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CH5115 ASSIGNMENT-4

(2) a) $$f(y;\theta) = \begin{cases} \frac{2y}{\theta^2} & 0 \le y \le \theta \\ 0 & \text{otherwise} \end{cases}$$

$$l = \prod_{k=1}^{N} \frac{2y[k]}{\theta^2} \quad \text{if } 0 \le y[k] \le \theta \ \forall k$$

$$= \frac{2^N}{\theta^{2N}} \prod_{k=1}^{N} y[k]$$

$$\Rightarrow L = \log\left(\frac{2^N}{\theta^{2N}} \prod_{k=1}^{N} y[k]\right)$$

$$= \Sigma \log y[k] + N \log 2 - 2N \log \theta$$

$$\Rightarrow \frac{\partial L}{\partial \theta} = \frac{-2N}{\theta}$$

We notice that there is no maxima/minima in the domain of theta. So we just have to check for the extreme values

Also $\frac{\partial L}{\partial \theta} < 0 \ \forall \theta$ $(\because \theta \in [0, \infty))$

$\Rightarrow$ Likelihood is a decreasing function of θ

$\Rightarrow$ We need to choose the minimum possible value of θ. —— (1) ~~(|θ| if θ can be -ve)~~

Also note $l = 0$ if $y[k]$ doesn't belong to $[0, \theta]$

For all $y[k]$ to be included in this range,

$\theta \ge \max(y[k])$ —— (2)

From ① & ②, $\boxed{\hat{\Theta}_{MLE} = \max(\underline{y}_N)}$ — ③

Checking for bias.

We need the PDF of $\max(\underline{y}_N)$ to estimate the mean.

$$F(\cancel{Y} \leq y) = \int_0^y \frac{2y}{\theta^2} = \frac{y^2}{\theta^2}$$

If $\max(y_N) \leq y$, then $y[k] \leq y \ \forall k$

$$\Rightarrow F(\max(y_N) \leq y) = (F(Y \leq y))^N$$

$$\Rightarrow F(\max(y_N) \leq y) = \left(\frac{y^2}{\theta^2}\right)^N = \frac{y^{2N}}{\theta^{2N}}$$

$$\Rightarrow f(\max(y_N) = y) = \frac{2N\, y^{2N-1}}{\theta^{2N}} \quad \text{— ④}$$

$$E(\max(y_N)) = \int_0^\theta 2N\, y \times \frac{y^{2N-1}}{\theta^{2N}}\, dy$$

$$= \left(\frac{2N}{2N+1}\right)\frac{\theta^{2N+1}}{\theta^{2N}} = \left(\frac{2N}{2N+1}\right)\theta$$

$$\Rightarrow E(\hat{\Theta}_{MLE}) = \left(\frac{2N}{2N+1}\right)\theta \quad \text{— ⑤}$$

So the MLE estimator is biased. Correcting for the bias, $\boxed{\hat{\Theta}_{unbiased} = \left(\frac{2N+1}{2N}\right)(\max(\underline{y}_N))}$

— ⑥

b) Let x be the median of Y

$$\int_0^x \frac{2y}{\theta^2}\, dy = \frac{1}{2} \quad \left(\because F(Y \le y) = F(Y \ge y) = \tfrac{1}{2} \text{ for median}\right)$$

$$\Rightarrow \frac{x^2}{\theta^2} = \frac{1}{2}$$

$$\Rightarrow \boxed{x = \frac{\theta}{\sqrt{2}}} \quad \text{— (7)}$$

$(\because x \ge 0)$ [Technically, the + or - sign depends on whether θ itself is +ve or negative. But without loss of generality we can assume $\theta > 0$ because it is present as θ^2 in the PDF]

Now, we know MLE has the following property,

if $\phi = g(\theta)$ then $\hat{\phi}_{MLE} = g(\hat{\theta}_{MLE})$ — (8)

$$\Rightarrow \hat{\phi}_{MLE} = \frac{\hat{\theta}_{MLE}}{\sqrt{2}},$$

$$\Rightarrow \hat{x}_{MLE} = \frac{\max(y_N)}{\sqrt{2}} \quad \text{— (9)}$$

However since the estimator is biased,

$$\left(E(\hat{x}_{MLE}) = E\left(\frac{\max(y_N)}{\sqrt{2}}\right) = \frac{2N}{2N+1}\frac{\theta}{\sqrt{2}} = \frac{2N}{2N+1} x \right)$$

we propose a modification,

$$\boxed{\hat{x} = \left(\frac{2N+1}{2N}\right)\left(\frac{\max(y_N)}{\sqrt{2}}\right)} \quad \text{— (10)}$$

c) We have already shown that the estimator is unbiased

$$\operatorname{var}(\hat{x}_{MLE}) = \left(\frac{2N+1}{2N}\right)^2 \times \frac{1}{2} \times \operatorname{var}(\max(\underline{y_N}))$$

$$\operatorname{var}(\max(y_N)) = E((\max(y_N))^2) - (E(\max(y_N)))^2$$

From the PDF in eqn (4),

$$E((\max(y_N))^2) = \int_0^{\theta} y^2 \left(\frac{2N\, y^{2N-1}}{\theta^{2N}}\right) dy$$

$$= \frac{2N}{\theta^{2N}} \int_0^{\theta} y^{2N+1}\, dy = \frac{2N \times \theta^{2N+2}}{(2N+2)\,\theta^{2N}}$$

$$\Rightarrow E((\max(y_N))^2) = \left(\frac{2N}{2N+2}\right)\theta^2$$

$$\therefore \operatorname{var}(\max(y_N)) = \theta^2\left(\left(\frac{2N}{2N+2}\right) - \left(\frac{2N}{2N+1}\right)^2\right)$$

$$= \theta^2\left(\frac{1}{1+\frac{1}{N}} - \left(\frac{1}{1+\frac{1}{2N}}\right)^2\right)$$

$$= \theta^2 \, \frac{1 + \frac{1}{4N^2} + \frac{1}{N} - 1 - \frac{1}{N}}{\left(1+\frac{1}{N}\right)\left(1+\frac{1}{2N}\right)^2}$$

$$= \theta^2 * \left(\frac{\frac{1}{4N^2}}{\left(1+\frac{1}{N}\right)\left(1+\frac{1}{2N}\right)^2}\right) \quad \text{—— (7)}$$

$$\lim_{N\to\infty} var(\max(y_N)) = \lim_{N\to\infty} \frac{\theta^2\left(\frac{1}{4N^2}\right)}{\left(1+\frac{1}{N}\right)\left(1+\frac{1}{2N}\right)^2}$$

$$\Rightarrow \lim_{N\to\infty} var(\max(y_N)) = 0 \quad (8)$$

$$\lim_{N\to\infty}\left(E\left((\hat{x}-x)^2\right)\right) = (\Delta\hat{x})^2 + var(\hat{x})$$

$$= \lim_{N\to\infty} 0 + \left(\frac{2N+1}{2N}\right)^2 \times \frac{1}{2} \times var(\max(y_N))$$

$$= \lim_{N\to\infty}\left(\frac{var(\max(y_N))}{2}\right)$$

$$\Rightarrow \lim_{N\to\infty} E\left((\hat{x}-x)^2\right) = 0$$

$\therefore$ the estimator exhibits mean squared convergence

$$\Rightarrow \boxed{\hat{x} = \left(\frac{2N+1}{2N}\right)\left(\frac{\max(y_N)}{\sqrt{2}}\right) \text{ is a consistent estimator}}$$

d) Consistency is verified numerically. However, we note that the estimates ~~doesn't~~ don't ~~doesn't~~ show asymptotic normality (asymptotically Gaussian) because the PDF doesn't satisfy all the regularity conditions

It violates the condition that the support of the PDF is independent of θ.

Here, the support: $[0, \theta]$ depends on θ itself.

So from theory we conclude that our observation from numerical simulation that the estimates are not asymptotically Gaussian is correct.

c) contd.. $\lim_{N\to\infty} E((x - \hat{x}_{MLE})^2) = \lim_{N\to\infty} (x - E(\hat{x}_{MLE}))^2 + var(\hat{x}_{MLE})$

Also, even if we consider the ~~un~~ biased estimator,

$$\hat{x}_{MLE} = \frac{\max(y_N)}{\sqrt{2}},$$

we still notice $\lim_{N\to\infty} var(\hat{x}_{MLE}) = \lim_{N\to\infty} \frac{var(\max(y_N))}{2} = 0$ — ①

& $\lim_{N\to\infty} x - E(\hat{x}_{MLE}) = \lim_{N\to\infty} x - \left(\frac{2N}{2N+1}\right)x.$

$$= \lim_{N\to\infty} \frac{x}{2N+1} = 0 \quad ②$$

① & ② ⟹ $\hat{x}_{MLE}$ is also consistent in the mean squared sense.

(because even though it is biased, it is asymptotically unbiased)