

Assignment-2

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Question-1

Setting derivatives to zero,

$$\begin{aligned}y + y^2 = 0 &\implies y(y + 1) = 0 \\&\implies y = 0, -1\end{aligned}$$

$$-x + \frac{1}{5}y - xy + \frac{6}{5}y^2 = 0$$

If $y = 0$,

$$\begin{aligned}-x + 0 - 0 + 0 &= 0 \\&\implies x = 0\end{aligned}$$

If $y = -1$,
 $-x - \frac{1}{5} + x - \frac{6}{5} = 0$
 \implies no solution.

Therefore only equilibrium point is $(0,0)$.

$$\begin{aligned}y + y^2 = 0 &\implies y(y + 1) = 0 \\&\implies y = 0, -1\end{aligned}$$

Linearisation expression,

$$\dot{x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \tilde{x} + \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{bmatrix} \tilde{u} \quad (1)$$

where $\tilde{x} = x - x^*$ and $\tilde{u} = u - u^*$ and x^* and u^* is the point around which we are linearising. (an equilibrium/steady state). The derivatives are evaluated at the point around which we are linearising.

$$A = \begin{bmatrix} 0 & 1 + 2y \\ -1 - y & \frac{1}{5} + \frac{12}{5}y^2 - x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & \frac{1}{5} \end{bmatrix}$$

$$\dot{x} = \tilde{y}$$

$$\dot{y} = -\tilde{x} + \frac{1}{5}\tilde{y}$$

Question-2

Setting $\dot{r} = 0$,

$$r(1 - r^2) = 0 \quad (2)$$

$$\implies r = 0, 1 \quad (3)$$

$$1 - \cos(\theta) = 0 \quad (4)$$

$$\implies \theta = 2n\pi \quad n \in Z \quad (5)$$

(note that r can't be negative.)

we get the jacobian (matrix A) as, $\begin{bmatrix} 1 - 3r^2 & 0 \\ 0 & \sin(\theta) \end{bmatrix}$

$\mathbf{r} = \mathbf{0}, \theta = 2n\pi$

Linearising around $\mathbf{r} = 0$, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$\dot{r} = \tilde{r}$$

$$\dot{\theta} = 0$$

$\mathbf{r} = \mathbf{1}, \theta = 2n\pi$

Linearising around $\mathbf{r} = 1$, $A = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$

$$\dot{r} = -2\tilde{r}$$

$$\dot{\theta} = 0$$

where, $\tilde{r} = r - r^*$

Question-3

Setting derivatives to 0,

$$\begin{aligned} \sin(y) &= 0 \\ \implies y &= \pm n\pi x - x^3 &= 0 \\ \implies x &= -1, 0, 1 \end{aligned}$$

To explain the behaviour around equilibrium points we need to find the eigen values of A matrix.

$$A = \begin{bmatrix} 0 & \cos(y) \\ 1 - 3x^2 & 0 \end{bmatrix}$$

Finding eigen values,

$$\begin{aligned} \begin{vmatrix} 0 - \lambda & \cos(y) \\ 1 - 3x^2 & 0 - \lambda \end{vmatrix} &= 0 \\ \implies \lambda^2 &= (1 - 3x^2)\cos(y) \end{aligned}$$

x, y	$y = 2n\pi$	$y = (2n + 1)\pi$
$x = 0$	$\lambda^2 > 0 \implies \lambda = \pm c \therefore \text{Saddle}$	$\lambda^2 < 0 \implies \lambda = \pm j \therefore \text{center focus}$
$x = 1$	$\lambda^2 < 0 \implies \lambda = \pm j \therefore \text{center focus}$	$\lambda^2 > 0 \implies \lambda = \pm c \therefore \text{Saddle}$
$x = -1$	$\lambda^2 < 0 \implies \lambda = \pm j \therefore \text{center focus}$	$\lambda^2 > 0 \implies \lambda = \pm c \therefore \text{Saddle}$

(6)

Question-4

Note that $x + y + z = 1$ (since they are fractions). So instead of using 3 dependent states, we can reduce the system to just 2 states.

$$\dot{x} = rx(1 - x - y) \quad (7)$$

$$\dot{y} = ry(1 - x - y) \quad (8)$$

$$(9)$$

Origin is an equilibrium point, and $x + y = 1$ is a equilibrium subspace. The Jacobian is,

$$\begin{bmatrix} r(1 - 2x - y) & -rx \\ -ry & r(1 - x - 2y) \end{bmatrix} \quad (10)$$

At origin

$$\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \quad (11)$$

Eigen values, $\lambda = r, r$. So origin is a stable equilibrium if $r < 0$ which means $x=y=0$. Therefore, the situation where everyone is a centrist is the stable equilibrium in this case. And if a small number of people become rightist or leftist, they are brought back to being centrist.

On the other hand if $r > 0$, the equilibrium is unstable. So if some people become rightist or leftist, the number of centrists starts reducing (moving away from equilibrium). And if $r = 0$ obviously, there is no dynamics.

Subspace $x + y = 1$

Parameterise the points as $\alpha, 1 - \alpha$.

$$\begin{bmatrix} -r\alpha & -r\alpha \\ -r(1 - \alpha) & -r(1 - \alpha) \end{bmatrix} \quad (12)$$

Eigen values of the matrix are $\lambda = 0, -r$. If $r > 0$ (hence, $\lambda < 0$) all the trajectories converge to the equilibrium space. If $r < 0$ (hence, $\lambda > 0$) diverge from the equilibrium space. This means for $r > 0$, the population evolves such that centrists population becomes 0 ($\because z = 1 - x - y = 0$). On the other hand, if $r < 0$, small perturbation from the equilibrium space causes continuous evolution away from it (resulting in everyone becoming centrists as deduced by observing stability of the equilibrium at origin).

Question-5

Let

$$x = \begin{bmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \end{bmatrix} \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

So the equations governing the dynamics are,

$$\dot{x}_1 = x_2 \quad (13)$$

$$\dot{x}_2 = x_1 x_4^2 - \frac{\beta}{x_1^2} + u_1 \quad (14)$$

$$\dot{x}_3 = x_4 \quad (15)$$

$$\dot{x}_4 = \frac{-2x_2 x_4}{x_1} + \frac{u_2}{x_1} \quad (16)$$

Computing the Jacobian (from eqn 1),

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \omega_0^2 + \frac{2\beta}{r_0^2} & 0 & 0 & 2r_0\omega_0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-2\omega_0}{r_0} & 0 & 0 \end{bmatrix} \quad (17)$$

where $\omega_0 = \sqrt{\frac{\beta}{r_0^3}}$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{r_0} \end{bmatrix} \quad (18)$$

The equations governing the linearized system are,

$$\dot{x} = A\tilde{x} + B\tilde{u} \quad (19)$$

$$x = \begin{bmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \end{bmatrix} u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (20)$$

$$\tilde{x} = x - x^* \quad \tilde{u} = u - u^* \quad (21)$$

$$\text{where } x^* = \begin{bmatrix} r_0 \\ 0 \\ \omega_0 t + \theta_0 \\ \dot{\theta} \end{bmatrix} \quad (22)$$

$$\text{and, } u^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (23)$$

Question-6

Only equilibrium point is origin. Jacobian at equilibrium point is found to be,

$$\begin{bmatrix} \sin(1) & 1 \\ -1 & \sin(1) \end{bmatrix}$$

Let's convert the equations to polar form.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \dot{r} &= \frac{x\dot{x}}{\sqrt{x^2 + y^2}} + \frac{y\dot{y}}{\sqrt{x^2 + y^2}} \\ &= (x^2 + y^2)(x^2 + y^2 - 1)\sin\left(\frac{1}{x^2 + y^2 - 1}\right)\frac{1}{r} \\ &= \frac{(r^2)(r^2 - 1)\sin\left(\frac{1}{r^2 - 1}\right)}{r} \\ \implies \dot{r} &= \frac{(r^2)(r^2 - 1)\sin\left(\frac{1}{r^2 - 1}\right)}{r} \end{aligned} \quad (24)$$

$$\begin{aligned}
\dot{\theta} &= \frac{d}{dt} \tan^{-1}\left(\frac{y}{x}\right) \\
&= \frac{1}{1 + \frac{y^2}{x^2}} \left(\frac{\dot{y}}{x} - \frac{\dot{x}y}{x^2} \right) \\
&= \frac{x^2}{r^2} \left(\frac{-r^2}{x^2} \right) \\
\Rightarrow \dot{\theta} &= -1
\end{aligned}$$

We see $\dot{r} = 0$ has infinite solutions because of the sine term, and $\dot{\theta}$ is always a constant. So there are infinitely many limit cycles (i.e., where $\dot{r} = 0$ and $\dot{\theta}$ is a constant).

Question-7

a)

Jacobian yields,

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\lambda = -1, -1$$

(The derivative of the log term goes to zero in the limit).
So the origin is a stable node.

b)

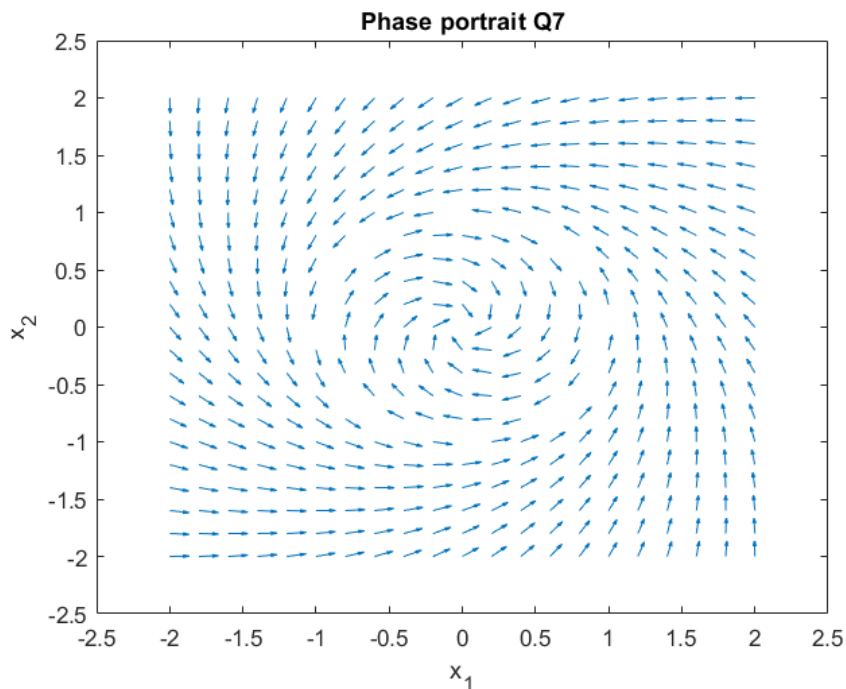


Figure 1: Phase Portrait Q7b

From the figure, we can see that trajectories close to origin spiral towards it, showing that it is a stable focus.

c)

Although the function is continuously differentiable, at origin, the derivatives converge to the respective values only in the limit. So, the Taylor series expansion is likely to not converge. This means the function is not in the neighbourhood of $x_1, x_2 = (0, 0)$. Since the function is not analytic, the linearization result differs from reality.

Question-8

a)

We can recast the system in state-space form (with y and \dot{y} as the respective states) as,

$$\dot{x}_1 = x_2 \dot{x}_2 = -x_1 - \epsilon(x_2)(1 - x_1^2 - x_2^2)$$

Jacobian is,

$$\begin{bmatrix} 0 & 1 \\ -1 & \epsilon \end{bmatrix} \implies \lambda = \epsilon \pm j$$

The equilibrium is unstable for $\epsilon > 0$. Consider,

$$V(x) = x_1^2 + x_2^2 = c. \quad (25)$$

$$\bar{\nabla} V = 2x_1 \hat{i} + 2x_2 \hat{j} \quad (26)$$

Now we find a c such that

$$\begin{aligned} \bar{\nabla} V \cdot f(x) &\leq 0 \\ \implies 2x_1 x_2 + 2x_2(-x_1 + \epsilon(1 - x_1^2 - x_2^2)) &\leq 0 \\ \implies \epsilon(1 - x_1^2 - x_2^2) &\leq 0 \\ \forall \epsilon > 0, \\ x_1^2 + x_2^2 \leq 1 &\implies c \leq 1 \end{aligned}$$

So, trajectories starting in $M = x_1^2 + x_2^2 \leq c$ (where c follows the above condition) stays inside it. Therefore, by Poincare-Benedixson criterion a stable orbit exists.

b)

Jacobian is,

$$\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \implies \lambda = 1, 1$$

So, the equilibrium is unstable. Consider,

$$V(x) = x_1^2 + x_2^2 = c. \quad (27)$$

$$\bar{\nabla} V = 2x_1 \hat{i} + 2x_2 \hat{j} \quad (28)$$

$$(29)$$

Now we find a c such that

$$\begin{aligned} \bar{\nabla} V \cdot f(x) &\leq 0 \\ \implies x_1 x_2 - x_1 x_2 + 2x_2^2(2 - 3x_1^2 - 2x_2^2) &\leq 0 \\ \implies (3x_1^2 + 2x_2^2) &\geq 2 \\ \iff 2x_1^2 + 2x_2^2 &\geq 2 \\ \implies c &\geq 1 \end{aligned}$$

So, trajectories starting in $M = x_1^2 + x_2^2 \leq c$ (where c follows the above condition) stays inside it. Therefore, by Poincare-Benedixson criterion a stable orbit exists.

c)

Jacobian is,

$$\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow \lambda = \frac{1 \pm \sqrt{3}j}{2}$$

The equilibrium is unstable for $\epsilon > 0$. Consider,

$$V(x) = 3x_1^2 + 2x_1x_2 + 2x_2^2 = c. \quad (30)$$

$$\bar{\nabla}V = (6x_1 + 2x_2)\hat{i} + (2x_1 + 4x_2)\hat{j} \quad (31)$$

Now we find a c such that

$$\begin{aligned} \bar{\nabla}V \cdot f(x) &\leq 0 \\ \Rightarrow (6x_1 + 2x_2)x_2 + (2x_1 + 4x_2)(-x_1 + x_2 - 2(x_1 + 2x_2)x_2^2) &\leq 0 \\ \Rightarrow 6x_1x_2 + 2x_2^2 - 2x_1^2 + 2x_1x_2 - 4x_1x_2 + 4x_2^2 - 4(x_1 + 2x_2)^2x_2^2 &\leq 0 \\ \Rightarrow -2x_1^2 + 4x_1x_2 + 6x_2^2 - 4(x_1 + 2x_2)^2x_2^2 &\leq 0 \\ \Rightarrow -2(x_1^2 + x_2^2) + 4x_2(x_1 + 2x_2) - 4(x_1 + 2x_2)^2x_2^2 &\leq 0 \\ \Rightarrow -2(x_1^2 + x_2^2) + 1 - (1 - 2x_2(x_1 + 2x_2))^2 &\leq 0 \\ -2(x_1^2 + x_2^2) + 1 - (1 - 2x_2(x_1 + 2x_2))^2 &\leq -2(x_1^2 + x_2^2) + 1 \leq 0 \\ \Rightarrow x_1^2 + x_2^2 &\geq -\frac{1}{2} \end{aligned}$$

So, trajectories starting in $M = 3x_1^2 + 2x_1x_2 + 2x_2^2 \leq c$, where c is such that the surface $V(x) = c$ contains the circle $x_1^2 + x_2^2 = \frac{1}{2}$ in its interior, stays inside it. Therefore, by Poincare-Benedixson criterion a stable orbit exists.

Question-9

Equilibrium point is origin.

Using the fact that $\frac{d}{dx} \tanh(x) = \text{sech}^2(x)$, and $\tanh(0) = 0$, the Jacobian is found to be,

$$\begin{bmatrix} \frac{-1}{\tau} + \lambda & -\lambda \\ \lambda & \frac{-1}{\tau} + \lambda \end{bmatrix}$$

Eigen values were found to be, $\lambda = -\frac{1}{\tau} \pm j$. The real part is greater than zero as long as $\lambda\tau > 1$. Therefore, the equilibrium is unstable. Consider,

$$V(x) = x_1^2 + x_2^2 = c. \quad (32)$$

$$\bar{\nabla}V = 2x_1\hat{i} + 2x_2\hat{j} \quad (33)$$

Now we find a c such that

$$\begin{aligned} \bar{\nabla}V \cdot f(x) &\leq 0 \\ \Rightarrow 2x_1\left(\frac{-x_1}{\tau} + \tanh(\lambda x_1) - \tanh(\lambda x_2)\right) + 2x_2\left(\frac{-x_2}{\tau} + \tanh(\lambda x_1) + \tanh(\lambda x_2)\right) &\leq 0 \\ \Rightarrow -x_1^2 - x_2^2 + x_1\tanh(\lambda x_1) - x_1\tanh(\lambda x_2) + x_2\tanh(\lambda x_1) + x_2\tanh(\lambda x_2) &\leq 0 \end{aligned}$$

Substituting $|\tanh(x)| = 1$, $x_1 = r\cos(\theta) \leq r$, and $x_2 = r\sin(\theta) \leq r$,

$$-\frac{r^2}{\tau} + r + r + r + r \leq 0$$

$$\implies r > 4\tau \implies c > 16\tau^2$$

So trajectories starting in $M = r \leq c$ (where c follows the above condition) stays inside it. Therefore, by Poincare-Benedixson criterion a stable orbit exists.

MATLAB code for q7

Contents

- derivatives
- Phase portrait
- function to evaluate derivatives in a mesh

```
clear; close all;
```

derivatives

```
f1 = @(x1,x2)(-x1 - x2/log(sqrt(x1^2+x2^2)));
f2 = @(x1,x2)(-x2 + x1/log(sqrt(x1^2+x2^2)));
```

Phase portrait

```
lim = 2; N = 21;
[X,Y,U,V] = derivatives(lim,N,f1,f2);
% Use quiver to plot; S = 0.75 to ensure arrows don't intersect
figure;
quiver(X,Y,U,V,0.5);
xlabel('x_1'); ylabel('x_2');
title('Phase portrait Q7');
```

function to evaluate derivatives in a mesh

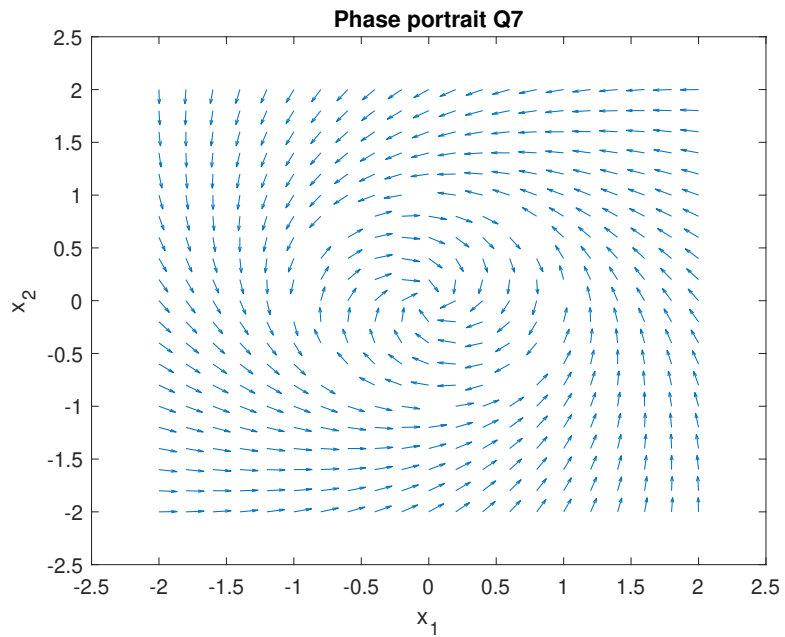
```
function [X,Y,U,V] = derivatives(lim,N,f1,f2)
% -----
% function to generate derivatives
% lim - limits of the x,y grid of the plot. derivatives will be
% evaluated at points between -lim to +lim
%
% N - number of points on each axis
%
% f1 - x derivative (xdot)
% f2 - y derivative (ydot)
% -----
% Generate mesh
x = linspace(-lim,lim,N);
y = linspace(-lim,lim,N);
[X,Y] = meshgrid(x,y);
% Variables to store derivatives
U = zeros(size(X));
V = zeros(size(X));
for i = 1:length(X)
    for j = 1:length(Y)
        % Evaluate the derivatives
```



```

        u = f1(X(i,j),Y(i,j));
        v = f2(X(i,j),Y(i,j));
        r = sqrt(u^2+v^2);
        % Normalizing them to 1 and storing
        U(i,j) = u/r;
        V(i,j) = v/r;
    end
end
end

```



References

- Students discussed with:
 1. Karthik S ME18B149
- Course notes used:
 1. Class notes
- Hassan Khalil