CH5115-Parameter and State Estimation
Quiz 1 solutions

Question 1

a

Given

$$f(y|x) = cy^2, -2 \leq y \leq 2$$
$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} - \infty \leq x \leq \infty$$

We know that

$$\int_{-\infty}^{\infty} f(y|x) = 1$$

We get

$$\int_{-2}^{2} cy^2 = 1$$
$$c = \frac{3}{16}$$

$$\mu_Y = E(y) = \int_{-2}^{2} \int_{-\infty}^{\infty} y f(x,y) dx dy$$
$$f(x,y) = f(y|x) f_X(x)$$
$$E(y) = \frac{3}{16\sqrt{2\pi}} \int_{-2}^{2} \int_{-\infty}^{\infty} y.y^2 e^{-x^2/2} dx dy$$
$$= 0$$

$$\sigma_Y^2 = E(y^2) - (E(y))^2$$
$$= E(y^2)$$
$$= \frac{3}{16\sqrt{2\pi}} \int_{-2}^{2} \int_{-\infty}^{\infty} y^2.y^2 e^{-x^2/2} dx dy$$
$$= \frac{12}{5}$$

b

Given zero mean jointly Gaussian distributed RVs X and Y and covariance matrix $\Sigma_X = \begin{bmatrix} 2 & 1.2 \\ 1.2 & 3 \end{bmatrix}$

$$\begin{aligned} E(Y|X) &= \mu_y + \rho\sigma_y \frac{x-\mu_x}{\sigma_x} \\ &= \frac{\sigma_{xy}}{\sigma_x}\frac{x-\mu_x}{\sigma_x} \end{aligned}$$

$$\sigma_x = 2, \sigma_{xy} = 1.2, \mu_y = 0 \text{ and } \mu_x = 0$$

$$E(Y|X) = 0.6x$$

Question 2

a

Given PDF of random variable Y

$$f(y;\theta) = \begin{cases} \theta y^{\theta-1}, & 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

MLE of of N observations is

$$\begin{aligned} L(\theta; y) &= ln \prod_{k=1}^{N} \theta y[k]^{\theta-1} \\ &= N ln\theta + (\theta - 1)\sum_{k=1}^{N} \ln y[k] \\ \frac{dL}{d\theta} &= \frac{N}{\theta} + \sum_{k=1}^{N} \ln y[k] \end{aligned}$$

For $\hat{\theta}$, we make $\frac{dL}{d\theta} = 0$ and we get

$$\hat{\theta} = -N/\sum_{k=1}^{N} \ln y[k]$$

Fisher information is given by

$$\begin{aligned} I(\theta) &= -E\left(\frac{d^2L}{d\theta^2}\right) \\ &= N/\theta^2 \end{aligned}$$

b

Given $\rho[\pm 1] = 0, \rho[\pm 2] = 0.6$ and $\rho[l]$ unknown for $|l| \geq 3$ for AR(2) process. Using Yule walker equations we get

$$\begin{bmatrix} 1 & d_1 & d_2 \\ d_1 & 1+d_2 & 0 \\ d_2 & d_1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_v^2 \\ \sigma_{vv}[1] \\ \sigma_{vv}[2] \end{bmatrix} = \begin{bmatrix} \sigma_e^2 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \rho[1] &= \sigma_{vv}[1]/\sigma_v^2 \\ \sigma_{vv}[1] &= 0 \\ \rho[2] &= \sigma_{vv}[2]/\sigma_v^2 \\ \sigma_{vv}[2] &= 0.6\sigma_v^2 \end{aligned}$$

$$\begin{bmatrix} 1 & d_1 & d_2 \\ d_1 & 1+d_2 & 0 \\ d_2 & d_1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_v^2 \\ 0 \\ 0.6\sigma_v^2 \end{bmatrix} = \begin{bmatrix} \sigma_e^2 \\ 0 \\ 0 \end{bmatrix}$$

On solving we get $d_1 = 0, d_2 = -0.6$ and $\sigma_v^2 = 1.562\sigma_e^2$