BY : S. VISHAL

CH18B020

CH5115 ASSIGNMENT - 3

① $f(y) = \begin{cases} e^{-(y-\theta)} & x > \theta, \quad -\infty < \theta < \infty \\ 0 & \text{otherwise} \end{cases}$

$$P_r(y[k] \geq y) = \int_y^\infty e^{-(y-\theta)}$$

Let $e^\theta = \beta$

$$\Rightarrow Pr(y[k] \geq y) = \begin{cases} \beta(e^{-y}) & \text{where } y > \theta \\ 1 & y \leq \theta \end{cases}$$

For $\min(y_N) \geq y$,

$$y[k] \geq y \ \forall k.$$

$$\Rightarrow P_r(\min(y_N) \geq y) = (\beta e^{-y})^N \qquad y > \theta$$

$$\Rightarrow Pr(\min(y_N) \leq y) = 1 - \beta^N e^{-Ny} \quad \text{——} ①$$

$$\Rightarrow f(\min(y_N) = y) = \frac{d}{dy}(1 - \beta^N e^{-Ny})$$

$$= \begin{cases} N\beta^N e^{-Ny} & y > \theta \\ 0 & y \leq \theta \end{cases} \quad \text{——} ②$$

① $\Rightarrow Pr(2\min(y_N) \le 2y) = 1 - \beta^N e^{-Ny}$

$\Rightarrow Pr(2\min(y_N) \le t) = 1 - \beta^N e^{-Nt/2}$

where $t = 2y$

$\rightarrow Pr(T \le t) = 1 - \beta^N e^{-Nt/2}$

$\Rightarrow f(T_N = t) = \dfrac{N\beta^N e^{-Nt/2}}{2}$

$$\boxed{f(T_N = t) = \begin{cases} \dfrac{N \exp(N(\theta - t/2))}{2} & t > 2\theta \\ 0 & \text{otherwise } (t \le 2\theta) \end{cases}}$$

$t \ge 2\theta$

b) $E(T_N) = \int_{\forall t} t f(t)\, dt$

$= \dfrac{N}{2} \int_{2\theta}^{\infty} t\, e^{N\theta - Nt/2}\, dt$

$= \dfrac{N\beta^N}{2} \int_{2\theta}^{\infty} t\, e^{-Nt/2}\, dt \qquad (\beta = e^{N\theta}$

$= \dfrac{N\beta^N}{2} \left[\dfrac{2te^{-Nt/2}}{-1 \times N} \Bigg|_{2\theta}^{\infty} + \int_{2\theta}^{\infty} \dfrac{2e^{-Nt/2}}{N}\, dt \right]$

$= \dfrac{N\beta^N}{2} \left[\dfrac{\cancel{2} \times 2\theta\, e^{-N\theta}}{N} + \dfrac{4e^{-N\theta}}{N^2} \right]$

$= 2\theta + \dfrac{2}{N}$

$E(T_N) \neq \theta \Rightarrow T_N$ is a biased estimator.

Correcting it: $\dfrac{T_N}{2} - \dfrac{1}{N} = T_N'$

$$\Rightarrow E(T_N') = E\left(\frac{T_N}{2} - \frac{1}{N}\right) = \frac{E(T_N)}{2} - \frac{1}{N}$$

$$= \theta + \frac{1}{N} - \frac{1}{N}$$

$$\Rightarrow E(T_N') = \theta \quad \checkmark \text{ unbiased}$$

$\therefore$ Corrected statistic:

$$\boxed{T_N' = \frac{T_N}{2} - \frac{1}{N}}$$

c) $T_N' \xrightarrow{p} \theta$ [To prove]

To prove: $\lim\limits_{N \to \infty} \Pr(|X_n - X| \geq \varepsilon) = 0$ where $\varepsilon > 0$

$$\Rightarrow \lim_{N \to \infty} \Pr(|T_N' - \theta| \geq \varepsilon) = 0$$

$$\Rightarrow \lim_{N \to \infty} \left[\Pr(T_N' - \theta \geq \varepsilon) + \Pr(T_N' - \theta \leq -\varepsilon)\right] = 0$$

(Because for the condition to satisfy either $T_N' - \theta \geq \varepsilon$ or $T_N' - \theta \leq -\varepsilon$)

Now we will compute the 2 probabilities individually and add them up

1st term: $\Pr\left(\frac{T_N}{2} - \frac{1}{N} - \theta \geq \varepsilon\right)$

$\Rightarrow \Pr\left(\frac{T_N}{2} \geq \varepsilon + \frac{1}{N} + \theta\right) = \Pr\left(T_N \geq 2\left(\varepsilon + \frac{1}{N} + \theta\right)\right)$

We now find out the CDF from PDF of T,

$$\Pr(T_N \leq t) = \int_{2\theta}^{t} \frac{N}{2} \exp\left(N\left(\theta - \frac{t}{2}\right)\right) dt$$

$$= 1 - \beta^N e^{-\frac{Nt}{2}} \quad \text{——} ①$$

$\Pr(T_N \geq t) = 1 - \Pr(T_N \leq t)$ $\quad$ [$\beta^N = e^{N\theta}$ as defined in part (a)]

$$= \beta^N e^{-Nt/2} \quad \text{——} ②$$

$$\therefore \Pr\left(T_N \geq 2\left(\varepsilon + \frac{1}{N} + \theta\right)\right) = \beta^N e^{-N\left(\varepsilon + \frac{1}{N} + \theta\right)}$$

$$= e^{-N\varepsilon - 1} \quad \text{——} ③$$

2nd term: $\Pr\left(\frac{T_N}{2} - \frac{1}{N} - \theta \leq -\varepsilon\right)$

using eqn ①,

$$\Rightarrow \Pr\left(T_N \leq 2\left(-\varepsilon + \frac{1}{N} + \theta\right)\right) = 1 - \beta^N e^{-N\left(-\varepsilon + \frac{1}{N} + \theta\right)}$$

$$= 1 - e^{N\varepsilon - 1}$$

However this eqn is valid ~~iff~~ only if,

$$2\left(-\varepsilon + \frac{1}{N} + \theta\right) \geq 2\theta \Rightarrow \frac{1}{N} - \varepsilon \geq 0$$

In the event $N \to \infty$, $-\varepsilon \geq 0$ But $\varepsilon > 0$

$\Rightarrow -\varepsilon < 0$

$\Rightarrow$ We can't use $1 - e^{+N\varepsilon - 1}$ expression in the given limit

$\therefore$ since $2(\theta + \frac{1}{N} - \varepsilon) < 2\theta$,

$F(T_N \leq t)$ at such $t = 0$

$(\because t_{min} = 2\theta)$

$$\Rightarrow \lim_{N \to \infty} \Pr\left(T_N \leq 2\left(-\varepsilon + \frac{1}{N} + \theta\right)\right) = 0 \quad \text{④}$$

Using ④ & ③ we have,

$$\lim_{N \to \infty} \left[\Pr(T_N' - \theta \geq \varepsilon) + \Pr(T_N' - \theta \leq -\varepsilon)\right]$$

$$= \lim_{N \to \infty} \left[e^{-N\varepsilon - 1} + 0\right]$$

$$= 0$$

$$\Rightarrow \lim_{N \to \infty} \left[\Pr(T_N' - \theta \geq \varepsilon) + \Pr(T_N' - \theta \leq -\varepsilon)\right] = 0$$

$$\therefore T_N' \xrightarrow{P} \theta$$

TJ