Assignment-2

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April 26, 2022

Question-1

$$
\begin{aligned}
\dot{y} &= \dot{x_1} = x_2 \\
\implies \ddot{y} &= \dot{x_2} \\
&= \cos(x_3) + x_1 + u
\end{aligned}
$$

Since u appears only in the second time derivative of y, relative degree, $\rho = 2$.

$$
\begin{aligned}
h(x) &= x_1 \\
\xi_1 &= x_1 \\
\dot{\xi_1} &= \dot{x_1} = x_2 \\
\implies \xi_2 &= x_2
\end{aligned}
$$

Also,

$$
\begin{aligned}
\dot{\xi_2} &= \dot{x_2} \\
&= \cos(x_3) + x_1 + u
\end{aligned}
$$

For finding η,

$$
\begin{aligned}
\begin{bmatrix} \frac{\partial \eta}{\partial x_1} & \frac{\partial \eta}{\partial x_2} & \frac{\partial \eta}{\partial x_3} & \frac{\partial \eta}{\partial x_4} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} &= 0 \\
\frac{\partial \eta}{\partial x_2} + \frac{\partial \eta}{\partial x_4} &= 0
\end{aligned}
$$

We need 4 total states - 2 observable states = 2 unobservable states.
We can have (ensuring T(.) is invertible),

$$
\begin{aligned}
\eta_1 &= x_3 && (1) \\
\eta_2 &= x_2 - x_4 && (2) \\
\implies \dot{\eta_1} &= \dot{x_3} && (3) \\
&= x_4 && (4) \\
\implies \dot{\eta_1} &= \xi_2 - \eta_2 && (5)
\end{aligned}
$$

And,

$$
\begin{aligned}
\implies \dot{\eta_2} &= \dot{x_2} - \dot{x_4} \\
\implies \dot{\eta_2} &= \cos(x_3) \\
&= \cos(\eta_1)
\end{aligned}
$$

$\therefore$ Normal form for the system is:
i) Observable part -

$$\dot{\xi} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u + \cos(\eta_1))$$

ii) Unobservable part -

$$\implies \dot{\eta_1} = \xi_2 - \eta_2$$
$$\implies \dot{\eta_2} = \cos(\eta_1)$$

A controller that drives $\xi \to 0$ is,

$$u = -\cos(\eta_1) - 2\xi_1 - 2\xi_2 \tag{6}$$

New feedback linearized system (observable part):

$$\dot{\xi} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

Eigen values are:

$$\lambda^2 + 2\lambda + 1 = 0$$
$$\lambda = -1, -1$$

So we can successfully drive ξ parts to zero. Let us now check the unobservable parts (zero dynamics),

$$\implies \dot{\eta_1} = -\eta_2$$
$$\implies \dot{\eta_2} = \cos(\eta_1)$$

Equilibrium at $(\frac{(2n+1)\pi}{2}, 0)$ Jacobian,

$$\begin{bmatrix} 0 & -1 \\ (\pm 1) & 0 \end{bmatrix}$$
$$\implies \lambda = \pm j \text{ [OR] } \pm 1$$

So the system is unstable. This is shown by the trajectories.
For the purpose of simulation, I assumed that $x_0 = [1, 1, 1, 1]$. We can see in fig. 1 that the observable states ξ_1, ξ_2 are taken to zero as desired by us, using the control law we just designed. However, if you look at fig. 2 corresponding to the unobservable states, we see that they don't die out even after $\xi \to 0$. Finally, looking at plot of each state in fig. 3, we can confidently state that state x_3 is diverging and not going to zero. This means that the system is unstable as predicted theoretically.

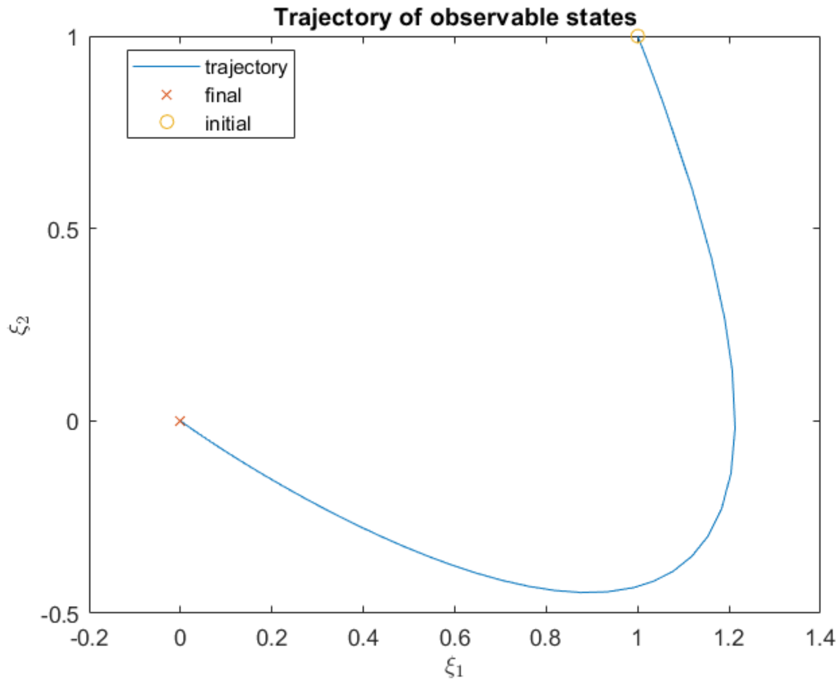

Figure 1: Trajectory of the observable part

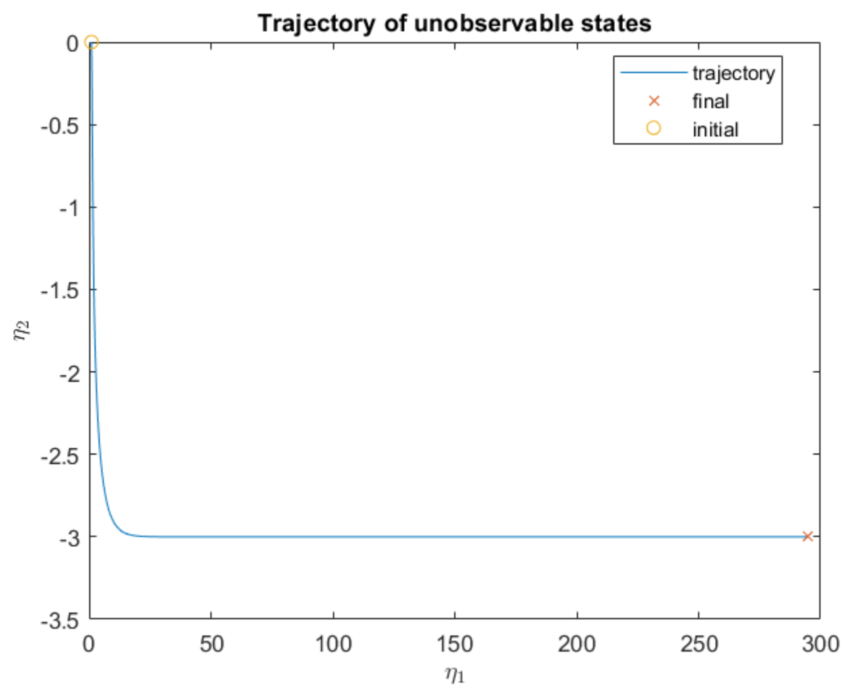

Figure 2: Trajectory of the unobservable part

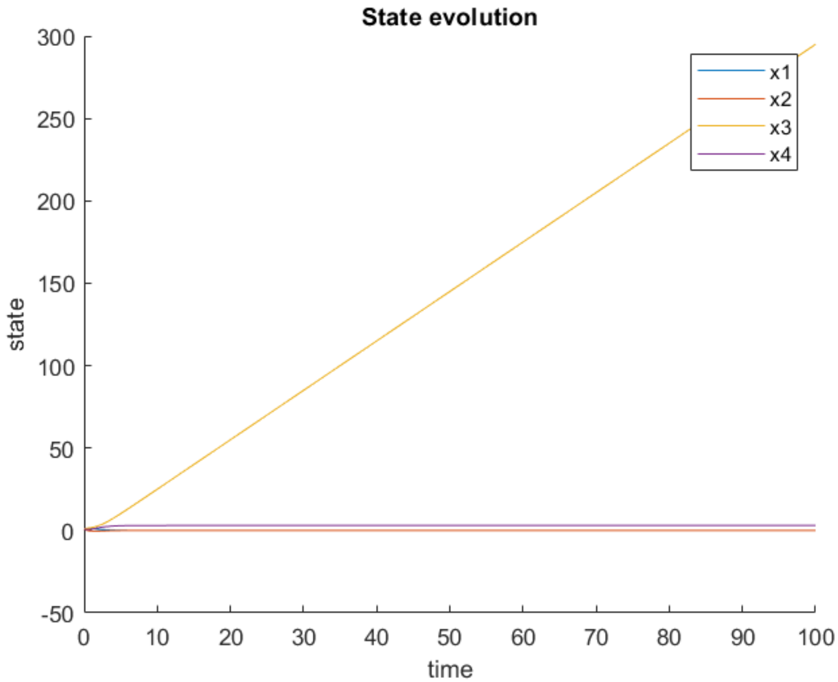

Figure 3: Evolution of states over time

Question-2

$$\dot{x} + a(t)\dot{x}^2 cos(5x) = b(t)u$$

Since this is a second order system we choose,

$$\begin{aligned} S &= \dot{\tilde{x}} + \lambda\tilde{x} && (7) \\ &= \dot{x} + \lambda x - sin(t) - \lambda cos(t) && (8) \end{aligned}$$

For both parts, let's fix $\lambda = 1$ We have,

$$\hat{f} = -\hat{a}\dot{x}^2 cos(5x)$$

where, $\hat{a} = \frac{1+2}{2} = 1.5$. Also,

$$\hat{b} = \sqrt{4 * 8} = \sqrt{32}$$

So we design,

$$\begin{aligned} u &= \frac{1}{\hat{b}}(\hat{a}\dot{x}^2 cos(5x) - K * sgn(S)) \\ &= \frac{1}{\sqrt{32}}(1.5\dot{x}^2 cos(5x) - K * sgn(S)) \end{aligned}$$

Use,

$$\begin{aligned} & K = |F| + \eta = |f - \hat{f}| + \eta \\ \implies & K = 0.5\dot{x}^2 + 1 \text{ (fixing K at the upper bound of cos)} \end{aligned}$$

where we fixed, $\eta = 1$. So, finally we get,

$$u = \frac{1}{\sqrt{32}}(1.5\dot{x}^2 cos(5x) - (0.5\dot{x}^2 + 1) * sgn(S))$$

Assuming the initial state is $x = 0, \dot{x} = 0$, and assuming a(t) and b(t) are uniform random variables sampled at each instant, we perform the simulation.

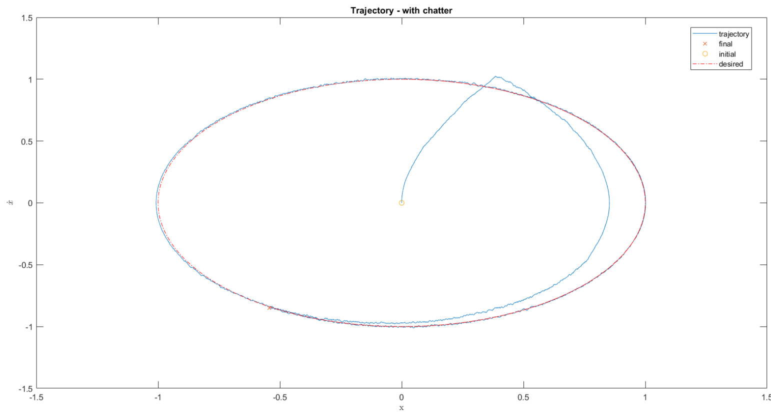

Figure 4: Trajectory - with chatter case

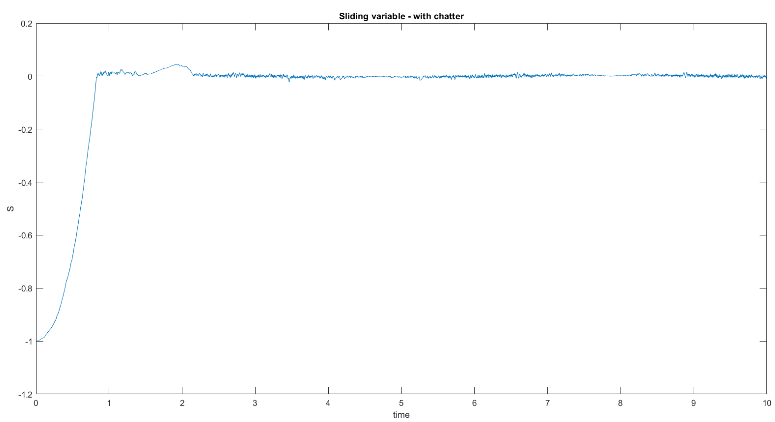

Figure 5: Sliding variable value - with chatter case

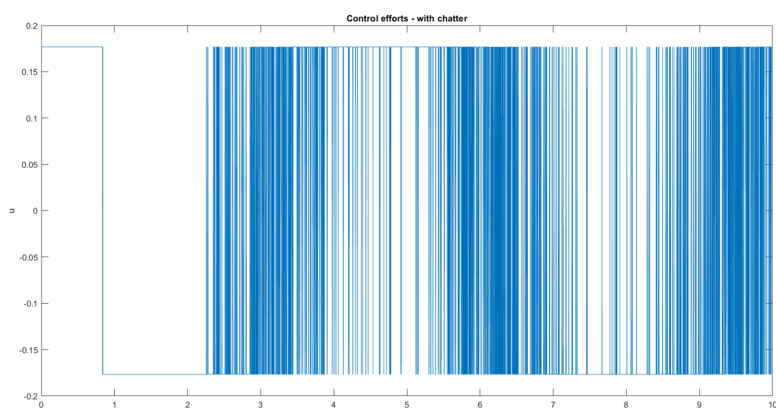

Figure 6: Control effort - with chatter case

From fig. 4, we observe that the trajectory we set is closely tracked, and fig. 5 shows that sliding variable is taken to 0, and is mostly contained in the region (small variations because parameters a, b are unknown). However, because of use of signum function, we see high frequency oscillations in the input (fig.6) after some point of time (ie after we get close to zero). To avoid this we can use a saturation function in place of the signum function, which will 'slow down' the system as we move closer to the region of S = 0 and ensures lesser oscillations of the input effort u.

Boundary Layer - avoiding chatter

Fixing boundary layer thickness as $\phi = 0.05$, we get,

$$u = \frac{1}{\sqrt{32}}(1.5\dot{x}^2 cos(5x) - (0.5\dot{x}^2 + 1) * sat(\frac{S}{0.05}))$$

where, sat(.) is the saturation function.
Once again, assuming the initial state is $x = 0, \dot{x} = 0$, and assuming a(t) and b(t) are uniform random variables sampled at each instant, we perform the simulation.

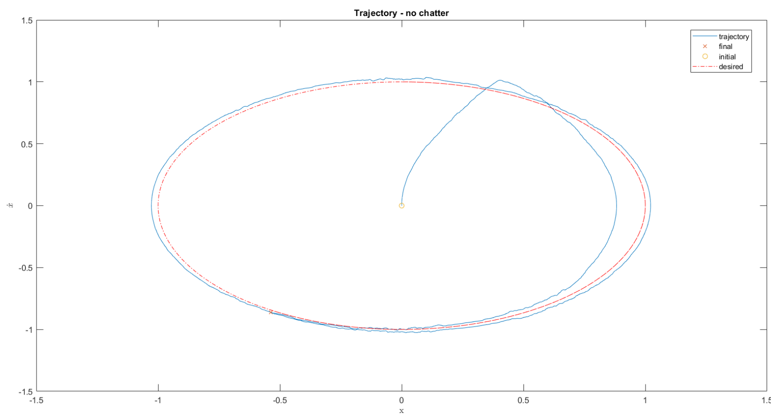

Figure 7: Trajectory - without chatter case

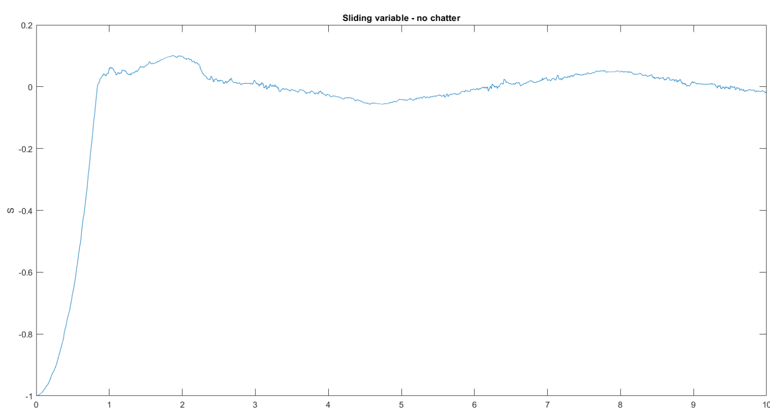

Figure 8: Sliding variable value - without chatter case

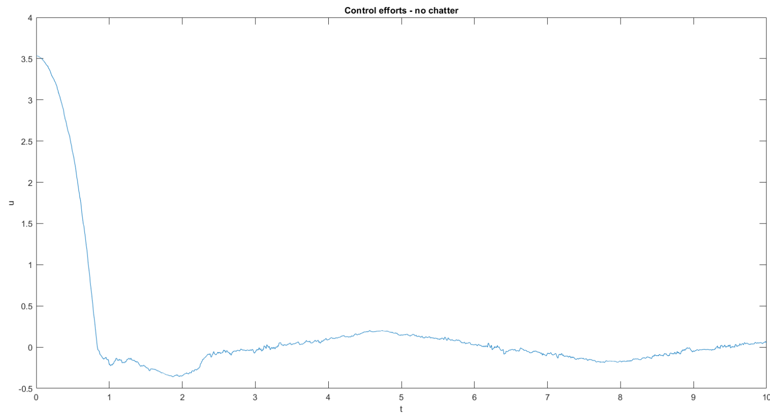

Figure 9: Control effort - without chatter case

From fig. 7, we observe that the trajectory tracking is not as good as before, and fig. 8 shows that sliding variable is taken to 0 but there are considerable oscillations around it (but approximately within the thickness of the boundary layer). And as expected, we have successfully reduced the oscillations in the input (fig.9).

References

- Students discussed with:

    1. Arvind Ragghav ME18B086
    2. Karthik Srinivasan ME18B149

- Class notes
- Hassan Khalil