EE6415: Nonlinear Systems Analysis

Jan-May, 2022

Tutorial 5

Instructions :

- **Submit on or before 11:59 PM, 15/03/2022**
- Any kind of plagiarism will be dealt with severely. Acknowledge any and every resource used, including any coursemates you may have discussed with.
- Include any plots/images you deem necessary
- Your submission must be named "RollNo.pdf". For example, if your roll number is EE17B158, your submission must have the name "EE17B158.pdf".
- Your submissions must be made on moodle. Any emailed submissions will not be accepted.
- It is required that you use LaTeX for writing your report. A template had been provided along with the first assignment for the same.

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1. Define $T : C[0,1] \to C[0,1]$ by $T(x)(t) = 1 + \int_0^t x(s)ds$ where the metric in C[0, 1] is defined as $d(f,g) = \max_{x\imath[0,1]} |f(x) - g(x)|$.
   (a) Is T a contraction?
   (b) If the space is changed to $C[0, \frac{1}{2}]$ will T be a contraction?
2. Let $f : [0,1] \to [0,1]$ be given by $f(x) = \frac{1}{1+x}$. Answer the following questions:
   (a) Is the map a contraction?
   (b) Does the function f has a unique fixed point?
3. For each of the functions $f(x)$ given, find whether f is (a) continuously differentiable (b) locally Lipschitz (c) continuous (d) globally Lipschitz.
   (a) $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$
   (b) $f(x) = \frac{x^3}{3} + |x|$
   (c) $f(x) = \begin{bmatrix} -x_1 + a|x_2| \\ -(a+b)x_1 + bx_1^2 - x_1 x_2 \end{bmatrix}$
4. Let $\|.\|_\alpha$ and $\|.\|_\beta$ be two different p-norms on $\mathbb{R}^n$. Show that $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz in $\|.\|_\alpha$ iff it is Lipschitz in $\|.\|_\beta$
5. The following result is known as the Gronwall-Bellman inequality. Prove the result.
   Let I denote an interval of the real line of the form $[a, \infty)$ or $[a, b]$ or $[a, b)$, with $a < b$. Let β and u be real-valued continuous functions defined on I. If u is differentiable in the interior I_o of I and satisfies the differential inequality

$$\dot{u}(t) \leq \beta(t)u(t), \quad t \in I_o$$

then, $u(t)$ is bounded by the solution of the corresponding differential equation $\dot{v}(t) = \beta(t)v(t)$.

$$u(t) \leq u(a) \exp\left(\int_a^t \beta(s) ds\right)$$

for all $t \in I$.

6. Let $f(t,x)$ be piecewise continuous in t, locally Lipschitz in x, and

$$\|f(t,x)\| \leq k_1 + k_2\|x\|, \quad \forall \quad (t,x) \in [t_0, \infty) \times \mathbb{R}^n$$

   (a) Show that the solution of

$$\dot{x} = f(t,x), \quad x(t_0) = x_0$$

   satisfies

$$\|x(t)\| \leq \|x_0\| \exp(k_2(t - t_0)) + \frac{k_1}{k_2}(\exp(k2(t - t0)) - 1), \quad \forall t \geq t_0$$

   for which the solution exists. [Hint: Use Gronwall-Bellman inequality]

   (b) Can the solution have a finite escape time

7. If the system $\dot{x} = f(t,x), \quad x(t_0) = x_0 = [a,b]^T$ is given by

$$\dot{x}_1 = -x_1 + \frac{2x_2}{1 + x_2^2}$$
$$\dot{x}_2 = -x_2 + \frac{2x_1}{1 + x_1^2}$$

show that the state equation has a unique solution defined for all $t \geq 0$.

8. The following result is known as the comparison lemma. Prove the lemma.

Consider the scalar differential equation

$$\dot{u} = f(t,u), \quad u(t_0) = u_0$$

where $f(t,u)$ is locally Lipschitz in u, and continuous in t, for all $t \geq 0$ and for all $u \in J \subseteq \mathbb{R}$. Let $[t_0, T)$ be the interval of existence for a solution $u(t) \in J$, for all $t \in [t_0, T)$. Let $v(t)$ be a continuous function whose upper right hand derivative is denoted by $D^+v(t)$, and satisfies,

$$D^+v(t) \leq f(t, v(t)), \quad v(t_0) \leq u_0$$

with $v(t) \in J$ for all $t \in [t_0, T)$. Then, $v(t) \leq u(t)$ for all $t \in [t_0, T)$.

9. Using the comparison lemma, show that the solution of the state equation

$$\dot{x}_1 = -x_1 + \frac{2x_2}{1 + x_2^2}$$
$$\dot{x}_2 = -x_2 + \frac{2x_1}{1 + x_1^2}$$

satisfies the inequality

$$\|x(t)\|_2 \leq e^{-t}\|x(0)\|_2 + \sqrt{2}(1 - e^{-t})$$