PHYS 209

Type Notation

$5 :: \text{Integer} \Rightarrow$ 5 is of type integers

sin -> function, sin(x) => output, (x) => input

functions are maps, they take inputs and spit out outputs.

$\sin : \mathbb{R} \longrightarrow \mathbb{R}$ "sin" is of type real to real

$sq : \mathbb{Z} \longrightarrow \mathbb{Z}$ (type) -> domain -> codomain

$sq = x \longrightarrow x^2$ (definition)

$sq(2) = 4$

$sq(-3) = 9$

x, y, z ... = dummy variable

$\text{abs} : \mathbb{R} \to \mathbb{R}^{\geq 0}$

$\text{abs} = x \to \sqrt{x^2}$

-> absolute value

$\text{abs}(-10) = 10$

T is operation which double the output of the function.

$f = x \longrightarrow x + 5$

$(T \cdot f) = x \to 2x + 10$

map to map both of them are function

$T : (A \to B) \longrightarrow (A \to C)$

$T = (x \to f(x)) \to (y \to 2f(y))$

$(T \cdot \sin)(x) = 2\sin(x)$

$$\frac{d}{dx} : (A \to B) \to (A \to B)$$

$$\frac{d}{dx} : (x \to f(x)) \to (x \to f'(x))$$

$$\frac{d}{dx} \cdot \sin = \cos$$ $\to$ it is a function not value

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$\int$ is computing an area of a curve

$$\int : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$$

$$\int = (x \to f(x)) \longrightarrow \int_{-\infty}^{\infty} f(y)\,dy$$

$\Rightarrow$ Force $f = x \to x^2 + 2$

$$W = \int_{-\infty}^{\infty} f(t)\,dt$$

$f : \mathbb{R} \to \mathbb{C}$

$f : x \to x^2 + i$

$\left(\frac{d}{dx} \cdot f\right) : \mathbb{R} \to \mathbb{R}$

$\frac{d}{dx} \cdot f = x \to 2x$

$2 : \mathbb{R}$

$\sin : \mathbb{R} \to \mathbb{R}$

$\underbrace{\sin(x)}_{\text{codomain}} : \mathbb{R}$

$$\frac{d^n}{dx^n} : (A \to B) \to (A \to B)$$

$$\left.\frac{d^{-1}}{dx^{-1}}\right\}$$ anti-derivative

$$\int \cdot \cos(x)\,dx = \sin(x) = \frac{d^{-1}}{dx^{-1}} \cdot \cos = \sin$$

$$\boxed{\begin{array}{l} f = \vec{v} \to \vec{u} \cdot \vec{v} \\ f(f(\vec{w})) \Rightarrow \text{ill-defined} \end{array}}$$

$$\frac{d}{dx} \cdot (A \to B) \to (C \to D) \qquad f : A \to B$$

$$\left(\frac{d}{dx} \cdot f\right) : A \to B$$

$$\left(\frac{d}{dx} \cdot \left(\frac{d}{dx} \cdot f\right)\right) : A \to B$$

Differential Equations

Definition: Any relation that contains the derivative higher order function $\frac{d}{dx}$ and an unknown function f as a differential equation

$$\frac{d}{dx} \cdot f = 0$$

Classification of D.E

$f(\lambda) \longrightarrow \lambda(f(x))$ $\boxed{\text{Linear}}$

$\lambda(\ldots) + (\ldots) = 0$ $\longrightarrow$ { general rule
quick way to check }

$f(x) f'(x) = 2$ $\boxed{\text{Non-linear}}$

$\lambda^2 (f(x) \cdot f'(x) + (-2)) = 0$

$\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial f(x,y)}{\partial y}$ $\boxed{\text{Partial}}$ others are ordinary

$\boxed{\text{order is the highest derivatives}}$

Let $p(x)$ satisfy $g\left(x, \frac{d}{dx}\right) p(x) = h(x)$

Let $f_h(x)$ satisfy $g\left(x, \frac{d}{dx}\right) f_h(x)$ $= 0 \Rightarrow$ trivial question

Define $s(x) = \underbrace{p(x)}_{\text{particular solution}} + c\, f_h(x) \longrightarrow$ homogenous solution

Observe: $g\left(x, \frac{d}{dx}\right) s(x) = g\left(x, \frac{d}{dx}\right) \left(p(x) + c\, f_h(x) \right)$

$$= g\left(x, \frac{d}{dx}\right) + c\, g\left(x, \frac{d}{dx}\right) f_h(x)$$

$$\left(\frac{d^2}{dx^2} + 4\right) f(x) = x$$

$$f''(x) + 4f(x) = x$$

homogenous part

$$\left(\frac{d^2}{dx^2} + 4\right) f(x)$$

$$f_h(x) = \sin(2x)$$

$$f_m(x) = \cos(2x)$$

General solution $= f_h(x) = c_1 \sin(2x) + c_2 \cos(2x)$

principle of superposition

$$p(x) = \frac{x}{4}$$

$$s(x) = \frac{x}{4} + \left(c_1 \cos(2x) + c_2 (\sin(2x)) \right)$$

We arrive at the conclusion "linear ordinary diff. eqn" are satisfied by a family of solutions of the form

$$s(x) = p(x) + \left(c_1 f_{h_1}(x) + c_2 f_{h_2}(x) + \ldots \right)$$

$$g\left(x, \frac{d}{dx}\right) f(x) = 0$$

$$\left(x^2 + \cos\left(\frac{d}{dx}\right) + \frac{1}{x} + \frac{d^2}{dx^2}\right) f(x) = 0$$

$$\left[c_n(x) \frac{d^n}{dx^n} + c_{(n-1)} \frac{d^{(n-1)}}{dx^{(n-1)}} + c_{(n-2)}(x) \frac{d^{n-2}}{dx^{n-2}} + \dots + c_0(x)\right] f(x) = 0$$

$$c_n(x) f^{n}(x) + c_{n-1} f^{(n-1)}(x) \dots\dots c_0(x) + f(x) = 0$$

Linear, ordinary, homogenous, order-n diff. eqn.

$$\sum_{i=0}^{n} c_i f^{(i)}(x) = 0$$

Linear, ordinary, homogenous, order-n diff. eqn. w/ constant coefficient

$$\boxed{\sum_{i=0}^{n} c_i f^{i}(x) = 0}$$

$$e^{d/dx} f(x) = x^2$$

↳ infinite order

$$\sum_{i=0}^{n} c_i f^{(i)}(x) = 0 \Rightarrow$$ most generic form

Linear Ordinary order-n Diff eqn. w/ constn coeff.

$$\boxed{\sum_{i=0}^{n} c_i f^{(i)}(x) = h(x)}$$ $\Rightarrow$ Non-homogenous

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$$g\left(x, \frac{d}{dx}\right) f(x) = 0$$

Kernel of a diff. operator is a set of functions which are annihilated by the differential operator.

$$\ker\left(\frac{d}{dx}\right) = \{x \to c\}$$ (functions are map)

↳ basically $f(x) = c$

Ex/

$$\ker\left(\frac{d^2}{dx^2}\right) = \{x \to c, x \to x\}$$

trivial solution is a element of kernel

all homogenous solutions are an elemt of kernel

$$\ker\left(g\left(x, \frac{d}{dx}\right)\right) = \{f_1, f_2, f_3 \ldots f_n\}$$

$$\ker\left(\frac{d^2}{dx^2} - 4\right) = \{x \to e^{2x}, x \to e^{-2x}\}$$

$$\left(\frac{d^2}{dx^2} - 4\right) e^{2x} = 0$$

$$\left(\frac{d^2}{dx^2} - 4\right) e^{-2x} = 0$$

$$\left(\frac{d^2}{dx^2} - 4\right) f(x) = 0$$

$\Updownarrow$

$$f''(x) - 4(f(x)) = 0$$

$$g\left(x, \frac{d}{dx}\right) f(x) = h(x)$$

$$\ker\left(g\left(x, \frac{d}{dx}\right)\right) = \{ f_1, \dots f_n)$$

$$f(x) = p(x) + c_1 f_1(x)\ c_2 f_2(x) + \dots + c_n f_n(x)$$

$$\sum_{i=0}^{n} c_i(x) f^{(i)}(x) = h(x)$$

$\ker\left(\sum_{i=0}^{n} c_i(x) \frac{d^i}{dx^i}\right)$ is always computable!

$$\frac{d^2}{dx^2} f(x) = 2 \qquad \ker\left(\frac{d^2}{dx^2}\right) = \{ x \to c,\ x \to x \}$$

$$f(x) = p(x) + c_1 + c_2 x$$

$$f(x) = x^2 + c_1 + c_2 x$$

$$\frac{d}{dx} f(x) = 0$$

$$f(x) = c$$

$$\frac{d^2}{dx^2} f(x) = 0$$

$$f(x) = ax + b$$

$$\frac{d^n}{dx^n} f(x) = 0$$

$$f(x) = \sum_{i=0}^{n-1} c_i x^i$$

$$\ker\left(\frac{d}{dx}-a\right) = ? = \{ x \to s(x) \}$$

number of elements is the order of d.e.

$$\left(\frac{d}{dx}-a\right)\cdot s(x) = 0 \Rightarrow \frac{d\,s(x)}{dx} = a\,s(x) \Rightarrow \frac{1}{a\,s(x)} = \frac{1}{\frac{d\,s(x)}{dx}}$$

$$\frac{d\,s(x)}{dx}\,\frac{dx}{ds(x)} = 1 \qquad \frac{1}{a\,s(x)} = \frac{dx}{ds(x)}$$

$$x = s^{-1}(y) \qquad \frac{1}{ay} = \frac{ds^{-1}(y)}{dy}$$

$$\int \frac{1}{ay}\,dy = \int \frac{ds^{-1}(y)}{ds}\,dy \Rightarrow s^{-1}(y) = \frac{1}{a}\ln y$$

$$ax = \ln(s(x))$$

$$\underline{s(x) = e^{ax}}$$

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① $\ln x = \int \frac{1}{x}\cdot dx$

② $\log(x) = y \Leftrightarrow x = e^{y}$

③ $\left(\frac{d}{dx}-a\right)e^{ax} = 0$

$$\left(\frac{d}{dx}-2\right)f(x) = 0$$

$$\ker\left(\frac{d}{dx}-2\right) = \{x \to e^{2x}\}$$

$$f(x) = ce^{2x}$$

$$\left(\frac{d}{dx}-a\right)\left(\frac{d}{dx}-b\right)f(x)=0 \qquad f(x)=c_1e^{bx}$$

$$\left(\frac{d}{dx}-a_1\right)\left(\frac{d}{dx}-a_2\right)\cdots\cdots\left(\frac{d}{dx}-a_n\right)=0$$

$$\Rightarrow f(x)=c_1e^{a_1x}+c_2e^{a_2x}\cdots\cdots+c_ne^{a_nx}$$

$\Rightarrow$ Complex numbers are algebraically closed

$$\left(b_n\frac{d^n}{dx^n}+b_{n-1}\frac{d^{n-1}}{dx^{n-1}}+b_{n-2}\frac{d^{n-2}}{dx^{n-2}}\cdots\cdots+b\frac{d}{dx}+b_0\right)f(x)=0$$

$$b_n\left(\frac{d}{dx}-a_1\right)\left(\frac{d}{dx}-a_2\right)\quad\left(\frac{d}{dx}-a_n\right)f(x)=0$$

$$b_nr^n+b_{n-1}r^{n-1}+b_1r+b_0=0$$

$$b_n(r-a_1)(r-a_2)\cdots\cdots(r-a_n)=0$$

$$\left(\frac{d}{dx}-a\right)f(x)=0$$

$$f(x)=e^{ax}$$

$$f'(x)-af(x)=0$$

$$\ker\left(\frac{d}{dx}-a\right)=\{x\to e^{ax}\}$$

$$\left(\frac{d}{dx}-a\right)\left(\frac{d}{dx}-b\right) f(x)=0$$

$$f''(x)-(a+b)f'(x)+ab\,f(x)=0$$

$$f(x)=C_1e^{ax}+C_2e^{bx}$$

$$\ker\left(\left(\frac{d}{dx}-a_1\right)\left(\frac{d}{dx}-a_2\right)\cdots\frac{d}{dx}-a_n\right)$$

$$=\ker\left(\frac{d}{dx}-a_1\right)\cup\ker\left(\frac{d}{dx}-a_2\right)\cup\cdots\ker\left(\frac{d}{dx}-a_n\right)$$

most general lin. ode's most general solution

$$f(x)=\sum_{i=1}^{n}c_i\cdot e^{ax}$$

with complex numbers we can find the roots
this called algebraically closed

$$\left(\frac{d}{dx}-1\right)\left(\frac{d}{dx}+1\right)\cdot f(x)=0$$

$$f(x)=C_1e^{x}+C_2e^{-x}$$

$$f''(x)-f(x)=0$$

$$\left(\frac{d}{dx} + 1\right)\left(\frac{d}{dx} + \delta + 1\right) f(x) = 0$$

$$f(x) = c_1 e^{-x} + c_2 e^{-(1+\delta)x}$$

$$c_2 = c_3 - \frac{c_1}{\delta} \qquad f(x) = c_3 e^{-(1+\delta)x} + c_1 \frac{e^{-x} + e^{-(1+\delta)x}}{\delta}$$

$$\lim_{\delta \to 0} f(x) = c_3 e^{-x} - c_1 \lim_{\delta \to 0} \frac{e^{-(1+\delta)} - e^{-x}}{\delta}$$

$$= \underline{c_3 e^{-x} - c_1 x e^{-x}}$$

$$f''(x) + 2f'(x) + f(x)$$

$$\left(\frac{d}{dx} + 1\right)\left(\frac{d}{dx} + 1\right) f(x) = 0$$

$$\left(\frac{d}{dx}\right)^2 f(x) = 0$$

$$f(x) = c_1 e^{-x} + c_2 x e^{-x}$$

$$\left(\frac{d}{dy} - i\right)^3 f(x) = 0$$

$$f(x) = \left(\sum_{\ell=0}^{2} c_\ell x^\ell\right) e^{ix}$$

$$f(x) = \left(c_0 + c_1 x + c_2 x^2\right) e^{ix}$$

$$\boxed{\begin{gathered}\left(\frac{d}{dx} - a_1\right)^{k_1}\left(\frac{d}{dx} - a_2\right)^{k_2} \cdots\cdots \left(\frac{d}{dx} a_n\right)^{k_n} f(x) = 0 \\ f(x) = \sum_{p=1}^{n} e^{a_p x}\left(\sum_{r=0}^{k_p-1} c_{pr} x^r\right)\end{gathered}}$$

$$\left(\frac{d}{dx} - 1\right)^2\left(\frac{d}{dx} + 2\right) f(x)$$

$$f'''(x) - 3f'(x) + 2f(x) = 0$$

$$r^3 - 3r + 2 = 0 \Longrightarrow (r+1)^2 (r-2) = 0$$

$$f(x) = \sum_{p=1}^{2} e^{a_p x}\left(\sum_{r=0}^{k_p\;1} c_{pr} x^r\right)$$

$$f(x) = e^x\left(\sum_{r=0}^{1} c_{1r} x^r\right)$$

$$e^{-2x}\left(\sum_{r=0}^{0} c_{2,r}\, x^{r}\right)$$

$$\boxed{f(x) = e^{x}\left(c_{1,0} + c_{1,1}x\right) + c_{2,0}\, e^{-2x}}$$

Linear, ode, homogenous, order-n diff. eqn. w/ constant coeff.

$$\left[b_n \frac{d^n}{dx^n} + b_{n-1}\frac{d^{n-1}}{dx^{n-1}} + \dots + b_1 \frac{d}{dx} + b_0\right] f(x) = 0$$

$$b_n \left(\frac{d}{dx} - a_1\right)^{k_1}\left(\frac{d}{dx} - a_2\right)^{k_2} \cdots\cdots \left(\frac{d}{dx} - a_m\right)^{k_m} f(x) = 0$$

$$\left(\frac{d}{dx} - 1\right) f(x) = 0 \implies f(x) = e^{x} c$$

$$\left(\frac{d}{dx} - 1\right)^2 f(x) = 0 \implies f(x)\; e^{x}\left(c_1 + c_2 x\right)$$

$$f(x) = \sum_{l=1}^{m} e^{a_l x}\left(\sum_{j=0}^{k_l - 1} c_{l,j}\, x^{j}\right)$$

$$\left(\frac{d}{dx} - 1\right)^2 \left(\frac{d}{dx} + 1\right) f(x) = 0$$

$$f'''(x) - f''(x) - f'(x) + f(x) = 0$$

$$f(x) = e^{x}(c_{1,0} + c_{1,1}x) + e^{-x} c_{(2,0)}$$

$c_{1,0} = 1$ $\quad f(x) = e^{x}$

$c_{1,1} = 0$

$c_{2,0} = 0$

$c_{1,0} = 0$ $\quad f(x) = e^{-x}$

$c_{1,1} = 0$

$c_{2,0} = 1$

$$\left(\frac{d}{dx} - 1\right)^2 \left(\frac{d}{dx} + 1\right) f(x) = 0 \Rightarrow f(x) = e^{x}(c_{1,0} + c_{1,1}x) + e^{-x} c_{2,0}$$

$$f(0) = c_{1,0} + c_{2,0}$$

$$f'(0) = c_{1,0} + c_{1,1} - c_{2,0}$$

$$f''(0) = c_{1,0} + 2c_{1,1} + c_{2,0}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} c_{1,0} \\ c_{1,1} \\ c_{20} \end{pmatrix} = \begin{pmatrix} f(0) \\ f'(0) \\ f''(0) \end{pmatrix}$$

$$f(x) = \frac{f(0) + f'(0)}{2} e^x + \frac{f''(0)}{3} e^x + \frac{f'(0) - f'(0)}{6} e^{-x}$$

$$\begin{pmatrix} c_{1,0} \\ c_{1,1} \\ c_{20} \end{pmatrix} = \begin{pmatrix} f(0) \\ f'(0) \\ f''(0) \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}^{-1}$$

$$f''(x) - 4f'(x) + 4f(x) = 0$$

$$\downarrow$$

$$\left(\frac{d}{dx} - 2\right)^2 f(x) = 0$$

$$f(x) = e^{2x}(c_0 + c_1 x)$$

$$f(0) = c_0 \qquad c_0 = f(0)$$

$$f'(0) = 2c_0 + c_1 \qquad c_1 = f'(0) - 2f(0)$$

$$f(x) = e^{2x}\left(f(0) + [f'(0) - 2f(0)]x\right)$$

$$f(x) = f(0)\left[e^{2x} - 2xe^{2x}\right] + f'(0)\left[xe^{2x}\right]$$

Free Fall

$$\vec{F} = m\frac{d^2(h(t))}{dt^2} = mg \qquad \frac{d^2(h(t))}{dt^2} = g$$

$$\boxed{h(x) = \frac{1}{2}gt^2 + c_1 t + c_0}$$

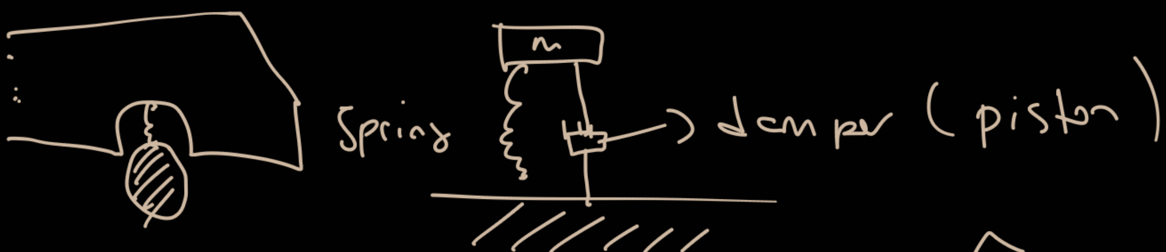
$$\left.\begin{matrix} h(0) \\ h'(0) \end{matrix}\right\} \text{we know}$$

$h(0) = c_0$

$h'(0) = c_1$

$$h(t) = \frac{1}{2}gt^2 + h'(0)t + h(0)$$

Mass-Spring Damper System Example



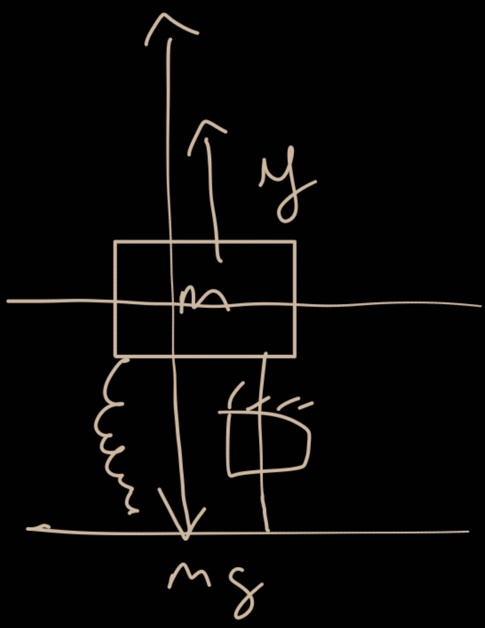
$F = kx \rightarrow$ spring

$F = c\dot{x} \rightarrow$ damper

$$F_{Total} = m\frac{d^2y(t)}{dt^2}$$

$$F_{spring} = -k\,y(t)$$

$$F_{Damper} = -c\frac{d(y(t))}{dt}$$



$$F_{Total} = F_{Spring} + F_{damper} - mg$$

$$-k\,y(t) - c\frac{dy(t)}{dt} - mg = m\frac{d^2y(t)}{dt^2}$$

$$\boxed{\left[\frac{d^2}{dt^2} + \frac{c}{m}\frac{d}{dt} + \frac{k}{m}\right] y(t) = -g}$$

$$y_p(t) = \frac{-gm}{k}$$

$$c.e = r^2 + \frac{c}{m} r + \frac{k}{m} = 0$$

$$y(t) = -\frac{gm}{k} + e^{-rt}\left(C_1 + C_2 t\right) \quad (\text{if } r_1 = r_2)$$

$$y(t) = -\frac{gm}{k} + C_1 e^{-r_1 t} + C_2 e^{-r_2 t}$$

$$\left(\frac{c}{m}\right)^2 - 4\frac{k}{m} \neq 0$$

$$y(0) = -\frac{gm}{k} + C_1 + C_2$$

$$y'(0) = -r_1 C_1 - r_2 C_2$$

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -r_1 & -r_2 \end{pmatrix} \begin{pmatrix} y(0) + \frac{gm}{k} \\ y'(0) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ -r_1 & -r_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y(0) + \frac{gm}{k} \\ y''(0 \end{pmatrix}$$

$$\left(\frac{e}{m}\right)^2 - \frac{4k}{m} = 0$$

$$y(0) = -\frac{gm}{k} + c_1$$

$$y'(0) = -rc_1 + c_2$$

$$y(t) = -\frac{gm}{k} + e^{-rt}\left(y(0) + \frac{gm}{k} + y^{r_0}(t) + ry(0)t + \frac{rgm}{k}\right)$$

$$y(t) = \frac{gm}{k}\left[e^{-rt} - 1 + rt\right] + y(0)e^{-rt}\left[1 + rt\right] + y'(0)e^{-rt}t$$

Laplace Transform

L.T is a map of function to function

$$\mathcal{L} : (\mathbb{C} \to \mathbb{C}) \to (\mathbb{C} \to \mathbb{C})$$

$$\mathcal{L} = \left(t \to f(t)\right) \to \left(s \to \int_0^\infty e^{-st} f(t)\, dt\right)$$

$$f_a = t \to e^{at}$$

$$f_a(t) = e^{-at}$$

$$\mathcal{L}(f_a) = s \to \int_0^\infty e^{-st} e^{at}\, dt$$

$$\mathcal{L}(f_a)(s) = \int_0^\infty e^{-(a+s)t}\, dt = \frac{1}{a+s}$$

$$f_a(t) = e^{-at} \xrightarrow{L.T} F_a(s) = \frac{1}{a+s}$$

$$F_a = \mathcal{L}(f_a)$$

$$\int_0^\infty e^{-(a+s)t}\,dt = \lim_{M\to\infty} \int_0^M e^{-(a+s)t}\,dt$$

$$= \lim_{M\to\infty} \left. \frac{e^{-(a+s)t}}{-(a+s)} \right|_0^M \frac{1}{a+s} - \lim_{M\to\infty} e^{-(a+s)M}$$

$$f(t) = e^{at} \Rightarrow \frac{1}{a+s}$$

$$f(t) = C$$

$$F(s) = \int_0^\infty C e^{-st}\,dt = C\int_0^\infty e^{-st}\cdot dt = \frac{C}{s}$$

$$f(t) + g(t) \xrightarrow{L.T} F(s) + G(s)$$

$$e^{t} + e^{-t} \xrightarrow{L.T} \frac{1}{1+s} + \frac{1}{s-1}$$

$$f(t) = t \Rightarrow F(s) = \int_0^\infty e^{-st}\,t\,dt$$

$$= \int_0^\infty \frac{d}{ds}\left(e^{-st}\right)dt$$

$$= -\frac{d}{ds}\int_0^\infty e^{-st} = \frac{1}{ds}\frac{1}{s} = \frac{1}{s^2} \quad = \frac{1}{s^2}$$

$$f(t) = t^2 \Rightarrow F(s) = \int_0^\infty e^{-st} t^2 \, dt = -\int_0^\infty \frac{d}{ds}\left(t e^{-st}\right) dt$$

$$= -\frac{d}{ds} \int_0^\infty t e^{-st} \, dt = -\frac{d}{ds}\left(\frac{1}{s^2}\right) = \frac{2}{s^3}$$

general rule $= f(t) = t^2 \Rightarrow F(s) = \dfrac{n!}{s^{n+1}}$

$$f(t) = \cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$$

$$= \frac{1}{2} e^{i\omega t} + \frac{1}{2} e^{-i\omega t}$$

$$F(s) = \frac{1}{2} \frac{1}{i\omega + s} + \frac{1}{2} \frac{1}{i\omega + s} = \frac{1}{2} \frac{2s}{s^2 + \omega^2} = \frac{s}{s^2 + \omega^2}$$

$$f(t) = \sin(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2i}$$

$$= \frac{1}{2i} e^{i\omega t} + \frac{1}{2} e^{-i\omega t}$$

$$F(s) = \frac{\omega}{\omega^2 + s^2}$$

$$F(s) = \frac{1}{2} \frac{1}{i\omega + s} + \frac{1}{2} \frac{1}{i\omega + s} = \frac{1}{2} \frac{2s}{s^2 + \omega^2} = \frac{s}{s^2 + \omega^2}$$

Laplace Transform:

$$\mathcal{L} : (\mathbb{C} \to \mathbb{C}) \longrightarrow (\mathbb{C} \longrightarrow \mathbb{C})$$

$$\mathcal{L} = (x \to f(x)) \longrightarrow \left(s \longrightarrow \int_0^\infty e^{-xs} f(x)\,dx\right)$$

$$\mathcal{L} \cdot f = F \qquad F(s) = \int_0^\infty e^{-sx} f(x)\,dx$$

$$f(x) = 1 \Rightarrow \quad F(s) = \frac{1}{s}$$

$$f(x) = e^{ax} \Rightarrow \quad F(s) = \frac{1}{s-a}$$

$$f(x) = x^n \Rightarrow \quad Fs = \frac{n!}{s^{n-1}}$$

$$f(x) \to F(s)$$

$$g(x) \to G(s)$$

$$c_1 f(x) + c_2 g(x) \longrightarrow c_1 F(s) + c_2 G(s)$$

$$\frac{d f(x)}{dx} \longrightarrow ??$$

$$\frac{d f(x)}{d x} \longrightarrow \int_0^{\infty} e^{-xs} \left(\frac{d f(x)}{dx} \right) dx$$

$$= \int_0^{\infty} \frac{d}{dx} \left(e^{-xs} f(x) \right) dx$$

$$= - \int_0^{\infty} (-s)\, e^{-xs} f(x) \cdot dx$$

$$\frac{d f(x)}{d x} \longrightarrow \lim_{x \to \infty} \left(e^{-xs} f(s) \right)$$

$$- \lim_{x \to 0} \left(e^{-xs} f(x) \right) + s \int_0^{\infty} e^{-sx} f(x)\, dx$$

$$\frac{d f(x)}{dx} \longrightarrow s F(s) - f(0)$$

$$f(x) \frac{d g(x)}{dx} = \frac{d}{dx} \left(f(x) \cdot g(x) \right) - d \frac{f(x)}{dx} g(x)$$

$$\frac{d^2 f(x)}{dx^2} \longrightarrow \int_0^\infty e^{-x}\left(\frac{d f(x)}{dx^2}\right) dx$$

$$= \int_0^\infty \frac{d}{dx}\left(e^{-xs}\frac{df(x)}{dx}\right) dx - \int_0^\infty (-s)\, e^{-xs}\left(\frac{d f(y)}{x}\right) dx$$

$$\frac{d^2 f(y)}{dx^2} \longrightarrow \cancelto{0}{\lim_{x\to\infty}\left(e^{-xs}\frac{d f(x)}{dx}\right)} - \lim_{x\to 0}\left(e^{-xs}\frac{d f(x)}{dx}\right)$$

$$+ s\int_0^\infty e^{-sx}\left(\frac{d f(x)}{dx}\right) dx$$

$$\frac{d^2 f(x)}{dx^2} \longrightarrow s\,\mathcal{L}\left(\frac{d f(y)}{dx}\right) - f'(0)$$

$$\frac{d^2 f(x)}{dy^2} \longrightarrow s^2 F(s) - s\, f(0) - f'(0)$$

$$\boxed{\frac{d^n f(x)}{dx^n} \longrightarrow s^n F(s) - \sum_{i=0}^{n-1} s^i f^{(n-1-i)}(0)}$$

$$f'(x) + 2f(x) = 5$$

$$\mathcal{L}(f'(x)) + 2\mathcal{L}(f(x)) = 5\mathcal{L}(1)$$

$$[sF(s) - f(0)] + 2F(s) = \frac{5}{s}$$

$$(s+2)F(s) = f(0) + \frac{5}{s}$$

$$F(s) = \frac{1}{s+2} f(0) + \frac{1}{s+2}\frac{5}{s}$$

$$f(x) = f(0)e^{-2x} + p(x)$$

$$\left(b_n \frac{d^n}{dx^n} + b_{n-1}\frac{d^{n-1}}{dx^{n-1}} + \dots + b_0\right) f(x) = h(x)$$

$$\underbrace{\left(b_n s^n + b_{n-1}s^{n-1} + \dots + b_0\right)}_{C(s)} F(s) = (\ )f_0 + (\ )(f'(0) + \dots + f^{n-1}(0))$$

$$F(s) = \frac{(\ ) + f(0)}{C(s)} + \frac{(\ldots)f'(0)}{C(s)} + \dots \frac{(\ )f^{(n-1)}}{C(s)} + \left(\frac{H'(s)}{C(s)}\right)$$

$$\frac{s-a}{(s-b)(s-c)} = \frac{\alpha}{s-b} + \frac{\beta}{s-c} = \left(\frac{ac-1}{b-c}\right)\frac{1}{s-b} + \left(\frac{1-ab}{b-c}\right)\frac{1}{s-c}$$

$$\frac{s^2+a_1s+a_2}{(s-r_1)(s-r_2)(s-r_3)} = \frac{\alpha_1}{s-r_1} + \frac{\alpha_2}{s-r_2} + \frac{\alpha}{s-r_3}$$

$$\frac{\alpha}{s-b} + \frac{\beta}{s-c} = \frac{\alpha(s-c)+\beta(s-b)}{(s-b)(s-c)} = \frac{(\alpha+\beta)s-(\alpha c+\beta b)}{(s-b)(s-c)}$$

$$\left.\begin{aligned} \alpha+\beta &= 1 \\ \alpha c+\beta b &= a \end{aligned}\right\} \begin{pmatrix} 1 & 1 \\ c & b \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ a \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ c & b \end{pmatrix}^{-1} = \frac{1}{b-c}\begin{pmatrix} -1 & c \\ 1 & -b \end{pmatrix}$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{b-c}\begin{pmatrix} -1 & c \\ -1 & -b \end{pmatrix}\begin{pmatrix} 1 \\ a \end{pmatrix} = \frac{1}{b-c}\begin{pmatrix} ac-1 \\ 1-ab \end{pmatrix}$$

$$\frac{s-5}{(s-1)(s+1)} = \frac{-2}{s-1} + \frac{3}{s+1}$$

L.T

L.T

$$-2e^{x}+3e^{-x}$$

$$\frac{2s-5}{(s-1)^2} = \frac{2(s-1)-3}{(s-1)^2} = \frac{2}{s-1} - \frac{3}{(s-1)^2}$$

$$\frac{2}{s-1} \xrightarrow{L.T} \# e^x \qquad \frac{3}{(s-1)^2} \longrightarrow \# x e^x$$

$$\frac{1}{s^n} \longleftrightarrow x^{(n+1)}$$

$$\frac{1}{(s-a)^n} \longleftrightarrow x^{(n+1)} e^{ax}$$

$$f(x) = e^{ax} \qquad F(s) = \int_0^\infty e^{-xs} e^{ax}\, dx = \frac{1}{s-a}$$

$$\int_0^\infty e^{-xs}(xe^{ax})\, dx = \frac{d}{da}\left(\frac{1}{s-a}\right) = \frac{1}{(s-a)^2}$$

$$f(x) = xe^{ax}$$

$$\downarrow$$

$$F(s) = \frac{1}{(s-a)^2}$$

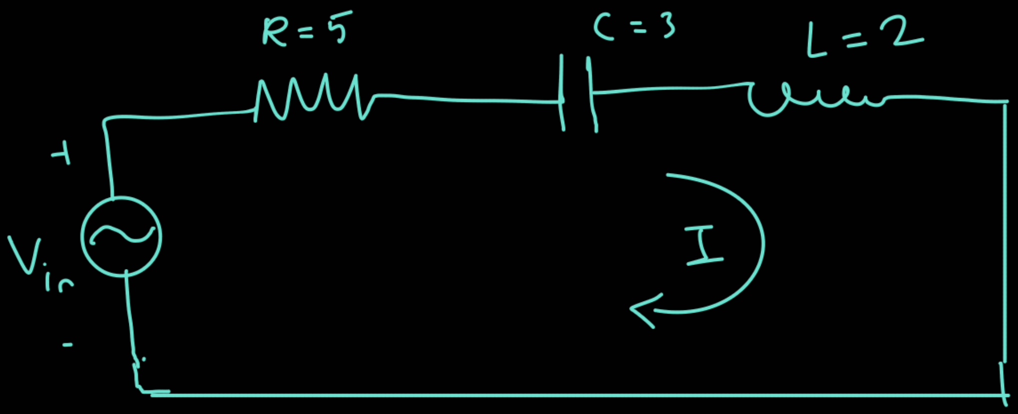
$$\int_0^\infty e^{-xs}(x^n e^{ax})\, dx = \frac{d^n}{da^n}\frac{1}{(s-a)} = \frac{n!}{(s-a)^{n+1}}$$

$$\frac{(\dots)}{(s-r_1)^{k_1}(s-r_2)^{k_2}\cdots(s-r_n)^{k_n}} = \frac{a_{1,1}}{s-r_1} + \frac{a_{1,2}}{(s-r_2)^2} + \dots \frac{a_{1,k_1}}{(s-r_n)^{k_n}}$$

$$+ \frac{a_{n,1}}{s-r_2} + \frac{a_{2,2}}{(s-r_2)^2} + \dots + \frac{a_{2,k_2}}{(s-r_2)^{k_2}}$$

$$+ \dots$$

$$+ \frac{a_{n,1}}{s-r_2} + \frac{a_{2,2}}{(s-r_2)^2} + \dots + \frac{a_{2,k_2}}{(s-r_n)^{k_n}}$$



$I(t) = ?$

$$V_{in}(t) = 10(\cos(t))$$

$V_R = R \cdot I$

$V_C = \frac{1}{C} I$

$L_L = L \frac{dI}{dt}$

$V_R + V_L + V_C = V_{in}$

$R \cdot \frac{dR}{dt} + L \cdot \frac{d^2 I}{dt^2} + \frac{1}{C} I = -10\sin(t)$

$$\left(5\frac{d^2}{dt^2} + 2\frac{d}{dt} + \frac{1}{3}\right) I(t) = -10\sin(t)$$

$$5(s^2 I(s) - s I(0) - I'(0)) + 2s(I(s)) - I_0 + \frac{1}{3} I(s)$$

$$= -10 \frac{1}{t^2+1}$$

$$\left(5s^2+2s+\frac{1}{3}\right) I(s) = (5s+2)\, I(0) + 5\, I'(0) - 10\,\frac{1}{t^2+1}$$

$$I(s) = \frac{5st^2}{5s^2+2s+\frac{1}{3}}\, I(0) + \frac{5}{5s^2+2s+\frac{1}{3}}\, I'(0) + \frac{10(s^2+1)}{5s^2+2s+\frac{1}{3}}$$

$$\mathcal{L} : (t \to i(t)) \longrightarrow (s \longrightarrow I(s))$$

where

$$I(s) = \int_0^\infty e^{-ts}\, i(t)\, dt$$

$$\int_0^\infty e^{-ts}\, i(t)\, dt = \int_0^\infty \frac{d}{dt}\left(e^{-ts}\, i(t)\right) dt + s\int_0^\infty e^{-ts}\, i(t)\, dt$$

$$= s\, I(s) - i(0)$$

$$\left(s^2 I(s) - s\,i(0) - i'(0)\right) + \frac{R}{L}\, s\left(I(s) - i(0)\right) + \frac{1}{LC}\, I(s)$$

$$= -\frac{V_0\,\omega}{L}\,\frac{\omega}{s^2+\omega^2}$$

$$\int_0^\infty e^{-st}\sin(\omega t)\, dt = \frac{1}{2i}\left[\int_0^\infty e^{-t(s-i\omega)}\, dt\right]$$

$$= \frac{1}{2i}\left[\frac{1}{s-i\omega} - \frac{1}{s+i\omega}\right] = \frac{\omega}{s^2+\omega^2}$$

Eulers Formula

$$e^{ix} = \cos(x) + i\sin(x)$$

$$\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right)I(s)=i'(0)+\left(s+\frac{R}{L}\right)i(0)-\frac{V_0\omega}{L}\frac{\omega}{s^2+\omega^2}$$

$$I(s)=\frac{i'(0)}{s^2+\frac{R}{L}s+\frac{1}{LC}}+\frac{s+\frac{R}{L}}{s^2+\frac{R}{L}s+\frac{1}{LC}}-\frac{V_0\omega^2}{L}\frac{1}{(s^2+\omega^2)\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right)}$$

$R=24 \qquad \omega=1$

$L=6 \qquad V_0=240$

$c=\frac{1}{6}$

$$\Downarrow$$

$$I(s)=\frac{i'(0)}{s^2+4s+1}+\frac{s+4}{s^2+4s+1}-\frac{40}{(s^2+1)(s^2+4s+1)}$$

$$\left(s^2+4s+1\right)=(s-4)(s-1) \qquad r\pm\ \frac{-4\mp\sqrt{16-4}}{2}$$

$$\left(s^2+4s+1\right)=\left(s+2+\sqrt{3}\right)\left(s+2-\sqrt{3}\right)$$

$$\frac{1}{s^2+4s+1}=\frac{1}{\left(s+2-\sqrt{3}\right)\left(s+2-\sqrt{3}\right)}=\frac{\alpha}{s+2+\sqrt{3}}-\frac{\alpha}{s+2-\sqrt{3}}$$

$$=\frac{1}{2\sqrt{3}}$$

$$\frac{i'(0)}{s^2+4s+1}=\frac{i'(0)}{2\sqrt{3}}\frac{1}{s+2\sqrt{3}}-\frac{i'(0)}{2\sqrt{3}}\frac{1}{s+2\sqrt{3}}$$

$$i(t) = \frac{i(0)}{2\sqrt{3}} \left(e^{-(2\sqrt{3})t} - e^{-2+\sqrt{3}t} \right)$$

$$+ \frac{i(0)}{2\sqrt{3}} \sqrt{3}-2\ e^{-2+\sqrt{3}t} + \sqrt{3+2}\ e^{-(2-\sqrt{3})t} \Big)$$

$$+ p(t)$$

$$\frac{1}{(s^2+1)(s^2+4s+1)} = \frac{c_1}{s+1} + \frac{c_2}{s-i} + \frac{c_3}{s+2+\sqrt{3}} + \frac{c_4}{s+2-\sqrt{3}}$$

$$\boxed{p(t) = c_1 e^{-it} + c_2 e^{it} + c_3 e^{-(2+\sqrt{3})t} + c_4 e^{-2+\sqrt{3}t}}$$

$$\begin{aligned} -40 = {} & c_1(s-i)(s+2+\sqrt{3})(s+2+\sqrt{3}) \\ & + c_2(s+i)(s+2+\sqrt{3})(s+2+\sqrt{3}) \\ & + c_3(s+i)(s+2+\sqrt{3})(s+2+\sqrt{3}) \\ & + c_4(s+i)(s+2+\sqrt{3})(s+2+\sqrt{3}) \end{aligned}$$

$$-40 = (\dots)s^3 + (\dots)s^2 + \dots\dots$$

$$g\left(x, \frac{d}{dx}\right) f(x) = h(x)$$

$$\downarrow$$

$$p(s) F(s) = (\dots) f(0) + (\dots) f'(0) + f^{n-1}(0) \dots + H(s)$$

$$F(s) = \frac{()}{p(s)} f(0) + \frac{(\dots)}{p(s)} + \frac{H(s)}{p(s)}$$

$$F_p(s) = \frac{1}{p(s)} \cdot H(s)$$

multiplication of $\frac{1}{p(s)}$ and $H(s)$

$\updownarrow$

???

Convolution

$$g\left(x, \frac{d}{dx}\right) i = \delta(x) \quad \text{with } i(0) = i'(0) \dots = 0$$

input

$\Downarrow$

System is completely zero

for example in RLC

$$p(s) I(s) = 1 \implies I(s) = \frac{1}{p(s)}$$

$\hookrightarrow$ its laplace transform is

$\left.\begin{array}{l} k=0 \\ i=0 \\ i=0 \\ v=0 \end{array}\right\}$ etc

$i(x)$ = impulse response = function that maps to $\frac{1}{p(s)}$ under $\mathcal{L}$

$\delta(x)$ = function that Maps to 1 under $\mathcal{L}$ transform

---

$$g\left(x, \frac{d}{dx}\right) f(x) = h(x)$$

$$P(s) F(s) = (\ ) f(0) + (\ ) f'(0) \cdots f^{(n-1)}(0) + H(s)$$

$$F(s) = \frac{(\ )}{p(s)} f(0) + \frac{(\ )}{p(s)} f^{(n-1)}(0) + \frac{H(s)}{P(s)}$$

$$(\mathcal{L} \cdot f_p)(s) = \frac{H(s)}{p(s)} = \frac{1}{p(s)} \cdot H(s)$$

What map. to muliplication under Laplace Transf.?

$$F(s) = \int_0^\infty e^{-sx} f(x)\,dx$$

$$G(s) = \int_0^\infty e^{-sx} g(y)\,dy$$

$z = x + y$

$dz = dy$

$$F(s) \cdot G(s) = \int_0^\infty dx \left(\int_0^\infty e^{-s(x+y)} f(x)\, g(y)\, dy \right)$$

$$= \int_0^\infty dx \int_0^\infty dz\, e^{-sz} f(x)\, g(z-x)$$

$$f(x) = 0 \quad \text{if } x < 0$$

$$g(x) = 0 \quad \text{if } x < 0$$

$$F(s)\, G(s) = \int_0^\infty dx \int_x^\infty dz\, e^{-sz} f(x)\, g(z-x)$$

$$+ \, 0$$

$$= \int_0^\infty dx \int_x^\infty dz\, e^{-sz} f(x)\, g(z-x)$$

$$+ \int_0^\infty dx \int_0^x dz\, e^{-sz} f(x)\, g(z-x)$$

$$F(s)\, G(s) = \int_0^\infty dx \int_0^\infty dz\, e^{-sz} f(x)\, g(z-x)$$

$$\underbrace{(f * g)}_{\text{convolution}}(z) = \int_0^\infty f(x)\, g(z-x)\, dx$$

$$\mathcal{L}(f * g) = \mathcal{L}(f)\, \mathcal{L}(g)$$

$$h(z) = (f * g)(z)$$

$$H(s) = F(s)\, G(s)$$

$$g\left(x, \frac{d}{dx}\right) f(x) = h(x) \Rightarrow f(x) = f_{h,1}(x) + f_{h,x} + f_g(x)$$

$$F_p(s) = \frac{1}{p(s)} H(s) \qquad i(x) \longrightarrow \frac{1}{p(s)}$$

$$f_p(x) = \int_0^x p(z)\, h(x-z)\, dz$$