(6) a) Jeffrey's prior, $\pi(\theta) \sim \{I(\theta)\}^{1/2}$

$$f(y[k] \mid \theta) = \begin{cases} \dfrac{\theta^{y[k]} e^{-\theta}}{(y[k])!} & \text{if } y[k] \text{ is a non-negative integer} \\ 0 & \text{otherwise.} \end{cases}$$

$$L = \log\left( \prod_{k=1}^{N} \frac{\theta^{y[k]} e^{-\theta}}{(y[k])!} \right)$$

$$= \log\left(e^{-\theta}\right)^{N} + \log \theta^{\sum y[k]} + c$$

where $c$ is a constant (term independent of $\theta$)

$$\Rightarrow L = \sum_{k=1}^{N} y[k] \log\theta - N\theta + c$$

$$S = \frac{\partial L}{\partial \theta} = \frac{\sum y[k]}{\theta} - N$$

$$\Rightarrow \frac{\partial^2 L}{\partial \theta^2} = \frac{-\sum y[k]}{\theta^2}$$

$$E\left(-\frac{\partial^2 L}{\partial \theta^2}\right) = \frac{\sum E(y[k])}{\theta^2} = \frac{N\theta}{\theta^2}$$

$(\because E(y(k)) = \theta)$

$$\Rightarrow I(\theta) = \frac{1}{\theta}.$$

$\therefore$ Jeffrey's prior, $\pi(\theta) \sim \left(\frac{1}{\theta}\right)^{1/2}$

$$\sim \boxed{\theta^{-1/2}}$$

So a Data Raja's prior is in the class of Jeffrey's priors.

b) $$f(\theta | y_N) = \frac{(f(y_N|\theta)) f(\theta)}{f(y_N)}$$

$$= c \times \prod_{k=1}^{N} \frac{\theta^{y[k]} e^{-\theta}}{(y[k])!} \times \theta^{-1/2}$$

$$= \frac{c \times \theta^{\sum y[k]} e^{-N\theta} \times \theta^{-1/2}}{\prod_{k=1}^{N} (y[k])!}$$

For given $y_N$, $\prod_{k=1}^{N} y[k]!$ is a constant

$$\Rightarrow f(\theta | y_N) = c' \times \theta^{\sum y[k]} \times e^{-N\theta} \times \theta^{-1/2}$$

Jeffrey's prior doesn't change upon transforming the RV $\Rightarrow f(2N\theta) \sim f(\theta) \sim \theta^{-1/2}$

$$f(2N\theta | y_N) = \frac{f(y_N | 2N\theta) f(2N\theta)}{f(y_N)}$$

$$= c' \times \left(\frac{2N\theta}{2N}\right)^{\sum y(k)} e^{-\frac{2N\theta}{2}} \times \left(\frac{2N\theta}{2N}\right)^{-1/2}$$

Let $x = 2N\theta$,

$$f(x \mid y_N) = c'' \times (x)^{\frac{(2\sum y(k))}{2}} e^{-x/2} (x)^{-1/2}$$

$$\bar{y} = \frac{\sum y(k)}{N} \Rightarrow \sum y(k) = N\bar{y}$$

$$\Rightarrow f(x \mid y_N) = c'' \left( (x)^{\left(\frac{2N\bar{y}+1-1}{2}\right)} e^{-x/2} (x)^{-1/2} \right)$$

$$= c'' (x)^{\left(\frac{2N\bar{y}+1}{2} - 1\right)} e^{-x/2}$$

This is of the $\chi^2$ distribution form.

dof $= (2N\bar{y}+1)$. So $c'' = \dfrac{2^{-\left(\frac{2N\bar{y}+1}{2}\right)}}{\Gamma\left(\frac{2N\bar{y}+1}{2}\right)}$

$$\Rightarrow f(x \mid y_N) = \frac{1}{2^{\frac{(2N\bar{y}+1)}{2}} \Gamma\left(\frac{2N\bar{y}+1}{2}\right)} \times x^{\left(\frac{2N\bar{y}+1}{2} - 1\right)} e^{-x/2}$$

$$= \frac{1}{2^{k/2} \Gamma(k/2)} \times x^{k/2 - 1} e^{-x/2}$$

where $k = \boxed{2N\bar{y}+1}$

Mean of this distribution, $E(x) = E(2N\theta)$ is the dof $k$.

$$\Rightarrow E(2N\theta) = (2N\bar{y}+1).$$

$$\therefore \boxed{f(2N\theta \mid y_N) \sim \chi^2(2N\bar{y}+1)}$$

c)

$$\hat{\theta}_{MMSE} = \min_{\theta} E((\theta - \hat{\theta})^2)$$

$$= \min_{\theta} E((2N\theta - 2N\hat{\theta})^2) \times \frac{1}{(2N)^2}$$

$$\Rightarrow \hat{\theta}_{MMSE} = \min_{\theta} (E(2N\theta - 2N\hat{\theta})^2) \times \frac{1}{(2N)^2}$$

( $\because 2N$ is a constant for given data)

$$\hat{\theta}_{MMSE} = \frac{1}{2N} \times \min_{x} \left( \frac{E((x - 2N\hat{\theta})^2)}{(2N)^2} \right)$$

$$\hat{x} = \frac{\hat{x}_{MMSE}}{2N} \Rightarrow \hat{\theta}_{MMSE} = \frac{\hat{x}_{MMSE}}{2N}$$

& $\hat{x}_{MMSE}$ is the mean of the posterior $\chi^2(k)$
$= k$

$$\Rightarrow \hat{x}_{MMSE} = 2N\bar{y} + 1$$

$$\therefore \boxed{\hat{\theta}_{MMSE} = \bar{y} + \frac{1}{2N}}$$

d) Let $a$ be the lower end and $b$ be the higher end of the credible interval for $2N\theta$. Let $F$ be the CDF of $\chi^2$

i) If a symmetric credible interval over the MMSE estimate is required:

$$F(b) - F(a) = (1 - \alpha) \quad \text{—— (1)}$$

$$\frac{a+b}{2} = \cancel{x}\left(2N\bar{y}+1\right) \quad (2)$$

Solving eqns (1) & (2) for a and b gives the symmetric credible interval around mean estimate of $2N\theta$

ii) Unsymmetric credible interval ( ~~when $\hat{\theta}$ - MM~~

→ This interval will be unsymmetric when the ~~Median esto~~ MMSE estimate is used

→ If Median estimate is used the interval is symmetric.

→ Solve the eqns (3) & (4)

$$F(a) = 0.025 \quad (3) \qquad F(b) = 0.975 \quad (4)$$

to get a & b which will represent the end points of the interval for $2N\theta$ estimate

$$\text{CDF}, \; F(x = 2N\theta) = \frac{\gamma\left(\frac{k}{2}, \frac{x}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} ; \; k = 2N\bar{y}+1$$

$\gamma(\cdot)$ → lower incomplete gamma fn. ; $\Gamma$ → upper incomplete gamma function

∴ Interval for $\theta$ is $\boxed{\left[\frac{a}{2N}, \frac{b}{2N}\right]}$

where a & b are obtained by solving (1) & (2) or (3) & (4) as needed.