Analytical solution for Sod shock tube

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This description shows the equations used in `sod_shock_exact.py`, *a script calculating the analytical solution of the Sod shock tube problem after a time* `t`. *Equations labelled here bear the same tags as in that script.*

The Sod shock tube contains two different regions separated by a thin barrier at $x = x_0$. Each region contains the same gas, but with different pressure and density.

The initial conditions for the left and right domains are

$$\begin{aligned} p_l &= 1\,, \quad p_r = 0.1 \\ \rho_l &= 1\,, \quad \rho_r = 0.125 \\ u_l &= 0\,, \quad u_r = 0 \end{aligned}$$

where p is pressure, ρ is mass density, and u is the gas velocity in the direction along the tube. At $t = 0$, the barrier is removed, and the two regions are allowed to mix.

When the barrier is removed, the tube is split into five domains.

- Left boundary (l): `pressure, density, and velocity are constant`
- Expansion fan (e): `pressure and density decreases`
- Contact area (2): `region where the initial regions meet`
- Shock (1): `shock front creates a discontinuity with the right boundary`
- Right boundary (r): `pressure, density, and velocity are constant`

The system is described with the one-dimensional Euler system of PDEs

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ \rho u \\ e\end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix}\rho u \\ \rho u^2 + p \\ (e+p)u\end{bmatrix} = 0\,,$$

where e is the internal energy.

We use the Equation Of State (EOS) to close the system of equations. The EOS is given by

$$p = \rho(\gamma - 1)e$$

with $\gamma = 1.4$. We define the constants

$$\Gamma = \frac{\gamma - 1}{\gamma + 1}, \quad \beta = \frac{\gamma - 1}{2\gamma},$$

which will be useful in our calculations.

Another equation needed before diving into the solution, is the equation for sound speed c. Without further ado,

$$c = \sqrt{\gamma \frac{p}{\rho}}.$$

First, the *shock* region is described by the Rankine–Hugoniot relations (wikipedia page). These relations can be simplified to

$$u_1 - u_2 = 0.$$

We can express the velocity in terms of pressure,

$$(p' - p_r)\sqrt{\frac{1-\Gamma}{\rho_r(p' + \Gamma p_r)}} - (p_l^\beta - p'^\beta)\sqrt{\frac{(1-\Gamma^2)p_l^{1/\gamma}}{\Gamma^2 \rho_l}} = 0,$$

and the p' that satisties this relation is in fact the pressure in the *shock* region. Furthermore,

$$\begin{aligned}
p_1 &= p', \\
\rho_1 &= \rho_r \frac{p_1 + \Gamma p_r}{p_r + \Gamma p_1}, \\
u_1 &= (p_1 - p_r)\sqrt{\frac{1-\Gamma}{\rho_r(p_1 + \Gamma p_r)}}.
\end{aligned} \tag{1}$$

Then, the physical quantities of the *contact* region are found by the pressure and velocity being continuous, and mass density follows from the adiabatic gas law. This gives

$$\begin{aligned}
u_2 &= u_1, \\
p_2 &= p_1, \\
\rho_2 &= \rho_l \left(\frac{p_2}{p_l}\right)^{1/\gamma}.
\end{aligned} \tag{2}$$

Expressions for *shock* and *contact* regions are inspired by Wikipedia's article on the Sod shock tube.

The *expansion fan* is solved according to Höfner's lecture notes (chapter 10). We calculate the region accordingly

$$
\begin{aligned}
u_e &= \frac{2}{\gamma+1}\left(c_l + \frac{x-x_0}{t}\right)\,, \\
p_e &= p_l\left(\frac{c_e}{c_l}\right)^{2\gamma/(\gamma-1)}\,, \\
\rho_e &= \gamma\frac{p_e}{c_e^2}\,.
\end{aligned}
\tag{e}
$$

The sound speed in the expansion fan is not constant, and found by

$$
c_e = c_l - (\gamma-1)\frac{u_e}{2}\,.
$$

Lastly, the boundaries of the different regions are calculated. There are four boundaries that need to be found. The boundaries between *left* boundary and *expansion fan* (le), *expansion fan* and *contact* region (e2), and between *contact* and *shock* region (21) are found using the same expressions as Höfner's lecture notes. The boundary between the *shock* region and the *right* boundary (1r) is inspired by Ibackus' github repository.

$$
\begin{aligned}
x_{le} &= x_0 - c_l t\,, && \text{(le)} \\
x_{e2} &= x_0 + (u_2 - c_2)t\,, && \text{(e2)} \\
x_{21} &= x_0 + u_2 t\,, && \text{(21)} \\
x_{1r} &= x_0 + wt\,. && \text{(1r)}
\end{aligned}
\tag{1}
$$

The factor w in equation (1r) is given by

$$
w = c_r\sqrt{1 + \frac{\gamma+1}{2\gamma}\left(\frac{p_2}{p_r} - 1\right)}\,.
$$