# Assignment-2

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## Question-1

### Part a)

Let,

$$P = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}; \tag{1}$$

(a symmetric matrix)
Substituting in the discrete time Lyapunov equation and simplifying,

$$\begin{align} A^T P A - A &= -Q \tag{2} \\ \text{A is invertible} \tag{3} \\ \implies A^T P - P(A)^{-1} &= -Q * (A)^{-1} \tag{4} \end{align}$$

LHS =

$$\begin{bmatrix} 0.9a - 2.7b - 1.1c & 0.36a - 0.66b - 0.36c - 1.0d - 0.4e & 0.32a - 0.63b - 0.13c - 1.0e - 0.4f \\ 0.2a + 3.2e - 3b - 0.2c - 1.7d - 0.71e & 0.56b - 1.6d - 0.56e & 0.32b + 0.2c - 0.63d - 1.0e - 0.2f \\ 0.2a - 0.4b + 0.3c - 1.7e - 0.71f & 0.2b + 0.36c - 0.4d - 1.3e - 0.36f & 0.52c - 1.0e - 0.73f \end{bmatrix}$$

RHS =

$$-1 * \begin{bmatrix} 0.7937 & -0.7143 & -0.6349 \\ 1.7063 & 1.9643 & 0.6349 \\ 2.1429 & 1.0714 & 4.2857 \end{bmatrix}$$

Taking a system of 6 equations,

$$\begin{align} 0.9 * a - 2.7 * b - 1.1 * c &= -0.7937 \tag{5} \\ 0.36 * a - 0.66 * b - 0.36 * c - 1.0 * d - 0.4 * e &= +0.7143 \tag{6} \\ 0.32 * a - 0.63 * b - 0.13 * c - 1.0 * e - 0.4 * f &= +0.6349 \tag{7} \\ 0.56 * b - 1.6 * d - 0.56 * e &= -1.9643 \tag{8} \\ 0.32 * b + 0.2 * c - 0.63 * d - 1.0 * e - 0.2 * f &= -0.6349 \tag{9} \\ 0.52 * c - 1.0 * e - 0.73 * f &= -4.2857 \tag{10} \\ \tag{11} \end{align}$$

Solving using MATLAB, we get a unique P as,

$$P = \begin{bmatrix} 62.4340 & 14.0704 & 17.2674 \\ 14.0704 & 5.4040 & 2.1379 \\ 17.2674 & 2.1379 & 15.2422 \end{bmatrix} \tag{12}$$

### Part b)

Eigenvalues of P are 1.7375, 10.2607 and 71.0820.

### Part c)

Since P is symmetric, unique and postive definite (all eigen values $> 0$), we can't comment on the stability. We need such P for all Q to confirm stability. If this P was not positive definite or wasn't unique, it would've meant the system is unstable. But being unique and positive definite for this Q alone doesn't guarantee stability. Instead, we can compute the eigen values of A (using MATLAB):

$$\lambda = 0.7, 0.8, 0.9$$

Since $|\lambda| < 1$, we conclude that it is a stable system.

## Question-2

### Part a)

We know that P must be a symmetric matrix. Let, $P = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. Substituting in the given equation,

$$\begin{bmatrix} 9.6 & 2.8 \\ 9.6 & -2.8 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} + \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} 9.6 & 2.8 \\ 9.6 & -2.8 \end{bmatrix} = \begin{bmatrix} -20 & 0 \\ 0 & -20 \end{bmatrix}$$
$$\implies \begin{bmatrix} 5.6a + 19.2b & 9.6a + 9.6c \\ 9.6a + 9.6c & 19.2b - 5.6c \end{bmatrix} = \begin{bmatrix} -20 & 0 \\ 0 & -20 \end{bmatrix} \implies$$

$$5.6a + 19.2b = -20 \tag{13}$$
$$a = -c \tag{14}$$
$$-5.6c + 19.2b = -20 \tag{15}$$

We end up the third equation is just second equation substituted in the first. So we end up with just 2 independent linear equations with 3 variables (eqns 13 and 14). Therefore, solution for P is not unique.

### Part b)

Eigen values of $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$$\implies \lambda^2 - (a + c)\lambda + ac - b^2 = 0$$
$$\implies \lambda^2 - a^2 - b^2 = 0 (\because \text{ using } 14)$$
$$\implies \lambda = \pm\sqrt{a^2 + b^2}$$
$$\implies \lambda = \pm\sqrt{a^2 + \frac{(5.6a + 20)^2}{(19.2)^2}}$$

### Part c)

We see that the value under the square root is always positive. So, we always get a positive and a negative real eigen value. Therefore, the matrix P won't be positive definite. The system will be unstable at origin.

## Question-3

Writing the equations in state space form (with $\omega = x_1$ and $\dot{\omega} = x_2$),

$$\dot{x_1} = x_2 \tag{16}$$
$$\dot{x_2} = -g(x_1)x_2 - x_1 \tag{17}$$

To check for stability we can find the eigen values of the jacobian matrix at the origin.

$$J = \begin{bmatrix} 0 & 1 \\ -x_2 g'(x_1) - 1 & -g(x_1) \end{bmatrix}$$
$$\implies J = \begin{bmatrix} 0 & 1 \\ -1 & -g(0) \end{bmatrix} \text{(at origin)}$$

Characterstic equations,

$$\lambda^2 + \lambda * g(0) + 1 = 0$$

Product of roots positive so both eigenvalues will be of same sign. We want both of them to be negative to guarantee asymptotic stability (there are no other equilibrium points too). So sum of roots is negative.

$$\implies -g(0) < 0$$
$$\implies g(0) > 0$$

# Question-4

## Part a)

Apart from origin we see that it has other equilibrium points: $x_1 = 1, x_2 \in R$. So it can never be globally asymptotically stable. Let us take our Lyapunov function to be,

$$V = 0.5(x_1^2 + x_2^2); \tag{18}$$

$$\begin{aligned} \dot{V} &= x_1\dot{x_1} + x_2\dot{x_2} \\ &= x_2x_1(1 - x_1^2) - x_2x_1(1 - x_1^2) - x_2^2(1 - x_1^2) \\ \implies \dot{V} &= -x_2^2(1 - x_1^2) \\ \implies \dot{V} &\leq 0 \forall (-1 < x_1 < 1 \text{ and } x_2 \in R) \end{aligned}$$

So within $D : \{x_1, x_2 : x_1^2 < 1\}$, we have

- $V(x) > 0 \ \forall \ x \neq 0$
- $V(0) = 0$
- $\dot{V} \leq 0$ (= 0 only at origin)
- Only one equilibrium point in D

Therefore, the system is locally asymptotically stable.

## Part b)

Origin is the only equilibrium point for this system. Let us take our Lyapunov function to be,

$$V = 0.5x_1^2 - 0.5x_2^2 \tag{19}$$

$$\begin{aligned} \dot{V} &= x_1\dot{x_1} - x_2\dot{x_2} \\ &= x_1^4 + x_1^3x_2 + x_2^2 + x_2^3 - x_1x_2^2 + x_1^3x_2 \\ &= x_1^4 + 2x_1^3x_2 + x_2^2 - x_2^3 - x_1x_2^2 \\ &= (x_1^2 + x_1x_2)^2 + x_2^2(1 - x_2 - x_1 - x_1^2) \end{aligned}$$

Consider $A = \{(x_1, x_2) : (1 - x_2 - x_1 - x_1^2) > 0\}$. This region exists around the origin since origin itself is a part of this region (and the region is also continuous). In this region,

$$\dot{V} = (x_1^2 + x_1 x_2)^2 + k x_2^2 \text{ where, } k > 0$$

Also $V > 0$ where $x_1 > x_2$. Let $B = \{(x_1, x_2) : (x_1 = 0 \text{ and } x_2 = 0) \text{ or } x_1 > x_2\}$. Now in $D = A \cap B$,

- $V(x) > 0 \; \forall \; x \neq 0$
- $V(0) = 0$
- $\dot{V} > 0$ ($= 0$ only at origin)
- Equilibrium point in D

So, the equilibrium is unstable.

### Part c)

Origin is the only equilibrium point for this system. Let us take our Lyapunov function to be,

$$V = 0.25 x_1^4 + 0.5 x_2^2 \tag{20}$$

(taken so that the derivatives cancel out some terms)

$$\begin{aligned} \dot{V} &= x_1^3 \dot{x_1} + x_2 \dot{x_2} \\ &= x_1^3 x_2 - x_1^3 x_2 - x_2^3 x_2 \\ \implies \dot{V} &= -x_2^4 \\ \implies \dot{V} &\leq 0 \forall (x_2 \in R) \\ &= 0 \forall (x_1 \in R) \text{ and } x_2 = 0 \end{aligned}$$

- $V(x) > 0 \; \forall \; x \neq 0$
- $V(0) = 0$
- $\dot{V} \leq 0$

From above, we can conclude the point is stable but for asymptotic stability, we can try checking if we can use LaSalle's Theorem.
Set of all points with $\dot{V} = 0$ E $= \{(x_1, x_2) : x_2 = 0\}$. Largest invariant set $M \in E$:

$$\begin{aligned} &\dot{x_1} = 0 \text{ and } \dot{x_2} = 0 \\ &\implies x_2 = 0 \text{ and } x_1 = -x_2 \\ &\implies x_1 = x_2 = 0 \end{aligned}$$

So the set M has just one point: the origin. By invoking LaSalle's theorem we can say that all trajectories will converge to M. But M contains only one point - the origin. Therefore, all trajectories converge to origin. Thus, the given system is globally asymptotically stable.

## Question-5

### Part a)

$$\begin{aligned} &\dot{x_1} = 0 \text{ and } \dot{x_2} = 0 \\ &\implies x_1 - x_1^3 + x_2 = 0 \text{ and } 3x_1 = x_2 \\ &\implies 4x_1 - x_1^3 = 0 \\ &\implies x_1 = 0, \pm 2 \end{aligned}$$

So points of equilibrium: (-2,-6), (0,0), and (2,6)

## Part b)

Jacobian is, $\begin{bmatrix} -3x_1^2+1 & 1 \\ 3 & -1 \end{bmatrix}$.

### Equilibrium point: (0,0)

$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$
Eigen values,

$$\begin{aligned} \lambda^2 - 1 - 3 &= 0 \\ \lambda &= \pm 2 \end{aligned}$$

Since there is a positive and a negative eigen value, the equilibrium point is saddle.

### Equilibrium points: (-2,-6) and (2,6)

Both have same Jacobian, hence we will see same type of behaviour around the equilibrium point. $\begin{bmatrix} -11 & 1 \\ 3 & -1 \end{bmatrix}$
Eigen values,

$$\begin{aligned} \lambda^2 + 12\lambda + 8 &= 0 \\ \lambda &= \frac{-12 \pm \sqrt{112}}{2} \\ \lambda &= -0.70851, -11.291 \end{aligned}$$

Since the eigenvalues are less than 0, both the points are asymptotically stable.

## Part c)

We need to shift coordinates for examining the region of attraction at each equilibrium point. Firstly shift to, $x_1 = 2, x_2 = 6$ as the origin.

$$\begin{aligned} \implies \alpha &= x_1 - 2 \\ \beta &= x_2 - 6 \\ \implies \dot{\alpha} = \dot{x_1} &= -\alpha^3 - 6\alpha^2 - 11\alpha + \beta \\ \dot{\beta} &= 3\alpha - \beta \end{aligned}$$

Similarly for other point,

$$\begin{aligned} \implies a &= x_1 + 2 \\ b &= x_2 + 6 \end{aligned}$$

In the derivative equation replace $\alpha, \beta$ with -a,-b (that will get equations for $a = -x_1 - 2$) and then do a whole minus. Since we won't have a constant term (equilibrium is origin), we will get the same set of equations. So by symmetry, results we show for one of the equilibrium points will hold for the other.
Now we focus just on (2,6). Linearizing the system, $A = \begin{bmatrix} -11 & 1 \\ 3 & -1 \end{bmatrix}$
Since the point is stable we can solve the Lyapunov equation to obtain a positive definite P for all positive definite Q. Putting Q = -I, we obtain,

$$P = \begin{bmatrix} 0.0938 & 0.1771 \\ 0.1771 & 0.6771 \end{bmatrix} \tag{21}$$

Consider $V = x^T P x$ as the lyapunov function. (Valid because P is positive definite). Now,

$\dot{V} = 0.0938 * (-\alpha^3 - 6\alpha^2 - 11\alpha + \beta) + 0.6771 * (3\alpha - \beta) + 0.1771 * ((-\alpha^3 - 6\alpha^2 - 11\alpha + \beta) * \beta + \alpha * (3\alpha - \beta))$

Plotting the level sets of P, along with the region $\dot{V} \leq 0$, we find that the contour V $= 0.05$ lies within the region $\dot{V} \leq 0$ (fig. 1). So in that region we have local asymptotic stability ($\dot{V} = 0$ only beyond the borders of the region apart from the origin). Therefore that region $\{V(x) < 0.05\}$ can be considered as a region of attractors.


Figure 1: V $= \beta = 0.05$ such that $\dot{V} < 0$

## Part d)

From fig. 2 we can see that our estimate is valid and is in fact, conservative and we can have a larger estimate of the level set of V.

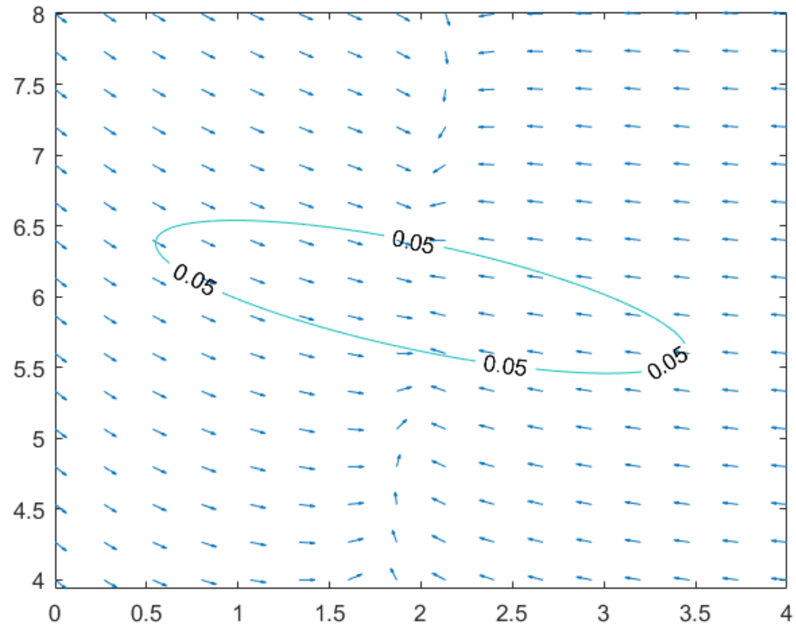


Figure 2: Region of attraction: V $\leq$ 0.5

## Question-6

### Parts a) and c)

$$V = 0.5(x_1^2 + x_2^2); \tag{22}$$

$$\begin{aligned}
\dot{V} &= x_1\dot{x_1} + x_2\dot{x_2} \\
&= x_1(h(t)x_2 - g(t)x_1^3) + x_2(-h(t)x_1 - g(t)x_2^3) \\
&= -g(t)(x_1^4 + x_2^4) \\
\implies \dot{V} &< 0 \forall x \neq 0 (\because g(t) > 0) \\
&= 0 \text{ if } x = 0
\end{aligned}$$

- $V(x) > 0 \ \forall \ x \neq 0$
- $V(0) = 0$
- $\dot{V} \leq 0$ ($= 0$ only at origin)
- Only one equilibrium point in $\mathbb{R}^n$
- V is radially unbounded

From the above points we conclude that the system is globally uniformly asymptotically stable. If it is globally uniformly asymptotically stable then it is locally uniformly asymptotically stable. So, the answers are:
**a)** Yes, it is locally uniformly asymptotically stable.
**c)** Yes, it is globally uniformly asymptotically stable.

### Parts b) and d)

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) = -g(t)(x_1^4 + x_2^4)$$

We can't bound this by some norm of x because g(t) is involved and we don't know whether it has a supremum. This rules out global exponential stability. We now need to check for local exponential stability. For local exponential stability, we can linearise and check for stability. For linear systems if it is stable, it means it is exponentially stable (because solution of $\dot{x} = Ax$ is $x = exp(At)$). And hence, we can use that to conclude local exponential stability of the original nonlinear system.
Jacobian at the origin, $\begin{bmatrix} 0 & h(t) \\ -h(t) & 0 \end{bmatrix}$.

$$\implies \lambda^2 + h(t)^2 = 0$$
$$\implies \lambda = \sqrt{-h(t)}$$

We know that h(t) is a real valued function, so $\sqrt{-h(t)}$ is an imaginary number of the form $\pm\omega j$.
$\implies$ Real part of $\lambda = 0$.
So the linearised system is not stable. This means that the nonlinear system is not exponentially stable. So, the answers for parts b) and d) are:
**b)** No. It is not exponentially stable.
**d)** No. It is not globally exponentially stable.

# Question-7

LaSalle's theorem states that as $\lim_{t\to\infty} x(t) \to M$. By the definition of limit,

$$\min_{m\in M} \|x(t) - m\| < \epsilon \forall t > T$$

(x keeps getting close to M means its minimum distance to the points in the set keeps decreasing before it finally becomes 0 in the limit).
In this case since we have isolated points, let's fix an $\epsilon < 0.5 \min_{m_i,m_j}(\|m_i - m_j\|)$ where $m_i, m_j inM$.
So, at time $t > T$, we have for some $x^* \in M$,

$$\|x(t) - x^*\| < \epsilon \tag{23}$$

By continuity of x(t), if it has to escape this epsilon bound of $x^*$ after this, at some $t_0 > T$ it has to go to $\|x(t_0) - x^*\| = \epsilon$. But, by definition of limit, $x(t_0)$ has to be in epsilon bound of the set M! Let,

$$\|x(t_0) - \tilde{x}\| < \epsilon \tag{24}$$

$$\begin{aligned} \|x(t_0) - \tilde{x}\| &= \|x(t_0) - x^* + x^* - \tilde{x}\| \\ &\geq \|x^* - \tilde{x}\| - \|x(t_0) - \tilde{x}\| \\ &\geq 2\epsilon - \epsilon \\ &\geq \epsilon \end{aligned}$$

But by definition of $\tilde{x}$ and limit, we had equation 24. We have a contradiction! Therefore, only way to resolve this is allowing,

$$\|x(t) - x^*\| < \epsilon \forall t > T \tag{25}$$
$$\implies \lim_{t\to\infty} x(t) = x^* \tag{26}$$

$\therefore \lim_{t\to\infty} x(t)$ exists and is equal to some point $x^* \in M$

# Question-8

## Part a) Checking radial unboundedness

Writing V in its polar form,

$$V(r,\theta) = \frac{r^2(cos(\theta) + sin(\theta))^2}{1 + r^2(cos(\theta) + sin(\theta))^2} + r^2(cos(\theta) - sin(\theta))^2$$

Consider moving along the line $cos(\theta) = sin(\theta)$

$$V = \frac{r^2(cos(\theta) + sin(\theta))^2}{1 + r^2(cos(\theta) + sin(\theta))^2}$$

Now, taking $\lim_{r\to\infty}$,

$$\begin{aligned}
\lim_{r\to\infty} V &= \lim_{r\to\infty} \frac{r^2(cos(\theta) + sin(\theta))^2}{1 + r^2(cos(\theta) + sin(\theta))^2} \\
&= \lim_{r\to\infty} \frac{(cos(\theta) + sin(\theta))^2}{\frac{1}{r^2} + (cos(\theta) + sin(\theta))^2} \\
&= 1 \\
\implies \lim_{r\to\infty} V &\neq \infty
\end{aligned}$$

Therefore, V is not radially unbounded. Hence, it is NOT a valid Lyapunov candidate function to deduce the global asymptotic stability of an equilibrium point.

## Part b) Plotting level sets

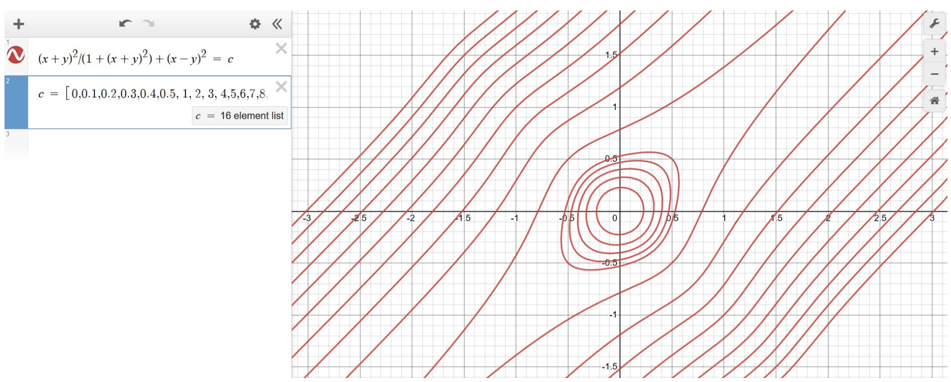


Figure 3: Level sets

From figure 3, we see that the level sets are open beyond a certain value (along $x_1 = x_2$). So, the function is radially bounded along the line and therefore, is NOT a valid Lyapunov candidate function to deduce the global asymptotic stability of an equilibrium point.

# References


- Students discussed with:
  1. Arvind Ragghav ME18B086
  2. Karthik Srinivasan ME18B149
- Course notes used:
  1. Class notes
  2. Linear systems slides
- Hassan Khalil