CH5115 QUIZ-1

BY: S. VISHAL
CH18B020

(1)

i) $\int_{\forall y} f(y|x)\,dy = 1$ $\left(\because f(y|x) = \frac{f(x,y)}{f(x)} \right.$
$\Rightarrow \int f(y|x)dy = \frac{\int f(x,y)dy}{f(x)}$
$\left. = \frac{f_X(x)}{f(x)} = 1 \right)$

$\Rightarrow \int_{-2}^{2} cy^2 = 1.$

$\Rightarrow c\left[\left(\frac{8}{3}\right) - \left(\frac{-8}{3}\right)\right] = 1$

$\Rightarrow c = 3/16$

ii) $f(y|x) = \frac{f(x,y)}{f(x)}$

$\Rightarrow f(x,y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \left(\frac{3}{16} y^2\right)$

$f_Y(y) = \int_{\forall x} \frac{f(x,y)}{\cancel{\sqrt{2\pi}}} \cancel{\left(\frac{3}{16} y^2\right)} dx$

$= \int \frac{3y^2}{16\sqrt{2\pi}} e^{-x^2/2} dx$

$= \frac{3y^2}{16} \int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$

which is nothing but integral of ~~pdf~~ ~~jt~~ univariate Gaussian ($\mu = 0$, $\sigma^2 = 1$)

$\Rightarrow f_Y(y) = \frac{3}{16} y^2$

$$E(Y) = \int_{-2}^{2} y \frac{3}{16} y^2 \, dy = \int_{-2}^{2} \frac{3}{16} y^3 \, dy = \frac{3}{64}(16 - 16)$$

$$= \frac{3}{16}\, 0$$

$$E(Y^2) = \int_{-2}^{2} y^2 \left(\frac{3}{16} y^2\right) dy = \int_{-2}^{2} \frac{3y^4}{16} \, dy$$

$$= \frac{3}{16}\left(\frac{32}{5} + \frac{32}{5}\right) = \frac{12}{5}$$

$$\sigma_Y^2 = E(Y^2) - (E(Y))^2$$

$$= \boxed{\frac{12}{5}}$$

$$\mu_Y = \boxed{0.}$$

(12) Assuming the distribution of X & Y is jointly Gaussian,

$$f(x,y) = \frac{1}{2\pi \sigma_Y \sigma_X \sqrt{1 - \rho_{XY}^2}} \int_{-\infty}^{\infty} y \exp\left(-\left[\frac{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - \frac{2\rho_{xy}(x-\mu_x)(y-\mu_y)}{\sigma_x \sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2}}{2(1-\rho^2_{xy})}\right]\right)$$

$$= \frac{1}{2\pi \sigma_Y \sigma_X \sqrt{1-\rho_{XY}^2}} \int_{-\infty}^{\infty} y \exp\left(-\left[\frac{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - \frac{2\rho_{xy}(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2}}{2(1-\rho^2_{xy})}\right]\right)$$

$$\Rightarrow E(Y|X=x) = \int_{-\infty}^{\infty} y \, \frac{P_{XY}(x,y)}{P_X(x)} \, dy$$

$$= \frac{1}{\sqrt{2\pi \sigma_Y^2 (1-\rho_{XY}^2)}} \int_{-\infty}^{\infty} y \exp\left(\left[\frac{(x-\mu_x)^2}{2\sigma_x^2}\right] - \left\{ \frac{\frac{(x-\mu_x)^2}{\sigma_x^2} - \frac{2\rho_{XY}(x-\mu_x)(y-\mu_y)}{\sigma_X \sigma_Y} - \frac{(y-\mu_y)^2}{\sigma_Y^2}}{2(1-\rho_{XY}^2)} \right\} \right) dy$$

$$= \cancel{\frac{1}{\sqrt{2\pi\sigma_Y}}} = \frac{1}{\sqrt{2\pi\sigma_Y^2(1-\rho_{XY}^2)}} \int_{-\infty}^{\infty} y \exp\left(- \left[\frac{\frac{\rho_{XY}(x-\mu_x)}{\sigma_x} + \frac{(y-\mu_y)^2}{\sigma_Y^2}}{2(1-\rho_{XY}^2)} \right] \right) dy$$

Let $z = \frac{-2\rho_{xy}(x-\mu_x)}{\sigma_x} + \frac{(y-\mu_y)^2}{\sigma_y^2}$

$\Rightarrow dy = \sigma_Y \, dz.$

$$\Rightarrow E(Y|X=x) = \frac{1}{\sqrt{2\pi(1-\rho_{XY}^2)}} \int_{-\infty}^{\infty} \left[z\sigma_Y + \frac{\sigma_Y \rho_{XY}(x-\mu_X)}{\sigma_X} + \mu_Y \right] \times \exp\left(- \frac{z^2}{2(1-\rho_{XY}^2)} \right) dz$$

$$= \frac{\sigma_{XY} \rho_{xy}(x-\mu_x)}{\sigma_x} + \mu_y$$

$$\Rightarrow E(Y|X=x) = \left(\frac{\sigma_{XY}}{\sigma_X^2} \right) x + \left(\mu_y - \frac{\sigma_{XY}}{\sigma_x^2} \mu_x \right).$$

Substituting $\sigma_{XY} = 1.2$, $\sigma_X^2 = 2$, $\sigma^2_Y = 3$

$$\Rightarrow E(\cancel{Y} \mid X = x) = \left(\frac{1.2}{2}\right)(x) + \left(\mu_y - \left(\frac{1.2}{2}\right)\mu_x\right)$$

$$= 0.6x + (\mu_y - 0.6\mu_x)$$

If $\mu_{y_x} = \mu_x = 0$

$$E(Y \mid X = x) = 0.6x$$