

# PHYS 209

## Type Notation

$S :: \text{Integer} \Rightarrow S$  is of type integers

$\sin \rightarrow$  function,  $\sin(+)$   $\Rightarrow$  output,  $(x) \Rightarrow$  input

functions are maps, they take inputs and spit out outputs.

$\sin: \mathbb{R} \rightarrow \mathbb{R}$  "sin" is of type real to real

$$Sg: \begin{matrix} \mathbb{Z} \xrightarrow{\text{domain}} \\ (\text{x type}) \end{matrix} \rightarrow \begin{matrix} \mathbb{Z} \xrightarrow{\text{codomain}} \\ (\text{y, z, ... = dummy}) \end{matrix}$$

$Sg = \begin{matrix} x \rightarrow x^2 \\ (\text{x definition}) \end{matrix}$

$$\begin{aligned} Sg(2) &= 4 \\ Sg(-3) &= 9 \end{aligned}$$

$y, m, z, \dots$  = dummy  
variables

$\text{abs}: \mathbb{R} \rightarrow \mathbb{R}^{>0}$

$\text{abs} = x \rightarrow \sqrt{x^2}$

$\hookrightarrow$  absolute value

$\text{abs}(-10) = 10$

$T$  is operation which double the output of the function.

$f = x \rightarrow x + 5$

$(T \cdot f) = x \rightarrow 2x + 10$

map to map both of them are function

$T: (A \rightarrow B) \rightarrow (A \rightarrow C)$

$T = (x \rightarrow f(x)) \rightarrow (y \rightarrow 2f(y))$

$(T \cdot \sin)(x) = 2 \sin(x)$

$$\frac{d}{dx} : (A \rightarrow B) \rightarrow (A \rightarrow B)$$

$$\frac{d}{dx} : (x \rightarrow f(x)) \rightarrow (x \rightarrow f'(x))$$

$$\frac{d}{dx} \cdot \sin = \cos \rightarrow \text{it is a function not value}$$

$\int$  is computing an area of a curve

$$\int : (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R}$$

$$\int = (x \rightarrow f(x)) \longrightarrow \int_{-\infty}^{\infty} f(y) dy$$

$$\Rightarrow \text{For } f = x \rightarrow x^2 + 2$$

$$W = \int_{-\infty}^{\infty} f(t) dt$$

$$f : \mathbb{R} \rightarrow \mathbb{C}$$

$$f : x \rightarrow x^2 - i$$

$$\left( \frac{d}{dx} \cdot f \right) : \mathbb{R} \rightarrow \mathbb{R}$$

$$\frac{d}{dx} \cdot f = x \rightarrow 2x$$

$$\sin : \mathbb{R} \rightarrow \mathbb{R}$$

$$\underbrace{\sin(x)}_{\text{codomain}} : \mathbb{R}$$

codomain

$$\frac{d^n}{dx^n} : (A \rightarrow B) \rightarrow (A \rightarrow B)$$

$$\frac{d^{-1}}{dx^{-1}} \quad \left. \right\} \text{anti-derivative}$$

$$\int \cdot \cos(x) dx = \sin(x) = \frac{d^{-1}}{dx^{-1}} \cdot \cos = \sin$$

$$\boxed{f = \vec{N} \rightarrow \vec{M} \cdot \vec{N} \\ f(f(\vec{w})) \Rightarrow \text{ill-defined}}$$

$$\frac{d}{dx} : (A \rightarrow B) \rightarrow (C \rightarrow D) \quad f : A \rightarrow B$$

$$\left( \frac{d}{dx} \cdot f \right) : A \rightarrow B$$

$$\left( \frac{d}{dx} \cdot \left( \frac{d}{dx} \cdot f \right) \right) : A \rightarrow B$$

# Differential Equations

Definition: Any relation that contains the derivative higher order function  $\frac{d}{dx}$  and an unknown function  $f$  as a differential equation

$$\frac{d}{dx} \cdot f = 0$$

## Classification of D.E

$$f(\lambda) \rightarrow \lambda(f(x)) \quad \boxed{\text{Linear}} \quad \left. \begin{array}{l} \text{general rule} \\ \text{quick way to check} \end{array} \right\}$$

$$\lambda(\dots) + (\dots) = 0 \quad \xrightarrow{\text{--- --- --- --- ---}}$$

$$f(x)f'(x) = 2 \quad \boxed{\text{Non-linear}}$$

$$\frac{\partial^2 (f(x) \cdot f'(x) + (-2))}{\partial x^2} = 0 \quad \xrightarrow{\text{--- --- --- --- --- ---}}$$

$$\frac{\partial^2 f(\text{---})}{\partial y^2} + \frac{\partial f(\text{---})}{\partial y} \quad \boxed{\text{Partial}} \quad \text{others are ordinary}$$

order is the highest derivatives

Let  $p(x)$  satisfy  $g\left(x, \frac{d}{dx}\right)p(x) = h(x)$

Let  $f_h(x)$  satisfy  $g\left(x, \frac{d}{dx}\right)f_h(x) = h(x)$

Define  $s(x) = p(x) + C f_h(x) \rightarrow$  homogeneous solution  
                   $\underbrace{\qquad}_{\text{particular solution}}$

Observe:  $g\left(x, \frac{d}{dx}\right)s(x) = g\left(x, \frac{d}{dx}\right)(p(x) + C f_h(x))$

$= 0 \Rightarrow$  trivial question

$$= g\left(x, \frac{d}{dx}\right) + g\left(x, \frac{d}{dx}\right) f_h(x)$$

$$\left( \frac{d^2}{dx^2} + L \right) f(x) = x$$

$$f''(x) + L f(x) = x$$

homogeneous part

$$\left( \frac{d^2}{dx^2} + L \right) f(x)$$

$$f(x) = \sin(\omega x)$$

$$f_m(x) = \cos(\omega x)$$

General solution  $= f_h(x) = C_1 \sin(\omega x) + C_2 \cos(\omega x)$

$$p(x) = \frac{x}{L}$$

principle of superposition

$$s(x) = \frac{x}{L} + (C_1 \cos(\omega x) + C_2 \sin(\omega x))$$

We arrive at the conclusion "linear ordinary diff. eqn" are satisfied by a family of solutions of the form

$$s(x) = p(x) + (C_1 f_{h1}(x) + C_2 f_{h2}(x) + \dots)$$

$$\delta \left( x, \frac{1}{dx} \right) f(x) = 0$$

$$\left( x^2 + \cos\left(\frac{1}{dx}\right) + \frac{1}{x} + \frac{d^2}{dx^2} \right) f(x) = 0$$

$$c_n(x) \frac{d^n}{dx^n} + c_{n-1} \frac{d^{(n-1)}}{dx^{(n-1)}} + c_{n-2} \frac{d^{n-2}}{dx^{n-2}} + \dots + c_0(x) f(x) = 0$$

$$c_n(x) f^n(x) + c_{n-1} f^{(n-1)}(x) + \dots + c_0(x) f(x) = 0$$

Linear, ordinary, homogeneous, order-n diff. eqn.

$$\sum_{i=0}^n c_i f^{(i)}(x) = 0$$

Linear, ordinary, homogeneous, order-n diff. eqn. w/ constant coefficients

$$\boxed{\sum_{i=0}^n c_i f^{(i)}(x) = 0}$$

$$e^{\int dx} f(x) = x^n$$

↳ infinite order

$$\sum_{i=0}^n c_i f^{(i)}(x) = 0 \Rightarrow \text{most generic form}$$

Linear Ordinary Order-n Diff eqn. w/ constn coeff.

$$\left[ \sum_{i=0}^n c_i f^{(i)}(x) = h(x) \right] \Rightarrow \text{Non-homogeneous}$$

$$g\left(x; \frac{d}{dx}\right) f(x) = 0$$

Kernel of a diff. operator is a set of functions which are annihilated by the differential operator.

$$\ker\left(\frac{d}{dx}\right) = \left\{ x \rightarrow c \right\} \quad (\text{functions are maps})$$

↳ basically  $f(x) = c$

~~Ex~~

$$\ker\left(\frac{d^2}{dx^2}\right) = \left\{ x \rightarrow c, x \rightarrow x \right\}$$

trivial solution  
is an element of  
kernel

all homogeneous  
solutions are elements  
of kernel

$$\ker\left(g\left(x; \frac{d}{dx}\right)\right) = \{f_1, f_2, f_3, \dots, f_n\}$$

$$\ker\left(\frac{d^2}{dx^2} - 4\right) = \{x \rightarrow e^{2x}, x \rightarrow e^{-2x}\}$$
$$\left(\frac{d^2}{dx^2} - 4\right) e^{2x} = 0$$
$$\left(\frac{d^2}{dx^2} - 4\right) e^{-2x} = 0$$
$$\left(\frac{d^2}{dx^2} - 4\right) f''(x) - 4(f(x)) = 0$$

$\uparrow\downarrow$

$$g\left(x, \frac{1}{dx}\right) f(x) = h(x)$$

$$\ker\left(g\left(x, \frac{1}{dx}\right)\right) = \{f_1, \dots, f_n\}$$

$$f(x) = p(x) + c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x)$$

$$\sum_{i=0}^n c_i(x) f^{(i)}(x) = h(x)$$

$\ker\left(\sum_{i=0}^n c_i(x) \frac{d^i}{dx^i}\right)$  is always computable!

$$\frac{d^2}{dx^2} f(x) = 2 \quad \ker\left(\frac{d^2}{dx^2}\right) = \{x \mapsto c, x \mapsto x\}$$

$$f(x) = p(x) + c_1 + c_2 x$$

$$\underbrace{f(x) = x^2 + c_1 + c_2 x}_{}$$

$$\left. \begin{array}{l} \frac{d}{dx} f(x) = 0 \\ f(x) = c \end{array} \right\} \quad \left. \begin{array}{l} \frac{d^2}{dx^2} f(x) = 0 \\ f(x) = ax + b \end{array} \right\} \quad \left. \begin{array}{l} \frac{d^n}{dx^n} f(x) = 0 \\ f(x) = \sum_{i=0}^{n-1} c_i x^i \end{array} \right\}$$

$$\ker \left( \frac{d}{dx} - a \right) = ? = \left\{ x \rightarrow s(x) \right\}$$

Number of elements is the order of d.e.

$$\left( \frac{d}{dx} - a \right) \cdot s(x) = 0 \Rightarrow \frac{ds(x)}{dx} = a s(x) \Rightarrow \frac{1}{a s(x)} = \frac{1}{\frac{ds(x)}{dx}}$$

$$\frac{ds(x)}{dx} = \frac{1}{a s(x)} \quad \frac{1}{a s(x)} = \frac{dx}{ds(x)}$$

$$x = s^{-1}(y) \quad \frac{1}{ay} = \frac{ds^{-1}(y)}{dy}$$

$$\int \frac{1}{ay} dy = \int \frac{ds^{-1}(y)}{ds} dy \Rightarrow s^{-1}(y) = \frac{1}{a} \ln y$$

$$ax = \ln(s(y))$$

$$\underline{s(x) = e^{ax}}$$

$$\textcircled{1} \quad \ln x = \int \frac{1}{x} dx$$

$$\textcircled{3} \quad \left( \frac{d}{dx} - a \right) e^{ax} = 0$$

$$\textcircled{2} \quad \log(x) = y \Leftrightarrow x = e^y$$

$$\left( \frac{d}{dx} - 2 \right) f(x) = 0$$

$$\ker \left( \frac{d}{dx} - 2 \right) = \left\{ x \rightarrow e^{2x} \right\}$$

$$f(x) = c e^{2x}$$

$$\left( \frac{d}{dx} - a \right) \left( \frac{d}{dx} - b \right) f(x) = 0 \quad f(x) = c_1 e^{bx}$$

$$\left( \frac{d}{dx} - a_1 \right) \left( \frac{d}{dx} - a_2 \right) \cdots \cdots \left( \frac{d}{dx} - a_n \right) = 0$$

$$\Rightarrow f(x) = c_1 e^{a_1 x} + c_2 e^{a_2 x} + \cdots + c_n e^{a_n x}$$

$\Rightarrow$  Complex numbers are algebraically closed

$$b_n \frac{d^n}{dx^n} + b_{n-1} \frac{d^{n-1}}{dx^{n-1}} + b_{n-2} \frac{d^{n-2}}{dx^{n-2}} + \cdots + b_1 \frac{d}{dx} + b_0 \right) f(x) = 0$$

$$b_n \left( \frac{d}{dx} - a_1 \right) \left( \frac{d}{dx} - a_2 \right) \cdots \left( \frac{d}{dx} - a_n \right) f(x) = 0$$

$$b_n r^n + b_{n-1} r^{n-1} + b_1 r + b_0 = 0$$

$$b_n (r - a_1) (r - a_2) \cdots (r - a_n) = 0$$

$$\left( \frac{d}{dx} - a \right) f(x) = 0$$

$$f(x) = e^{ax}$$

$$f'(x) - af(x) = 0$$

$$\ker \left( \frac{d}{dx} - a \right) = \{ x \rightarrow e^{ax} \}$$

$$\left( \frac{d}{dx} - a \right) \left( \frac{d}{dx} - b \right) f(x) = 0$$

$$f''(x) - (a+b)f'(x) + abf(x) = 0$$

$$f(x) = c_1 e^{ax} + c_2 e^{bx}$$

$$\ker \left( \left( \frac{d}{dx} - a_1 \right) \left( \frac{d}{dx} - a_2 \right) \cdots \left( \frac{d}{dx} - a_n \right) \right)$$

$$= \ker \left( \frac{d}{dx} - a_1 \right) \cup \ker \left( \frac{d}{dx} - a_2 \right) \cup \cdots \cup \ker \left( \frac{d}{dx} - a_n \right)$$

most general lin.ode's most general solution

$$f(x) = \sum_{i=1}^n c_i \cdot e^{a_i x}$$

with complex numbers we can find the roots  
this called algebraically closed

$$\left( \frac{d}{dx} - 1 \right) \left( \frac{d}{dx} + 1 \right) f(x) = 0$$

$$f(x) = c_1 e^x + c_2 e^{-x}$$

$$f''(x) - f(x) = 0$$

$$\left( \frac{d}{dx} + 1 \right) \left( \frac{d}{dx} + \delta + 1 \right) f(v) = 0$$

$$f(x) = C_1 e^{-x} + C_2 e^{-(1+\delta)x}$$

$$C_2 = C_3 - \frac{C_1}{\delta}$$

$$f(x) = C_3 e^{-(1+\delta)x} + \frac{C_1 e^{-x} + e^{-(1+\delta)x}}{\delta}$$

$$\lim_{\delta \rightarrow 0} f(x) = C_3 e^{-x} - C_1 \lim_{\delta \rightarrow 0} \frac{e^{-(1+\delta)x} - e^{-x}}{\delta}$$

$$= C_3 e^{-x} - C_1 x e^{-x}$$

$$f''(x) + 2f'(x) + f(x)$$

$$\left( \frac{d}{dx} + 1 \right) \left( \frac{d}{dx} + 1 \right) f(v) = 0$$

$$\left( \frac{d}{dx} \right)^2 f(x) = 0$$

$$f(x) = C_1 e^{-x} + C_2 x e^{-x}$$

$$\left( \frac{d}{dx} - i \right)^3 f(x) = 0$$

$$f(x) = \left( \sum_{k=0}^r c_k x^k \right) e^{ix}$$

$$f(x) = (c_0 + c_1 x + c_2 x^2) e^{ix}$$

$$\left( \frac{d}{dx} - \alpha_1 \right)^{k_1} \left( \frac{d}{dx} - \alpha_2 \right)^{k_2} \cdots \left( \frac{d}{dx} \alpha_n \right)^{k_n} f(x) = 0$$

$$f(y) = \sum_{p=1}^r e^{\alpha_p y} \left( \sum_{r=0}^{k_p-1} c_{pr} x^r \right)$$

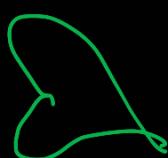
$$\left( \frac{d}{dx} - 1 \right)^2 \left( \frac{d}{dx} + 2 \right) f(x)$$

$$f'''(x) - 3f'(x) + 2f(x) = 0$$

$$r^3 - 3r + 2 = 0 \Rightarrow (r+1)^2(r-2) = 0$$

$$f(x) = \sum_{p=1}^2 e^{\alpha_p x} \left( \sum_{r=0}^{k_p-1} c_{pr} r x^r \right)$$

$$f(x) = e^x \left( \sum_{r=0}^1 c_1 r x^r \right)$$



$$e^{-2x} \left( \sum_{r=0}^6 c_{2,r} x^r \right)$$

$$f(x) = e^x (c_{1,0} + c_{1,1}x) + c_{2,0} e^{-2x}$$

Linear ODE, homogeneous, order -n diff. eqn. w/ const. coeff.

$$\left[ b_n \overbrace{\frac{d^n}{dx^n}} + b_{n-1} \overbrace{\frac{d^{n-1}}{dx^{n-1}}} + \dots + b_1 \overbrace{\frac{d}{dx}} + b_0 \right] f(x) = 0$$

$$b_n \left( \frac{d}{dx} - a_n \right)^{k_1} \left( \frac{d}{dx} - a_n \right)^{k_2} \dots \left( \frac{d}{dx} - a_m \right)^{k_m} f(x) = 0$$

$$\left( \frac{d}{dx} - 1 \right) f(x) = 0 \Rightarrow f(x) = e^x C$$

$$\left( \frac{d}{dx} - 1 \right)^2 f(x) = 0 \Rightarrow f(x) e^x (C_1 + C_2 x)$$

$$f(x) = \sum_{k=1}^m e^{a_k x} \left( \sum_{j=0}^{k-1} c_{k,j} x^j \right)$$

$$\left(\frac{d}{dx} - 1\right)^2 \left(\frac{d}{dx} + 1\right) f(x) = 0$$

$$f'''(x) - f''(x) - f'(x) + \underline{\underline{f''(x)}} = 0$$

$$f(x) = e^x (c_{1,0} + c_{1,1}x) + e^{-x} c_{(-1,0)}$$

$$c_{1,0} = 1 \quad f(x) = e^x$$

$$c_{1,1} = 0$$

$$c_{(-1,0)} = 0$$

$$\begin{aligned} c_{1,0} &= 0 \\ c_{1,1} &= 0 \\ c_{(-1,0)} &= 1 \end{aligned}$$

$$\left(\frac{d}{dx} - 1\right)^2 \left(\frac{d}{dx} + 1\right) f(x) = 0 \Rightarrow \begin{aligned} f(x) &= e^x (c_{1,0} + c_{1,1}x) \\ &\quad + e^{-x} c_{(-1,0)} \end{aligned}$$

$$f(0) = c_{1,0} + c_{(-1,0)}$$

$$f'(0) = c_{1,0} + c_{1,1} - c_{(-1,0)}$$

$$f''(0) = c_{1,0} + 2c_{1,1} + c_{(-1,0)}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} c_{1,0} \\ c_{1,1} \\ c_{2,0} \end{pmatrix} = \begin{pmatrix} f(0) \\ f'(0) \\ f''(0) \end{pmatrix}$$

$$f(x) = \frac{f(0) + f'(0)}{2} e^x + \frac{f''(0)}{3} e^{2x} + \frac{f'(0) - f(0)}{6} e^{3x}$$

$$\begin{pmatrix} c_{1,0} \\ c_{1,1} \\ c_{2,0} \end{pmatrix} = \begin{pmatrix} f(0) \\ f'(0) \\ f''(0) \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}^{-1}$$

$$f''(x) - 4f'(x) + 4f(x) = 0$$

↓

$$\left(\frac{d}{dx} - 2\right)^2 f(x) = 0$$

$$f(x) = e^{2x} (c_0 + c_1 x)$$

$$f(0) = c_0 \quad \xrightarrow{\hspace{1cm}} \quad c_0 = f(0)$$

$$f'(0) = 2c_0 + c_1 \quad \xrightarrow{\hspace{1cm}} \quad c_1 = f'(0) - 2f(0)$$

$$f(x) = e^{2x} \left( f(0) + [f'(0) - 2f(0)]x \right)$$

$$f(x) = f(0) \left[ e^{2x} - 2x e^{2x} \right] + f'(0) \left[ x e^{2x} \right]$$

Free Fall

$$\vec{F} = m \frac{\ddot{h}(t)}{dt^2} = mg \quad \frac{d^2(h(t))}{dt^2} = g$$

$$h(x) = \frac{1}{2} gt^2 + c_1 t + c_0$$

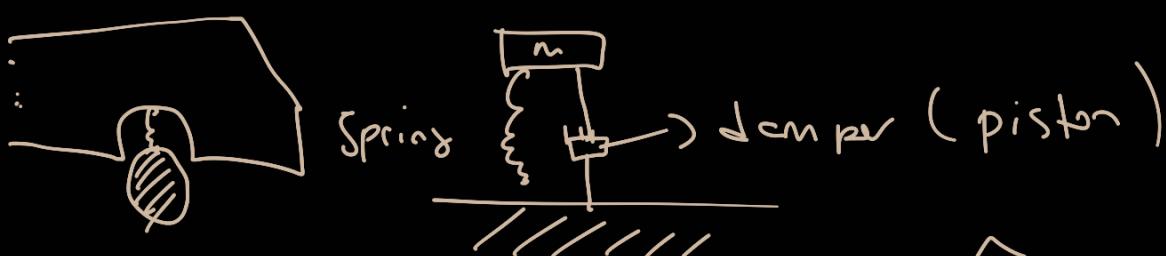
$\dot{h}(0)$ 
 $\ddot{h}(0)$ 
}
We know

$$h(0) = c_0$$

$$\dot{h}(0) = c_1$$

$$h(t) = \frac{1}{2} gt^2 + h'(0)t + h(0)$$

Mass-Spring-Damper System Example



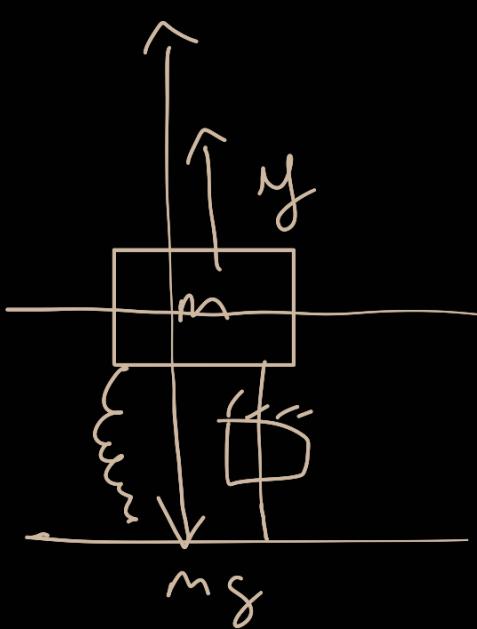
$$F = kx \rightarrow \text{Spring}$$

$$F = cx \rightarrow \text{damper}$$

$$F_{\text{Total}} = m \frac{d^2y(t)}{dt^2}$$

$$F_{\text{Spring}} = -k y(t)$$

$$F_{\text{Damper}} = -c \frac{dy(t)}{dt}$$



$$F_{\text{Total}} = F_{\text{Spring}} + F_{\text{Dumper}} - mg$$

$$-k y(t) - \left( \frac{dy(t)}{dt} \right) - mg = m \frac{d^2 y(t)}{dt^2}$$

$$\left[ \frac{d^2}{dt^2} + \frac{c}{m} \frac{d}{dt} + \frac{k}{m} \right] y(t) = -g$$

$$y_p(t) = \frac{-gm}{k}$$

$$C \cdot e = r^2 + \frac{c}{m} r + \frac{k}{m} = 0$$

$$y^{(+)} = -\frac{gm}{k} + e^{-rt} (C_1 + C_2 t) \quad (\text{if } r_1 = r_2)$$

$$y(t) = -\frac{gm}{k} + C_1 e^{-r_1 t} + C_2 e^{-r_2 t}$$

$$\left(\frac{c}{m}\right)^2 - 4 \frac{k}{m} \neq 0$$

$$y(0) = -\frac{gm}{k} + C_1 + C_2$$

$$y'(0) = -r_1 C_1 - r_2 C_2$$

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -r_1 & -r_2 \end{pmatrix} \begin{pmatrix} y(0) + \frac{gm}{k} \\ y'(0) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ -r_1 & -r_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y(0) + \frac{g^m}{k} \\ y'(0) \end{pmatrix}$$

$$\left(\frac{c}{m}\right)^2 - \frac{4k}{m} = 0 \quad y(0) = -\frac{g^m}{k} + c_1$$

$$y'(0) = -rc_1 + c_2$$

$$y(t) = -\frac{g^m}{k} + e^{-rt} \left( y(0) + \frac{g^m}{k} + y^{r_0}(t) + cy(\omega)t + \frac{c^m}{k} \right)$$

$$y(t) = \frac{g^m}{k} \left[ e^{-rt} - 1 + rt \right] + y(0) e^{-rt} [1 + rt] + y'(0) e^{-rt} t$$

# Laplace Transform

L.T is a map of function to function

$$\mathcal{L} : (\mathbb{C} \rightarrow \mathbb{C}) \rightarrow (\mathbb{C} \rightarrow \mathbb{C})$$

$$\mathcal{L} = (t \mapsto f(t)) \rightarrow \left( s \mapsto \int_0^\infty e^{-st} f(t) dt \right)$$

$$f_a = t \mapsto e^{at}$$

$$f_a(t) = e^{-at}$$

$$\mathcal{L}(f_a) = s \mapsto \int_0^\infty e^{-st} e^{at} dt$$

$$\mathcal{L}(f_a)(s) = \int_0^\infty e^{-(a+s)t} dt = \frac{1}{a+s}$$

$$f_n(t) = e^{-at} \xrightarrow{\text{L.T}} F_a(s) = \frac{1}{a+s}$$

$$F_a = \mathcal{L}(f_a)$$

$$\int_0^\infty e^{-(\alpha+\gamma)t} dt = \lim_{M \rightarrow \infty} \int_0^M e^{-(\alpha+\gamma)t} dt$$

$$= \lim_{M \rightarrow \infty} \frac{e^{-(\alpha+\gamma)t}}{-(\alpha+\gamma)} \Big|_0^M - \lim_{M \rightarrow \infty} \frac{e^{-(\alpha+\gamma)M}}{\alpha+\gamma}$$

$$f(t) = e^{\alpha t} \Rightarrow \frac{1}{\alpha+\gamma}$$

$$f(t) = C$$

$$F(s) = \int_0^\infty C e^{-st} dt = C \int_0^\infty e^{-st} dt = \frac{C}{s}$$

$$f(t) + g(t) \xrightarrow{L.T} F(s) + G(s)$$

$$e^t + e^{-t} \xrightarrow{L.T} \frac{1}{1+s} + \frac{1}{s-1}$$

$$f(t) = t \Rightarrow F(s) = \int_0^\infty e^{-st} t dt$$

$$= \int_0^\infty \frac{1}{s} \frac{d}{ds} (e^{-st}) dt$$

$$= - \frac{d}{ds} \int_0^\infty e^{-st} = \frac{1}{s} \frac{1}{s} = \frac{1}{s^2}$$

=  $\frac{1}{s^2}$

$$f(t) = t^2 \Rightarrow F(s) = \int_0^\infty e^{-st} t^2 dt = - \int_0^\infty \frac{d}{ds} (t e^{-st}) dt$$

$$= - \frac{d}{ds} \int_0^\infty t e^{-st} dt = - \frac{d}{ds} \left( \frac{1}{s^2} + \boxed{\frac{2}{s^3}} \right)$$

General rule  $f(t) = t^n \Rightarrow F(s) = \frac{n!}{s^{n+1}}$

$$f(t) = \cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$$

$$= \frac{1}{2} e^{i\omega t} + \frac{1}{2} e^{-i\omega t}$$

$$F(s) = \frac{1}{2} \frac{1}{i\omega + s} + \frac{1}{2} \frac{1}{i\omega - s} = \frac{1}{2} \frac{2s}{s^2 + \omega^2} = \frac{s}{s^2 + \omega^2}$$

$$f(t) = \sin(\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$$

$$= \frac{1}{2i} e^{i\omega t} - \frac{1}{2i} e^{-i\omega t}$$

$$F(s) = \frac{\omega}{\omega^2 + s^2}$$

$$- (s) = \frac{1}{2} \frac{1}{i\omega + s} + \frac{1}{2} \frac{1}{i\omega - s} = \frac{1}{2} \frac{2s}{s^2 + \omega^2} = \frac{s}{s^2 + \omega^2}$$

Laplace Transform:

$$\mathcal{L} : (\mathbb{C} \rightarrow \mathbb{C}) \rightarrow (\mathbb{C} \rightarrow \mathbb{C})$$

$$\mathcal{L} = (x \rightarrow f(x)) \rightarrow (s \rightarrow \int_0^{\infty} e^{-sx} f(x) dx)$$

$$\mathcal{L} \cdot f = F \quad F(s) = \int_0^{\infty} e^{-sx} f(x) dx$$

$$f(x) = 1 \Rightarrow F(s) = \frac{1}{s}$$

$$f(x) = e^{ax} \Rightarrow F(s) = \frac{1}{s-a}$$

$$f(x) = x^n \Rightarrow F(s) = \frac{n!}{s^{n+1}}$$

$$f(x) \rightarrow F(s)$$

$$g(x) \rightarrow G(s)$$

$$c_1 f(x) + c_2 g(x) \rightarrow c_1 F(s) + c_2 G(s)$$

$$\frac{d f(x)}{dx} \rightarrow ??$$

$$\begin{aligned}
 \frac{d f(x)}{dx} &\rightarrow \int_0^\infty e^{-xs} \left( \frac{d f(x)}{dx} \right) dx \\
 &= \int_0^\infty \frac{d}{dx} \left( e^{-xs} f(x) \right) dx \\
 &= - \int_0^\infty (-s) e^{-xs} f(x) \cdot dx
 \end{aligned}$$

$$\begin{aligned}
 \frac{d f(x)}{dx} &\xrightarrow{x \rightarrow \infty} \lim_{x \rightarrow \infty} \left( e^{-xs} f(x) \right) \\
 &\curvearrowleft - \lim_{x \rightarrow 0} \left( e^{-xs} f(x) \right) + s \int_0^\infty e^{-sx} f(x) dx
 \end{aligned}$$

$$\frac{d f(x)}{dx} \rightarrow s F(s) - f(0)$$

$$f(x) \frac{d g(x)}{dx} = \frac{d}{dx} \left( f(x) \cdot g(x) \right) - \frac{d f(x)}{dx} g(x)$$

$$\frac{\frac{d^2 f(x)}{dx^2}}{x^2} \rightarrow \int_0^\infty e^{-sx} \left( \frac{d f(x)}{dx} \right) dx$$

$$= \int_0^\infty \frac{d}{dx} \left( e^{-xs} \frac{df(x)}{dx} \right) dx - \int_0^\infty (-s) e^{-xs} \left( \frac{df(x)}{dx} \right) dx$$

$$\frac{\frac{d^2 f(x)}{dx^2}}{x^2} \rightarrow \lim_{x \rightarrow \infty} \left( e^{-xs} \frac{df(x)}{dx} \right) \xrightarrow{x \rightarrow 0} - \lim_{x \rightarrow 0} \left( e^{-xs} \frac{df(x)}{dx} \right)$$

$$+ s \int_0^\infty e^{-sx} \left( \frac{df(x)}{dx} \right) dx$$

$$\frac{\frac{d^2 f(x)}{dx^2}}{x^2} \rightarrow s \int_0^\infty \left( \frac{df(x)}{dx} \right) - f'(0)$$

$$\frac{\frac{d^2 f(x)}{dx^2}}{x^2} \rightarrow s^2 F(s) - s f(0) - f'(0)$$

$$\boxed{\frac{\frac{d^n f(x)}{dx^n}}{x^n} \rightarrow s^n F(s) - \sum_{i=0}^{n-1} s^i f^{(n-i)}(0)}$$

$$f'(x) + 2f(x) = 5$$

$$f'(y) + 2 f(y) = 5 f(1)$$

$$[SF(s) - f(0)] + 2F(s) = \frac{5}{s}$$

$$(s+2)F(s) = f(0) + \frac{5}{s}$$

$$F(s) = \frac{1}{s+2} f(0) + \frac{1}{s+2} \frac{5}{s}$$

$$f(x) = f(0) e^{-2x} + p(x)$$

$$\left( b_n \frac{d^n}{dx^n} + b_{n-1} \frac{d^{n-1}}{dx^{n-1}} + \dots + b_0 \right) f(x) = h(x)$$

$$(b_n s^n + b_{n-1} s^{n-1} + \dots + b_0) F(s) = ( ) f_0 + ( ) (f'_0) + \dots + f^{n-1}(0)$$

$$F(s) = \frac{() + f(0)}{c(s)} + \frac{(..) - f'(0)}{c(s)} + \dots + \frac{() f^{(r-1)}}{c(s)} + \frac{f^{(r)}(s)}{c(s)}$$

$$\frac{s-a}{(s-b)(s-c)} = \frac{\alpha}{s-b} + \frac{\beta}{s-c} = \left( \frac{a^{c-1}}{b-c} \right) \frac{1}{s-b} + \left( \frac{1-ab}{b-c} \right) \frac{1}{s-c}$$

$$\frac{s^2 + \alpha_1 s + \alpha_2}{(s-\tau_1)(s-\tau_2)(s-\tau_3)} = \frac{\alpha_1}{s-\tau_1} + \frac{\alpha_2}{s-\tau_2} + \frac{\alpha_3}{s-\tau_3}$$

$$\frac{\alpha}{s-b} + \frac{\beta}{s-c} = \alpha \frac{(s-c) + \beta(s-b)}{(s-b)(s-c)} = \frac{(\alpha + \beta)s - (\alpha c + \beta b)}{(s-b)(s-c)}$$

$$\left. \begin{array}{l} \alpha + \beta = 1 \\ \alpha c + \beta b = a \end{array} \right\} \begin{pmatrix} 1 & 1 \\ c & b \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ a \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ c & b \end{pmatrix}^{-1} = \frac{1}{b-c} \begin{pmatrix} -1 & c \\ 1 & -b \end{pmatrix}$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{b-c} \begin{pmatrix} -1 & c \\ -1 & -b \end{pmatrix} \begin{pmatrix} 1 \\ a \end{pmatrix} = \frac{1}{b-c} \begin{pmatrix} a - 1 \\ 1 - ab \end{pmatrix}$$

$$\frac{s-5}{(s-1)(s+1)} = \frac{-2}{s-1} + \frac{3}{s+1}$$

L.T      L.T

-2e<sup>x</sup> + 3e<sup>-x</sup>

$$\frac{2s-5}{(s-1)^2} = \frac{2(s-1)-3}{(s-1)^2} = \frac{2}{s-1} - \frac{3}{(s-1)^2}$$

L.T  
↓      ↓

$$\# e^x + \# x e^x$$

$$\frac{1}{s^n} \longleftrightarrow x^{(n+1)}$$

$$\frac{1}{(s-a)^n} \longleftrightarrow x^{(n+1)} e^{ax}$$

$$f(x) = e^{ax} \quad F(s) = \int_0^\infty e^{-xs} e^{ax} dx = \frac{1}{s-a}$$

$$\int_0^\infty e^{-xs} (xe^{ax}) dx = \frac{d}{da} \left( \frac{1}{s-a} \right) = \frac{1}{(s-a)^2}$$

$$f(x) = xe^{ax}$$



$$F(s) = \frac{1}{(s-a)^2}$$

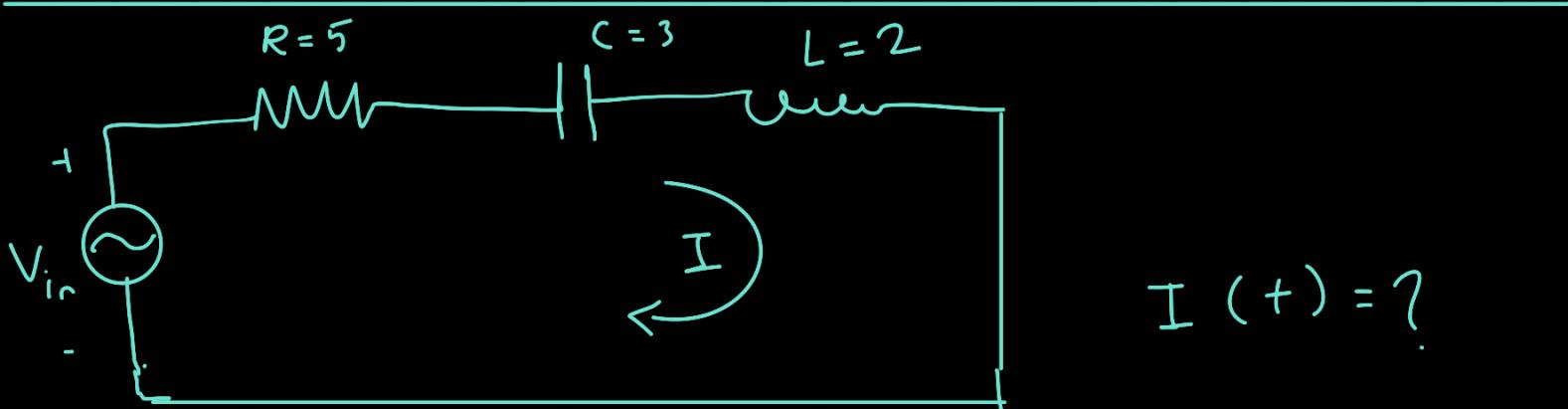
$$\int_0^\infty e^{-xs} (x^n e^{ax}) dx = \frac{d^n}{da^n} \left( \frac{1}{s-a} \right) = \frac{n!}{(s-a)^{n+1}}$$

$$\frac{(\dots)}{(s-r_1)^{k_1} (s-r_2)^{k_2} \dots (s-r_n)^{k_n}} = \frac{a_{1,1}}{s-r_1} + \frac{a_{1,2}}{(s-r_2)^2} + \dots + \frac{a_{1,k_1}}{(s-r_n)^{k_1}}$$

$$+ \frac{a_{n,1}}{s-r_2} + \frac{a_{2,2}}{(s-r_3)^2} + \dots + \frac{a_{2,k_2}}{(s-r_n)^{k_2}}$$

$$+ \dots$$

$$+ \frac{a_{n,1}}{s-r_2} + \frac{a_{2,2}}{(s-r_3)^2} + \dots + \frac{a_{2,k_2}}{(s-r_n)^{k_2}}$$



$$I(+)=?$$

$$V_{in}(+) = 10(\cos(t))$$

$$V_R = R \cdot I$$

$$V_R + V_L + V_C = V_{in}$$

$$V_C = \frac{1}{C} I$$

$$R \cdot \frac{dI}{dt} + L \cdot \frac{d^2I}{dt^2} + \frac{1}{C} I = -10\sin(t)$$

$$L \cdot \frac{dI}{dt}$$

$$\left( 5 \frac{d^2}{dt^2} + 2 \frac{d}{dt} + \frac{1}{3} \right) I(+) = -10\sin(t)$$

$$5(s^2 I(s) - s I(0) - I(0)) + 2 s(I(s)) - I_0 + \frac{1}{3} I(s)$$

$$= -10 \frac{1}{t^2 + 1}$$

$$\left( 5s^2 + 2s + \frac{1}{3} \right) I(s) = (ss+2) J(s) + sI'(s) - 10 \frac{1}{s^2+1}$$

$$I(s) = \frac{5s^2}{5s^2 + 2s + \frac{1}{3}} J(s) + \frac{s}{5s^2 + 2s + \frac{1}{3}} I'(s) + \frac{10(s^2 + 9)}{5s^2 + 2s + \frac{1}{3}}$$

$$\mathcal{L} : (t \rightarrow i(t)) \longrightarrow (s \rightarrow I(s))$$

where

$$I(s) = \int_0^\infty e^{-ts} i(t) dt$$

$$\begin{aligned} \int_0^\infty e^{-ts} i(t) dt &= \int_0^\infty \frac{d}{dt} (e^{-ts} i(t)) dt + s \int_0^\infty e^{-ts} i(t) dt \\ &= s I(s) - i(0) \end{aligned}$$

$$\begin{aligned} & (s^2 I(s) - s i(0) - i'(0)) + \frac{R}{L} s (I(s) - i(0)) - \frac{1}{LC} s I(s) \\ &= -\frac{V_0 \omega}{L} \frac{\omega}{s^2 + \omega^2} \end{aligned}$$

$$\int_0^\infty e^{-st} \sin(\omega t) dt = \frac{1}{2i} \left[ \int_0^\infty e^{-t(s-i\omega)} dt \right]$$

$$= \frac{1}{2i} \left[ \frac{1}{s-i\omega} - \frac{1}{s+i\omega} \right] = \frac{\omega}{s^2 + \omega^2}$$

$$\begin{aligned} & \text{Euler's Formula} \\ & e^{it} = \cos(x) + i \sin(x) \end{aligned}$$

$$\left( s^2 + \frac{R}{L} s + \frac{1}{LC} \right) I(s) = i'(0) + \left( s + \frac{R}{L} \right) i(0) - \frac{V_0 \omega}{L} \frac{\omega}{s^2 + \omega^2}$$

$$I(s) = \frac{i'(0)}{s^2 + \frac{R}{L}s + \frac{1}{LC}} + \frac{s + \frac{R}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} - \frac{V_0 \omega}{L} \frac{1}{(s^2 + \omega^2)(s^2 + \frac{R}{L}s + \frac{1}{LC})}$$

$$R = 2 \text{ h} \quad \omega = 1$$

$$L = 6 \quad V_0 = 2 \text{ V}$$

$$C = \frac{1}{6}$$

↓↓

$$I(s) = \frac{i'(0)}{s^2 + 4s + 1} + \frac{s+4}{s^2 + 4s + 1} - \frac{4s}{(s^2 + 1)(s^2 + 4s + 1)}$$

$$(s^2 + 4s + 1) = (s-1)(s+1) \quad r \pm \frac{-4 \mp \sqrt{16-4}}{2}$$

$$(s^2 + 4s + 1) = (s+2+\sqrt{3})(s+2-\sqrt{3})$$

$$\frac{1}{s^2 + 4s + 1} = \frac{1}{(s+2+\sqrt{3})(s+2-\sqrt{3})} = \frac{\alpha}{s+2+\sqrt{3}} - \frac{\beta}{s+2-\sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}}$$

$$\frac{i'(0)}{s^2 + 4s + 1} = \frac{i'(0)}{2\sqrt{3}} \frac{1}{s+2+\sqrt{3}} - \frac{i'(0)}{2\sqrt{3}} \frac{1}{s+2-\sqrt{3}}$$

$$i(t) = \frac{i(0)}{2\sqrt{3}} \left( e^{-(2\sqrt{3})t} - e^{-2+\sqrt{3}t} \right)$$

$$+ \frac{i(0)}{2\sqrt{3}} \sqrt{2-2} e^{-2-\sqrt{3}t} + \sqrt{42} e^{-(2-\sqrt{3})t}$$

$$+ p(t)$$

$$\frac{1}{(s^2+1)(s^2+4s+1)} = \frac{c_1}{s+1} + \frac{c_2}{s-i} + \frac{c_3}{s+2+\sqrt{3}} + \frac{c_4}{s+2-\sqrt{3}}$$

$$p(t) = c_1 e^{-it} + c_2 e^{it} + c_3 e^{-(2+\sqrt{3})t} + c_4 e^{-2+\sqrt{3}t}$$

}

$$-40 = c_1(s-i)(s+2+\sqrt{3})(s+2-\sqrt{3})$$

$$+ c_2(s+i)(s+2+\sqrt{3})(s+2-\sqrt{3})$$

$$+ c_3(s+i)(s+2+\sqrt{3})(s+2-\sqrt{3})$$

$$+ c_4(s+i)(s+2+\sqrt{3})(s+2-\sqrt{3})$$

$$-40 = (\dots)s^3 + (\dots)s^2 + \dots$$

$$g(x, \frac{d}{dx}) f(x) = h(x)$$

↓

$$P(s) F(s) = (\dots) f(0) + (\dots) f'(0) + f^{n-1}(0) \dots + H(s)$$

$$F(s) = \frac{(\cdot)}{P(s)} f(0) + \frac{(\cdot)}{P(s)} f'(0) + \frac{H(s)}{P(s)}$$

$$F_p(s) = \frac{1}{P(s)} \cdot H(s)$$

Multiplication of  $\frac{1}{P(s)}$  and  $H(s)$

↑

???

Convolution

$$g(x, \frac{d}{dx}) i = \delta(x) \text{ with } i(0) = i'(0) \dots = 0$$

↓

System is causal zero

input

$$P(s) I(s) = 1 \Rightarrow I(s) = \frac{1}{P(s)}$$

↳ its Laplace transform is

for even  
in RCL  
 $k=0$   
 $v=0$  } etc

$i(v) = \text{impulse response} = \text{function that maps to } \frac{1}{p(s)}$   
under  $\mathcal{L}$

$\mathcal{G}(x) = \text{function that maps to } 1 \text{ under } \mathcal{L} \text{ transform}$

$$g(x, \frac{d}{dx}) f(x) = h(x)$$

$$P(s) f(s) = (0)f(0) + (1)f'(0) + \dots + \int^{(n-1)}(0) H(s)$$

$$F(s) = \frac{(0)}{P(s)} f(0) + \frac{(1)}{P(s)} f'(0) + \frac{H(s)}{P(s)}$$

$$(\mathcal{L} \cdot f_p)(s) = \frac{H(s)}{P(s)} = \frac{1}{P(s)} \cdot H(s)$$

What map to multiplication under Laplace Transf.?

$$F(s) = \int_0^\infty e^{-sx} f(v) dv$$

$$G(s) = \int_0^\infty e^{-sy} g(y) dy$$

$$z = x+y$$

$$F(s) \cdot G(s) = \int_0^\infty \left( \int_0^\infty e^{-s(x+y)} f(x) g(y) dy \right) dx \quad dt = dy$$

$$= \int_0^\infty dx \int_0^\infty dz e^{-sz} f(x) g(z-x)$$

$$f(x) = 0 \quad \text{if } x < 0$$

$$g(x) = 0 \quad \text{if } x < 0$$

$$F(s) G(s) = \int_0^\infty dx \int_0^\infty dz e^{-sz} f(x) g(z-x)$$

+ 0

$$= \int_0^\infty dy \int_x^\infty dz e^{-sz} f(x) g(z-y)$$

$$+ \int_0^\infty dx \int_0^x dz e^{-sz} f(y) g(z-x)$$

$$F(s) G(s) = \int_0^\infty dx \int_0^\infty dz e^{-sz} f(x) g(z-x)$$

$$\underbrace{(f * g)(z)}_{\text{Convolution}} = \int_0^\infty f(x) g(z-x) dx$$

Convolution

$$\mathcal{L}(f * g) = \mathcal{L}(f) \mathcal{L}(g)$$

$$h(z) = (f * g)(z)$$

$$H(s) = F(s) G(s)$$

$$g(x, \frac{d}{dx}) f(x) = h(x) \Rightarrow f(x) = f_{h,1}(x) + f_{h,x} + f_g(x)$$

$$F_p(s) = \frac{1}{p(s)} H(s) \quad i(x) \rightarrow \frac{1}{p(s)}$$

$$F_p(x) = \int_0^x p(z) h(x-z) dz$$

$$g(x, \frac{d}{dx}) i(x) = \delta(x) \quad i(0) = i'(0) = 0$$

$$p(s) I(s) = 1 \Rightarrow I(s) = \frac{1}{p(s)}$$

$$1 = \int_0^\infty e^{-sx} \delta(x) dx$$

$$1 = \int_0^\infty \left( \sum_{n=0}^\infty \frac{(-sx)^n}{n!} \right) \delta(x) dx$$

$$1 = \sum_{n=0}^\infty s^n \left( \frac{(-1)^n}{n!} \right) \int_0^\infty x^n \delta(x) dx$$

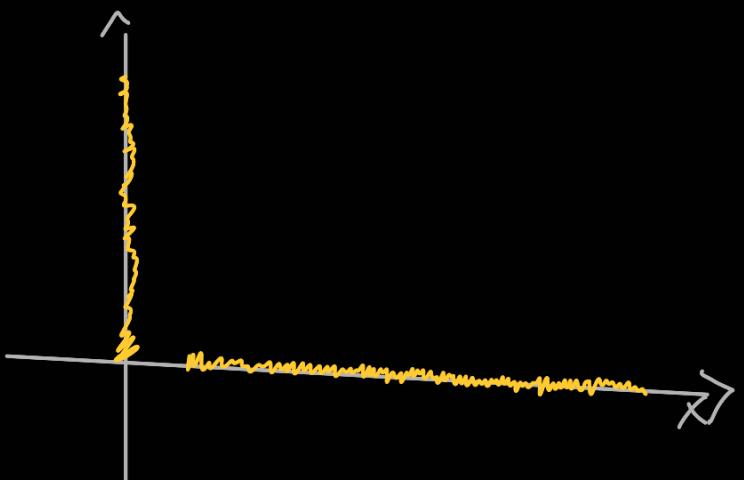
$$\int_0^\infty x^n \delta(x) dx = \begin{cases} 0 & \text{for } n \in \mathbb{Z}^+ \\ 1 & \text{for } n = 0 \end{cases}$$

$$1 s^0 + 0 s^1 + 0 s^2 + \dots =$$

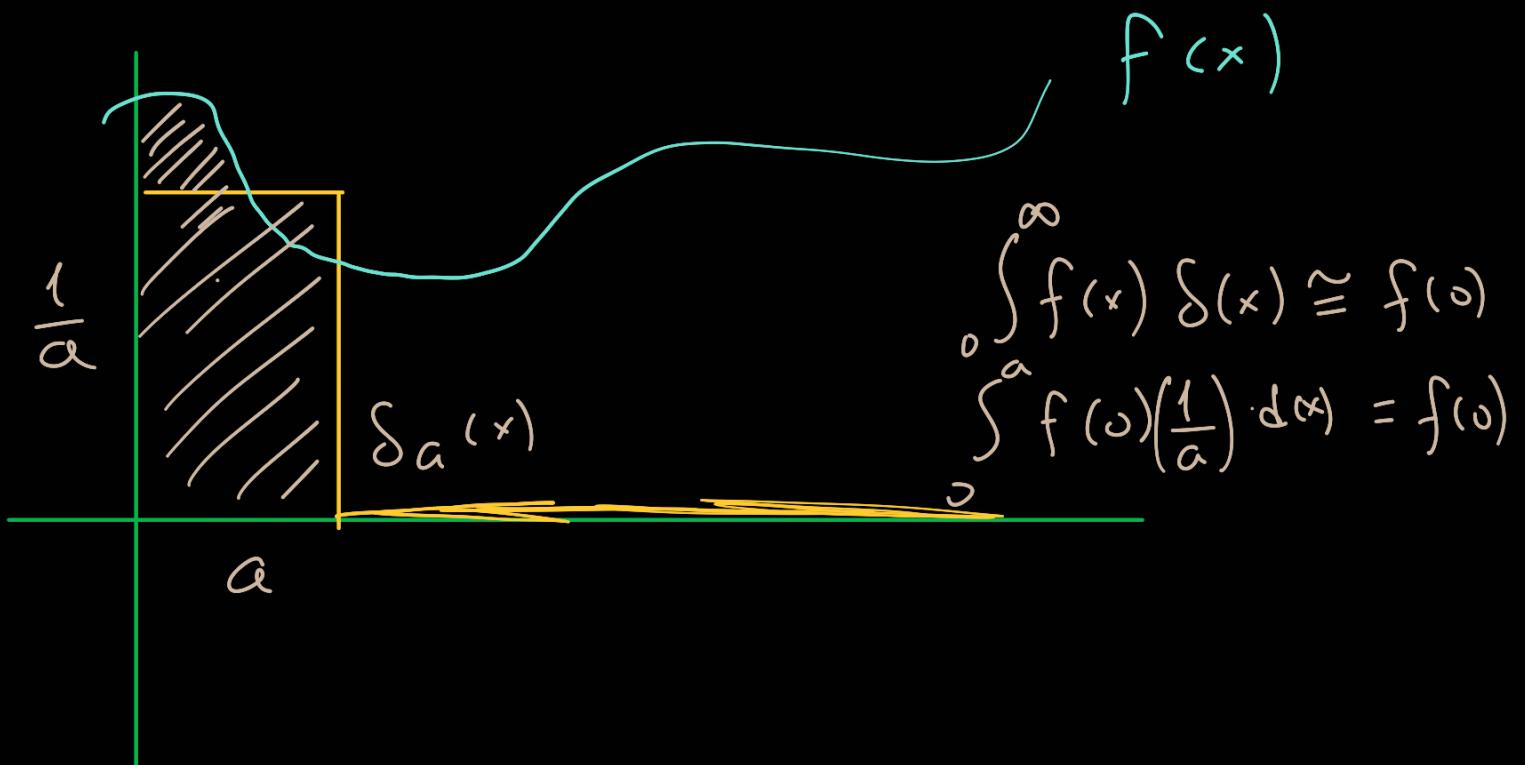
$$\left( \int_0^\infty \delta(x) dx \right) s^0 + \underbrace{\left( - \int_0^\infty x \delta(x) dx \right)}_0 s^1 + \underbrace{\left( \frac{1}{2!} \delta(x) dx \right)}_0 s^2 + \dots$$

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ ?? & \text{if } x = 0 \end{cases}$$

$$\int_0^\infty \delta(x) dx = 1 \Rightarrow \text{Dirac-Delta Function}$$



$$\int_0^\infty f(x) \delta(x-a) dx = f(a)$$



$$\int_0^\infty x^n \delta(x) dx = \begin{cases} 0 & \text{if } n \notin \mathbb{Z}^+ \\ 1 & \text{if } n=0 \end{cases}$$

$$\int_0^\sigma e^{-xs} \delta(x) dx = e^{-x\sigma} \Big|_{x=0} = 1$$

$$I(s) = \frac{1}{P(s)} = \frac{1}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0}$$

$$= \frac{c_1}{s - r_1} + \frac{c_2}{s - r_2} + \frac{c_3}{(s - r_3)^2} + \dots$$

$$\cancel{f'(x) + \cos(2x) = f(x)}$$

$$f(0) = 0$$

$$f'(0) = 0$$

$$\left(\frac{d}{dx} - 1\right) f(x) = -\cos(2x)$$

$$P(s) = s-1 \quad I(x) = e^x$$

$$\bar{I}(s) = \frac{1}{s-1}$$

$$f_p(x) = \int_0^x i(z) h(x-z) dz$$

$$= \int_0^x e^z (-\cos(2x-2z)) dz$$

$$f_p(x) = - \int_0^x e^z \cos(2x-2z) dz$$

$$= \frac{1}{2} \int_0^x e^{z+2ix - z^2} dz - \frac{1}{2} \int_0^x e^{z-2ix + z^2} dz$$

$$= \frac{e^{2ix}}{2} \int_0^x e^{z(1-2i)} dz - \frac{e^{-2ix}}{2} \int_0^x e^{z(1+2i)} dz$$

$$= -\frac{e^{2ix}}{2} \cdot \frac{e^{z(1-2i)}}{1-2i} \Big|_0^x - \frac{e^{-2ix}}{2} \cdot \frac{e^{z(1+2i)}}{1+2i} \Big|_0^x$$

$$\cos \alpha = \frac{e^{ix} + e^{-ix}}{2}$$

$$f_p(x) = -\frac{e^{2ix}}{z(1-z)} \left( e^{x(1-z)-1} \right) \cdot \frac{e^{-2iz}}{z(1-z)} \left( e^{x_i(m)} - 1 \right)$$

$$= \frac{1}{z(1-z)} \cdot (e^{2ix} - e^x) + \frac{1}{z(1-z)} \left( e^{2ix} \right) \left( e^{-2iz} - e^x \right)$$

$$f_p(x) = -\frac{1}{5} e^x + \left( \frac{e^{2ix}}{z(1-z)} + \dots \right)$$

$$\left( \frac{1}{dx} - 1 \right) f(x) = -\cos(2x)$$