(2) a) $$V(f_n) = \sum_{k=0}^{N-1} v[k] \exp(-j 2\pi f_n k)$$

$$= \sum_{k=0}^{N-1} v[k] \cos(2\pi f_n k) - j\, v[k] \sin(2\pi f_n k)$$

hence, $$a_n = \sum_{k=0}^{N-1} v[k] \cos(2\pi f_n k)$$

and $$b_n = -\sum_{k=0}^{N-1} v[k] \sin(2\pi f_n k)$$

Notice that, a_n and b_n are sum of i.i.d Gaussian variables ($\because v$ is GWN) which are scaled by a constant factor ($\cos(2\pi f_n k)$ for kth term of a_n and $\sin(2\pi f_n k)$ for kth term of b_n)

Such a product remains Gaussian with a change in σ^2.

So a_n and b_n are just sum of independent Gaussian Random Variables

$\Rightarrow$ a_n and b_n themselves should be Gaussian distributed.

To prove that sum of Gaussian random variables is Gaussian, I will use the moment generating fn

i) For Gaussian X, $MGF = E(e^{sX})$

let $\alpha \sim N(0,1)$ $\Rightarrow MGF(X) = E\left(e^{s(\sigma Y + \mu)}\right)$

$$= \int \frac{1}{\sqrt{2\pi}} e^{s\sigma Y} \cdot e^{\mu s} e^{-\frac{1}{2}y^2} dy$$
$\forall y$

$$= \frac{e^{\mu s}}{} \int_{-\infty}^{\infty} \frac{e^{s\sigma y - \frac{1}{2}y^2}}{\sqrt{2\pi}} dy$$

$$= e^{\mu s} e^{\frac{\sigma^2 s^2}{2}} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(y^2 - 2\sigma s + \sigma^2 s^2)}}{\sqrt{2\pi}} dy$$

$$\Rightarrow M_x(s) = e^{\mu s + \frac{\sigma^2 s^2}{2}}$$

ii) Consider sum of i.i.d RVs $Y = \sum_{i=1}^{N} X_i$

$$M_Y(s) = E(e^{sY}) = E(e^{s\sum X_i}) = E\left(\prod_{i=1}^{N} e^{sX_i}\right)$$

$$\Rightarrow \because \text{ i.i.d}, \quad M_Y(s) = \prod_{i=1}^{N} E(e^{sX_i})$$

$$\Rightarrow M_Y(s) = \prod_{i=1}^{N} E(e^{sX_i})$$

From (i), $M_Y(s) = \prod e^{\mu_i s + \frac{\sigma_i^2 s^2}{2}}$

$$= \exp\left(\left(\sum \mu_i\right) s + \frac{\sum \sigma_i^2}{2} s^2\right)$$

So Y is Gaussian distributed with

$$\mu = \sum_{i=1}^{N} \mu_i \quad \& \quad \sigma^2 = \frac{\sum \sigma_i^2}{2}$$

$\therefore$ Sum of i.i.d. Gaussian RVs is a Gaussian RV.

b) $\text{corr}(a_n, b_n) = \dfrac{\text{cov}(a_n, b_n)}{\sqrt{\sigma_{a_n}^2 \sigma_{b_n}^2}}$

$\Rightarrow \text{corr}(a_n, b_n) = \dfrac{E(a_n b_n) - E(a_n)E(b_n)}{\sqrt{\sigma_{a_n}^2 \sigma_{b_n}^2}}$ —— ①

Let's evaluate these moments one by one,

$a_n = \sum v(k) \cos 2\pi f_n k$ —— ② $\Rightarrow E(a_n) = \sum E(v(k)\cos 2\pi f_n k)$

$= 0$ —— ③

$\cdot \ \text{var}(a_n) = \text{var}\left(\sum v(k) \cos 2\pi f_n k\right)$

$= \sum (\text{var}(v(k))(\cos(2\pi f_n k))^2$ (because v_k is i.i.d)

$\Rightarrow \text{var}(a_n) = \sum_{k=0}^{N-1} \cos^2(2\pi f_n k)$ —— ④ $(\sigma_v^2 = 1)$

Similarly, $b_n = -\sum v(k) \sin 2\pi f_n k$ —— ⑤

$\Rightarrow E(b_n) = 0$ —— ⑥ & $\text{var}(b_n) = \sum \sin^2 2\pi f_n k$ —— ⑦

$E(a_n b_n) = E\left(\left(\sum v(k) \cos(2\pi f_n k)\right)\left(\sum v(k) \sin(2\pi f_n k)\right)\right)$ (from ② & ⑤)

Let $2\pi f_n = \omega_n$

$\Rightarrow E(a_n b_n) = E\left(\left(\sum v(k) \cos \omega_n k\right)\left(\sum v(k) \sin(\omega_n k)\right)\right)$

Multiplying the summations,

$= E\left(\left(\sum v^2(k) \cos \omega_n k \sin \omega_n k\right) + \sum_{k=0}^{N-1} \sum_{\substack{l=0 \\ l \neq k}}^{N-1} v(k) v(l) \cos \omega_n k \sin \omega_n l\right)$

P.T.O.

$$= E\left(\sum v^2(k) \cos w_n k \sin w_n k + \sum_{\substack{k=0}}^{N-1} \sum_{\substack{l=0 \\ l \neq k}}^{N-1} v(k) v(l) \cos w_n k \sin w_n l \right)$$

$\because$ $v[k]$ is white noise (GWN) $\sim N(0,1)$

$E(v^2(k)) = 1$ & $E(v(k)\,v(l)) = \sigma_{vv}[k-l]$

$= 0 \;\forall\; l \neq k$

$$\Rightarrow E(a_n b_n) = \sum E(v^2(k)) \cos w_n k \sin w_n k + \sum_{k=0}^{N-1} \sum_{\substack{l=0 \\ l \neq k}}^{N-1} E(v(k)\,v(l)) \cos w_n k \sin w_n l$$

$$= \sum \cos w_n k \sin w_n k \quad — ⑧$$

From eqns ①, ③, ④, ⑥, ⑦, ⑧ we have

$$\boxed{\operatorname{corr}(a_n, b_n) = \frac{\sum_{k=0}^{N-1} \cos(2\pi f_n k) \sin(2\pi f_n k)}{\sqrt{\left(\sum_{k=0}^{N-1} \cos^2(2\pi f_n k)\right)\left(\sum_{k=0}^{N-1} \sin^2(2\pi f_n k)\right)}}}$$

c) $$\xi_n = \frac{2 P_{vv}(f_n)}{\gamma(f_n)} = \frac{2(a_n^2 + b_n^2)}{N \gamma(f_n)}$$

$$\gamma(f_n) = \sum_{l=-\infty}^{\infty} \sigma_{vv}[l] \exp(-j 2\pi f l)$$

$$\because \sigma_{v}[l] = \begin{cases} 1 & l = 0 \\ 0 & l \neq 0 \end{cases}$$

$$\Rightarrow \gamma(f_n) = 1$$

P.T.O.

$\therefore\ G_n = \dfrac{2(a_n^2 + b_n^2)}{N}$

Substitute a_n & b_n (eqns ② & ⑤)

$$= \frac{2}{N}\Big[\sum v^2(k)\cos^2(2\pi f_n k) + \sum\sum v(k)v(l)\cos(2\pi f_n k)\cos 2\pi f_n l$$
$$+ \sum v^2(k)\sin^2(2\pi f_n k) + \sum\sum v(k)v(l)\sin(2\pi f_n k)\sin(2\pi f_n l)\Big]$$

group the $\sin^2$ & $\cos^2$ terms ($\cos^2\theta + \sin^2\theta = 1$

$$\Rightarrow G = \frac{2}{N}\Big[\sum v^2(k) + \sum\sum v(k)v(l)\big[\cos(\omega_n k)\cos\omega_n l + \sin\omega_n k \sin\omega_n l\big]\Big]$$

$$\Rightarrow G = \frac{2}{N}\Big[\sum_{k=0}^{N-1} v^2(k) + \sum_{k=0}^{N-1}\sum_{\substack{l=0 \\ l\neq k}}^{N-1} v(k)v(l)\cos(\omega_n(k-l))\Big]$$

Sum of square of N independent Gaussian RV

form the X^2 distribution.

Also product of 2 independent Gaussian RVs

XY can be represented as $\frac{1}{4}\left((X+Y)^2 - (X-Y)^2\right)$ sum

of 2 X^2 RVs

$\Rightarrow G_n$ is a weighted sum of X^2 distributed R.V.s

$\Rightarrow G_n$ is a X^2 distributed random variable

In limit $N \to \infty$ when cross terms go to zero

$$G_n \xrightarrow{d} X^2(2)$$

i) Mean of e_n

$$E(e_n) = \frac{2}{N}\left[\sum E(v^2) + 2\sum_{k=0}^{N-1}\sum_{\substack{l=0\\ l\neq k}}^{N-1} E(v(k)v(l))(\cos(\omega_n(l-k))\right]$$

$$E(v^2) = \sigma^2 = 1$$

$$E(v(k)v(l)) = \sigma_{vv}[l-k] = \begin{cases} \sigma^2 & l=k \\ 0 & l\neq k \end{cases}$$

$$= \frac{2}{N} \times N \Rightarrow \boxed{E(e_n) = 2}$$

ii) Variance of e_n.

$$var(e_n) = \frac{4}{N^2} var\left(\sum v_k^2 + \sum_{k=0}^{N-1}\sum_{\substack{l=0\\ l\neq k}}^{N-1} v(k)v(l)\cos(\omega_n(k-l))\right)$$

$$var(X+Y) = var(X) + 2cov(X,Y) + var(Y)$$

here first term is X and second term is Y.

- $\underline{var(X)} = var(\sum v_k^2)$; $v(k)$ is i.i.d

$$\Rightarrow var(X) = \sum \sigma^4 = N\sigma_v^4 = N\times 1$$

$$\Rightarrow \underline{var(X) = N}$$

$$(\because var(AB) = var(A)var(B) + var(A)(E(B))^2 + var(B)(E(A))^2$$

$$\Rightarrow var(v^2) = (\sigma^2)^2 = \sigma^4)$$

- $\underline{cov(X,Y)} = E(XY) - E(X)E(Y)$;

$E(Y) = 0$ because $E(v(k)v(l)) = 0$ for $k\neq l$.

$$= E\{\sum(v(k))^2 \sum\sum v(k)v(l)\cos\omega_n(k-l))$$

Notice that any term will be of the form $(v(k))^2 v(l)$ or $v(j)v(k)v(l)$

$E(v(i)v(k)v(l)) = E(v(i))E(v(k))E(v(l)) = 0$

$E((v(k))^2 v(l)) = E((v(k))^2)\cancel{E(v(l))}^{0} = 0$

(This is because all vk are 0 mean &
v(k), v(j) are uncorrelated for all $k \neq j$)

$\Rightarrow \underline{\text{cov}(X,Y) = 0}$

$\text{var}(Y) = \text{var}\left(\sum\sum v(k)v(l)\cos(\omega n(k-l))\right)$

We know that $E(Y) = 0$

$\Rightarrow \underline{\text{var}(Y)} = E\left(\left(\sum\sum v(k)v(l)\cos(\omega n(k-l))\right)^2\right)$

$= E\left(4\sum_{k=0}^{N-1}\sum_{\substack{l=0\\k\neq l}}^{k-1}(v(k)v(l))^2\cos^2(\omega n(k-l)) + 8\sum\sum_{\substack{(i,j)\neq(k,l)\\(i\neq j),(k\neq l)}}\sum\sum v(i)v(j)v(k)v(l)\cos(\omega n(i-j))\cos(\omega n(k-l))\right)$

The second term will never have all variables having even powers, i.e. $(v(k)v(l))^2$

So because of the i.i.d & $E(v) = 0$, we can say that $\underline{E(\cdot)\text{ of second term goes to zero}}$.

$\Rightarrow \text{var}(Y) = 4E\left(\sum_{k=0}^{N-1}\sum_{\substack{l=0\\k\neq l}}^{k-1}(v(k)v(l))^2\cos^2(\omega n(k-l))\right)$

Once again we invoke, $\text{var}(AB) = \text{var}(A)\text{var}(B) + \text{var}(A)(E(B))^2 + \text{var}(B)(E(A))^2$

$\Rightarrow \text{var}(v(k)v(l)) = \text{var}(v(k))\text{var}(v(l))$

$= \sigma_v^4 = 1$

Since $E(v(k)v(l)) = 0$, $(\forall k \neq l)$,

$$E((v(k)v(l))^2) = \mathrm{var}(v(k)v(l))$$

$$= 1$$

$$\therefore \mathrm{var}(Y) = 4 \sum_{k=0}^{N-1} \sum_{l=0}^{k-1} \cancel{\mathrm{var}}\, E(v(k)v(l)) \cos^2(w_n(k-l))$$

$$= 4 \sum_{k=0}^{N-1} \sum_{\substack{l=0 \\ k \neq l}}^{k-1} \cos^2(w_n(k-l))$$

$$= \overset{2}{\cancel{4}} \sum_{k=0}^{N-1} \sum_{\substack{l=0 \\ k \neq l}}^{N-1} \frac{(1 + \cos(2w_n(k-l))}{\cancel{2}}$$

$$\sum \sum \cos(2w_n(k-l)) = \sum \sum \cos(4\pi f_n(k-l))$$

$$= \sum_{k \neq l} \sum^{N-1} \cos\left(4\pi \frac{n}{N}(k-l)\right)$$

Using the fact that $\sum_{k=0}^{N-1} \cos\left(2\pi \frac{n}{N} k\right) = 0$,

i.e. that $\cos 0 = 1$ will be missing from ~~the~~ sides the sum $(\because k \neq l.)$

So we get

$$\sum_{k=0}^{N-1} \sum_{\substack{k \neq l \\ l=0}}^{k-1} \cos\left(4\pi \frac{n}{N}(k-l)\right) = \frac{-N}{2}$$

(I also verified this in MATLAB)

$$\Rightarrow \mathrm{var}(Y) = 2 \times \left[N \times \left(\frac{N-1}{2}\right) - \frac{N}{2}\right] = N(N-2)$$

$\therefore \operatorname{var}(e_{1n}) = \frac{4}{N} + \frac{4}{N^2} \times N(N-2)$

$= 4\left(\frac{1}{N} + \frac{N-2}{N}\right)$

$$\boxed{\operatorname{var}(e_{1n}) = 4\frac{(N-1)}{N}}$$

As expected $\lim_{N\to\infty} \operatorname{var}(e_{1n}) = 2 \times 2 = \lim_{N\to\infty} \left(\frac{1-\frac{1}{N}}{1}\right)4$

$= \boxed{4} \to \operatorname{var}(X^2(2))$

d) Mean squared convergence to be checked

$\Rightarrow \lim_{N\to\infty} E\left(\left(P(f_n) - \gamma_{vv}(f_n)\right)^2\right)$

We already obtained that $\gamma_{vv}(f_n) = 1$,

also $E(q_n) = 2 \Rightarrow E\left(\frac{2P(f_n)}{\gamma(f_n)}\right) = 2$

$\Rightarrow E(P(f_n)) = \gamma(f_n) = 1$

So the RV $P(f_n) - \gamma_{vv}(f_n)$ has zero mean

$\therefore \lim_{N\to\infty} E\left(\left(P(f_n) - \gamma(f_n)\right)^2\right)$

$= \operatorname{var}((P(f_n) - 1)) + \cancel{(E(P(f_n) - \gamma(f_n)))^2}^{\,0}$

$= \lim_{N\to\infty} \operatorname{var}(P(f_n) - 1) = \lim_{N\to\infty} \operatorname{var}(P(f_n))$

PTO

from part (c)

$$var(e_n) = \frac{4(N-1)}{N}$$

$$\Rightarrow var\left(\frac{2P(f_n)}{\gamma(f_n)}\right) = \frac{4(N-1)}{N}$$

$$\Rightarrow var(P(f_n)) = \frac{(N-1)}{N}$$

$$\Rightarrow \lim_{N \to \infty} var(P(f_n)) = \frac{\cancel{N}-1}{\cancel{N}} \; 1$$

$$\therefore \lim_{N \to \infty} E\left(\left(P(f_n) - \gamma_{vv}(f_n)\right)^2\right) = 1 \neq 0$$

$\Rightarrow$ $P(f_n)$ does not exhibit mean squared convergence to $\gamma(f_n)$

However it is an unbiased estimator.