BY: S. VISHAL
CH18B020

CH5120 ASSIGNMENT - 1

I. Change of Basis.

1 · $\underline{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\underline{x} = \alpha_1 \underline{v_1} + \alpha_2 \underline{v_2}$

$\Rightarrow \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_2 \\ \alpha_1 - \alpha_2 \end{bmatrix}$

$\Rightarrow \alpha_1 = \alpha_2 = 1 \Rightarrow \underline{x} = \underline{v_1} + \underline{v_2}$

2 · $y = Kx$.

$x : \begin{bmatrix} F_1 \\ F_2 \\ F_{out} \end{bmatrix} \quad y : \begin{bmatrix} h \\ x_B \end{bmatrix}$

Assuming the initial basis to be natural basis,

$\hat{x} = V^{-1}x$ and $\hat{y} = W^{-1}y$.

where $W = [\underline{w_1} \ \underline{w_2}]$ and $V = [\underline{v_1} \ \underline{v_2} \ \underline{v_3}]$

$\because y = Kx \Rightarrow W^{-1}y = W^{-1}KV(V^{-1}x)$

$(\because VV^{-1} = I)$

$\Rightarrow \hat{y} = (W^{-1}KV)(x)$

$W = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad V = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

$\Rightarrow V^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$, $W^{-1} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}$

$K_{new} = W^{-1} K V$

$= \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 4 & 2 & 4 \\ 0.5 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 4 & 6 & -2 \\ 0.5 & 1.5 & 1 \end{bmatrix}$

$\Rightarrow$ the new gain $= \begin{bmatrix} 2.25 & 3.75 & -0.5 \\ 1.75 & 2.25 & -1.5 \end{bmatrix}$.

Problem 2: Linear Dependence

(3) If p vectors are linearly independent, then their dimension $n \geq p$.

$\therefore$ The correct options: $n = 3, 4, 5$.
(c), (d), (e).

(4) If possible let $\underline{u}, \underline{v}, \underline{w}$ be linearly dependent.

$\Rightarrow$ there exists non-zero $\alpha_1, \alpha_2, \alpha_3$ ~~such~~ such that

$\alpha_1 \underline{u} + \alpha_2 \underline{v} + \alpha_3 \underline{w} = 0$ and at least one of $\alpha_1, \alpha_2, \alpha_3$ is non-zero.

$\Rightarrow \alpha_1(\underline{x}_1 + \underline{x}_2) + \alpha_2(\underline{x}_1 + \underline{x}_3) + \alpha_3(\underline{x}_2 + \underline{x}_3) = 0$

$\Rightarrow (\alpha_1 + \alpha_2)\underline{x}_1 + (\alpha_1 + \alpha_3)\underline{x}_2 + (\alpha_2 + \alpha_3)\underline{x}_3 = 0$

$\Rightarrow (\alpha_1 + \alpha_2)\underline{x_1} + (\alpha_1 + \alpha_3)\underline{x_2} + (\alpha_2 + \alpha_3)\underline{x_3} = 0$

We know that $\underline{x_1}, \underline{x_2}, \underline{x_3}$ are independent

$\Rightarrow \sum_i \beta_i \underline{x_i} = 0$ iff $\beta_i = 0 \ \forall i$

$\therefore \alpha_1 + \alpha_2 = 0, \ \alpha_1 + \alpha_3 = 0$ & $\alpha_2 + \alpha_3 = 0$

$$\left.\begin{array}{l} \alpha_1 + \alpha_3 = 0 \Rightarrow \alpha_1 = -\alpha_3 \\ \alpha_1 + \alpha_2 = 0 \Rightarrow \alpha_1 = -\alpha_2 \\ \alpha_2 + \alpha_3 = 0 \Rightarrow \alpha_2 = -\alpha_3 \end{array}\right\} \Rightarrow \begin{array}{l} \alpha_1 = \alpha_2 \\ \alpha_1 = -\alpha_2 \end{array}$$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$

But we assumed that at least one of $(\alpha_1, \alpha_2, \alpha_3)$ are non-zero.

$\therefore$ there is a contradiction

$\Rightarrow \underline{u}, \underline{v}$ and $\underline{w}$ are linearly independent

Problem 3 : Null and Image Spaces

$L = \begin{bmatrix} 2 & 0 & 4 \\ 0.5 & 1 & 0 \end{bmatrix}$ ($\because$ roll-no = "CH18B020")

(5) Null Space : $Lx = 0 \Rightarrow \begin{bmatrix} 2 & 0 & 4 \\ 0.5 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} 2x_1 + 4x_3 \\ \frac{x_1}{2} + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_3 = \frac{\cancel{2}.x_1}{-2}$ and $x_2 = \frac{-x_1}{2}$

$\underline{x} = \begin{bmatrix} x_1 \\ \frac{-x_1}{2} \\ -\frac{x_1}{2} \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1/2 \\ -1/2 \end{bmatrix}$

$\therefore$ all points lying on the line along the vector $(1, -0.5, -0.5)$ belong to the NULLSPACE

Normalised value : $[-0.8165 \quad 0.4082 \quad 0.4082]$

Image space : vector space formed by columns of L

$\Rightarrow x_1 \begin{pmatrix} 2 \\ 0.5 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 4 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 2x_1 + 4x_3 \\ x_1 + 0.5x_2 \end{pmatrix}$

We observe that the 2 values can be manipulated independently by modifying x_2 & x_3

$\Rightarrow$ Image space is whole of $\mathbb{R}^2$ space.

(Spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$)

Problem 4: Eigenvalue decomposition and Matrix Exponent.

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$$B = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$

(6) $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ 2 & 0-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - \lambda = 0$. - characterstic equation.

$$\boxed{\lambda = 0, 1}$$

(7) $LHS = B^2 - B$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = RHS$$

$\therefore$ The Cayley Hamilton theorem is verified.

(8) $B\underline{x} = \lambda \underline{x}$

i) $\lambda = 0 \Rightarrow \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 = 0. \\ x_2 = 1 \end{array}$

ii) $\lambda = 1 \Rightarrow \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow 2x_1 = x_2$

$\Rightarrow$ the eigen vectors are $\boxed{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1/\sqrt{5} \\ \frac{2}{\sqrt{5}} \end{bmatrix}}$

$$V = \begin{bmatrix} 0 & \frac{1}{\sqrt{5}} \\ 1 & \frac{2}{\sqrt{5}} \end{bmatrix} \Rightarrow V^{-1} = \begin{bmatrix} \frac{-2}{\sqrt{5}} & 1 \\ \sqrt{5} & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \sqrt{5} & 0 \end{bmatrix}$$

$$B = V \Lambda V^{-1} = \begin{bmatrix} 0 & \frac{1}{\sqrt{5}} \\ 1 & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ \sqrt{5} & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/\sqrt{5} \\ 1 & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ \sqrt{5} & 0 \end{bmatrix}$$

eigen vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ corresponds to $\lambda = 0$

and eigen vector $\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ corresponds to $\lambda = 1$

(9) $B = V \Lambda V^{-1}$

$\Rightarrow e^{B} = V e^{\Lambda} V^{-1}$

$$e^{\Lambda} = \begin{bmatrix} e^{0} & 0 \\ 0 & e^{1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e \end{bmatrix}$$

$$\therefore e^{B} = \begin{bmatrix} 0 & 1/\sqrt{5} \\ 1 & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e \end{bmatrix} \begin{bmatrix} -2 & 1 \\ \sqrt{5} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1/\sqrt{5} \\ 1 & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} -2 & 1 \\ \sqrt{5}e & 0 \end{bmatrix}$$

$$\Rightarrow e^{B} = \begin{bmatrix} e & 0 \\ -2+2e & 1 \end{bmatrix} = \begin{bmatrix} 2.7183 & 0 \\ 3.4366 & 1 \end{bmatrix}$$

(10) Problem 5: Jordan decomposition.

$$C = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \qquad |C - \lambda I| = 0$$

$$\Rightarrow (1-\lambda)(3-\lambda) + 1 = 0$$

$$\Rightarrow 3 + \lambda^2 - 4\lambda + 1 = 0.$$

$$\Rightarrow \lambda = 2, 2 \text{ (repeated roots)}$$

$C\underline{x} = \lambda \underline{x}$ (eigen vector)

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

$$\Rightarrow \left.\begin{array}{l} x_1 + x_2 = 2x_1 \\ -x_1 + 3x_2 = 2x_2 \end{array}\right\} \Rightarrow x_1 = x_2$$

∴ The eigen. vector is $\underline{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Generalised eigen vector: $(A - \lambda I)\,\underline{v_2} = \underline{v_1}$

$$\Rightarrow \left(\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -x_1 + x_2 \\ x_2 - x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow x_2 - x_1 = 1$$

$\Rightarrow$ the family of vectors satisfying the generalised eigen vector eqn: $\begin{bmatrix} \beta \\ \beta + 1 \end{bmatrix}$

where β is an arbitrary ~~value~~ real number.

For simplicity, let $\beta = 0$, $\underline{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\therefore \quad \underline{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \underline{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

1b) $X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow X^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$\Rightarrow C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

This is the "Jordan Canonical Form".

If $C = X \Lambda X^{-1}$, it means,

$$e^C = X e^{\Lambda} X^{-1}$$

$\Rightarrow e^C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} e^{\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

also $\exp\left(\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\right) = \begin{bmatrix} e^{\lambda} & e^{\lambda} \\ 0 & e^{\lambda} \end{bmatrix}$.

$\Rightarrow e^C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\lambda} & e^{\lambda} \\ 0 & e^{\lambda} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^2 & e^2 \\ 0 & e^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$$\Rightarrow \exp(C) = \begin{bmatrix} e^2 & e^2 \\ e^2 & 2e^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & e^2 \\ -e^2 & 2e^2 \end{bmatrix}$$

$$\Rightarrow \boxed{\exp(C) = \begin{bmatrix} 0 & 7.3891 \\ -7.3891 & 14.7781 \end{bmatrix}}$$