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CH18B020

CH5115 ASSIGNMENT - 2

① a) $v[k] = A\cos^2(2\pi f k + \phi)$

$E(v[k]) = E(A\cos^2(2\pi f k + \phi))$

$= \cos^2(2\pi f k + \phi)\, E(A)$ $\quad$ ($\because$ A is the R.V.)

$\Rightarrow E(v[k]) = 0 \quad (\because E(A) = 0)$ $= \mu$.

$\text{var}(v[k]) = E((v[k] - E(v[k]))^2)$

$= E((v[k])^2) = E(A^2\cos^4(2\pi f k + \phi))$

$= (\cos^4(2\pi f k + \phi))\, E(A^2)$

We know that $\sigma^2_A = E(A^2) - (\cancel{E(A)})^2$

$\Rightarrow E(A^2) = \sigma_A^2$

$= 1$

$\therefore \text{var}(v[k]) = \cos^4(2\pi f k + \phi)$

$\Rightarrow \text{var}(v[k])$ is a function of k (time step)

$\Rightarrow$ The random process is ~~covariance~~ non stationary

Autocovariance $= E((v[k-l] - E(v[k-l]))(v[k] - E(v[k])))$

$= E(v[k-l]\,v[k])$

$= E(A^2\cos^2(2\pi f k + \phi)\cos^2(2\pi f(k-l) + \phi))$

$= (E(A^2))(\cos^2(2\pi f k + \phi)\cos^2(2\pi f(k-l) + \phi))$

$$= \left(\cos\left(4\pi f k - 2\pi f l + 2\phi\right) + \cos\left(2\pi l f\right)\right)^2$$

$\Rightarrow$ ACVF of $v[k]$ at any ~~only~~ lag l is a function of k.

~~$\Rightarrow$ $v[k]$ covariance of $v[k]$~~.

$\Rightarrow$ $v[k]$ is covariance non-stationary

1b)

$$v[k] - v[k-1] = e[k]$$

$$v[k-1] - v[k-2] = e[k-1]$$

$$\vdots$$

$$v[1] - v[0] = e[1]$$

$$v[0] = 0 \text{ (given)}$$

Add all

$$v[k] = \sum_{i=0}^{k} e[i]$$

$$e[i] \sim N(0, \sigma^2)$$

$$\Rightarrow E(v[k]) = E\left(\sum e[k]\right)$$

$$= \sum \left(E(e[i])\right) = \sum 0$$

$$\Rightarrow E(v[k]) = 0$$

$$\text{var}(v[k]) = E\left(\left(v[k] - E(v[k])\right)^2\right)$$

$$= E\left((v[k])^2\right) = E\left(\left(\sum_{i=1}^{k} e[i]\right)^2\right)$$

$$= E\left(\sum_{i=1}^{k} (e[i])^2 + 2\sum_{j=1}^{k} \sum_{i=1}^{j-1} e[i]e[j]\right)$$

$$= \sum_{i=1}^{k} E\left(e[i]^2\right) + 2\sum\sum E\left(e[i]e[j]\right)$$

$$\Rightarrow E(e[i]^2) = E\left((e[i])^2\right) - \left(E(e[i])\right)^2 = \sigma_e^2$$

$$E(e[i]e[j]) = E(e[i])E(e[j])$$

$\because$ Gaussian white noise process $\Rightarrow$ i.i.d.

$$\Rightarrow E(e[i]e[j]) = 0$$

$\therefore$ var $(v[k]) = k\sigma^2 \Rightarrow$ var$(v[k])$ varies with k

$\Rightarrow$ non - stationary.

From the MATLAB plot we can see that var$(v[k])$ is continuously increasing as k increases.

$\Rightarrow$ var $(v[k])$ is non- stationary

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(2) a) Given :

$\rightarrow u[k] \sim WN(0, \sigma_u^2) \Rightarrow E(u[k]) = 0$

$E(u^2[k]) = \sigma_u^2$

$E(u[k]\, u[k-l]) = 0 \quad \forall\, l \neq 0$

$\rightarrow e[k] \sim WN(0, \sigma_e^2) \Rightarrow E(e[k]) = 0$

$E(e^2[k]) = \sigma_e^2$

$E(e[k]\, e[k-l]) = 0 \quad \forall\, l \neq 0.$

$\rightarrow \sigma_{eu}[l] = 0 \Rightarrow E(e[k]\, u[k-l]) = 0$

$$y^*_o[k] = \frac{b_2^o q^{-2}}{1 + f_1^o q^{-1}} u[k]$$

$$\Rightarrow y^*[k] + f_1^o q^{-1}(y^*[k]) = b_2^o q^{-2}(u[k])$$

$$\Rightarrow y^*[k] + f_1^o y[k-1] = b_2^o u[k-2]$$

$$\Rightarrow y^*[k] = b_2^o u[k-2] - f_1^o y[k-1]$$ — (4) (using the property of backward shift operator)

$E(y^*[k]) = E(\cancel{u[k-2]}) - f_1^o E(y[k-1])$ (Assuming mean stationarity)

$E(y^*) = 0$ — (1)

$$E(y^{*2}(k)) = \sigma^2_{y^*}$$

$$\Rightarrow \sigma^2_{y^*} = E\left(\left(b_2^o u(k-2) - f_1 y^*(k-1)\right)^2\right)$$

$$= E\Big(b_2^{o2} u^2[k-2] + f_1^{o2}(y^*[k-1])^2 \cancel{\quad} \; - 2 f_1^o b_2^o u[k-2] y^*[k-1]\Big)$$

Expectation of sum is sum of $E(\cdot)$

$$= b_2^{o2} \sigma_u^2 + f_1^{o2} \sigma^2_{y^*} - 2 f_1^o b_2^o \sigma_{y^*u}[1] \quad \text{—— (2)}$$

(y^* is assumed to be variance stationary)

$$\sigma_{y^*u}[0] = E(y^*[k] u[k])$$

$$= 0 \quad (\because u[k] \text{ doesn't affect } y[j] \;\; \forall j < k+2)$$

$$\sigma_{y^*u}[1] = E(y^*[k] u[k-1])$$

$$= E\left(\left(b_2^o u[k-2] - f_1^o y^*[k-1]\right) u[k-1]\right)$$

$$= b_2^o E(u[k-2] u[k-1]) - E(f_1^o y^*[k-1] u[k-1])$$

$$= b_2^o \sigma_{uu}[1] - \sigma_{y^*u}[0]$$

$$\Rightarrow \sigma_{y^*u}[1] = 0. \quad \text{—— (3)}$$

Using this,

$$(2) \Rightarrow \sigma^2_{y^*} = b_2^{o2} \sigma_u^2 + f_1^{o2} \sigma^2_{y^*}$$

$$\Rightarrow \sigma^2_{y^*} = \left(\frac{b_2^{o2} \sigma_u^2}{1 - f_1^{o2}}\right) \quad \text{—— (4)}$$

$$\sigma_{y^*e}[l] = E\left(y^*[k]\, e[k-l]\right)$$

$$= E\left(\left(b_2^o\, u[k-2] - f_1^o\, y^*[k-1]\right) e[k-l]\right)$$

$$= b_2^o \cancel{\sigma_{ue}[l-2]}^{\,0} - f_1^o\, \sigma_{y^*e}[l-1]$$

Assuming $\sigma_{y^*e}[0] = 0$ (Truth is not correlated with the noise)

$$\Rightarrow \sigma_{y^*e}[l] = 0 \quad \text{— (5)}$$

i) To find σ_y^2 : ~~$\sigma_y^2 = E$~~

~~$E(y^2)$~~ $E(y) = E(y^*(k) + e) = 0$

$$\Rightarrow E(y(k)) = 0.$$

$$\therefore \sigma_y^2 = E\left((y[k])^2\right)$$

$$= E\left((y^*[k] + e[k])^2\right)$$

$$= \sigma_{y^*}^2 + \sigma_e^2 + 2\cancel{\sigma_{y^*e}}^{\,0}$$

$$\Rightarrow \boxed{\sigma_y^2 = \frac{b_2^{o\,2}\,\sigma_u^2}{(1 - f_1^{o\,2})} + \sigma_e^2}$$

ii) $\sigma_{y^*y^*}[1] = E\left(y^*[k]\, y^*[k-1]\right)$

$$= E\left(\left(b_2^o\, u[k-2] - f_1^o\, y^*[k-1]\right)\left(y^*[k-1]\right)\right)$$

$$= E\left(b_2^o\, u[k-2]\, y^*[k-1] - f_1^o\left(y^*[k-1]\right)^2\right)$$

$$= b_2^o\, \sigma_{y^*u}[1] - f_1^o\, \sigma_{y^*}^2$$

$$\Rightarrow \sigma_{y^* y^*}[1] = -f_1^\circ \sigma_{y^*}^2$$

$$= -f_1^\circ \left(\frac{\sigma_u^2 b_2^{\circ 2}}{1 - f_1^{\circ 2}} \right) \quad \text{—— (6)}$$

$$\sigma_{yy}[1] = E\left((y^*(k) + e(k)) \, (y^*(k) + e(k)) \right)$$

$$= E(y^{*2}(k)) + E(e^2(k)) + 2E(u(k)\, e(k))$$

$$= \sigma_{y^*}^2 + \sigma_e^2 + 2\cancel{\sigma_{eu}}^{0}$$

$$\Rightarrow \boxed{\sigma_{yy}[1] = -\frac{f_1^\circ \sigma_u^2 b_2^{\circ 2}}{(1 - f_1^{\circ 2})} + \sigma_e^2}$$

ii) $$\sigma_{yy}[1] = E\left((y^*(k) + e[k])(y^*(k-1) + e(k-1)) \right)$$

$$= \sigma_{y^* y^*}[1] + \sigma_{ee}[1] + \cancel{\sigma_{y^* e}[1]}^{0} + \cancel{\sigma_{e y^*}[1]}^{0}$$

$$\Rightarrow \boxed{\sigma_{yy}[1] = -f_1^\circ \left(\frac{\sigma_u^2 b_2^{\circ 2}}{1 - f_1^{\circ 2}} \right)}$$

iii) $$\sigma_{yu}[1] = E\left((y^*(k) + e(k))(u(k-1)) \right)$$

$$= E(y^*(k)\, u(k-1)) + E(e(k)\, u(k-1))$$

$$= \sigma_{y^* u}[1] + \cancel{\sigma_{eu}(1)}^{0}$$

$$= 0 \quad (\text{from eqn } (4))$$

$$\Rightarrow \boxed{\sigma_{yu}[1] = 0}$$

iv) $\sigma_{y^*u}[2] = E\left(\left(b_2^0 u(k-2) - f_1 y^*(k-1)\right) u(k-2)\right)$

$= E\left(b_2^0 u^2(k-2) - f_1^0 y_1^* (k-1) u(k-2)\right)$

$= b_2^0 \sigma_{u^2} - f_1^0 \sigma_{y^*u}[1]$

From ④, $\sigma_{y^*u}[1] = 0$

$\Rightarrow \sigma_{y^*u}[2] = b_2^0 \sigma_{u^2}$ ——— ⑥

$\cancel{\sigma_{yu}(2) = E\left(\left(y^*(k) + e(k)\right)\left(y^*(k-2) + e(k-2)\right)\right)}$

$\cancel{= E\, \sigma_{y^*y^*}(2) + \sigma}$

$\sigma_{yu}(2) = E\left(\left(y^*(k) + e(k)\right)\left(u(k-2)\right)\right)$

$= E\left(y^*(k) u(k-2) + e(k) u(k-2)\right)$

$= \sigma_{y^*u}[2] + \cancel{\sigma_{eu}(2)}^{\,0}$

From ⑥, $\sigma_{y^*u}(2) = b_2^0 \sigma_{u^2}$

$$\boxed{\sigma_{yu}(2) = b_2^0 \sigma_{u^2}}$$

Summary.

i) $\sigma_y^2 = \dfrac{b_2^{o2}\,\sigma_u^2}{1-f_1^{o2}} + \sigma_e^2$

ii) $\sigma_{yy}[1] = -f_1^o \left(\dfrac{\sigma_u^2\, b_2^{o2}}{1-f_1^{o2}}\right)$

iii) $\sigma_{yu}(1) = 0$

iv) $\sigma_{yu}(2) = b_2^o \sigma_u^2$

MATLAB Simulation

→ σ_{y*}^2 is obtained from σ_u^2, b_0 & f_0 using eqn ④

$\sigma_{y*}^2 = 10.6667$

→ σ_e^2 is obtained using SNR = 10 ⇒ $\sigma_e^2 = 0.9375$

→ $y*$ is generated using the given relation with u and assuming $y*[1] = y*[2] = 0$

→ estimates of variance is obtained using var() and estimates of auto & cross covariances is obtained using xcov().

→ It is ensured that UNBIASED estimators are used.

	True value	Estimate	Error %
σ_y^2	11.604	11.5168	0.753 %
$\sigma_{yy}[1]$	-5.333	-5.546	+ 3.1 %
$\sigma_{yu}[1]$	0	-0.0891	0.089 (absolute)
$\sigma_{yu}[2]$	4	3.938	1.55 %

④ a) $L(\mu; \underline{y_N}) = \ln f(\underline{y_N} \mid \mu)$

$$= \ln\left(\prod_{k=1}^{N}\left(\frac{1}{\sqrt{2\pi}\sigma}\right)\left(\exp\left(-\frac{1}{2}\left(\frac{y[k]-\mu}{\sigma}\right)^2\right)\right)\right)$$

Since it is a GWN process we have used the property that the joint pdf is simply the product of marginal pdfs.

$$\Rightarrow L = \sum_{k=1}^{n} \frac{-1}{2}\left(\frac{y[k]-\mu}{\sigma_e}\right)^2 + C$$

C — A constant term, independent of $y[k]$

$$\frac{\partial L}{\partial \mu} = 0$$ (Likelihood should be maximised)

$$\Rightarrow \sum_{k=1}^{N} \frac{+2}{2}\left(\frac{y[k]-\mu}{\sigma_e^2}\right) = 0$$

$$\Rightarrow N\mu = \sum_{k=1}^{N} y[k]$$

$$\Rightarrow \hat{\mu} = \frac{\sum_{k=1}^{N} y[k]}{N}$$

Thus, the ML estimate of mean is simply the samplemean.

$$FI = E\left(\left(\frac{\partial L}{\partial \theta}\right)^2\right) = -E\left(\frac{\partial^2 L}{\partial \theta^2}\right)$$

$$\frac{\partial^2 L}{\partial \mu^2} = \frac{\partial}{\partial \mu}\left(+\sum\left(\frac{y[k]-\mu}{\sigma_e^2}\right)\right)$$

$$\Rightarrow \frac{\partial^2 L}{\partial \mu^2} = -\sum_{k=1}^{N} \frac{1}{\sigma_e^2} = \frac{-N}{\sigma_e^2}.$$

$$\therefore \text{F.I.}, \quad I(\mu) = -E\left(\frac{\partial^2 L}{\partial \mu^2}\right)$$

$$\Rightarrow \boxed{I(\mu) = \frac{N}{\sigma_e^2}}$$

b) $Y = aX + b + \varepsilon. \Rightarrow \varepsilon = Y - aX - b.$

$\varepsilon \sim N(0, \sigma_e^2) \qquad = y[k] - ax[k] - b$

$$\Rightarrow L = \log\left(\prod_{k=1}^{N} \left(\frac{1}{\sqrt{2\pi}\sigma_e}\right)\left(\exp \frac{-(y - ax - b)^2}{2\sigma_e^2}\right)\right)$$

$$= \underset{\substack{\downarrow \\ \text{term without} \\ a \,\&\, b}}{C} + \frac{-1}{2}\sum_{k=1}^{N} \frac{(y[k] - ax[k] - b)^2}{\sigma_e^2}$$

Maximising $L \rightarrow$ objective.

$$\frac{\partial L}{\partial a} = \frac{2}{2} \sum \frac{x[k]\,(y[k] - ax[k] - b)}{\sigma^2}$$

$$= \frac{1}{\sigma_e^2}\left(\sum x[k]\,y[k] - a\sum (x[k])^2 - b\sum x[k]\right)$$

$$\frac{\partial^2 L}{\partial a^2} = \frac{-1}{\sigma_e^2} \sum_{k=1}^{N} (x(k))^2$$

$$\frac{\partial^2 L}{\partial b\, \partial a} = -\frac{\sum x[k]}{\sigma_e^2}$$

$$\frac{\partial L}{\partial b} = \frac{-2}{2}\,\frac{\sum \left(y[k] - a\,x[k] - b \right)}{\sigma_e^2}$$

$$= \frac{1}{\sigma_e^2} \sum y[k] - \frac{\sum x[k]\; a}{\sigma_e^2} \; \frac{-Nb}{\sigma_e^2}$$

$$\frac{\partial^2 L}{\partial b^2} = \frac{-1}{\sigma_e^2} \times N = \frac{-N}{\sigma_e^2}$$

$$\frac{\partial L}{\partial a\,\partial b} = -\frac{\sum x[k]}{\sigma_e^2}$$

Fisher's information matrix: $\begin{bmatrix} -E\left(\frac{\partial^2 L}{\partial a^2}\right) & -E\left(\frac{\partial L}{\partial b\,\partial a}\right) \\ -E\left(\frac{\partial L}{\partial a\,\partial b}\right) & -E\left(\frac{\partial^2 L}{\partial b^2}\right) \end{bmatrix}$

Since $x[k]$ is fixed, we get

$$\text{F.I. matrix} = \begin{bmatrix} \frac{1}{\sigma_e^2}\sum (x(k))^2 & \frac{\sum x(k)}{\sigma_e^2} \\ \frac{\sum x[k]}{\sigma_e^2} & \frac{N}{\sigma^2} \end{bmatrix}$$

$$\boxed{\begin{aligned} I(a) &= \frac{1}{\sigma_e^2} \sum (x(k))^2 \\ I(b) &= \frac{N}{\sigma_e^2} \end{aligned}}$$

b) ~~MAP~~ ML estimate:

Solve the eqns $\frac{\partial L}{\partial a} = 0$; $\frac{\partial L}{\partial b} = 0$ (linear equations in a & b)

to get $\hat{a}$ & $\hat{b}$

$$\Rightarrow \left(\sum (x(k))^2\right) a + \left(\sum (x(k))\right) b = \frac{\sum x(k)\, y(k)}{\sigma^2}$$

and $\left(\sum x(k)\right) a + N b = \sum y(k)$

Solution:
$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum (x(k))^2 & \sum x(k) \\ \sum x(k) & N \end{bmatrix}^{-1} \begin{bmatrix} \sum x(k) y(k) \\ \sum y(k) \end{bmatrix}$$

unique solution exists as long as

$$N \sum (x(k))^2 - \left(\sum x(k)\right)^2 \neq 0$$

Numerical estimate is obtained by searching the area:

$a \in [1, 3]$ (in steps of 0.01) $b \in (2, 4]$ (in steps of 0.01)

$\hat{a}$, numerical estimate = 2.03; $\hat{a}$ analytical estimate = 2.027

$\hat{b}$, numerical estimate = 2.78; $\hat{b}$ analytical = 2.7893.

X is generated randomly.