INDIAN INSTITUTE OF TECHNOLOGY MADRAS

Department of Chemical Engineering

CH5115 : Parameter and State Estimation (Jul-Nov 2020)
Solutions to Assignment 1

Marks Distribution

	Question 1	Question 2	Question 3	Question 4
(a)	20	20	10	20
(b)	10	–	10	10

Question 1

(a)

$$f(x,y) = \begin{cases} K\frac{e^{-x/y}e^{-y}}{y} & x>0, y>0 \\ 0 & \text{elsewhere} \end{cases}$$

(i)

We know that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$$
$$K\int_{0}^{\infty}\int_{0}^{\infty} \frac{e^{-x/y}e^{-y}}{y}dxdy = 1$$
$$K = 1$$

(ii)

Marginal density of Y is

$$f_Y(y) = \int_0^\infty \frac{e^{-x/y}e^{-y}}{y} dx$$
$$= e^{-y}, \quad y{>}0$$

(iii)

$$Pr(0 < X < 1, 0.2 < Y < 0.4) = \int_0^1 \int_{0.2}^{0.4} f(x,y)dxdy$$
$$= \int_0^1 \int_{0.2}^{0.4} \frac{e^{-x/y}e^{-y}}{y} dxdy$$
$$= 0.1429$$

(iv)

$$E(X|Y) = \int_{-\infty}^{\infty} x f_{X|Y=y}(x)dx$$
$$= \int_{-\infty}^{\infty} x \frac{f(x,y)}{f_Y(y)} dx$$
$$= \int_0^{\infty} x \frac{f(x,y)}{f_Y(y)} dx$$
$$= y$$

(b)

Let

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix}$$

Given that the random variables X and Y follows Gaussian distribution. Therefore, the joint pdf is

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y(\sqrt{(1-\rho_{XY}^2)}} \int_{-\infty}^{\infty} y \exp\left(- \left[\frac{\frac{(x-\mu_x)^2}{\sigma_X^2} - \frac{2\rho_{XY}(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} - \frac{(y-\mu_y)^2}{\sigma_Y^2}}{2(1-\rho_{XY}^2)} \right] \right) dy$$

Conditional expectation $E(Y|X=x)$ is given by,

$E(Y|X=x) = \int_{-\infty}^{\infty} y \frac{P_{XY(x,y)}}{P_X(x)} dy$

$$= 1/\sqrt{2\pi\sigma_Y^2(1-\rho_{XY}^2)} \int_{-\infty}^{\infty} y \exp\left(\left[\frac{(x-\mu_x)^2}{2\sigma_X^2}\right] - \left[\frac{\frac{(x-\mu_x)^2}{\sigma_X^2} - \frac{2\rho_{XY}(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} - \frac{(y-\mu_y)^2}{\sigma_Y^2}}{2(1-\rho_{XY}^2)}\right]\right) dy$$

$$= 1/\sqrt{2\pi\sigma_Y^2(1-\rho_{XY}^2)} \int_{-\infty}^{\infty} y \exp\left(-\left[\frac{-\frac{\rho_{XY}(x-\mu_x)}{\sigma_x} + \frac{(y-\mu_y)^2}{\sigma_Y^2}}{2(1-\rho_{XY}^2)}\right]\right) dy$$

let $z = -\frac{2\rho_{XY}(x-\mu_x)}{\sigma_x} + \frac{(y-\mu_y)^2}{\sigma_Y^2}$

$\implies dy = \sigma_Y dz$

$$\implies 1/\sqrt{2\pi(1-\rho_{XY}^2)} \int_{-\infty}^{\infty} \left[z\sigma_Y + \frac{\sigma_Y\rho_{XY(x-\mu_X)}}{\sigma_x} + \mu_y\right] \exp\left[-\frac{z^2}{2(1-\rho_{XY}^2}\right] dz$$

$$= \frac{\sigma_Y\rho_{XY}(x-\mu_x)}{\sigma_x} + \mu_y$$

$$= \left[\frac{\sigma_{XY}}{\sigma_X^2}\right]x + \left[\mu_y - \frac{\sigma_{XY}}{\sigma_x^2}\mu_x\right]$$

Thus $E(Y|X)$ is a linear function of X.

Question 2

Function in `MATLAB` to calculate sample covariance matrix given samples of two random variables,

```
% Function to calculate Sample Covariance Matrix given
% samples of two random variables
% October 4, 2020
% Kanchan Aggarwal

function[covmat] = SampleCov(x,y) % defining function name
    covmat = zeros(2,2); % define matrix with all zeros
    N = length(x); % calculate size of vector x
    p1 = sum(((y-mean(y)).*(x-mean(x)))/N);
    p2 = sum(((x-mean(x)).*(x-mean(x)))/N);
    p3 = sum(((y-mean(y)).*(y-mean(y)))/N);
    covmat(1,1) = p2;
    covmat(1,2) = p1;
    covmat(2,1) = p1;
    covmat(2,2) = p3;
    covmat % returns the value of covmat
end
```

Testing the function: Given $X \sim \mathcal{N}(1, 2)$ and $Y = 3X^2 + 5X$

```
% Testing the SampleCov function
% October 4, 2020
% Kanchan Aggarwal

% Generate 1000 samples of X using rnorm
mean_x = 1;
var_x = 2;
X =  mean_x + sqrt(var_x)*randn(1000,1); % Generarting the white Gaussian
    noise

% Generate Y from X
Y = 3*X.^2 + 5*X;

% Sample covariance matrix using SampleCov function
Sigma_yx1 = SampleCov(Y,X);

% Sample covariances using the routine cov
sigma_yx2 = cov(Y,X);

%% Results
sigma_yx2 =

```

```
   279.3756     20.1700
    20.1700      1.9194

Sigma_yx1 =

    279.0962     20.1498
    20.1498      1.9175
```

The sample covariance matrix obtained using user defined function is

$$\hat{\Sigma}_{yx} = \begin{pmatrix} 279.0962 & 20.1498 \\ 20.1498 & 1.9175 \end{pmatrix}$$

and using the routine `cov` is

$$\hat{\Sigma}_{yx} = \begin{pmatrix} 279.3756 & 20.1700 \\ 20.1700 & 1.9194 \end{pmatrix}$$

The estimated covariance between X and Y using the cov routine and user defined function are similar but not identical. The reason is cov routine estimates the covariance using $\dfrac{1}{N-1}\sum_{k=1}^{N}\big((x[k]-\bar{x})(y[k]-\bar{y})\big)$.
If we change the user-defined function to $\hat{\sigma}_{xy} = \dfrac{1}{N-1}\sum_{k=1}^{N}\big((x[k]-\bar{x})(y[k]-\bar{y})\big)$ instead of the expression given in the question, then both the cov routine and the user-defined function will give identical results.
Modified function in `MATLAB` to calculate sample covariance matrix given samples of two random variables,

```
% Function to calculate Sample Covariance Matrix given
% samples of two random variables
% October 4, 2020
% Kanchan Aggarwal

function[covmat] = SampleCov(x,y) % defining function name
    covmat = zeros(2,2); % define matrix with all zeros
    N = length(x); % calculate size of vector x
    p1 = sum(((y-mean(y)).*(x-mean(x)))/(N-1));
    p2 = sum(((x-mean(x)).*(x-mean(x)))/(N-1));
    p3 = sum(((y-mean(y)).*(y-mean(y)))/(N-));
    covmat(1,1) = p2;
    covmat(1,2) = p1;
```

```
    covmat(2,1) = p1;
    covmat(2,2) = p3;
    covmat % returns the value of covmat
end
```

The sample covariance matrix obtained using *modified use-defined* function is

$$\hat{\Sigma}_{yx} = \begin{pmatrix} 279.3756 & 20.1700 \\ 20.1700 & 1.9194 \end{pmatrix}$$

and using the routine `cov` is

$$\hat{\Sigma}_{yx} = \begin{pmatrix} 279.3756 & 20.1700 \\ 20.1700 & 1.9194 \end{pmatrix}$$

The estimates are identical now.

Theoretical covariance matrix:

Given, $Y = 3X^2 + 5X, \mu_x = E(X) = 1$ and $\sigma_x^2 = 2$

$$\begin{aligned} E(x^2) &= 3 \\ E(y) = \mu_y &= E(3X^2 + 5X) \\ &= 14 \end{aligned}$$

Since X follow Gaussian distribution, the third central moment will be zero, *i.e,*

$$\begin{aligned} E((X - \mu_X)^3) &= 0 \\ E(X^3) &= 7 \end{aligned}$$

Also, the fourth central moment is $3\sigma^4$,

$$\begin{aligned} E((X - \mu_X)^4) &= 12 \\ E(X^4) &= 25 \end{aligned}$$

The covariance between X and Y is

$$\begin{aligned}\sigma_{xy} &= E((X-\mu_x)(Y-\mu_y)) \\ &= E((X-1)(3X^2+5X-14)) \\ &= 22\end{aligned}$$

Similarly, $\sigma_{yx} = 22$ and σ_{yy} is

$$\begin{aligned}\sigma_{yy} = \sigma_y^2 &= E((Y-\mu_y)(Y-\mu_y)) \\ &= E((3X^2+5X-14)(3X^2+5X-14) \\ &= 314\end{aligned}$$

Therefore, the theoretical covariance matrix is

$$\Sigma_{yx} = \begin{pmatrix} 314 & 22 \\ 22 & 2 \end{pmatrix}$$

MATLAB script to show that as N increases, estimated covariance $\hat{\sigma}_{XY}$ tends to theoretical value.

```
% Consistency of the estimated covariance matrix
% October 4, 2020
% Kanchan Aggarwal
% Run through increasing data lengths
for k = 1:100000
sig_xy = cov(Y(1:k),X(1:k));
sigma_xy(:,k) = reshape(sig_xy,[],1);
end
figure;subplot(3,1,1)
plot(sigma_xy(1,:))
hold on;plot(314*ones(length(sigma_xy(1,:)),1))
ylabel('Estimated \sigma_{Y}^2');
subplot(3,1,2)
plot(sigma_xy(2,:))
hold on;plot(22*ones(length(sigma_xy(1,:)),1))
ylabel('Estimated \sigma_{XY}');
subplot(3,1,3)
plot(sigma_xy(4,:))
hold on;plot(2*ones(length(sigma_xy(1,:)),1))
ylabel('Estimated \sigma_{X}^2');xlabel('Sample size (N)')
```

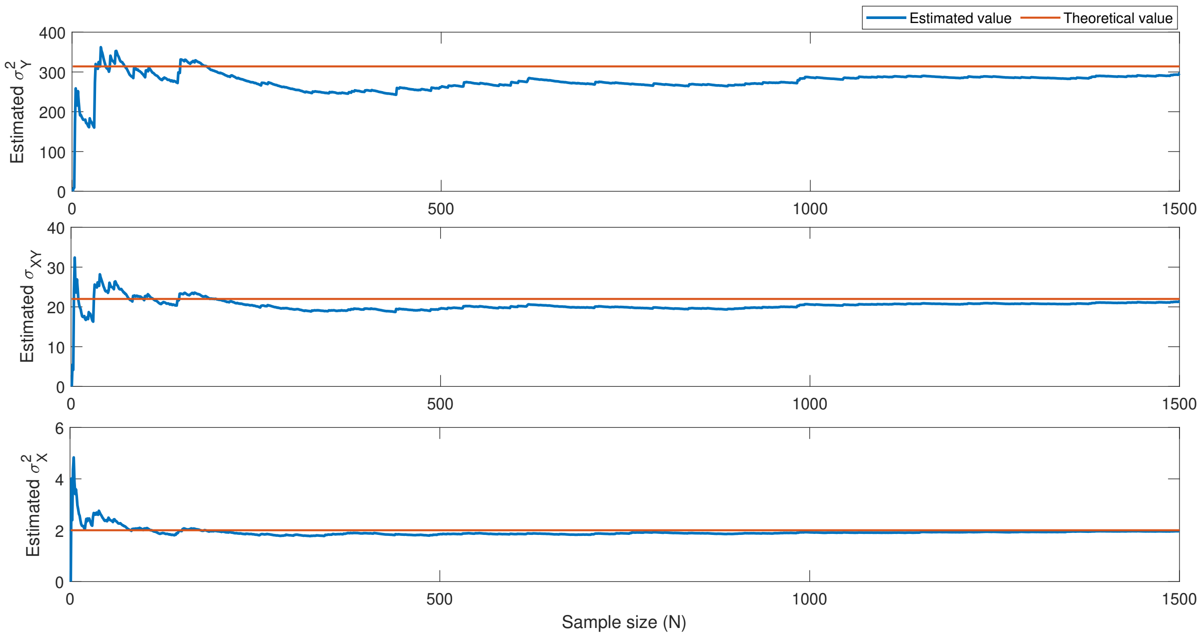


Figure 1: Estimated covariance tends to theoretical value as N increases.

As observed from Figure 1, as the sample size increases, bias in the estimated parameters tends to 0 and the parameter tends to the theoretical values.

Question 3

a

The given covariance matrix is,

$$\Sigma = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 9 & -3 \\ 2 & -3 & 25 \end{bmatrix}$$

We know that $\rho(X_i X_j) = \frac{Cov(X_i X_j)}{\sigma_{X_i} \sigma_{X_j}}$

$$\rho = \begin{bmatrix} 1 & 0.1667 & 0.2 \\ 0.`1667 & 1 & -0.2 \\ 0.2 & -0.2 & 1 \end{bmatrix}$$

b

From the properties of variance we know that,

$$\sigma^2(aX_1 + bX_2) = a^2\sigma^2(X_1) + b^2\sigma^2(X_2) + 2ab\sigma(X_1X_2)$$
$$\sigma^2(X_2/2 + X_3/2) = 7$$
$$\sigma(X_1X_4) = E(X_1 - \mu_{X_1})(X_4 - \mu_{X_4})$$

substituting $X_4 = X_2/2 + X_3/2$

$$\sigma(X_1X_4) = \frac{1}{2}E(X_1X_2) + \frac{1}{2}E(X_1X_3) = 1.5$$
$$\rho(X_1X_4) = \frac{Cov(X_1X_4)}{\sigma_{X_1}\sigma_{X_4}} = 0.2835$$

Question 4

a

`MATLAB` script to determine the optimal MAE predictor of random variable $X \sim \mathcal{X}^2(10)$

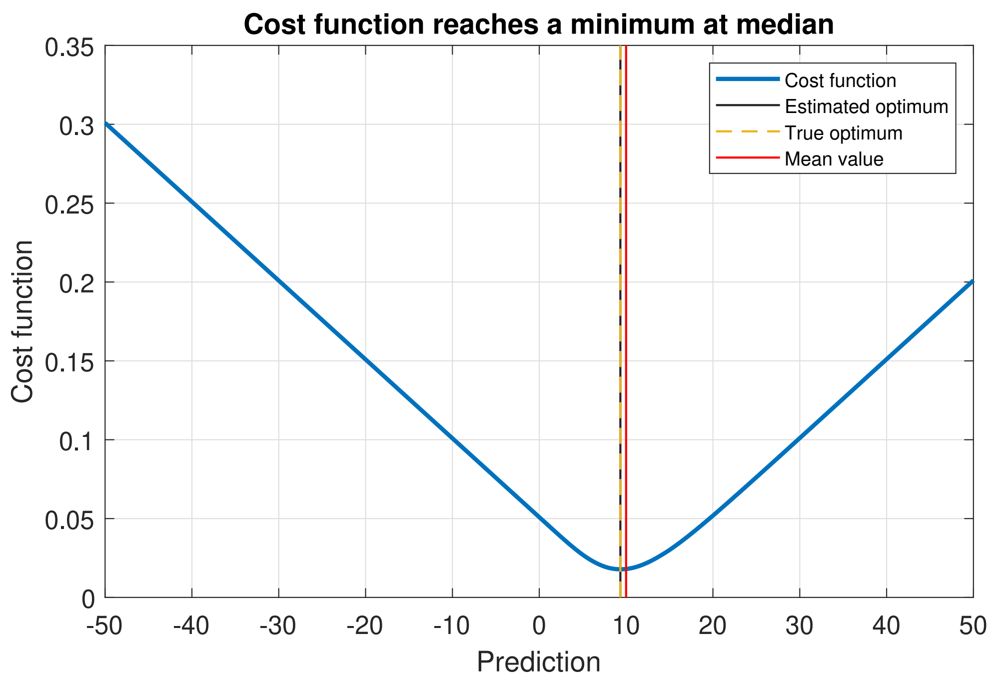
```
% optimal MAE predictor
% October 4, 2020
% Kanchan Aggarwal

R = 200; N = 5000;
% Generating data
X = chi2rnd(10,N,R);
% Estimate of mean and median for 200 realizations
med_X = median(X);
mean_X = mean(X);
mean_med = mean(med_X);
mean_mean = mean(mean_X);
% Guess for best prediction
xhat_vec = (-50:0.01:50)';
L = length(xhat_vec);
% Cost function
Jvals = zeros(L,1);
for i = 1: L
    Jvals(i) = norm(X - xhat_vec(i),1)/( N*R );
end

% Find the minimum
```

```
[mval,ival] = min(Jvals);
xhat_opt = xhat_vec(ival) ;

%Plotting the cost function
figure;plot(xhat_vec,Jvals)
hold on;plot([xhat_opt xhat_opt],[0,0.35])
hold on;plot([mean_med mean_med],[0,0.35])
hold on;plot([mean_mean mean_mean],[0,0.35])
xlabel('Prediction')
ylabel('Cost function')
```



Optimal MAE prediction is 9.3400, median is 9.3379 and mean value is 9.9995. Therefore, median is the optimal MAE predictor for random variable X.

b

The pdf of the random variable X follows $\chi^2(n) = \frac{x^{n/2-1} - e^{-x/2}}{2^{n/2}\Gamma(\frac{n}{2})}$

```
%prob_xstar = integral(@(x)chi2pdf(x,10),0.9*9.34,1.1*9.34);
%prob_xmean = integral(@(x)chi2pdf(x,10),0.9*10,1.1*10);
```

)

$\Pr(\mu_X)$= 0.1746
$\Pr(X^*)$= 0.1725

The MMSE is better predictor in this case since the probability of the random variable taking values around MMSE is higher than MMAE,