- CH5120 Assignment - 1 Solutions :-

→ 1.) Roll number :- CH16B001

$\Rightarrow a = 0 , b = 1$

$$\Rightarrow \underline{X} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

New basis vectors :- $\underline{V_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\underline{V_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\Rightarrow$ Let $c_1 \underline{V_1} + c_2 \underline{V_2} = \underline{X}$

$$\Rightarrow c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$$

$\Rightarrow$ we can write $\underline{X} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as → $\underline{0.5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \underline{0.5} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

2.) Inputs → Flow rates → F_1, F_2, F_{out}

Outputs → h, x_B

Gain matrix : $K = \begin{bmatrix} 4 & 2 & 4 \\ 0.5 & 1 & 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} h^{ss} \\ x_B^{ss} \end{bmatrix} = \begin{bmatrix} 4 & 2 & 4 \\ 0.5 & 1 & 0 \end{bmatrix} \begin{bmatrix} F_1^{ss} \\ F_2^{ss} \\ F_{out}^{ss} \end{bmatrix} \rightarrow (1)$$

'ss' denotes steady state value.

⇒ domain space (input space) is expressed in terms of $\underline{v_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{v_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\underline{v_3} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Co-domain space (output space) is expressed in terms of :- $\underline{w_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\underline{w_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Let $\begin{bmatrix} h^{ss} \\ x_B^{ss} \end{bmatrix} = \tilde{h}^{ss} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \tilde{x}_B^{ss} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$$\Rightarrow \begin{bmatrix} h^{ss} \\ x_B^{ss} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \tilde{h}^{ss} \\ \tilde{x}_B^{ss} \end{bmatrix} \rightarrow (2)$$

Let $\begin{bmatrix} F_1^{ss} \\ F_2^{ss} \\ F_{out}^{ss} \end{bmatrix} = \tilde{F}_1^{ss}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \tilde{F}_2^{ss}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \tilde{F}_{out}^{ss}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} F_1^{ss} \\ F_2^{ss} \\ F_{out}^{ss} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \tilde{F}_1^{ss} \\ \tilde{F}_2^{ss} \\ \tilde{F}_{out}^{ss} \end{pmatrix} \quad - (3)$$

Substitute eqn^s (2) and (3) in (1)

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \tilde{h}^{ss} \\ \tilde{X}_B^{ss} \end{pmatrix} = \begin{pmatrix} 4 & 2 & 4 \\ 0.5 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \times \begin{pmatrix} \tilde{F}_1^{ss} \\ \tilde{F}_2^{ss} \\ \tilde{F}_{out}^{ss} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \tilde{h}^{ss} \\ \tilde{X}_B^{ss} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 4 & 2 & 4 \\ 0.5 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \tilde{F}_1^{ss} \\ \tilde{F}_2^{ss} \\ \tilde{F}_{out}^{ss} \end{pmatrix}$$

$$\Rightarrow \boxed{\begin{pmatrix} \tilde{h}^{ss} \\ \tilde{X}_B^{ss} \end{pmatrix} = \underbrace{\begin{pmatrix} 2.25 & 3.75 & -0.5 \\ 1.75 & 2.25 & -1.5 \end{pmatrix}}_{\text{new } K \text{ (gain) matrix.}}\begin{pmatrix} \tilde{F}_1^{ss} \\ \tilde{F}_2^{ss} \\ \tilde{F}_{out}^{ss} \end{pmatrix}}$$

Problem 2: Linear Dependence

$\underline{x_1}, \underline{x_2}, \underline{x_3} \in \mathbb{R}^n \rightarrow$ Linearly independent

→ 3.) Possible values of n :- $3, 4, 5$.

→ 4.) $\underline{u} = \underline{x_1} + \underline{x_2}$

$\underline{v} = \underline{x_1} + \underline{x_3}$

$\underline{w} = \underline{x_2} + \underline{x_3}$

Consider eq^n :- $\alpha_1 \underline{u} + \alpha_2 \underline{v} + \alpha_3 \underline{w} = \underline{0}$

$\Rightarrow \alpha_1 (\underline{x_1} + \underline{x_2}) + \alpha_2 (\underline{x_1} + \underline{x_3}) + \alpha_3 (\underline{x_2} + \underline{x_3}) = \underline{0}$

$\Rightarrow (\alpha_1 + \alpha_2) \underline{x_1} + (\alpha_1 + \alpha_3) \underline{x_2} + (\alpha_2 + \alpha_3) \underline{x_3} = 0$

Let $\alpha_1 + \alpha_2 = c_1$

$\alpha_1 + \alpha_3 = c_2$

$\alpha_2 + \alpha_3 = c_3$

Since $\underline{x_1}, \underline{x_2}, \underline{x_3}$ are Linearly independent

$\Rightarrow c_1 = c_2 = c_3 = 0$

$$\Rightarrow \left.\begin{aligned} (\alpha_1 + \alpha_2) &= 0 \\ (\alpha_1 + \alpha_3) &= 0 \\ (\alpha_2 + \alpha_3) &= 0 \end{aligned}\right\} \Rightarrow \boxed{\alpha_1 = \alpha_2 = \alpha_3 = 0}$$

Since $\alpha_1 \underline{u} + \alpha_2 \underline{v} + \alpha_3 \underline{w} = \underline{0}$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$

$\Rightarrow$ $\underline{u}$, $\underline{v}$, $\underline{w}$ are linearly independent vectors

Problem 3: Null & Image Spaces

CH16B001 $\Rightarrow$ $a = 0$, $b = 1$

$$L = \begin{bmatrix} 0 & 1 & 4 \\ 0.5 & 1 & 0 \end{bmatrix}$$

for Null space $\Rightarrow$ $\underline{L}\,\underline{x} = \underline{0}$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 4 \\ 0.5 & 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad x_2 + 4x_3 = 0 \quad - (1)$$

$$x_1 + 2x_2 = 0 \quad - (2)$$

Let $x_1 = -2 \Rightarrow x_2 = 1$, $x_3 = -\frac{1}{4}$

Normalize $\begin{pmatrix} -2 \\ 1 \\ -1/4 \end{pmatrix} \longrightarrow \begin{pmatrix} -0.8889 \\ 0.4444 \\ -0.1111 \end{pmatrix}$

$$\boxed{\text{Null space} = \left\{ \alpha \begin{pmatrix} -0.8889 \\ 0.4444 \\ -0.1111 \end{pmatrix} \right\}, \ \alpha \in \mathbb{R}}$$

• Image space / Column space

$$\cancel{\underline{y} = \underline{A}\,\underline{x}}$$

$$\underline{y} = \underline{L}\,\underline{x}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 0.5 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1 \begin{pmatrix} 1 \\ 0.5 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

Vectors :- $\underbrace{\left\{ \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \end{pmatrix} \right\}}_{\downarrow}$ → Linearly dependent

Only 2 vectors are linearly independent

Image space of $\underline{L}$ → $\mathbb{R}^2$

We could even write $\underline{L}$ as

$$\underbrace{\left\{ c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}}_{\text{spans } \mathbb{R}^2}, \quad c_1, c_2 \in \mathbb{R}$$

SVD of L :- Columns of U :- Image space

Last column of V :- Null space.

svd(L) : $U = \begin{pmatrix} -0.998 & 0.0631 \\ -0.0631 & 0.998 \end{pmatrix}$

$$\boxed{\text{Image space} = \left\{ c_1 \begin{pmatrix} -0.998 \\ -0.0631 \end{pmatrix} + c_2 \begin{pmatrix} 0.0631 \\ 0.998 \end{pmatrix} \right\} \quad c_1, c_2 \in \mathbb{R}.}$$

- Last column of V :- $\begin{pmatrix} -0.8889 \\ 0.4444 \\ -0.1111 \end{pmatrix}$

$$\boxed{\text{Null space} :- \left\{ \alpha \begin{pmatrix} -0.8889 \\ 0.4444 \\ -0.1111 \end{pmatrix} \right\}, \ \alpha \in \mathbb{R}}$$

$\{ \ \}$ → represents set.

Image space computed by MATLAB and that computed by hand can be compared using the measure of <u>subspace angle</u>.

• Problem 4 :- ~~E~~ Eigenvalue Decomposition :-

Let $\underline{B} = \begin{bmatrix} 1 & 0 \\ 8 & 7 \end{bmatrix}$

Characteristic

Equation :- $\det(B - \lambda I) = 0$

$\Rightarrow (\lambda - 1)(\lambda - 7) = 0$

$\Rightarrow \boxed{\lambda^2 - 8\lambda + 7 = 0}$

$\downarrow$

characteristic

equation

$\boxed{\lambda = 1, 7} \rightarrow$ eigenvalues.

$\Rightarrow \underline{A}^2 - 8\underline{A} + 7I_{2\times 2}$:-

$= \begin{bmatrix} 1 & 0 \\ 8 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 8 & 7 \end{bmatrix} - 8\begin{bmatrix} 1 & 0 \\ 8 & 7 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 64 & 49 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ 64 & 56 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \underline{0}_{2\times 2}$

(Cayley - Hamilton Theorem satisfied)

$\underline{B} = \underline{V}\,\underline{\Lambda}\,\underline{V}^{-1}$

$$\underline{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix}$$

Eigenvectors for λ_1, λ_2 :- $\underline{v}_1$, $\underline{v}_2$

$\Rightarrow (B - \lambda_1 I)\,\underline{v}_1 = \underline{0} \qquad (\lambda_1 = 1)$

$$\Rightarrow \begin{pmatrix} 0 & 0 \\ 8 & 6 \end{pmatrix}\begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow 8v_{11} + 6v_{12} = 0$

$\Rightarrow 4v_{11} + 3v_{12} = 0$

Choose $v_{11} = 3 \Rightarrow v_{12} = -4$

$$\Rightarrow \boxed{\underline{v}_1 = \begin{pmatrix} 3/5 \\ -4/5 \end{pmatrix} = \begin{pmatrix} 0.6 \\ -0.8 \end{pmatrix}} \quad [\,\|\underline{v}_1\|_2 = 1\,]$$

$\Rightarrow (B - \lambda_2 I)\,\underline{v}_2 = \underline{0} \qquad (\lambda_2 = 7)$

$$\Rightarrow \begin{pmatrix} -6 & 0 \\ 8 & 0 \end{pmatrix}\begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \boxed{v_{21} = 0} \Rightarrow \boxed{\underline{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}} \quad [\,\|v_2\|_2 = 1\,]$$

$\Rightarrow$ Eigenvalue Decomposition of $\underline{B}$:

$$\begin{bmatrix} 1 & 0 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 0.6 & 0 \\ -0.8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 0.6 & 0 \\ -0.8 & 1 \end{bmatrix}^{-1}$$

$$\underline{B} = \underline{V}\,\underline{\Lambda}\,\underline{V}^{-1}$$

$$e^{B} = \underline{V}\, e^{\Lambda}\, \underline{V}^{-1}$$

$$e^{\Lambda} = \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix} = \begin{bmatrix} e^{1} & 0 \\ 0 & e^{7} \end{bmatrix}$$

$$\Rightarrow e^{B} = \begin{bmatrix} 0.6 & 0 \\ -0.8 & 1 \end{bmatrix} \begin{bmatrix} e^{1} & 0 \\ 0 & e^{7} \end{bmatrix} \begin{bmatrix} 0.6 & 0 \\ -0.8 & 1 \end{bmatrix}^{-1}$$

$$e^{B} = \begin{bmatrix} 2.7183 & 0 \\ 1458.6 & 1096.6 \end{bmatrix}$$

Problem - 5 : Jordan Decomposition

$$C = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$$

for eigenvalues,

$$\det(C - \lambda I) = 0$$

$$\Rightarrow (\lambda - 1)(\lambda - 3) + 1 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 4 = 0$$

$\boxed{\lambda = 2}$ → repeated eigenvalue

for eigenvector,

$$\Rightarrow (C - \lambda I)\underline{v} = \underline{0}$$

$$\Rightarrow \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow v_{11} = v_{21}$$

$\Rightarrow$ ~~Ee~~ Eigenvector :- $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$\Rightarrow$ For Generalized eigenvector :-

$$\Rightarrow (C - \lambda I)\underline{w} \neq 0, \quad (C - \lambda I)^2 \underline{w} = \underline{0}$$

However notice that $(C - \lambda I)^2 \underline{w} = \underline{0}$ is satisfied for any $\underline{w}$, as $(C - \lambda I)^2 = \underline{0}_{2\times 2}$

Jordan Decomposition

$$C = X \Lambda X^{-1}$$

Let $\underline{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ be the generalized eigenvector of C, s.t. $\|\underline{w}\|_2 = 1$

or $w_1^2 + w_2^2 = 1$ — ①

$$\Rightarrow C = [\underline{v} \mid \underline{w}] \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} [\underline{v} \mid \underline{w}]^{-1}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & w_1 \\ \frac{1}{\sqrt{2}} & w_2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & w_1 \\ \frac{1}{\sqrt{2}} & w_2 \end{pmatrix}^{-1}$$

$\Rightarrow$ We get a single equation :-

$$\boxed{w_2 - w_1 = \frac{1}{\sqrt{2}}} \quad - ②$$

Solve ① and ② :-

$$w_1 = \frac{\sqrt{6} - \sqrt{2}}{4} = 0.2588$$

$$w_2 = \frac{\sqrt{6} + \sqrt{2}}{4} = 0.9659$$

$$\Rightarrow \underline{w} = \begin{pmatrix} 0.2588 \\ 0.9659 \end{pmatrix} \rightarrow \text{generalized eigenvector.}$$

Jordan Decomposition :-

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{6}-\sqrt{2}}{4} \\ \frac{1}{\sqrt{2}} & \frac{\sqrt{6}+\sqrt{2}}{4} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{6}-\sqrt{2}}{4} \\ \frac{1}{\sqrt{2}} & \frac{\sqrt{6}+\sqrt{2}}{4} \end{pmatrix}^{-1}$$

$$C = X \Lambda X^{-1}$$

$$\Rightarrow e^{C} = X e^{\Lambda} X^{-1}$$

$$\Rightarrow e^{\Lambda} = \begin{bmatrix} e^{\lambda} & e^{\lambda} \\ 0 & e^{\lambda} \end{bmatrix} = \begin{bmatrix} e^{2} & e^{2} \\ 0 & e^{2} \end{bmatrix}$$

$$\Rightarrow e^{C} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{6}-\sqrt{2}}{4} \\ \frac{1}{\sqrt{2}} & \frac{\sqrt{6}+\sqrt{2}}{4} \end{pmatrix} \begin{pmatrix} e^{2} & e^{2} \\ 0 & e^{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{6}-\sqrt{2}}{4} \\ \frac{1}{\sqrt{2}} & \frac{\sqrt{6}+\sqrt{2}}{4} \end{pmatrix}^{-1}$$

$$e^{C} \approx \begin{pmatrix} 0 & 7.3891 \\ -7.3891 & 14.7781 \end{pmatrix}$$

$\downarrow$

Matrix Exponent