BY: S.VISHAL

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(2) $$CRLB = \frac{1}{I(\theta)}$$

$$L = \log \prod_{i=1}^{N} \theta^{y_i}(1-\theta)^{1-y_i} \quad \forall\, y_i \in \{0,1\}$$

$$= \sum y_i \log\theta + \sum(1-y_i)\log(1-\theta)$$

$$\Rightarrow \frac{\partial L}{\partial \theta} = \frac{\sum y_i}{\theta} - \frac{\sum(1-y_i)}{(1-\theta)}$$

$$\Rightarrow \frac{\partial^2 L}{\partial \theta^2} = \frac{-\sum y_i}{\theta^2} - \frac{\sum(1-y_i)}{(1-\theta)^2}$$

$$\Rightarrow E\left(-\frac{\partial^2 L}{\partial \theta^2}\right) = I(\theta) = \frac{E(\sum y_i)}{\theta^2} + \frac{E(\sum(1-y_i))}{(1-\theta)^2}$$

$$= \frac{N\theta}{\theta^2} + \frac{N-N\theta}{(1-\theta)^2} = N\left[\frac{1}{\theta} + \frac{1}{1-\theta}\right]$$

$$\Rightarrow I(\theta) = \frac{N}{\theta(1-\theta)}$$

$$\Rightarrow CRLB = var(MVUE) = \cancel{\frac{\theta(1-\theta)}{N}} \quad \frac{1}{I(\theta)}$$

$$\Rightarrow var(MVUE) = \boxed{\frac{\theta(1-\theta)}{N}}$$

$$\text{var}(\hat{\Theta}_{\text{realizable}}) = \text{var}\left(\frac{T_i}{N}\right) = \text{var}\left(\frac{\sum Y_i}{N}\right)$$

$$= \frac{1}{N^2}\sum \text{var}(Y_i) \quad \because \text{All } Y_i\text{'s are iid.}$$

$$= \frac{1}{N^2} \times \sum_{i=1}^{N} \theta(1-\theta) = \frac{\theta(1-\theta)}{N}$$

$$\Rightarrow \text{var}(\hat{\Theta}_{\text{realizable}}) = \boxed{\frac{\theta(1-\theta)}{N}}$$

We observe that $\text{var}(\hat{\Theta}_{\text{realizable}}) = \text{var}(\text{MVUE})$

To verify this let's calculate $\hat{\Theta}^*$ ~~through~~ CRLB theorem ~~theom~~

$$\hat{\Theta}^* = \frac{S}{I(\theta)} + \theta. \quad \text{where } S = \frac{\partial L}{\partial \theta}$$

$$\Rightarrow \hat{\Theta}^* = \frac{\frac{\sum y_i}{\theta} - \frac{\sum(1-y_i)}{(1-\theta)}}{\frac{N}{\theta(1-\theta)}} + \theta$$

$$= \frac{1}{N}\left[\frac{\sum y_i - N\theta}{\cancel{(1-\theta)}}\right] + \theta$$

$$\Rightarrow \hat{\Theta}^* = \frac{\sum y_i}{N}$$ ~~Θ̂ realizable~~ which is realizable (independent of θ).

Since in this case, $\hat{\Theta}^*$ is realisable,

$\text{var}(\text{MVUE}) = \text{var}(\hat{\Theta}^*)$

The precision of the estimators obtained through CRLB and Rao Blackwell theorem are the same.

This is because CRLB estimator is realizable and is the same as the one obtained through Rao Blackwell theorem.

However CRLB need not be always realizable.

In such cases,

$$var(\text{Rao Blackwell } \hat{\theta}) > var(\text{MVUE})$$

MVUE → hypothetical - given by CRLB.

That is there will be a compromise of precision on $\hat{\theta}$ obtained through Rao Blackwell.

This is necessary in order to get a realizable estimator.

Also CRLB requires the PDF to follow regularity conditions. This is not required in Rao-Blackwell - Lehmann - Scheffe theorem.