Assignment-2

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March 7, 2022

Question-1

Each element can be either one or zero, so totally there will be 2*2*2 = 8 triples.
If $x = x_1x_2x_3$ and $y = y_1y_2y_3$, the metric can be represented as

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| \tag{1}$$

Conditions which have to be followed by a metric:

$$d(x, y) \geq 0 \text{ and } = 0 \text{ iff } x = y \tag{2}$$
$$d(x, y) = d(y, x) \tag{3}$$
$$d(x, z) \leq d(x, y) + d(y, z) \tag{4}$$

The metric is obviously positive for all elements in X. And, it is equal to zero only when $x_i = y_i \forall i$. So equation 2 is followed. Also,

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| \quad = |y_1 - x_1| + |y_2 - x_2| + |y_3 - x_3| = d(y, x) \implies d(x, y) \quad = d(y, x)$$

So equation 3 is obeyed.

$$\begin{aligned} d(x, y) + d(y, z) &= |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| + |z_1 - y_1| + |z_2 - y_2| + |z_3 - y_3| \\ &= (|x_1 - y_1| + |-z_1 + y_1|) + (|x_2 - y_2| + |-z_2 + y_2|) + (|x_3 - y_3| + |-z_3 + y_3|) \\ &\geq |x_1 - z_1| + |x_2 - z_2| + |x_3 - z_3| \end{aligned}$$

$\implies d(x, z) \leq d(x, y) + d(y, z)$

$\therefore$ d(.) is indeed a metric.

Question-2

Integral of a positive value is always positive, $\implies d(x, y) > 0$

$$\begin{aligned} d(x, y) &= \int_a^b |x(t) - y(t)|\, dt \\ &= \int_a^b |y(t) - x(t)|\, dt \\ &= d(y, x) \end{aligned}$$

$$\begin{aligned} d(x, x) &= 0 \text{ iff } |x - y| = 0 \\ \implies d(x, x) &= 0 \text{ iff } x = y \end{aligned}$$

First two conditions satisfied.

$$d(x, y) + d(y, z) = \int_a^b |x(t) - y(t)| + |y(t) - z(t)|\, dt \tag{5}$$
$$>= \int_a^b |x(t) - y(t) + y(t) - z(t)|\, dt \because |a| + |b| \geq |a + b| \tag{6}$$
$$>= d(x, z) \tag{7}$$

$\therefore$ d(.) is indeed a metric.

Question-3

Since $\sum_{j=1}^{\infty} \frac{1}{2^j}$ converges (GP), it is just a subset of the second case. If we prove for any r_j, we prove for $\frac{1}{2^j}$ also. So, let's prove the general case itself. d(x,x) = 0, and d(.) is non-zero otherwise, because it is a sum of positive terms.

$$d(x,y) = \sum_{j=1}^{\infty} r_j \frac{|\zeta_j - \eta_j|}{1 + |\zeta_j - \eta_j|} \tag{8}$$

$$\leq \sum_{j=1}^{\infty} r_j \because \frac{|\zeta_j - \eta_j|}{1 + |\zeta_j - \eta_j|} < 1 \tag{9}$$

$$\tag{10}$$

Since we are given $\sum_{j=1}^{\infty} r_j$ converges, d(x,y) must be finite (must converge).
Consider the sequences, $x = (\alpha_j)$, $y = (\beta_j)$, $z = (\gamma_j)$. Let us compare the individual terms of d(x,y) + d(y,z) and d(x,z). If we can prove an inequality for the individual positive terms, the same inequality will be followed by the sum.

$$d(x,y) + d(y,z) = \sum_{j=1}^{\infty} r_j \frac{|\alpha_j - \beta_j|}{1 + |\alpha_j - \beta_j|} + \sum_{j=1}^{\infty} r_j \frac{|\beta_j - \gamma_j|}{1 + |\beta_j - \gamma_j|} \tag{11}$$

$$= \sum_{j=1}^{\infty} r_j \frac{|\alpha_j - \beta_j| + |\beta_j - \gamma_j| + |\alpha_j - \beta_j||\beta_j - \gamma_j|}{1 + |\alpha_j - \beta_j| + |\beta_j - \gamma_j| + |\alpha_j - \beta_j||\beta_j - \gamma_j|} \tag{12}$$

Note that,

$$\frac{a}{1+a} > \frac{b}{1+b} \text{ when } a > b \tag{13}$$

$$|\alpha_j - \beta_j| + |\beta_j - \gamma_j| + |\alpha_j - \beta_j||\beta_j - \gamma_j| \tag{14}$$

$$>= |\alpha_j - \gamma_j| + |\alpha_j - \beta_j||\beta_j - \gamma_j| \because \text{ triangle inequality} \tag{15}$$

So,

$$12, 13, 15 \implies d(x,z) \leq d(x,y) + d(y,z) \tag{16}$$

$\therefore$ d(.) is indeed a metric.

Question-4

$$d(x,y) \geq 0 (\because \sqrt{a} \geq 0) \& d(x,y) = 0 \text{ iff } x = y$$
$$d(x,y) = d(y,x) (\because \text{ mod is symmetric})$$

Note that

$$\sqrt{a} + \sqrt{b} \leq \sqrt{a+b} \tag{17}$$

$$d(x,y) + d(y,z) = \sqrt{|x-y|} + \sqrt{|y-z|}$$
$$\geq \sqrt{|x-y| + |y-z|} \because 17$$
$$\geq \sqrt{|x-z|} \because \text{ triangle inequality}$$

$\therefore$ d(.) is indeed a metric.

Question-5

Cauchy implies Boundedness

Since the sequence is Cauchy, for some ϵ we have,

$$|x_m - x_n| < \epsilon, \forall m, n \geq N$$

Fix n = N

$$\implies x_N - \epsilon < x_m < x_N + \epsilon \ \forall \ m, n \geq N$$

So terms after x_N are bounded as above. So the entire sequence is bounded by,

$$max(x_1, x_2, x_3, ..., x_N, x_N - \epsilon, x_N + \epsilon)$$

and

$$min(x_1, x_2, x_3, ..., x_N, x_N - \epsilon, x_N + \epsilon)$$

. $\therefore$ Cauchy sequences are bounded.

Boundedness for Cauchy and Convergence

No. Boundedness is not sufficient to prove that the sequence is Cauchy and that the sequence converges. Look at the sequence $x_n = (-1)^n$. It is bounded by -1 and 1 but is not Cauchy (difference between consecutive elements is always 2) and doesn't converge.

Question-6

a) kd

Repeatedly using d(.) is a metric,

$$kd(x,y) \geq 0 \implies k > 0 \text{ and } = 0 \text{ iff } x = y (\text{property of d}) \tag{18}$$

$$kd(x,y) = kd(y,x) \forall k (\because d(x,y) = d(y,x)) \tag{19}$$

$$kd(x,z) \leq kd(x,y) + kd(y,z) \forall k (\because d(x,z) \leq d(x,y) + d(y,z)) \tag{20}$$

So, kd is a metric $\forall k > 0$

b) k + d

$$k + d(x,y) \geq 0 \implies k > 0 \tag{21}$$

$$\text{and } d(x,y) = 0 \text{ iff } x = y (\text{property of d}) \implies k = 0 \tag{22}$$

If we put k = 0, then the metric is same as d. So rest of the properties will be satisfied. Only allowable k is, k = 0.

Question-7

Conditions which have to be followed by a metric:

$$d(x,y) \geq 0 \text{ and } = 0 \text{ iff } x = y \tag{23}$$

$$d(x,y) = d(y,x) \tag{24}$$

$$d(x,z) \leq d(x,y) + d(y,z) \tag{25}$$

Conditions 23 and 24 are satisfied by the given norm, since square of reals is non-negative and $(x-y)^2 = (y-x)^2$. However, condition 25 is violated. We can show a counter example.

Consider x = 0, z = 2, y = 1.
$RHS = (0-2)^2 = 4$
$LHS = (0-1)^2 + (1-2)^2 = 1 + 1 = 2$

$$\implies LHS \geq RHS$$

So 25 is violated. Therefore, $(x-y)^2$ is not a metric.

Question-8

We will first prove the b) part and use the result to prove a).

b)

$$\begin{aligned} d(y,z) &\leq d(x,y) + d(x,z) \text{ (property of metric space)} && (26) \\ \implies d(y,z) - d(x,z) &\leq d(x,y) && (27) \\ \implies -d(x,y) &\leq d(x,z) - d(y,z) && (28) \end{aligned}$$

We also see,

$$\begin{aligned} d(x,z) &\leq d(x,y) + d(y,z) \text{ (property of metric space)} && (29) \\ d(x,z) - d(y,z) &\leq d(x,y) && (30) \end{aligned}$$

From eqns 28, 30 we have,

$$|d(x,z) - d(y,z)| \leq d(x,y) \tag{31}$$

Hence, proved.

a)

$$\begin{aligned} |d(x,y) - d(z,w)| &= |d(x,y) - d(y,z) + d(y,z) - d(z,w)| \\ &\leq |d(x,y) - d(y,z)| + |d(z,w) - d(y,z)| (\because \text{ traingle inequality }) \end{aligned}$$

Using the inequality from eqn 31 once for each of the mod, we have,

$$|d(x,y) - d(z,w)| \leq d(x,z) + d(y,w)$$

Hence, proved.

Question-9

a)

Convergence

As we take $n \to \infty$,

$$\lim_{n\to\infty} a_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n}$$

The sum of sets of 2^n terms (i.e. 1st and 2nd term, second, third, fourth and fifth term, etc) is at least $\frac{2^n - 2^{n-1}}{2^n} = \frac{1}{2}$. So when we have infinite terms, the sum diverges to infinity.
We see that RHS diverges, so a_n diverges.

Cauchy Sequence

Without loss of generality, assume, $m > n$. Then, we see,

$$\begin{aligned}|x_m - x_n| &= |\frac{1}{n} + \frac{1}{n+1} + ... + \frac{1}{m-1} + \frac{1}{m}| \\ &\geq |\frac{m-n}{m}| (\because m > n \implies \frac{1}{m} < \frac{1}{n})\end{aligned}$$

We see that as we keep increasing m, the lower bound keeps increasing, so we can't have an N such that,

$$|x_m - x_n| \leq \epsilon \forall m, n > N$$

Therefore, the sequence is not Cauchy.

b)

Convergence

We know,

$$\begin{aligned}F_n &= F_{n-1} + F_{n-2} \\ \frac{F_n}{F_{n-1}} &= 1 + \frac{F_{n-2}}{F_{n-1}} \\ \implies a_n &= 1 + \frac{1}{a_{n-1}} \\ \implies \lim_{n\to\infty} a_n &= \lim_{n\to\infty} 1 + \frac{1}{a_{n-1}} \\ \implies a^* &= 1 + \frac{1}{a^*} \\ \implies a^* &= \frac{1+\sqrt{5}}{2} (\because a^* > 0)\end{aligned}$$

So if the limit exists, it is irrational, i.e. $a^* \notin Q$. Therefore, the sequence doesn't converge in Q

Cauchy Sequence

One way to show this is, since the sequence converges in the Real space, then it must be Cauchy, because Cauchy is a necessary condition for convergence.

Given the Fibonacci sequence where $F_{n+1} = F_n + F_{n-1}$, we have with $a_n = \frac{F_{n+1}}{F_n}$,

$$|a_{n+1} - a_n| = \left| \frac{F_{n+2}}{F_{n+1}} - \frac{F_{n+1}}{F_n} \right| \tag{32}$$

$$= \left| \frac{F_{n+2}F_n - F_{n+1}^2}{F_{n+1}F_n} \right| \tag{33}$$

$$= \left| \frac{F_{n+1}F_n + F_n^2 - F_{n+1}F_n - F_{n+1}F_{n-1}}{F_n^2 + F_nF_{n-1}} \right|. \tag{34}$$

Note that the sequence is increasing and $F_n^2 + F_nF_{n-1} > 2F_nF_{n-1}$.

Hence,

$$|a_{n+1} - a_n| < \left| \frac{F_{n-1}^2 - F_nF_{n-2}}{2F_{n-1}F_{n-2}} \right| \tag{35}$$

$$= \frac{1}{2} \left| \frac{F_n}{F_{n-1}} - \frac{F_{n-1}}{F_{n-2}} \right| \qquad = \frac{1}{2} |a_n - a_{n-1}| \tag{36}$$

Repeatedly using this for n-1 times, we will arrive at, $|a_{n+1} - a_n| < \left(\frac{1}{2}\right)^{n-1} \left|\frac{F_2}{F_1} - \frac{F_1}{F_0}\right|$

$$\implies \lim_{n\to\infty} |a_{n+1} - a_n| \to 0 \tag{37}$$

Since the difference between terms goes to zero, the sequence is Cauchy.

References

- Students discussed with:
  1. Arvind Ragghav ME18B086
  2. Karthik Srinivasan ME18B149
- Course notes used:
  1. Class notes
- Math stack exchange (golden ratio proof)