# CH5120 – Modern Control Theory
## Assignment - 7 - Solutions

P1: i) $$\frac{dx}{dt} = \underbrace{\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{A_c} x + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{B_c} u$$

Char eqn of $A_c$: $|A_c - \lambda I_3| = 0 \Rightarrow \begin{vmatrix} -1-\lambda & 0 & 1 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$

$$= (-\lambda)\left((-1-\lambda)(-\lambda) - 0\right) = 0$$

$$\Rightarrow \lambda^2(1+\lambda) = 0 \quad \therefore \lambda = 0, 0, -1$$

For continuous-time systems: a system is stable if the Real part of every eigenvalue is less than zero i.e

If $Re(\lambda_i) < 0$ for all $\lambda_i$, then System is stable.

Conversely, one can say, If $Re(\lambda_i) > 0$ for atleast one $\lambda_i$, then the overall system is unstable.

Here, 2 out of 3 eigenvalues are equal to zero & the other is $< 0$

$\Rightarrow$ Overall system is neither stable nor unstable i.e it is marginally stable

[Qualitatively, one can say that This result won't change irrespective of whether one's analysing the system in continuous-time or discrete-time, as the act of "sampling" has no effect on the "dynamics" of the system]

2) Discrete-time model: $x_{k+1} = Ax_k + Bu_k$ ; $h = 0.2$

* $A = e^{A_c h}$ & $B = \left\{ \int_0^h e^{A\tau} d\tau \right\} B_c$ (From prev. modules)

$$A_c = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda = 0, 0, -1$$

* To compute matrix exponential, we need to first "decompose" $A_c$ :-

Eigenvectors of $A_c$ : $(\lambda = 0)$ $\Rightarrow$ $A_c \underset{\sim}{v}_1 = 0$ $\Rightarrow$ $\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$[(A_c - \lambda I)\underset{\sim}{v} = 0]$

$\Rightarrow -x_1 + x_3 = 0$ ; $x_3 = 0$, $0 = 0$

$$\Rightarrow \underset{\sim}{v}_1 = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ $\underset{\sim}{v}_1 = [0 \ 1 \ 0]^T$ is an eigenvector.

$\lambda = 0$ has a multiplicity of 2 $\rightarrow$ Generalised eigenvector also needed.

$$\Rightarrow (A_c - 0I)\underset{\sim}{v}_2 = \underset{\sim}{v}_1$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} -x_1 + x_3 &= 0 \\ x_3 &= 1 \\ 0 &= 0 \end{aligned}$$

$$\Rightarrow \underset{\sim}{v}_2 = \begin{bmatrix} 1 \\ x_2 \\ 1 \end{bmatrix} = [1, 0, 1]^T + x_2 [0, 1, 0]^T$$

Choose $x_2 = 0$ $\Rightarrow$ $\underset{\sim}{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

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(3)

$\lambda = -1$ : $(A_c + I)\underset{\sim}{V_3} = 0$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_3 = 0 \\ x_2 + x_3 = 0 \\ x_3 = 0. \end{array}$$

$$\Rightarrow \underset{\sim}{V_3} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \text{Choose } x_1 = 1$$

$\therefore$ We can write $A_c$ as :

$$\underset{A_c}{\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}} = \underset{V}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \underset{J}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}} \underset{V^{-1}}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}$$

$$\Rightarrow e^{A_c} = V e^{J} V^{-1}$$

$$\left[ \text{If } J = \begin{bmatrix} \lambda & a & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix} \Rightarrow e^{J} = \begin{bmatrix} e^{\lambda} & ae^{\lambda} & 0 \\ 0 & e^{\lambda} & 0 \\ 0 & 0 & e^{\mu} \end{bmatrix} \right]$$

$$\Rightarrow e^{J} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-1} \end{bmatrix} \Rightarrow e^{A_c} = V e^{J} V^{-1} = \begin{bmatrix} e^{-1} & 0 & 1-e^{-1} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A = e^{A_c h} = e^{A_c \times 0.2}$$

$$= \begin{bmatrix} 0.8187 & 0 & 0.1813 \\ 0 & 1 & 0.2 \\ 0 & 0 & 1 \end{bmatrix}$$

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(4)

$$B = \left\{ \int_0^{0.2} e^{A_c \tau} d\tau \right\} B_c = \begin{bmatrix} 0.2173 \\ 0 \\ 0 \end{bmatrix}$$ (Similarly compute matrix exponent & integrate).

Chareqn of A : $|A - \lambda I_3| = 0 \Rightarrow \begin{vmatrix} 0.8187-\lambda & 0 & 0.1813 \\ 0 & 1-\lambda & 0.2 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda = 1, 1, 0.8187$

For discrete time systems, a system is stable if all poles lie inside the unit circle

Here, 2 poles lie on the unit circle $(\lambda = 1)$
& 1 pole is inside the unit circle $(\lambda = 0.8187)$.

$\therefore$ System is marginally stable [Qualitatively stated a while back]

~~[ " " zero eigen~~

3) Controllability Grammian, $W_c = [B \ AB \ A^2B]$ $(\because A \Rightarrow 3\times3)$

$B = \begin{bmatrix} 0.2173 \\ 0 \\ 0 \end{bmatrix}$ ; $AB = \begin{bmatrix} 0.8187 & 0 & 0.1813 \\ 0 & 1 & 0.2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.2173 \\ 0 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 0.1779 \\ 0 \\ 0 \end{bmatrix}$

$A^2B = \begin{bmatrix} 0.8187 & 0 & 0.1813 \\ 0 & 1 & 0.2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1779 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.1457 \\ 0 \\ 0 \end{bmatrix}$

$$\therefore W_c = \begin{bmatrix} 0.2173 & 0.1779 & 0.1457 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank}(W_c) = 1 \neq 3$$

(not full rank)

$\therefore$ The system is not controllable.

4) Hautus condition is checked for $\lambda = 1$ poles. (Unstable eigvals)

i.e $[A - \lambda I \,|\, B] = [A - I \,|\, B]$

$$= \left[ \begin{bmatrix} 0.8187 & 0 & 0.1813 \\ 0 & 1 & 0.2 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| \begin{bmatrix} 0.2173 \\ 0 \\ 0 \end{bmatrix} \right]$$

$$= \begin{bmatrix} -0.1813 & 0 & 0.1813 & 0.2173 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \text{rank} = 2 \neq 3$$

(not full rank)

$\therefore$ The system is not stabilisable

5). Qualitatively, we can conclude the given system is asymptotically stable by simply looking at the step response plots – It achieves saturation after a certain time instant.

Also, we can also say that, since we have a bounded output for a 'bounded' input (step), the system is BIBO stable which implies asymptotic stability.

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P2: Stability & System response:

CH16B004 ⇒ $A = \begin{bmatrix} 0 & -0.4 \\ 0.4 & 0.25 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

p q

& $x(k+1) = Ax(k) + Bu(k)$

6) + Eigenvalues of A ⇒ $\lambda_{1,2} = 0.125 \pm 0.38i$ ⇒ Both $\lambda$'s have a magnitude $< 1$

⇓

$|\lambda_i| \approx 0.4$

∴ System is asymptotically stable.

7) $W_c = [B \ \ AB]$ & $AB = \begin{bmatrix} 0 & -0.4 \\ 0.4 & 0.25 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.4 \end{bmatrix}$

⇒ $W_c = \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix}$ ⇒ rank$(W_c) = 2$ ⇒ full rank

∴ System is controllable.

8) $u = 0$ ⇒ $x(k+1) = Ax(k)$

⇓

$x(1) = Ax_0$ ⇒ $x(2) = Ax(1) = A^2(x_0)$

⇓

⋮

$\underline{x(10) = A^{10} x_0}$

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Given $x_0 = [1 \quad 1]^T$ ; $A^{10} = \begin{bmatrix} 0.1061 & 0.004 \\ -0.004 & 0.1035 \end{bmatrix} \times 10^{-3}$

(Using MATLAB)

$\Rightarrow x(10) = A^{10} x_0$

* Can also iterate from $k = 0$ to $9$ & solve the same *

$= \begin{bmatrix} 0.1061 & 0.004 \\ -0.004 & 0.1035 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \times 10^{-3}$

$= \begin{bmatrix} 0.1101 \\ 0.0995 \end{bmatrix} \times 10^{-3}$

P3. Controllability, Observability & Subspaces.

(Refer to the accompanying MATLAB code for this Problem)

9) $W_c = [B \ AB \ A^2B \ A^3B] = \begin{bmatrix} 0.5 & 0.375 & 0.3375 & 0.3263 \\ 0.5 & 0.375 & 0.3375 & 0.3262 \\ 0 & 0.125 & 0.1625 & 0.1738 \\ 0 & -0.125 & -0.1625 & -0.1738 \end{bmatrix}$ $\Rightarrow \text{rank}(W_c) = 2 \neq 4$

$\Downarrow$

Not controllable

10) $W_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & -2 \\ 1.7 & 0.3 & 0.3 & -1.7 \\ 1.55 & 0.45 & 0.45 & -1.55 \\ 1.475 & 0.525 & 0.525 & -1.475 \end{bmatrix}$ $\Rightarrow \text{rank}(W_o) = 2 \neq 4$

$\Downarrow$

Not Observable

Eigenvalues of $A = \{1, 0, 0.5, 0.3\}$

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$\therefore$ Applying Hautus Condition, we get:

11) $\ast$ rank$\left([A-\lambda_i I \,;\, B]\right) = 4$ only for $\lambda_i = 1$ & $0.3$

$\Rightarrow$ System is not controllable but it is stabilisable

( $\because$ full rank for $\lambda = 1$ )

12) $\ast$ rank$\left(\begin{bmatrix} A-\lambda_i I \\ C \end{bmatrix}\right) = 4$ only for $\lambda_i = 1$ & $0.5$

$\Rightarrow$ System is not observable.

13) $$\hat{A} = U^{-1} A U = \begin{bmatrix} 0.9136 & -0.351 & -0.2281 & 0.1579 \\ -0.0151 & 0.3864 & -0.1139 & -0.0088 \\ 0 & 0 & 0.3 & -0.3 \\ 0 & 0 & -0.2 & 0.2 \end{bmatrix}$$

$$\& \; \hat{B} = U^{-1} B = \begin{bmatrix} -0.6823 \\ -0.1857 \\ 0 \\ 0 \end{bmatrix}$$

$\ast$ $W_c = U \Sigma V^T$

& $x = Uz$

(coordinate transform).

Using svd( ) in MATLAB,

$$U = \begin{bmatrix} -0.6823 & -0.1857 & 0.7071 & 0 \\ -0.6823 & -0.1857 & 0.7071 & 0 \\ -0.1857 & 0.6823 & 0 & 0.7071 \\ 0.1857 & -0.6823 & 0 & 0.7071 \end{bmatrix}$$

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$S = \text{diag}(\{1.1438, 0.2412, 0, 0\})$

& $V = \begin{bmatrix} -0.5965 & -0.7699 & -0.0925 & 0.2069 \\ -0.4880 & 0.1296 & 0.5090 & -0.6971 \\ -0.4554 & 0.3995 & -0.7764 & -0.1738 \\ -0.4457 & 0.4804 & 0.36 & 0.6641 \end{bmatrix}$

→ Implies that first two columns of U (corr. to non-zero singular values) form a "controllable" subspace.

& the other 2 columns of U form a separate "Uncontrollable" subspace

Basically, performing SVD on $W_c$ allows us to identify the controllable subspace for the system, in case the overall system happens to be uncontrollable