Assignment-2

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Question-1

$$\dot{x_1} = y - 2x \tag{1}$$
$$\dot{x_2} = \mu + x^2 - y \tag{2}$$

Finding equilibria,

$$\begin{aligned} eqn1 &\implies y = 2x \\ eqn2 &\implies \mu + x^2 - y = 0 \\ &\implies x^2 - 2x + \mu = 0 \\ &\implies x = \frac{2 \pm \sqrt{4 - 4\mu}}{2} \\ &\implies x = 1 \pm \sqrt{1 - \mu} \end{aligned}$$

We see that when,

- $\mu > 1$: No points of equilibria
- $\mu = 1$: One equilibrium point
- $\mu < 1$: 2 equilibria

The Jacobian is, $\begin{bmatrix} -2 & 1 \\ 2x^* & -1 \end{bmatrix}$
where x^* is the eqbm point of interest.

The characterstic equation will be

$$\begin{vmatrix} -2 - \lambda & 1 \\ 2(1 \pm \sqrt{1 - \mu}) & -1 - \lambda \end{vmatrix} = 0$$

$$\implies \lambda^2 + 3\lambda \mp 2\sqrt{1 - \mu} = 0$$

Eqbm i:

$$\lambda^2 + 3\lambda - 2\sqrt{1 - \mu} = 0$$

Notice sum of roots is negative and product of roots is also negative. This means real parts of both roots are negative, therefore it is a stable equilibrium
Eqbm ii:

$$\lambda^2 + 3\lambda + 2\sqrt{1 - \mu} = 0$$

Notice sum of roots is negative and product of roots is positive. This means real parts of one root is negative and other is positive, therefore it is a saddle. Therefore, this is a saddle node bifurcation.

Question-2

Setting derivatives to zero,

$$x = y \tag{3}$$
$$\mu x + x + sin(x) = 0 \tag{4}$$

Using Taylor series expansion for sin(x),

$$\begin{aligned} \mu x + x + x - \frac{x^3}{6} = 0 \\ \implies x = 0 \text{when} \mu < 0; \\ x = 0, \pm\sqrt{6(\mu+2)} \text{otherwise} \end{aligned}$$

Jacobian, $\begin{bmatrix} \mu + 1 - \frac{x^2}{2} & 1 \\ 1 & -1 \end{bmatrix}$
Characteristic equation, $\lambda^2 - (\mu - \frac{x^2}{2})\lambda - \mu - 1 + \frac{x^2}{2} - 1 = 0$
For origin,

$$\lambda^2 - (\mu)\lambda - \mu - 2 = 0 \implies D = \mu^2 + 4(\mu+2) = (\mu+2)^2 + 4 > 0 \forall \mu$$

Product of roots changes sign at $\mu = -2$ and sum of roots changes sign at $\mu = 0$. We have,

- $\mu < -2$: Sum of roots is negative and product of roots are positive, so we have 2 negative real eigen values. So origin is a stable node.
- $0 > \mu \geq -2$: Sum of roots and product of roots both negative. So one real and one imaginary eigen value. Therefore, origin is a saddle point.
- $\mu \geq 0$: Sum of roots is positive, product of roots is negative. So one real and one imaginary eigen value. Therefore, origin is a saddle point. By the last two cases we infer, it is saddle for $\mu \geq -2$

For the other roots, since $\mu < -2$ and $3(\mu+2) + \mu > 0$ the sum of eigenvalues is negative and product of eigenvalues is positive, meaning the real parts of both roots are negative. So the other equilibria, whenever they exist, are stable.

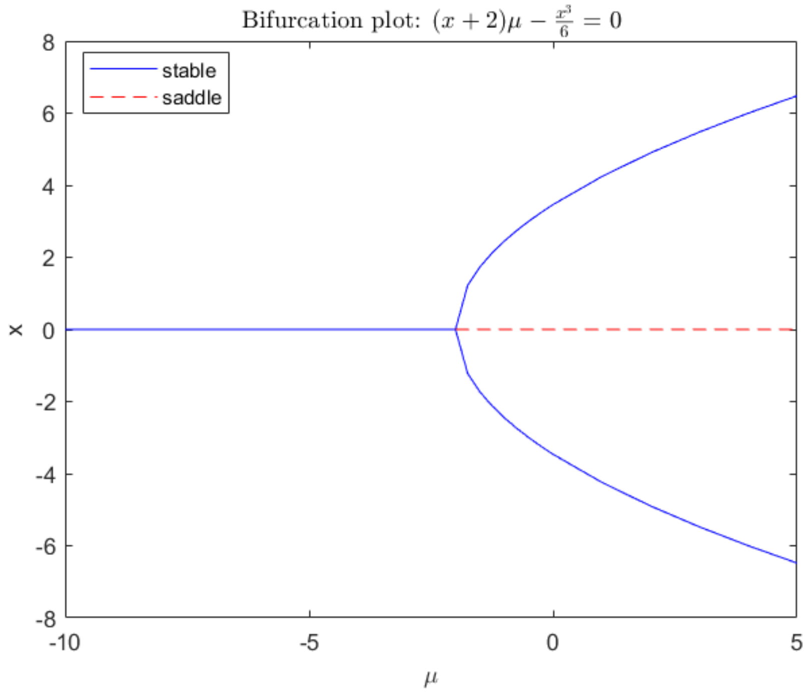

Figure 1: Bifurcation plot

Conclusion: $\mu = -2$ is a critical point and it has supercritical pitchfork bifurcation.

Question-3

a)

Setting derivatives to zero,

$$x_2 = 0$$
$$\implies \mu * x_1 - x_1^3 = 0 \implies x_1 = \pm\sqrt{mu}, 0 \text{when} \mu \geq 0 \text{and} x_1 = 0 \mu < 0$$

Jacobian,

$$\begin{bmatrix} 0 & 1 \\ \mu - 3x_1^2 - 6x_1x_2 & \mu - 1 - 3x_1^2 \end{bmatrix}$$

At each of the equilibrium points, $\begin{bmatrix} 0 & 1 \\ \mu & \mu - 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ -2\mu & -2\mu - 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ -2\mu & -2\mu - 1 \end{bmatrix}$ The corresponding eigen values,

- $\lambda = -1, \mu$: stable when $\mu < 0$ and saddle when $\mu > 0$
- $\lambda = -1, -2\mu$: stable (because $\mu > 0$ for the root to exist)
- $\lambda = -1, -2\mu$: stable (because $\mu > 0$ for the root to exist)

One root becomes 3 roots after $\mu_{critical} = 0$ and origin transforms from stable to saddle, and other two roots are stable. Hence this is a Supercritical Pitchfork bifurcation.

b)

Once again setting derivatives to zero,

$$\begin{aligned} x_2 &= 0 \\ \mu - x_1^2 = 0 \implies x_1 &= \pm\sqrt{\mu}\text{when,}\mu > 0; \text{no roots otherwise} \end{aligned}$$

Jacobian is, $\begin{bmatrix} 0 & 1 \\ -2(x_1 + x_2) & -1 - 2x_1 \end{bmatrix}$
At each of the equilibrium points,
$\begin{bmatrix} 0 & 1 \\ -2\sqrt{\mu} & -1 - 2\sqrt{\mu} \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 2\mu & 2\mu - 1 \end{bmatrix}$
Eigenvalues are, $\lambda = -1, -2\sqrt{\mu}$ and $\lambda = -1, 2\sqrt{\mu}$. So the first point is unstable and other is stable. Therefore, this is a saddle node bifurcation (after critical 2 eqbm points emerge of which one is stable and other is unstable).

c)

Once again setting derivatives to zero,

$$\begin{aligned} x_2 &= 0 \\ \mu x_1 - x_1^3 = 0 \implies x_1 &= 0, \pm\sqrt{\mu}\text{when,}\mu > 0; 0, \text{otherwise} \end{aligned}$$

Jacobian yields, $\begin{bmatrix} 0 & 1 \\ \mu - 3x_1^2 + 6x_1x_2 & \mu - 1 + 3x_1^2 \end{bmatrix}$
Characteristic equation, $\lambda^2 - (\mu - 1 + 3x_1^2)\lambda - (\mu - 3x_1^2) = 0$.

- (0,0), $\lambda = \mu, -1$. Stable for $\mu < 0$ and saddle otherwise.
- ($\pm\sqrt{\mu}$,0):
  Characteristic equation is
  $$\lambda^2 - (4\mu - 1)\lambda + 2\mu = 0$$
  We note that roots change from real to complex at $(4\mu - 1)^2 - 8\mu = 0 \implies \mu = \frac{2\pm\sqrt{3}}{4} = 0.067, 0.933$. Since the discriminant is an upward parabola, it is negative between these roots (so complex eigen values) and positive elsewhere (real eigen values). We note that the real part is +ve when $2\mu > 0$ and $(4\mu - 1) > 0 \implies \mu > 0.25$. Combining these facts we arrive at these conclusions:
  1. $0 < \mu < 0.067$: Stable nodes
  2. $0.067 < \mu < 0.25$: Stable foci
  3. $0.25 < \mu < 0.933$: Unstable foci
  4. $0.933 < \mu$: Unstable nodes

At $\mu = 0$, there is a change from single stable root at origin to saddle at origin and two stable roots elsewhere. So **Supercritical Pitchfork bifurcation at** $\mu = 0$. At $\mu = 0.25$, the two equilibria apart from origin switch from being stable foci to unstable foci. So **Supercritical Hopf bifurcation at** $\mu = 0.25$

Question-4

Setting derivatives to 0,

$$y = ax \tag{5}$$

$$\frac{x^2}{1+x^2} - by = 0 \tag{6}$$

$$\frac{x^2}{1+x^2} - bax = 0 \tag{7}$$

$$abx^2 - x + ab = 0 \tag{8}$$

$$x = 0 \text{ and } x = \frac{1 \pm \sqrt{1-4a^2b^2}}{2ab} \text{ when } a \in [\frac{-1}{2b}, \frac{1}{2b}] \tag{9}$$

Jacobian,

$$\begin{bmatrix} -a & 1 \\ \\ \frac{2x}{(x^2+1)^2} & -b \end{bmatrix}$$

At origin, $\lambda = -a, -b$. Both < 0, hence origin is a stable node irrespective of a.
From the characteristic equation we obtain, sum of roots $= (a+b)$ and product of roots $= ab - \frac{2x}{(1+x^2)^2}$.
Sum of roots is always positive because $a, b > 0$.
For simplification purpose, let $\mu = \frac{1}{2ab}$.
$\implies x_{eqbm} = \mu \pm \sqrt{\mu^2 - 1}$ Now,

$$\begin{aligned} (1 + x^2_{eqbm}) &= 1 + \mu^2 + \mu^2 - 1 \pm \mu\sqrt{\mu^2-1} \\ &= 2(\mu^2 \pm \mu\sqrt{\mu^2-1}) \\ &= 2x_{eqbm} \\ \implies \text{product of roots} &= \frac{1}{2\mu} - \frac{1}{2(\mu \pm \sqrt{\mu^2-1})} \\ &= \pm\frac{\sqrt{\mu^2-1}}{2(\mu \pm \sqrt{\mu^2-1})} \end{aligned}$$

Denominator is always positive ($\because \mu > 1$ for the roots to exist and $\mu > \sqrt{\mu^2 - 1}$). Depending on the equilibrium point, numerator is positive or negative. So for the equilibrium point with higher value has both sum and product are positive leading it to be a stable equilibrium, whereas the other equilibrium point has a negative product making it have one positive and one negative eigen values and hence it is a saddle point.
Conclusion: Since $a > 0$, from eqn 9 we infer that $a_c = \frac{1}{2b}$. Before the critical point we have 3 equilibria (two positive values of x and origin) with origin as a stable node, the equilibrium closer to the origin is a saddle point and the equilibrium away from the origin is a stable node. After the critical point we just have one equilibrium point that is origin.
It resembles pitchfork bifurcation since in one side of the critical point we have a saddle point in the middle which has two stable nodes on its sides. However, it is important to note that in pitchfork the middle equilibrium exists irrespective of other equilibrium and inverts its stability based on parameter value. So, it is perhaps more appropriate to term it as a kind of saddle node bifurcation, since a saddle point and a stable node exist before the parameter crosses the critical value and suddenly disappear after the critical point.

Question-5

a)

(We used to look for regions such that, $V \leq c$. But here, $V \geq c$, so accordingly we can pose the problem as $-x_2 \leq 0$ to bring to our usual form.)

$$\begin{aligned} V(x) &= -x_2 = 0 \\ \implies \overline{\nabla} V &= -\hat{j} \end{aligned}$$

Checking the dot product of phase vector with normal to the curve,

$$\begin{aligned} &\overline{\nabla} V \cdot (f_1\hat{i} + f_2\hat{j}) \\ &\qquad = -\hat{j} \cdot ((a_1 - x_1x_2)\hat{i} + (bx_1^2 - cx_2)\hat{j}) \qquad\qquad = 0 \\ &\qquad = (-bx_1^2 + cx_2)|_{x_2=0} \\ &\qquad = -bx_1^2 \\ &\qquad \leq 0 \forall x_1 \end{aligned}$$

So trajectories starting in D, stay in D forever.

b)

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = a - x_2 - c$$

We know $c > a$ and $x_2 \geq 0$ in D. So,

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \leq 0$$

always in D. Eg. at $x_2 = 0 \in D$, it has a nonzero value of a - c. So the expression doesn't have a sign change and it is also not identically zero everywhere in D and D is fully connected. By Benedixson criterion, no stable orbit exists in D.

Question-6

To prove that there is no stable orbit (note that all limit cycles are stable orbits), we need to invoke the Benedixson criterion. It states that there are no stable orbits fully contained in a simply connected region D if:

1. $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ has no sign changes and
2. $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is not identically zero

Since we are required to prove that no stable orbits exist at all in this question, we can take the simply connected region to be the real plane $\mathbb{R}^2$. Note that the criterion doesn't demand a bounded set.

a)

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -1 + a$$

$\because a \neq 1$, it is not identically zero. It has no sign changes because it is a constant. So there are no stable orbits.

b)

Converting to polar,

$$\begin{aligned}\dot{r} &= \frac{d}{dt}\sqrt{(x_1^2 + x_2^2)} \\ &= \frac{x_1\dot{x_1}}{\sqrt{(x_1^2 + x_2^2)}} + \frac{x_2\dot{x_2}}{\sqrt{(x_1^2 + x_2^2)}} \\ &= \frac{-x_1^2 + x_1^2 r - x_2^2 + x_2^2 r}{r} \\ &= \frac{r^2(r^2 - 1)}{r} \\ \dot{\theta} &= \frac{1}{1 + \frac{x_2^2}{x_1^2}}\left(\frac{\dot{x_2}x_1 - \dot{x_1}x_2}{x_1^2}\right)\end{aligned}$$

$$\dot{\theta} = 0$$

We observe that $\dot{\theta} = 0$, so there is no movement in tangential direction at all and dynamics always occurs on the radial line. So there are no stable orbits.

c)

$$\begin{aligned}&\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \\ &= -x_2^2\end{aligned}$$

It is not identically zero (eg. it is -1 for $x_2 = 1$). It has no sign changes because $-x_2^2 \leq 0 \ \forall x_2$. So there are no stable orbits.

Question-7

a)

Setting derivatives to zero, (and noting $x_1 = 0$ doesnt have a solution)

$$\begin{aligned}x_2 &= 1 + x_1^2 \\ a - x_1 - \frac{4x_1x_2}{1 + x_1^2} &= 0 \implies a - x_1 - 4x_1 && = 0 \\ \implies x_1 &= \frac{a}{5}, x_2 = 1 + \frac{a^2}{25}\end{aligned}$$

Jacobian, $\begin{bmatrix} -1 - \frac{4x_2}{1+x_1^2} + \frac{8x_2}{1+x_1^2} & -\frac{4x_1}{1+x_1^2} \\ \\ b(1 - \frac{x_2}{1+x_1^2}) + \frac{2bx_1^2x_2}{(1+x_1^2)^2} & \frac{-bx_1}{1+x_1^2} \end{bmatrix}$

Substituting the roots, $\begin{bmatrix} \frac{-5+(a/5)^2}{1+(a/5)^2} & -4\frac{a}{5} \\ \\ 2b(\frac{a}{5})^2 & -b\frac{a}{5} \end{bmatrix}$

Characteristic equation, $\lambda^2 + (5 - 3(\frac{a}{5})^2 + b(\frac{a}{5}))\lambda + ab(1 + (\frac{a}{5})^2)$. Product of roots is positive because a and b are positive. If sum of roots is negative then negative real parts for both roots and hence, stable equilibrium. If the sum is positive, then the equilibrium is unstable. For stable orbit to exist, we require the equilibrium contained in the set to be unstable, so, sum of roots < 0.

1. $b \geq \frac{3a}{5} - \frac{25}{a} \implies$ stable equilibrium

2. $b < \frac{3a}{5} - \frac{25}{a} \implies$ unstable equilibrium

For Poincare Benedixson we need an unstable equilibrium. So $b < \frac{3a}{5} - \frac{25}{a}$. Consider a rectangle bound by the lines, $x_1 = 0, x_2 = 0, x_1 = a, x_2 = 1 + a^2$ as set M.

1. $x_1 = 0 \implies \dot{x_1} > 0$, so doesn't move out through this boundary into the negative plane.

2. $x_2 = 0 \implies \dot{x_2} > 0$ for $0 < x_1 < a$, so doesn't move out through this boundary into the negative plane.

3. $x_1 = a \implies \dot{x_1} < 0$ so doesn't move out through this boundary into higher x_1 region.

4. $x_2 = 1 + a^2 \implies \dot{x_2} < 0$ for $0 < x_1 < a$ so doesn't move out through this boundary into higher x_2 region.

Trajectories starting in the bounded set M, stays within the set forever and the equilibrium contained within the set is unstable. Therefore by Poincare-Benedixson criterion, a stable orbit exists (within M), given $b < \frac{3a}{5} - \frac{25}{a}$.

b)

At $b = \frac{3a}{5} - \frac{25}{a}$ we saw a switch in the nature of the equilibrium. Note that when $b = \frac{3a}{5} - \frac{25}{a}$, we have purely imaginary roots (equation is of the form $\lambda^2 + c = 0, c > 0$). So in a neighbourhood around that region we see a tranistion from complex roots with negative real parts to positive real parts passing through 0. A stable focus becomes an unstable focus. Therefore the system has a Supercritical Hopf bifurcation (with $b_{critical} = \frac{3a}{5} - \frac{25}{a}$).

Question-8

a)

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -(2b - g(x_1))a$$
$$a > 0, b > 0$$

case i) $|x_1| > 1$

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -2ba \tag{10}$$
$$< 0 \tag{11}$$

case ii) $|x_1| <= 1$

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -(2b - k)a \tag{12}$$
$$> 0 \text{ if } 2b < k \tag{13}$$
$$< 0 \text{ if } 2b > k \tag{14}$$

$\implies$ for no sign change and not identically zero,
$k < 2b$
So when $k < 2b$ by Benedixson criterion there is no stable orbit for the given system. Therefore by Poincare-Benedixson criterion, a stable orbit exists (within M), given, $k > 2b$.

b)

Note that if $k > 2b$, Jacobian gives, $\begin{bmatrix} 0 & 1 \\ -a^2 & (k-2b) \end{bmatrix}$ Sum of roots is > 0 and product of roots is also positive. So the roots have positive real parts and hence, origin is an unstable equilibrium.
For convenience, replacing $x_1 and x_2$ with x and y. To find a suitable set M, lets try to move along a trajectory with the y-axis serving as an additional boundary.

- First start at some point (0,s). So g(x) = k. Both rates are positive and we move up till we reach x = 1.
- When you have x = 1, g vanishes and hence we start moving downwards. Note that we still will be moving along positive x axis till we hit the x-axis itself.
- Once we hit the x-axis, $\dot{x}$ changes its sign and starts reducing. Now both x and y are reducing, and they continue to reduce till we hit x = 1. Note that y can start increasing in some middle point but it can never hit y = 0. x is definitely decreasing but positive. y has to be $< -ax/b$ for it to increase to 0. So it can never be 0 because that would demand x = 0.
- Now we are in negative y and positive x (at x=1). Using $k > 2b$ we can say, for sure y will start decreasing.
- At the y-axis trajectory will anyway point leftwards so we can move down arbitrarily such that if we start a trajectory we end up reaching a point below (0,s) (there, x will increase pointing towards positive x-axis). This completes a closed set out of which a trajectory can't come out

Any trajectory starting in M stays in M for all future time and the equilibrium point contained in it is unstable. Therefore by Poincare-Benedixson theorem, a stable orbit exists (within M).

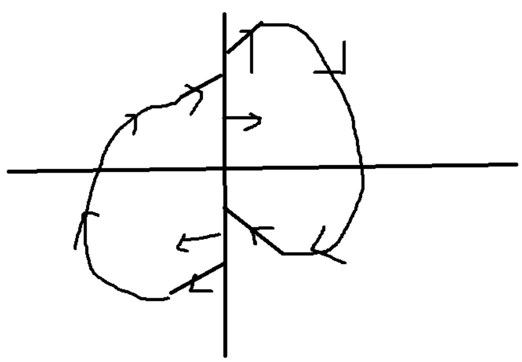

Figure 2: sample trajectory

Question-9

Videos in the drive folder (given access to TAs' smail): link. These were the systems used.

a) Saddle Node

$$\dot{x} = \mu - x^2 \tag{15}$$
$$\dot{y} = -y \tag{16}$$

b) Transcritical

$$\dot{x} = \mu x - x^2 \tag{17}$$
$$\dot{y} = -y \tag{18}$$

c) Supercritical Pitchfork

$$\dot{x} = \mu x - x^3 \tag{19}$$
$$\dot{y} = -y \tag{20}$$

d) Subcritical Pitchfork

$$\dot{x} = \mu x + x^3 \tag{21}$$
$$\dot{y} = -y \tag{22}$$

e) Supercritical Hopf

$$\dot{x} = x(\mu - x^2 - y^2) + y \tag{23}$$
$$\dot{y} = -x + y(\mu - x^2 - y^2) \tag{24}$$

References

- Students discussed with:
  1. Arvind Ragghav ME18B086
  2. Karthik Srinivasan ME18B149
- Course notes used:
  1. Class notes/NPTEL lectures
- Hassan Khalil textbook