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CH5120 : MODERN CONTROL THEORY CH18B020

① Problem 1 : Model Conversion.

$\tau = 0.5(1+a) = 0.5(1+0) = 0.5$

$\alpha = \frac{1}{\tau} = 2.$

$$\frac{dx}{dt} = (-\alpha)x + (5\alpha)u \Rightarrow \frac{dx}{dt} + (\alpha)x = (5\alpha)u$$

$$\Rightarrow \left(\frac{dx}{dt} + \alpha x\right) e^{\alpha t} = (5\alpha)u(e^{\alpha t})$$

$$\Rightarrow \int_{x(k)\exp(k\Delta t\alpha)}^{x(k+1)\exp((k+1)\Delta t)} d(x e^{\alpha t}) = \int_{k\Delta t}^{(k+1)\Delta t} 5\alpha (e^{\alpha t}) u \, dt.$$

$$\Rightarrow (x(k+1)) \exp[(k+1)\alpha\Delta t] - x(k)\exp[k\Delta t\alpha] = 5\alpha \int_{k\Delta t}^{(k+1)\Delta t} e^{\alpha t} u \, dt$$

If the system follows a zero order hold,

$u = u(k) =$ constant for the entire time from $k\Delta t$ to $(k+1)\Delta t$

$$\Rightarrow (x(k+1)) \exp((k+1)\alpha\Delta t) - x(k)\exp(k\alpha\Delta t)$$

$$= 5\left(-\exp(k\Delta t\alpha) + \exp((k+1)\alpha\Delta t)\right) u(k)$$

Dividing both sides by $\exp((k+1)\Delta t\alpha)$

$$\Rightarrow x(k+1) - x(k)\exp(-\alpha\Delta t)$$

$$= 5(1 - \exp(-\alpha\Delta t))\, u(k)$$

Substituting $\alpha = 2$ and $\Delta t = 0.5$,

$$x(k+1) = (x(k))\, e^{-0.1} + 5\left(1 - e^{-0.1}\right) u(k)$$

$$= (0.9048)\, x(k) + (0.0952)5\, u(k)$$

$$= (0.9048)\, x(k) + (0.4758)\, u(k)$$

comparing with $x(k+1) = A\,x(k) + B\,u(k)$,

$A = 0.9048$ and $B = 0.4758$.

$y = (1)x \Rightarrow y(k) = x(k)$

$\Rightarrow C = C^{\text{continous}} = \underline{1}$

$\therefore$ $$\boxed{\begin{aligned} x(k+1) &= 0.9048\, x(k) + 0.4758\, u(k) \\ y(k) &= x(k) \end{aligned}}$$

(2) Input delay $= 1.5$ sec. $= \dfrac{1.5}{\Delta t}$ steps $= \dfrac{1.5}{0.5}$

= 3 time steps.

Since there is a delay of 3 time steps (exactly) we replace $u(k)$ as $u(k-3)$ in the expression obtained in the previous question

$$\Rightarrow x(k+1)\exp((k+1)\Delta t\,\alpha) - x(k)\exp(k\Delta t\,\alpha) = \left(5\alpha \int_{k\Delta t}^{(k+1)\Delta t} e^{\alpha t}\, dt\right) u(k-3)$$

$$\Rightarrow x(k+1)\, e^{(k+1)\Delta t\alpha} - x(k)\, e^{k\Delta t\alpha} = 5\left(e^{(k+1)\Delta t\alpha} - e^{k\Delta t\alpha}\right) u(k-3)$$

$$\Rightarrow x(k+1) = (x(k))\, e^{-\alpha \Delta t} + 5\left(1 - e^{-\alpha \Delta t}\right) u(k-3)$$

$$\Rightarrow \boxed{\begin{array}{l} x(k+1) = 0.9048\, x(k) + 0.4758\, u(k-3) \\ y(k) = x(k). \\ A = 0.9048,\ B = 0.4758,\ C = 1 \end{array}}$$

Problem 2: Model Linearisation

$C_{in} = 5 + 0.2 \times a = 5 + 0.2 \times 0.$

$\Rightarrow C_{in} = 5$ units.

(3) $A = 0.2$, $K = 0.5$, $k = 1.5$, $C_{in} = 5$, $F_{in} = 0.5$

$$\frac{dh}{dt} = \frac{1}{A}\left(F_{in} - K\sqrt{h}\right) \quad \text{--- (1)}$$

$$\frac{dc}{dt} = \frac{F_{in}}{Ah}\left(C_{in} - c\right) - k C^2. \quad \text{--- (2)}$$

At steady state $\frac{dh}{dt} = 0$ and $\frac{dc}{dt} = 0.$

(1) $\Rightarrow \frac{1}{A}\left(F_{in} - K\sqrt{h_s}\right) = 0.$

$$\Rightarrow h_{steady\text{-}state} = \left(\frac{F_{in}}{K}\right)^2$$

$$\Rightarrow h_{ss} = \left(\frac{0.5}{0.5}\right)^2 = \boxed{1}$$

subscript ss denotes steady state.

(2) $\Rightarrow \frac{F_{in}}{A h_{ss}}\left(C_{in} - C_{ss}\right) - k C_{ss}^2 = 0$

$$\Rightarrow \frac{0.5}{0.2 \times 1}\left(5 - c_{ss}\right) - \frac{3}{2} c_{ss}^2 = 0$$

$\Rightarrow 3c_{ss}^2 + 5c_{ss} - 25 = 0.$

Taking only the +ve root ($\because c_s \geq 0$),

$$c_{ss} = \frac{-5 \pm \sqrt{25+300}}{6} = 2.171$$

$$\Rightarrow \boxed{c_{ss} = 2.171}$$

4 $$\frac{dh}{dt} = f(x,u) = \frac{1}{A}\left(F_{in} - K\sqrt{h}\right)$$

$$f(x,u) = \cancel{f(x_{ss}, u_{ss})}^{0} + \frac{\partial f}{\partial F_{in}}\left(F_{in} - F_{in,ss}\right) + \frac{\partial f}{\partial h}(h - h_{ss})$$

$$= \frac{(F_{in} - F_{in,ss})}{A} - \frac{K}{2\sqrt{h_{ss}}A}(h - h_{ss})$$

writing in terms of deviation variables,

$$(\tilde{\beta} = \beta - \beta_{ss})$$

$$\frac{dh}{dt} = \frac{1}{A}\tilde{F}_{in} + \left(-\frac{K}{2\sqrt{h_{ss}}A}\right)\tilde{h}$$

Note that $\frac{d\tilde{h}}{dt} = \frac{dh}{dt}$

$$\Rightarrow \frac{d\tilde{h}}{dt} = \frac{1}{A}\tilde{F}_{in} + \left(\frac{-K}{2A\sqrt{h_{ss}}}\right)\tilde{h} \quad \text{--- (1)}$$

$$\frac{dc}{dt} = g(x,u) = \frac{F_{in}}{Ah}(c_{in} - c) - (kc^2)$$

$$\Rightarrow g(x,y) = \cancel{g(x_{ss}, u_{ss})}^{0} + \frac{\partial g}{\partial F_{in}}(F_{in} - F_{in\,ss})$$

$$+ \frac{\partial g}{\partial c}(c - c_{ss}) + \frac{\partial g}{\partial h}(h - h_{ss})$$

$$\frac{dc}{dt} = \left(\frac{C_{in} - C_{ss}}{A h_{ss}}\right)(F_{in} - F_{in,ss}) + \left(\frac{-F_{in}}{A h_{ss}^2}\right)(C_{in} - C_{ss}) \times (h - h_{ss})$$

$$+ \left(-2kC_{ss} \cdot \cancel{\ } \frac{F_{in,ss}}{A h_{ss}}\right)(C - C_{ss})$$

In terms of deviation variables,

$$\Rightarrow \frac{d\tilde{c}}{dt} = \left(\frac{C_{in} - C_{ss}}{A h_{ss}}\right)\tilde{F}_{in} + \left(\frac{-F_{in,ss}(C_{in} - C_{ss})}{A h_{ss}^2}\right) \times \tilde{h}$$

$$+ \left(-2kC_{ss} - \frac{F_{in,ss}}{A h_{ss}}\right)\tilde{c} \qquad \text{—— (2)}$$

Substituting the values,

$$(1) \Rightarrow \frac{d\tilde{h}}{dt} = 5\tilde{F}_{in} + (-1.25)\tilde{h}$$

$$= \cancel{5} -1.25\tilde{h} + \cancel{5}0\tilde{c} + 5\tilde{F}_{in} \qquad \text{—— (3)}$$

$$(2) \Rightarrow \frac{d\tilde{c}}{dt} = 14.1435\tilde{F}_{in} \cancel{+} 7.072\tilde{h} - 9.01\cancel{4}\tilde{c} \qquad \text{—— (4)}$$

$$\underline{\tilde{x}} = \begin{bmatrix} \tilde{h} \\ \tilde{c} \end{bmatrix}, \quad \underline{\tilde{u}} = \begin{bmatrix} \tilde{F}_{in} \end{bmatrix}$$

~~Defining deviation~~.

Denoting deviation variables with a ~~*~~ prime "$'$" rather than tilde '$\sim$',

P. T. O.

$$\frac{dx'}{dt} = \begin{bmatrix} -1.25 & 0 \\ -7.072 & -9.014 \end{bmatrix} x' + \begin{bmatrix} 5 \\ 14.145 \end{bmatrix} u'$$

$$y' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x'$$

$$\boxed{A^C = \begin{bmatrix} -1.25 & 0 \\ -7.072 & -9.014 \end{bmatrix} ; \quad B^C = \begin{bmatrix} 5 \\ 14.145 \end{bmatrix}}$$

Problem 3: MIMO system.

(5) $G_{11} \Rightarrow x_{11}(k+1) = A_{11}\, x_{11}(k) + B_{11}\, u(k)$ & $y_{11}(k) = C_{11}\, x_{11}(k)$

Similarly for G_{12}, G_{21}, G_{22} we can convert to state-space

$x_{ij}(k)$ is a 2×1 vector $\Rightarrow$ total no. of states $= 4 \times 2 = 8$

$u(k)$ is a 2×1 vector (2 inputs)

$y(k)$ is a 2×1 vector (2 outputs)

$$\begin{bmatrix} x_{11}(k+1) \\ x_{12}(k+1) \\ x_{21}(k+1) \\ x_{22}(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & 0 & 0 \\ 0 & A_{12} & 0 & 0 \\ 0 & 0 & A_{21} & 0 \\ 0 & 0 & 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_{11}(k) \\ x_{12}(k) \\ x_{21}(k) \\ x_{22}(k) \end{bmatrix} + \begin{bmatrix} B_{11} & 0 \\ 0 & B_{12} \\ B_{21} & 0 \\ 0 & B_{22} \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$$

$$y(k) = \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 \\ 0 & 0 & C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{bmatrix}$$

Combining all the eqns to a single MIMO model,

$$A\ \cancel{x(k+1)} = \begin{bmatrix} 0.42 & -0.27 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.17 & 0.96 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.43 & -0.28 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.35 & 0.92 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.27 & -0.47 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.3 & 0.86 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.48 & -0.32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.18 & 0.96 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.68 & 0 \\ 0.097 & 0 \\ 0 & 0.35 \\ 0 & 0.1 \\ 0.59 & 0 \\ 0.18 & 0 \\ 0 & 0.72 \\ 0 & 0.1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0.8 & 0 & 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.96 & 0 & 0.44 \end{bmatrix}$$

$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = Cx(k)$$

$y \in \mathbb{R}^2$, $u \in \mathbb{R}^2$, $x \in \mathbb{R}^8$

A : 8 x 8 matrix B : 8x2 matrix

C : 2x8 matrix