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CH5120 MODERN CONTROL THEORY ASSIGNMENT-2

Problem 1: Step-response Model.

① $Y(s) = G(s)\,u(s)$

Step input in laplace domain: $u(s) = 1/s$

$\Rightarrow Y(s) = \left(\frac{5}{\tau s+1}\right)\frac{1}{s}$

$\Rightarrow y(t) = L^{-1}\left(5\left(\frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}\right)\right)$

$\Rightarrow y(t) = 5\left(1-e^{-t/\tau}\right)$

For a finite ~~impulse~~ step we know that,

$y_i(k+1) = y_{i+1}(k) + \Delta u_k S_{i+1}$

But $\Delta u_k = 0$ (only one step)

$\Rightarrow y_i(k+1) = y_{i+1}(k) = y_0(k+i+1)$

The step response coefficients (parameters) are simply the response at that instant.

$S_k = y(k) = 5\left(1-e^{-\frac{\Delta t\, k}{\tau}}\right)$

$\Delta t = 0.5$, $\tau = 0.5(1+9)$ (rollno: CH18B020)
$= 0.5$

$\Rightarrow \boxed{S_k = 5\left(1-e^{-k}\right)}$

$$\Rightarrow \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ \vdots \\ \vdots \\ S_n \end{bmatrix} = \begin{bmatrix} 3.1606 \\ 4.3233 \\ 4.7511 \\ \vdots \\ 5(1-e^{-n}) \end{bmatrix}$$

$S_4 = 4.9084$
$S_5 = 4.966$
$S_6 = 4.987$
$S_7 = 4.995$
$S_8 = 4.998$

(2) $$y(s) = G(s)\,u(s) = \frac{5e^{-0.15s}}{(\tau s+1)}\left(\frac{1}{s}\right)$$

$$\Rightarrow y(t) = 5\left(L^{-1}\left(\frac{e^{-\theta s}}{s} - \frac{e^{-\theta s}}{s+\frac{1}{\tau}}\right)\right)$$

$$\Rightarrow y(t) = \begin{cases} 0 & t < \theta \\ 5\left(1-e^{-\left(\frac{t-\theta}{\tau}\right)}\right) & t \geq \theta \end{cases} \quad \text{— (1)}$$

<u>derivation</u>: $$L^{-1}\left(\frac{5e^{-\theta}}{s} - \frac{5e^{-\theta s}}{s+\frac{1}{\tau}}\right) = L^{-1}\left(\frac{5e^{-\theta}}{s}\right) - L^{-1}\left(\frac{5e^{-\theta s}}{s+\frac{1}{\tau}}\right)$$

<u>Time domain shifting property</u>: $$L\{f(t-T)\,u(t-T)\} = e^{-sT}F(s)$$

$$\Rightarrow L^{-1}\left(\frac{e^{-\theta}}{s}\right) = \begin{cases} 0 & t < \theta \\ 5 & t \geq \theta \end{cases} \quad \text{— (2)}$$

Combine (2) and (3) to get (1)

$$L^{-1}\left(\frac{e^{-\theta s}}{s+\frac{1}{\tau}}\right) = \begin{cases} 0 & t < \theta \\ 5e^{-(t-\theta)/\tau} & t \geq \theta \end{cases} \quad \text{— (3)}$$

((1) = (2) - (3))

$$y(t) = \mathcal{L}^{-1}\left(\frac{5}{s}e^{-\theta s}\right) - \mathcal{L}^{-1}\left(\frac{5e^{-\theta s}}{s+\frac{1}{\tau}}\right)$$

From ② & ③,

$$y(t) = \begin{cases} 5\,0 - 0 & t < \theta. \\ 5 - 5e^{-\frac{(t-\theta)}{\tau}} & t \geq \theta \end{cases}$$

$$\Rightarrow \quad y(t) = \begin{cases} 0 & t < \theta \\ 5\left(1 - e^{-\left(\frac{t-\theta}{\tau}\right)}\right) & t \geq \theta \end{cases}$$

$$\Rightarrow y[k] = \begin{cases} 0 & k\Delta t < \theta \\ 5\left(1 - e^{-\left(\frac{k\Delta t - \theta}{\tau}\right)}\right) & k\Delta t \geq \theta \end{cases}$$

$\underline{\theta = 0.15 \text{ seconds}}$

$$\Rightarrow y[k] = \begin{cases} 0 & k < 0.3 \\ 5\left(1 - e^{+0.3 - k}\right) & k \geq 0.3 \end{cases}$$

$$\Rightarrow \boxed{S_k = \begin{cases} 0 & k < 0.3 \\ 5\left(1 - e^{0.3 - k}\right) & k \geq 0.3 \end{cases}}$$

$k \in \{\cancel{0}\, 1, 2, 3, \ldots n\}$

$S_1 = 2.5171$, $S_2 = 4.0816$, $S_3 = 4.66$, $S_4 = 4.88$

$S_5 = 4.95$, $S_6 = 4.98$, $S_7 = 4.9946$, $S_8 = 4.998 \ldots$

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(3) From the previous part we know that,

$$y[k] = \begin{cases} 0 & k < \theta/\Delta t \\ 5\left(1 - e^{-\left(\frac{k\Delta t - \theta}{\tau}\right)}\right) & k\Delta t \geq \theta \end{cases}$$

$\underline{\theta = 1.5 s}$

$$\Rightarrow y[k] = \begin{cases} 0 & k < 3. \\ 5\left(1 - e^{-(k-3)}\right) & k \geq 3 \end{cases}$$

$$\boxed{S_k = \begin{cases} 0 & k < 3 \\ 5\left(1 - e^{-(k-3)}\right) & k \geq 3 \end{cases}}$$

$\Rightarrow S_1 = S_2 = S_3 = 0$

$S_4 = 3.16,\ S_5 = 4.32,\ S_6 = 4.75,\ S_7 = 4.91,$

$S_8 = 4.966,\ S_9 = 4.988,\ S_{10} = 4.995$

Part-2 : Impulse Response Model

(4) $Y(s) = G(s)\,u(s)$

$$= \frac{5}{\tau s+1}\left(\frac{1}{s}\right)\left(1-e^{-\Delta t s}\right)$$

$$\Rightarrow y(t) = L^{-1}\left[\left(\frac{5}{s} - \frac{5}{s+\frac{1}{\tau}}\right) - \left(\frac{e^{-\Delta t s}\times 5}{s} - \frac{e^{-\Delta t s}\times 5}{s+\frac{1}{\tau}}\right)\right]$$

$$L^{-1}\left(\frac{5}{s}\right) = 5;\quad L^{-1}\left(\frac{5}{s+\frac{1}{\tau}}\right) = 5e^{-t/\tau}$$

$$L^{-1}\left(\frac{e^{-\Delta t s}\times 5}{s}\right) = 5u(t-\Delta t) = \begin{cases} 0 & t < \Delta t \\ 5 & t \geq \Delta t \end{cases}$$

$$L^{-1}\left(\frac{e^{-\Delta t s}\times 5}{s+\frac{1}{\tau}}\right) = 5u(t-\Delta t)\left(e^{-\left(\frac{t-\Delta t}{\tau}\right)}\right)$$

$$= \begin{cases} 0 & t < \Delta t \\ 5e^{-\left(\frac{t-\Delta t}{\tau}\right)} & t \geq \Delta t \end{cases}$$

$$y(t) = \begin{cases} 5\left(1 - e^{-t/\tau}\right) & t < \Delta t \\ 5\left(\exp\left(-\left(\dfrac{t - \Delta t}{\tau}\right)\right) - \exp\left(\dfrac{-t}{\tau}\right)\right) & t \geq \Delta t \end{cases}$$

$\tau = \Delta t = 0.5\,s$; $t = k\,\Delta t$.

$$\Rightarrow y[k] = \begin{cases} 5(1 - \exp(-k)) & k < \dfrac{\Delta t}{\Delta t} \\ 5(\exp(-(k-1)) - \exp(-k)) & k \geq \dfrac{\Delta t}{\Delta t} \end{cases}$$

$$\Rightarrow h_k = \begin{cases} 5(1 - \exp(-k)) ; & k < 1 \\ 5(\exp(-(k-1)) - \exp(-k)) ; & k \geq 1 \end{cases}$$

$$\Rightarrow \boxed{h_k = 5(\exp(-(k-1)) - \exp(-k)) ; \quad k \geq 1}$$

$h_1 = 3.161$, $h_2 = 1.1627$, $h_3 = 0.42777$, $h_4 = 0.157$, $h_5 = 0.0579$, $h_6 = 0.021$, $h_7 = 0.0078$, $h_8 = 0.0029$, $h_9 = 0.0011$, $h_{10} = 0.00039$

(5) Relationship between step and impulse response coefficients : $h_k = S_k - S_{k-1}$.

From question (1),

$$S_k = 5(1 - \exp(-k)).$$

$$\therefore h_k = \begin{cases} 5(1 - \exp(-k)) & k = 1 \\ 5(\exp(-(k-1)) - \exp(-k)) & k > 1 \end{cases}$$

$$\Rightarrow h_k = 5(\exp(-(k-1)) - \exp(-k))$$ ($\because$ function is continuous at $k = 1$)

∴ The expressions obtained in q 4 and q 5 are same.

⇒ Our derivations in q 1, q 4 are valid.

$h_1 = 3.1606$, $h_2 = 1.1627$, $h_3 = 0.4277$, $h_4 = 0.1574$

$h_5 = 0.0579$, $h_6 = 0.0213$, $h_7 = 0.0078$, $h_8 = 0.0029$.

$h_9 = 0.0011$, $h_{10} = 3.9 \times 10^{-4}$

Hence verified.

In vectorial form,

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \\ h_{10} \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \\ S_8 \\ S_9 \\ S_{10} \end{bmatrix} - \begin{bmatrix} 0 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \\ S_8 \\ S_9 \end{bmatrix} = \begin{bmatrix} 3.1606 \\ 1.1627 \\ 0.4277 \\ 0.1574 \\ 0.0579 \\ 0.0213 \\ 0.0078 \\ 0.0029 \\ 0.0011 \\ 3.9 \times 10^{-4} \end{bmatrix}$$

(S_k values are obtained from qn 1)

these h_k values are same as obtained in q 4.

Final coefficients for all questions

(1)

$$\left[\begin{array}{c} 3.1606 \\ 4.323 \\ 4.751 \\ 4.908 \\ 4.966 \\ 4.987 \\ 4.995 \\ 4.998 \\ 4.9994 \\ 4.9998 \end{array}\right] \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \\ s_8 \\ s_9 \\ s_{10} \end{array}$$

(Step response parameters)

(2)

$$\left[\begin{array}{c} 2.5171 \\ 4.086 \\ 4.664 \\ 4.876 \\ 4.954 \\ 4.983 \\ 4.994 \\ 4.997 \\ 4.9992 \\ 4.9997 \end{array}\right] \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \\ s_8 \\ s_9 \\ s_{10} \end{array}$$

Step response with delay $\theta = 0.15$

(3)

$$\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 3.1606 \\ 4.323 \\ 4.751 \\ 4.908 \\ 4.966 \\ 4.988 \\ 4.995 \\ 4.998 \\ 4.9994 \\ 4.9998 \end{array}\right] \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \\ s_8 \\ s_9 \\ s_{10} \\ s_{11} \\ s_{12} \\ s_{13} \end{array}$$

Step response with delay $\theta = 1.5$

(4)

$$\left[\begin{array}{c} 3.1606 \\ 1.163 \\ 0.428 \\ 0.157 \\ 0.0579 \\ 0.0213 \\ 0.0078 \\ 0.0029 \\ 0.0011 \\ 3.9005 \times 10^{-4} \end{array}\right] \begin{array}{c} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \\ h_{10} \end{array}$$

Impulse response parameters

(5)

$$\left[\begin{array}{c} 3.1606 \\ 1.163 \\ 0.428 \\ 0.157 \\ 0.0579 \\ 0.0213 \\ 0.0078 \\ 0.0029 \\ 0.0011 \\ 3.9005 \times 10^{-4} \end{array}\right] \begin{array}{c} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \\ h_{10} \end{array}$$

Impulse response parameters calculated using step response parameters. h values same as the ones calculated in q4.