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CH5115 QUIZ-3.

CH18B020

② i) $f(y) = \frac{\lambda^{\cancel{y}} e^{-\lambda}}{\cancel{y}!} = \frac{\lambda^{y} e^{-\lambda}}{y!}$ $y > 0$.

$y \in \mathbb{Z}^+$ (+ve integers)

$$l = \prod_{k=1}^{N} \frac{\lambda^{y[k]} e^{-\lambda}}{(y[k])!}$$

$$\Rightarrow L = \ln l(\theta | y)$$

$$= \ln\left( \prod_{k=1}^{N} \frac{\lambda^{y[k]} e^{-\lambda}}{y[k]!} \right).$$

$$= \sum_{k=1}^{N} y(k) \ln \lambda - N\lambda - \sum \cancel{\log}(y[k]!)$$

$$\Rightarrow L = \sum_{k=1}^{N} y[k] \ln \lambda - N\lambda - \sum_{k=1}^{N} \ln (y[k]!)$$

$$\Rightarrow \frac{\partial L}{\partial \lambda} = \frac{\sum y[k]}{\lambda} - N$$

Maximise $L \Rightarrow \frac{\partial L}{\partial \lambda} = 0$

$\Rightarrow \lambda = \frac{\sum y[k]}{N}$ is the maxima of fn.

$$\frac{\partial^2 L}{\partial \lambda^2} = -\frac{\sum y[k]}{\lambda^2}$$

Since $\lambda^2 > 0$ $\forall \lambda \in \mathbb{R}^+$ and $y[k] \in \mathbb{Z}^+$ for all $k$, we have $\frac{\partial^2 L}{\partial \lambda^2} < 0$ at $\lambda = \frac{\sum y[k]}{N}$

$\therefore$ The point is indeed a maxima.

Also, $\forall \lambda > \frac{\sum y(k)}{N}$ the function is decreasing

Given constraint : $\lambda \geq b$

$$\Rightarrow \hat{\lambda} = \begin{cases} \frac{\sum y[k]}{N} & \frac{\sum y[k]}{N} \geq b \\ b. & \frac{\sum y[k]}{N} < b \end{cases}$$

This can be simplified to

$$\boxed{\hat{\lambda}_{MLE} = \max\left(\frac{\sum y[k]}{N}, b\right)}$$

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ii) $Y \sim \Gamma(1, \theta) \Rightarrow f(y) = \frac{1}{\Gamma(1)\theta^1} y^{1-1} e^{-y/\theta}$

$$\Rightarrow f(y) = \frac{e^{-y/\theta}}{\theta} \quad \forall x \geq 0$$

(exponential PDF)

~~$\forall \theta > 0\}$~~

where $\theta > 0$

$$\Rightarrow f(y) = \frac{e^{-y/\theta}}{\theta} \quad \text{——— (1)}$$

$$l = \prod \frac{e^{-y/\theta}}{\theta} = \prod_{k=1}^{N} \frac{e^{\frac{-y[k]}{\theta}}}{\theta} \quad \text{——— (2)}$$

$$\Rightarrow \mathcal{L} = -\frac{\sum y[k]}{\theta} - N \ln \theta$$

$$\Rightarrow \frac{\partial L}{\partial \theta} = \frac{+\sum_{k=1}^{N} y[k]}{\theta^2} - \frac{N}{\theta}$$

$$\Rightarrow \frac{\partial^2 L}{\partial \theta^2} = \frac{-2\sum_{k=1}^{N} y[k]}{\theta^3} + \frac{N}{\theta^2} \quad \text{--- (3)}$$

$$I(\theta) = -E\left(\frac{\partial^2 L}{\partial \theta^2}\right) = \frac{2\sum E(y[k])}{\theta^3} - \frac{N}{\theta^2} \quad \text{--- (4)}$$

[Fisher - Information]

Mean of exponential distribution $\lambda e^{-\lambda} = \frac{1}{\lambda}$

$$\Rightarrow E(y[k]) = \frac{1}{1/\theta} = \theta \quad \text{--- (5)}$$

Substituting (5) in (4) we.

$$I(\theta) = \frac{2N\theta}{\theta^3} - \frac{N}{\theta^2} \Rightarrow I(\theta) = \frac{N}{\theta^2}$$

$\Rightarrow$ Jeffrey's prior $\pi(\theta) \propto (I(\theta))^{1/2}$.

$$\Rightarrow \pi(\theta) \propto \sqrt{\frac{1}{\theta^2}}$$

$$\Rightarrow \pi(\theta) \propto \frac{1}{\theta}$$

No. Parigyna's choice is not in the list of Jeffry's prior

Correct prior : $\boxed{\pi(\theta) \propto \frac{1}{\theta}}$

b) $$f(\theta \mid y_N) = \frac{f(y_N \mid \theta)\, f(\theta)}{f(y_N)} = \frac{\pi(\theta)\, \overbrace{f(y_N \mid \theta)}^{}}{f(y_N)}$$

$\downarrow$ likelihood, $l$.

$$= \frac{C \cancel{\theta}}{\theta} \prod_{k=1}^{N} \frac{e^{-\frac{y[k]}{\theta}}}{\theta}$$

$$= \frac{C \cancel{\theta}}{\theta} \cdot \frac{\exp\left(-\sum_{k=1}^{N} \frac{y[k]}{\theta}\right)}{\theta^{N}}.$$

$$\boxed{f(\theta \mid y_N) = \frac{c}{\theta^{N+1}} \times \exp\left(-\sum \frac{y[k]}{\theta}\right)} \quad \text{— (6)}$$

where $c$ is the proportionality constant which should be adjusted to make $f(\theta \mid y_N)$ a valid pdf.

c) Consider 2 RVs $X$ & $Y$ and $Y = \frac{1}{X}$.

$$\Rightarrow F(Y \leq y) = F\left(\frac{1}{X} \leq y\right)$$

$$= F\left(X \geq \frac{1}{y}\right)$$

$$\Rightarrow F(Y \leq y) = 1 - F\left(X \leq \frac{1}{y}\right)$$

differentiate both sides wrt $y$.

$$\Rightarrow f_Y(y) = \frac{1}{y^2} f_X\left(\frac{1}{y}\right)$$

$y \to T$, $x \to \theta$ $\left(\because \text{In this problem } T = \frac{1}{\theta}\right)$

$$\Rightarrow f_T(T \mid y_N) = \frac{1}{T^2} f_X\left(\frac{1}{T}\right)$$

$$= -C \frac{T^{N+1}}{T^2} \exp\left(-T \sum_{k=1}^{N} y[k]\right)$$

$$\Rightarrow f(T \mid y_N) = -C\, T^{N-1} \exp\left(-T \sum y[k]\right)$$

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The MMSE is the mean of the above PDF.

$$\hat{T} = \int_0^\infty T f(T \mid y_N)\, dT \quad \text{—— (8)}$$

$$1 = \int_0^\infty f(T \mid y_N)\, dT \quad \text{—— (9)}$$

$$\frac{(8)}{(9)} \Rightarrow \frac{\int_0^\infty T^N \exp(-T \sum y[k]) (-c)\, dT}{\int_0^\infty T^{N-1} \exp(-T \sum y[k]) (-c)\, dT} = \hat{T} \quad \text{—— (10)}$$

Since c is a constant we can simply cancel it. We also recognise that.

$$\int_0^\infty x^N e^{-ax}\, dx = \frac{\Gamma(n+1)}{a^{n+1}}$$

(replace variable $ax$ as $u$ to get the result)

$$\Rightarrow \int_0^\infty x^N e^{-ax}\, dx = \frac{N!}{a^{N+1}} \quad \text{—— (11)}$$

$a = \sum y[k]$ ; Substituting (11) in (10)

$$\frac{N!}{(\sum y(k))^{n+1}} \times \frac{(\sum y(k))^n}{(N-1)!} = \hat{T}$$

$$\Rightarrow \hat{T} = \frac{N}{\sum_{k=1}^{N} y[k]}$$

Thus the MMSE estimator of $T$ is

$$\boxed{\hat{T} = \frac{N}{\sum_{k=1}^{N} y[k]}}$$