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CH5120 - Modern Control Theory
Assignment - 3 - Solutions.

P1. Model Conversion.

Q1 $G(s) = \frac{5\alpha}{s+\alpha}$; $\alpha = \frac{1}{\tau}$ $\Leftrightarrow$ $\frac{dx}{dt} = -\alpha x + 5\alpha u$; $y = (1)x$

CH16B004 $\Rightarrow$ $\tau = 0.5(1+4) = 2.5$ & $\alpha = \frac{1}{\tau} = 0.4$

$\frac{dx}{dt} = ax + bu$ $\quad (a = -\alpha,\ b = 5\alpha)$

$= \frac{dx}{dt} - ax = bu$ ⟶ Multiply both sides by I.F $= e^{-at}$
& integrate from $k\Delta t$ to $(k+1)\Delta t$
$k \in \mathbb{Z}$

• $$\int_{k\Delta t}^{(k+1)\Delta t} \frac{dx}{dt}\cdot e^{-at} - ax\cdot e^{-at}\, dt = \int_{k\Delta t}^{(k+1)\Delta t} bu e^{-at} dt \quad -\text{(I)}$$

$$* \ \frac{d}{dt}[x\cdot e^{-at}] = \frac{dx}{dt} e^{-at} - axe^{-at}$$

* Applying ZOH (Zero Order Hold) for $u(t)$: for $t \in [k\Delta t, (k+1)\Delta t)$
$\Rightarrow u(t) = u(k\Delta t)$
$\equiv u_k$

$\Rightarrow$ (I) becomes: $$\int_{k\Delta t}^{(k+1)\Delta t} \frac{d}{dt}[x\cdot e^{-at}]dt = \int_{k\Delta t}^{(k+1)\Delta t} bu_k e^{-at} dt.$$

$$= \underbrace{x((k+1)\Delta t)}_{x_{k+1}} e^{-a(k+1)\Delta t} - \underbrace{x(k\Delta t)}_{x_k} e^{-ak\Delta t} = bu_k \int_{k\Delta t}^{(k+1)\Delta t} e^{-at} dt.$$

$$= x_{k+1}\, e^{-a(k+1)\Delta t} - x_k e^{-ak\Delta t} = b u_k \left[\frac{-1}{a} e^{-at}\right]_{k\Delta t}^{(k+1)\Delta t}$$

$\times$ both sides by $e^{a(k+1)\Delta t}$

$$= x_{k+1} - x_k e^{a\Delta t} = b u_k \left(\frac{-1}{a}\right)\left[e^{-a(k+1)\Delta t} - e^{-ak\Delta t}\right] e^{a(k+1)\Delta t}$$

$$= x_{k+1} - x_k e^{a\Delta t} = -\frac{b}{a} u_k \left[1 - e^{a\Delta t}\right]$$

$\Rightarrow$ Writing in standard form : $\boxed{x_{k+1} = e^{a\Delta t} x_k - \frac{b}{a}(1 - e^{a\Delta t}) u_k \quad \& \quad y_k = (1)\, x_k}$ (No change in output eq'n)

Subs:

$a = -\alpha = -0.4$

$\& \; b = 5\alpha = 2 \qquad \Delta t = 0.5$

$$\Rightarrow x_{k+1} = e^{-0.4\times 0.5} x_k - \frac{2}{-0.4}(1 - e^{-0.4\times 0.5}) u_k \;;\; y_k = x_k$$

$$\Rightarrow \boxed{\begin{aligned} x_{k+1} &= 0.8187\, x_k + 0.9063\, u_k \\ y_k &= x_k \end{aligned}}$$ Discrete time State space model

Q2. Input delay = 1.5 = 3 × 0.5 = $\underline{3\Delta t}$, ∴ existing SS model is modified as:

$$x_{k+1} = 0.8187\, x_k + 0.9063\, u_{\underline{k-3}}$$

$$y_k = x_k$$

Define a new state variable $\hat{x}_k = \begin{bmatrix} x_k \\ u_{k-1} \\ u_{k-2} \\ u_{k-3} \end{bmatrix}$ ('Augmented' state variable)

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∴ We can express same model using $\hat{x}_k$ as:

$$\underbrace{\begin{bmatrix} x_{k+1} \\ u_k \\ u_{k-1} \\ u_{k-2} \end{bmatrix}}_{\hat{x}_{k+1}} = \underbrace{\begin{bmatrix} 0.8187 & 0 & 0 & 0.9063 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\hat{A}} \underbrace{\begin{bmatrix} x_k \\ u_{k-1} \\ u_{k-2} \\ u_{k-3} \end{bmatrix}}_{\hat{x}_k} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\hat{B}} u_k$$

$$\& \quad y_k = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}}_{\hat{C}} \underbrace{\begin{bmatrix} x_k \\ u_{k-1} \\ u_{k-2} \\ u_{k-3} \end{bmatrix}}_{\hat{x}_k}$$

(or) $\hat{x}_{k+1} = \hat{A}\hat{x}_k + \hat{B}u_k \quad ; \quad y_k = \hat{C}\hat{x}_k$

$$\hat{A} = \begin{bmatrix} 0.8187 & 0 & 0 & 0.9063 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} ; \quad \hat{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} ; \quad \hat{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

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④

P2. Model Linearization

Q.

$$\frac{dh}{dt} = \frac{1}{A}\left(F_{in} - K\sqrt{h}\right) = f(\overbrace{h, C}^{x}, \overbrace{F_{in}}^{u})$$

$$\frac{dC}{dt} = \frac{F_{in}}{Ah}(C_{in} - C) - kC^2 = g(h, C, F_{in})$$

* Input, u : F_{in} ; States, x : h & C

Parameters : $A = 0.2$, $K = 0.5$, $k = 1.5$, $C_{in} = 5 + 0.2 \times 4$
$= 5.8$

Q3. $h_{ss} = ?$ & $C_{ss} = ?$; Operating pt ⇒ $F_{in} = 0.5$

@ Steady state, $\frac{dh}{dt} = 0$ & $\frac{dC}{dt} = 0$

$$\left.\frac{dh}{dt}\right|_{@ss} = \frac{1}{A}\left(F_{in} - K\sqrt{h_{ss}}\right) = 0 \Rightarrow h_{ss} = \left(\frac{F_{in}}{K}\right)^2 = \left(\frac{0.5}{0.5}\right)^2 = \underline{1}$$

$$\left.\frac{dC}{dt}\right|_{@ss} = \frac{F_{in}}{Ah_{ss}}(C_{in} - C_{ss}) - kC_{ss}^2 = 0$$

$$= \frac{0.5}{0.2 \times 1}(5.8 - C_{ss}) - 1.5C_{ss}^2 = 0$$

$$\Rightarrow 1.5C_{ss}^2 + 2.5C_{ss} - 14.5 = 0$$

⇓

$C_{ss} = 2.3855$ ✓ or $C_{ss} = -4.0522$ ✗ not physical.

$$\boxed{\therefore \text{ Steady state values: } h_{ss} = 1 \; ; \; C_{ss} = 2.3855}$$

@ op. pt for $F_{in} = 0.5$

Q4. Linearising around steady state values using Taylor series expansion

$$\frac{dh}{dt} = f(h, c, F_{in}) = f(h_{ss}, C_{ss}, F_{in,ss}) + \left.\frac{\partial f}{\partial h}\right|_{ss}(h - h_{ss}) + \left.\frac{\partial f}{\partial c}\right|_{ss}(c - c_{ss})$$
$$+ \frac{\partial f}{\partial F_{in}}(F_{in} - F_{in,ss})$$

$$f(h, c, F_{in}) = \frac{1}{A}(F_{in} - K\sqrt{h})$$

$$\Rightarrow \frac{\partial f}{\partial h} = \frac{-k}{A} \times \frac{1}{2\sqrt{h}} \Rightarrow \left.\frac{\partial f}{\partial h}\right|_{ss} = \frac{-0.5}{0.2} \times \frac{1}{2\sqrt{1}} = -1.25$$

$$\frac{\partial f}{\partial c} = 0 \;;\; \frac{\partial f}{\partial F_{in}} = \frac{1}{A} = \frac{1}{0.2} = 5$$

$$\therefore \frac{dh}{dt} = f(h, c, F_{in}) = 0 + (-1.25)(h - h_{ss}) + 0(c - C_{ss})$$
$$+ 5(F_{in} - F_{in,ss})$$

$$\left(f(h_{ss}, C_{ss}, F_{in,ss}) = 0 \;\because \text{steady state} \right)$$

Define: $h' = h - h_{ss}$, $c' = c - C_{ss}$ & $F_{in}' = F_{in} - F_{in,ss}$

$$\Downarrow$$

$$\frac{dh'}{dt} = \frac{dh}{dt}$$

$$\Rightarrow \boxed{\frac{dh'}{dt} = -1.25\,h' + 0\,c' + 5F_{in}'} \quad -(1)$$

$$\frac{dC}{dt} = g(h, c, F_{in}) = \cancel{g(h_{ss}, C_{ss}, F_{in,ss})}^{\,0} + \left.\frac{\partial g}{\partial h}\right|_{ss} h' + \left.\frac{\partial g}{\partial c}\right|_{ss} c' + \left.\frac{\partial g}{\partial F_{in}}\right|_{ss} \cdot F_{in}'$$

(same procedure as Before)

$$g(h, c, F_{in}) = \frac{F_{in}}{Ah}(C_{in} - c) - kc^2$$

$$\Rightarrow \frac{\partial g}{\partial h} = \frac{-F_{in}\,(C_{in} - C)}{A h^2} \Rightarrow \left.\frac{\partial g}{\partial h}\right|_{ss} = \frac{-0.5}{0.2 \times 1^2}\,(5.8 - 2.3855)$$

$$= -8.5362$$

$$\frac{\partial g}{\partial C} = \frac{-F_{in}}{A h} - 2kC \Rightarrow \left.\frac{\partial g}{\partial c}\right|_{ss} = \frac{-0.5}{0.2 \times 1} - 2 \times 1.5 \times 2.3855$$

$$= -9.6565$$

$$\frac{\partial g}{\partial F_{in}} = \frac{C_{in} - C}{A h} \Rightarrow \left.\frac{\partial g}{\partial F_{in}}\right|_{ss} = \frac{5.8 - 2.3855}{0.2 \times 1} = 17.0725$$

$$\Rightarrow \frac{dc'}{dt} = -8.5362\,h' - 9.6565\,c' + 17.0725\,F_{in}' \quad \text{—(2)}$$

Combining (1) & (2) we have for $x' = \begin{bmatrix} h' \\ c' \end{bmatrix}$

$$\frac{dx'}{dt} = \frac{d}{dt}\begin{bmatrix} h' \\ c' \end{bmatrix} = \begin{bmatrix} -1.25 & 0 \\ -8.5362 & -9.6565 \end{bmatrix}\begin{bmatrix} h' \\ c' \end{bmatrix} + \begin{bmatrix} 5 \\ 17.0725 \end{bmatrix} F_{in}'$$

& $y' = x'$ (Output variables same as state variables)

(or) $\dfrac{dx'}{dt} = A^c x' + B^c u'$; $y' = x'$, where

$$A^c = \begin{bmatrix} -1.25 & 0 \\ -8.5362 & -9.6565 \end{bmatrix} \quad B^c = \begin{bmatrix} 5 \\ 17.0725 \end{bmatrix}$$

Linearised State space model

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P3. MIMO System: $\longrightarrow$ Given: 2 input - 2 output system.

Q5
$$A_{11} = \begin{bmatrix} 0.42 & -0.27 \\ 0.17 & 0.96 \end{bmatrix} \quad B_{11} = \begin{bmatrix} 0.68 \\ 0.097 \end{bmatrix} \quad C_{11} = \begin{bmatrix} 0 & 0.8 \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} 0.43 & -0.28 \\ 0.35 & 0.92 \end{bmatrix} \quad B_{12} = \begin{bmatrix} 0.35 \\ 0.1 \end{bmatrix} \quad C_{12} = \begin{bmatrix} 0 & 0.4 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 0.27 & -0.47 \\ 0.30 & 0.86 \end{bmatrix} \quad B_{21} = \begin{bmatrix} 0.59 \\ 0.18 \end{bmatrix} \quad C_{21} = \begin{bmatrix} 0 & 0.96 \end{bmatrix}$$

$$A_{22} = \begin{bmatrix} 0.48 & -0.32 \\ 0.18 & 0.96 \end{bmatrix} \quad B_{22} = \begin{bmatrix} 0.72 \\ 0.1 \end{bmatrix} \quad C_{22} = \begin{bmatrix} 0 & 0.44 \end{bmatrix}$$

Define: state x_{ij} of s-s (A_{ij}, B_{ij}, C_{ij}) & $x = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{bmatrix}$

$*$ $x_{11}(k+1) = A_{11} x_{11}(k) + B_{11} u_1(k)$

$x_{12}(k+1) = A_{12} x_{12}(k) + B_{12} u_2(k)$ & $y_1(k) = C_{11} x_1(k) + C_{12} x_2(k)$

$x_{21}(k+1) = A_{21} x_{21}(k) + B_{21} u_1(k)$ $\quad y_2(k) = C_{21} x_1(k) + C_{22} x_2(k)$

$x_{22}(k+1) = A_{22} x_{22}(k) + B_{22} u_2(k)$

[$G_{ij} \longrightarrow$ Transfer f'n b/w i^{th} output variable & j^{th} input variable]

$$\Rightarrow \begin{bmatrix} x_{11}(k+1) \\ x_{12}(k+1) \\ x_{21}(k+1) \\ x_{22}(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & 0 & 0 \\ 0 & A_{12} & 0 & 0 \\ 0 & 0 & A_{21} & 0 \\ 0 & 0 & 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_{11}(k) \\ x_{12}(k) \\ x_{21}(k) \\ x_{22}(k) \end{bmatrix} + \begin{bmatrix} B_{11} & 0 \\ 0 & B_{12} \\ B_{21} & 0 \\ 0 & B_{22} \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$$

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 \\ 0 & 0 & C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} x_{11}(k) \\ x_{12}(k) \\ x_{21}(k) \\ x_{22}(k) \end{bmatrix}$$

* '0''s are zero matrices of appropriate sizes.

∴ Assembling these matrices, we get :
using given data

$$A = \begin{bmatrix} 0.42 & -0.27 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.17 & 0.96 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.43 & -0.28 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.35 & 0.92 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.27 & -0.47 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.30 & 0.86 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.48 & -0.32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.18 & 0.96 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.68 & 0 \\ 0.097 & 0 \\ 0 & 0.35 \\ 0 & 0.1 \\ 0.59 & 0 \\ 0.18 & 0 \\ 0 & 0.72 \\ 0 & 0.1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0.8 & 0 & 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.96 & 0 & 0.44 \end{bmatrix}$$

$$x(k+1) = A\,x(k) + B\,u(k)$$
$$y(k) = C\,x(k)$$

$$\left(y(k) = [y_1(k) \;\; y_2(k)]^T; \; u(k) = [u_1(k) \;\; u_2(k)]^T \right)$$