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CH18B020

CH5115 ASSIGNMENT - 2

1. a) $v[k] = A\cos^2(2\pi f k + \phi)$

$$E(v[k]) = E(A\cos^2(2\pi f k + \phi))$$
$$= \cos^2(2\pi f k + \phi)\, E(A) \qquad (\because A \text{ is the R.V.})$$
$$\Rightarrow E(v[k]) = 0 \qquad (\because E(A) = 0)$$

$$var(v[k]) = E((v[k] - E(v[k]))^2)$$
$$= E((v[k])^2) = E(A^2\cos^4(2\pi f k + \phi))$$
$$= (\cos^4(2\pi f k + \phi))\, E(A^2)$$

We know that $\sigma^2_A = E(A^2) - (\cancel{E(A)})^2$

$$\Rightarrow E(A^2) = \sigma_A^2$$
$$= 1$$

$$\therefore var(v[k]) = \cos^4(2\pi f k + \phi)$$

$\Rightarrow var(v[k])$ is a function of k. (time step)

$\Rightarrow$ The random process is ~~covariance~~ variance non stationary

$$\text{Autocovariance} = E((v[k-l] - E(v[k-l]))(v[k] - E(v[k])))$$
$$= E(v[k-l]\, v[k])$$
$$= E(A^2\cos^2(2\pi f k + \phi)\cos^2(2\pi f(k-l) + \phi))$$
$$= (E(A^2))(\cos^2(2\pi f k + \phi)\cos^2(2\pi f(k-l) + \phi))$$

$$= \left(\cos\left(4\pi f k - 2\pi f l + 2\phi\right) + \cos\left(2\pi l f\right) \right)^2$$

$\Rightarrow$ ACVF of $v[k]$ at ~~only~~ any lag l is a function of k.

~~$\Rightarrow$ v[t. covariance of $v[k]$~~

$\Rightarrow$ $v[k]$ is covariance non-stationary

1b)

$$v[k] - v[k-1] = e[k]$$

$$v[k-1] - v[k-2] = e[k-1]$$

$$\vdots$$

$$v[1] - v[0] = e[1]$$

& $v[0] = 0$ (given)

Add all

$$v[k] = \sum_{i=0}^{k} e[i]$$

$$e[i] \sim N(0, \sigma^2)$$

$$\Rightarrow E(v[k]) = E\left(\sum e[k]\right)$$

$$= \sum \left(E(e[i])\right) = \sum 0$$

$$\Rightarrow E(v[k]) = 0$$

$$var(v[k]) = E\left(\left(v[k] - E(v[k])\right)^2\right)$$

$$= E\left((v[k])^2\right) = E\left(\left(\sum_{i=1}^{k} e[i]\right)^2\right)$$

$$= E\left(\sum_{i=1}^{k}(e[i])^2 + 2\sum_{j=1}^{k}\sum_{i=1}^{j-1} e[i]e[j]\right)$$

$$= \sum_{i=1}^{k} E\left(e[i]^2\right) + 2\sum\sum E\left(e[i]e[j]\right)$$

$$E(e[i]^2) = E((e[i])^2) - (E(e[i]))^2 = \sigma^2$$

$$E(e[i]e[j]) = E(e[i])E(e[j])$$

$\because$ Gaussian white noise process $\Rightarrow$ i.i.d.

$$\Rightarrow E(e[i]e[j]) = 0$$

$\therefore$ var $(v[k]) = k\sigma^2 \Rightarrow$ var$(v[k])$ varies with k

$\Rightarrow$ non-stationary.

From the MATLAB plot we can see that var$(v[k])$ is continously increasing as k increases.

$\Rightarrow$ var $(v[k])$ is non-stationary

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② a) Given:

$\rightarrow u[k] \sim WN(0, \sigma_u^2) \Rightarrow E(u[k]) = 0$

$E(u^2[k]) = \sigma_u^2$

$E(u[k]\, u[k-l]) = 0 \quad \forall l \neq 0$

$\rightarrow e[k] \sim WN(0, \sigma_e^2) \Rightarrow E(e[k]) = 0$

$E(e^2[k]) = \sigma_e^2$

$E(e[k]\, e[k-l]) = 0 \quad \forall l \neq 0.$

$\rightarrow \sigma_{eu}[l] = 0 \Rightarrow E(e[k]\, u[k-l]) = 0$

$$y^*_o[k] = \frac{b_2^o q^{-2}}{1 + f_1^o q^{-1}} u[k]$$

$\Rightarrow y^*[k] + f_1^o q^{-1}(y^*[k]) = b_2^o q^{-2}(u[k])$

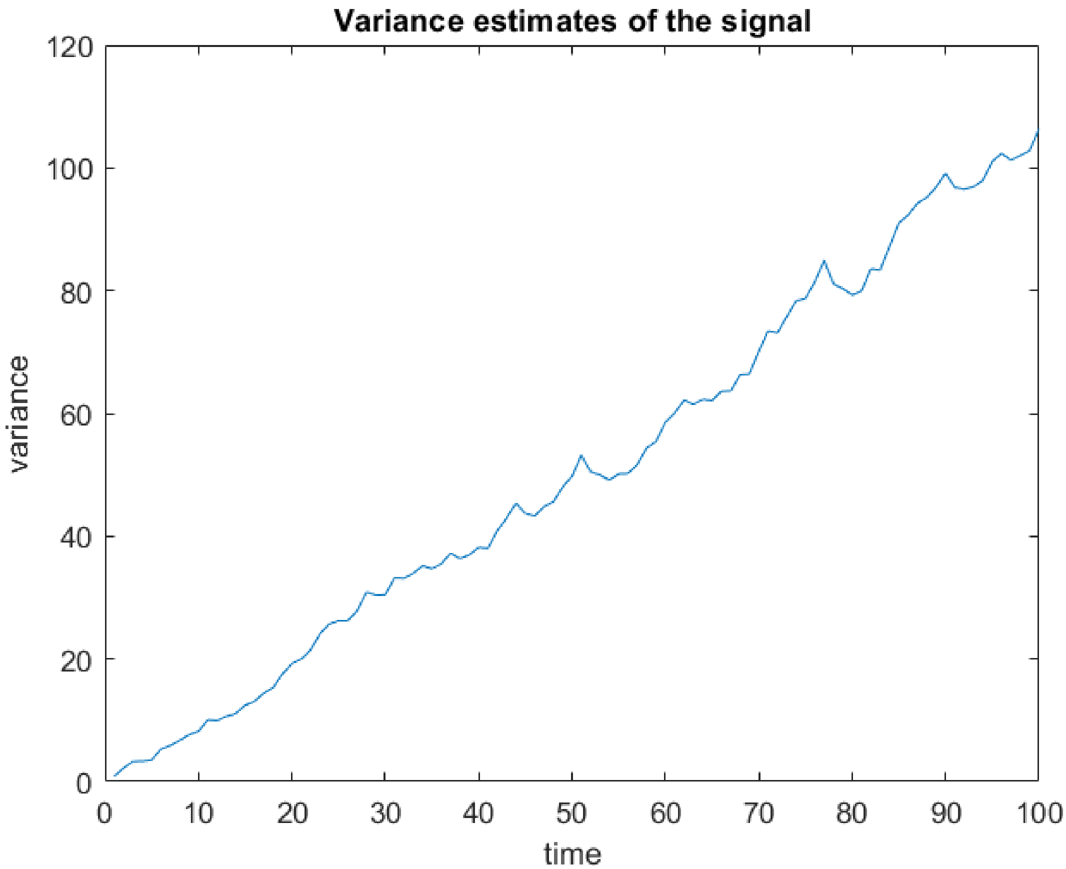
$\Rightarrow y^*[k] + f_1^o y[k-1] = b_2^o u[k-2]$

$\Rightarrow y^*[k] = b_2^o u[k-2] - f_1^o y[k-1]$ — ⑭ (using the property of backward shift operator)

$E(y^*[k]) = E(\cancel{u[k-2]}) - f_1^o E(y[k-1])$ (Assuming mean stationarity)

$E(y^*) = 0$ — ①

Question 1



As expected, we see variance of the process keep increasing with time.

- ⇨ **The process is variance non-stationary.**

$E(y^{*2}(k)) = \sigma^2_{y^*}$

$\Rightarrow \sigma^2_{y^*} = E\left(\left(b_2^\circ u(k-2) - f_1 y^*(k-1)\right)^2\right)$

$= E\left(b_2^{\circ 2} u^2[k-2] + f_1^{\circ 2}(y^*[k-1])^2 \cancel{\quad} - 2f_1^\circ b_2^\circ u[k-2] y^*[k-1]\right)$

Expectation of sum is sum of $E(\cdot)$

$= b_2^{\circ 2}\sigma_u^2 + f_1^{\circ 2}\sigma^2_{y^*} - 2f_1^\circ b_2^\circ \sigma_{y^*u}[1]$ — ②

(y^* is assumed to be variance stationary)

$\sigma_{y^*u}[0] = E(y^*[k]u[k])$

$= 0$ ($\because$ $u[k]$ doesn't affect $y[j]$ $\forall j < k+2$)

$\sigma_{y^*u}[1] = E(y^*[k]u[k-1])$

$= E\left(\left(b_2^\circ u[k-2] - f_1^\circ y^*[k-1]\right)u[k-1]\right)$

$= b_2^\circ E(u[k-2]u[k-1]) - E(f_1^\circ y^*[k-1]u[k-1])$

$= b_2^\circ \sigma_{uu}[1] - \sigma_{y^*u}[0]$

$\Rightarrow \sigma_{y^*u}[1] = 0.$ — ③

Using this,

② $\Rightarrow \sigma^2_{y^*} = b_2^{\circ 2}\sigma_u^2 + f_1^{\circ 2}\sigma^2_{y^*}$

$\Rightarrow \sigma^2_{y^*} = \left(\dfrac{b_2^{\circ 2}\sigma_u^2}{1 - f_1^{\circ 2}}\right)$ — ④

$$\sigma_{y^*e}[l] = E\left(y^*[k]\, e[k-l]\right)$$

$$= E\left(\left(b_2^o\, u[k-2] - f_1^o\, y[k-1]\right) e[k-l]\right)$$

$$= b_2^o \cancel{\sigma_{ue}[l-2]}^{\,0} - f_1^o\, \sigma_{y^*e}[l-1]$$

Assuming $\sigma_{y^*e}[0] = 0$ (Truth is not correlated with the noise)

$$\Rightarrow \sigma_{y^*e}[l] = 0 \quad — (5)$$

i) To find σ_y^2 :

$E(y) = E(y^* + e) = 0$

$\Rightarrow E(y(k)) = 0.$

$$\therefore \sigma_y^2 = E\left((y[k])^2\right)$$

$$= E\left((y^*[k] + e[k])^2\right)$$

$$= \sigma_{y^*}^2 + \sigma_e^2 + 2\cancel{\sigma_{y^*e}}^{\,0}$$

$$\Rightarrow \boxed{\sigma_y^2 = \frac{b_2^{o\,2}\, \sigma_u^2}{(1 - f_1^{o\,2})} + \sigma_e^2}$$

ii) $\sigma_{y^*y^*}[1] = E\left(y^*[k]\, y^*[k-1]\right)$

$$= E\left(\left(b_2^o\, u[k-2] - f_1^o\, y^*[k-1]\right)\left(y^*[k-1]\right)\right)$$

$$= E\left(b_2^o\, u[k-2]\, y^*[k-1] - f_1^o\left(y^*[k-1]\right)^2\right)$$

$$= b_2^o\, \sigma_{y^*u}[1] - f_1^o\, \sigma_{y^*}^2$$

$$\Rightarrow \sigma_{y^* y^*}[1] = -f_1^0 \sigma_{y^*}^2$$

$$= -f_1^0 \left(\frac{\sigma_u^2 b_2^{0\,2}}{1 - f_1^{0\,2}} \right) \quad \text{—— (6)}$$

$$\sigma_{yy}[1] = E\left(\left(y^*(k) + e(k) \right) \left(y^*(k) + e(k) \right) \right)$$

$$= E\left(y^{*2}(k) \right) + E\left(e^2(k) \right) + 2E\left(u(k) e(k) \right)$$

$$= \sigma_{y^*}^2 + \sigma_e^2 + 2 \cancelto{0}{\sigma_{eu}}$$

$$\Rightarrow \boxed{\sigma_{yy}[1] = -\frac{f_1^0 \sigma_u^2 b_2^{0\,2}}{(1 - f_1^{0\,2})} + \sigma_e^2}$$

ii) $$\sigma_{yy}[1] = E\left(\left(y^*(k) + e[k] \right) \left(y^*(k-1) + e(k-1) \right) \right)$$

$$= \sigma_{y^* y^*}[1] + \cancelto{0}{\sigma_{ee}[1]} + \cancelto{0}{\sigma_{y^* e}[1]} + \cancelto{0}{\sigma_{e y^*}[1]}$$

$$\Rightarrow \boxed{\sigma_{yy}[1] = -f_1^0 \left(\frac{\sigma_u^2 b_2^{0\,2}}{1 - f_1^{0\,2}} \right)}$$

iii) $$\sigma_{yu}[1] = E\left(\left(y^*(k) + e(k) \right) \left(u(k-1) \right) \right)$$

$$= E\left(y^*(k) u(k-1) \right) + E\left(e(k) u(k-1) \right)$$

$$= \sigma_{y^* u}[1] + \cancelto{0}{\sigma_{eu}(1)}$$

$$= 0 \quad (\text{from eqn (4)})$$

$$\Rightarrow \boxed{\sigma_{yu}[1] = 0}$$

iv) $\sigma_{y^*u}[2] = E\left(\left(b_2^0 u(k-2) - f_1^0 y^*(k-1)\right) u(k-2)\right)$

$= E\left(b_2^0 u^2(k-2) - f_1^0 y_1^*(k-1) u(k-2)\right)$

$= b_2^0 \sigma u^2 - f_1^0 \sigma_{y^*u}[1]$

From ④, $\sigma_{y^*u}[1] = 0$

$\Rightarrow \sigma_{y^*u}[2] = b_2^0 \sigma u^2$ —— ⑥

$\cancel{\sigma_{yu}(2) = E\left(\left(y^*(k) + e(k)\right)\left(y^*(k-2) + e(k-2)\right)\right)}$

$\cancel{= E\, \sigma_{y^*y^*}(2) + \sigma}$

$\sigma_{yu}(2) = E\left(\left(y^*(k) + e(k)\right)\left(u(k-2)\right)\right)$

$= E\left(y^*(k) u(k-2) + e(k) u(k-2)\right)$

$= \sigma_{y^*u}[2] + \cancel{\sigma_{eu}(2)}^{\,0}$

From ⑥, $\sigma_{y^*u}(2) = b_2^0 \sigma u^2$

$$\boxed{\sigma_{yu}(2) = b_2^0 \sigma u^2}$$

Summary.

i) $\sigma_y^2 = \frac{b_2^{o2} \sigma_u^2}{1 - f_1^{o2}} + \sigma_e^2$

ii) $\sigma_{yy}[1] = -f_1^o \left(\frac{\sigma_u^2 b_2^{o2}}{1 - f_1^{o2}} \right)$

iii) $\sigma_{yu}(1) = 0$

iv) $\sigma_{yu}(2) = b_2^o \sigma_u^2$

MATLAB Simulation

→ $\sigma_{y^*}^2$ is obtained from σ_u^2, b_0 & f_0 using eqn (4)

$\sigma_{y^*}^2 = 10.6667$

→ σ_e^2 is obtained using SNR = 10 ⇒ $\sigma_e^2 = 0.9375$

→ y^* is generated using the given relation with u and assuming $y^*[1] = y^*[2] = 0$

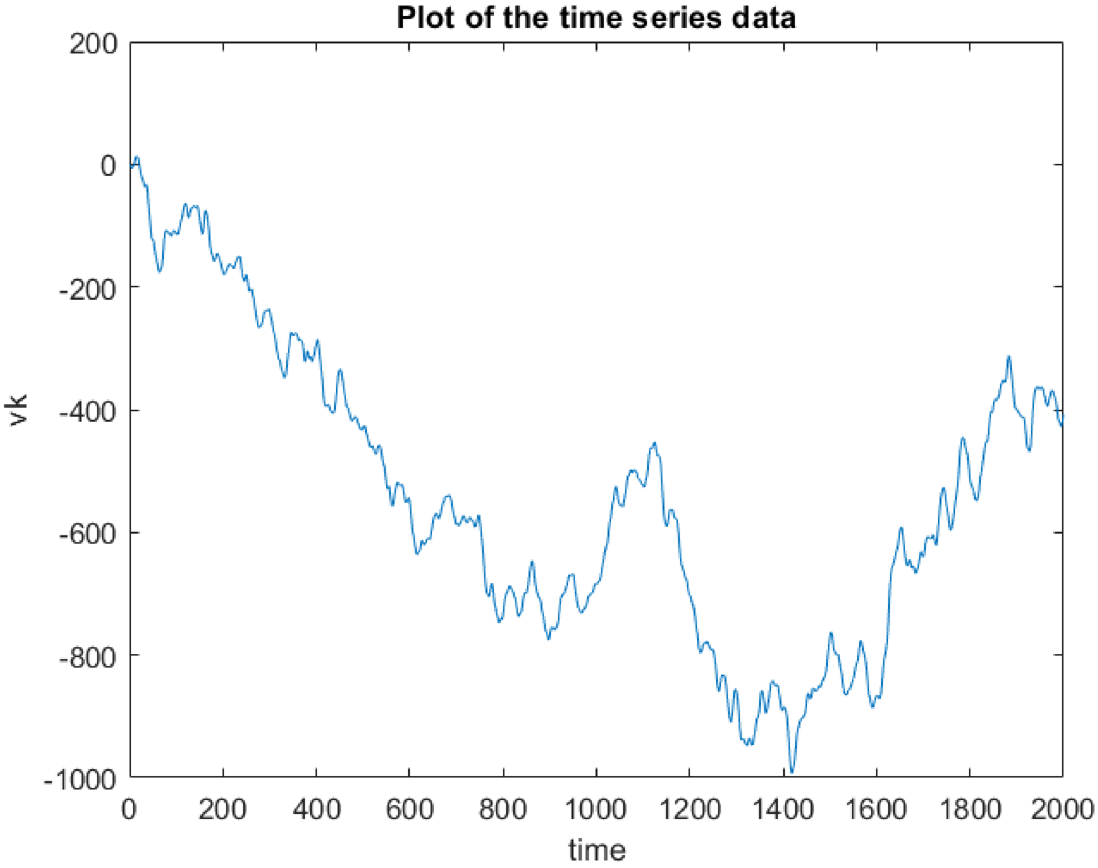
→ estimates of variance is obtained using var() and estimates of auto & cross covariances is obtained using xcov().

→ It is ensured that UNBIASED estimators are used.

	True value	Estimate	Error %
σ_y^2	11.604	11.5168	0.753 %
$\sigma_{yy}[1]$	-5.333	-5.546	+3.1 %
$\sigma_{yu}[1]$	0	-0.0891	0.089 (absolute)
$\sigma_{yu}[2]$	4	3.938	1.55 %

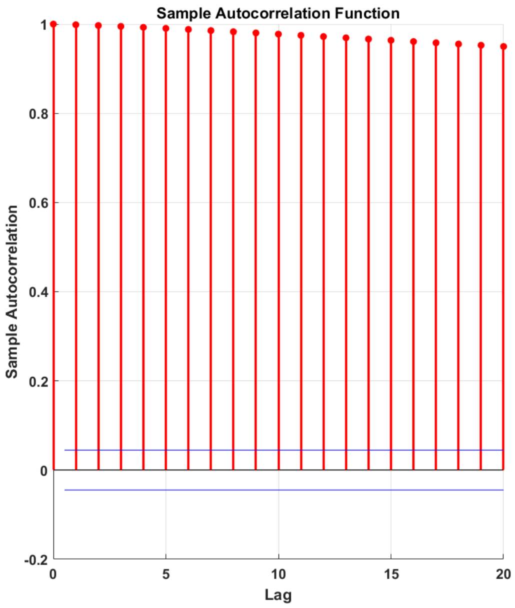
Question 3

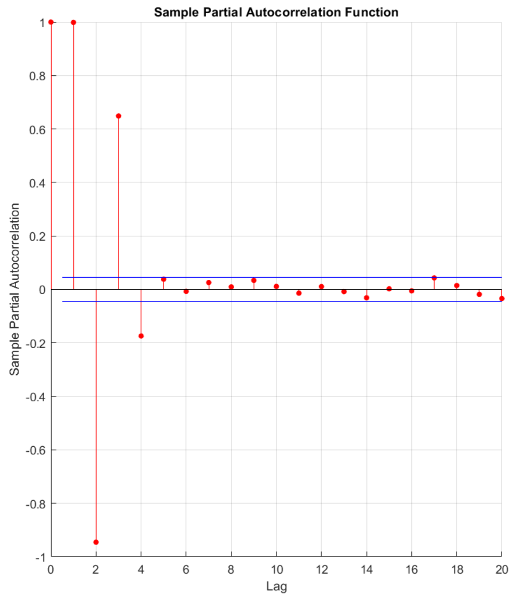
Investigating the presence of random walk effects



Plot of the data indicates the presence of non-stationarity which is not of trend-type.

To determine the presence of integrating effects, let us examine the ACF and PACFs of the given data.





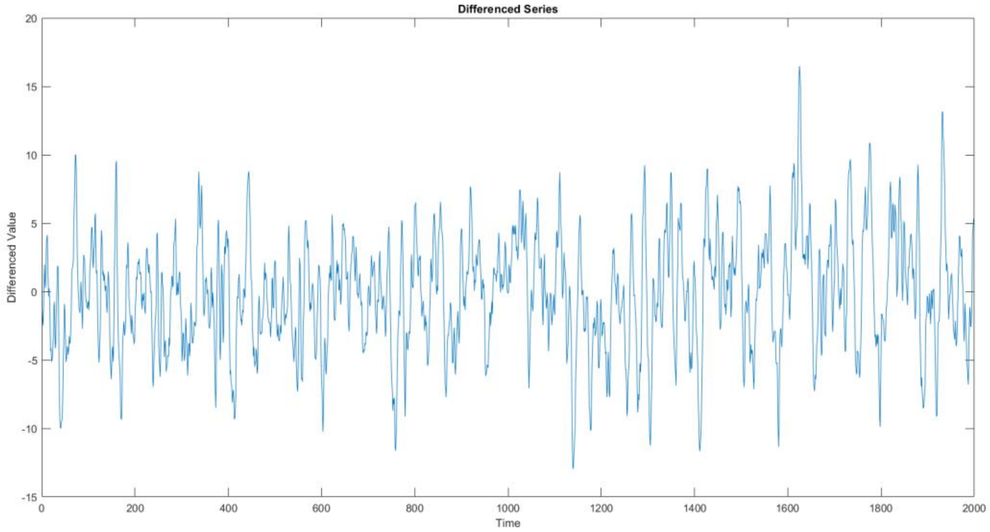
ACF decays **very very slowly**, while PACF[1] is 1. These 2 observations strongly indicate the presence of random walk effect. We shall now perform the **ADF test** to solidify our conclusion.

```
ADF Test
h =
   0

pval =
     0.8455
```

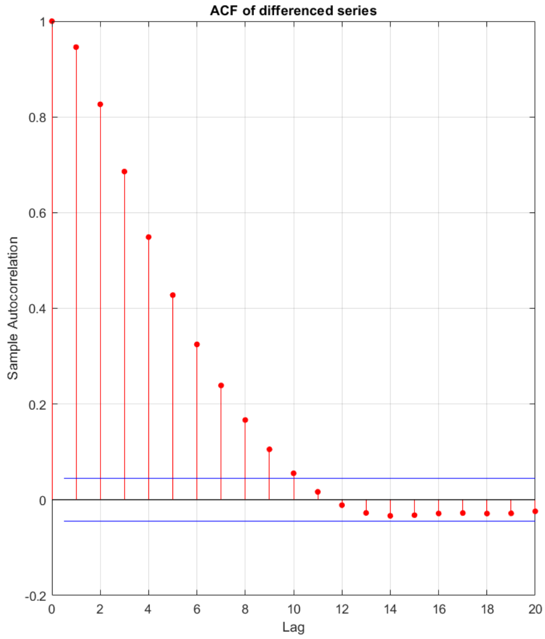
We obtain a high p-value of 0.845>alpha=0.05. This implies the null hypothesis of a unit root in the time series is rejected.

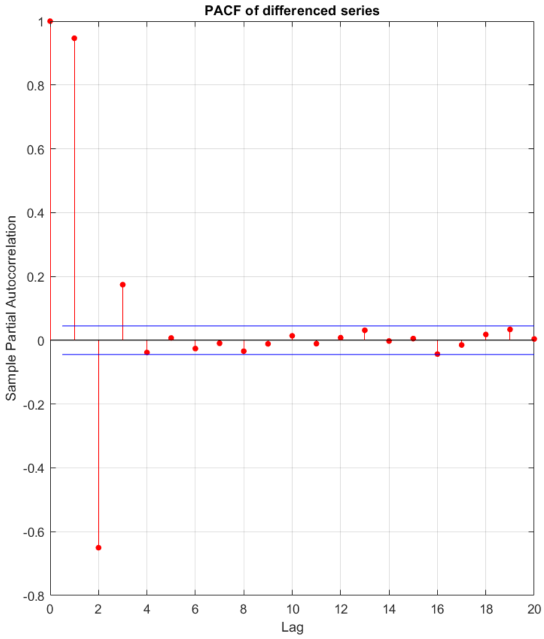
Therefore, the series contains random walk effects.

Differencing the series and finding the order of integration effects



Unlike the original series, the differenced series looks to have some sense of stationarity.





We can see an exponential decay of ACF indicating AR component and no abrupt change to 0 in PACF indicating MA component. This could mean that the integrating effects have been removed.

Conduct the **ADF test on the differenced series:**

```
ADF test on differenced series
h =
    1

pval =
    1.0000e-03
```

pvalue = 0.001<alpha=0.05

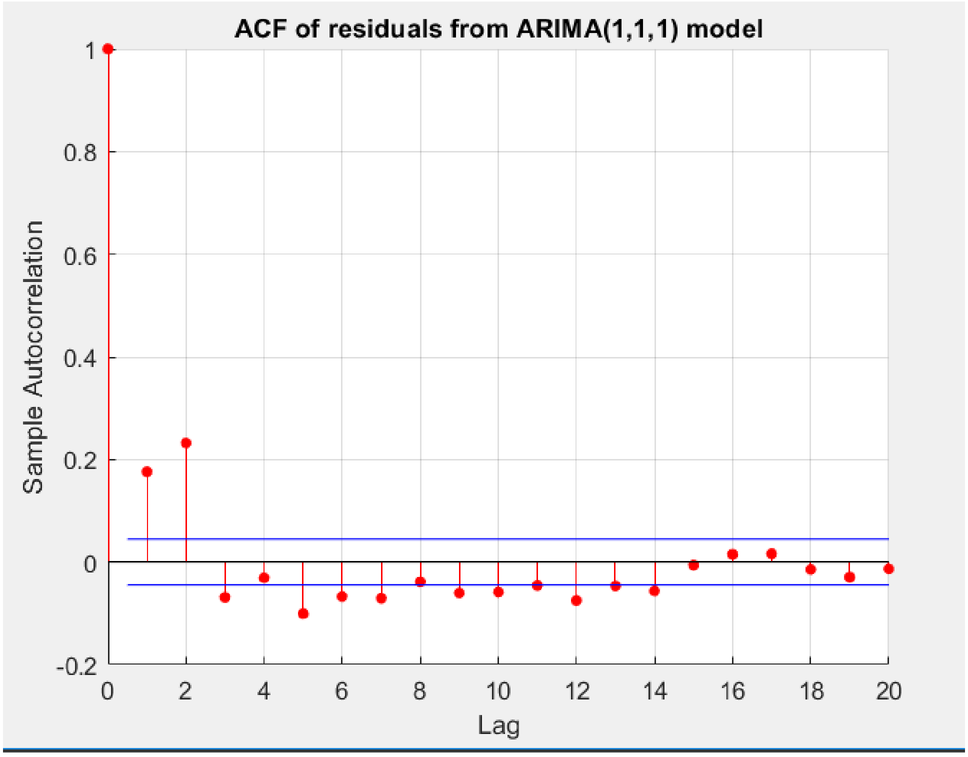
=> Null hypothesis is not rejected.

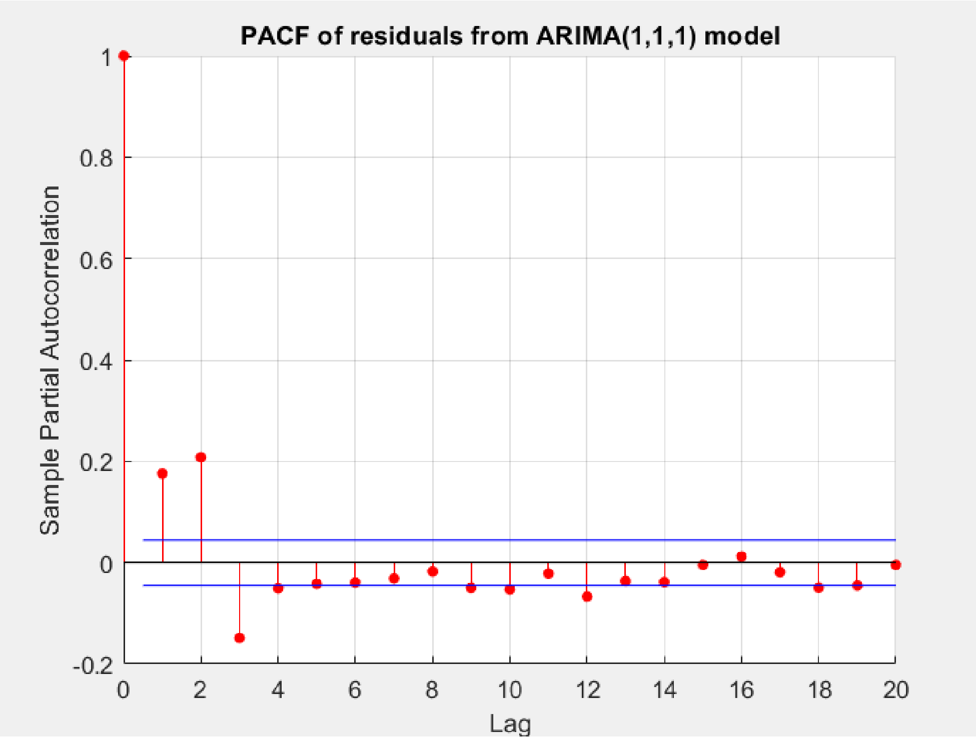
Therefore, we can say that the **differenced series doesn't contain random walk effects**. Since it seems to contain both AR and MA effects, we go for an ARIMA model. (Or we can model the differenced series as an ARMA model.)

ARIMA model-1: AR-1, I-1, MA-1

Since with a single differencing we are able to remove the random walk effects, we can say that, the **order of integrating process is 1**. That fixes the ‘I’ value of the ARIMA model.

Let us now examine the model residuals.





We can see that the ACF and PACF are still far away from zero at small lags. So, the residuals don't appear to be White Noise. We will confirm this using the lbq test.

```
Whiteness Test for Residuals results
   1

     0
```

With pvalue of 0, the lbqtest indicates **the residuals are not white.** This means that the model is underfitting. Errors in the parameter estimates:

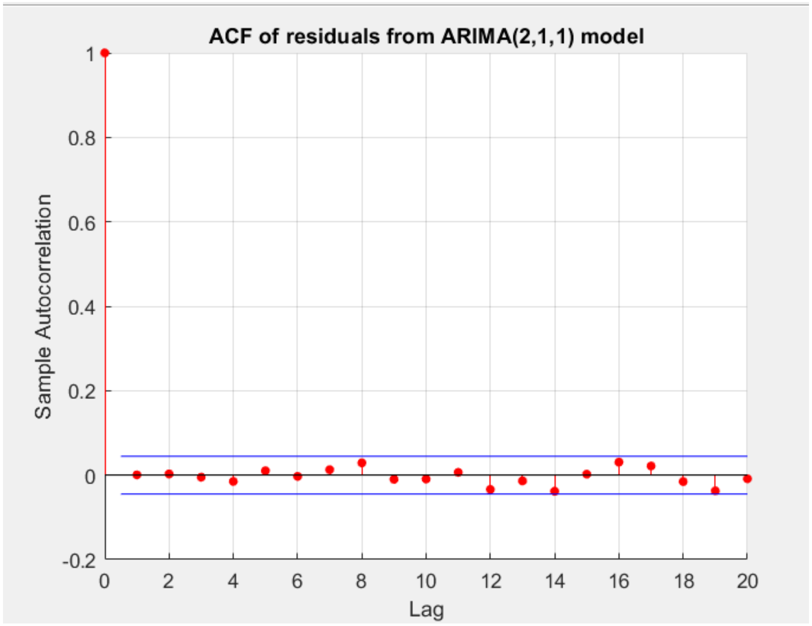
```
ARIMA(1,1,1) Model (Gaussian Distribution)

 Effective Sample Size: 2000
 Number of Estimated Parameters: 3
 LogLikelihood: -2960.14
 AIC: 5926.28
 BIC: 5943.08
```

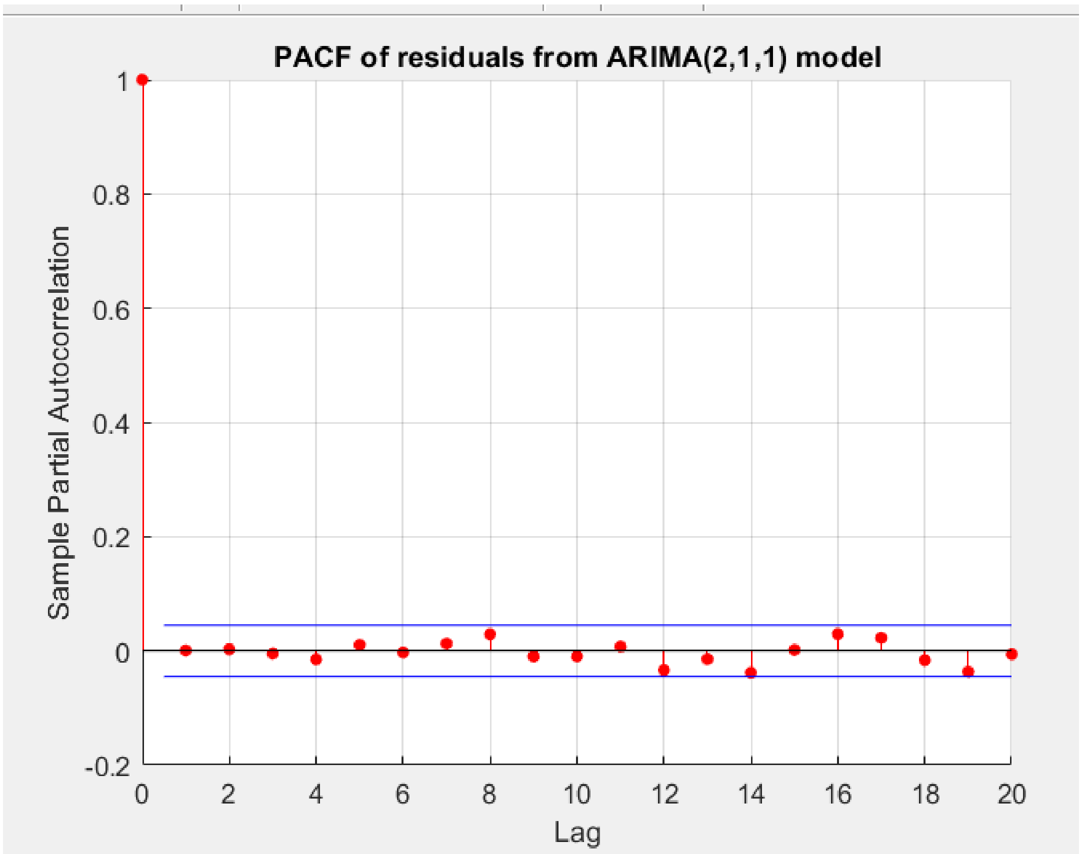
	Value	StandardError	TStatistic	PValue
Constant	0	0	NaN	NaN
AR{1}	0.91364	0.0090083	101.42	0
MA{1}	0.62323	0.018013	34.598	2.6672e-262
Variance	1.1301	0.033596	33.637	4.9076e-248

The ACF seems to decay and the PACF seems to cut off to zero after a set of initial lags. So let's increase the AR component and build a new ARIMA model.

ARIMA model-1: AR-2, I-1, MA-1

Examining the residuals:





PACF and ACF are extremely close to zero at all lags != 0. So, the residuals could be white noise!!

Performing the lbq test:

```
Whiteness Test for Residuals results
   0

    0.7668
```

Model passes the whiteness test! (Implies the model is not under-fitting)

Errors in parameter estimates:

```
ARIMA(2,1,1) Model (Gaussian Distribution)

  Effective Sample Size: 2000
  Number of Estimated Parameters: 4
  LogLikelihood: -2843.6
  AIC: 5695.2
  BIC: 5717.61
```

	Value	StandardError	TStatistic	PValue
Constant	0	0	NaN	NaN
AR{1}	1.4103	0.02846	49.554	0
AR{2}	-0.5069	0.027996	-18.106	2.8568e-73
MA{1}	0.27114	0.031749	8.5402	1.3396e-17
Variance	1.0057	0.030717	32.742	3.9533e-235

We further observe that **all the coefficients are statistically significant**. (Implies the model is not over-fitting).

Also, ARIMA(2,1,1) has a **lower Akaike Information Criterion (AIC) value** compared to ARIMA(1,1,1). This backs our observation that the ARIMA(2,1,1) is the better model.

Conclusion: The built ARIMA (2,1,1) model satisfactorily represents the given data.

ARIMA (1,1,2) model was also tried but its residuals did not pass the whiteness test.

④ a) $L(\mu; \underline{y_N}) = \ln f(\underline{y_N} | \mu)$

$$= \ln \left(\prod_{k=1}^{N} \left(\frac{1}{\sqrt{2\pi}\sigma} \right) \left(\exp\left(-\frac{1}{2} \left(\frac{y[k]-\mu}{\sigma} \right)^2 \right) \right) \right)$$

Since it is a GWN process we have used the property that the joint pdf is simply the product of marginal pdfs.

$$\Rightarrow L = \sum_{k=1}^{n} \frac{-1}{2} \left(\frac{y[k]-\mu}{\sigma_e} \right)^2 + C$$

$C \downarrow$ A constant term, independent of $y[k]$

$\frac{\partial L}{\partial \mu} = 0$ (Likelihood should be maximised)

$$\Rightarrow \sum_{k=1}^{N} \frac{+2}{2} \left(\frac{y[k]-\mu}{\sigma_e^2} \right) = 0$$

$$\Rightarrow N\mu = \sum_{k=1}^{N} y[k]$$

$$\Rightarrow \hat{\mu} = \frac{\sum_{k=1}^{N} y[k]}{N}$$

Thus, the ML estimate of mean is simply the sample mean.

$$FI = E\left(\left(\frac{\partial L}{\partial \theta} \right)^2 \right) = -E\left(\frac{\partial^2 L}{\partial \theta^2} \right)$$

$$\frac{\partial^2 L}{\partial \mu^2} = \frac{\partial}{\partial \mu} \left(+\sum \left(\frac{y[k]-\mu}{\sigma_e^2} \right) \right)$$

$$\Rightarrow \frac{\partial^2 L}{\partial \mu^2} = -\sum_{k=1}^{N} \frac{1}{\sigma_e^2} = \frac{-N}{\sigma_e^2}.$$

$$\therefore \text{F.I.}, \ I(\mu) = -E\left(\frac{\partial^2 L}{\partial \mu^2}\right)$$

$$\Rightarrow \boxed{I(\mu) = \frac{N}{\sigma_e^2}}$$

b) $Y = aX + b + \varepsilon. \Rightarrow \varepsilon = Y - aX - b.$

$\varepsilon \sim N(0, \sigma_e^2) \qquad = y[k] - ax[k] - b$

$$\Rightarrow L = \log\left(\prod_{k=1}^{N}\left(\frac{1}{\sqrt{2\pi}\sigma_e}\right)\left(\exp \frac{-(y-ax-b)^2}{2\sigma_e^2}\right)\right)$$

$$= C + \frac{-1}{2}\sum_{k=1}^{N} \frac{(y[k]-ax[k]-b)^2}{\sigma_e^2}$$

$C \downarrow$ term without a & b

maximising $L \rightarrow$ objective.

$$\frac{\partial L}{\partial a} = \frac{2}{2}\sum \frac{x[k](y[k]-ax[k]-b)}{\sigma^2}$$

$$= \frac{1}{\sigma_e^2}\left(\sum x[k]y[k] - a\sum (x[k])^2 - b\sum x[k]\right)$$

$$\frac{\partial^2 L}{\partial a^2} = \frac{-1}{\sigma_e^2}\sum_{k=1}^{N}(x(k))^2$$

$$\frac{\partial^2 L}{\partial b\, \partial a} = -\frac{\sum x[k]}{\sigma_e^2}$$

$$\frac{\partial L}{\partial b} = \frac{-2}{2} \frac{\sum (y[k] - a\, x[k] - b)}{\sigma_e^2}$$

$$= \frac{1}{\sigma_e^2} \sum y[k] - \frac{\sum x[k]\, a}{\sigma_e^2} \frac{-Nb}{\sigma_e^2}$$

$$\frac{\partial^2 L}{\partial b^2} = \frac{-1}{\sigma_e^2} \times N = \frac{-N}{\sigma_e^2}$$

$$\frac{\partial L}{\partial a\, \partial b} = -\frac{\sum x[k]}{\sigma_e^2}$$

Fisher's information matrix: $\begin{bmatrix} -E\left(\frac{\partial^2 L}{\partial a^2}\right) & -E\left(\frac{\partial L}{\partial b\, \partial a}\right) \\ -E\left(\frac{\partial L}{\partial a\, \partial b}\right) & -E\left(\frac{\partial^2 L}{\partial b^2}\right) \end{bmatrix}$

Since $x[k]$ is fixed, we get

$$\text{F.I. matrix} = \begin{bmatrix} \frac{1}{\sigma_e^2}\sum (x(k))^2 & \frac{\sum x(k)}{\sigma_e^2} \\ \frac{\sum x[k]}{\sigma_e^2} & \frac{N}{\sigma^2} \end{bmatrix}$$

$$\boxed{\begin{aligned} I(a) &= \frac{1}{\sigma_e^2} \sum (x(k))^2 \\ I(b) &= \frac{N}{\sigma_e^2} \end{aligned}}$$

b) ~~MAT~~ ML estimate :

Solve the eqns $\frac{\partial L}{\partial a} = 0 ; \frac{\partial L}{\partial b} = 0$ (linear equations in a & b)

to get $\hat{a}$ & $\hat{b}$

$$\Rightarrow \left(\sum (x(k))^2\right) a + \left(\sum (x(k))\right) b = \frac{\sum x(k)\, y(k)}{}$$

and $\left(\sum x(k)\right) a + N b = \sum y(k)$

Solution :
$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum (x(k))^2 & \sum x(k) \\ \sum x(k) & N \end{bmatrix}^{-1} \begin{bmatrix} \sum x(k) y(k) \\ \sum y(k) \end{bmatrix}$$

unique solution exists as long as

$$N \sum (x(k))^2 - \left(\sum x(k)\right)^2 \neq 0$$

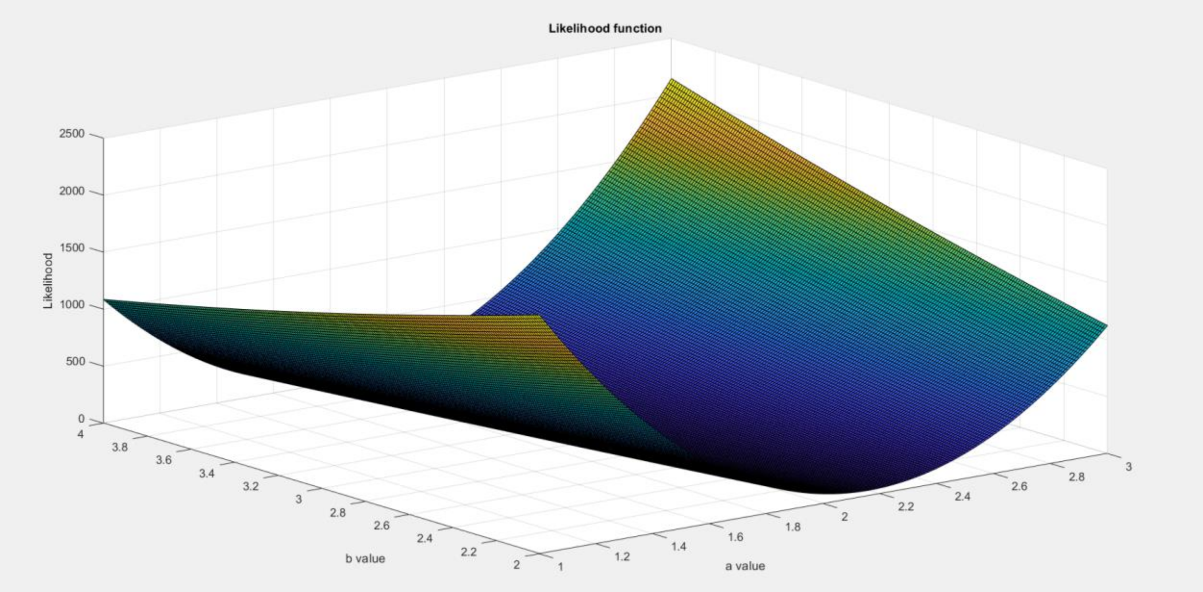
Numerical estimate is obtained by searching the area :
$a \in [1, 3]$ (in steps of 0.01) $b \in (2, 4]$ (in steps of 0.01)

$\hat{a}$, numerical estimate = 2.03 ; $\hat{a}$ analytical estimate = 2.027
$\hat{b}$, numerical estimate = 2.78 ; $\hat{b}$ ~~best~~ analytical = 2.7893.

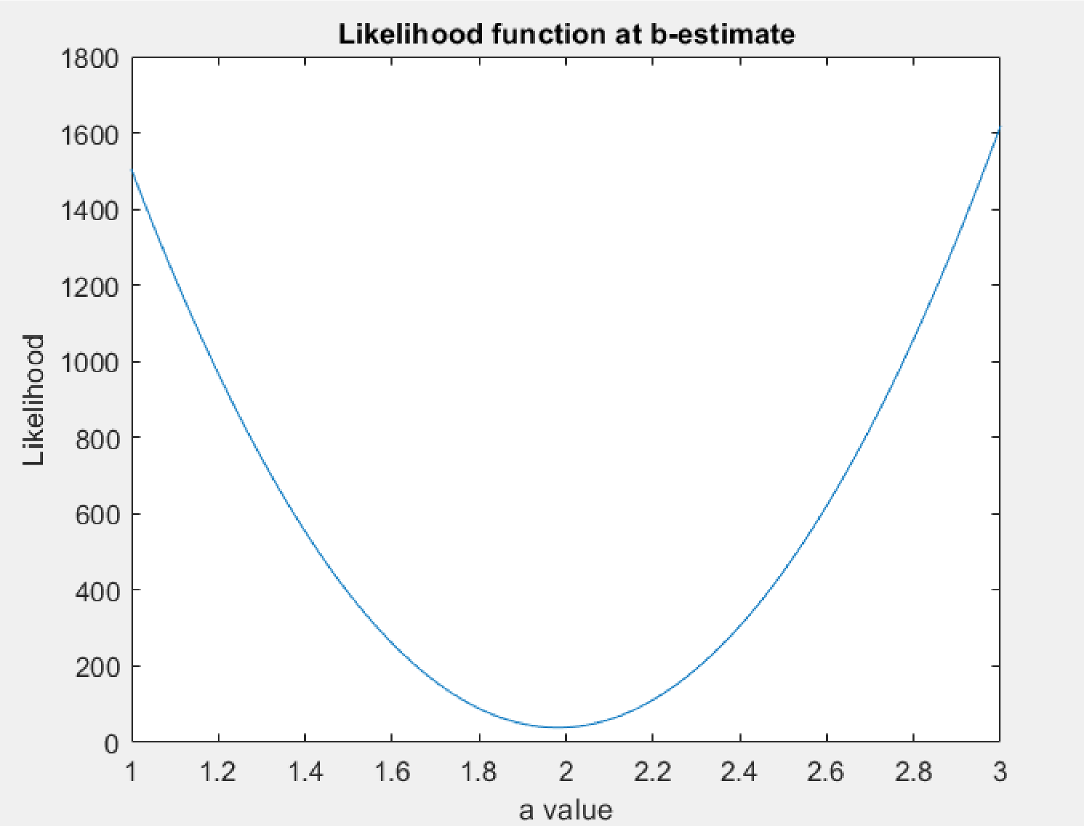
X is generated randomly.

Question 4

From the graphs we can see that, the minimum of the likelihood function is close to the true values of a and b (2 and 3) and also to the analytical estimate.



Likelihood evaluated at different a, for a fixed b(=MLE).



Likelihood evaluated at different b, for a fixed a (=MLE).