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① a)

$$y^* = \alpha_0 x^* \quad \text{—— ①}$$

$$y(k) = y^*(k) + \varepsilon_y(k) \quad \text{—— ②}$$

$$x(k) = x^*(k) + \varepsilon_x(k) \quad \text{—— ③}$$

① ⇒ $$\mu_{y^*} = \alpha_0 \mu_{x^*} \quad \text{—— ④}$$

$$\sigma_{y^*}^2 = \alpha_0^2 \sigma_{x^*}^2 \quad \text{—— ⑤}$$

② ⇒ $E(y(k)) = E(y^*(k)) + E(\varepsilon_y(k))$

⇒ Replace theoritical mean with ~~actual~~ sample mean in LHS,

$$\bar{y} = \mu_{y^*} + 0 = \mu_{y^*} \quad \text{—— ⑥}$$

$$\text{var}(y(k)) = \text{var}(y^*(k)) + \text{var}(\varepsilon_y(k))$$

$$\Rightarrow \hat{\sigma}_y^2 = \sigma_{y^*}^2 + \lambda_y \quad \text{—— ⑦}$$

where $\hat{\sigma}_y^2 = \dfrac{\sum (y_k - \bar{y})^2}{N}$

Similarly ③ ⇒ $$\bar{x} = \mu_{x^*} \quad \text{—— ⑧}$$

$$\hat{\sigma}_x^2 = \sigma_{x^*}^2 + \lambda_x \quad \text{—— ⑨}$$

~~Solving the 6 eqns ④ to ⑨, we can obtain the 6 unknowns — $\alpha_0, \lambda_x, \lambda_y, \mu_{x^*}, \mu_{y^*}$,~~

$$E(x^* y^*) = \alpha_0 E(X_*^2)$$

$$= \alpha_0 \left(\sigma_{x^*}^2 + \mu_{x^*}^2\right)$$

$$\Rightarrow E\left(\left(x(k) - \varepsilon_x(k)\right)\left(y(k) - \varepsilon_y(k)\right)\right)$$

$$= \alpha_0 \sigma_{x^*}^2 + \mu_{x^*}$$

$$\Rightarrow E\left(x(k)\, y(k)\right) = \alpha_0 \sigma_{x^*}^2 + \mu_{x^*}^2$$

$$\left(\because E(x\varepsilon_y) = 0,\ E(y\varepsilon_x) = 0,\ E(\varepsilon_x \varepsilon_y) = 0\right)$$

$$\Rightarrow \frac{\sum x(k)\, y(k)}{N} = \alpha_0 \sigma_{x^*}^2 + \mu_{x^*}^2$$

$$\Rightarrow \alpha_0 \sigma_{x^*}^2 + \mu_{x^*}^2 = \overline{xy} \quad \text{—— (10)}$$

Solving 7 eqns: (4) - (10) we can ~~et~~ get

the 7 unknowns: $\alpha_0, \lambda_x, \lambda_y, \mu_{x^*}, \mu_{y^*}, \sigma_{x^*}^2, \sigma_{y^*}^2$

$\mu_{x^*} = \bar{x}$ (eqn (8)) & ~~also~~ $\mu_{y^*} = \bar{y}$ (eqn (8))

$$\Rightarrow \alpha_0 \sigma_{x^*}^2 + \bar{x} = \overline{xy}$$

$$\Rightarrow \sigma_{x^*}^2 = \frac{\overline{xy} - (\bar{x})^2}{\alpha_0}$$

$$\text{eqn (4)} \Rightarrow \alpha_0 = \frac{\mu_{y^*}}{\mu_{x^*}} = \frac{\bar{y}}{\bar{x}}$$

$$\therefore \sigma_{x^*}^2 = \frac{\overline{xy} - (\bar{x})^2}{\bar{y}} \times \bar{x}$$

$$\lambda_x = \hat{\sigma}_x^2 - \sigma_{x^*}^2$$

$$= \hat{\sigma}_x^2 - \left(\frac{\overline{xy} - (\bar{x})^2}{\bar{y}}\right) \cdot \bar{x}$$

$$\sigma_{y^*}^2 = \alpha_0^2 \sigma_{x^*}^2 = \alpha_0^2 \sigma_{x^*}^2$$

$$\sigma_{y^*}^2 = \left(\frac{\bar{y}}{\bar{x}}\right)\left(\overline{xy} - (\bar{x})^2\right)$$

$$\therefore \lambda_y = \hat{\sigma}_y^2 - \left(\frac{\bar{y}}{\bar{x}}\right)\left(\overline{xy} - (\bar{x})^2\right)$$

Final answers:

i) $\hat{\alpha}_0 = \dfrac{\bar{y}}{\bar{x}}$ ii) $\hat{\mu}_x^* = \bar{x}$ iii) $\hat{\sigma}_{x^*}^2 = \dfrac{(\overline{xy} - (\bar{x})^2)}{(\bar{y})} \times (\bar{x})$

iv) $\hat{\lambda}_x = \hat{\sigma}_x^2 - \left(\dfrac{\overline{xy} - (\bar{x})^2}{\bar{y}}\right)\bar{x}$ v) $\hat{\lambda}_y = \hat{\sigma}_y^2 - \left(\dfrac{\bar{y}}{\bar{x}}\right)\left(\overline{xy} - (\bar{x})^2\right)$

Consistency:

$$\lim_{N \to \infty} \hat{\alpha}_0 = \frac{\frac{\sum y(k)}{N}}{\frac{\sum x(k)}{N}} = \frac{\lim_{N\to\infty} \frac{\sum y^*(k) + \varepsilon_y(k)}{N}}{\lim_{N\to\infty} \frac{\sum x^*(k) + \varepsilon_x(k)}{N}}$$

But,

$$\lim_{N \to \infty} \frac{\sum \gamma(k)}{N} = E(\gamma)$$

$$\Rightarrow \lim_{N\to\infty} \hat{\alpha}_0 = \frac{E(y^*) + 0}{E(x^*) + 0}$$

$$= \frac{\mu_y^*}{\mu_x^*} = \alpha_0$$

So the Method of moments estimator is consistent

ii) If $\mu_x^* = 0$,

④ $\Rightarrow \mu_y^* = 0$ (as long as α_0 is finite)

eqns ⑥ & ⑧ become redundant

we are left with $\alpha_0, \lambda_x, \lambda_y, \sigma_{x^*}^2, \sigma_{y^*}^2$ -

5 unknowns but only (⑤, ⑦, ⑨, ⑩) - 4 equations

If we are restricted to first & second order moment eqns the only other option we have is to use the cross covariances:

$E(x^* \varepsilon_y) = 0$; $E(y^* \varepsilon_x) = 0$

$\Rightarrow E(x \varepsilon_y) = 0$; $E(y \varepsilon_x) = 0$

however these can't be computed as sample averages because ε_y, ε_x are themselves unknown.

So we need one more parameter to be given to proceed further.
(or one more relation)

I assume that $\frac{\lambda_y}{\lambda_x} = \beta$ is a known parameter

$$\Rightarrow \lambda_y = \beta \lambda_x. \quad \text{— (11)}$$

(10) $\Rightarrow$ $\sigma_x^2 = \frac{\overline{xy}}{\alpha_0}$; (5) $\Rightarrow$ $\sigma_y^2 = \alpha_0 \overline{xy}$

subst. in (9) & (7),

$$\hat{\sigma}_x^2 = \frac{\overline{xy}}{\alpha_0} + \lambda_x. \quad \text{— (12)}$$

$$\hat{\sigma}_y^2 = \alpha_0 \overline{xy} + \beta \lambda_x. \quad \text{— (13)}$$

$$\hat{\sigma}_y^2 - \beta \hat{\sigma}_x^2 = \overline{xy} \left(\alpha_0 - \frac{\beta}{\alpha_0} \right)$$

~~$\Rightarrow \overline{xy} = f(\alpha$~~ Since this results in a quadratic,

I assume λ_x is known.

(9), (11) $\Rightarrow$ $\hat{\sigma}_x^2 - \lambda_x = \frac{\overline{xy}}{\alpha_0}$

$$\Rightarrow \hat{\alpha}_0 = \frac{\overline{xy}}{\hat{\sigma}_x^2 - \lambda_x}.$$

In $\lim\limits_{N\to\infty} \hat{\alpha}_0 = \dfrac{\lim\limits_{N\to\infty} \overline{xy}}{\lim\limits_{N\to\infty} \hat{\sigma}_x^2 - \lambda x}$

$$\lim_{N\to\infty} \overline{xy} = E(xy)$$

$$= E\left((x^* + \varepsilon x)(y^* + \varepsilon y)\right)$$

$$= E(x^* y^*)$$

$$= \alpha_0 \left(\sigma_{x^*}^2 + \cancel{\mu_x^2}^{0}\right)$$

$$= \alpha_0 \sigma_{x^*}^2 = \alpha_0 \sigma_{x^*}^2$$

$$\lim_{N\to\infty} \hat{\sigma} x^2 - \lambda x = \lim_{N\to\infty} \frac{\Sigma (x - \bar{x})^2}{N} - \lambda x$$

$$= \lim_{N\to\infty} E\left(\frac{\Sigma (x - \bar{x})^2}{N}\right) - \lambda x$$

$$= \lim_{N\to\infty} = \frac{N}{N-1} \sigma_x^2 - \lambda x$$

$$= \frac{N}{N-1}\left(\sigma_{x^*}^2 + \lambda x\right) - \lambda x$$

$$= \sigma_{x^*}^2 + \lambda x - \lambda x = \sigma_{x^*}^2$$

$$\therefore \lim_{N\to\infty} \hat{\alpha}_0 = \frac{\alpha_0 \sigma_{x^*}^2}{\sigma_{x^*}^2}$$

$$= \alpha_0$$

$\therefore$ <u>It is consistent.</u>

other values,

$$\boxed{\sigma_x^2 = \frac{\overline{xy}}{t} \quad \hat{\sigma}_x^2 - \lambda_x.}$$

$$\lambda_y = -a_0^2 \sigma_x^2 + \hat{\sigma}_y^2$$

$$= \hat{\sigma}_y^2 - \cancel{\left(\frac{\hat{\sigma}_x^2 - \lambda_x}{\overline{xy}} \cdot \frac{\overline{xy}}{\hat{\sigma}_x^2 - \lambda_x}\right)\sigma_x^2}$$

$$\Rightarrow \boxed{\lambda_y = \hat{\sigma}_y^2 - \overline{xy}}$$

b) $\Phi = x$ LS: $\hat{y} = \Phi \theta$

$$\Phi = x^T$$

Solution to the least squares problem is obtained as

$$\cancel{\theta = (\Phi^T x)^{-1} x}$$

$$\theta = (\Phi^T \Phi)^{-1} \Phi^T y$$

$$\Rightarrow \hat{\alpha} = (x^T x)^{-1} x^T y$$

$$\Rightarrow \hat{\alpha} = \frac{\sum_{k=1}^{N} x(k) y(k)}{\sum_{k=1}^{N} (x(k))^2}$$

$$\Rightarrow \hat{\alpha} = \frac{\sum x(k)\, y(k)}{\sum x(k)^2}$$

Consistency,

$$\lim_{N \to \infty} \hat{\alpha} = \lim_{N\to\infty} \frac{\sum^{N} (x^{o}(k) + \varepsilon_x(k))(y^{*} + \varepsilon_y(k))}{\sum^{N} (x + \varepsilon_x(k))^2}$$

$$= \lim_{N \to \infty} \frac{\dfrac{\sum (x^{o}(k) y^{o}(k) + \varepsilon_x(k)\varepsilon_y(k) + \varepsilon_x(k) y^{*}(k) + \varepsilon_y(k) x^{o}(k))}{N}}{\sum (x + \varepsilon_x(k))^2 / N}$$

$$\text{Denominator}^* = \lim_{N\to\infty} \frac{\sum x^{*2} + 2x^*\varepsilon_x + \varepsilon_x^2}{N}$$

$\frac{\sum x^{*2}}{N}$ is an asymptotically unbiased estimator of $E(x^{*2})$

$$\Rightarrow \lim_{N\to\infty} \frac{\sum x^{*2}}{N} = E(x^{*2}) = \sigma_{x^*}^2 + \mu_{x^*}$$

$$\lim_{N\to\infty} \frac{\sum 2x^*\varepsilon_x}{N} = 2E(x^*\varepsilon_x) = 0$$

$$\lim_{N\to\infty} \frac{\sum \varepsilon_x^2}{N} = E(\varepsilon_x^2) = \lambda_x$$

$\therefore$ Denominator $= \sigma_{x^*}^2 + \mu_{x^*} + \lambda_x$

Similarly Numerator $= (\alpha_0 \sigma_{x^*}^2 + \alpha_0 \mu_{x^*}) + 0 + 0 + 0$

$$\therefore \lim_{N\to\infty} \hat{\alpha}_{LS} = \frac{\alpha_0(\sigma_{x^*}^2 + \mu_{x^*})}{\sigma_{x^*}^2 + \mu_{x^*} + \lambda_x} \neq 0$$

$\therefore$ The LS estimator is inconsistent.

This is as expected because,

$$y^* = \alpha x^* \Rightarrow y - \varepsilon_y = \alpha_0 x - \alpha_0 \varepsilon_x$$
$$\Rightarrow y = \alpha_0 x + (\varepsilon_y - \alpha_0 \varepsilon_x)$$

We note that the error itself is a fn of α_0

also $E\left(x(n)(\epsilon_y - \alpha_0 \epsilon_x)\right)$

$= E\left((x^* - \epsilon_x)(\epsilon_y - \alpha_0 \epsilon_x)\right)$

$= \alpha_0 \lambda_x$

$\Rightarrow$ The error is correlated with the regressor.

this violates the Least Squares assumption.

So, the estimator is inconsistent.

If we let $\lambda_x \to 0$ (error is uncorrelated with x),

we get $\lim_{N\to\infty} \hat{\alpha}_{LS} = \dfrac{\alpha_0 (\sigma^2_{x^*} + \mu^2_{x^*})}{\sigma^2_{x^*} + \mu^2_{x^*}}$

$= \alpha_0$

Then as expected LS is consistent