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CH18B020

CH5115 QUIZ-2

(3) No.! MVUE of $\beta \neq \frac{1}{\hat{\alpha}^*}$

We can verify this by examining the CRLB of the 2 cases and check whether $\hat{\beta}^* = \frac{1}{\hat{\alpha}^*}$

$$y[k] = \alpha u[k] + e[k]$$

$$\Rightarrow e[k] = y[k] - \alpha u[k]$$

Likelihood function, $L = \log\left(\prod_{k=1}^{n}\left(\frac{1}{\sqrt{2\pi}\sigma_e}\right)\left(\exp\left(\frac{-(y(k)-\alpha u(k))^2}{2\sigma_e^2}\right)\right)\right)$

$$\Rightarrow L = \underbrace{c}_{\text{constant term}} + \sum_{k=1}^{n}\left(\frac{-1}{2\sigma_e^2}\right)(y[k]-\alpha u[k])^2$$

$$\Rightarrow S = \frac{\partial L}{\partial \theta} = \frac{\partial L}{\partial \alpha} = \boxed{\frac{\sum u(k)(y(k)-\alpha u(k))}{\sigma_e^2}}$$

also, $\frac{\partial^2 L}{\partial \alpha^2} = \frac{-\sum (u(k))^2}{\sigma_e^2}$

$$I(\alpha) = -E\left(\frac{\partial^2 L}{\partial \alpha^2}\right) = \boxed{\frac{\sum_{k=1}^{N}(u(k))^2}{\sigma_e^2}}$$

($\because u(k)$ is a ~~constant~~ deterministic)

From Cramer Raos we have

$$\hat{\alpha}^* = \frac{S}{I(\theta)} + \alpha.$$

$$= \frac{\sum u(k)(y(k)-\alpha u(k))}{\sum (u(k))^2} \times \frac{\sigma_e^2}{\sigma_e^2} + \alpha.$$

$$= \frac{\sum [u(k)y(k) - \alpha (u(k))^2]}{\sum u^2(k)} + \alpha.$$

$$\Rightarrow \boxed{\hat{\alpha}^* = \frac{\sum u(k)y(k)}{\sum u^2(k)}}$$

(Independent of α $\Rightarrow$ valid estimator)

β is estimated from this as

$$\hat{\beta} = \frac{\sum u^2(k)}{\sum u(k)y(k)}$$

However if we sets up the problem as,

$$y(k) = \frac{1}{\beta} u(k) + e(k)$$

$$L = C + \sum_{k=1}^{N} \left(\frac{-1}{2\sigma_e^2}\right)\left(y(k) - \frac{1}{\beta}u(k)\right)^2$$

$$\Rightarrow S = \frac{\partial L}{\partial \beta} = -\frac{\sum u(k)\left(y(k) - \frac{u(k)}{\beta}\right) \times \frac{1}{\beta^2}}{\sigma_e^2}$$

$$\frac{\partial^2 L}{\partial \beta^2} = \frac{2}{\beta^3 \sigma_e^2} \sum u(k)y(k) - \frac{3}{\beta^4 \sigma_e^2} \sum (u(k))^2$$

$$I(\theta) = -E\left(\frac{\partial^2 L}{\partial \beta^2}\right) \quad ; \quad y(k) = \frac{u(k)}{\beta} + e$$

$$\Rightarrow E(y(k)) = \frac{u(k)}{\beta}.$$

$$= \frac{3}{\sigma_e^2 \beta^4} \sum (u(k))^2 - \frac{2}{\sigma_e^2 \beta^3} \frac{\sum u^2(k)}{\beta}$$

$$\Rightarrow I(\theta) = \frac{\sum (u(k))^2}{\sigma_e^2 \beta^4}$$

$$\Rightarrow I(\beta) = \frac{\Sigma (u(k))^2}{\sigma_e^2 \beta^4}$$

<u>CRLB</u>

$$\hat{\beta}^* = \frac{S}{I(\beta)} + \beta$$

$$= \frac{-\Sigma u(k)\left(y(k) - \frac{u(k)}{\beta}\right) \times \frac{1}{\sigma_e^2 \beta^2}}{\frac{\Sigma u(k)^2}{\sigma_e^2 \beta^4}} + \beta$$

$$= \frac{\Sigma u(k)(-y(k)\beta + u(k))\beta}{\Sigma u(k)^2} + \beta$$

$$= \cancel{2\left(\Sigma (u(k))^2\right)\beta} - \beta$$

$$= \frac{-\beta^2 \left[\Sigma u(k) y(k)\right] + 2\beta \, \Sigma (u(k))^2}{\Sigma (u(k))^2}$$

$\Rightarrow$ In terms of β

$\therefore$ In fact, the MVUE given by CRLB doesn't exist! (because $\hat{\beta}^*$ is dependent on β itself)

So the given $\hat{\beta} = \frac{1}{\hat{\alpha}^*} \neq \hat{\beta}_{MVUE}$

It is not the most precise estimator.

$$\hat{\beta} = \frac{1}{\hat{\alpha}^*} = \frac{\sum u^2(k)}{\sum u(k)\,y(k)}$$

This estimator was used and its consistency (in terms of MSE) was tested numerically.

The estimator was found to be CONSISTENT in the sense of MSE.

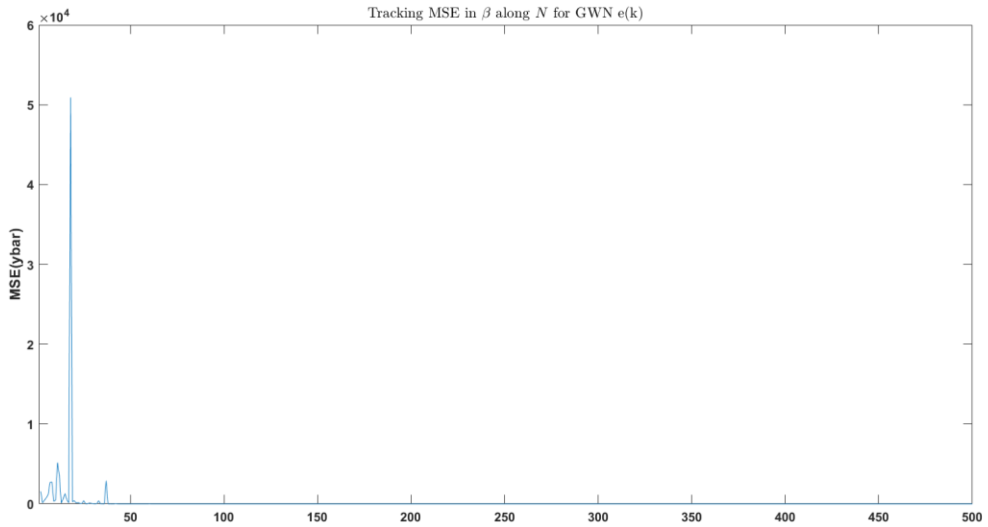
i.e. for large N, the $\mathrm{var}(\hat{\beta}) + (\hat{\beta} - \beta_0)^2$ was close to zero (mse $\approx$ 0.03 for ~~record~~ sample size N of 500)

Also, the convergence was faster for the cases where σ_e^2 was lower.

(Theoritical method was not used because we might need to evaluate a cumbersome integral to define $E\left(\frac{\sum u^2(k)}{\sum u(k)\,y(k)}\right)$)

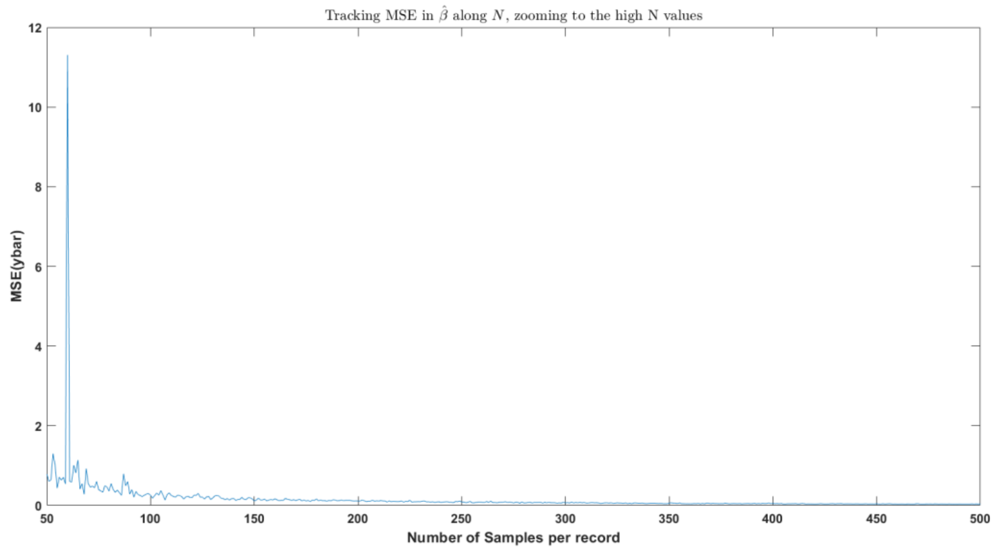
Numerical verification of consistency

MSE along N



Since the initial MSEs are very high, the y-scale is also bit high. Hence, I am attaching a graph of the MSEs after the errors fall down to a reasonable range. This is after N = 50.

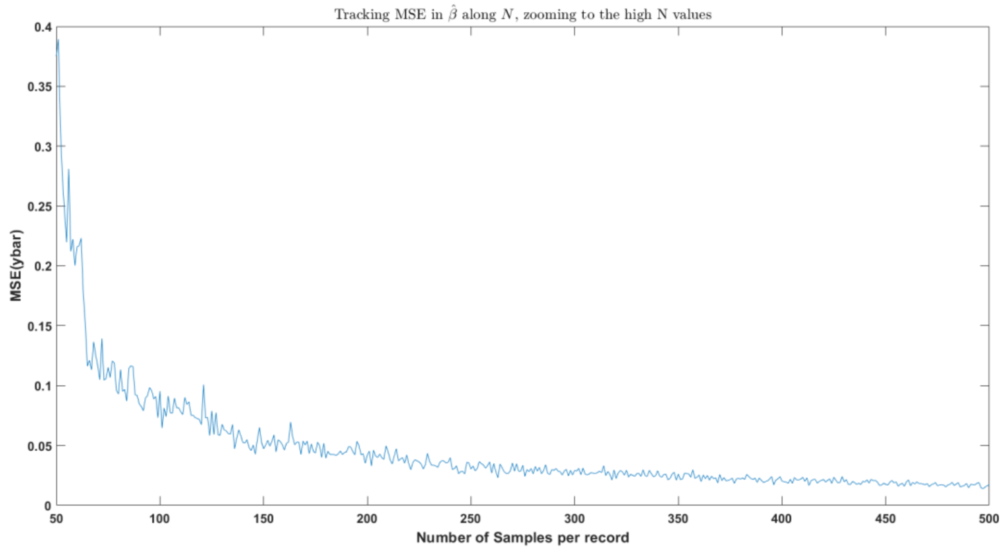
MSE along N (zoomed)



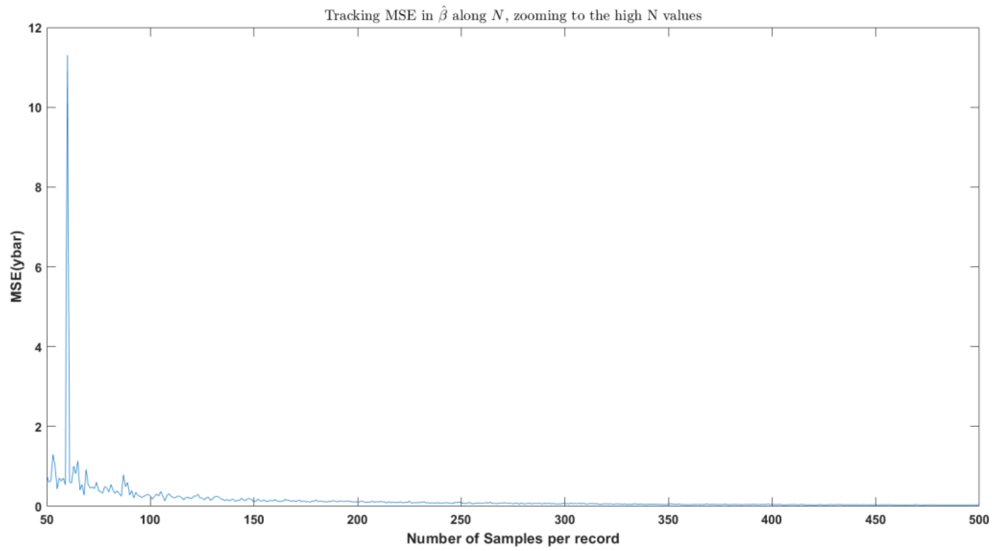
By examining the actual values, I saw that MSE<0.05 for sufficiently large number of samples.

Comparing MSE at different σ_e^2

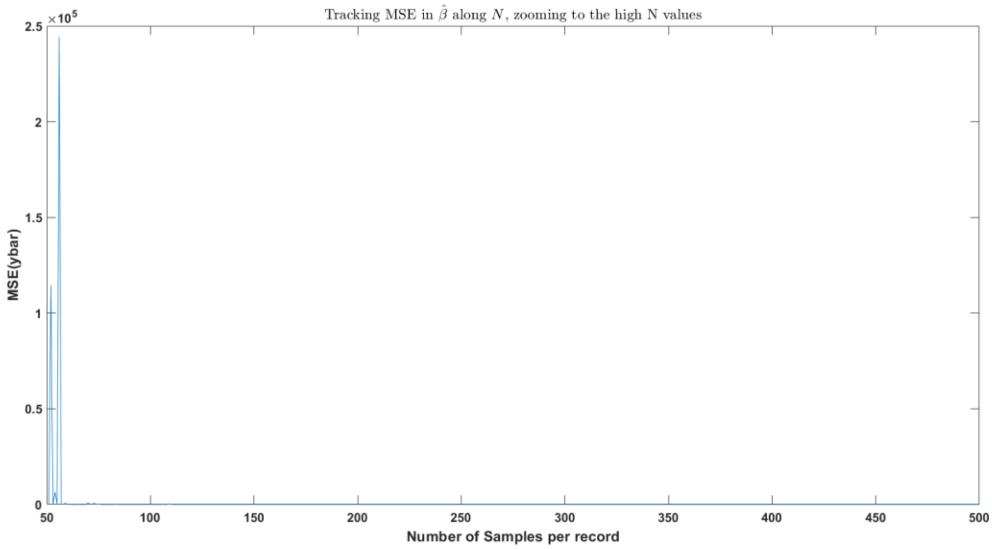
$\sigma_e^2 = 0.5$



$\sigma_e^2 = 1$



$\sigma_e^2 = 2$



From the above graphs we can see that the convergence is faster if the variance of the White Noise term is lower.

CODE

```
close all;
N = 500; % Max number of samples in a record
R = 300; % Number of records
u = randn(1,N); % Randomly Generate and fix uk
beta0 = 2; % True value of beta
var_e = 1; % Variance in the error term
mserror = zeros(N-1,1); % Vector for storing the MSE at
each sample size
beta_vec = zeros(R,1); % Vector for storing beta estimate
from each record
for n = 2:N
    e_mat = sqrt(var_e)*randn(R,n); % Each row represents
one record
    for r = 1:R
        y = u(1:n)/beta0 + e_mat(r,:);
        % Estimate beta for record r
        beta_vec(r) = sum(u(1:n).^2)./sum(u(1:n).*y);
    end
    beta_hat = mean(beta_vec);
    mserror(n-1)  = var(beta_vec) + ((mean(beta_vec)-
beta0)^2);
end
plot((2:N),mserror);
set(gca,'fontsize',12,'fontweight','bold','xlim',[1 N]);
ylabel('MSE(ybar)','fontsize',14,'fontweight','bold');
title('Tracking MSE in $\beta$ along $N$ for GWN
e(k)','fontsize',14,'fontweight','bold','interpreter','late
x');
figure();
plot((50:N),mserror(49:end));
set(gca,'fontsize',12,'fontweight','bold','xlim',[50 N]);
ylabel('MSE(ybar)','fontsize',14,'fontweight','bold');
xlabel('Number of Samples per
record','fontsize',14,'fontweight','bold');
title('Tracking MSE in $\hat{\beta}$ along $N$, zooming to
the high N
values','fontsize',14,'fontweight','bold','interpreter','la
tex');
```