Learn to Rank with Soft Condorcet (CIFAR Image Example)

April 25, 2025

[2]:
```
# Add this line in Google Colab!
# !pip install POT
```

[3]:
```
import os
import requests

import numpy as np
import matplotlib.pyplot as plt
import tensorflow as tf
from tensorflow import keras as kr

from PIL import Image
from skimage.morphology import dilation,erosion,opening,closing,square,disk

import ot
from sklearn.metrics import pairwise_distances
from sklearn.metrics.pairwise import paired_distances

import seaborn as sns
import pandas as pd
```

```
2025-04-25 11:26:47.866936: I tensorflow/core/util/port.cc:153] oneDNN custom
operations are on. You may see slightly different numerical results due to
floating-point round-off errors from different computation orders. To turn them
off, set the environment variable `TF_ENABLE_ONEDNN_OPTS=0`.
2025-04-25 11:26:47.874029: E
external/local_xla/xla/stream_executor/cuda/cuda_fft.cc:477] Unable to register
cuFFT factory: Attempting to register factory for plugin cuFFT when one has
already been registered
WARNING: All log messages before absl::InitializeLog() is called are written to
STDERR
E0000 00:00:1745591207.882134   24429 cuda_dnn.cc:8310] Unable to register cuDNN
factory: Attempting to register factory for plugin cuDNN when one has already
been registered
E0000 00:00:1745591207.884603   24429 cuda_blas.cc:1418] Unable to register
cuBLAS factory: Attempting to register factory for plugin cuBLAS when one has
already been registered
2025-04-25 11:26:47.893435: I tensorflow/core/platform/cpu_feature_guard.cc:210]
```

```
This TensorFlow binary is optimized to use available CPU instructions in
performance-critical operations.
To enable the following instructions: AVX2 AVX_VNNI FMA, in other operations,
rebuild TensorFlow with the appropriate compiler flags.
```

1 Check if the folder Figures exists:

[5]:
```python
folder_name = 'Figures'
if not os.path.exists(folder_name):
    try:
        os.makedirs(folder_name)
        print(f"Folder '{folder_name}' created successfully.")
    except OSError as e:
            print(f"Error creating folder '{folder_name}': {e}")
else:
    print(f"Folder '{folder_name}' already exists.")
```

```
Folder 'Figures' already exists.
```

2 Read the CIFAR dataset

[7]:
```python
# Load the CIFAR-10 dataset
(x_train, y_train), (x_test, y_test) = kr.datasets.cifar10.load_data()

x_train = x_train/255
```

[8]:
```python
# Select the first Ntr training images;
Ntr = 100
Nval = 100

Xtr = x_train[0:Ntr]
Xval = x_train[Ntr:Ntr+Nval]

# Display the image
for i in range(5):
    plt.imshow(Xtr[i])
    plt.axis(False)
    plt.savefig(folder_name+'/img'+str(i)+'.png', bbox_inches='tight', dpi=300)
    # plt.show()
```

3 Define some lexicographical orderings

[10]:
```
# def h_RG(img):
#     return (img[...,0]-img[...,1])**2/(1.e-6+np.sum(img,2))

# def h_GB(img):
#     return (img[...,1]-img[...,2])**2/(1.e-6+np.sum(img,2))

# def h_BR(img):
#     return (img[...,2]-img[...,0])**2/(1.e-6+np.sum(img,2))

# rho_list = [h_RG, h_GB, h_BR]
# rho_list_names = ["h_RG", "h_GB", "h_BR"]

def LexRGB(img):
    return np.dot(img[...,:3], [255, 1, 1/255])

def LexGBR(img):
    return np.dot(img[...,:3], [1/255, 255, 1])

def LexBRG(img):
    return np.dot(img[...,:3], [1, 1/255, 255])
```

```python
rho_list = [LexRGB, LexGBR, LexBRG]
rho_list_names = ["Lex_RGB", "Lex_GBR", "Lex_BRG"]
```

4 Apply the reduced mappings to the images for the training phase.

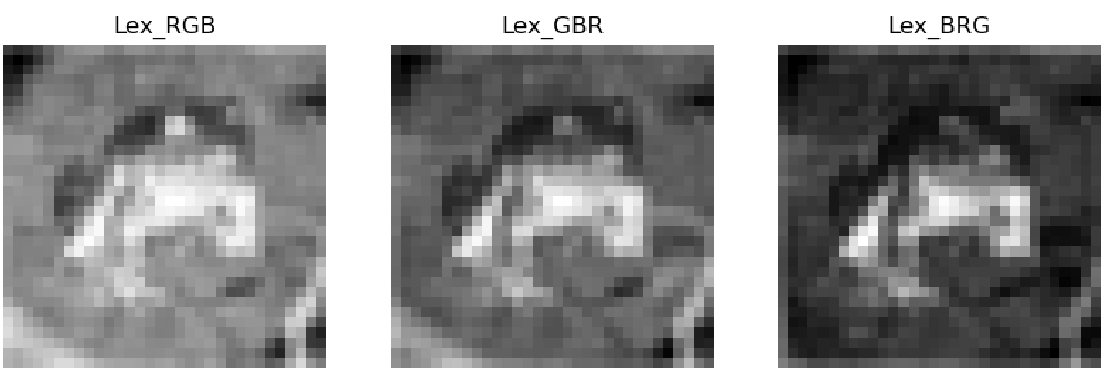
[12]:

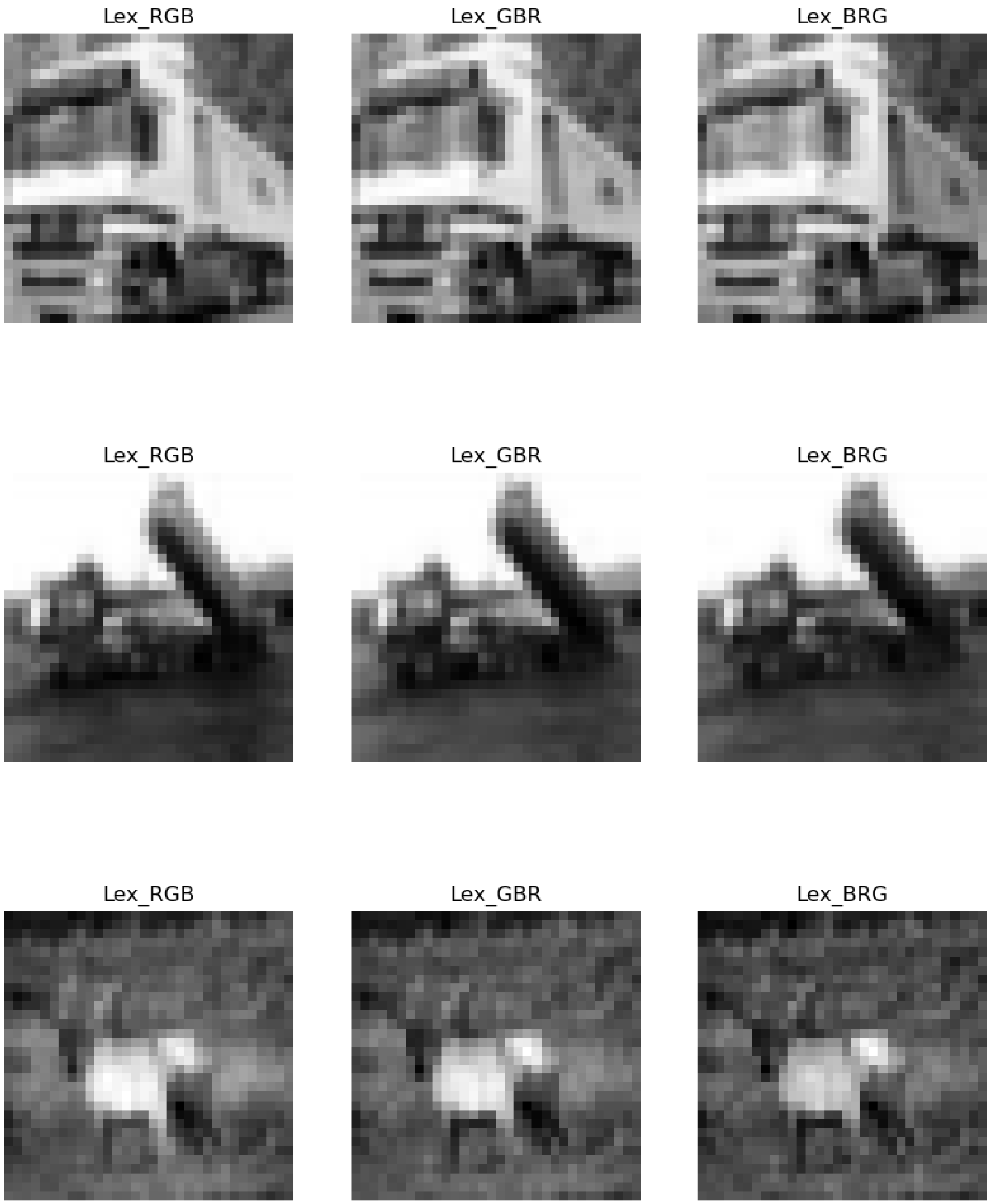
```python
Ytr = np.stack([np.stack([rho(x) for rho in rho_list],axis=2) for x in Xtr],axis=0)
Yval = np.stack([np.stack([rho(x) for rho in rho_list],axis=2) for x in Xval],axis=0)
```

5 Show some images

[14]:

```python
for i in range(5):
    fig, axs = plt.subplots(1, 3, figsize=(10, 5))
    for j in range(len(rho_list)):
        axs[j].imshow(Ytr[i][:,:,j],cmap="gray")
        axs[j].axis(False)
        axs[j].set_title(rho_list_names[j])
    plt.show()
```




Lex_RGB
Lex_GBR
Lex_BRG
Lex_RGB
Lex_GBR
Lex_BRG
Lex_RGB
Lex_GBR
Lex_BRG

6 Compute the (centred) Kendall coding from a list of color values

[16]:
```
def Kendall_coding(values):
  # Convert the values to gray and reshape to a column vector
  x = values.reshape(-1,1)
  # Compute the kendall coding
  return 1.0*(x<=x.T)-(x.T<=x)

def Average_Kendall_coding(list_values):
  return np.stack([Kendall_coding(list_values[:,i]) for i in range(list_values.
 ↪shape[1])],axis=2).mean(axis=2)
```

[17]:
```
Average_Kendall_coding(Ytr[0].reshape(-1,3))[:3,:3]
```

[17]:
```
array([[ 0.        , -1.        , -1.        ],
       [ 1.        ,  0.        ,  0.33333333],
       [ 1.        , -0.33333333,  0.        ]])
```

7 Define a MLP network that converts a color values to a gray-scale values

The network architecture is (3-64-1) with relu activation function in the hidden layer, no activation function and no bias in the output layer. However, because we are working with images, we implemented the using Conv2D layers with (1,1) kernels.

[19]:
```
model = kr.Sequential([
    kr.Input(shape=(3,)),
    kr.layers.Dense(64, activation='relu'),
    kr.layers.Dense(1, activation=None, use_bias = False),
    ])
```

```
I0000 00:00:1745591213.697543   24429 gpu_device.cc:2022] Created device
/job:localhost/replica:0/task:0/device:GPU:0 with 9170 MB memory:  -> device: 0,
```

```
name: NVIDIA GeForce RTX 4070 SUPER, pci bus id: 0000:01:00.0, compute
capability: 8.9
```

[20]:
```python
model.summary()
```

```
Model: "sequential"
```

Layer (type)	Output Shape	Param #
dense (Dense)	(None, 64)	256
dense_1 (Dense)	(None, 1)	64

```
Total params: 320 (1.25 KB)

Trainable params: 320 (1.25 KB)

Non-trainable params: 0 (0.00 B)
```

[21]:
```python
aux = Xtr[:5]

y_pred = model.predict(aux.reshape(-1,3)).reshape(aux.shape[0],aux.shape[1],aux.
 ↪shape[2])

fig, axs = plt.subplots(1, 5, figsize=(15, 5))

for i in range(5):
    axs[i].imshow(y_pred[i],cmap="gray")
    axs[i].axis(False)
plt.show()
```

```
160/160           0s 284us/step
WARNING: All log messages before absl::InitializeLog() is called are written to
STDERR
I0000 00:00:1745591214.220070   24586 service.cc:148] XLA service 0x7aad10003a00
initialized for platform CUDA (this does not guarantee that XLA will be used).
Devices:
I0000 00:00:1745591214.220089   24586 service.cc:156]   StreamExecutor device
(0): NVIDIA GeForce RTX 4070 SUPER, Compute Capability 8.9
2025-04-25 11:26:54.223651: I
tensorflow/compiler/mlir/tensorflow/utils/dump_mlir_util.cc:268] disabling MLIR
crash reproducer, set env var `MLIR_CRASH_REPRODUCER_DIRECTORY` to enable.
I0000 00:00:1745591214.229358   24586 cuda_dnn.cc:529] Loaded cuDNN version
```

```
90501
I0000 00:00:1745591214.254098   24586 device_compiler.h:188] Compiled cluster
using XLA!  This line is logged at most once for the lifetime of the process.
```

8 Define the Kemeny-Young loss

[23]:
```python
def Kemeny_Young_Loss(y_true, y_pred):
    tau = 1.0
    # Compute the Kendal Average Coding
    K = y_true.shape[1]
    Delta = []
    for i in range(K):
      x =  tf.reshape(y_true[:,i],(-1,1))
      Delta.append(tf.cast(x<=tf.transpose(x),tf.float32)-tf.cast(tf.transpose(x)<=x,tf.float32))
    Delta = tf.reduce_mean(tf.stack(Delta,axis=2),axis=2)
    # Compute Y
    Y = tf.math.sigmoid((y_pred - tf.transpose(y_pred)) / tau)
    return tf.reduce_mean(Delta * Y)
```

[24]:
```python
model.compile(optimizer="Adam", loss=Kemeny_Young_Loss)
# model.compile(optimizer=kr.optimizers.Adam(learning_rate=0.0001), loss=Kemeny_Young_Loss)
```

[25]:
```python
hist = model.fit(Xtr.reshape(-1,3), Ytr.reshape(-1,3), validation_data=(Xval.reshape(-1,3),Yval.reshape(-1,3)), batch_size=32*32, epochs=100,verbose=True)
```

```
Epoch 1/100
100/100              1s 2ms/step -
loss: -0.0535 - val_loss: -0.1626
Epoch 2/100
100/100              0s 561us/step -
loss: -0.2155 - val_loss: -0.2648
Epoch 3/100
100/100              0s 620us/step -
loss: -0.3035 - val_loss: -0.3119
Epoch 4/100
100/100              0s 1ms/step -
```

```
loss: -0.3390 - val_loss: -0.3376
Epoch 5/100
100/100             0s 577us/step -
loss: -0.3569 - val_loss: -0.3540
Epoch 6/100
100/100             0s 617us/step -
loss: -0.3678 - val_loss: -0.3652
Epoch 7/100
100/100             0s 564us/step -
loss: -0.3745 - val_loss: -0.3734
Epoch 8/100
100/100             0s 585us/step -
loss: -0.3809 - val_loss: -0.3798
Epoch 9/100
100/100             0s 558us/step -
loss: -0.3835 - val_loss: -0.3849
Epoch 10/100
100/100             0s 635us/step -
loss: -0.3866 - val_loss: -0.3891
Epoch 11/100
100/100             0s 1ms/step -
loss: -0.3888 - val_loss: -0.3927
Epoch 12/100
100/100             0s 770us/step -
loss: -0.3902 - val_loss: -0.3957
Epoch 13/100
100/100             0s 551us/step -
loss: -0.3919 - val_loss: -0.3984
Epoch 14/100
100/100             0s 605us/step -
loss: -0.3927 - val_loss: -0.4008
Epoch 15/100
100/100             0s 682us/step -
loss: -0.3940 - val_loss: -0.4029
Epoch 16/100
100/100             0s 710us/step -
loss: -0.3953 - val_loss: -0.4048
Epoch 17/100
100/100             0s 2ms/step -
loss: -0.3962 - val_loss: -0.4065
Epoch 18/100
100/100             0s 1ms/step -
loss: -0.3971 - val_loss: -0.4080
Epoch 19/100
100/100             0s 743us/step -
loss: -0.3971 - val_loss: -0.4095
Epoch 20/100
100/100             0s 2ms/step -
```

```
loss: -0.3981 - val_loss: -0.4108
Epoch 21/100
100/100            0s 709us/step -
loss: -0.3987 - val_loss: -0.4120
Epoch 22/100
100/100            0s 756us/step -
loss: -0.3993 - val_loss: -0.4132
Epoch 23/100
100/100            0s 1ms/step -
loss: -0.3997 - val_loss: -0.4143
Epoch 24/100
100/100            0s 767us/step -
loss: -0.3995 - val_loss: -0.4153
Epoch 25/100
100/100            0s 940us/step -
loss: -0.4006 - val_loss: -0.4162
Epoch 26/100
100/100            0s 2ms/step -
loss: -0.4009 - val_loss: -0.4171
Epoch 27/100
100/100            0s 599us/step -
loss: -0.4012 - val_loss: -0.4179
Epoch 28/100
100/100            0s 600us/step -
loss: -0.4014 - val_loss: -0.4187
Epoch 29/100
100/100            0s 976us/step -
loss: -0.4010 - val_loss: -0.4195
Epoch 30/100
100/100            0s 945us/step -
loss: -0.4022 - val_loss: -0.4202
Epoch 31/100
100/100            0s 1ms/step -
loss: -0.4021 - val_loss: -0.4209
Epoch 32/100
100/100            0s 795us/step -
loss: -0.4022 - val_loss: -0.4215
Epoch 33/100
100/100            0s 1ms/step -
loss: -0.4026 - val_loss: -0.4222
Epoch 34/100
100/100            0s 1ms/step -
loss: -0.4027 - val_loss: -0.4228
Epoch 35/100
100/100            0s 1ms/step -
loss: -0.4029 - val_loss: -0.4233
Epoch 36/100
100/100            0s 606us/step -
```

```
loss: -0.4027 - val_loss: -0.4239
Epoch 37/100
100/100            0s 636us/step -
loss: -0.4028 - val_loss: -0.4244
Epoch 38/100
100/100            0s 691us/step -
loss: -0.4028 - val_loss: -0.4249
Epoch 39/100
100/100            0s 654us/step -
loss: -0.4036 - val_loss: -0.4254
Epoch 40/100
100/100            0s 1ms/step -
loss: -0.4039 - val_loss: -0.4259
Epoch 41/100
100/100            0s 1ms/step -
loss: -0.4033 - val_loss: -0.4263
Epoch 42/100
100/100            0s 729us/step -
loss: -0.4038 - val_loss: -0.4267
Epoch 43/100
100/100            0s 583us/step -
loss: -0.4037 - val_loss: -0.4272
Epoch 44/100
100/100            0s 546us/step -
loss: -0.4040 - val_loss: -0.4276
Epoch 45/100
100/100            0s 564us/step -
loss: -0.4040 - val_loss: -0.4279
Epoch 46/100
100/100            0s 557us/step -
loss: -0.4042 - val_loss: -0.4283
Epoch 47/100
100/100            0s 567us/step -
loss: -0.4046 - val_loss: -0.4287
Epoch 48/100
100/100            0s 889us/step -
loss: -0.4047 - val_loss: -0.4291
Epoch 49/100
100/100            0s 630us/step -
loss: -0.4050 - val_loss: -0.4294
Epoch 50/100
100/100            0s 1ms/step -
loss: -0.4051 - val_loss: -0.4297
Epoch 51/100
100/100            0s 869us/step -
loss: -0.4048 - val_loss: -0.4301
Epoch 52/100
100/100            0s 702us/step -
```

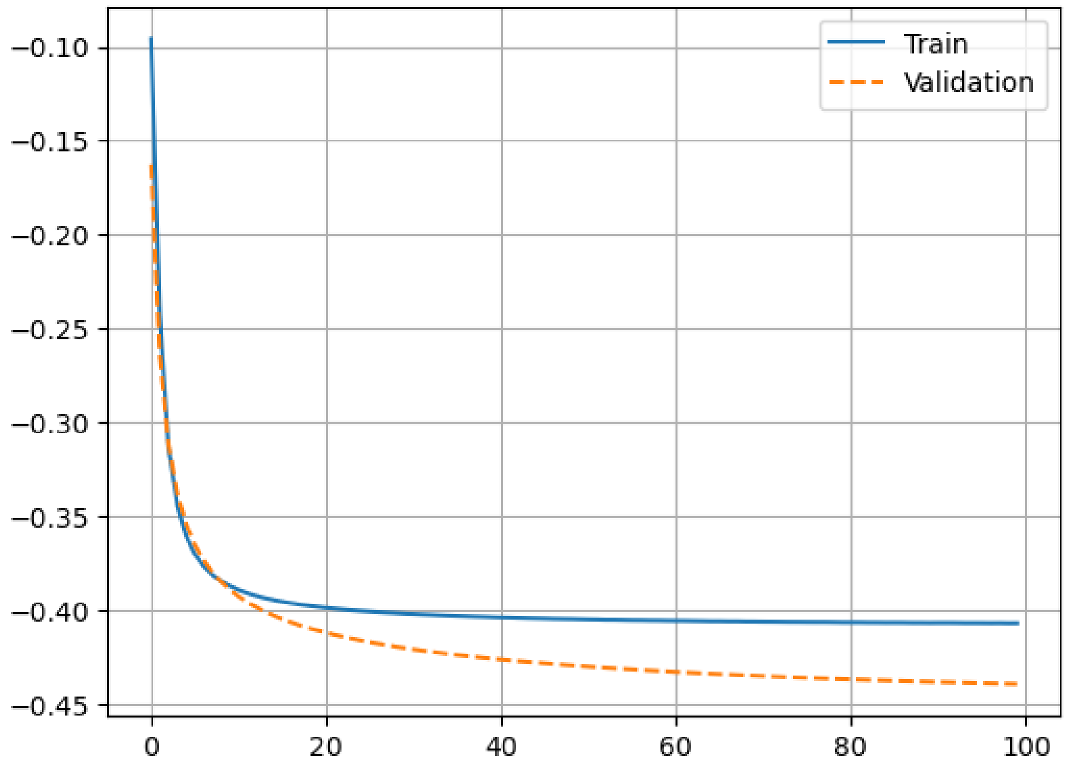
```
loss: -0.4054 - val_loss: -0.4304
Epoch 53/100
100/100             0s 1ms/step -
loss: -0.4047 - val_loss: -0.4307
Epoch 54/100
100/100             0s 570us/step -
loss: -0.4054 - val_loss: -0.4310
Epoch 55/100
100/100             0s 829us/step -
loss: -0.4055 - val_loss: -0.4313
Epoch 56/100
100/100             0s 622us/step -
loss: -0.4052 - val_loss: -0.4316
Epoch 57/100
100/100             0s 556us/step -
loss: -0.4054 - val_loss: -0.4319
Epoch 58/100
100/100             0s 729us/step -
loss: -0.4057 - val_loss: -0.4321
Epoch 59/100
100/100             0s 798us/step -
loss: -0.4053 - val_loss: -0.4324
Epoch 60/100
100/100             0s 1ms/step -
loss: -0.4055 - val_loss: -0.4327
Epoch 61/100
100/100             0s 586us/step -
loss: -0.4061 - val_loss: -0.4329
Epoch 62/100
100/100             0s 2ms/step -
loss: -0.4053 - val_loss: -0.4332
Epoch 63/100
100/100             0s 804us/step -
loss: -0.4059 - val_loss: -0.4334
Epoch 64/100
100/100             0s 1ms/step -
loss: -0.4057 - val_loss: -0.4336
Epoch 65/100
100/100             0s 582us/step -
loss: -0.4055 - val_loss: -0.4339
Epoch 66/100
100/100             0s 942us/step -
loss: -0.4056 - val_loss: -0.4341
Epoch 67/100
100/100             0s 1ms/step -
loss: -0.4064 - val_loss: -0.4343
Epoch 68/100
100/100             0s 614us/step -
```

```
loss: -0.4057 - val_loss: -0.4345
Epoch 69/100
100/100             0s 551us/step -
loss: -0.4068 - val_loss: -0.4347
Epoch 70/100
100/100             0s 540us/step -
loss: -0.4064 - val_loss: -0.4349
Epoch 71/100
100/100             0s 556us/step -
loss: -0.4061 - val_loss: -0.4351
Epoch 72/100
100/100             0s 667us/step -
loss: -0.4060 - val_loss: -0.4353
Epoch 73/100
100/100             0s 1ms/step -
loss: -0.4061 - val_loss: -0.4355
Epoch 74/100
100/100             0s 700us/step -
loss: -0.4060 - val_loss: -0.4357
Epoch 75/100
100/100             0s 1ms/step -
loss: -0.4059 - val_loss: -0.4359
Epoch 76/100
100/100             0s 830us/step -
loss: -0.4064 - val_loss: -0.4361
Epoch 77/100
100/100             0s 715us/step -
loss: -0.4064 - val_loss: -0.4363
Epoch 78/100
100/100             0s 1ms/step -
loss: -0.4060 - val_loss: -0.4364
Epoch 79/100
100/100             0s 894us/step -
loss: -0.4068 - val_loss: -0.4366
Epoch 80/100
100/100             0s 872us/step -
loss: -0.4065 - val_loss: -0.4367
Epoch 81/100
100/100             0s 932us/step -
loss: -0.4061 - val_loss: -0.4369
Epoch 82/100
100/100             0s 812us/step -
loss: -0.4068 - val_loss: -0.4371
Epoch 83/100
100/100             0s 660us/step -
loss: -0.4064 - val_loss: -0.4372
Epoch 84/100
100/100             0s 1ms/step -
```

```
loss: -0.4060 - val_loss: -0.4374
Epoch 85/100
100/100              0s 825us/step -
loss: -0.4067 - val_loss: -0.4375
Epoch 86/100
100/100              0s 905us/step -
loss: -0.4064 - val_loss: -0.4376
Epoch 87/100
100/100              0s 671us/step -
loss: -0.4067 - val_loss: -0.4378
Epoch 88/100
100/100              0s 826us/step -
loss: -0.4063 - val_loss: -0.4379
Epoch 89/100
100/100              0s 717us/step -
loss: -0.4067 - val_loss: -0.4381
Epoch 90/100
100/100              0s 687us/step -
loss: -0.4066 - val_loss: -0.4382
Epoch 91/100
100/100              0s 690us/step -
loss: -0.4070 - val_loss: -0.4383
Epoch 92/100
100/100              0s 710us/step -
loss: -0.4071 - val_loss: -0.4384
Epoch 93/100
100/100              0s 662us/step -
loss: -0.4071 - val_loss: -0.4385
Epoch 94/100
100/100              0s 561us/step -
loss: -0.4064 - val_loss: -0.4387
Epoch 95/100
100/100              0s 566us/step -
loss: -0.4069 - val_loss: -0.4388
Epoch 96/100
100/100              0s 719us/step -
loss: -0.4073 - val_loss: -0.4389
Epoch 97/100
100/100              0s 711us/step -
loss: -0.4065 - val_loss: -0.4390
Epoch 98/100
100/100              0s 803us/step -
loss: -0.4069 - val_loss: -0.4391
Epoch 99/100
100/100              0s 613us/step -
loss: -0.4070 - val_loss: -0.4392
Epoch 100/100
100/100              0s 937us/step -
```

loss: -0.4071 - val_loss: -0.4393

[26]:
```
plt.plot(hist.history['loss'],'-',label="Train")
plt.plot(hist.history['val_loss'],'--',label="Validation")
plt.grid()
plt.legend()
plt.savefig(folder_name+'/'+'loss_values.png', bbox_inches='tight', dpi=300)
plt.show()
```



9 Evaluating the learning h-mapping in some images

[28]:
```
fig, axs = plt.subplots(1, 5, figsize=(15, 5))

for i in range(5):
    y_pred = model.predict(Xtr[i].reshape(-1,3))
    axs[i].imshow(y_pred.reshape(32,32),cmap="gray")
    axs[i].axis(False)
plt.show()
```

```
32/32                 0s 278us/step
32/32                 0s 246us/step
32/32                 0s 233us/step
```

```
32/32                 0s 334us/step
32/32                 0s 288us/step
```

[29]:
```python
fig, axs = plt.subplots(1, 5, figsize=(15, 5))

for i in range(5):
    y_pred = model.predict(Xval[i].reshape(-1,3))
    axs[i].imshow(y_pred.reshape(32,32),cmap="gray")
    axs[i].axis(False)
plt.show()
```

```
32/32                 0s 243us/step
32/32                 0s 262us/step
32/32                 0s 253us/step
32/32                 0s 233us/step
32/32                 0s 229us/step
```

10 Morphological Operators for Color Images

[31]:
```python
def h_model(model):
    def aux(img):
        return model.predict(img.reshape(-1,3)).reshape(img.shape[0],img.
↪shape[1])
    return aux

def h_average(rho_list):
    def aux(img):
        return np.mean(np.stack([rho(img) for rho in rho_list],axis=2),axis=2)
    return aux
```

```python
# h-moprhological operator

def hMM(I, h = h_model(model), mm_op = dilation, SE = square(7)):
    nr,nc,nb = I.shape
    Data=np.reshape(I,[nr*nc,nb])
    horder = h(I).flatten()
    order = np.argsort(horder[:])
    latt = np.zeros(Data.shape[0],dtype=np.int32)
    latt[order] = np.arange(Data.shape[0])
    latt = np.reshape(latt,[nr,nc])
    mm_latt = mm_op(latt,SE)
    imOut=Data[order[mm_latt],:]
    return np.reshape(imOut,[nr,nc,nb])
```

```
/tmp/ipykernel_24429/4017338546.py:13: FutureWarning: `square` is deprecated
since version 0.25 and will be removed in version 0.27. Use
`skimage.morphology.footprint_rectangle` instead.
  def hMM(I, h = h_model(model), mm_op = dilation, SE = square(7)):
```

11 Irregularity Index

[33]:
```python
def global_irregularity(imI,imJ,metric='euclidean'):
    M,N,K = imI.shape
    a = imI.reshape(M*N,K)
    b = imJ.reshape(M*N,K)
    vu,hu = np.unique(a,return_counts=True,axis=0)
    vv,hv = np.unique(b,return_counts=True,axis=0)
    C = pairwise_distances(vu,vv,metric=metric)
    return 1-ot.emd2(1.0*hu,1.0*hv,C)/np.
 ↪sum(paired_distances(a,b,metric=metric))
```

12 Show morphological operation applied on images from the training set

[35]:
```python
SE_size = 3
SE = square(SE_size)

rho_list = [LexRGB, LexGBR, LexBRG, h_model(model)]
rho_list_names = ["Lex_RGB", "Lex_GBR", "Lex_BRG","Model"]

Irreg = np.zeros((5,len(rho_list)))

for i,x in enumerate(Xtr[:5]):
    for j,rho in enumerate(rho_list):
        img = hMM(x,h=rho, mm_op = opening, SE = SE)
```

```
        Irreg[i,j] = 100*global_irregularity(x,img)
        plt.imshow(img)
        plt.axis(False)
        plt.savefig(folder_name+'/open_'+rho_list_names[j]+str(i)+'.png',␣
↪bbox_inches='tight', dpi=300)
        # plt.show()
```

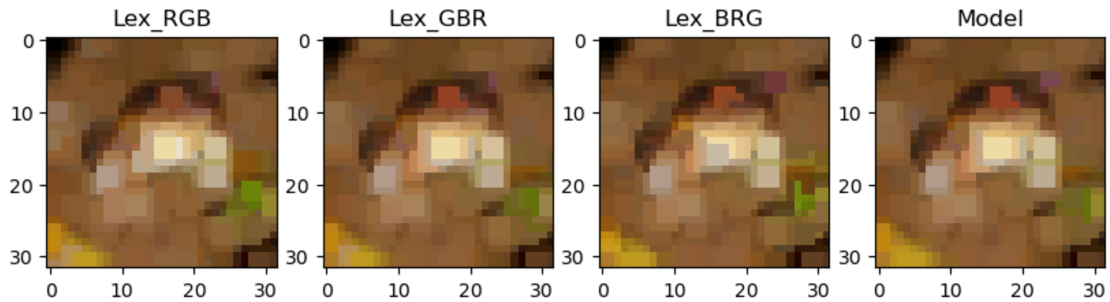
```
/tmp/ipykernel_24429/3596821883.py:2: FutureWarning: `square` is deprecated
since version 0.25 and will be removed in version 0.27. Use
`skimage.morphology.footprint_rectangle` instead.
  SE = square(SE_size)
```

```
32/32                 0s 273us/step
32/32                 0s 260us/step
32/32                 0s 238us/step
32/32                 0s 228us/step
32/32                 0s 363us/step
```

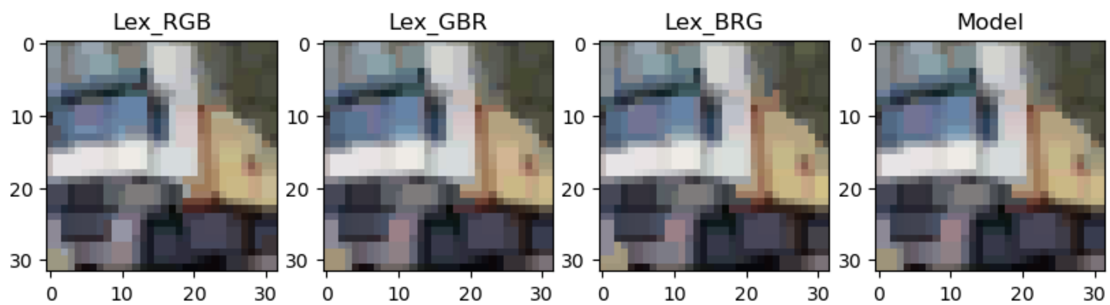
[36]:

```
for x in Xtr[:5]:
    fig, axs = plt.subplots(1, 4, figsize=(10, 5))
    for i,rho in enumerate(rho_list):
      axs[i].imshow(hMM(x,h=rho, mm_op = opening, SE = SE))
      axs[i].set_title(rho_list_names[i])
    plt.show()
```

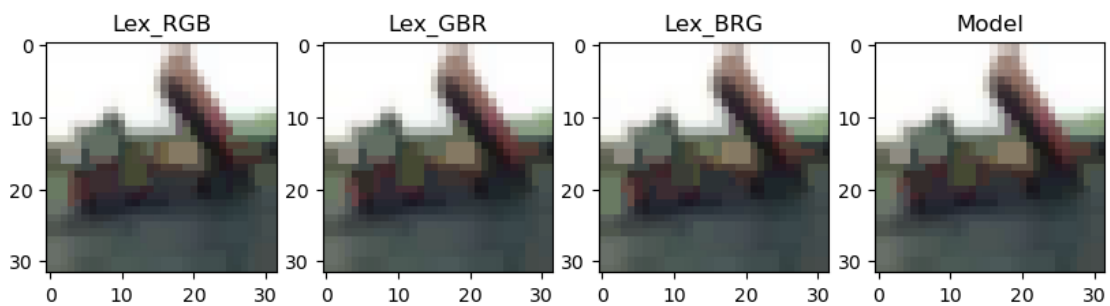
```
32/32                0s 309us/step
```



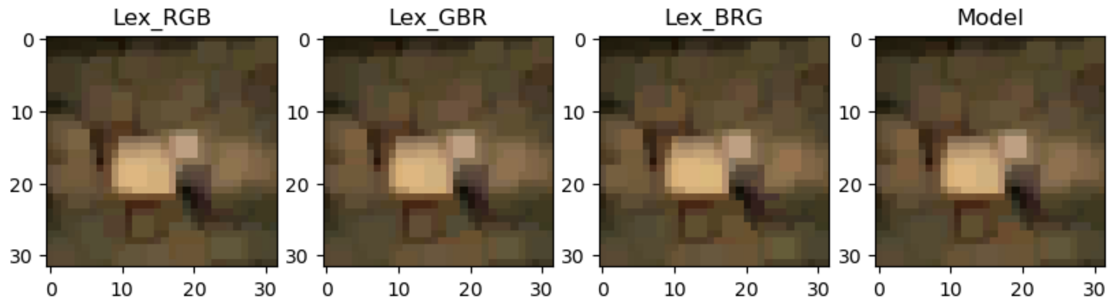

```
32/32                0s 295us/step
```



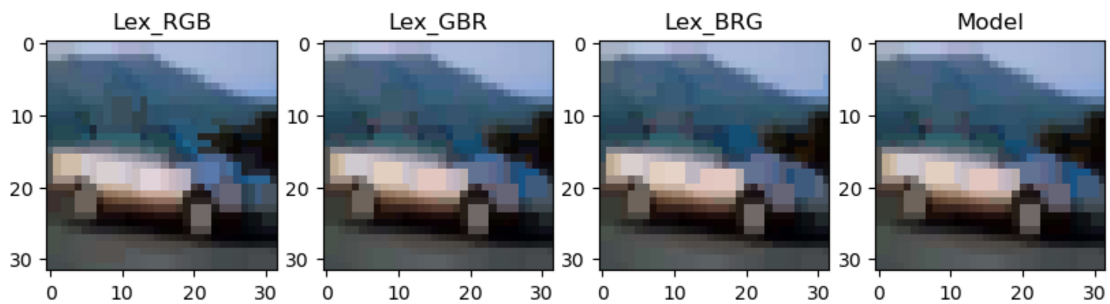

```
32/32                0s 420us/step
```




```
32/32                0s 272us/step
```



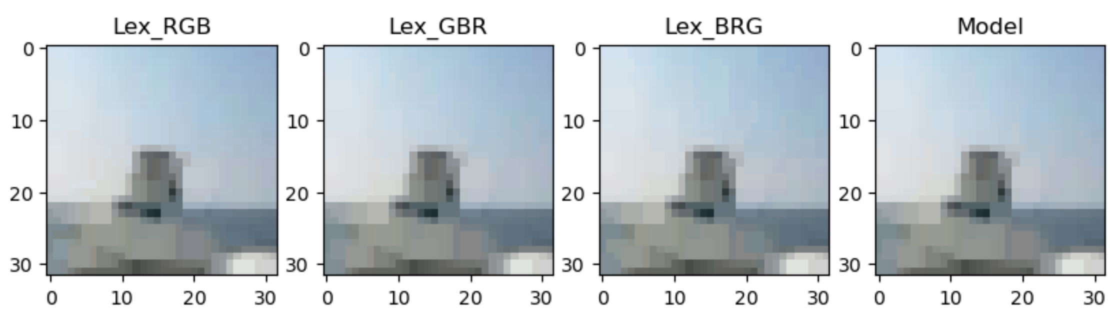

```
32/32                 0s 276us/step
```



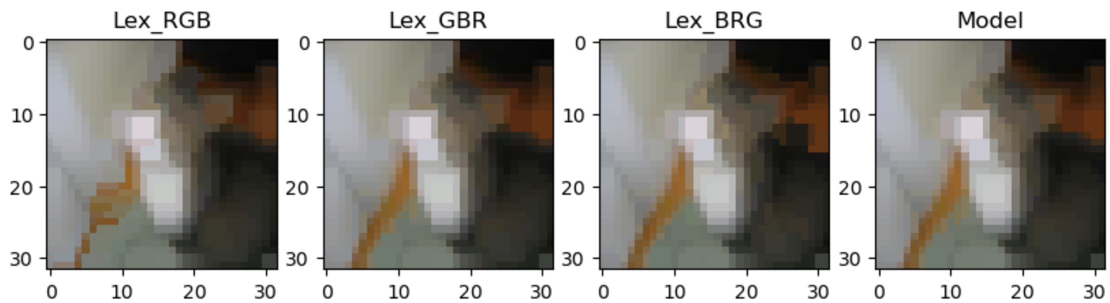
13 Show some images from validation set

[38]:
```python
for x in Xval[:5]:
    fig, axs = plt.subplots(1, 4, figsize=(10, 5))
    for i,rho in enumerate(rho_list):
      axs[i].imshow(hMM(x,h=rho, mm_op = opening, SE = SE))
      axs[i].set_title(rho_list_names[i])
    plt.show()
```

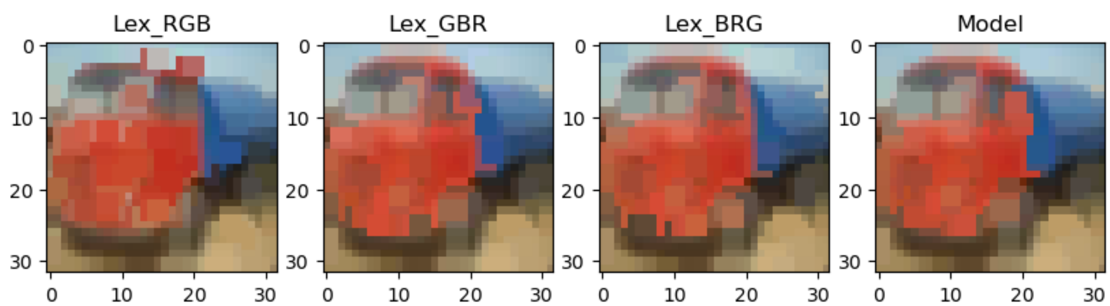
```
32/32                 0s 325us/step
```



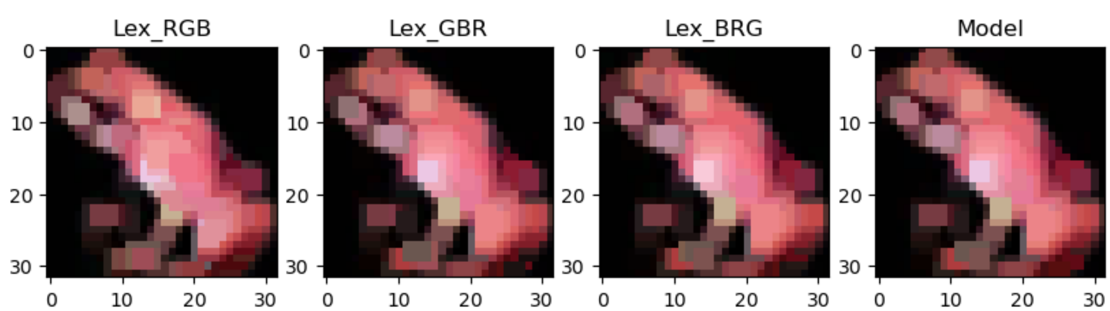
```
32/32            0s 313us/step
```



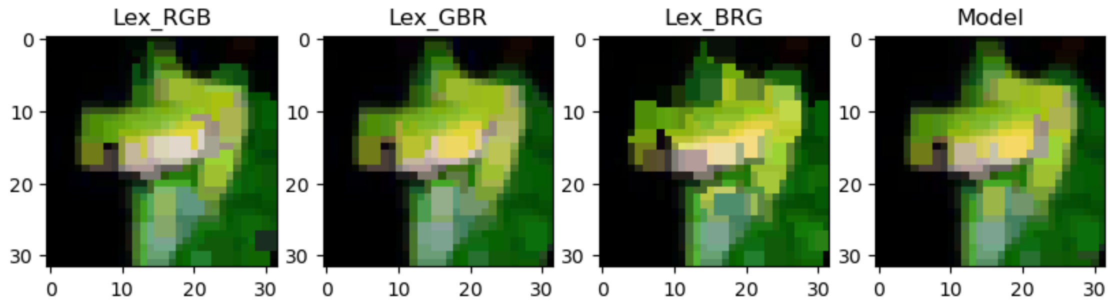

```
32/32            0s 287us/step
```




```
32/32            0s 286us/step
```




```
32/32            0s 272us/step
```



14 Compute the Irregularity index in the validation set

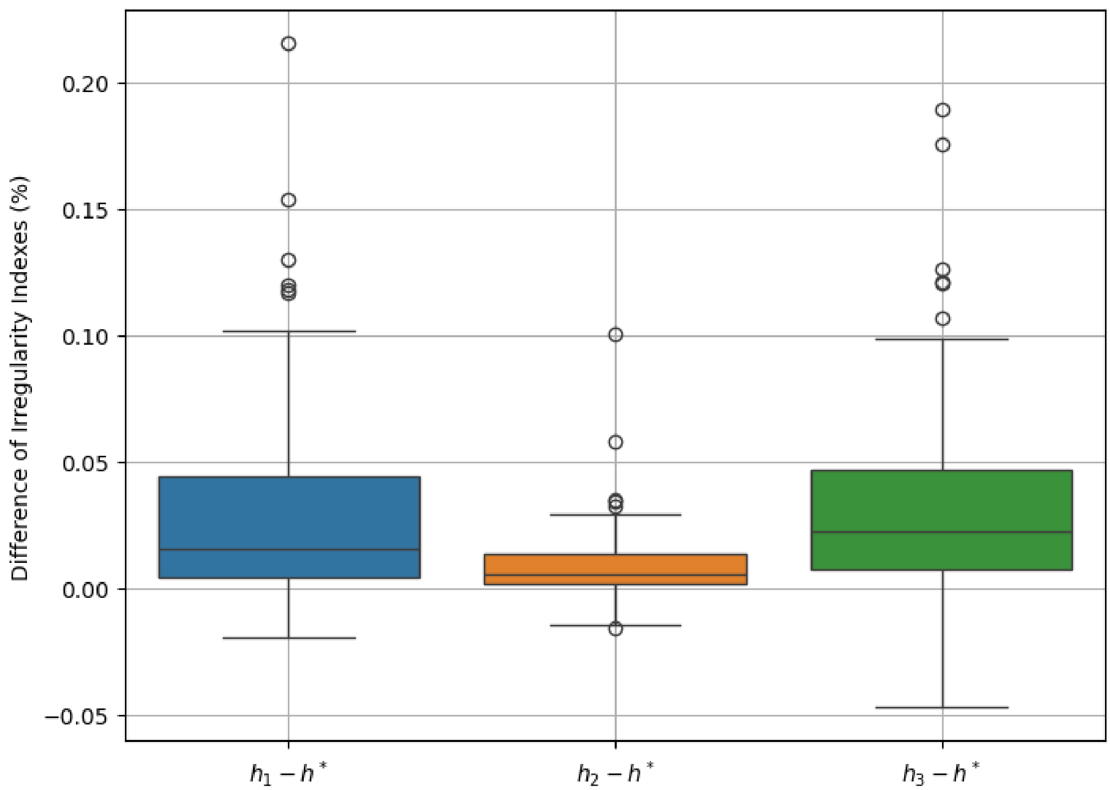
[40]:
```
Irreg = np.zeros((Xval.shape[0],len(rho_list)))

for i,x in enumerate(Xval):
    for j,rho in enumerate(rho_list):
        img = hMM(x,h=rho, mm_op = opening, SE = SE)
        Irreg[i,j] = global_irregularity(x,img)
```

```
32/32            0s 568us/step
32/32            0s 382us/step
32/32            0s 416us/step
32/32            0s 500us/step
32/32            0s 543us/step
32/32            0s 522us/step
32/32            0s 538us/step
32/32            0s 537us/step
32/32            0s 535us/step
32/32            0s 522us/step
32/32            0s 408us/step
32/32            0s 528us/step
32/32            0s 538us/step
32/32            0s 452us/step
32/32            0s 447us/step
32/32            0s 373us/step
32/32            0s 602us/step
32/32            0s 522us/step
32/32            0s 411us/step
32/32            0s 518us/step
32/32            0s 553us/step
32/32            0s 556us/step
32/32            0s 601us/step
32/32            0s 606us/step
32/32            0s 600us/step
32/32            0s 593us/step
```

```
32/32                  0s 609us/step
32/32                  0s 600us/step
32/32                  0s 626us/step
32/32                  0s 622us/step
32/32                  0s 581us/step
32/32                  0s 602us/step
32/32                  0s 711us/step
32/32                  0s 528us/step
32/32                  0s 521us/step
32/32                  0s 574us/step
32/32                  0s 520us/step
32/32                  0s 495us/step
32/32                  0s 439us/step
32/32                  0s 515us/step
32/32                  0s 435us/step
32/32                  0s 517us/step
32/32                  0s 421us/step
32/32                  0s 429us/step
32/32                  0s 598us/step
32/32                  0s 555us/step
32/32                  0s 444us/step
32/32                  0s 566us/step
32/32                  0s 560us/step
32/32                  0s 491us/step
32/32                  0s 403us/step
32/32                  0s 448us/step
32/32                  0s 536us/step
32/32                  0s 448us/step
32/32                  0s 540us/step
32/32                  0s 603us/step
32/32                  0s 600us/step
32/32                  0s 554us/step
32/32                  0s 583us/step
32/32                  0s 570us/step
32/32                  0s 535us/step
32/32                  0s 638us/step
32/32                  0s 588us/step
32/32                  0s 576us/step
32/32                  0s 583us/step
32/32                  0s 633us/step
32/32                  0s 652us/step
32/32                  0s 529us/step
32/32                  0s 657us/step
32/32                  0s 573us/step
32/32                  0s 571us/step
32/32                  0s 568us/step
32/32                  0s 598us/step
32/32                  0s 642us/step
```

```
32/32              0s 581us/step
32/32              0s 604us/step
32/32              0s 616us/step
32/32              0s 622us/step
32/32              0s 668us/step
32/32              0s 594us/step
32/32              0s 558us/step
32/32              0s 594us/step
32/32              0s 539us/step
32/32              0s 676us/step
32/32              0s 561us/step
32/32              0s 555us/step
32/32              0s 412us/step
32/32              0s 442us/step
32/32              0s 536us/step
32/32              0s 548us/step
32/32              0s 512us/step
32/32              0s 518us/step
32/32              0s 499us/step
32/32              0s 521us/step
32/32              0s 673us/step
32/32              0s 549us/step
32/32              0s 540us/step
32/32              0s 622us/step
32/32              0s 393us/step
32/32              0s 485us/step
```

[41]:
```
# Convert the numpy array to a pandas DataFrame for seaborn
IrregDif = np.stack([Irreg[:,i]-Irreg[:,3] for i in range(3)],axis=1)

df = pd.DataFrame(IrregDif, columns=["$h_1-h^*$","$h_2-h^*$","$h_3-h^*$"])

plt.figure(figsize=(8, 6))

# Create the box plot using seaborn
sns.boxplot(data=df)

# Add labels and title
plt.ylabel('Difference of Irregularity Indexes (%)')
# plt.title('Box Plot of Features (Seaborn)')
plt.grid()
plt.savefig(folder_name+'/'+'irregularity_values.png', bbox_inches='tight',␣
↪dpi=300)
plt.show()
```



15 Example using a color image from the BSD dataset

[43]:
```
# Create the Images folder if it doesn't exist
folder_name = 'Images'
if not os.path.exists(folder_name):
    os.makedirs(folder_name)

image_url = 'https://www.ime.unicamp.br/~valle/Gallery/124084.jpg'  # Corrected␣
↪URL

try:
    response = requests.get(image_url, stream=True)
    response.raise_for_status()  # Raise an exception for bad status codes

    image = Image.open(response.raw)
    image_path = os.path.join(folder_name, '124084.jpg')
    image.save(image_path)
    print(f"Image downloaded and saved to {image_path}")

except requests.exceptions.RequestException as e:
```

```
    print(f"Error downloading image: {e}")
except Exception as e:
    print(f"Error saving image: {e}")
```

```
Image downloaded and saved to Images/124084.jpg
```

[44]:
```
# Load a color image:
image_path = "Images/124084.jpg"  # Replace with the actual path to your image
image = Image.open(image_path)
image = image.resize((image.size[0],image.size[1]))
x = np.array(image)/255

# Display the image
plt.imshow(x)
plt.axis(False)
plt.savefig(folder_name+'/flowers.png', bbox_inches='tight', dpi=300)
```

[45]:
```
plt.imshow(h_model(model)(x),cmap="gray")
```

```
4826/4826              1s 257us/step
```

[45]: <matplotlib.image.AxesImage at 0x7aaeee18abd0>

[46]:
```
SE = disk(10)
for i,rho in enumerate(rho_list):
    plt.figure(figsize=(10,5))
    plt.imshow(hMM(x,h=rho, mm_op = closing, SE = SE))
    plt.axis(False)
    plt.savefig(folder_name+'/closing_'+rho_list_names[i]+'.png',␣
 ↪bbox_inches='tight', dpi=300)
    # plt.show()
```

```
4826/4826               1s 232us/step
```

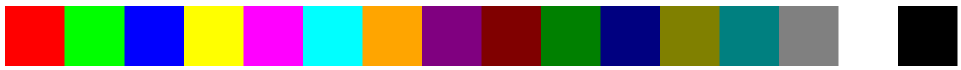
16 Show the color orderings

[48]:

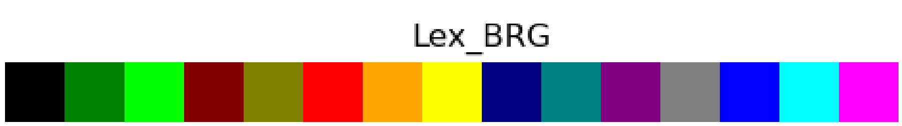
```
# Generate a 4x4 color image
image = np.zeros((4, 4, 3))

# Assign colors to each pixel
image[0, 0] = [255, 0, 0]       # Red
image[0, 1] = [0, 255, 0]       # Light Green
image[0, 2] = [0, 0, 255]       # Blue
image[0, 3] = [255, 255, 0]     # Yellow
image[1, 0] = [255, 0, 255]     # Magenta
image[1, 1] = [0, 255, 255]     # Cyan
image[1, 2] = [255, 165, 0]     # Orange
image[1, 3] = [128, 0, 128]     # Purple
image[2, 0] = [128, 0, 0]       # Maroon
image[2, 1] = [0, 128, 0]       # Dark Green
image[2, 2] = [0, 0, 128]       # Navy
image[2, 3] = [128, 128, 0]     # Olive
image[3, 0] = [0, 128, 128]     # Teal
image[3, 1] = [128, 128, 128]   # Gray
image[3, 2] = [255, 255, 255]   # White
image[3, 3] = [0, 0, 0]        # Black

color_names = [
    "Red",
    "Light Green",
    "Blue",
    "Yellow",
    "Magenta",
    "Cyan",
    "Orange",
    "Purple",
    "Maroon",
    "Dark Green",
    "Navy",
    "Olive",
    "Teal",
    "Gray",
    "White",
    "Black"
]


image = image/255  # Normalize pixel values to the range [0, 1]

# Display the color elements
print("Color elements - NOT ORDERED!")
```

```python
[M,N,K] = image.shape
plt.imshow(image.reshape(1,M*N,K))
plt.axis('off') # Hide axes
plt.savefig(folder_name+'/colors.png', bbox_inches='tight', dpi=300)
plt.show()
```

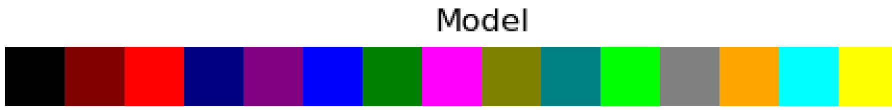
```
Color elements - NOT ORDERED!
```



```python
[49]: # fig, axs = plt.subplots(4, 1, figsize=(10, 5))

x = image
[M,N,K] = x.shape
for i,rho in enumerate(rho_list):
    ind = np.argsort(rho(x).flatten())
    im_sorted = (x.reshape(M*N,3)[ind,:]).reshape(1,M*N,K)
    plt.imshow(im_sorted)
    plt.axis(False)
    plt.savefig(folder_name+'/colors_'+rho_list_names[i]+'.png', bbox_inches='tight', dpi=300)
    plt.title(rho_list_names[i])
    plt.show()
```



```
1/1          0s 27ms/step
```




```
[50]: ind = np.argsort(h_model(model)(x).flatten())
      print("h^* color ranking:")
      [color_names[i] for i in ind]
```

```
1/1          0s 9ms/step
h^* color ranking:
```

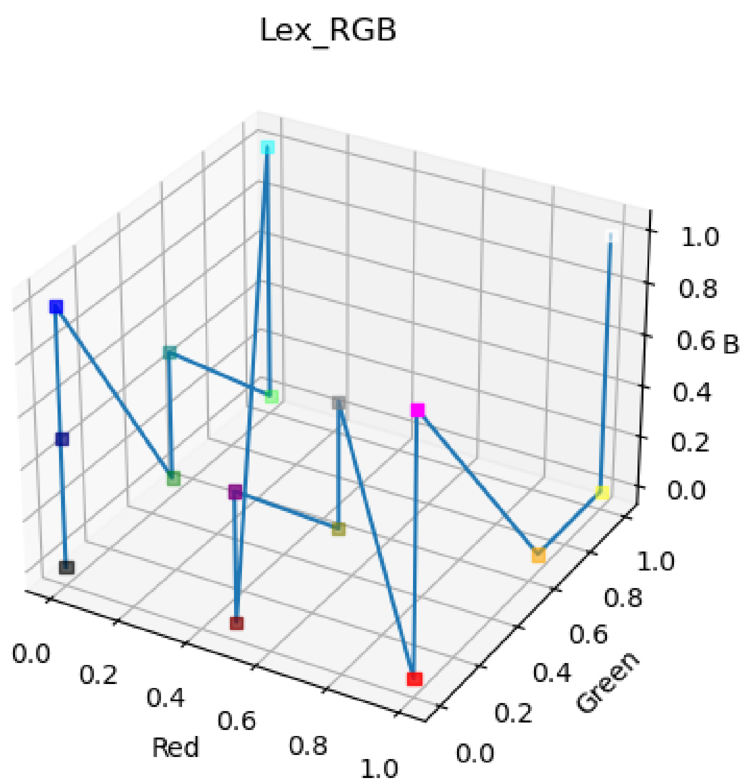
```
[50]: ['Black',
       'Maroon',
       'Red',
       'Navy',
       'Purple',
       'Blue',
       'Dark Green',
       'Magenta',
       'Olive',
       'Teal',
       'Light Green',
       'Gray',
       'Orange',
       'Cyan',
       'Yellow',
       'White']
```

17 Show the path of the color rank in the 3d space

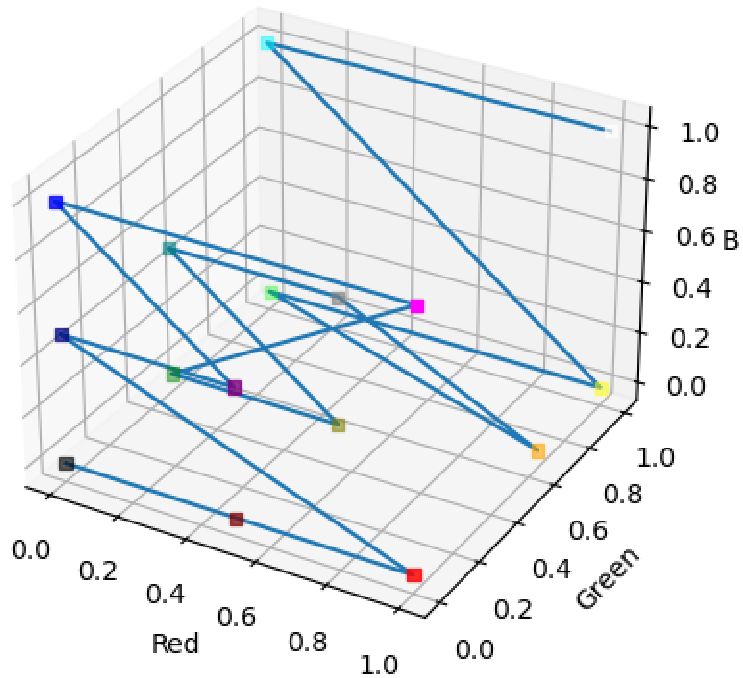
This is nice but difficult to interpret!

```
[52]: x = image
      [M,N,K] = x.shape
      for i,rho in enumerate(rho_list):
          ind = np.argsort(rho(x).flatten())
          im_sorted = (x.reshape(M*N,3)[ind,:]).reshape(M*N,K)
          fig = plt.figure()
          ax = fig.add_subplot(111, projection='3d')

          # Plot of the colors
          ax.plot(im_sorted[:, 0], im_sorted[:, 1], im_sorted[:, 2])
```

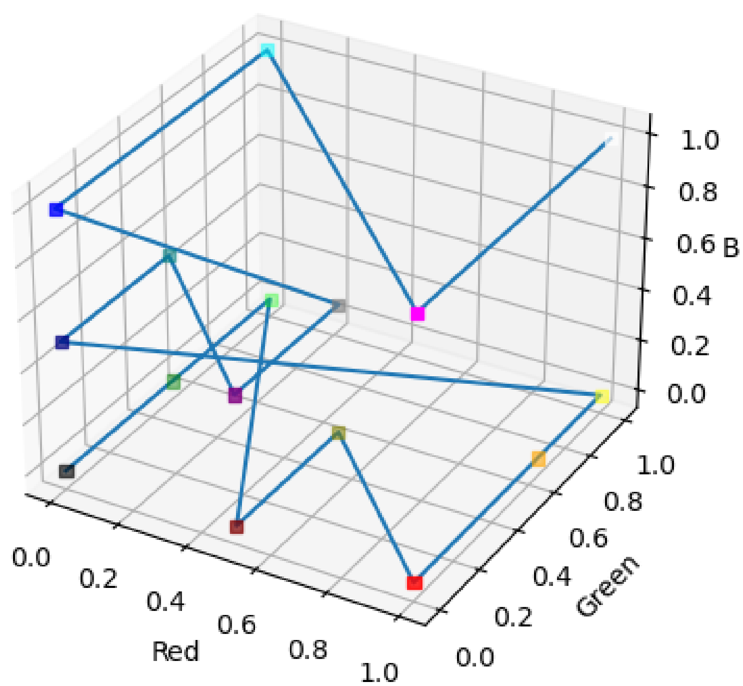
```
    ax.scatter(im_sorted[:, 0], im_sorted[:, 1], im_sorted[:, 2],␣
↪color=im_sorted, marker='s')

    # Set axis labels
    ax.set_xlabel('Red')
    ax.set_ylabel('Green')
    ax.set_zlabel('Blue')
    ax.set_title(rho_list_names[i])

    # Show the plot
    plt.show()
```



Lex_GBR



Lex_BRG



```
1/1                0s 9ms/step
```

Model