(3) a) $y[k] = A\sin(2\pi f_0 k) + e[k]$

$e[k] \sim N(0, \sigma_e^2)$

$$l = \prod_{k=0}^{N-1} \left( \frac{y[k] - A\sin(2\pi f_0 k)}{(\sqrt{2\pi}\,\sigma_e)} \right)$$

$$\Rightarrow L = \log \frac{1}{(2\pi\sigma_e^2)^{N/2}} - \frac{\sum (y(k) - A\sin(2\pi f_0 k))^2}{2\sigma_e^2} \quad \text{---} (1)$$

$$\Rightarrow \frac{\partial L}{\partial f_0} = \frac{2\pi A\cos(2\pi f_0 k)}{\sigma_e^2} \sum (y(k) - A\sin(2\pi f_0 k))\,k$$

$$\Rightarrow \frac{\partial^2 L}{\partial f_0^2} = \left(\frac{4\pi^2 A^2}{\sigma_e^2}\right)\left(\sum -y(k)k^2\sin(2\pi f_0 k) + A k^2 \sin^2(2\pi f_0 k) - A\cos^2(2\pi f_0 k)k^2\right)$$

$$\Rightarrow E\left(-\frac{\partial^2 L}{\partial f_0^2}\right) = \frac{-4\pi^2 A^2}{\sigma_e^2}\left(\sum\left(-A\sin^2(2\pi f_0 k) + A\sin^2(2\pi f_0 k) - A\cos^2(2\pi f_0 k)\right)k^2\right)$$

$\because (E(y(k)) = A\sin 2\pi f_0 k + 0 = A\sin(2\pi f_0 k))$

$$\Rightarrow E\left(\frac{-\partial^2 L}{\partial f_0^2}\right) = \frac{4\pi^2 A^2}{\sigma_e^2}\sum_{k=0}^{N-1} k^2\cos^2(2\pi f_0 k) \quad \text{---} (2)$$

$$(1) \Rightarrow \frac{\partial L}{\partial A} = -\frac{\sum \sin 2\pi f_0 k\,(A\sin(2\pi f_0 k) - y(k))}{\sigma_e^2} \quad \text{---} (3)$$

$$\Rightarrow \frac{\partial^2 L}{\partial A^2} = -\frac{\sum \sin^2 2\pi f_0 k}{\sigma_e^2} \quad (4)$$

$$\Rightarrow -E\left(\frac{\partial^2 L}{\partial A^2}\right) = \frac{\sum \sin^2(2\pi f_0 k)}{\sigma e^2} \quad \text{—— (4)}$$

$$(3) \Rightarrow \cancel{\frac{\partial L}{\partial f \partial A}} = 2\pi k \left(\cancel{\frac{A - y(k)}{\sigma e^2}}\right)$$

$$(3) \Rightarrow \frac{\partial L}{\partial f \partial A} = -\sum \left(\frac{2\pi k}{\sigma e^2}\right)\left[A\, 2 \sin 2\pi f_0 k \cos 2\pi f_0 k - y(k)\cos(2\pi f_0 k)\right]$$

$$-E\left(\frac{\partial L}{\partial f \partial A}\right) = \frac{2\pi}{\sigma e^2}\sum\left[2 A k \sin 2\pi f_0 k \cos(2\pi f_0 k) - A k \sin 2\pi f_0 k \cos 2\pi f_0 k\right]$$

$$\Rightarrow -E\left(\frac{\partial L}{\partial f \partial A}\right) = \frac{2\pi A}{\sigma e^2}\sum_{k=0}^{N-1} k \sin(2\pi f_0 k)\cos(2\pi f_0 k) \quad \text{—— (5)}$$

---

$$I(\underline{\theta}) = \begin{bmatrix} -E\left(\frac{\partial^2 L}{\partial A^2}\right) & -E\left(\frac{\partial L}{\partial f \partial A}\right) \\ -E\left(\frac{\partial L}{\partial f \partial A}\right) & -E\left(\frac{\partial^2 L}{\partial f^2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4\pi^2 A^2}{\sigma e^2}\sum k^2 \cos^2(2\pi f_0 k) & \frac{2\pi A}{\sigma e^2}\sum k \sin(2\pi f_0 k)\cos(2\pi f_0 k) \\ \frac{2\pi A}{\sigma e^2}\sum k \sin(2\pi f_0 k)\cdot\cos(2\pi f_0 k) & \frac{\sum \sin^2(2\pi f_0 k)}{\sigma e^2} \end{bmatrix}$$

For simplicity, let

$$a = \frac{4\pi^2 A^2}{\sigma_e^2} \sum k^2 \cos^2(2\pi f_0 k)$$

$$b = \frac{2\pi A}{\sigma_e^2} \sum k \sin(2\pi f_0 k) \cos(2\pi f_0 k)$$

$$d = \sum \frac{\sin^2(2\pi f_0 k)}{\sigma_e^2}$$

$$\therefore I(\theta) = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

The CRLB is given by

$$\Sigma_{\hat{\theta}} \geq (I(\theta))^{-1}$$

$$I(\theta)^{-1} = \begin{bmatrix} d & -b \\ -b & a \end{bmatrix} \frac{1}{(ad - b^2)}$$

For a +ve semidefinite matrix

$$a_{jj} \geq 0$$

$$\Rightarrow [\Sigma_{\hat{\theta}}]_{jj} \geq [(I(\theta))^{-1}]_{jj}$$

Specifically $\operatorname{var}(\hat{\theta}_1) \geq \dfrac{d}{ad - b^2}$ — (1)

&

$\operatorname{var}(\hat{\theta}_2) \geq \dfrac{a}{ad - b^2}$ — (2)

$\hat{\theta}_1 \rightarrow f_0$ $\hat{\theta}_2 \rightarrow$ Amplitude

In the case where ~~the~~ one of the parameters is known ( single unknown case),

$$\operatorname{var}(\hat{\theta}) \geq \frac{1}{-E\left(\frac{\partial^2 L}{\partial \theta^2}\right)}$$

$\Rightarrow$ the bounds are

for $\hat{\theta}_1$ (frequency ~~Amplitude~~) $= \dfrac{1}{a} = \dfrac{d}{ad}$ —— ③

& for $\hat{\theta}_2$ (Amplitude) $= \dfrac{1}{d.} = \dfrac{a}{ad}$ —— ④

① > ③ $\left( ad > ad - b^2 \Rightarrow \dfrac{d}{ad} < \dfrac{d}{ad-b^2} \right)$

② > ④ $\left( ad > ad - b^2 \Rightarrow \dfrac{a}{ad} < \dfrac{a}{ad-b^2} \right)$

We see that the lower bound is lower for both parameters in the case of single parameter unknown. (other is known).

This is intuitive because, if we already know the true value of the other parameter, the entire data is used to just get information on the single unknown. However, if both are unknown, from the same data we need to estimate 2 unknowns, resulting in decrease in information individually ~~about~~ ~~each~~ ~~one of~~ compared to earlier case

③ b) We know that $E\left((y-\mu)^2\right) = \sigma^2$, here $\mu = 0$

$\Rightarrow E(y^2) = \sigma^2$.

So we can assume that the transformed data $y \rightarrow y^2$ is coming out of a DGP such that the true mean is $\sigma^2$

$$y^2[k] = \sigma^2 + e[k] \quad \text{—— ①}$$

where $e[k]$ is an uncorrelated ($\because \rho(y[k], y[l]) = 0 \;\forall k \neq l$) and zero mean Random variable

$\sigma^2$ is the true variance of $y[k]$

In vectorial form,

$$\underline{y}_N = L\,\sigma^2 + \underline{e} \quad \text{—— ②}$$

$L$ is an $N \times 1$ vector of ones

$$L = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{N \times 1}$$

To ~~estimate~~ determine BLUE we need $\Sigma_e^*$.

Since $y[k]$ has uniform variance $\forall k$, $e[k]$ should also be a homoskedastic variable.

And $\Sigma_e$ should be diagonal, since the error terms are not correlated.

$$var(y^2) = var(\cancel{\sigma^2}^{0}) + var(e)$$

$$\Rightarrow var(e) = var(y^2) = E((y^2 - \sigma^2)^2)$$

$$= E(y^4) - (\sigma^2)^2 \quad \text{— (3)} \quad [E(X^2) - (E(X))^2]$$

Since $y$ is Gaussian we use the Moment Generating function as derived in Q2.

$$MGF: \exp\left(s\cancel{\mu}^{0} + \frac{s^2\sigma^2}{2}\right) = \exp\left(\frac{s^2\sigma^2}{2}\right)$$

$$\frac{\partial M_y(s)}{\partial s} = \frac{2\sigma^2 s}{2}\exp\left(\frac{s^2\sigma^2}{2}\right) = \sigma^2 s \exp\left(\frac{s^2\sigma^2}{2}\right)$$

$$\Rightarrow \frac{\partial^2 M}{\partial s^2} = \sigma^2 \exp\left(\frac{s^2\sigma^2}{2}\right)(1 + s^2\sigma^2)$$

$$\Rightarrow \frac{\partial^3 M}{\partial s^3} = \sigma^2 \exp\left(\frac{s^2\sigma^2}{2}\right)(2s\sigma^2 + s^3\sigma^4)$$

$$= \sigma^4 \exp\left(\frac{s^2\sigma^2}{2}\right)(\cancel{2}3s + s^3\sigma^2 + \cdots)$$ other terms don't contribute

$$\Rightarrow \frac{\partial^4 M}{\partial s^4} = \sigma^4 \exp\left(\frac{s^2\sigma^2}{2}\right)(\cancel{2}3 + 3s^2\sigma^2 + \cdots)$$

$$\therefore E(y^4) = \left.\frac{\partial^4 M}{\partial s^4}\right|_{s=0} = 3\sigma^4 \quad \text{— (4)}$$

Substitute (4) in (3),

$$var(e) = 3\sigma^4 - \sigma^4$$

$$= 2\sigma^4$$

$$\therefore \Sigma_e = \begin{bmatrix} 2\sigma^4 & & 0 \\ & 2\sigma^4 & \\ 0 & \ddots & 2\sigma^4 \end{bmatrix}_{N \times N}$$

$$\Sigma_e^{-1} = \begin{bmatrix} \frac{1}{2\sigma^4} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{2\sigma^4} \end{bmatrix}_{N\times N}.$$

We know that the solution of BLUE is,

$$A = (L^T \Sigma_e^{-1} L)^{-1} (\Sigma_e^{-1} L)$$

$$= \left( [1 \cdots 1] \begin{bmatrix} \frac{1}{2\sigma^4} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{2\sigma^4} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right)^{-1} [\Sigma_e^{-1} L]$$

$$= \left( \begin{bmatrix} \frac{1}{2\sigma^4} & \cdots & \frac{1}{2\sigma^4} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right)^{-1} \left( \begin{bmatrix} \frac{1}{2\sigma^4} \\ \frac{1}{2\sigma^4} \\ \vdots \\ \frac{1}{2\sigma^4} \end{bmatrix} \right)$$

$$= \left( \frac{2\sigma^4}{N} \right) \begin{bmatrix} \frac{1}{2\sigma^4} \\ \vdots \\ \frac{1}{2\sigma^4} \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} \frac{1}{N} \\ \frac{1}{N} \\ \vdots \\ \frac{1}{N} \end{bmatrix}_{N\times 1}$$

$$\therefore \hat{\theta}^*_{BLUE} = A\underline{y^2} = \frac{\sum^N y^2(k)}{N} = \frac{\sum_{k=0}^{N-1} y^2[k]}{N}.$$

$$\Rightarrow \boxed{\hat{\sigma}^2 = \frac{\sum y^2(k)}{N}}$$

The expression might look similar to the biased sample variance case. However there we used the sample mean, here we utilise the true mean.

So $E\left(\frac{\sum y^2}{N}\right) = \frac{N\sigma^2}{N} = \sigma^2$,

the given estimator is indeed unbiased and since it is linear and we solved the optimisation problem to get A, it is the Best Linear Unbiased Estimator

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4 a) $N = 100$ samples

$$\hat{\mu} = 14578$$

We need the confidence for $\mu$ lying in the interval

$$12000 \leq \mu \leq 16000$$

We know that

$$z_1 \leq \frac{\hat{\mu} - \mu}{\frac{\sigma}{\sqrt{N}}} \leq z_2$$

where $z_1$ & $z_2$ are dictated by the confidence level.

$$\Rightarrow \hat{\mu} - \frac{z_2 \sigma}{\sqrt{N}} \leq \mu \leq -z_1 \frac{\sigma}{\sqrt{N}} + \hat{\mu}$$

$$\frac{\hat{\mu} - z_2 \sigma}{\sqrt{N}} = 12000 \qquad \hat{\mu} = 14578 \qquad N = 100$$
$$\hat{\sigma} = 1845$$

Since $\sigma$ is not known, I am substituting the given sample deviation (assuming it is unbiased)

$$\Rightarrow z_2 = \frac{(14578 - 12000) \times 10}{1845}$$
$$= 13.97$$

$$\frac{\hat{\mu} - z_1 \sigma}{\sqrt{N}} = 16000$$
$$\Rightarrow z_1 = \frac{(14578 - 16000) \times 10}{1845}$$
$$= -7.7073$$

$\Rightarrow$ The confidence region : $-7.707 \leq z \leq 13.97$

$\therefore$ The confidence $= F(z \leq 13.97) - F(z \leq -7.707)$

$z \in N(0,1)$

is nearly 1 (very very high confidence)

We can say that the average molecular weight of the polymer is in between 12000 and 16000 with nearly 100% confidence (but not exactly 100%.)

b) $N_1 = 60$, $\bar{x}_1 = 85.2$, $s_1 = 6.8$

$N_2 = 55$, $\bar{x}_1 = 87.2$, $s_1 = 8.8$

Since both the distributions are similar, let us assume ~~both~~ unknown but equal population variance for the 2 groups

Pooled variance, $S_p^2 = \dfrac{(N_1 - 1)S_1^2 + (N_2 - 1)S_2^2}{N_1 + N_2 - 2}$ → (∵ we estimate $\bar{x}_1$ & $\bar{x}_2$ - 2 values)

$\Rightarrow S_p = 7.82$

Consider the statistic $\bar{x}_1 - \bar{x}_2$

$$var(\bar{x}_1 - \bar{x}_2) = var(\bar{x}_1) + var(\bar{x}_2) + \cancel{2cov(\bar{x}_1, \bar{x}_2)}^{0}$$

($\bar{x}_1$ and $\bar{x}_2$ are uncorrelated)

$$= \frac{S_p^2}{N_1} + \frac{S_p^2}{N_2}$$

First we will test whether $\bar{x}_1 - \bar{x}_2 = 0$ or not.

$H_0: \bar{x}_1 - \bar{x}_2 = 0 \quad \alpha = 0.05$

$H_1: \bar{x}_1 - \bar{x}_2 \neq 0$

Critical value approach: $Z \geq 1.96$ or $Z \leq -1.96$ to Reject $H_0$

$$\Rightarrow -1.96 < \frac{(\bar{x}_1 - \bar{x}_2) - 0}{\sqrt{\frac{S_p^2}{N_1} + \frac{S_p^2}{N_2}}} < 1.96$$

for $H_0$ ~~to~~ not be rejected

$$\frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{S_p^2}{N_1} + \frac{S_p^2}{N_2}}} = \frac{85.2 - 87.2}{7.82\sqrt{\frac{1}{N_2} + \frac{1}{N_1}}} = -1.37$$

Since $-1.96 \leq -1.37 \leq 1.96$, we cannot reject the Null Hypothesis that the average scores of the schools are same

∴ We conclude that on an average the institutions perform equally well.

Also note that since $\sigma^2$ is unknown we should have resorted to the t-test but since the dof is $N_1 + N_2 - 2 = 98$ very high, approximately the we can use the standard normal distribution