EE6415 ENDSEM EXAM

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Question-1

For a system to be passive, PR lemma demands the existence of a positive definite P, L and W such that,

$$A^T P + PA = -L^T L \tag{1}$$
$$PB = C^T - L^T W \tag{2}$$
$$W^T W = D + D^T \tag{3}$$
$$\tag{4}$$

This can be posed equivalently as,

$$\begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -D - D^T \end{bmatrix} \leq 0 \tag{5}$$
$$P > 0 \tag{6}$$

That is, the matrix is negative definite and P is positive definite.
Reference for the matrix form: Passivity lecture linked in the qp

Part a)

System	P from YALMIP	Passivity
G_0	10	Passive
G_1	12.0605 17.3906 34.5737 17.3906 106.4264 102.4985 34.5737 102.4985 434.9530	Not Passive
G_2	1.0000 2.0000 2.0000 5.0000	Passive
G_3	3.7463 2.4192 2.4192 42.8553	Strictly Passive

Not printing P for the rest of the sections because the number of states becomes too large.

Part b)

All 6 permutations were tested out (one of the transfer functions as the system and the other as the feedback, so totally ${}^3P_2$ permutations). It was found that,

1. **passive system:** G_2 in feedback with G_3
2. **strictly passive system:** G_3 in feedback with G_2

This makes sense, because these two were the only passive systems, and passive systems in feedback is also a passive system. Rest of the combinations were not passive.

Part c)

None of the cascaded systems were found to be passive.

Part d)

We can verify the results through Nyquist plots of the system. Nyquist plots of Strictly Passive systems lie strictly in the Right Half plane. Nyquist plots of Passive systems lie in the right half plane and may touch the y axis. If the nyquist plot at some point is in the left half plane, then the system is not passive.
This was verified for part-a) alone and the interpretation from Nyquist plot matches with the one given by the PR lemma solver function. One can also check the results using MATLAB's isPassive function.

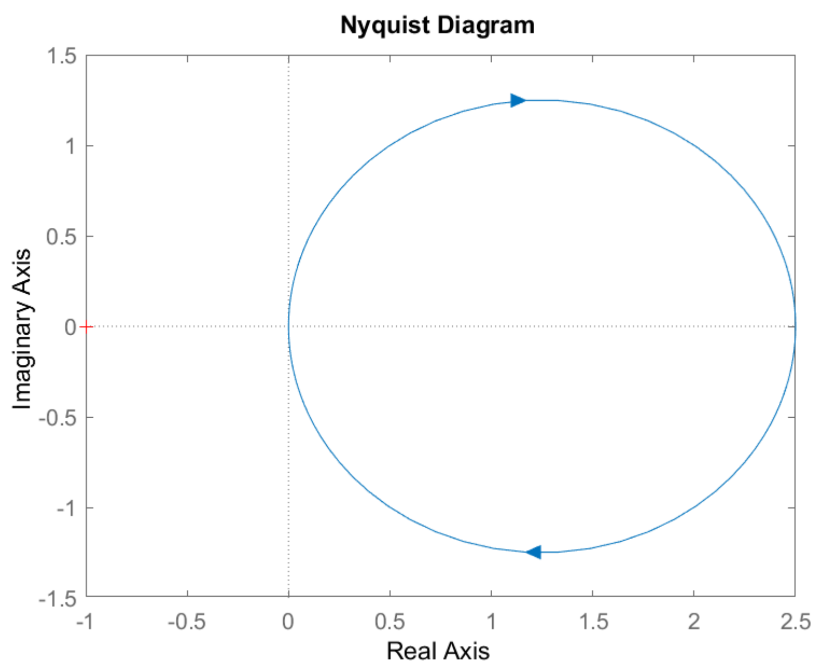

Figure 1: G_0 - lies in RHP - passive

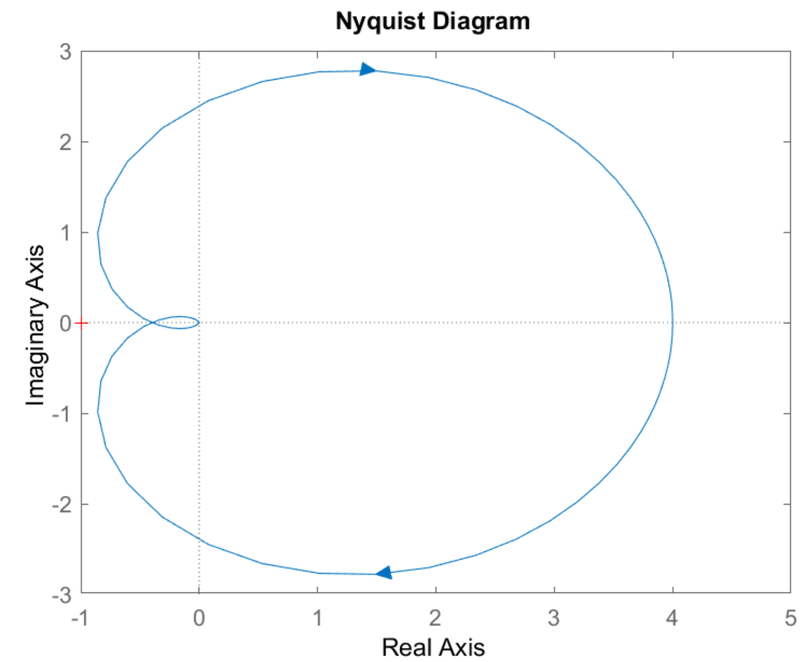

Figure 2: G_1 - crosses over to LHP - not passive

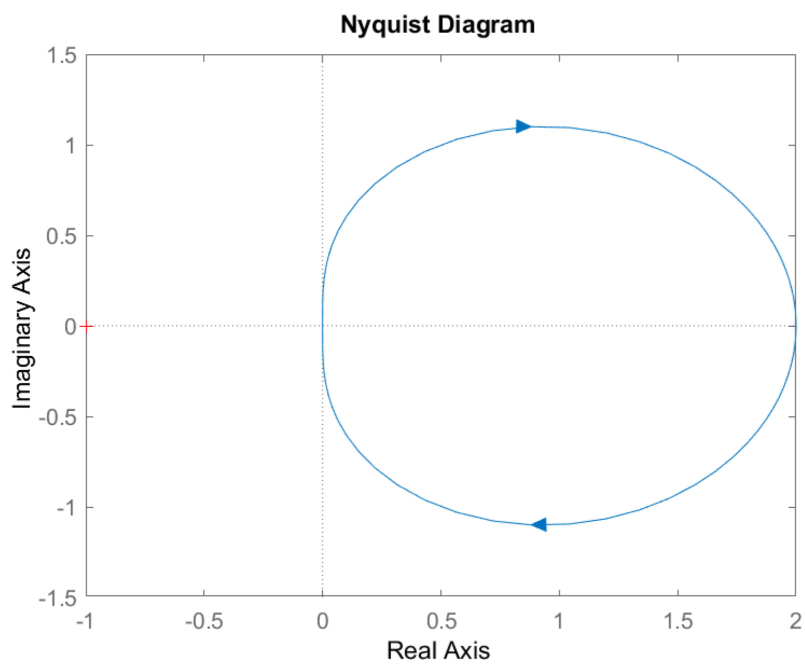

Figure 3: G_2 - lies in RHP - passive

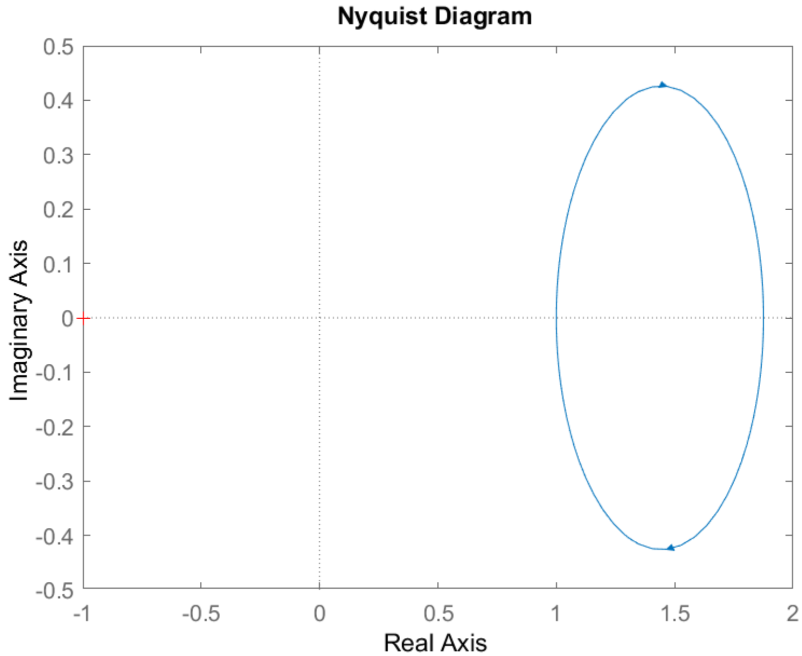

Figure 4: G_3 - strictly lies in RHP - strictly passive

Question-2

Part a)

Substituting the constants and writing the system equations,

$$\dot{x_1} = x_2 \tag{7}$$
$$\dot{x_2} = -10\sin x_1 - (x_1 - x_3) \tag{8}$$
$$\dot{x_3} = x_4 \tag{9}$$
$$\dot{x_4} = x_1 - x_3 + u \tag{10}$$

Order of zero dynamics = System order - relative degree. Here system order is 4, so for zero dynamics of order 0, 1, 2, and 3, we will need a relative degree of 4, 3, 2, and 1 respectively.
In a system of relative degree of r, u appears in the rth time derivative of output y. Here, u appears in $\dot{x_4}$, and in turn in $\ddot{x_3}$, $\dddot{x_2}$, and $\ddddot{x_1}$. So the designed outputs are,

- Order of Zero dynamics = 0 $\implies$ $y = x_1$
- Order of Zero dynamics = 1 $\implies$ $y = x_2$
- Order of Zero dynamics = 2 $\implies$ $y = x_3$
- Order of Zero dynamics = 3 $\implies$ $y = x_4$

Part b)

$$f = \begin{bmatrix} x_2 \\ -10\sin x_1 - (x_1 - x_3) \\ x_4 \\ x_1 - x_3 \end{bmatrix} \tag{11}$$

$$g = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{12}$$

$$adf(g) = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g \tag{13}$$

In our case, g is not a function of x, so the first term is 0.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -10\cos(x_1) - 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

$$\implies adf(g) = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$adf^2(g) = [f, [f, g]] \tag{14}$$

$$\implies adf^2(g) = \frac{\partial (adf(g))}{\partial x} f - \frac{\partial f}{\partial x} adf(g)$$

$$\implies adf^2(g) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

Similarly computing,$adf^3(g)$

$$adf^3(g) = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Using MATLAB one finds rank(g,$adf(g)$,$adf^2(g)$,$adf^3(g)$) $= 4 =$ number of states. So it is full rank. Since all these are constant vectors, the distribution will obviously be involutive.
$\therefore$ The system is fully feedback linearizable.

$\xi_1 = h(x) = x_1$
$\xi_2 = L_f h = x_2$
$\xi_3 = L_f^2 h = -10\sin(x_1) - (x_1 - x_3)$
$\xi_4 = L_f^3 h = -10x_2\cos(x_1) - (x_2 - x_4)$
$\implies \dot{\xi_4} = -10(-10sin(x_1) - (x_1 - x_3))cos(x_1) + 10x_2^2 sin(x_1) - (-10\sin(x_1) - (x_1 - x_3) - (x_1 - x_3 + u))$
$\implies \dot{\xi_4} = -\xi_3(10\cos(\xi_1) + 2) + 10\sin(\xi_1)(\xi_2^2 - 1) + u$

No zero order dynamics since we have chosen y such that the relative order is 4. So,

$$u = \xi_3(10\cos(\xi_1) + 2) - 10\sin(\xi_1)(\xi_2^2 - 1) + v \tag{15}$$

Also note that $\dddot{y} = \dot{\xi}_4$

$$
\begin{aligned}
y_{set-point} &= r(t) \\
&= sin(t) \\
\implies \xi_1 &= sin(t) \\
\implies \xi_2 &= cos(t) \\
\implies \xi_3 &= -sin(t) \\
\implies \xi_4 &= -cos(t)
\end{aligned}
$$

To track these, we can set v as a proportional controller with,

$$
v = -(k_1(\xi_1 - sin(t)) + k_2(\xi_2 - cos(t)) + k_3(\xi_3 + sin(t)) + k_4(\xi_4 + cos(t)) \tag{16}
$$

where, k_1, k_2, k_3, k_4 are positive constants decided by where we want to place the poles of the linearized system. For eg. if we want to place them at -1,-2,-5,-10, the corresponding values of k are (100.0000, 180.0000, 97.0000, 18.0000)

To obtain x from ξ,

$$
\begin{aligned}
x_1 &= \xi_1 \\
x_2 &= \xi_2 \\
x_3 &= \xi_3 + 10\sin(\xi_1) + \xi_1 \\
x_4 &= \xi_4 + 10\xi_2\cos(\xi_1) + \xi_2
\end{aligned}
$$

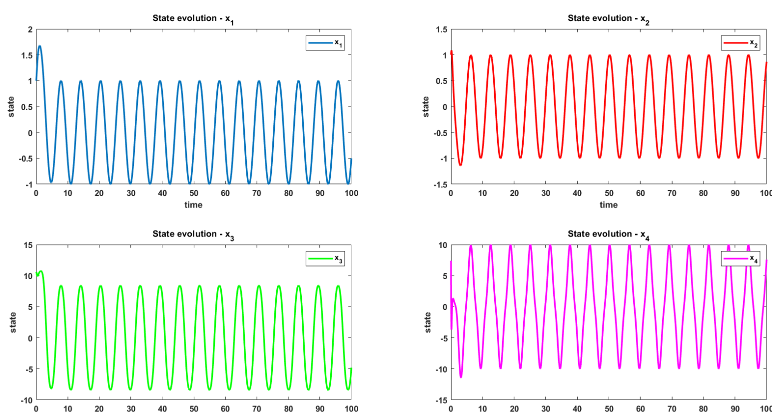

Figure 5: State evolution

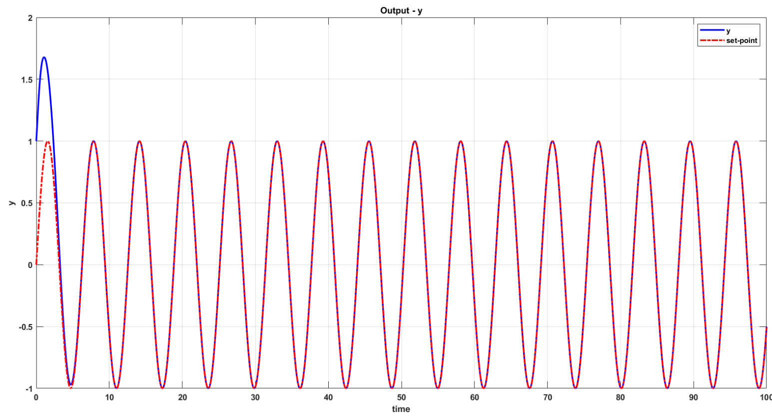

Figure 6: Output tracking

Initial states (in ξ coordinates) was taken to be (1,1,1,1). As defined earlier, $y = x_1$. From fig. 6 we see that $y = \sin(t)$ is tracked well.

Part c)

$$
\begin{aligned}
y &= x_3 \\
\dot{y} &= x_4 \\
\ddot{y} &= x_1 - x_3 + u
\end{aligned}
$$

So, relative degree is 2. There will be 2 observable and 2 non observable states.

$$
\begin{aligned}
\xi_1 &= x_3 \\
\xi_2 &= x_4
\end{aligned}
$$

For finding η,

$$
\begin{bmatrix} \frac{\partial \eta}{\partial x_1} & \frac{\partial \eta}{\partial x_2} & \frac{\partial \eta}{\partial x_3} & \frac{\partial \eta}{\partial x_4} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0
$$

We can have (ensuring T(.) is invertible),

$$
\begin{aligned}
\eta_1 &= x_1 && (17) \\
\eta_2 &= x_2 && (18) \\
\implies \dot{\eta_1} &= \dot{x_1} && (19) \\
&= x_2 = \eta_2 && (20) \\
\dot{\eta_2} &= -10\sin(x_1) - x_1 + x_3 && (21) \\
&= -10\sin(\eta_1) - \eta_1 + \xi_1 && (22)
\end{aligned}
$$

Also,

$$
\begin{aligned}
\dot{\xi_1} &= \xi_2 \\
\dot{\xi_2} &= \eta_1 - \xi_1 + u
\end{aligned}
$$

After ξ states are driven to 0,

$$
\begin{aligned}
\dot{\eta_1} &= \eta_2 && (23) \\
\dot{\eta_2} &= -10\sin(\eta_1) - \eta_1 && (24)
\end{aligned}
$$

Let us take our Lyapunov function to be,

$$V = 0.5(\eta_1^2 + \eta_2^2) \tag{25}$$

$$\begin{aligned}\dot{V} &= \eta_1\dot{\eta_1} + \eta_2\dot{\eta_2} \\ &= -10\eta_2\sin(\eta_1)\end{aligned}$$

Define set $D = \{\{(\eta_1, \eta_2) : 0 < \eta_1 < \frac{\pi}{2}, \eta_2 < 0\} \cup \{(\eta_1, \eta_2) : (0 > \eta_1 > \frac{-\pi}{2}, \eta_2 > 0\}\}$

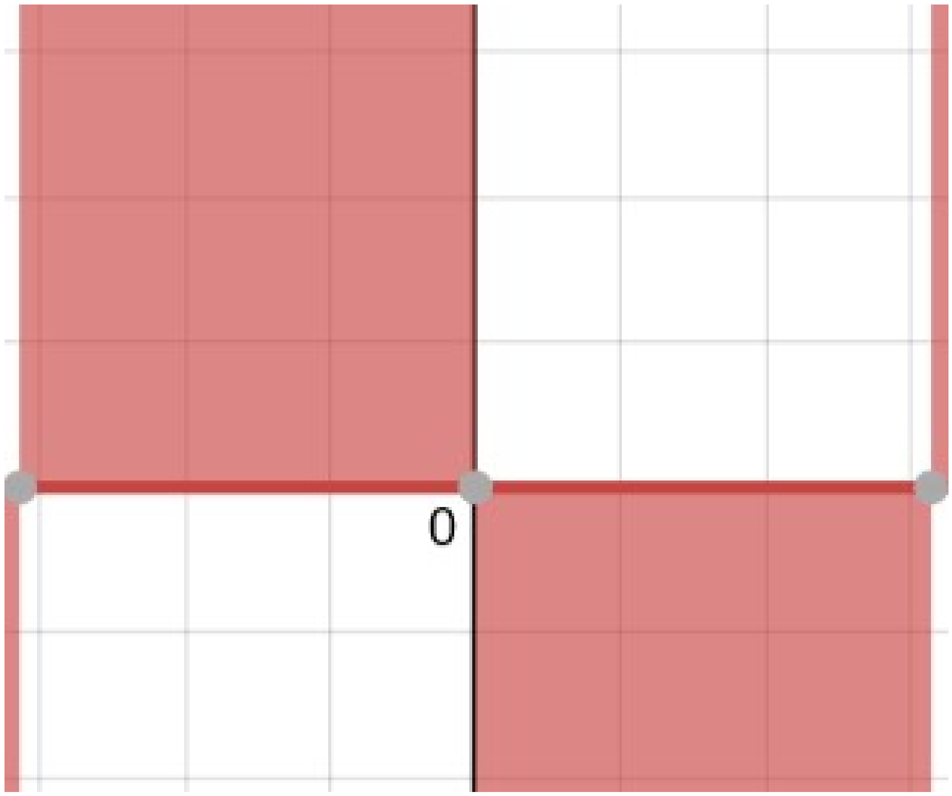

Figure 7: Set D - shaded region

- $V(x) > 0 \; \forall \; x \neq 0$
- V(x) is continuous and differentiable
- $\dot{V} > 0$ in D
- Origin $(V(0) = 0)$ is a part of the boundary of both components of D

So, by theorem 4.3, the zero dynamics is unstable. Therefore, the system is not minimum phase.

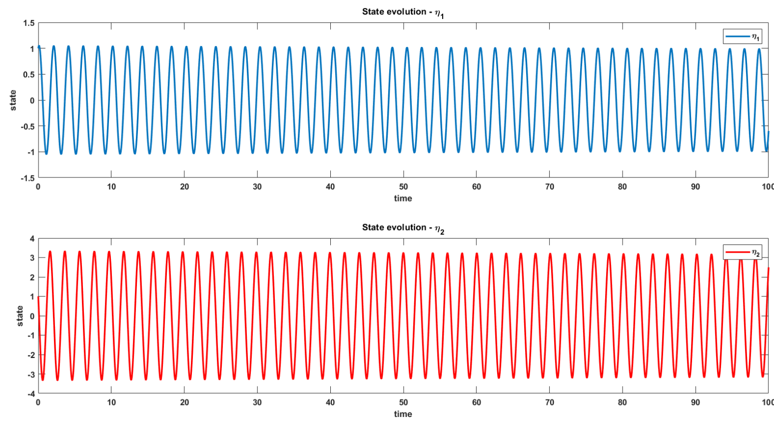

Figure 8: Simulation response of η starting from (1,1)

Fig. 8 also confirms that the zero dynamics is not asymptotically stable (in fact, it seems to give an oscillatory response). The phase portrait in figure 9 has arrows pointing away from the equilibrium at origin showing that it is unstable. Hence, the system is not minimum phase.

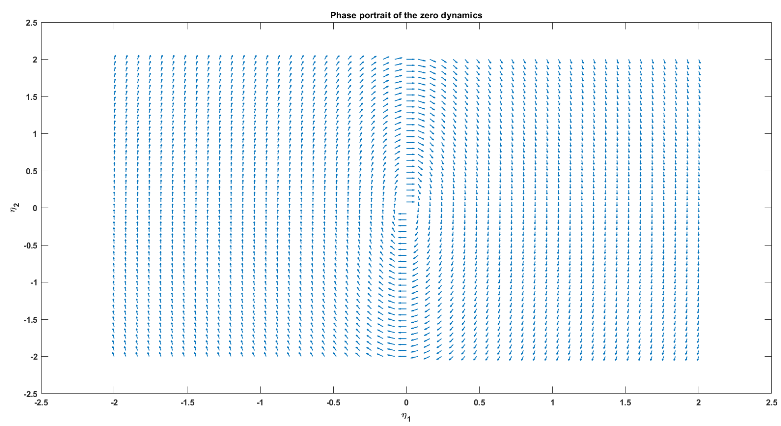

Figure 9: Phase Portrait of η

Question-3

Writing the equations in terms of y alone (input-output form),

$$
\begin{aligned}
y &= x_1 \\
\implies \dot{y} &= x_1 + (1-a) * x_2 \\
\implies x_2 &= \frac{\dot{y} - x_1}{(1-a)} \\
\implies x_2 &= \frac{\dot{y} - y}{(1-a)}
\end{aligned}
$$

Once again taking a time derivative,

$$\implies bx_2^2 + x_1 + u = \frac{\ddot{y} - \dot{y}}{(1-a)}$$
$$\implies b(\frac{\dot{y} - y}{(1-a)})^2 + y + u = \frac{\ddot{y} - \dot{y}}{(1-a)}$$
$$\implies \ddot{y} = \dot{y} + (1-a)(b(\frac{\dot{y} - y}{(1-a)})^2 + y) + (1-a)(u)$$

where, a = 0.5 and $\hat{b} = 1.5$
Since this is a second order system we choose,

$$S = \dot{\tilde{y}} + \lambda \tilde{y} \tag{26}$$
$$= \dot{y} + 0.5y \text{ (fix } \lambda = 0.5) \tag{27}$$

So we design,

$$\begin{aligned} u &= \frac{1}{(1-a)}(-\hat{f} - K * sgn(S)) \\ &= \frac{1}{(1-a)}(-\dot{y} - (1-a)(\hat{b}(\frac{\dot{y} - y}{(1-a)})^2 + y) - K * sgn(S)) \\ &= \frac{1}{0.5}(-\dot{y} - 0.5(1.5(\frac{\dot{y} - y}{0.5})^2 + y) - K * sgn(S)) \end{aligned}$$

Use,

$$K = |f - \hat{f}| + \eta$$
$$\implies K = |f - \hat{f}| + 1$$

where we fixed, $\eta = 1$.

Assuming the initial state is $y = 1, \dot{y} = 1$, and assuming b is a uniform random variable between 1 and 2 sampled before starting each simulation, we perform the simulation.

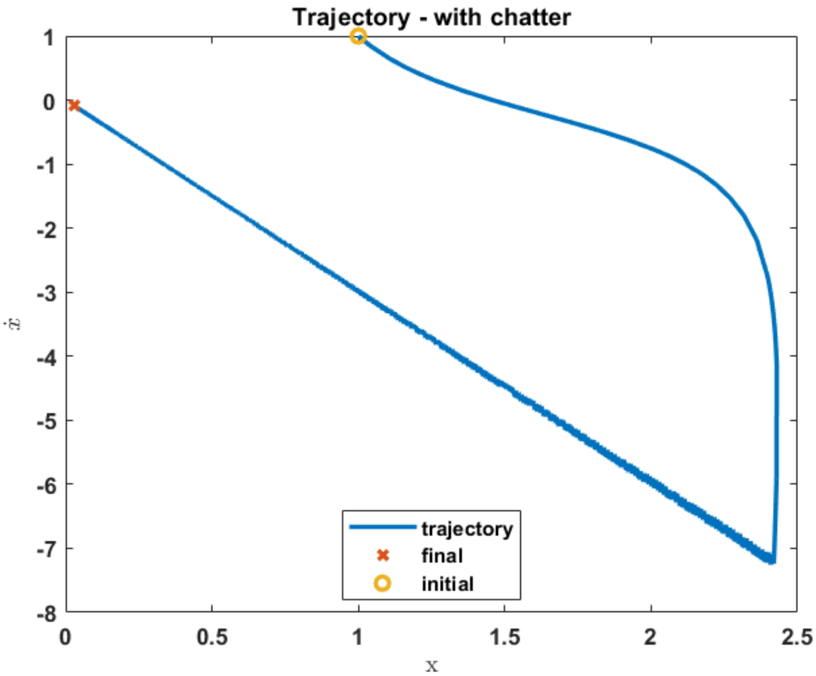

Figure 10: Trajectory - with chatter case

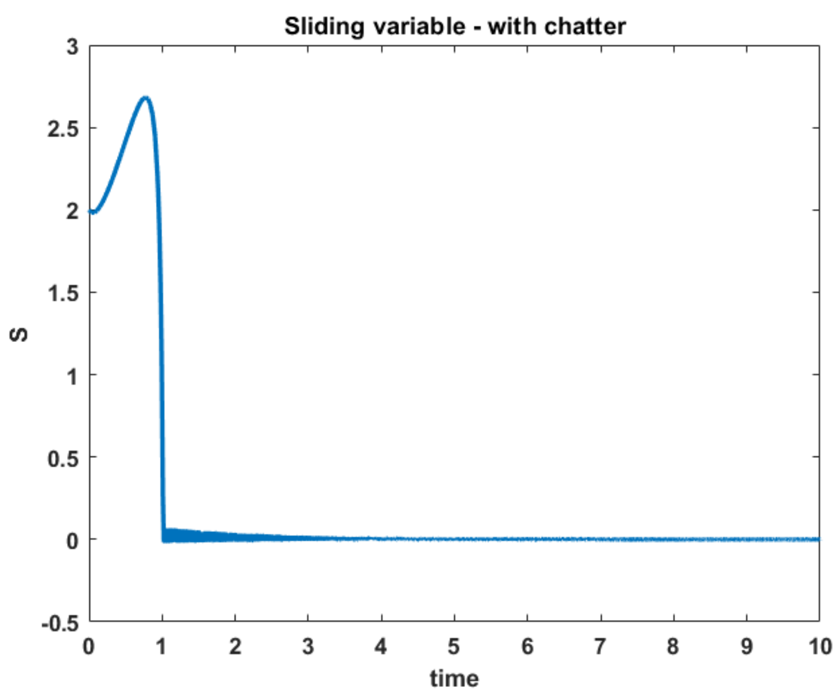

Figure 11: Sliding variable value - with chatter case

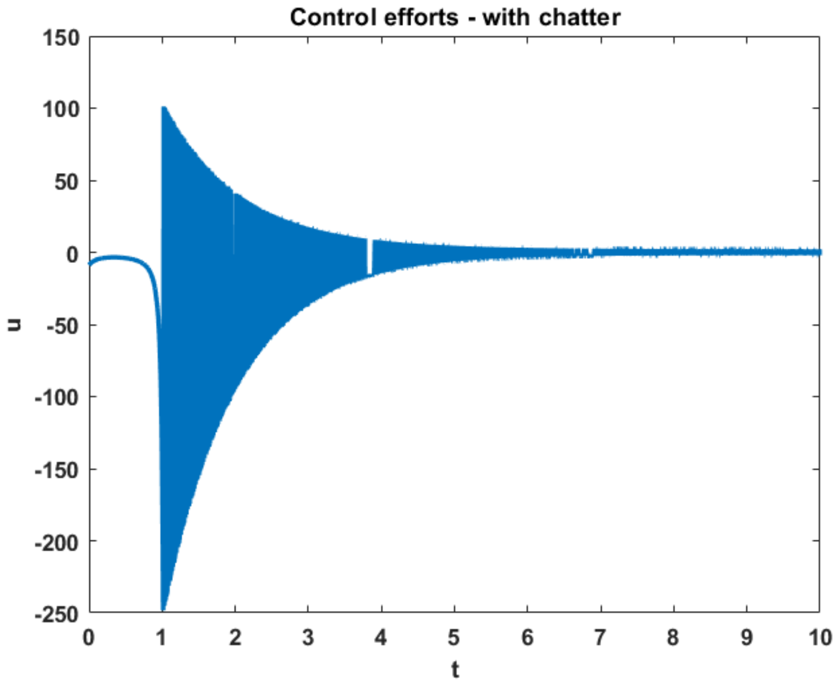

Figure 12: Control effort - with chatter case

From fig. 10, we observe that the trajectory we set is closely tracked, and fig. 11 shows that sliding variable is taken to 0, and is mostly contained in the region. However, because of use of signum function, we see high frequency oscillations in the input in fig.12 (so chatter is present). To avoid this we can use a saturation function in place of the signum function, which will 'slow down' the system as we move closer to the region of S $= 0$ and ensure lesser oscillations of the input effort u. So we remove chatter by using the saturation function with $\phi = 0.5$.

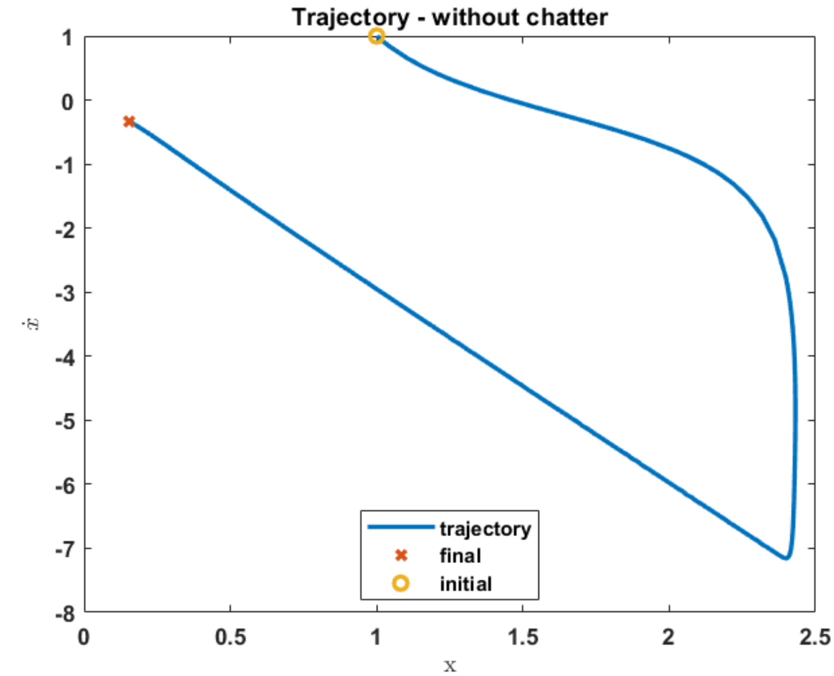

Figure 13: Trajectory - without chatter case

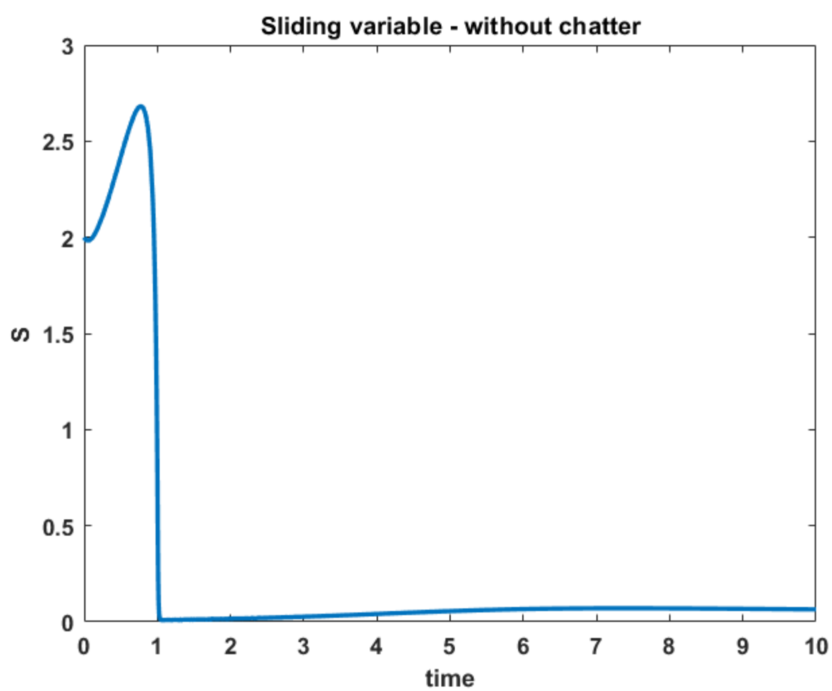

Figure 14: Sliding variable value - without chatter case

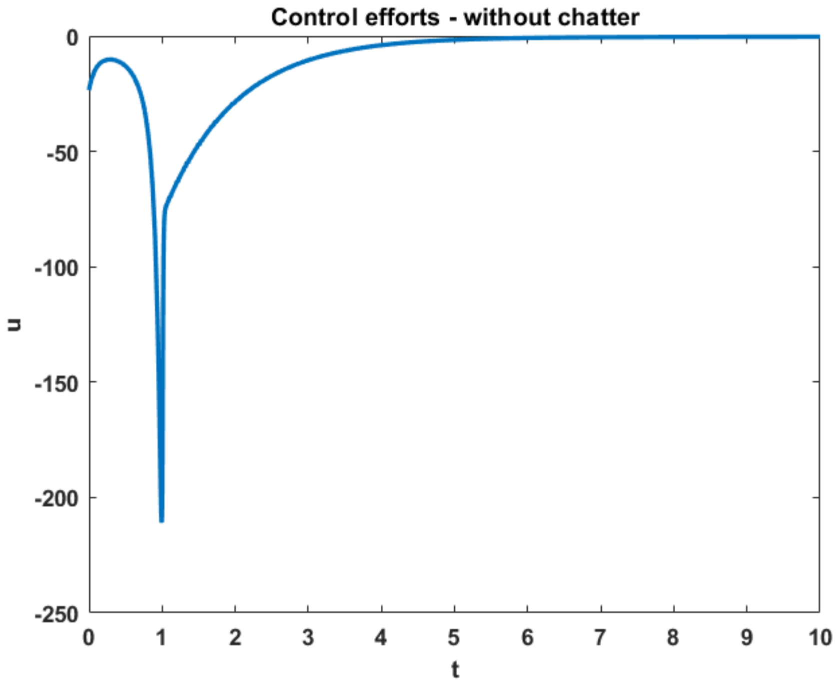

Figure 15: Control effort - without chatter case

Fig. 14 shows that sliding variable is taken to 0 but there are considerable oscillations around it (but approximately within the thickness of the boundary layer). And as expected, we have successfully removed the chatter in the input (fig. 15).

Question-4

Yes, the relative degree and zero dynamics won't be affected by a feedback transformation of the form u = $\alpha(x)+\beta(x)v$. Relative degree (number of time derivatives after which u appears) and zero dynamics (unobservable states-by definition, the input terms aren't present in their dynamics), are both entirely determined by f(x) and g(x) and have nothing to do with u. So they remain invariant irrespective of u.
For a system to be made passive by feedback:

- Relative degree of the system is 1
- Weakly minimum phase

Part a)

$$
\begin{aligned}
& y = x_1 \\
\implies & \dot{y} = \dot{x_1} \\
& \quad = x_3 - x_2^2 \\
\implies & \ddot{y} = x_1^2 - x_3 + u - 2x_2(-x_2 - u)
\end{aligned}
$$

Relative degree = 2. So, it can **NOT** be made passive through feedback.

Part b)

Define new x_2 (to make sure origin is an equilibrium point),

$$x_2^* = x_2 - \frac{\pi}{2}$$

$$
\begin{aligned}
y &= x_2^* + \frac{\pi}{2} \\
\implies \dot{y} &= \dot{x_2} \\
&= cos(x_1)cos(x_2^* + \frac{\pi}{2}) + u
\end{aligned}
$$

So relative degree is 1.

$$\xi_1 = x_2^* + \frac{\pi}{2} \tag{28}$$

$$\tag{29}$$

For finding η,

$$\begin{bmatrix} \frac{\partial \eta}{\partial x_1} & \frac{\partial \eta}{\partial x_2^*} & \frac{\partial \eta}{\partial x_3} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 0$$

We can have (ensuring T(.) is invertible),

$$
\begin{aligned}
\eta_1 &= x_1 + x_2^* \\
\eta_2 &= x_3 \\
\implies \dot{\eta_1} &= \dot{x_1} + \dot{x_2^*} \\
&= x_1^3 + \cos(x_1)\cos(x_2^* + \frac{\pi}{2}) \\
\implies \dot{\eta_1} &= (\eta_1 - \xi_1 + \frac{\pi}{2})^3 + \cos(\eta_1 - \xi_1 + \frac{\pi}{2})\cos(\xi_1) \\
\dot{\eta_2} &= x_2 \\
&= \xi_1
\end{aligned}
$$

As $\xi_1 \to 0$,

$$
\begin{aligned}
\implies \dot{\eta_1} &= (\eta_1)^3 - \sin(\eta_1) \\
\dot{\eta_2} &= 0
\end{aligned}
$$

Let us take our Lyapunov function to be,

$$V = 0.5(\eta_1^2 + \eta_2^2) \tag{30}$$

$$
\begin{aligned}
\dot{V} &= \eta_1\dot{\eta_1} + \eta_2\dot{\eta_2} \\
&= \eta_1((\eta_1 + \frac{\pi}{2})^3 - \sin(\eta_1))
\end{aligned}
$$

Let the set U be first quadrant.

- $V(x) > 0 \; \forall \; x \neq 0$
- V(x) is continuous and differentiable
- $\dot{V} > 0$ in U
- Origin ($V(0) = 0$) is a part of the boundary of U

So, by theorem 4.3, the zero dynamics is unstable. Therefore, the system is not minimum phase. Hence, it can **NOT** be made passive through feedback.

Part c)

Let the storage function be

$$E = 0.5(x_1^2 + x_2^2) \tag{31}$$

We define,

$$y = x_2$$

With this definition, the system is already in its normal form (x_2 - observable, x_1 - unobservable and doesn't have an input term in its derivative). The relative degree is 1 and the system is weakly minimum phase ($\dot{x_1} = 0$ when $x_2 = 0$). So the system can be made passive through feedback.

To cancel the $x_1^3 x_2$ term that will come due to $\dot{x_1} x_1$, we can use a

$$u = v - x_1^3$$

$$\dot{x_1} = x_1^2 x_2 \tag{32}$$

$$\dot{x_2} = -x_1^3 + v \tag{33}$$

Checking for passivity with V as storage function,

$$\begin{aligned} v^T y - \frac{dE}{dt} &= ux_2 - (x_1^3 \dot{x_1} + x_2 \dot{x_2}) \\ &= vx_2 - (x_1^3 x_2 - x_1^3 x_2 + vx_2) \\ &= 0 \end{aligned}$$

So, the system is passive with a radially unbounded positive definite storage function.

Also, $\because y(t) = 0, v(t) = 0 \implies (x_1, x_2) = (0, 0)$, the output is zero state observable.

If we provide a input $-\phi(y)$ such that,

$$y\phi(y) > 0 \tag{34}$$

we can stabilize the system.
Choose, $\phi(y) = y \implies y\phi(y) = y^2 > 0$

$$\begin{aligned} \phi(y) &= y \\ \implies v &= -y \end{aligned}$$

$\therefore u = -x_1^3 - y$ achieves global asymptotic stability at (0,0).

Question-5

Part a)

For computational simplicity, we substitute the value of the constants to obtain the system dynamics as,

$$\dot{x} = v \tag{35}$$

$$\dot{v} = \frac{4u + 40\cos(\theta)\sin(\theta) + 4(2\omega^2 \sin(\theta) - v)}{4(6 - \cos^2(\theta))} \tag{36}$$

$$\dot{\theta} = \omega \tag{37}$$

$$\dot{\omega} = \frac{-120\sin(\theta) - 2\cos(\theta)(2\omega^2\sin(\theta) - v) - 2u\cos(\theta)}{4(6 - \cos^2(\theta)} \tag{38}$$

We are given,

$$y = \theta \tag{39}$$

So,

$$\begin{aligned}
\dot{y} &= \dot{\theta} = \omega \\
\ddot{y} &= \dot{\omega} \\
&= \frac{-120\sin(\theta) - 2\cos(\theta)(2\omega^2\sin(\theta) - v) - 2u\cos(\theta)}{4(6-\cos^2(\theta))}
\end{aligned}$$

Second derivative of y contains input term u. So, **relative degree = 2**

Part b)

Observable states:

$$\xi_1 = h(x) = \theta$$
$$\xi_2 = L_f h = \omega$$

Dynamics of observable states,

$$\dot{\xi_1} = \xi_2 \tag{40}$$

$$\dot{\xi_2} = \frac{-120\sin(\theta) - 2\cos(\theta)(2\omega^2\sin(\theta) - v) - 2u\cos(\theta)}{4(6-\cos^2(\theta))} \tag{41}$$

We need this system to be asymptotically stable and go to zero. First we will cancel all non-linear terms, and replace it with dynamics such that the new feedback linearized system is,

$$\dot{\xi} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

Eigen values are:

$$\lambda^2 + 2\lambda + 1 = 0$$
$$\lambda = -1, -1$$

Making it an asymptotically stable linear system. Hence we design u as,

$$u = ((-\theta - 2\omega)\frac{(4(6-\cos^2(\theta)))}{2cos(\theta)}) + \frac{1}{2\cos(\theta)}(-120\sin(\theta) - 2\cos(\theta)(2\omega^2\sin(\theta) - v)) \tag{42}$$

$$= ((-\xi_1 - 2\xi_2)\frac{(4(6-\cos^2(\xi_1)))}{2cos(\xi_1)}) + \frac{1}{2\cos(\xi_1)}(-120\sin(\xi_1) - 2\cos(\xi_1)(2\xi_2^2\sin(\xi_1) - v)) \tag{43}$$

where the first term ensures the required dynamics and the second term cancels out the existing non-linear dynamics. So we can successfully drive ξ parts to zero.

Part c)

Yes, the system exhibits zero dynamics since relative degree < number of states. We need 4 total states - 2 observable states = 2 unobservable states.
Unobservable states,

$$\begin{bmatrix} \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial v} & \frac{\partial \eta}{\partial \theta} & \frac{\partial \eta}{\partial \omega} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{4}{4(6-\cos^2(\theta))} \\ 0 \\ \frac{-2\cos(\theta)}{4(6-\cos^2(\theta))} \end{bmatrix} = 0$$

$$\implies \frac{2}{\cos(\theta)}\frac{\partial \eta}{\partial v} = \frac{\partial \eta}{\partial \omega}$$

Possible states satisfying the above equation (while ensuring T(.) is invertible):

$$\eta_1 = x$$
$$\eta_2 = \frac{v\cos(\theta)}{2} + \omega$$

Their derivatives are,

$$\begin{aligned}
\dot{\eta_1} &= v \\
&= \frac{L\eta_2}{\cos(\theta)} - \omega \\
\dot{\eta_2} &= \frac{\dot{v}}{L} - \omega(\sin(\theta))\frac{v}{L} + \dot{\omega}
\end{aligned}$$

$$\text{Numerator of } \dot{\eta_2} = -2g\cos(\theta)sin(\theta) + 2(2\omega^2\sin(\theta) - v) + 12g\sin(\theta) - 4\cos(\theta)\omega^2\sin(\theta) + 2v\cos(\theta)$$

Substituting constants and When $\xi \to 0$, $\theta, \omega \to 0$

$$\begin{aligned}
\implies \dot{\eta_1} &= 2\eta_2 \\
\dot{\eta_2} &= 0
\end{aligned}$$

Let us take our Lyapunov function to be,

$$V = 0.5(\eta_1^2 + \eta_2^2) \tag{44}$$

$$\begin{aligned}
\dot{V} &= \eta_1\dot{\eta_1} + \eta_2\dot{\eta_2} \\
&= 2\eta_1\eta_2
\end{aligned}$$

Define set $U = \{(\eta_1, \eta_2) : \text{points in first and third quadrants}\}$

- $V(x) > 0 \ \forall \ x \neq 0$
- V(x) is continuous and differentiable
- $\dot{V} > 0$ in U
- Origin ($V(0) = 0$) is a part of the boundary of both components of U

So, by theorem 4.3, the zero dynamics is unstable. Therefore, the system is not minimum phase.

Part d)

We can see the state evolution in figure 16. As expected, our designed input drives the observable variables θ and ω to zero. Thus, it accomplishes the desired objective. One more observation is that the system is moving with a constant velocity - which is expected, because acceleration would cause the pendulum to change its angle.

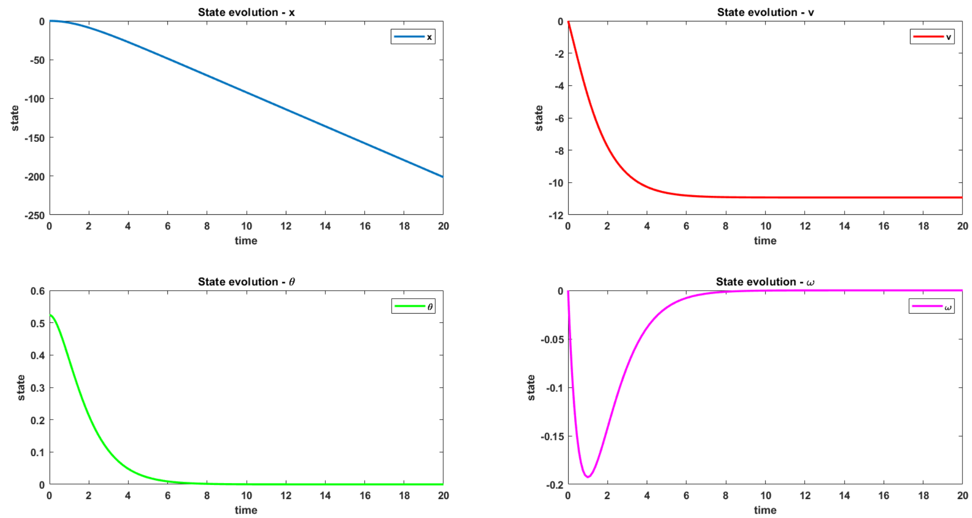

Figure 16: State evolution. $x_0 = (0, 0, \frac{\pi}{6}, 0)$ - released at an angle

Question-6

Part a)

Let us take our Lyapunov function to be,

$$V = 0.5(x_1^2 + x_2^2) > 0 \tag{45}$$

$$\begin{aligned} \dot{V} &= x_1\dot{x_1} + x_2\dot{x_2} \\ &= x_1x_2 - x_2x_1 + x_2^2(1 - x_1^2 - x_2^2) \\ &= x_2^2(1 - x_1^2 - x_2^2) \end{aligned}$$

We observe that in the set $U = \{(x_1, x_2) : (x_1^2 + x_2^2) < 1 \text{ and } x_2 > 0\}$,

- $V(x) > 0 \; \forall \; x \neq 0$
- V(x) is continuous and differentiable
- $\dot{V} > 0$ in the set U
- Origin $(V(0) = 0)$ is a part of the boundary $(x_2 = 0)$

So by theorem 4.3 we conclude that the origin is unstable.

Part b)

Consider,

$$V = 0.5(x_1^2 + x_2^2) \tag{46}$$

$$\tilde{V} = -0.5(x_1^2 + x_2^2) \tag{47}$$

Points within the unit circle

Consider the set, $D = \{(x_1, x_2) : c \leq (x_1^2 + x_2^2) \leq 1\}, 0 < c < 1$. This set doesn't comprise origin but c can be made arbitrarily small to include points near origin.

$$V(x) = x_1^2 + x_2^2 \tag{48}$$

$$\overline{\nabla} V = 2x_1\hat{i} + 2x_2\hat{j} \tag{49}$$

We find sign of

$$\overline{\nabla}V.f(x)$$

at the boundaries. If it is > 0, then radius of the position (x_1, x_2) is increasing and if it is < 0, then radius of the position (x_1, x_2) is decreasing.
At $(x_1^2 + x_2^2) = c$,

$$\begin{aligned}\overline{\nabla}V.f(x) &\leq 0 \\ &= x_1x_2 + x_2(-x_1 + (1 - x_1^2 - x_2^2)) \\ &= x_2^2(1 - x_1^2 - x_2^2)\end{aligned}$$

Checking its value at the boundaries of the set D,
at $x_1^2 + x_2^2 = c < 1$,

$$\begin{aligned}x_2^2(1 - x_1^2 - x_2^2) &> 0 \\ \overline{\nabla}V.f(x) &> 0\end{aligned}$$

at $x_1^2 + x_2^2 = 1$,

$$x_2^2(1 - x_1^2 - x_2^2) = 0$$

So it does not go out (or in) at the unit circle, it just follows a circular path. And it goes towards the set D at $x_1^2 + x_2^2 = c$ ($\because$ radius increases (becomes more than c) and $c < r < 1$ which lies inside the set). Therefore, we conclude that D is an invariant set. By appropriately adjusting c we can define a compact set D such that it includes any arbitrary point apart from origin in or on the unit circle.
Now, consider $\tilde{V}$,

$$\begin{aligned}\dot{\tilde{V}} &= -(x_1\dot{x_1} + x_2\dot{x_2}) \\ &= -x_2^2(1 - x_1^2 - x_2^2)\end{aligned}$$

Inside the unit circle, $(1 - x_1^2 - x_2^2) > 0$

$$\implies \dot{\tilde{V}} \leq 0$$

And it is equal to zero at the unit circle alone.
So, using La Salle's theorem we can argue that any trajectory starting in D will end up in set containing points where $\dot{\tilde{V}} = 0$. The set is nothing but the unit circle itself. So all trajectories starting on or within the unit circle, apart from origin, will end up in the unit circle (in fact, the unit circle is a limit cycle).

Points outside the unit circle

Consider the set, $D = \{(x_1, x_2) : 1 \leq (x_1^2 + x_2^2) \leq c\}, c > 1$.

$$V(x) = x_1^2 + x_2^2 \tag{50}$$

$$\overline{\nabla}V = 2x_1\hat{i} + 2x_2\hat{j} \tag{51}$$

We find sign of

$$\overline{\nabla}V.f(x)$$

at the boundaries. If it is > 0, then radius of the position (x_1, x_2) is increasing and if it is < 0, then radius of the position (x_1, x_2) is decreasing.
At $(x_1^2 + x_2^2) = c$,

$$
\begin{aligned}
\overline{\nabla} V.f(x) &\leq 0 \\
&= x_1x_2 + x_2(-x_1 + (1 - x_1^2 - x_2^2)) \\
&= x_2^2(1 - x_1^2 - x_2^2)
\end{aligned}
$$

Checking its value at the boundaries of the set D,
at $x_1^2 + x_2^2 = c > 1$,

$$
\begin{aligned}
x_2^2(1 - x_1^2 - x_2^2) < 0 \\
\overline{\nabla} V.f(x) < 0
\end{aligned}
$$

at $x_1^2 + x_2^2 = 1$,

$$
x_2^2(1 - x_1^2 - x_2^2) = 0
$$

So it does not go out (or in) at the unit circle, it just follows a circular path. And it goes towards the set D at $x_1^2 + x_2^2 = c$ ($\because$ radius decreases (becomes less than c) and $1 < r < c$ which lies inside the set). Therefore, we conclude that D is an invariant set. By appropriately adjusting c we can define a compact set D such that it includes any arbitrary point apart from origin outside or on the unit circle.
Now, consider V,

$$
\begin{aligned}
\dot{V} &= (x_1\dot{x_1} + x_2\dot{x_2}) \\
&= x_2^2(1 - x_1^2 - x_2^2)
\end{aligned}
$$

Inside the unit circle, $(1 - x_1^2 - x_2^2) > 0$

$$
\implies \dot{V} \leq 0
$$

And it is equal to zero at the unit circle alone.
So, using La Salle's theorem we can argue that any trajectory starting in D will end up in a set containing points where $\dot{V} = 0$. The set is nothing but the unit circle itself. So all trajectories starting outside the unit circle, apart from origin, will end up in the unit circle.

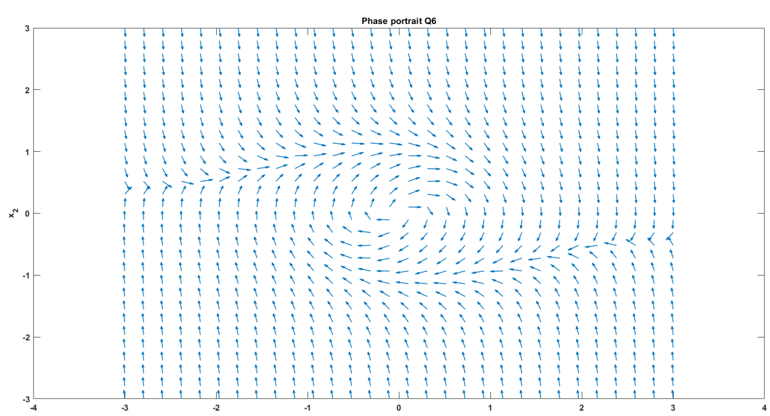

Figure 17: Phase Portrait Q6

The phase portrait agrees with our theoretical interpretation.

Conclusion: Trajectories starting from points other than origin end up in the unit circle