV(\Omega) \cap C^\infty(\Omega) : ||u||_{BV} \leq 1\}$ is totally bounded. In other words, we have to prove that for each $\epsilon > 0$ there exist functions $u_1, \ldots, u_M \in S$ such that S is covered by the balls B_k with center in u_k and radius ϵ, where $k = 1, \ldots, M$.

Let $\epsilon > 0$ and let Ω_ϵ be an open set which approximates Ω in the sense that $\overline{\Omega_\epsilon} \subset \Omega$ and

$$C_P\frac{\mathrm{diam}(\Omega)}{|\Omega|^{1/4}}|\Omega \setminus \Omega_\epsilon|^{1/4} + \frac{|\Omega \setminus \Omega_\epsilon|}{|\Omega|} < \frac{\epsilon}{6}, \tag{5.24}$$

where C_P is a constant as in Lemma 5.17. For any $u \in S$, using Lemma 5.17 and Hölder's inequality, we have the estimate

$$\begin{aligned}
\int_{\Omega\setminus\Omega_\epsilon}|u| &\leq \int_{\Omega\setminus\Omega_\epsilon}|u-u_\Omega| + \int_{|\Omega\setminus\Omega_\epsilon|}|u_\Omega| \\
&\leq \left(\int_{\Omega\setminus\Omega_\epsilon} 1dx\right)^{1/4}\left(\int_\Omega |u-u_\Omega|^{4/3}\right)^{3/4} + |\Omega\setminus\Omega_\epsilon||u_\Omega| \\
&\leq \left(\int_\Omega |u-u_\Omega|^{4/3}\right)^{3/4}|\Omega\setminus\Omega_\epsilon|^{1/4} + |\Omega\setminus\Omega_\epsilon||u_\Omega| \\
&\leq C_P\frac{\mathrm{diam}(\Omega)}{|\Omega|^{1/4}}|\Omega\setminus\Omega_\epsilon|^{\frac{1}{4}}\mathrm{Var}_{\mathbb{H}}(u,\Omega) + \frac{|\Omega\setminus\Omega_\epsilon|}{|\Omega|}\int_\Omega |u|.
\end{aligned} \tag{5.25}$$

Since $u \in S$, (5.24) and (5.25) yield

$$\int_{\Omega\setminus\Omega_\epsilon} |u| < \frac{\epsilon}{6}. \tag{5.26}$$

We prove now that there exists a positive integer M so that for any sufficiently small $\delta > 0$ one can construct a family of balls $\{B_j\}$, $j = 1, \dots, N$ with the following properties:

1. the diameter of each B_j is less than or equal to $\delta\epsilon$,
2. $\Omega_\epsilon \subset \bigcup_{j=1}^N B_j \subset \Omega$,
3. $\sum_{j=1}^N \chi_{C_P B_j} \le M\chi_\Omega$.

To show these statements, let $r := \delta\epsilon$ be such that $0 < C_P r < \mathrm{dist}(\bar{\Omega}_\epsilon, \mathbb{H}\setminus\Omega)$. Let $\mathcal{S}$ be a set of points in Ω_ϵ such that the pairweise mutual distance between them is at least r and let $\mathcal{S}$ be maximal with respect to this property.

Firstly let us prove that $\mathcal{S}$ is finite: by definition the family of balls $\{B(x, r/2) : x \in \mathcal{S}\}$ is disjoint and $\bigcup_{x\in\mathcal{S}} B(x, r/2) \subset \Omega$. Thus, if $N = \sharp\mathcal{S}$,

$$N|B(0,1)|\left(\frac{r}{2}\right)^4 = \sum_{x\in\mathcal{S}} |B(x, \frac{r}{2})| \le |\Omega|$$

which proves that N is finite. Let us denote the balls in $\{B(x, r) : x \in \mathcal{S}\}$ as $B_1 = B(x_1, r), \dots, B_N = B(x_N, r)$. Again, by definition,

$$\Omega_\epsilon \subset \bigcup_{j=1}^N B_j \subset \Omega.$$

Now let us prove that there exists M independent of r such that

$$\sum_{j=1}^N \chi_{C_P B_j} \le M\chi_\Omega.$$

To this end, let $x \in \Omega$. Eventually by translation, we may suppose that x corresponds to the origin. Suppose that $0 \in C_P B_{j_1} \cap \ldots \cap C_P B_{j_M}$, then $C_P B_{j_k} \subset B(0, 2C_P r)$ for each $k = 1, \dots, M$ and

$$M|B(0, \frac{r}{2})| = \left|\bigcup_{k=1}^M B(x_{j_k}, \frac{r}{2})\right| \le \left|\bigcup_{k=1}^M B_j\right| \le \left|\bigcup_{k=1}^M C_P B_j\right| \le |B(0, 2C_P r)|.$$

Thus
$$M \leq \frac{|B(0, 2C_P r)|}{|B(0, r/2)|} = 2^8 C_P^4,$$
and we get (1), (2) and (3).

Let $\delta > 0$ so that
$$C_P M \delta < \frac{1}{6}. \tag{5.27}$$
Next we consider any function $v \in BV(\Omega) \cap C^\infty(\Omega)$ and, using (5.23), we estimate
$$\begin{aligned}
\int_{\Omega_\epsilon} |v| &\leq \sum_{j=1}^N \int_{B_j} |v| dx \leq \sum_{j=1}^N \int_{B_j} |v - v_{B_j}| dx + |B_j||v_{B_j}| \\
&\leq \sum_{j=1}^N \left(C_P \delta \epsilon \int_{C_P B_j} |\nabla_0 v| dx + \left| \int_{B_j} v \ dx \right| \right) \\
&\leq C_P K \delta \epsilon \mathrm{Var}_{\mathbb{H}}(v, \Omega) + \sum_{j=1}^N \left| \int_{B_j} v \ dx \right| \\
&\leq \frac{\epsilon}{6} \mathrm{Var}_{\mathbb{H}}(v, \Omega) + \sum_{j=1}^N \left| \int_{B_j} v \ dx \right|.
\end{aligned} \tag{5.28}$$
At this point we define a linear operator $T : BV(\Omega) \to \mathbb{R}^N$ as follows:
$$T(u) = \left(\int_{B_1} u \ dx, \ldots, \int_{B_N} u \ dx \right).$$
Notice that the image of the bounded set S under T is bounded in $\mathbb{R}^N$. Then there exist functions $u_1, \ldots, u_M \in S$ so that for any $u \in S$,
$$|Tu - Tu_j| = \sum_{k=1}^N \int_{B_k} |u - u_j| dx < \frac{\epsilon}{3} \tag{5.29}$$
for some $j \in \{1, \ldots, M\}$. Moreover,
$$||u - u_j||_{L^1(\Omega)} \leq \int_{\Omega_\epsilon} |u - u_j| dx + \int_{\Omega \setminus \Omega_\epsilon} |u - u_j| dx = I + II. \tag{5.30}$$
From (5.28) and (5.29) we have
$$\begin{aligned}
I &\leq \frac{\epsilon}{6} \mathrm{Var}_{\mathbb{H}}(u - u_j) + \sum_{k=1}^N \left| \int_{B_k} (u - u_j) dx \right| \\
&\leq \frac{\epsilon}{6} (||u||_{BV(\Omega)} + ||u_j||_{BV(\Omega)}) + \frac{\epsilon}{3} \leq \frac{2\epsilon}{3}.
\end{aligned} \tag{5.31}$$

For the second term in (5.30) we use (5.26) to estimate

$$II \leq \int_{\Omega \setminus \Omega_\epsilon} |u| dx + \int_{\Omega \setminus \Omega_\epsilon} |u_j| dx \leq \frac{\epsilon}{3}. \tag{5.32}$$

From (5.30), (5.31) and (5.32) we obtain that S is totally bounded. □

5.3 First variation of the perimeter

In this section we present two derivations of the first variation formula for the perimeter of surfaces in $\mathbb{H}$. First we consider parametric representations of the surface and variations which vanish in a neighborhood of the characteristic set. Next, we present an argument in which variations over full surface are allowed.

We start by introducing some useful notation: Let $\Omega \subset \mathbb{H}$ be a bounded region enclosed by a surface S, and consider variations Ω_t and S_t along a given vector field $Z \in C^1(\mathbb{H}, \mathbb{R}^3)$. We say that Ω is *perimeter stationary* if

$$\frac{d}{dt} P_{\mathbb{H}}(\Omega_t)\bigg|_{t=0} = 0 \quad \text{for all choices } Z \in C^1(\mathbb{H}, \mathbb{R}^3). \tag{5.33}$$

We say that Ω is *volume-preserving perimeter stationary* if

$$\frac{d}{dt} P_{\mathbb{H}}(\Omega_t)\bigg|_{t=0} = 0 \quad \forall Z \in C^1(\mathbb{H}, \mathbb{R}^3) \text{ s.t. } \frac{d}{dt}|\Omega_t|\bigg|_{t=0} 0.$$

Since a flow line of a vector field Z is a path whose velocity field lies in Z, if Z is tangent to S at every point, then variations along Z do not change $S = S_t$, hence for tangential variations condition (5.33) always holds.

5.3.1 Parametric surface and noncharacteristic variations

We compute the first variation of the perimeter wih respect to the Euclidean normal $\vec{n}$ and then with respect to a horizontal frame $\{X_1, X_2\}$. Throughout the section the emphasis is on explicit computations using the underlying Euclidean structure of $\mathbb{R}^3$.

Let $\Omega \subset \mathbb{R}^2$ be a domain, let $\epsilon > 0$, and let $\Xi : \Omega \times (-\epsilon, \epsilon) \to \mathbb{H}$ represent a flow $S_t = \Xi(\Omega, t)$ of noncharacteristic surface patches in $\mathbb{H}$, with $\Xi(\Omega, 0) = S$. Denote by A the 2×3 matrix of coefficients of the horizontal frame X_1, X_2 and recall that the horizontal perimeter of S_t may be calculated using Proposition 5.8. Our goal is to evaluate

$$\begin{aligned} \frac{d}{dt} P_{\mathbb{H}}(S_t)|_{t=0} &= \frac{d}{dt} \int_{S_t} |A\vec{n}| d\sigma \bigg|_{t=0} \\ &= \frac{d}{dt} \int_{\Omega} |A(\Xi_u \times \Xi_v)| du dv \bigg|_{t=0} \\ &= \int_{\Omega} \frac{1}{|AV|} \left(\left\langle AV, A\frac{dV}{dt} \right\rangle + \left\langle AV, \left(\frac{dA}{dt}\right) V \right\rangle \right) \bigg|_{t=0}, \end{aligned} \tag{5.34}$$

where we have set $V = \Xi_u \times \Xi_v$ and $\vec{n} = V/|V|$. Here the second equality follows from the change of variable formula. The underlying metric in use here is the Euclidean inner product and norm.

To simplify the notation we will always ignore the higher order terms in the Taylor expansion of Ξ, writing

$$\Xi(u, v, t) = \Xi(u, v, 0) + t\frac{d\Xi}{dt}(u, v, 0) =: X(u, v) + t\lambda\vec{n}(u, v, 0).$$

A simple computation yields $V = X_u \times X_v + t(X_u \times (\lambda\vec{n})_v + (\lambda\vec{n})_u \times X_v)$, and consequently

$$\frac{dV}{dt}\bigg|_{t=0} = \lambda(X_u \times \vec{n}_v + \vec{n}_u \times X_v) + \lambda_v X_u \times \vec{n} + \lambda_u \vec{n} \times X_v.$$

Recall the definition of the shape operator given in Definition 1.24 and denote by b_α^β the coefficients of its matrix in the basis given by X_u, X_v. Then

$$\vec{n}_\alpha = -b_\alpha^u X_u - b_\alpha^v X_v$$

for every $\alpha \in \{u, v\}$. Consequently $X_u \times \vec{n}_v = -b_v^v X_u \times X_v$ and $\vec{n}_u \times X_v = -b_u^u X_u \times X_v$. Endow S with the Riemannian metric induced by the Euclidean metric in $\mathbb{C} \times \mathbb{R}$. Thus we have

$$\frac{dV}{dt}\bigg|_{t=0} = -\lambda\mathcal{H} X_u \times X_v + \lambda_v X_u \times \vec{n} + \lambda_u \vec{n} \times X_v, \tag{5.35}$$

where $\mathcal{H} = b_u^u + b_v^v$ denotes the mean curvature of S.

Since

$$A|_\Xi = \begin{pmatrix} 1 & 0 & -\frac{1}{2}\Xi_2 \\ 0 & 1 & \frac{1}{2}\Xi_1 \end{pmatrix}$$

we find

$$\frac{dA}{dt}\bigg|_\Xi = \begin{pmatrix} 0 & 0 & -\frac{1}{2}\lambda n_2 \\ 0 & 0 & \frac{1}{2}\lambda n_1 \end{pmatrix}, \quad \text{with } \vec{n} = (n_1, n_2, n_3).$$

Thus

$$\frac{dA}{dt}V = \frac{1}{2}\lambda V_3(-n_2, n_1), \quad \text{where } V = (V_1, V_2, V_3). \tag{5.36}$$

By virtue of (5.34), (5.35) and (5.36) we obtain

$$\begin{aligned} \frac{d}{dt}P_{\mathbb{H}}(S_t)|_{t=0} = \int_\Omega \frac{1}{|AV|} \Big\langle AV, -\lambda\mathcal{H}A(X_u \times X_v) \\ +\lambda_v A(X_u \times \vec{n}) + \lambda_u A(\vec{n} \times X_v) + \frac{1}{2}\lambda V_3(-n_2, n_1)\Big|_{t=0} \Big\rangle \, dudv. \end{aligned} \tag{5.37}$$

An integration by parts in u and v gives the following expression for the first variation of the perimeter:

$$\begin{aligned} \int_\Omega \lambda \Bigg[-\mathcal{H}|AV| - \left(\frac{\langle AV, A(X_u \times \vec{n})\rangle}{|AV|} \right)_v - \left(\frac{\langle AV, A(X_v \times \vec{n})}{|AV|} \right)_u \\ + \frac{1}{2}\frac{V_3}{|AV|}\langle AV, (-n_2, n_1)\rangle \Bigg] \Bigg|_{t=0} dudv. \end{aligned} \tag{5.38}$$

In essence, (5.38) is the derivative of the perimeter functional in the direction $Y = \lambda\vec{n}$. Since variations along purely tangential directions are zero, then normal variations represent the complete “gradient”of the perimeter functional.

We now want to generalize the preceding to the case of general perturbations of the original surface S. Thus let us take a basis of $\mathbb{R}^3$ given by the standard left invariant basis of the horizontal bundle $X_1 = (1, 0, -\frac{1}{2}x_2)$ and $X_2 = (0, 1, \frac{1}{2}x_1)$ together with the vector $T = (\frac{1}{2}x_2, -\frac{1}{2}x_1, 1)$ (note that T differs from the standard left invariant vector field $X_3 = (0, 0, 1)$) and consider a general variation of the form

$$\Xi = \mathcal{X} + t(aX_1 + bX_2 + cT) + o(t),$$

where $\mathcal{X} = (x_1, x_2, x_3)$ is a parameterization of S. Our goal is still to evaluate (5.34), which remains unchanged. In order to compute $\Xi_u \times \Xi_v$ we make here all the calculations

$$
\begin{aligned}
\mathcal{X}_\alpha \times X_1 &= \left(-\frac{1}{2}x_2x_{2,\alpha}, \frac{1}{2}x_2x_{1,\alpha} + x_{3,\alpha}, -x_{2,\alpha}\right), \\
\mathcal{X}_\alpha \times X_2 &= \left(-x_{3,\alpha} + \frac{1}{2}x_1x_{2,\alpha}, -\frac{1}{2}x_1x_{1,\alpha}, x_{1,\alpha}\right), \\
X_{1,\alpha} \times \mathcal{X}_\beta &= \left(\frac{1}{2}x_{2,\alpha}x_{2,\beta}, -\frac{1}{2}x_{2\alpha}x_{1,\beta}, 0\right), \\
X_{2,\alpha} \times \mathcal{X}_\beta &= \left(-\frac{1}{2}x_{1,\alpha}x_{2,\beta}, \frac{1}{2}x_{1,\alpha}x_{1,\beta}, 0\right), \\
\mathcal{X}_\alpha \times T &= \left(x_{2,\alpha} + \frac{1}{2}x_1x_{3,\alpha}, \frac{1}{2}x_2x_{3,\alpha} - x_{1,\alpha}, -\frac{1}{2}x_1x_{1,\alpha} - \frac{1}{2}x_2x_{2,\alpha}\right), \\
\mathcal{X}_\beta \times T_\alpha &= \left(\frac{1}{2}x_{1,\alpha}x_{3,\beta}, \frac{1}{2}x_{2,\alpha}x_{3,\beta}, -\frac{1}{2}x_{1,\alpha}x_{1,\beta} - \frac{1}{2}x_{2,\alpha}x_{2,\beta}\right).
\end{aligned}
$$

Here $\alpha, \beta \in \{u, v\}$ and we have written $x_{i,\alpha} = \partial x_i/\partial\alpha$, etc. These in turn yield

$$
\begin{aligned}
A(\mathcal{X}_\alpha \times X_1) &= (0, \omega(\mathcal{X}_\alpha)), \\
A(\mathcal{X}_\alpha \times X_2) &= (-\omega(\mathcal{X}_\alpha), 0), \\
A(X_{1,\alpha} \times \mathcal{X}_\beta) &= \frac{1}{2}x_{2,\alpha}\mathbf{i}z_\beta, \\
A(X_{2,\alpha} \times \mathcal{X}_\beta) &= -\frac{1}{2}x_{1,\alpha}\mathbf{i}z_\beta, \\
A(\mathcal{X}_\alpha \times T) &= \frac{1}{4}(2\omega(\mathcal{X}_\alpha)z - (4 + |z|^2)\mathbf{i}z_\alpha) \\
A(\mathcal{X}_\beta \times T_\alpha) &= \frac{1}{4}(2\omega(\mathcal{X}_\beta)z_\alpha - \frac{1}{2}(|z|^2)_\alpha\mathbf{i}z_\beta),
\end{aligned} \tag{5.39}
$$

where we have let $z = x_1 + \mathbf{i}x_2$ and $\omega(\mathcal{X}_\alpha) = x_{3,\alpha} + \frac{1}{2}(x_2x_{1,\alpha} - x_1x_{2,\alpha})$.

Thanks to (5.39) we have

$$
\begin{aligned}
A\frac{d}{dt}(\Xi_u \times \Xi_v) &= a_v(0, \omega(\mathcal{X}_u)) - a_u(0, \omega(\mathcal{X}_v)) + \frac{1}{2}a\mathbf{i}(x_{2u}z_v - x_{2v}z_u) \\
&\quad + b_v(-\omega(\mathcal{X}_u), 0) - b_u(-\omega(\mathcal{X}_v), 0) + \frac{1}{2}b\mathbf{i}(x_{1v}z_u - x_{1u}z_v) \\
&\quad + c\left[\frac{1}{2}\omega(\mathcal{X}_u)z_v - \frac{1}{2}\omega(\mathcal{X}_v)z_u - \frac{1}{8}(|z|^2)_v\mathbf{i}z_u + \frac{1}{8}(|z|^2)_u\mathbf{i}z_v\right] \\
&\quad + c_v\left[(1 + \frac{1}{4}|z|^2)\mathbf{i}z_u + \frac{1}{2}(\omega(\mathcal{X}_u)z\right] - c_u\left[(1 + \frac{1}{4}|z|^2)\mathbf{i}z_v + \frac{1}{2}(\omega(\mathcal{X}_v)z\right].
\end{aligned} \tag{5.40}
$$

Set $\mathcal{F} = \{X_1, X_2\}$ the horizontal, orthonormal frame and express the horizontal normal ν_H in this frame as the coordinate vector

$$[\nu_H]_{\mathcal{F}} = \frac{A(\mathcal{X}_u \times \mathcal{X}_v)}{|A(\mathcal{X}_u \times \mathcal{X}_v)|} = (\nu_H^1, \nu_H^2). \tag{5.41}$$

Since

$$\begin{aligned} A(\mathcal{X}_u \times \mathcal{X}_v) &= \big(x_{2u}x_{3v} - x_{3u}x_{2v} - \frac{1}{2}x_2(x_{1u}x_{2v} - x_{2u}x_{1v}), \\ &\qquad x_{3u}x_{1v} - x_{1u}x_{3v} + \frac{1}{2}x_1(x_{1u}x_{2v} - x_{2u}x_{1v})\big) \\ &= \big(- x_{2v}\omega(\mathcal{X}_u) + x_{2u}\omega(\mathcal{X}_v)\,,\, x_{1v}\omega(\mathcal{X}_u) - x_{1u}\omega(\mathcal{X}_v)\big) \\ &= \mathbf{i}[\omega(\mathcal{X}_u)z_v - \omega(\mathcal{X}_v)z_u]. \end{aligned} \tag{5.42}$$

using (5.40) we have (after several cancellations)

$$\begin{aligned} \Big\langle [\nu_H]_{\mathcal{F}}, A\frac{d}{dt}(\Xi_u \times \Xi_v) \Big\rangle &= a\Big(\langle [\nu_H]_{\mathcal{F}}, (0, \omega(\mathcal{X}_v))\rangle_u - \langle [\nu_H]_{\mathcal{F}}, (0, \omega(\mathcal{X}_u))\rangle_v \Big) \\ &\quad + b\Big(\langle [\nu_H]_{\mathcal{F}}, (\omega(\mathcal{X}_u), 0)\rangle_v - \langle [\nu_H]_{\mathcal{F}}, (\omega(\mathcal{X}_v), 0)\rangle_u \Big) \\ &\quad + c\Big(\langle \partial_u [\nu_H]_{\mathcal{F}}, (1 + \frac{1}{4}|z|^2)\mathbf{i}z_v + \frac{1}{2}\omega(\mathcal{X}_v)z\rangle \\ &\qquad - \langle \partial_v [\nu_H]_{\mathcal{F}}, (1 + \frac{1}{4}|z|^2)\mathbf{i}z_u + \frac{1}{2}\omega(\mathcal{X}_u)z\rangle\Big). \end{aligned} \tag{5.43}$$

On the other hand,

$$\left(\frac{dA}{dt}\right)\mathcal{X}_u \times \mathcal{X}_v = \frac{1}{2}V_3(-Y_2, Y_1),$$

where we have set $V = \mathcal{X}_u \times \mathcal{X}_v$ and $Y = (Y_1, Y_2, Y_3) \in \mathbb{R}^3$ such that

$$Y_1 = \frac{d}{dt}\Xi_1 = a + \frac{1}{2}cx_2 \qquad Y_2 = \frac{d}{dt}\Xi_2 = b - \frac{1}{2}cx_1.$$

We have

$$(-Y_2, Y_1) = a(0,1) + b(-1,0) + \frac{1}{2}cz.$$

Combining the latter with (5.34) and (5.43) we finally obtain the first variation of the perimeter. With respect to every single direction it is:
Variation along X_1 ($a = 1, b = 0, c = 0$):

$$[\nu_H^2\omega(\mathcal{X}_v)]_u - [\nu_H^2\omega(\mathcal{X}_u)]_v + \frac{1}{2}V_3\nu_H^2. \tag{5.44}$$

Variation along X_2 $(a = 0, b = 1, c = 0)$:

$$[\nu_H^1 \omega(\mathcal{X}_u)]_v - [\nu_H^1 \omega(\mathcal{X}_v)]_u - \frac{1}{2} V_3 \nu_H^1. \tag{5.45}$$

Variation along X_3 $(a = 0, b = 0, c = 1)$:

$$\left\langle \partial_u [\nu_H^2]_{\mathcal{F}}, \left(1 + \frac{1}{4}|z|^2\right) \mathbf{i} z_v + \frac{1}{2}\omega(\mathcal{X}_v)] z \right\rangle - \left\langle \partial_v [\nu_H^2]_{\mathcal{F}}, \left(1 + \frac{1}{4}|z|^2\right) \mathbf{i} z_u + \frac{1}{2}\omega(\mathcal{X}_u)] z \right\rangle + \frac{1}{4} V_3 \left\langle [\nu_H]_{\mathcal{F}}, z \right\rangle. \tag{5.46}$$

At this point we restrict our attention to horizontal variations by setting $c = 0$. Given a horizontal variation, determined by the flow in direction $xX_1 + yX_2$, the corresponding variation can be dermined by the scalar product between the vectors (x, y) and (a, b), where (a, b) is chosen according to (5.44) and (5.45), i.e., it is given by

$$\left([\nu_H^2 \omega(\mathcal{X}_v)]_u - [\nu_H^2 \omega(\mathcal{X}_u)]_v + \frac{1}{2} V_3 \nu_H^2 ; [\nu_H^1 \omega(\mathcal{X}_u)]_v - [\nu_H^1 \omega(\mathcal{X}_v)]_u - \frac{1}{2} V_3 \nu_H^1 \right). \tag{5.47}$$

Now, since the Schwarz inequality gives

$$(x, y) \cdot (a, b) \le ||(x, y)||\,||(a, b)||,$$

the variation corresponding to a flow in the direction $aX_1 + bX_2$ corresponds to the maximum variation, among horizontal variations. We summarize the preceding computations in the following proposition.

Proposition 5.19. *The maximum variation of the perimeter among horizontal variations and outside the characteristic locus is obtained along the direction aX_1+bX_2, where (a, b) is chosen as in* (5.47)*, and is equal to $I+II$, where*

$$I = \left(\omega(\mathcal{X}_u)_v - \omega(\mathcal{X}_v)_u - \frac{1}{2} V_3 \right) \boldsymbol{i} [\nu_H]_{\mathcal{F}}$$

and

$$II = \omega(\mathcal{X}_u) \partial_v (\boldsymbol{i}[\nu_H]_{\mathcal{F}}) - \omega(\mathcal{X}_v) \partial_u (\boldsymbol{i}[\nu_H]_{\mathcal{F}}), \tag{5.48}$$

here $[\nu_H]_{\mathcal{F}}$ is as in (5.41)*. Moreover, the component corresponding to I is tangent to the surface.*

Proof. We are only left with the proof of the last statement. Since $[\nu_H]_{\mathcal{F}}$ is a unit vector, the factors I and II are orthogonal. The vector

$$-\nu_H^2 X_1 + \nu_H^1 X_2 = A^T \mathbf{i} \frac{A(\mathcal{X}_u \times \mathcal{X}_v)}{|A(\mathcal{X}_u \times \mathcal{X}_v)|}$$

is in the same direction of I, then we conclude by observing that

$$\langle \vec{n}, -\nu_H^2 X_1 + \nu_H^1 X_2 \rangle = \left\langle \frac{(\mathcal{X}_u \times \mathcal{X}_v)}{|(\mathcal{X}_u \times \mathcal{X}_v)|}, \frac{A^T \mathbf{i} A(\mathcal{X}_u \times \mathcal{X}_v)}{|A(\mathcal{X}_u \times \mathcal{X}_v)|} \right\rangle = 0.$$

□

Remark 5.20. Observe that if we can choose a parameterization of the surface so that z_u and z_v are orthonormal and use the fact that $V_3 = \langle \mathbf{i} z_u, z_v \rangle$, we can rewrite I more explicitly in the following form:

$$I = \left[\langle \mathbf{i} z_u, z_v \rangle - \frac{1}{2} V_3 \right] \mathbf{i}[\nu_H]_{\mathcal{F}} = \frac{1}{2} V_3 \mathbf{i}[\nu_H]_{\mathcal{F}}. \tag{5.49}$$

Since we just proved that the component of the variation corresponding to I is horizontal and *tangent* to the surface, it thus corresponds only to a reparameterization of the surface. In the following we ignore it and focus on the component II.

Proposition 5.21. *The non-tangential component of the maximal variation of the perimeter $P_{\mathbb{H}}$ occurs along the vector $Z = aX_1 + bX_2$, where*

$$(a, b) = -\mathcal{H}_0 |A(\mathcal{X}_u \times \mathcal{X}_v)| [\nu_H]_{\mathcal{F}},$$

i.e., $Z = -\mathcal{H}_0 \nu_H |\pi_H(\nu_1)|_1$ with π_H denoting the (Euclidean) orthogonal projection on the horizontal bundle.

Proof. To identify II, we consider the Legendrian foliation on the surface. Recall that this foliation is composed of horizontal curves $\tilde{\gamma}$ lying on the surface which are flow lines of the horizontal vector field

$$-\mathbf{i}[\nu_H]_{\mathcal{F}} \cdot \nabla_0 := \nu_H^2 X_1 - \nu_H^1 X_2.$$

Since we are using a parametric representation of the surface S, we consider a curve $\gamma = (u, v) : [0, L] \to \mathbb{R}^2$, such that $\tilde{\gamma}(s) = \mathcal{X}(\gamma(s)) = \mathcal{X}(u(s), v(s))$.

Note that

$$\frac{d}{ds} \pi_z \tilde{\gamma} = z_u u' + z_v v'.$$

On the other hand, by (5.42), and by the definition of the Legendrian foliation we also have

$$\frac{d}{ds} \pi_z \tilde{\gamma} = -\mathbf{i}[\nu_H]_{\mathcal{F}} = -\frac{\omega(\mathcal{X}_v) z_u - \omega(\mathcal{X}_u) z_v}{|\omega(\mathcal{X}_v) z_u - \omega(\mathcal{X}_u) z_v|}.$$

As a consequence,

$$u' = -\frac{\omega(\mathcal{X}_v)}{|\omega(\mathcal{X}_v)z_u - \omega(\mathcal{X}_u)z_v|}, \quad \text{and} \quad v' = \frac{\omega(\mathcal{X}_u)}{|\omega(\mathcal{X}_v)z_u - \omega(\mathcal{X}_u)z_v|}.$$

Now, by virtue of Proposition 4.6 we have that the curvature of $\pi_z\tilde{\gamma}$ is given by $\mathcal{H}_0[\nu_H]_{\mathcal{F}}$, hence

$$\begin{aligned} \mathcal{H}_0[\nu_H]_{\mathcal{F}} = \frac{d^2}{ds^2}\pi_z\tilde{\gamma} &= \frac{d}{ds}\left(-\mathbf{i}[\nu_H]_{\mathcal{F}}(u(s),v(s))\right) \\ &= -(\mathbf{i}[\nu_H]_{\mathcal{F}})_u u' - (\mathbf{i}[\nu_H]_{\mathcal{F}})_v v' \\ &= \frac{(\mathbf{i}[\nu_H]_{\mathcal{F}})_u\omega(\mathcal{X}_v) - (\mathbf{i}[\nu_H]_{\mathcal{F}})_v\omega(\mathcal{X}_u)}{|A(\mathcal{X}_u \times \mathcal{X}_v)|} \\ &= \frac{-II}{|A(\mathcal{X}_u \times \mathcal{X}_v)|}. \end{aligned} \tag{5.50}$$

□

Formula (5.50) provides an explicit representation of the horizontal mean curvature for noncharacteristic parametric surface.

Proposition 5.22. *Let $S \subset \mathbb{H}$ be a C^2 surface, which is parametrized by the map $\mathcal{X} : \Omega \to \mathbb{H}$. Then outside the characteristic set $\Sigma(S)$ one has*

$$\mathcal{H}_0[\nu_H]_{\mathcal{F}} = \frac{\omega(\mathcal{X}_v)\partial_u(\boldsymbol{i}[\nu_H]_{\mathcal{F}}) - \omega(\mathcal{X}_u)\partial_v(\boldsymbol{i}[\nu_H]_{\mathcal{F}})}{|A(\mathcal{X}_u \times \mathcal{X}_v)|}, \tag{5.51}$$

where $[\nu_H]_{\mathcal{F}}$ is given by (5.41) *and $\omega = dx_3 - \frac{1}{2}(x_1dx_2 - x_2dx_1)$.*

Remark 5.23. Since tangential variations result in reparametrizations of the surface, which do not modify the perimeter, it is important that we consider only the normal component of the maximal horizontal variation identified above, i.e.,

$$\begin{aligned} &\left\langle A^T\left(-\mathcal{H}_0|A(\mathcal{X}_u \times \mathcal{X}_v)|[\nu_H]_{\mathcal{F}} + \frac{1}{2}\langle \mathbf{i}z_u, z_v\rangle \mathbf{i}[\nu_H]_{\mathcal{F}}\right), \vec{n}\right\rangle \\ &\qquad = -\mathcal{H}_0\frac{|A(\mathcal{X}_u \times \mathcal{X}_v)|}{|(\mathcal{X}_u \times \mathcal{X}_v)|} = -\mathcal{H}_0|A\vec{n}| = -\mathcal{H}_0|\pi_H(\nu_1)|_1. \end{aligned} \tag{5.52}$$

5.3.2 General variations

In the next proposition we give an extension of the first variation formula which can be applied also to variations across the characteristic locus.

Proposition 5.24. *Let $S \subset \mathbb{H}$ be an oriented C^2 immersed surface with g_1-Riemannian normal ν_1 and horizontal normal ν_H. Suppose that U is a C^1 vector field with compact support on S, let $\phi_t(p) = \exp_p(tU)$ and let S_t be the surface $\phi_t(S)$. Then*

$$\frac{d}{dt}P_{\mathbb{H}}(S_t)\bigg|_{t=0} = \int_{S\setminus\Sigma(S)} [u(\mathrm{div}_S\nu_H) - \mathrm{div}_S(u(\nu_H)_{\mathrm{tang}})]\, d\sigma, \tag{5.53}$$

where $u = \langle U, \nu_1\rangle_1$. Moreover, if $\mathrm{div}_S\nu_H \in L^1(S, d\sigma)$, then

$$\frac{d}{dt}P_{\mathbb{H}}(S_t)\bigg|_{t=0} = \int_S u(\mathrm{div}_S\nu_H)d\sigma - \int_S \mathrm{div}_S(u(\nu_H)_{\mathrm{tang}})d\sigma, \tag{5.54}$$

where $u = \langle U, \nu_1\rangle_1$.

Proof. Let us denote by v_{tang} the tangential component of a vector v and by $d\sigma$ the Riemannian surface area element on S. Let N_H be the g_1 projection of ν_1 on the horizontal distribution, that is $N_H = \sum_{i=1}^2 \langle \nu_1, X_i\rangle_1 X_i$. Then N_H is the component of ν_1 in the direction of ν_H and $|N_H| = \langle \nu_1, \nu_H\rangle$. Recall that in view of Corollary 5.9, the perimeter measure is $d\mu = |\pi_H(\nu_1)|d\sigma = |N_H|d\sigma$. Let $d\sigma_t$ be the Riemannian surface area element on the surface S_t.

Let ν_1 be the unitary vector field normal to the surfaces S_t. Note that, since the differential of ϕ_t sends the tangent space to S at p in the tangent space to $\phi_t(S)$ at $\phi_t(p)$, the vector field ν_1 is such that $\nu_1(\phi_t(p)) = d\phi_t(\nu_1(p))$ and is therefore a C^1 vector field. By taking a partition of the unity we can extend this vector field defined a priori on the surfaces S_t to the whole $\mathbb{H}$. Let us still denote this extension by ν_1.

By Corollary 5.9 and using the Rimannian co-area formula 1.44 we obtain

$$\begin{aligned} P_{\mathbb{H}}(S_t) &= \int_{S_t} d\mu = \int_{S_t} |N_H| d\sigma_t \\ &= \int_S (|N_H| \circ \phi_t)|J_{\phi_t}| d\sigma \\ &= \int_{S\setminus\Sigma(S)} (|N_H| \circ \phi_t)|J_{\phi_t}| d\sigma \end{aligned}$$

where the last equality follows because the Riemannian surface measure of $\Sigma(S)$ is zero. This result is due to Derridj. We just observe here that $\Sigma(S)$ is nowhere dense, that is it doesn't admit proper open subsets. In fact, $\Sigma(S)$ is the set of points where the tangent space coincides with the horizontal plane. The horizontal plane, however, is not involutive, because it isn't closed under the commutator law. Then by Frobenius theorem 1.6, the distribution constituted by the horizontal planes cannot be the tangent space of a submanifold. Since these are already tangent to $\Sigma(S)$, $\Sigma(S)$ can't be a submanifold.

In the next computation, we use the divergence identity (1.24)

$$\frac{d}{dt}|J_{\phi_t}|\bigg|_{t=0} = \mathrm{div}_S U \tag{5.55}$$

and the standard formula

$$\mathrm{div}_S(fV) = f\mathrm{div}_S V + V_{\mathrm{tang}}(f) \tag{5.56}$$

for $f : S \to \mathbb{R}$ and a C^1 vector field V on S, computed in equation (1.20). Here, as before, we will use the decomposition $V = V_{\mathrm{norm}} + V_{\mathrm{tang}}$, where V_{norm} and V_{tang} denote the (Riemannian) normal and the tangential components to S of V respectively. Let us use equations (5.55) and (5.56) to compute the differential of $P_{\mathbb{H}}(S_t)$ with respect to t, then

$$\begin{aligned}
\frac{d}{dt}P_{\mathbb{H}}(S_t)\bigg|_{t=0} &= \int_{S\setminus\Sigma(S)} \frac{d(|N_H|\circ\phi_t)|J_{\phi_t}|}{dt}\bigg|_{t=0} d\sigma \\
&= \int_{S\setminus\Sigma(S)} \frac{d(|N_H|\circ\phi_t)}{dt}\bigg|_{t=0} |J_{\phi_t}||_{t=0} + (|N_H|\circ\phi_t)|_{t=0}\frac{d|J_{\phi_t}|}{dt}\bigg|_{t=0} d\sigma \\
&= \int_{S\setminus\Sigma(S)} \langle\nabla|N_H|, \frac{d\phi_t}{dt}\rangle_1\bigg|_{t=0} + |N_H|\mathrm{div}_S U d\sigma \\
&= \int_{S\setminus\Sigma(S)} U(|N_H|) + \mathrm{div}_S(|N_H|U) - U_{\mathrm{tang}}(|N_H|)d\sigma \\
&= \int_{S\setminus\Sigma(S)} U_{\mathrm{norm}}(|N_H|) + \mathrm{div}_S(|N_H|U_{\mathrm{norm}}) + \mathrm{div}_S(|N_H|U_{\mathrm{tang}})d\sigma
\end{aligned}$$

Since U_{tang} is tangent to the surface, we can apply the divergence theorem 1.29 and obtain that $\int_{S\setminus\Sigma(S)} \mathrm{div}_S(|N_H|U_{\mathrm{tang}}))d\sigma = \int_S \mathrm{div}_S(|N_H|U_{\mathrm{tang}}))d\sigma = 0$, because U is Lipschitz and has compact support in S. Hence

$$\begin{aligned}
\frac{d}{dt}P_{\mathbb{H}}(S_t)\bigg|_{t=0} &= \int_{S\setminus\Sigma(S)} U_{\mathrm{norm}}(|N_H|) + \mathrm{div}_S(|N_H|(U_{\mathrm{norm}}))d\sigma \\
&= \int_{S\setminus\Sigma(S)} U_{\mathrm{norm}}(|N_H|) + |N_H|\mathrm{div}_S(U_{\mathrm{norm}})d\sigma
\end{aligned} \tag{5.57}$$

Notice that $N_H = \nu_1 - \langle\nu_1, X_3\rangle_1 X_3$ and $|N_H| = \langle N_H, N_H\rangle_1/|N_H| = \langle N_H, \nu_H\rangle_1$. Then, for any C^1 vector field V, since the Levi-Civita connection is compatible with the metric (see equation (1.4)), one has

$$V(|N_H|) = \langle D_V N_H, \nu_H\rangle_1 + \langle N_H, D_V\nu_H\rangle_1 = \langle D_V N_H, \nu_H\rangle_1 .$$

The last equality follows because $|\nu_H| = 1$ and then

$$\langle N_H, D_V \nu_H \rangle_1 = |N_H| \langle \nu_H, D_V \nu_H \rangle_1 = \frac{1}{2} |N_H| V \langle \nu_H, \nu_H \rangle_1 = 0. \tag{5.58}$$

Continuing the calculation, by using the properties of the affine connection, we have

$$\begin{aligned} V(|N_H|) &= \langle D_V N_H, \nu_H \rangle_1 \\ &= \langle D_V \nu_1, \nu_H \rangle_1 - \langle \nu_1, X_3 \rangle_1 \langle D_V X_3, \nu_H \rangle_1 - V(\langle \nu_1, X_3 \rangle_1) \langle X_3, \nu_H \rangle_1 \\ &= \langle D_V \nu_1, \nu_H \rangle_1 - \langle \nu_1, X_3 \rangle_1 \langle D_V X_3, \nu_H \rangle_1 , \end{aligned} \tag{5.59}$$

where the last equality holds because X_3 is orthogonal to ν_H.

We now make use of the following formula, that we will prove in Lemma 5.27:

$$D_{U_{\text{norm}}} \nu_1 = -\nabla_S u, \tag{5.60}$$

where ∇_S is the gradient on S.

Let us write $\nu_1 = aX_1 + bX_2 + cX_3$, then by Definition 1.10 and equations (3.46) we have

$$\langle D_{\nu_1} X_3, X_3 \rangle_1 = a \langle \nabla_{X_1} X_3, X_3 \rangle + b \langle \nabla_{X_2} X_3, X_3 \rangle_1 + c \langle \nabla_{X_3} X_3, X_3 \rangle_1 = 0$$

and

$$\langle \nabla_{\nu_1} X_3, \nu_1 \rangle_1 = a \langle \nabla_{X_1} X_3, bX_2 \rangle_1 + b \langle \nabla_{X_2} X_3, aX_1 \rangle_1 = ab - ba = 0.$$

From these computations it follows that

$$\langle D_{U_{\text{norm}}} X_3, \nu_H \rangle_1 = 0. \tag{5.61}$$

Using Equations (5.59), (5.60) and (5.61) we obtain that

$$U_{\text{norm}}(|N_H|) = \langle D_{U_{\text{norm}}} \nu_1, \nu_H \rangle_1 - \langle \nu_1, X_3 \rangle_1 \langle D_{U_{\text{norm}}} X_3, \nu_H \rangle_1 = - \langle \nabla_S u, \nu_H \rangle_1 .$$

Thus the integrand in the last line of (5.57) is

$$\begin{aligned} &U_{\text{norm}}(|N_H|) + |N_H| \mathrm{div}_S(U_{\text{norm}}) \\ &\quad = \langle -\nabla_S u, \nu_H \rangle_1 + |N_H| \mathrm{div}_S(u\nu_1) \\ &\quad = -(\nu_H)_{\text{tang}}(u) + u |N_H| \mathrm{div}_S \nu_1 \\ &\quad = -\mathrm{div}_S(u(\nu_H)_{\text{tang}}) + u \mathrm{div}_S((\nu_H)_{\text{tang}}) + u(\mathrm{div}_S(|N_H|\nu_1) - (\nu_1)_{\text{tang}}(|N_H|)) \\ &\quad = -\mathrm{div}_S(u(\nu_H)_{\text{tang}}) + u \mathrm{div}_S \nu_H, \end{aligned} \tag{5.62}$$

where the last equality holds because $(\nu_H)_{\text{norm}} = \langle \nu_H, \nu_1 \rangle_1 \nu_1 = |N_H|\nu_1$.

Since $U_{\text{norm}}(|N_H|)+|N_H|\text{div}_S(U_{\text{norm}})$ is compactly supported and bounded on S (because it is continuous on a compact), it is in $L^1(S, d\sigma)$ and we obtain equation (5.53). Under the additional assumption $\text{div}_S \nu_H \in L^1(S, d\sigma)$, by linearity also $\text{div}_S(u(\nu_H)_{\text{tang}}) \in L^1(S, d\sigma)$. Then we obtain (5.54) and the proof is concluded. □

Remark 5.25. The hypothesis of Theorem 5.24 can be slightly weakened: we may consider compactly supported, bounded vector fields U on S such that $|N_H|\text{div}_S(U_{\text{norm}}) \in L^1(S)$ (so the final implication still holds), $|N_H|U_{\text{tang}}$ is Lipschitz continuous (so that we can apply the divergence theorem) and $\text{div}_S \nu_H \in L^1(S, ud\sigma)$.

This observation is important as it allows us to consider horizontal variations $U = a\nu_H$, with $a \in C^1_0(S)$, which correspond to $ud\sigma = ad\mu$; in this case all the assumptions hold except the last one, which, by equation (4.9), is essentially reduced to the requirement $\mathcal{H}_0 \in L^1(S, d\mu)$. In view of (4.14), the latter condition is always true for C^2 surfaces. We will use such variations in Proposition 6.7.

Remark 5.26. If U is compactly supported in $S \setminus \Sigma(S)$, by an application of the divergence theorem, equation (5.53) reduces to

$$\frac{d}{dt} P_{\mathbb{H}}(S_t) \bigg|_{t=0} = \int_S u(\text{div}_S \nu_H) d\sigma$$

with $u = \langle U, \nu_1 \rangle_1$.

With the following Lemma, which was pointed out to me by Prof. Ritoré, we prove formula (5.60) in a more general setting.

Lemma 5.27. *Let M be a Riemannian manifold, $S \subset M$ a hypersurface in M such that it is well defined its unit normal N, U a smooth vector field with compact support in M and $\{\phi_t\}_{t\in\mathbb{R}}$ the associated flow. Let us denote by $A(v) = -D_v N$ for $v \in T_pS$ the Weingarten operator. Then*

$$D_U N = -\nabla_S u - A(U^T), \tag{5.63}$$

where (by abuse of notation) N is the unit normal to the hypersurface $\phi_t(S)$ (an extension of the original one) and $u := \langle U, N \rangle|_S$.

Using the decomposition $D_U N = D_{U^T} N + D_{U^\perp} N$, the fact that $D_{U^T} N = -A(U^T)$ and (5.63) we obtain (5.60).

Proof. Equality (5.63) holds trivially in the interior of $\{U = 0\}$ and, by continuity of the formula, extends to the boundary of this set. So we only need to establish (5.63) at a point $p \in S$ where $U_p \neq 0$.

We note first that $\langle D_{U_p} N, N_p \rangle = 0$ since $|N| = 1$. Pick some $v \in T_p S$ and extend it to a vector field Z in a neghborhood of p which is invariant by the flow of U, that is $Z(\phi_t(q)) = d\phi_t(Z_q)$ for each q in the neighborhood of p. By the geometrical interpretation of the Lie bracket (see Proposition 1.7) this is equivalent to $[U, Z] = 0$. Moreover, the differential $d\phi_t$ sends the tangent space to S at p in the tangent space to $\phi_t(S)$ at $\phi_t(p)$. Then the restriction of Z to the integral curve $t \mapsto \phi_t(p)$, i.e. $Z(\phi_t(p)) = d\phi_t(v)$, is tangent to $\phi_t(S)$ and hence $\langle N, Z \rangle = 0$. We have

$$\begin{aligned}
\langle D_{U_p} N, v \rangle &= \langle D_{U_p} N, Z_p \rangle = U_p \langle N, Z \rangle - \langle N_p, D_{U_p} Z \rangle \\
&= -\langle N_p, D_{Z_p} U \rangle = -Z_p \langle N, U \rangle + \langle D_{Z_p} N, U_p \rangle \\
&= -\langle \nabla_S u, Z_p \rangle - \langle A(Z_p), U_p^T, \rangle \\
&= -\langle \nabla_S u, v \rangle - \langle A(U^T), v \rangle,
\end{aligned}$$

where in the last equality we have used that A is self-adjoint in $T_p S$. So (5.63) is proven. $\square$

Chapter 6

The Isoperimetric problem in the Sub-Riemannian Heisenberg group

The solution of the isoperimetric inequality in $\mathbb{H}$ is still an open question in mathematics: recent studies have picked out the "best" constant for the isoperimetric inequality and have proved the existence of minimizers, nevertheless, up to now, none has yet characterized the sets which realize the minimum for the inequality in the largest possible class of Cacciopoli sets.

This chapter is the core of this survey. We introduce the isoperimetric inequality in $\mathbb{H}$ in the setting of C^1 sets and give a proof which make use of the Sobolev inequality (5.7). Next we define the isoperimetric profile of $\mathbb{H}$ and present Pansu's 1982 conjecture. Finally we show a proof of the existence of an isoperimetric profile and descrive some of the existing literature on the isoperimetric problem.

6.1 The Isoperimetric Inequality in $\mathbb{H}$

The isoperimetric inequality in $\mathbb{H}$ with respect to the horizontal perimeter has first been proved by Pansu. We begin by stating it in the setting of C^1 sets.

Theorem 6.1 (Pansu's isoperimetric theorem in $\mathbb{H}$). *There exists a constant $C > 0$ so that*

$$|E|^{\frac{3}{4}} \leq C P_{\mathbb{H}}(E) \tag{6.1}$$

for any bounded open set $E \subset \mathbb{H}$ with C^1 boundary.

Let us focus our attention on the exponent in (6.1). Comparing it with equation (2.22) we note that the right dimension that is involved in the isoperimetric problem is the homogeneous dimension, which in the Heisenberg group is equal to 4. In the Euclidean case we didn't noticed it, since the topological and homogeneous dimension are the same in this case.

Now we present a proof of Theorem 6.1 which isn't the original one given by Pansu, but is based on the geometric Sobolev embedding $S^{1,1} \subset L^{4/3}$ viewed in Section 5.2.

Proof. Let us recall the geometric inequality (5.7): there exists a constant $C_1(\mathbb{H}) < \infty$ such that

$$||f||_{\frac{4}{3}} \leq C_1(\mathbb{H})||\nabla_0 f||_1$$

for each $f \in S^{1,1}(\mathbb{H})$. If we show the equivalence between this inequality and (6.1), the theorem will be proved.

Let $E \subset \mathbb{H}$ be a bounded, open, C^1 set, and let $R > 0$ be such that the closure $\bar{E}$ of E is contained in $B(0,R)$. Choose $\delta > 0$ such that $2\delta < \operatorname{dist}(\bar{E}, \partial B(0,R))$, where $\operatorname{dist}(\cdot, \bar{E})$ denotes the Euclidean distance from $\bar{E}$. Define the function $g_\delta(x) = 1 - \frac{\operatorname{dist}(x,\bar{E})}{\delta}$ and let $f_\delta(x)$ be its positive part, i.e.

$$f_\delta(x) = \begin{cases} 1 & \text{if } x \in \bar{E} \\ 1 - \frac{\operatorname{dist}(x,\bar{E})}{\delta} & \text{if } \operatorname{dist}(x,\bar{E}) < \delta \\ 0 & \text{otherwise.} \end{cases}$$

Notice that f_δ is an (Euclidean) Lipschitz function, indeed let A_δ be the intersection of $B(0,R)$ with an (Euclidean) tubular neighborhood of E of radius δ. Then it is sufficient to prove Lipschitz condition only for $x, y \in A_\delta$. In this case:

$$\begin{aligned} |f_\delta(x) - f_\delta(y)| &= \left|1 - \frac{\operatorname{dist}(x,\bar{E})}{\delta} - \left(1 - \frac{\operatorname{dist}(y,\bar{E})}{\delta}\right)\right| \\ &= \frac{1}{\delta}\left|\operatorname{dist}(x,\bar{E}) - \operatorname{dist}(y,\bar{E})\right| \leq \frac{1}{\delta}|x-y|, \end{aligned}$$

that is Lipschitz condition for f_δ with Lipschitz constant $1/\delta$.

Applying the weak-type Sobolev-Gagliardo-Nirenberg inequality (Proposition 5.11) to f_δ we obtain

$$\begin{aligned} |E|^{\frac{3}{4}} &\leq |\{x \in B(0,R) : f_\delta(x) > t\}|^{\frac{3}{4}} \\ &\leq \frac{C}{t}\int_{B(0,R)} |\nabla_0 f_\delta(y)| dy \\ &\leq \frac{C}{t}\int_{A_\delta} \left|\nabla_0\left(\frac{\operatorname{dist}(y,\bar{E})}{\delta}\right)\right| dy = \frac{C}{\delta t}\int_{B(0,R)} |\nabla_0 \operatorname{dist}(y,\bar{E})| dy \quad (6.2) \end{aligned}$$

for each $t < 1$. Letting $t \to 1$, by an application of the co-area formula 1.44, we obtain

$$|E|^{\frac{3}{4}} \leq \frac{C}{\delta} \int_0^{\delta} \int_{\{y \in B(0,R):\mathrm{dist}(y,\bar{E})=s\}} \frac{|\nabla_0 \mathrm{dist}(\cdot, \bar{E})|}{|\nabla \mathrm{dist}(\cdot, \bar{E})|} d\mathcal{H}^{n-1} ds,$$

where $d\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure with respect to the backgroud Euclidean metric. The proof is concluded once we let $\delta \to 0$ and apply Corollary 5.9. $\square$

6.2 The Isoperimetric Profile of $\mathbb{H}$ and Pansu's conjecture

We introduce in this section the "isoperimetric profile" of $\mathbb{H}$. This is a family of sets which realize the equality in the isoperimetric inequality. To present it we need to define the best isoperimetric constant of the Heisenberg group, which is the best constant $C_{\mathrm{iso}}(\mathbb{H})$ for which the isoperimetric inequality

$$\min\{|\Omega|^{3/4}, |\mathbb{H} \setminus \Omega|^{3/4}\} \leq C_{\mathrm{iso}}(\mathbb{H}) P_{\mathbb{H}}(\Omega) \tag{6.3}$$

holds for each Cacciopoli set in $\mathbb{H}$. We can define it as

$$C_{\mathrm{iso}}(\mathbb{H}) = \sup_{\Omega} \frac{\min\{|\Omega|^{3/4}, |\mathbb{H} \setminus \Omega|^{3/4}\}}{P_{\mathbb{H}}(\Omega)}, \tag{6.4}$$

where the supremmum is taken on all Cacciopoli subsets of the Heisenberg group; in a dual manner we may present it as

$$\Big(C_{\mathrm{iso}}(\mathbb{H})\Big)^{-1} = \inf\{P_{\mathbb{H}}(E) : E \subset \mathbb{H} \text{ is a bounded Cacciopoli set and } |E| = 1\}. \tag{6.5}$$

Definition 6.2. An *isoperimetric profile* for $\mathbb{H}$ is a family of bounded Cacciopoli sets $\Omega_{\mathrm{profile}} = \Omega_{\mathrm{profile}}(V)$, with $V > 0$, such that $|\Omega_{\mathrm{profile}}(V)| = V$ and

$$|\Omega_{\mathrm{profile}}|^{3/4} = C_{\mathrm{iso}}(\mathbb{H}) P_{\mathbb{H}}(\Omega_{\mathrm{profile}}).$$

The invariance and scaling properties of the Haar measure and perimeter measure clearly imply that the sets comprising the isoperimetric profile are closed under the operations of left translation and group dilation. Pansu conjectured that any set in the isoperimetric profile of $\mathbb{H}$ is, up to translation and dilation, a bubble set $\mathcal{B}(0, R)$. In Section 3.3 we have studied that $\mathcal{B}(0, R)$ is obtained by rotating around the x_3-axis a geodesic joining two points at height $\pm\pi R^2/2$. More precisely, these cylindrically simmetric surfaces have profile curve

$$x_3 = f_R(r) = \pm\frac{1}{4}\left(r\sqrt{R^2 - r^2} + R^2 \arccos\frac{r}{R}\right).$$

Setting $u(x) = f_R(|z|) - x_3$, where $x = (z, x_3)$, we compute the volume of $\mathcal{B}(0, R)$ in cylindrical coordinates as

$$|\mathcal{B}(0, R)| = 2\pi \int_0^R r2f(r)dr = \frac{3}{16}\pi^2 R^4 \tag{6.6}$$

and

$$P_{\mathbb{H}}(\mathcal{B}(0, R)) = \int_{\partial\mathcal{B}(0,R)} \frac{|\nabla_0 u|}{|\nabla u|} d\sigma = 2\int_{\partial\mathcal{B}(0,R)^+} \frac{|\nabla_0 u|}{|\nabla u|} d\sigma$$

where $\partial\mathcal{B}(0, R)^+$ denotes the half part of the surface over the $\mathbb{C}$ plane, corresponding to $x_3 > 0$. Applying the area formula 1.42 we then obtain

$$P_{\mathbb{H}}(\mathcal{B}(0, R)) = 2\int_{B(0,R)} |\nabla_0 u| = 4\pi \int_0^R r\sqrt{f'(r)^2 + \frac{r^2}{4}}dr = \frac{1}{2}\pi^2 R^3,$$

where $B(0, R)$ is the ball with center in 0 and radius R in the $\mathbb{C}$-plane and the second equality follows calculating $|\nabla_0 u| = \sqrt{f'(r)^2 + r^2/4}$.

We can therefore write explicitely the value of the Heisenberg isoperimetric constant and the isoperimetric profile, conjectured by Pansu:

Conjecture 6.3 (Pansu)**.**

$$C_{\text{iso}}(\mathbb{H}) = \frac{|\mathcal{B}(0, R)|^{3/4}}{P_{\mathbb{H}}(\mathcal{B}(0, R))} = \frac{3^{3/4}}{4\sqrt{\pi}} \tag{6.7}$$

for any R, and the equality in (6.3) *is obtained if and only if Ω is a bubble set.*

6.3 Existence of minimizers

In this section we establish the existence of an isoperimetric profile.

Theorem 6.4 (Leonardi-Rigot)**.** *The Heisenberg group $\mathbb{H}$ admits an isoperimetric profile. More precisely, for any $V > 0$, there exists a bounded set $\Omega \subset \mathbb{H}$ with finite perimeter so that $|\Omega| = V$ and*

$$|\Omega|^{3/4} = C_{\text{iso}}(\mathbb{H}) P_{\mathbb{H}}(\Omega).$$

Proof. The proof of this theorem follows classical lines: one considers a sequence of sets $\Omega_i \subset \mathbb{H}$ with $|\Omega_i| = 1$ whose isoperimetric ratios

$$C_i = \frac{|\Omega_i|^{3/4}}{P_{\mathbb{H}}(\Omega_i)} = \frac{1}{P_{\mathbb{H}}(\Omega_i)}$$

converge to $C_{\text{iso}}(\mathbb{H})$ as $i \to \infty$ and examines the existence and properties of a subconvergent limit. It sufficies to show that

1. the sequence (Ω_i) subconverges to a set Ω_∞, and

2. $|\Omega_\infty| = 1$.

Step (1) requires the following compactness theorem.

Theorem 6.5 (Garofalo-Nhieu). *Let (Ω_i) be a sequence of measurable sets so that*

$$\sup_i P_{\mathbb{H}}(\Omega_i) < \infty.$$

Then (Ω_i) subconverges in $L^1_{\text{loc}}(\mathbb{H})$ to a measurable set Ω_∞ with finite perimeter.

Proof. In view of Lemma 5.15, $\Omega = B(x, R)$ are John domains and we can apply Theorem 5.16, to conclude that $BV(\Omega)$ is compactly embedded in $L^1(\Omega)$. If (Ω_i) is a sequence of Caccioppoli sets as in the statement of Theorem 6.5, then the functions $f_i = \chi_{\Omega_i \cap \Omega}$ lie in $BV(\Omega)$ and

$$\begin{aligned} ||f_i||_{BV(\Omega)} &= ||f_i||_{L^1(\Omega)} + \text{Var}_{\mathbb{H}}(f_i, \Omega) \\ &\leq |B(x, R)| + P_{\mathbb{H}}(\Omega_i, \Omega) \leq M < \infty, \end{aligned}$$

where M is a suitable constant. In view of Theorem 5.16 there exists a subsequence of $\{f_i\}_i$ that converges in $L^1(\Omega)$ to an $f_\infty \in BV(\Omega)$. Then, considering pointwise convergence, $f_\infty = \chi_{\Omega_\infty}$ for some set $\Omega_\infty \subset \Omega$. The lower-semicontinuity of the perimeter functional with respect to L^1_{loc} convergence guaranties that $P_{\mathbb{H}}(\Omega_\infty) \leq M$. This completes the proof of Theorem 6.5. □

Let us prove step (2): we have to show that the limit set Ω_∞ has volume equal to 1. A priori, in fact, the sets Ω_i may become very thin, spread out, and in the limit lose volume at infinity. But this is not the case, because, as we are going to see, a fixed amount of volume must lie within a ball of radius 1. We need the following Lemma.

Lemma 6.6. *Let* $\omega_{\mathbb{H}} = |B(0,1)|$, *and let* A *be a bounded set with finite perimeter. Assume that* $m \in (0, \omega_{\mathbb{H}}/2)$ *is such that* $|A \cap B(x,1)| < m$ *for all* $x \in \mathbb{H}$. *Then there exists a constant* $c > 0$ *so that*

$$c\left(\frac{|A|}{P_{\mathbb{H}}(A)}\right)^4 \leq m. \tag{6.8}$$

Proof. Consider the points $x \in \mathbb{H}$ such that $|A \cap B(x, \frac{1}{2})| > 0$. Let $\mathcal{S}$ be a set of such points such that the pairwise mutual distance between them is at least $1/2$ and let $\mathcal{S}$ be maximal with this property. By definition of $\mathcal{S}$ the union of balls $B(x, 1/4)$ with $x \in \mathcal{S}$ is a disjoint union and is contained in the $1/2$-neighborhoods of A, $A_{1/2}$, then, since $|A_{1/2}| < \infty$,

$$\left|\bigcup_{x\in\mathcal{S}} B\left(x, \frac{1}{4}\right)\right| = \sum_{x\in\mathcal{S}}\left|B\left(x, \frac{1}{4}\right)\right| \leq |A_{\frac{1}{2}}|.$$

Hence $\mathcal{S}$ is a finite set and we denote by N its cardinality.

By the maximality of $\mathcal{S}$, $A \subset \cap_{x\in\mathcal{S}} B(x,1)$; in fact, by contradiction, if there exists $y \in A \setminus \cap_{x\in\mathcal{S}} B(x,1)$, then $\mathcal{S} \cup \{y\}$ were a set which properly contains $\mathcal{S}$ and with the same properties of $\mathcal{S}$; this contradicts the maximality of $\mathcal{S}$. Hence

$$\begin{aligned}
|A| &\leq \sum_{x\in\mathcal{S}} |A \cap B(x,1)| \\
&= \sum_{x\in\mathcal{S}} |A \cap B(x,1)|^{1/4} |A \cap B(x,1)|^{3/4} \\
&\leq m^{1/4} \sum_{x\in\mathcal{S}} |A \cap B(x,1)|^{3/4}.
\end{aligned}$$

By the relative isoperimetric inequality for balls (Lemma 5.18)

$$|A \cap B(x,1)|^{3/4} < C P_{\mathbb{H}}(A, B(x,1)),$$

(note that $\min\{|A \cap B(x,1)|, |B(x,1) \setminus A|\} = |A \cap B(x,1)| < m$). Hence

$$\begin{aligned}
|A| &\leq Cm^{1/4} \sum_{x\in\mathcal{S}} P_{\mathbb{H}}(A, B(x,1)) \\
&\leq Cm^{1/4} \sum_{x\in\mathcal{S}} P_{\mathbb{H}}(A) = CNm^{1/4} P_{\mathbb{H}}(A).
\end{aligned}$$

□

Using this lemma, we achieve the proof of Theorem 6.4. Recall that

$$\Big(C_{\text{iso}}(\mathbb{H})\Big)^{-1} = \inf\{P_{\mathbb{H}}(E) : E \subset \mathbb{H} \text{ is a bounded Cacciopoli set and } |E| = 1\}$$

and consider a minimizing sequence $\{\Omega_i\}$ satisfying $|\Omega_i| = 1$ and

$$P_{\mathbb{H}}(\Omega_i) \le C_{\text{iso}}(\mathbb{H})^{-1}\left(1 + \frac{1}{i}\right). \tag{6.9}$$

By Lemma 6.6 we say that there exist $x_i \in \Omega_i$ for $i \in \mathbb{N}$, such that

$$|\Omega_i \cap B(x_i, 1)| \ge m_0 \tag{6.10}$$

for some absolute constant $m_0 > 0$. Indeed, if no such constant existed, then for each $k \in \mathbb{N}$, there exists i_k such that $|\Omega_{i_k} \cap B(x, 1)| < 1/k$ for each $x \in \Omega_{i_k}$ and therefore for each $y \in \mathbb{H}$. So, by Lemma 6.6 there exists $c > 0$ such that

$$c\left(\frac{|\Omega_{i_k}|}{P_{\mathbb{H}}(\Omega_{i_k})}\right)^4 \le 1/k.$$

Hence

$$\frac{1}{k^{1/4}} \ge \frac{|\Omega_{i_k}|}{P_{\mathbb{H}}(\Omega_{i_k})} \ge \frac{|\Omega_{i_k}|}{C_{\text{iso}}^{-1}(\mathbb{H})\left(1 + \frac{1}{i_k}\right)} = C_{\text{iso}}(\mathbb{H})\frac{i_k}{i_k + 1} \ge \frac{C_{\text{iso}}(\mathbb{H})}{2}.$$

Letting $k \to \infty$ we reach the contradiction $C_{\text{iso}}(\mathbb{H}) = 0$.

Return to equation (6.10); right translating each Ω_i by x_i, we may assume that Ω_i contains the origin and that

$$|\Omega_i \cap B(0, 1)| \ge m_0. \tag{6.11}$$

Theorem 6.5 assures the existence of Ω_∞ with finite perimeter such that Ω_i subconverges to Ω_∞ in $L^1_{\text{loc}}(\mathbb{H})$. Lower semi-continuity of the perimeter and inequality (6.9) yield

$$P_{\mathbb{H}}(\Omega_\infty) \le \liminf_{i\to\infty} P_{\mathbb{H}}(\Omega_i) \le \liminf_{i\to\infty} C_{\text{iso}}(\mathbb{H})^{-1}\left(1 + \frac{1}{i}\right) = C_{\text{iso}}(\mathbb{H})^{-1}.$$

The choice of m_0 implies $m_0 \le \lim_{i\to\infty} |\Omega_i \cap B(0, 1)| = |\Omega_\infty \cap B(0, 1)|$ and lower semi-continuity gives

$$|\Omega_\infty| \le \liminf_{i\to\infty} |\Omega_i| \le 1. \tag{6.12}$$

Taking all of this together yields

$$m_0 \le |\Omega_\infty| \le 1.$$

We complete the proof by showing that Ω_∞ is essentially bounded, i.e. by proving that

(3) if $r \geq 2$ is such that $|\Omega_\infty \cap B(0,r)| < 1$ then $r \leq R_0 < \infty$.

Indeed, if we prove (3), then, by (6.12) and since (Ω_i) subconvergies to Ω_∞ in L^1_{loc}, we have

$$1 \geq |\Omega_\infty| \geq |\Omega_\infty \cap B(0,R_0)| = 1$$

and step (2) is proven.

To prove (3), let us introduce $m_i(\rho) = |\Omega_i \cap B(0,\rho)|$ and let $m_\infty(\rho) = |\Omega_\infty \cap B(0,R)|$. By assumption, $m_\infty(r) < 1$. Using the relation between the (local) perimeter and the rate of change of the (local) volume of a set (see Theorem 2.12)

$$m_i(\rho)^{3/4} \leq C_{\text{iso}}(\mathbb{H}) P_{\mathbb{H}}(\Omega_i \cap B(0,\rho)) \leq C_{\text{iso}}(\mathbb{H})(P_{\mathbb{H}}(\Omega_i, B(0,\rho)) + m_i'(\rho))$$

and

$$\begin{aligned}(1-m_i(\rho))^{3/4} &\leq C_{\text{iso}}(\mathbb{H}) P_{\mathbb{H}}(\Omega_i \cap \overline{B(0,\rho)}^c) \\ &\leq C_{\text{iso}}(\mathbb{H}) \left(P_{\mathbb{H}}(\Omega_i, \mathbb{H} \setminus \overline{B(0,\rho)}) + m_i'(\rho)\right)\end{aligned}$$

for almost every $\rho > 0$. Using (6.9) we deduce

$$\begin{aligned}m_i(\rho)^{3/4} + (1-m_i(\rho))^{3/4} &\leq C_{\text{iso}}(\mathbb{H}) \left(P_{\mathbb{H}}(\Omega_i) + 2m_i'(\rho)\right) \\ &\leq 1 + \frac{1}{i} + 2C_{\text{iso}}(\mathbb{H}) m_i'(\rho)\end{aligned} \tag{6.13}$$

for almost every $\rho > 0$.

Let us analyze the general function $\Phi(x) = x^{3/4} + (1-x)^{3/4} - 1$. Let $\epsilon = \min\{m_0, \frac{1-m_\infty(r)}{2}\}$. Since $m_\infty(r) < 1$ by assumption, then

$$m_\infty(r) < \frac{m_\infty(r)+1}{2}.$$

Hence, since $|\Omega_i|$ locally convergies to $|\Omega_\infty|$, for sufficiently large i we have

$$m_i(r) = |\Omega_i \cap B(0,r)| \leq \frac{m_\infty(r)+1}{2} \leq 1 - \epsilon.$$

Moreover, by choosing a greater i if necessary, we can suppose

$$\Phi(x) \geq \frac{1}{i} \quad \text{for all } x \in [\epsilon, 1-\epsilon].$$

Since

$$\epsilon \leq m_0 \leq m_i(1) \leq m_i(\rho) \leq m_i(r) \leq 1 - \epsilon$$

for all such i and all $1 \leq \rho \leq r$ (see (6.11)), we may rewrite the differential inequality (6.13) in the form

$$1 \leq C \frac{m_i'(\rho)}{\Phi(m_i(\rho)) - \frac{1}{i}}, \qquad \text{a.e. } \rho \in [1, r].$$

Integrating from $r/2$ to r gives

$$\begin{aligned} \frac{r}{2} &\leq C \int_{\frac{r}{2}}^{r} \frac{m_i'(\rho) d\rho}{\Phi(m_i(\rho)) - \frac{1}{i}} = C \int_{m_i(r/2)}^{m_i(r)} \frac{dx}{\Phi(x) - \frac{1}{i}} \\ &\leq C \int_{\epsilon}^{1-\epsilon} \frac{dx}{\Phi(x) - \frac{1}{i}} \longrightarrow C \int_0^1 \frac{dx}{\Phi(x)} < \infty \end{aligned}$$

as $i \to \infty$. Then $r \leq 2C \int_0^1 \Phi(x)^{-1} dx =: R_0$ as desired. $\square$

6.4 A C^2 isoperimetric profile has constant horizontal mean curvature

In this section we prove that if the isoperimetric profile of $\mathbb{H}$ is C^2 smooth then it necessarily has constant horizontal mean curvature away from the characteristic set. The proof is based on the first variation of the perimeter seen is Section 5.3, with an additional volume constraint. We refer to the paper [17].

Proposition 6.7. *Let $\Omega \subset \mathbb{H}$ be a bounded open set enclosed by an oriented C^2 immersed surface S with g_1-Riemannian normal ν_1. Denote by $d\mu$ the perimeter measure defined in Corollary 5.9, and by $\mathcal{H}_0$ the horizontal mean curvature of S. If S is volume-preserving and perimeter stationary, then it has constant horintal mean curvature in $S \setminus \Sigma(S)$ equal to*

$$\mathcal{H}_0 = \frac{\int_S \mathcal{H}_0 d\mu}{P_{\mathbb{H}}(S)}. \tag{6.14}$$

Proof. We use the notation introduced in Section 5.3.2 and consider variations $\Omega_t = \phi_t(\Omega)$ along a vector field U defined on S. The hyphotesis that S is volume-preserving and perimeter stationary means that

$$\frac{d}{dt} P_{\mathbb{H}}(S_t)\bigg|_{t=0} = \int_S u(\mathrm{div}_S \nu_H) d\sigma - \int_S \mathrm{div}_S(u(\nu_H)_{\mathrm{tang}}) d\sigma = 0 \tag{6.15}$$

for each U such that

$$\frac{d}{dt}|\Omega_t|\bigg|_{t=0} = \int_\Omega \frac{d}{dt}|J_{\phi_t}|\bigg|_{t=0} dV = \int_\Omega \mathrm{div}_S U dV = -\int_S \langle U, \nu_1 \rangle d\sigma = 0, \tag{6.16}$$

where we have used formula (5.55) for the calculation of $d|J_{\phi_t}|/dt$ and the divergence theorem.

Define $\tilde{\mathcal{H}}_0 := \int_S \mathcal{H}_0 d\mu / P_{\mathbb{H}}(S)$. By the fundamental lemma of the calculus of variations, we complete the proof if we show that

$$\int_{S\setminus\Sigma(S)} (\mathcal{H}_0 - \tilde{\mathcal{H}}_0) a d\mu = 0 \quad \text{for each } a \in C_0^\infty(S \setminus \Sigma(S)). \tag{6.17}$$

To each function $a \in C_0^\infty(S \setminus \Sigma(S))$ is associated a vector field U with compact support in $S\setminus\Sigma(S)$, by defining $U = a\nu_H$. Then by equations (6.15) and (6.16) and using Remark 5.26, we have

$$\frac{d}{dt} P_{\mathbb{H}}(S_t)\bigg|_{t=0} = \int_S u(\mathrm{div}_S \nu_H) d\sigma = 0$$

whenever

$$\frac{d}{dt}|\Omega_t|\bigg|_{t=0} = -\int_S \langle a\nu_H, \nu_1\rangle d\sigma = -\int_S a d\mu = 0.$$

We observe that $\mathcal{H}_0 = -\mathrm{div}_S \nu_H$. Indeed let e_1, e_2 be an orthonormal base of T_pS such that e_1, e_2, ν_1 is positive oriented. Calculating the divergence with respect to such basis, since $\nu_1 = |N_H|\nu_H + \langle \nu_1, X_3\rangle X_3$, we obtain

$$\mathcal{H}_0 = -\mathrm{div}\nu_H = -\mathrm{div}_S \nu_H - \langle D_{\nu_1}\nu_H, \nu_1\rangle = -\mathrm{div}_S \nu_H.$$

By this observation, we have that if $a \in C_0^\infty(S\setminus\Sigma(S))$ is such that $\int_S a d\mu = 0$ then

$$\int_{S\setminus\Sigma(S)} (\mathcal{H}_0 - \tilde{\mathcal{H}}_0) a d\mu = -\int_{S\setminus\Sigma(S)} \mathrm{div}_S \nu_H \, a \, d\mu - \int_{S\setminus\Sigma(S)} \tilde{\mathcal{H}}_0 \, a \, d\mu = 0$$

and (6.17) is verified in this case.

Take now a general function $a \in C_0^\infty(S \setminus \Sigma(S))$ and the associated vector field $U = a\nu_H$. Let $\{S_n\}_{n\in\mathbb{N}}$ be approximating open sets such that $S_n \subset S_{n+1} \subset S\setminus\Sigma(S)$ and $\bigcup_n S_n = S\setminus\Sigma(S)$ and take functions $\varphi_n \in C_0^\infty(S\setminus\Sigma(S))$ such that $\varphi_n = 1$ on S_n. Let n be such that $\mathrm{spt}(a) \subset S_n$ and define

$$\tilde{a} := a - \varphi_n \frac{\int_{S\setminus\Sigma(S)} a d\mu}{P_{\mathbb{H}}(S)}.$$

In this way we have $\int_S \tilde{a} d\mu = 0$, so that (6.17) holds for $\tilde{a}$. Hence

$$\begin{aligned} \int_{S\setminus\Sigma(S)} (\mathcal{H}_0 - \tilde{\mathcal{H}}_0) a d\mu &= \int_{S\setminus\Sigma(S)} (\mathcal{H}_0 - \tilde{\mathcal{H}}_0)\varphi_n \frac{\int_{S\setminus\Sigma(S)} a d\mu}{P_{\mathbb{H}}(S)} d\mu \\ &= \frac{\int_{S\setminus\Sigma(S)} a d\mu}{P_{\mathbb{H}}(S)} \int_{S\setminus\Sigma(S)} (\mathcal{H}_0 - \tilde{\mathcal{H}}_0)\varphi_n d\mu \end{aligned}$$

which tends to 0 as $n \to \infty$. This completes the proof of (6.17). $\square$

6.5 Recent results

We collect in this section some of the recent results towards the solution of the isoperimetric problem in $\mathbb{H}$. They are partial solutions, which prove Pansu's conjecture in some restricted class of sets and give some evidence to support Pansu's thesis. We don't give the proofs here, which can be found in the respective papers or, sketched, in [6]. Our goal is to cite the recent developments in the area, without pretend to be exhaustive, but with intent to put the basis for a future deeper study.

6.5.1 Existence and characterization of minimizers with additional symmetries

Danielli, Garofalo and Nhieu have given in [8] a solution of the isoperimetric problem in a special class of domains, having suitable symmetry and regularity properties.

Let us introduce the half-spaces $\mathbb{H}_+ = \{(z, x_3) \in \mathbb{H} : x_3 > 0\}$ and $\mathbb{H}_- = \{(z, x_3) : x_3 < 0\}$ and consider the collection

$$\mathcal{E} = \{E \subset \mathbb{H} : E \text{ satisfies properties (i) and (ii)}\}$$

where

(i) $|E \cap \mathbb{H}_+| = |E \cap \mathbb{H}_-|$, and

(ii) there exist $R > 0$, and functions $u, v : \overline{B_R} \to [0, \infty)$, with $u, v \in C^1(B_R) \cap C(\overline{B_R})$, $u = v = 0$ on ∂B_R, and such that

$$\begin{aligned} \partial E \cap \mathbb{H}_+ &= \{(z, x_3) \in \mathbb{H}_+ : |z| < R, x_3 = u(z)\} \\ \text{and} \quad \partial E \cap \mathbb{H}_- &= \{(z, x_3) \in \mathbb{H}_- : |z| < R, x_3 = -v(z)\}. \end{aligned}$$

Here $B_R = B((0, 0), R)$ denotes the ball of radius R in $\mathbb{R}^2$.

Note that the functions u and v which describe the upper and lower portions of a set $E \in \mathcal{E}$ may be different. Besides C^1 smoothness, and the fact that their common domain is a metric ball, no additional assumptions are made on them.

Theorem 6.8 (Danielli-Garofalo-Nhieu). *Let $V > 0$, and define $R > 0$ so that $V = |\mathcal{B}(0, R)|$ (see formula (6.6)). Then the variational problem*

$$\min_{E \in \mathcal{E} : |E| = V} P_{\mathbb{H}}(E)$$

has a unique solution in $\mathcal{E}$ given by the bubble set $\mathcal{B}(0, R)$.

6.5.2 The C^2 isoperimetric profile in $\mathbb{H}$

Ritoré and Rosales proved in [17] that the bubble sets are the isoperimetric minimizers among the sets with C^2 boundaries.

Theorem 6.9 (Ritoré-Rosales). *If Ω is an isoperimetric region in $\mathbb{H}$ which is bounded by a C^2 smooth surface S, then S is congruent (i.e. equivalent by the composition of a Heisenberg isometry and dilation) to the boundary of a bubble set.*

6.5.3 The convex isoperimetric profile of $\mathbb{H}$

Monti and Rickly have proved in [15] Pansu's conjecture among Euclidean convex sets.

Theorem 6.10 (Monti-Rickly). *If Ω is an (Euclidean) convex region in $\mathbb{H}$ which is an isoperimetric set, then Ω is congruent to a bubble set.*

Note that from the point of view of the regularity the assumption in Theorem 6.10 is stricktly weaker than that in Theorem 6.9, since it is required only that the sets are convex, without assuming any smoothness properties.

6.5.4 The isoperimetric profile in a reduced symmetry setting

We cite a very recent result due to Ritoré, that can be found in [16]. We denote by D a closed disk in the $\mathbb{C}$-plane centered in the origin and by C the relative vertical cylinder over D in $\mathbb{H}$.

Theorem 6.11 (Ritoré). *Let $E \subset \mathbb{H}$ be a bounded set with finite perimeter such that*

$$D \subset E \subset C.$$

Then its perimeter is greater or equal to that of the bubble set with the same volume as E. The equality holds only for bubble sets.

Notice that this new result generalizes the preceding Theorem 6.8, indeed any set E, for which properties (i) and (ii) of section 6.5.1 hold, satisfies the hyphothesis of Theorem 6.11 with $D = B_R$, but any smoothness assumption is required now.

Conclusions

In this survey we have collected, in a synthetical and possibly ordered way, the existent literature on the isoperimetric problem in $\mathbb{H}$. At the beginning we have approached the question in the Euclidean space, with the intent to give a motivation for the successive generalization and to introduce some of the classical notions, which we have transferred then on the Heisenberg group.

The most interesting study is developed in the second part of the thesis. Here we have introduced the first Heisenberg group and the geometric measure theory on it, reaching the limit of what is known about the isoperimetric problem in $\mathbb{H}$.

Nowadays we have a lot of solutions of the problem in several special cases. What we need now is something like the contribution realized by De Giorgi in 1954 in the Euclidean space: a kind of revolution which will allow us to go back from the most general case to those already known.

I declair herewith that I have written this work indipendently and by using the indicated resources.

Trento, July 7, 2011.

Elisa Paoli

Bibliography

[1] L. Ambrosio, *Corso introduttivo alla Teoria Geometrica della Misura ed alle Superfici Minime*, Appunti, Scuola Normale Superiore, Pisa, 2010.

[2] L. Ambrosio, N. Fusco, D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Mathematical Monographs, Oxford University Press, 2000.

[3] L. Ambrosio, P. Tilli, *Topics on Analysis in Metric Spaces*, vol. 25 in Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, 2004.

[4] V. Blåsjö, *The Evolution of the Isoperimetric Problem*, The American Mathematical Monthly, vol. 112, (June-July 2005), 526-566.

[5] W. M. Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry*, Pure and Applied Mathematics, Academic Press, 1975.

[6] L. Capogna, D. Danielli, S. D. Pauls, J. T. Tyson, *An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem*, Progress in Mathematics, Birkhäuser Verlag, 2007.

[7] I. Chavel, *Isoperimetric Inequalities, Differential geometric and analytic perspectives*, Cambridge Tracts in Mathematics vol. 145, Cambridge University Press, Cambridge, 2001.

[8] D. Danielli, N. Garofalo, D.-M. Nhieu, *A partial solution of the isoperimetric problem for the Heisenberg group*, Forum Mathematicum 20, 1 (2008), 99-143.

[9] E. De Giorgi, *Selected papers*, Springer, The Netherlands, 2006.

[10] M. P. Do Carmo, *Riemannian Geometry*, Mathematics: Theory & Applications, Birkhäuser, Boston, 1992.

[11] L. C. Evans, R. F. Gariepy, *Measure Theory and Fine Porperties of Functions*, Studies in Advanced Mathematics, CRC Press, Inc., 1992.

[12] B. C. Hall, *Lie groups, Lie algebras and representations : an elementary introduction*, Graduate Texts in Mathematics, Vol. 222, Springer-Verlag, 2003.

[13] J. Jost, *Riemannian Geometry and Geometric Analysis*, fifth edition, Springer, 2000.

[14] S. G. Krantz, H. R. Parks, *Geometric Integration Theory*, Birkhäuser, Boston, 2008.

[15] R. Monti, M. Rickly, *Convex isoperimetric sets in the Heisenberg group*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 8 (2009), no. 2, 391–415.

[16] M. Ritoré, *A proof by calibration of an isoperimetric inequality in the Heisenberg group* $\mathbb{H}^n$, Calc. Var., DOI 10.1007/s00526-011-0425-8.

[17] M. Ritoré, C. Rosales, *Area-stationary surfaces in the Heisenberg group* $\mathbb{H}^1$, Adv. Math. 219 (2008), no. 2, 633-671.

[18] M. Ritoré, C. Rosales, *Rotationally invariant hypersurfaces with constant mean curvature in the Heisenberg group* $\mathbb{H}^n$, J. Geom. Anal. 16, no. 4 (2006), 703-720.

[19] L. Simon, *Lectures on geometric measure theory*, Proceedings of the Centre of Mathematical Analysis, Volume 3, Australian National University Centre for Mathematical Analysis, 1983.

[20] F. W. Warner, *Foundations of differentiable manifolds and Lie groups*, Springer Verlag, USA, 1983.