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CH18B020

CH5115 ASSIGNMENT - 3

(1) $f(y) = \begin{cases} e^{-(y-\theta)} & x > \theta, \quad -\infty < \theta < \infty \\ 0 & \text{otherwise} \end{cases}$

$$P_r(y[k] \geq y) = \int_y^\infty e^{-(y-\theta)}$$

Let $e^{\theta} = \beta$

$$\Rightarrow P_r(y[k] \geq y) = \begin{cases} \beta(e^{-y}) & \text{where } y > \theta \\ 1 & y \leq \theta \end{cases}$$

For $\min(y_N) \geq y$,

$$y[k] \geq y \quad \forall k.$$

$$\Rightarrow P_r(\min(y_N) \geq y) = (\beta e^{-y})^N \qquad y > \theta$$

$$\Rightarrow P_r(\min(y_N) \leq y) = 1 - \beta^N e^{-Ny} \quad \text{--- (1)}$$

$$\Rightarrow f(\min(y_N) = y) = \frac{d}{dy}(1 - \beta^N e^{-Ny})$$

$$= \begin{cases} N\beta^N e^{-Ny} & y > \theta \\ 0 & y \leq \theta \end{cases} \quad \text{--- (2)}$$

① $\Rightarrow \Pr(2\min(y_N) \le 2y) = 1 - \beta^N e^{-Ny}$

$\Rightarrow \Pr(2\min(y_N) \le t) = 1 - \beta^N e^{-Nt/2}$

where $t = 2y$

$\rightarrow \Pr(T \le t) = 1 - \beta^N e^{-Nt/2}$

$$\Rightarrow f(T_N = t) = \frac{N\beta^N e^{-Nt/2}}{2}$$

$$\boxed{f(T_N = t) = \begin{cases} \frac{N \exp(N(\theta - t/2))}{2} & \overset{t > 2\theta}{\uparrow} \quad t \cancel{\ge} 2\theta \\ 0 & \text{otherwise } (t \le 2\theta) \end{cases}}$$

b) $E(T_N) = \int_{\forall t} t f(t)\, dt$

$$= \frac{N}{2} \int_{2\theta}^{\infty} t\, e^{N\theta - Nt/2}\, dt$$

$$= \frac{N\beta^N}{2} \int_{2\theta}^{\infty} t\, e^{-Nt/2}\, dt \quad (\beta = e^{N\theta}$$

$$= \frac{N\beta^N}{2} \left[\left. \frac{2t e^{-Nt/2}}{-1 \times N} \right|_{2\theta}^{\infty} + \int_{2\theta}^{\infty} \frac{2 e^{-Nt/2}}{N} dt \right]$$

$$= \frac{N\beta^N}{2} \left[\frac{2 \times 2\theta\, e^{-N\theta}}{N} + \frac{4 e^{-N\theta}}{N^2} \right]$$

$$= 2\theta + \frac{2}{N}$$

$E(T_N) \neq \theta \Rightarrow T_N$ is a biased estimator.

Correcting it : $\frac{T_N}{2} - \frac{1}{N} = T_N'$

$\Rightarrow E(T_N') = E\left(\frac{T_N}{2} - \frac{1}{N}\right) = \frac{E(T_N)}{2} - \frac{1}{N}$

$= \theta + \frac{1}{N} - \frac{1}{N}$

$\Rightarrow E(T_N') = \theta$ ✓ unbiased

$\therefore$ Corrected statistic :

$$\boxed{T_N' = \frac{T_N}{2} - \frac{1}{N}}$$

c) $T_N' \xrightarrow{p} \theta$ [To prove]

To prove: $\lim_{N \to \infty} Pr(|X_n - X| \geq \varepsilon) = 0$ where $\varepsilon > 0$

$\Rightarrow \lim_{N \to \infty} Pr(|T_N' - \theta| \geq \varepsilon) = 0$

$\Rightarrow \lim_{N \to \infty} \left[Pr(T_N' - \theta \geq \varepsilon) + Pr(T_N' - \theta \leq -\varepsilon)\right] = 0$

(Because for the condition to satisfy either $T_N' - \theta \geq \varepsilon$ or $T_N' - \theta \leq -\varepsilon$)

Now we will compute the 2 probabilities individually and add them up

1st term: $Pr\left(\frac{T_N}{2} - \frac{1}{N} - \theta \geq \varepsilon\right)$

$\Rightarrow Pr\left(\frac{T_N}{2} \geq \varepsilon + \frac{1}{N} + \theta\right) = Pr\left(T_N \geq 2\left(\varepsilon + \frac{1}{N} + \theta\right)\right)$

We now find out the CDF from PDF of T,

$$Pr(T_N \leq t) = \int_{2\theta}^{t} \frac{N}{2} \exp\left(N\left(\theta - \frac{t}{2}\right)\right) dt$$

$$= 1 - \beta^N e^{-\frac{Nt}{2}} \quad \text{—— (1)}$$

$Pr(T_N \geq t) = 1 - Pr(T_N \leq t)$ $\quad [\beta^N = e^{N\theta}$ as defined in part (a)$]$

$$= \beta^N e^{-Nt/2} \quad \text{—— (2)}$$

$$\therefore Pr\left(T_N \geq 2\left(\varepsilon + \frac{1}{N} + \theta\right)\right) = \beta^N e^{-N\left(\varepsilon + \frac{1}{N} + \theta\right)}$$

$$= e^{-N\varepsilon - 1} \quad \text{—— (3)}$$

2nd term: $Pr\left(\frac{T_N}{2} - \frac{1}{N} - \theta \leq -\varepsilon\right)$

using eqn (1),

$$\Rightarrow Pr\left(\frac{T_N}{2} \leq 2\left(-\varepsilon + \frac{1}{N} + \theta\right)\right) = 1 - \beta^N e^{-N\left(-\varepsilon + \frac{1}{N} + \theta\right)}$$

$$= 1 - e^{N\varepsilon - 1}$$

However this eqn is valid ~~iff~~ only if,

$$2\left(-\varepsilon + \frac{1}{N} + \theta\right) \geq 2\theta \Rightarrow \frac{1}{N} - \varepsilon \geq 0$$

In the event $N \to \infty$, $\quad -\varepsilon \geq 0$ But $\varepsilon > 0$

$\Rightarrow -\varepsilon < 0$

$\Rightarrow$ We can't use $1 - e^{+N\varepsilon - 1}$ expression in the given limit

$\therefore$ since $2\left(\theta + \frac{1}{N} - \varepsilon\right) < 2\theta$,

$F(T_N \le t)$ at such $t = 0$

$(\because t_{max} = 2\theta)$

$$\Rightarrow \lim_{N \to \infty} \Pr\left(T_N \le 2\left(-\varepsilon + \frac{1}{N} + \theta\right)\right) = 0 \quad \text{--- (4)}$$

Using (4) & (3) we have,

$$\lim_{N \to \infty} \left[\Pr(T_N' - \theta \ge \varepsilon) + \Pr(T_N' - \theta \le -\varepsilon)\right]$$

$$= \lim_{N \to \infty} \left[e^{-N\varepsilon - 1} + 0\right]$$

$$= 0$$

$$\Rightarrow \lim_{N \to \infty} \left[\Pr(T_N' - \theta \ge \varepsilon) + \Pr(T_N' - \theta \le -\varepsilon)\right] = 0$$

$$\therefore T_N' \xrightarrow{P} \theta$$

2 a) $V(f_n) = \sum_{k=0}^{N-1} v[k] \exp(-j 2\pi f_n k)$

$= \sum_{k=0}^{N-1} v[k] \cos(2\pi f_n k) - j\, v[k] \sin(2\pi f_n k)$

hence, $a_n = \sum_{k=0}^{N-1} v[k] \cos(2\pi f_n k)$

and $b_n = -\sum_{k=0}^{N-1} v[k] \sin(2\pi f_n k)$

Notice that, a_n and b_n are sum of i.i.d Gaussian variables ($\because$ v is GWN) which are scaled by a constant factor ($\cos(2\pi f_n k)$ for kth term of a_n and $\sin(2\pi f_n k)$ for kth term of b_n) such a product remains Gaussian with a change in σ^2. So a_n and b_n are just sum of independent Gaussian Random Variables

$\Rightarrow$ a_n and b_n themselves should be Gaussian distributed.

To prove that sum of Gaussian random variables is Gaussian, I will use the moment generating fn

i) For Gaussian X, $MGF = E(e^{sX})$

let $\alpha \sim N(0,1)$ $\Rightarrow MGF(X) = E\left(e^{s(\sigma Y + \mu)}\right)$

$= \int \frac{1}{\sqrt{2\pi}} e^{s\sigma Y} \cdot e^{\mu s} e^{-\frac{1}{2}y^2} dy$
$\forall y$

$$= \frac{e^{\mu s}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{s\sigma y - \frac{1}{2}y^2} \, dy$$

$$= e^{\mu s} e^{\frac{\sigma^2 s^2}{2}} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(y^2 - 2\sigma s + \sigma^2 s^2)}}{\sqrt{2\pi}} \, dy$$

$$\Rightarrow M_x(s) = e^{\mu s + \frac{\sigma^2 s^2}{2}}$$

ii) Consider sum of i.i.d RVs $Y = \sum_{i=1}^{N} X_i$

$$M_Y(s) = E(e^{sY}) = E(e^{s\sum X_i}) = E\left(\prod_{i=1}^{N} e^{sX_i}\right)$$

$$\Rightarrow \because \text{ i.i.d}, \; M_Y(s) = \prod_{i=1}^{N} E(e^{sX_i})$$

$$\Rightarrow M_Y(s) = \prod_{i=1}^{N} E(e^{sX_i})$$

From (i), $M_Y(s) = \prod e^{\mu_i s + \frac{\sigma_i^2 s^2}{2}}$

$$= \exp\left(\left(\sum \mu_i\right) s + \frac{\sum \sigma_i^2}{2} s^2 \right)$$

So Y is Gaussian distributed with

$$\mu = \sum_{i=1}^{N} \mu_i \quad \& \quad \sigma^2 = \frac{\sum \sigma_i^2}{2}$$

$\therefore$ Sum of i.i.d. Gaussian RVs is a Gaussian RV.

b) $\text{corr}(a_n, b_n) = \dfrac{\text{cov}(a_n, b_n)}{\sqrt{\sigma_{a_n}^2 \sigma_{b_n}^2}}$

$\Rightarrow \text{corr}(a_n, b_n) = \dfrac{E(a_n b_n) - E(a_n)E(b_n)}{\sqrt{\sigma_{a_n}^2 \sigma_{b_n}^2}}$ —— (1)

Let's evaluate these moments one by one,

$a_n = \sum v(k) \cos 2\pi f_n k$ —— (2) $\Rightarrow E(a_n) = \sum E(v(k)\cos 2\pi f_n k$

$= 0$ —— (3)

$\cdot \text{var}(a_n) = \text{var}\left(\sum v(k) \cos 2\pi f_n k\right)$

$= \sum (\text{var}(v(k))(\cos(2\pi f_n k))^2$ (because v_k is i.i.d)

$\Rightarrow \text{var}(a_n) = \sum_{k=0}^{N-1} \cos^2(2\pi f_n k)$ —— (4) $(\sigma_v^2 = 1)$

Similarly, $b_n = -\sum v(k) \sin 2\pi f_n k$ —— (5)

$\Rightarrow E(b_n) = 0$ —— (6) & $\text{var}(b_n) = \sum \sin^2 2\pi f_n k$ —— (7)

$E(a_n b_n) = E\left(\left(\sum v(k) \cos(2\pi f_n k)\right)\left(\sum v(k) \sin(2\pi f_n k)\right)\right)$ (from (2) & (5))

Let $2\pi f_n = \omega_n$

$\Rightarrow E(a_n b_n) = E\left(\left(\sum v(k) \cos \omega_n k\right)\left(\sum v(k) \sin(\omega_n k)\right)\right)$

Multiplying the summations,

$= E\left(\left(\sum v^2(k) \cos \omega_n k \sin \omega_n k\right) + \sum_{k=0}^{N-1} \sum_{\substack{l=0 \\ l \neq k}} v(k) v(l) \cos \omega_n k \sin \omega_n k\right)$

P.T.O.

$$= E\left(\sum v^2(k) \cos wnk \sin wnk + \sum_{\substack{k=0}}^{N-1} \sum_{\substack{l=0 \\ l \neq k}}^{N-1} v(k)v(l) \cos wnk \sin wnl \right)$$

$\because$ $v[k]$ is white noise (GWN) $\sim N(0,1)$

$E(v^2(k)) = 1$ & $E(v(k)v(l)) = \sigma_{vv}[k-l]$

$= 0 \ \forall \ l \neq k$

$$\Rightarrow E(a_n b_n) = \sum E(v^2(k)) \cos wnk \sin wnk + \sum_{k=0}^{N-1} \sum_{\substack{l=0 \\ l \neq k}}^{N-1} E(v(k)v(l)) \cos wnk \sin wnl)$$

$$= \sum \cos wnk \sin wnk \quad \text{— (8)}$$

From eqns (1), (3), (4), (6), (7), (8) we have

$$\boxed{\mathrm{corr}(a_n, b_n) = \frac{\sum_{k=0}^{N-1} \cos(2\pi f_n k) \sin(2\pi f_n k)}{\sqrt{\left(\sum_{k=0}^{N-1} \cos^2(2\pi f_n k)\right)\left(\sum_{k=0}^{N-1} \sin^2(2\pi f_n k)\right)}}}$$

c) $$\xi_n = \frac{2 P_{vv}(f_n)}{\gamma(f_n)} = \frac{2(a_n^2 + b_n^2)}{N \gamma(f_n)}$$

$$\gamma(f_n) = \sum_{l=-\infty}^{\infty} \sigma_{vv}[l] \exp(-j 2\pi f l)$$

$$\because \sigma_v[l] = \begin{cases} 1 & l = 0 \\ 0 & l \neq 0 \end{cases}$$

$$\Rightarrow \gamma(f_n) = 1$$

P.T.O.

$\therefore G_n = \frac{2(a_n^2 + b_n^2)}{N}$

Substitute a_n & b_n (eqns ② & ⑤)

$$= \frac{2}{N}\Big[\sum v^2(k)\cos^2(2\pi f_n k) + \sum\sum v(k)v(l)\cos(2\pi f_n k)\cos 2\pi f_n l$$
$$+ \sum v^2(k)\sin^2(2\pi f_n k) + \sum\sum v(k)v(l)\sin(2\pi f_n k)\sin(2\pi f_n l)\Big]$$

group the $\sin^2$ & $\cos^2$ terms ($\cos^2\theta + \sin^2\theta = 1$

$$\Rightarrow G = \frac{2}{N}\Big[\sum v^2(k) + \sum\sum v(k)v(l)\big[\cos(w_n k)\cos w_n l + \sin w_n k \sin w_n l\big]\Big]$$

$$\Rightarrow G = \frac{2}{N}\Big[\sum_{k=0}^{N-1} v^2(k) + \sum_{k=0}^{N-1}\sum_{\substack{l=0\\ l\neq k}}^{N-1} v(k)v(l)\cos(w_n(k-l))\Big]$$

Sum of square of N independent Gaussian RV form the X^2 distribution.

Also product of 2 independent Gaussian RVs XY can be represented as $\frac{1}{4}((X+Y)^2 - (X-Y)^2)$ sum of 2 X^2 RVs

$\Rightarrow G_n$ is a weighted sum of X^2 distributed R.V.s

$\Rightarrow G_n$ is a X^2 distributed random variable

In limit $N \to \infty$ when cross terms go to zero

$$G_n \xrightarrow{d} X^2(2)$$

i) Mean of e_n

$$E(e_n) = \frac{2}{N}\left[\sum E(v^2) + 2\sum_{k=0}^{N-1}\sum_{\substack{l=0\\ l\neq k}}^{N-1} E(v(k)v(l))(\cos(\omega_n(l-k)))\right]$$

$$E(v^2) = \sigma^2 = 1$$

$$E(v(k)v(l)) = \sigma_{vv}[l-k] = \begin{cases} \sigma^2 & l=k \\ 0 & l\neq k \end{cases}$$

$$= \frac{2}{N}\times N \Rightarrow \boxed{E(e_n) = 2}$$

ii) Variance of e_n.

$$\mathrm{var}(e_n) = \frac{4}{N^2}\,\mathrm{var}\left(\sum v_k^2 + \sum_{k=0}^{N-1}\sum_{\substack{l=0\\ l\neq k}}^{N-1} v(k)v(l)\cos(\omega_n(k-l))\right)$$

$$\mathrm{var}(X+Y) = \mathrm{var}(X) + 2\mathrm{cov}(X,Y) + \mathrm{var}(Y)$$

here first term is X and second term is Y.

- $\mathrm{var}(X) = \mathrm{var}(\sum v_k^2)$; $v(k)$ is i.i.d

$$\Rightarrow \mathrm{var}(X) = \sum \sigma^4 = N\sigma_v^4 = N\times 1$$

$$\Rightarrow \mathrm{var}(X) = N$$

$$(\because \mathrm{var}(AB) = \mathrm{var}(A)\,\mathrm{var}(B) + \mathrm{var}(A)(E(B))^2 + \mathrm{var}(B)(E(A))^2$$

$$\Rightarrow \mathrm{var}(v^2) = (\sigma^2)^2 = \sigma^4 \;)$$

- $\mathrm{cov}(X,Y) = E(XY) - E(X)E(Y)$;

$E(Y)=0$ because $E(v(k)v(l))=0$ for $k\neq l$.

$$= E\{\sum(v(k))^2\sum\sum v(k)v(l)\cos\omega_n(k-l))$$

Notice that any term will be of the form $(v(k))^2v(l)$ or $v(j)v(k)v(l)$

$E(v(i)v(k)v(l)) = E(v(i))E(v(k))E(v(l)) = 0$

$E((v(k))^2 v(l)) = E((v(k))^2)\, \cancel{E(v(l))}^{0} = 0$

(this is because all v_k are 0 mean & $v(k), v(j)$ are uncorrelated for all $k \neq j$)

$\Rightarrow \underline{\mathrm{cov}(X, Y) = 0}$

- $\mathrm{var}(Y) = \mathrm{var}\left(\sum\sum v(k)\, v(l) \cos(\omega n(k-l))\right)$

We know that $E(Y) = 0$

$\Rightarrow \underline{\mathrm{var}(Y)} = E\left(\left(\sum\sum v(k)\, v(l) \cos(\omega n(k-l))\right)^2\right)$

$= E\left(4\sum_{k=0}^{N-1}\sum_{\substack{l=0 \\ k\neq l}}^{k-1}(v(k)v(l))^2 \cos^2(\omega n(k-l)) + 8\sum_{\substack{(i,j)\neq(k,l) \\ (i\neq j),(k\neq l)}}\sum\sum\sum v(i)v(j)v(k)v(l) \cos(\omega n(i-j)) \cos(\omega n(k-l))\right)$

The second term will never have all variables having even powers, i.e. $(v(k)v(l))^2$

So because of the i.i.d & $E(v) = 0$, we can say that $\underline{E(\cdot)\text{ of second term goes to zero}}$.

$\Rightarrow \mathrm{var}(Y) = 4\, E\left(\sum_{k=0}^{N-1}\sum_{\substack{l=0 \\ k\neq l}}^{k-1}(v(k)\, v(l))^2 \cos^2(\omega n(k-l))\right)$

Once again we invoke, $\mathrm{var}(AB) = \mathrm{var}(A)\mathrm{var}(B) + \mathrm{var}(A)(E(B))^2 + \mathrm{var}(B)(E(A))^2$

$\Rightarrow \mathrm{var}(v(k)v(l)) = \mathrm{var}(v(k))\mathrm{var}(v(l))$

$= \sigma_v^4 = 1$

Since $E(v(k)\,v(l)) = 0$, (if $k \neq l$),

$$E((v(k)\,v(l))^2) = \text{var}(v(k)\,v(l))$$

$$= 1$$

$$\therefore \text{var}(Y) = 4 \sum_{k=0}^{N-1} \sum_{l=0}^{k-1} \cancel{\text{var}}\, E\,(v(k)\,v(l)) \cos^2(w n(k-l))$$

$$= 4 \sum_{k=0}^{N-1} \sum_{l=0}^{k-1} \cos^2(w n(k-l))$$

$$= \overset{2}{\cancel{4}} \sum_{k=0}^{N-1} \sum_{\substack{l=0 \\ k\neq l}}^{N-1} \frac{(1 + \cos(2 w n(k-l)))}{\cancel{2}}$$

$$\sum \sum \cos(2 w n (k-l)) = \sum \sum \cos(4\pi f n(k-l))$$

$$= \sum_{k\neq l} \sum^{N-1} \cos\left(4\pi \frac{n}{N} (k-l)\right)$$

Using the fact that $\sum_{k=0}^{N-1} \cos\left(2\pi \frac{n}{N} k\right) = 0$,

& that $\cos 0 = 1$ will be missing from ~~the~~ sides the sum ($\because k \neq l$.)

So we get

$$\sum_{k=0}^{N-1} \sum_{\substack{k\neq l \\ l=0}}^{k-1} \cos\left(4\pi \frac{n}{N} (k-l)\right) = \frac{-N}{2}$$ (I also verified this in MATLAB)

$$\Rightarrow \text{var}(Y) = 2\times\left[N \times \frac{(N-1)}{2} - \frac{N}{2}\right] = N(N-2)$$

$\therefore \; var(e_{1n}) = \frac{4}{N} + \frac{4}{N^2} \times N(N-2)$

$= \frac{4}{N} 4\left(\frac{1}{N} + \frac{N-2}{N}\right)$

$$\boxed{var(e_{1n}) = 4\frac{(N-1)}{N}}$$

As expected $\lim_{N\to\infty} var(e_{1n}) = 2 \times 2 = \lim_{N\to\infty} \left(\frac{1-\frac{1}{N}}{1}\right) 4$

$= \boxed{4} \to var(X^2(2))$

d) Mean squared convergence to be checked

$\Rightarrow \lim_{N\to\infty} E\left(\left(P(f_n) - \gamma_{vv}(f_n)\right)^2\right)$

We already obtained that $\gamma_{vv}(f_n) = 1$,

also $E(q_n) = 2 \Rightarrow E\left(\frac{2P(f_n)}{\gamma(f_n)}\right) = 2$

$\Rightarrow E(P(f_n)) = \gamma(f_n) = 1$

So the RV $P(f_n) - \gamma_{vv}(f_n)$ has zero mean

$\therefore \lim_{N\to\infty} E\left(\left(P(f_n) - \gamma(f_n)\right)^2\right)$

$= var((P(f_n) - 1)) + \cancel{\left(E(P(f_n) - \gamma(f_n))\right)^2}^{\,0}$

$= \lim_{N\to\infty} var(P(f_n) - 1) = \lim_{N\to\infty} var(P(f_n))$

PTO

from part (c)

$$\text{var}(e_n) = \frac{4(N-1)}{N}$$

$$\Rightarrow \text{var}\left(\frac{2P(f_n)}{\gamma(f_n)}\right) = \frac{4(N-1)}{N}$$

$$\Rightarrow \text{var}(P(f_n)) = \frac{(N-1)}{N}$$

$$\Rightarrow \lim_{N\to\infty} \text{var}(P(f_n)) = \cancel{\frac{N-1}{N}}\ 1$$

$$\therefore \lim_{N\to\infty} E\left(\left(P(f_n) - \gamma_{vv}(f_n)\right)^2\right) = 1 \neq 0$$

$\Rightarrow P(f_n)$ does not exhibit mean squared convergence to $\gamma(f_n)$

However it is an unbiased estimator.

③ a) $y[k] = A\sin(2\pi f_0 k) + e[k]$

$e[k] \sim N(0, \sigma_e^2)$

$$l = \prod_{k=0}^{N-1}\left(\frac{y[k] - A\sin(2\pi f_0 k)}{(\sqrt{2\pi}\,\sigma_e)}\right)$$

$$\Rightarrow L = \log\frac{1}{(2\pi\sigma_e^2)^{N/2}} - \frac{\sum\left(y(k) - A\sin(2\pi f_0 k)\right)^2}{2\sigma_e^2} \quad \text{———} ①$$

$$\Rightarrow \frac{\partial L}{\partial f_0} = \frac{2\pi A\cos(2\pi f_0 k)}{\sigma_e^2}\sum\left(y(k) - A\sin(2\pi f_0 k)\right)k$$

$$\Rightarrow \frac{\partial^2 L}{\partial f_0^2} = \left(\frac{4\pi^2 A^2}{\sigma_e^2}\right)\left(\sum -y(k)k^2\sin(2\pi f_0 k) + A k^2\sin^2(2\pi f_0 k) - A\cos^2(2\pi f_0 k)k^2\right)$$

$$\Rightarrow E\left(-\frac{\partial^2 L}{\partial f_0^2}\right) = \frac{-4\pi^2 A^2}{\sigma_e^2}\left(\sum\left(-A\sin^2(2\pi f_0 k) + A\sin^2(2\pi f_0 k) - A\cos^2(2\pi f_0 k)\right)k^2\right)$$

$\because (E(y(k)) = A\sin 2\pi f_0 k + 0 = A\sin(2\pi f_0 k))$

$$\Rightarrow E\left(\frac{-\partial^2 L}{\partial f_0^2}\right) = \frac{4\pi^2 A^2}{\sigma_e^2}\sum_{k=0}^{N-1} k^2\cos^2(2\pi f_0 k) \quad \text{———} ②$$

$$① \Rightarrow \frac{\partial L}{\partial A} = -\frac{\sum \sin 2\pi f_0 k\,(A\sin(2\pi f_0 k) - y(k))}{\sigma_e^2} \quad \text{———} ③$$

$$\Rightarrow \frac{\partial^2 L}{\partial A^2} = -\frac{\sum \sin^2 2\pi f_0 k}{\sigma_e^2} \quad ④$$

$$\Rightarrow -E\left(\frac{\partial^2 L}{\partial A^2}\right) = \frac{\sum \sin^2(2\pi f_0 k)}{\sigma_e^2} \quad \text{———} \; (4)$$

$$(3) \Rightarrow \cancel{\frac{\partial L}{\partial f \, \partial A}} = 2\pi k \left(\cancel{\frac{A - y(k)}{\sigma_e^2}}\right)$$

$$(3) \Rightarrow \frac{\partial L}{\partial f \, \partial A} = -\sum \left(\frac{2\pi k}{\sigma_e^2}\right)\left[A \, 2 \sin 2\pi f_0 k \cos 2\pi f_0 k - y(k) \cos(2\pi f_0 k)\right]$$

$$-E\left(\frac{\partial L}{\partial f \, \partial A}\right) = \frac{2\pi}{\sigma_e^2} \sum \left[2 A k \sin 2\pi f_0 k \cos (2\pi f_0 k) - A k \sin 2\pi f_0 k \cos 2\pi f_0 k\right]$$

$$\Rightarrow -E\left(\frac{\partial L}{\partial f \, \partial A}\right) = \frac{2\pi A}{\sigma_e^2} \sum_{k=0}^{N-1} k \sin(2\pi f_0 k) \cos(2\pi f_0 k) \quad \text{———} \; (5)$$

$$I(\underline{\theta}) = \begin{bmatrix} -E\left(\frac{\partial^2 L}{\partial A^2}\right) & -E\left(\frac{\partial L}{\partial f \, \partial A}\right) \\ -E\left(\frac{\partial L}{\partial f \, \partial A}\right) & -E\left(\frac{\partial^2 L}{\partial f^2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4\pi^2 A^2}{\sigma_e^2} \sum k^2 \cos^2(2\pi f_0 k) & \frac{2\pi A}{\sigma_e^2} \sum k \sin(2\pi f_0 k) \cos(2\pi f_0 k) \\ \frac{2\pi A}{\sigma_e^2} \sum k \sin(2\pi f_0 k) \cdot \cos(2\pi f_0 k) & \frac{\sum \sin^2(2\pi f_0 k)}{\sigma_e^2} \end{bmatrix}$$

For simplicity, let

$$a = \frac{4\pi^2 A^2}{\sigma_e^2} \sum k^2 \cos^2(2\pi f_0 k)$$

$$b = \frac{2\pi A}{\sigma_e^2} \sum k \sin(2\pi f_0 k) \cos(2\pi f_0 k)$$

$$d = \sum \frac{\sin^2(2\pi f_0 k)}{\sigma_e^2}$$

$$\therefore I(\theta) = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

The CRLB is given by

$$\Sigma_{\hat{\theta}} \geq (I(\theta))^{-1}$$

$$I(\theta)^{-1} = \begin{bmatrix} d & -b \\ -b & a \end{bmatrix} \frac{1}{(ad - b^2)}$$

For a +ve semidefinite matrix

$$a_{jj} \geq 0$$

$$\Rightarrow [\Sigma_{\hat{\theta}}]_{jj} \geq [(I(\theta))^{-1}]_{jj}$$

Specifically $\operatorname{var}(\hat{\theta}_1) \geq \dfrac{d}{ad - b^2}$ —(1)

& $\operatorname{var}(\hat{\theta}_2) \geq \dfrac{a}{ad - b^2}$ —(2)

$\hat{\theta}_1 \rightarrow f_0$ $\hat{\theta}_2 \rightarrow$ Amplitude

In the case where ~~the~~ one of the parameters is known (single unknown case),

$$\mathrm{var}(\hat{\theta}) \geq \frac{1}{-E\left(\frac{\partial^2 L}{\partial \theta^2}\right)}$$

$\Rightarrow$ the bounds are for $\hat{\theta}_1$ (frequency ~~Amplitude~~) $= \dfrac{1}{a} = \dfrac{d}{ad}$ — ③

& for $\hat{\theta}_2$ (Amplitude) $= \dfrac{1}{d} = \dfrac{a}{ad}$ — ④

① > ③ $\left(ad > ad - b^2 \Rightarrow \dfrac{d}{ad} < \dfrac{d}{ad-b^2} \right)$

② > ④ $\left(ad > ad - b^2 \Rightarrow \dfrac{a}{ad} < \dfrac{a}{ad-b^2} \right)$

We see that the lower bound is lower for both parameters in the case of single parameter unknown. (other is known).

This is intuitive because, if we already know the true value of the other parameter, the entire data is used to just get information on the single unknown. However, if both are unknown, from the same data we need to estimate 2 unknowns, resulting in decrease in information individually ~~about~~ ~~each~~ ~~one of~~ compared to earlier case

(3) b) We know that $E\left((y-\mu)^2\right) = \sigma^2$, here $\mu = 0$

$\Rightarrow E(y^2) = \sigma^2$.

So we can assume that the transformed data $y \to y^2$ is coming out of a DGP such that the true mean is σ^2

$$y^2[k] = \sigma^2 + e[k] \quad \text{—— (1)}$$

where $e[k]$ is an uncorrelated ($\because \rho(y[k], y[l]) = 0 \ \forall k \neq l$) and zero mean Random variable

σ^2 is the true variance of $y[k]$

In vectorial form,

$$\underline{y}_N = L\sigma^2 + \underline{e} \quad \text{—— (2)}$$

L is an $N \times 1$ vector of ones

$$L = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{N \times 1}$$

To ~~estimate~~ determine BLUE we need Σ_e^*.

Since $y[k]$ has uniform variance $\forall k$, $e[k]$ should also be a homoskedastic variable.

And Σ_e should be diagonal, since the error terms are not correlated.

$$var(y^2) = \cancel{var(\sigma^2)}^{0} + var(e)$$

$$\Rightarrow var(e) = var(y^2) = E\left((y^2 - \sigma^2)^2\right)$$

$$= E(y^4) - (\sigma^2)^2 \quad — (3) \quad \left[E(X^2) - (E(X))^2\right]$$

Since y is Gaussian we use the Moment Generating function as derived in Q2.

$$MGF: \exp\left(s\cancel{y}^{0} + \frac{s^2\sigma^2}{2}\right) = \exp\left(\frac{s^2\sigma^2}{2}\right)$$

$$\frac{\partial M_y(s)}{\partial s} = \frac{2\sigma^2 s}{2}\exp\left(\frac{s^2\sigma^2}{2}\right) = \sigma^2 s \exp\left(\frac{s^2\sigma^2}{2}\right)$$

$$\Rightarrow \frac{\partial^2 M}{\partial s^2} = \sigma^2 \exp\left(\frac{s^2\sigma^2}{2}\right)\left(1 + s^2\sigma^2\right)$$

$$\Rightarrow \frac{\partial^3 M}{\partial s^3} = \sigma^2 \exp\left(\frac{s^2\sigma^2}{2}\right)\left(2s\sigma^2 + s^3\sigma^4\right)$$

$$= \sigma^4 \exp\left(\frac{s^2\sigma^2}{2}\right)\left(3s + s^3\sigma^2 + \cdots\right)$$ other terms don't contribute

$$\Rightarrow \frac{\partial^4 M}{\partial s^4} = \sigma^4 \exp\left(\frac{s^2\sigma^2}{2}\right)\left(3 + 3s^2\sigma^2 + \cdots\right)$$

$$\therefore E(y^4) = \left.\frac{\partial^4 M}{\partial s^4}\right|_{s=0} = 3\sigma^4 \quad — (4)$$

Substitute (4) in (3),

$$var(e) = 3\sigma^4 - \sigma^4$$

$$= 2\sigma^4$$

$$\therefore \Sigma_e = \begin{bmatrix} 2\sigma^4 & & 0 \\ & 2\sigma^4 & \\ 0 & & 2\sigma^4 \end{bmatrix}_{N \times N}$$

$$\Sigma_e^{-1} = \begin{bmatrix} \frac{1}{2\sigma^4} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{2\sigma^4} \end{bmatrix}_{N \times N}.$$

We know that the solution of BLUE is,

$$A = (L^T \Sigma_e^{-1} L)^{-1} (\Sigma_e^{-1} L)$$

$$= \left([1 \cdots 1] \begin{bmatrix} \frac{1}{2\sigma^4} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{2\sigma^4} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right)^{-1} [\Sigma_e^{-1} L]$$

$$= \left(\begin{bmatrix} \frac{1}{2\sigma^4} & \cdots & \frac{1}{2\sigma^4} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} \frac{1}{2\sigma^4} \\ \frac{1}{2\sigma^4} \\ \vdots \\ \frac{1}{2\sigma^4} \end{bmatrix} \right)$$

$$= \left(\frac{2\sigma^4}{N} \right) \begin{bmatrix} \frac{1}{2\sigma^4} \\ \vdots \\ \frac{1}{2\sigma^4} \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} \frac{1}{N} \\ \frac{1}{N} \\ \vdots \\ \frac{1}{N} \end{bmatrix}_{N \times 1}$$

$$\therefore \hat{\theta}^*_{BLUE} = A\underline{y}^2 = \frac{\sum^N y^2(k)}{N} = \frac{\sum_{k=0}^{N-1} y^2[k]}{N}.$$

$$\Rightarrow \boxed{\hat{\sigma}^2 = \frac{\sum y^2(k)}{N}}$$

The expression might look similar to the biased sample variance case. However there we used the sample mean, here we utilise the true mean.

So $E\left(\frac{\sum y^2}{N}\right) = \frac{N\sigma^2}{N} = \sigma^2$,

the given estimator is indeed unbiased and since it is linear and we solved the optimisation problem to get A, it is the Best Linear Unbiased Estimator

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(4) a) $N = 100$ samples

$$\hat{\mu} = 14578$$

We need the confidence for μ lying in the interval

$$12000 \leq \mu \leq 16000$$

We know that

$$z_1 \leq \frac{\hat{\mu} - \mu}{\frac{\sigma}{\sqrt{N}}} \leq z_2$$

where z_1 & z_2 are dictated by the confidence level.

$$\Rightarrow \hat{\mu} - \frac{z_2\sigma}{\sqrt{N}} \leq \mu \leq -z_1\frac{\sigma}{\sqrt{N}} + \hat{\mu}$$

$$\hat{\mu} - \frac{z_2 \sigma}{\sqrt{N}} = 12000 \qquad \hat{\mu} = 14578 \qquad N = 100$$
$$\hat{\sigma} = 1845$$

Since σ is not known, I am substituting the given sample deviation (assuming it is unbiased)

$\Rightarrow$ ~~$\sigma =$~~

$$\Rightarrow z_2 = \frac{(14578 - 12000) \times 10}{1845}$$
$$= 13.97$$

$$\hat{\mu} - \frac{z_1 \sigma}{\sqrt{N}} = 16000$$
$$\Rightarrow z_1 = \frac{(14578 - 16000) \times 10}{1845}$$
$$= -7.7073$$

$\Rightarrow$ The confidence region : $-7.707 \leq z \leq 13.97$

$\therefore$ The confidence $= F(z \leq 13.97) - F(z \leq -7.707)$

$z \in N(0,1)$

is nearly 1 (very very high confidence)

We can say that the average molecular weight of the polymer is in between 12000 and 16000 with nearly 100% confidence (but not exactly 100%.)

b) $N_1 = 60$, $\bar{x}_1 = 85.2$, $s_1 = 6.8$

$N_2 = 55$, $\bar{x}_1 = 87.2$, $s_1 = 8.8$

Since both the distributions are similar, let us assume ~~but~~ unknown but equal population variance for the 2 groups

Pooled variance, $S_p^2 = \dfrac{(N_1-1)S_1^2 + (N_2-1)S_2^2}{N_1+N_2-2}$ ↳ (2 ~~use~~ estimate $\bar{x}_1$ & $\bar{x}_2$ - 2 values)

$\Rightarrow S_p = 7.82$

Consider the statistic $\bar{x}_1 - \bar{x}_2$

$$var(\bar{x}_1 - \bar{x}_2) = var(\bar{x}_1) + var(\bar{x}_2) + \underbrace{2\,cov(\bar{x}_1, \bar{x}_2)}_{0}$$

($\bar{x}_1$ and $\bar{x}_2$ are uncorrelated)

$$= \frac{S_p^2}{N_1} + \frac{S_p^2}{N_2}$$

First we will test whether $\bar{x}_1 - \bar{x}_2 = 0$ or not.

$H_0 : \bar{x}_1 - \bar{x}_2 = 0$ $\quad \alpha = 0.05$

$H_1 : \bar{x}_1 - \bar{x}_2 \neq 0$

Critical value approach: $Z \geq 1.96$ or $Z \leq -1.96$ to Reject H_0

$$\Rightarrow -1.96 < \frac{(\bar{x}_1 - \bar{x}_2) - 0}{\sqrt{\frac{S_p^2}{N_1} + \frac{S_p^2}{N_2}}} < 1.96$$

for H_0 ~~to~~ not be rejected

$$\frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{S_p^2}{N_1} + \frac{S_p^2}{N_2}}} = \frac{85.2 - 87.2}{7.82\sqrt{\frac{1}{N_2} + \frac{1}{N_1}}} = -1.37$$

Since $-1.96 \leq -1.37 \leq 1.96$, we cannot reject the Null Hypothesis that the average scores of the schools are same

$\therefore$ We conclude that on an average the institutions perform equally well.

Also note that since σ^2 is unknown we should have resorted to the t-test but since the dof is $N_1+N_2-2 = 98$ very high, approximately the we can use the standard normal distribution

⑤a) Consider the likelihood fn,

$$l = \begin{cases} \prod_{k=1}^{N} e^{-(y(k)-\theta)} & \text{if } \min(y_k) > \theta \\ 0 & \text{if } \min(y_k') < \theta \end{cases}$$

$$\Rightarrow l = \begin{cases} \exp\left(\sum_{k=1}^{N} \theta - y[k]\right) & \min(y(k)) > \theta \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \exp(N\theta)\exp\left(\sum - y(k)\right) & \min(y(k)) > \theta \\ 0 & \text{otherwise} \end{cases}$$

~~Using the Heaviside step fn,~~

$$\cancel{H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \end{cases}}$$

Consider the function,

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$$\Rightarrow l = \left[\exp(N\theta)\left(\exp\left(\sum - y(k)\right)\right)\left(H(\min(y_N) - \theta)\right)\right]$$

$$= \left[H(\min(y_N) - \theta)\exp(N\theta)\right]\left[\exp\left(-\sum y(k)\right)\right]$$

consider term 1 as $\phi(T(y_N), \theta)$ $\left(\because T(\underline{y}_N) = \min(\underline{y}_N)\right)$

term 2 as $K(\underline{y}_N)$

$\therefore$ By the Neyman-Fisher factorisation theorem,

$T_N = \min(y_N)$ is a sufficient statistic.

Further we know that it is a complete statistic

By Rao-Blackwell theorem, we need to construct an unbiased estimator using T_N to get the MVUE

from Q1 b), we know an unbiased estimator of Θ as,

$$T_N' = \frac{T_N}{2} - \frac{1}{N}$$

$$\Rightarrow \text{MVUE of } \Theta = \frac{T_N}{2} - \frac{1}{N}$$

$$= \underline{\min(y_N)} - \frac{1}{N}$$

$$\Rightarrow \boxed{\hat{\Theta}_{MVUE} = \min(y_N) - \frac{1}{N}}$$