Assignment 10

Due: Friday, 27th November at 11:5p pm

- Please submit your work on MATLAB Grader
- Your solutions will be manually graded. We will **not use auto-grading** in Grader

Problem Description

Consider a distillation column which has state $x \in \mathcal{R}^{20}$. Consider the following definitions:

$$u = \begin{bmatrix} L \\ V \end{bmatrix}, \qquad w = \begin{bmatrix} F \\ x_F \end{bmatrix}, \qquad y_c = \begin{bmatrix} x_D \\ x_B \end{bmatrix}, \qquad y_m = \begin{bmatrix} T_D \\ T_B \end{bmatrix}$$

- Manipulated inputs (u): Reflux and reboiler flowrates
- Unmeasured disturbances (w): Feed flowrate and composition
- Measured outputs (y_m): Temperatures
- Unmeasured outputs (y_c): Distillate and bottoms compositions

A MATLAB file is uploaded, which provides system matrices for the model:

$$x(k+1) = Ax(k) + Bu(k) + B_w w(k)$$
$$y_m(k) = Cx(k) + v(k)$$
$$y_c(k) = Hx(k)$$

The *unmeasured* disturbances, $w(k)$, are non-stationary in nature. In Problem-1, we will setup state estimator using Kalman Filter and in Problem-2, we will simulate LQR control

Problem 1: State Estimation **(14 points)**

Problem 1a: Kalman Filter with $w(k)$ assumed to be white noise sequences **(6 points)**

Assume $w(k)$ is a white noise sequence with the covariances R_1 and R_2 as given in the uploaded file. With this, use an unsteady-state Kalman filter:

$$\hat{x}(k+1|k) = Ax(k|k) + Bu(k)$$
$$\hat{x}(k+1|k+1) = \hat{x}(k+1|k) + K_k\big(y_m(k) - C\hat{x}(k+1|k)\big)$$

Assume $\hat{x}(0|0) = \mathbf{0}$ and $P(0|0) = \alpha I$ (where α is a small number).

Start at $k = 0$, and at each time k:

- Compute $\hat{x}(k+1|k)$ and $\bar{P}(k+1|k)$ using previous $\hat{x}(k|k),\ P(k|k)$
- Compute Kalman gain, K_{k+1}. Also compute covariance $P(k+1|k+1)$
- Use K_{k+1} to compute $\hat{x}(k+1|k+1)$
- Use $\hat{x}(k+1|k+1)$ to compute $\hat{y}_c(k+1|k+1)$

1. Make a plot of $y_c(k)$ and $\hat{y}_c(k+1|k+1)$ vs. time.

2. Compute and return the `SSE` = $\sum_{k=1}^{200}(y_c - \hat{y}_c)^T(y_c - \hat{y}_c)$

Problem 1b: Kalman Filter with $w(k)$ assumed to be integrating white noise **(8 points)**

With $w(k)$ as integrating white noise (IWN) sequence, you will have to use one of the approaches discussed in class to handle non-white noise sequences. Note that for IWN,

$$w(k) = w(k-1) + \varepsilon(k) \quad \text{or} \quad \Delta w(k) = \varepsilon(k)$$

where, $\mathrm{cov}\{\varepsilon\} = R_1$.

With $\hat{x}(0|0) = \mathbf{0}$ and $P(0|0) = \alpha I$ (α is a small number), repeat all the steps in Part-1.

3. Make a plot of $y_c(k)$ and $\hat{y}_c(k+1|k+1)$ vs. time.

4. Compute and return the `SSE` $= \sum_{k=1}^{200} (y_c - \hat{y}_c)^T (y_c - \hat{y}_c)$ for this case as well

Problem 2: Output Feedback LQG Control **(6 points)**

The control objective is to reject the unmeasured disturbances, $w(k)$, and control y_c at the origin. The control objective is given as:

$$\min_{u(\cdot)} \sum_{i=0}^{\infty} y_c^T \bar{Q}_y y_c + u^T R u$$

where, $Q = I$ and $R = I$. Note that from the output model, the above objective is written in standard LQR form as:

$$\min_{u(\cdot)} \sum_{i=0}^{\infty} x^T \left(H^T \bar{Q}_y H\right) x + u^T R u$$

The rate form of the equation is given by the following augmentation

$$z(k) = \begin{bmatrix} \Delta x(k) \\ y_m(k) \\ y_c(k) \end{bmatrix}$$

Since the equations are

$$\Delta x(k+1) = A\Delta x(k) + B\Delta u(k) + B_w \varepsilon(k)$$
$$y(k) = C\Delta x(k) + y(k-1) + v(k)$$

we will be able to write the equations in the following form:

$$z(k+1) = \Phi z(k) + \Gamma u(k) + \Psi \varepsilon(k)$$
$$y_m(k) = \Xi z(k) + v(k)$$

Part-1: Provide the matrices

Provide the values of `phi`, `gamma`, `psi` and `xi` matrices. Ensure that

$$Q = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \bar{Q}_y \begin{bmatrix} 0 & 0 & I \end{bmatrix}, \qquad R = I$$

Part-2: Design Infinite-Horizon LQR Gain

Start with $S = I$

Use the Ricatti difference equation using $\Phi, \Gamma, \Psi, \Xi, Q, R$ from Part-1. Iterate for 50 iterations.

Report the value of S after 50^{th} iteration.

Report the corresponding value of L_∞ if S has converged

Part-3: Design Kalman Filter

Start with initial $P(0|0) = I$. Hence compute $\bar{P}(1|0)$

Use the Ricatti difference equation using $\Phi, \Gamma, \Psi, \Xi, Q, R$ to compute Kalman gains K and $\bar{K}$.

Hence compute $P(1|1)$. Repeat for 50 iterations and verify if P and K have converged.

If so, please report P_∞ and K_∞ in `Pinf` and `Kinf`, respectively.

Problem for Practice (not graded): LQG Simulations

Assume a unit step-change in F and step-change of magnitude 0.5 in x_F. Since these are unmeasured disturbances, you will also need to design Kalman Filter. Note that only the temperatures (i.e., y_m) are measured. The augmented state $\hat{z}(k|k)$ must be estimated from these measurements.

Perform simulations of the LQG to ensure regulation of the controlled outputs.