Assignment-1

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Question 1

Setting the derivatives to zero to find the equilibrium points, we have

$$\begin{aligned} y &= 0 \\ and&, \\ -x - \beta y &= 0 \\ \implies x &= 0 \end{aligned}$$

So, the system has only one equilibrium point at $(0, 0)$.

a) Phase portraits

Limits of plot has been set as -1 and 1, since origin is our point of interest.

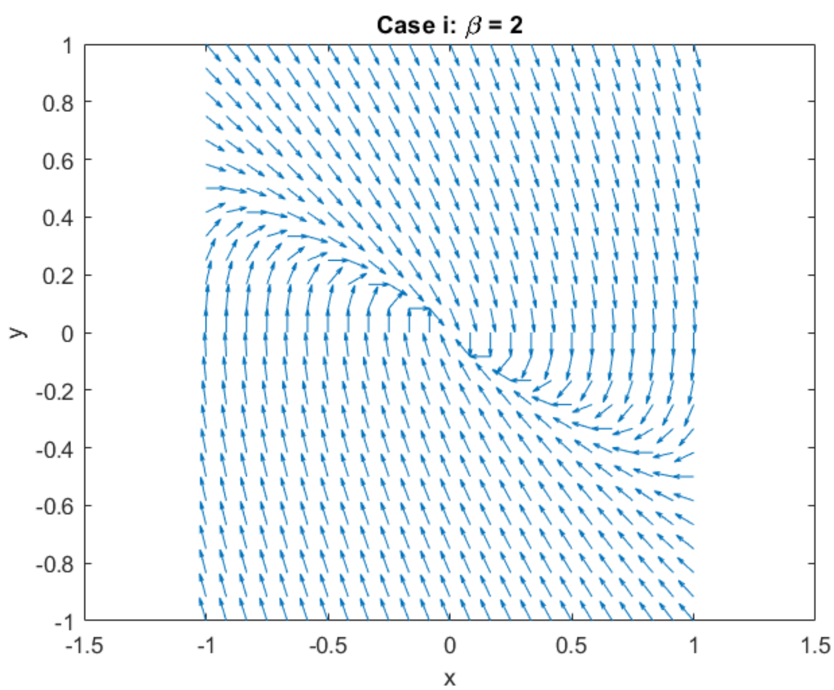

Figure 1: Phase portrait when $\beta = 2$

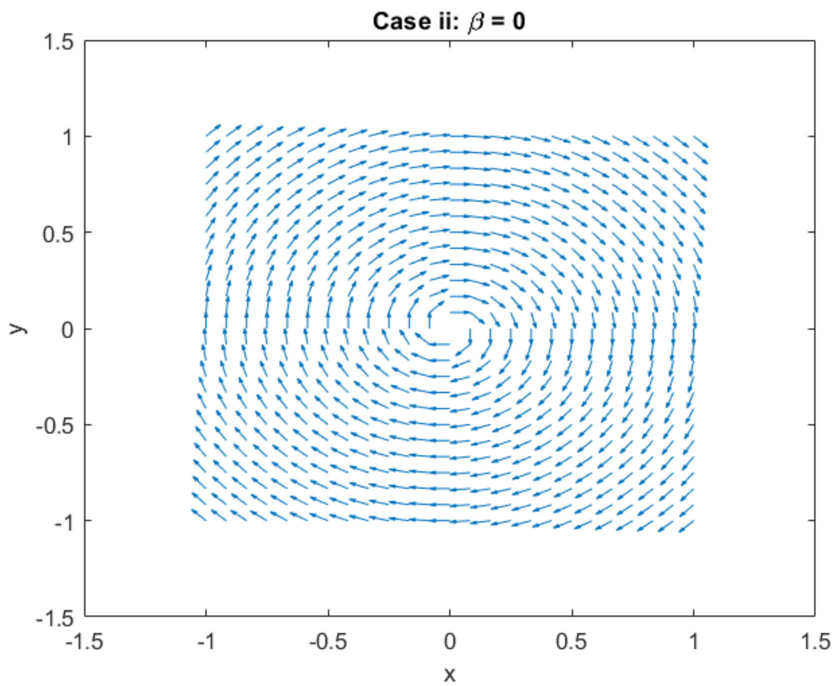

Figure 2: Phase portrait when $\beta = 0$

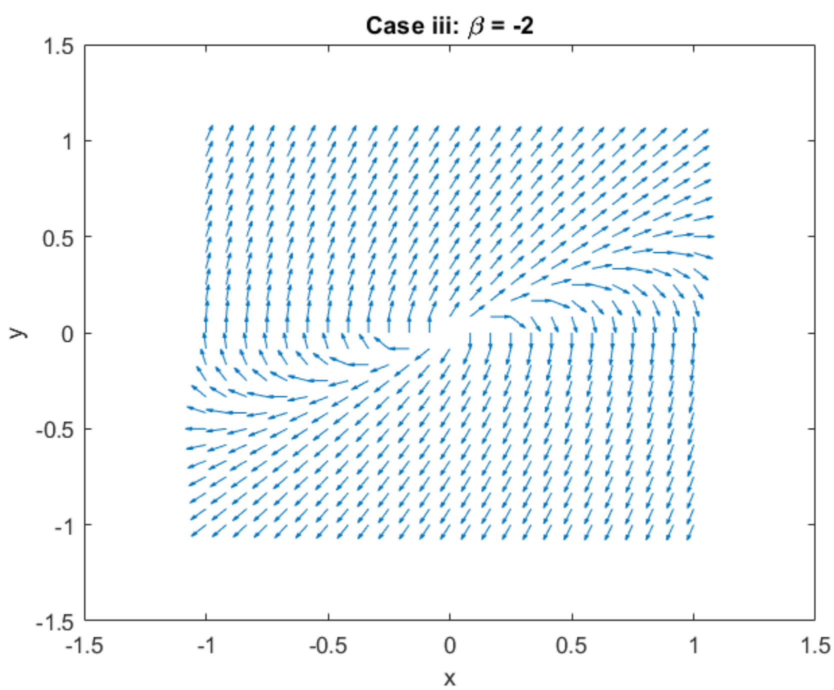

Figure 3: Phase portrait when $\beta = -2$

b) Inferences on stability

$\beta = 2$
From figure 1, we observe that a small perturbation at the origin will cause the dynamics to proceed such that the equilibrium is restored. So, the equilibrium is stable.
$\beta = 0$
From figure 2, we observe that the trajectories are concentric ellipses around the origin. So, the equilibrium point is a center focus.
$\beta = -2$
From figure 3, we observe that a small perturbation at the origin will cause the dynamics to proceed such that the equilibrium the system moves away from the equilibrium. So, the equilibrium is unstable.

Question 2

Part a)

$$\dot{x} = x(5 - x - 2y) \tag{1}$$
$$\dot{y} = y(4 - x - y) \tag{2}$$

Setting the derivatives to 0, we get the solutions as $(0,0), (0,4), (5,0)$, and $(3,1)$. These are the equilibrium points of the system.

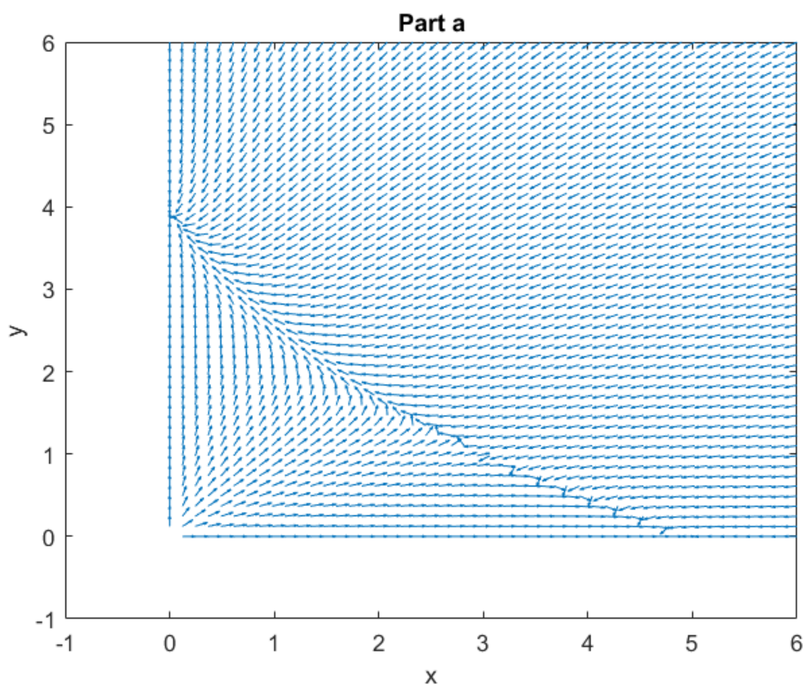

Figure 4: Phase Portrait Q2a

Stability:
i. (0,0) Unstable equilibrium as a small perturbation takes it away from the point (arrows pointing outward).
ii. (0,4) Stable equilibrium since for a small perturbation the system will evolve such that the equilibrium is restored (arrows point towards the eqbm point). No limit cycle.
iii. (5,0) Stable equilibrium since for a small perturbation the system will evolve such that the equilibrium is restored (arrows pointing inward-slight deviation to left pushes it back to right, slight deviation to right pushes it back to left).

iv. (3,1) Saddle point as a small perturbation in some directions restores equilibrium, and perturbation in other directions takes it away from the equilibrium.
No limit cycle found.

Part b)

Once again equilibrium point is origin. If we move slightly down equilibrium is restored, if we move slightly up the system goes away from equilibrium. So this is a saddle point. No limit cycle.

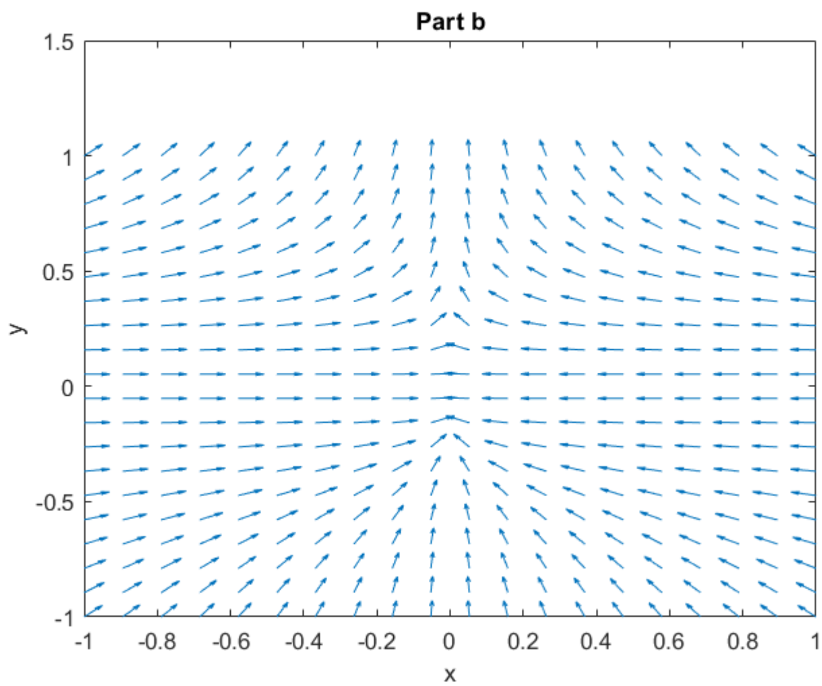

Figure 5: Phase Portrait Q2b

Part c)

$$\dot{x} = y \tag{3}$$
$$\dot{y} = x - x^3 \tag{4}$$

Setting the derivatives to zero we get the solutions as $(0, 0)$, $(1, 0)$ and $(-1, 0)$. These are the equilibrium points.

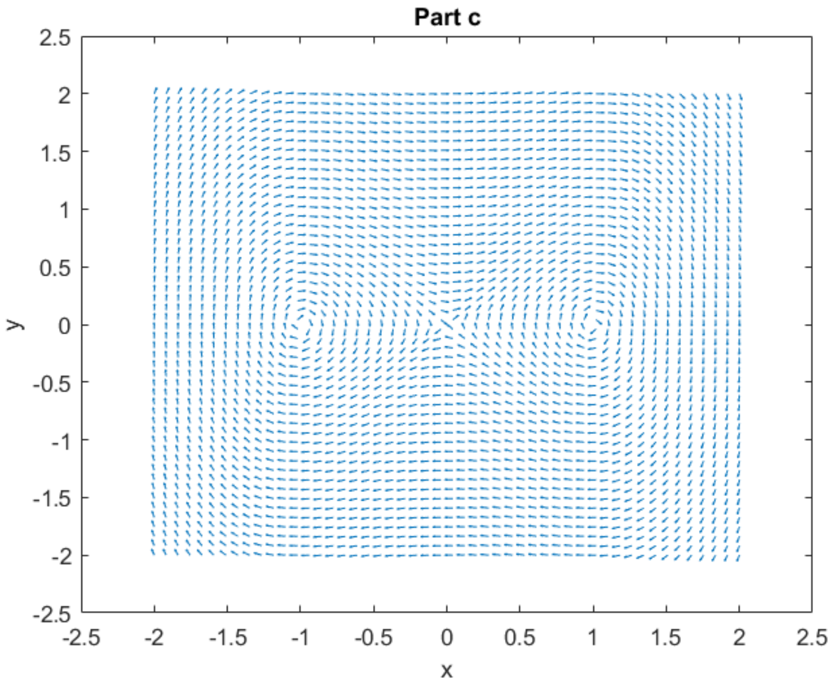

Figure 6: Phase Portrait Q2c

We observe that: **i. (0,0)** Saddle point since in some directions it points towards the origin (restoration of equilibrium) and in some directions it points away from the origin ($\hat{i} - \hat{j}$ and $\hat{i} + \hat{j}$ directions respectively).
ii. (1,0) We observe that the trajectories are concentric circles around the equilibrium point (1,0). So, the equilibrium point is a center focus.
iii. (-1,0) We observe that the trajectories are concentric circles around the equilibrium point (-1,0). So, the equilibrium point is a center focus.
No limit cycles found.

Part d)

By setting derivatives to zero we get the solution as: $(n\pi, m\pi)$

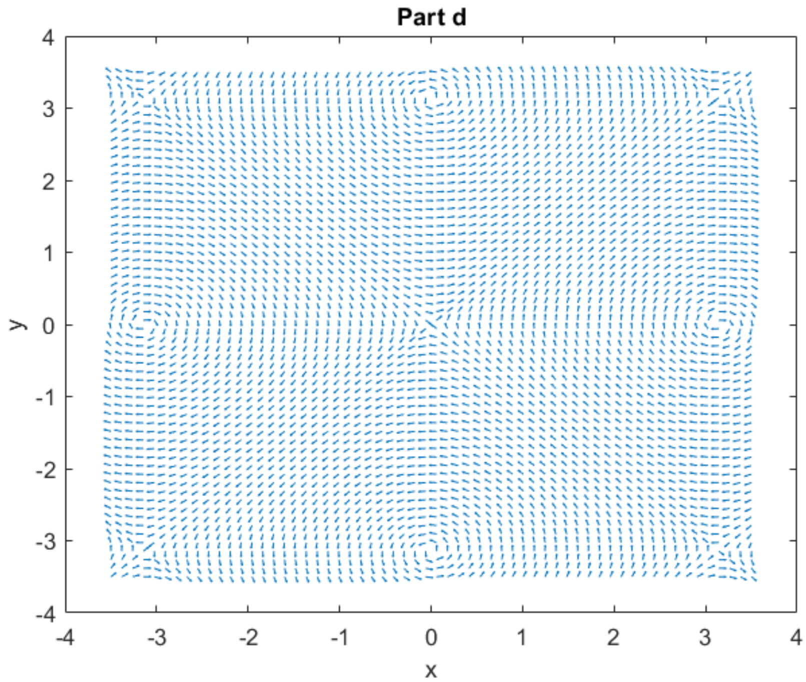

Figure 7: Phase Portrait Q2d

From figure 7, we find that:

1. n and m both are even $(0, 0)$ or n and m both are odd (π, π) $\implies$ saddle point since in some directions it points towards the origin (restoration of equilibrium) and in some directions it points away from the origin.
2. n is odd and m is even (or vice versa) $\implies$ center focus since trajectories concentric ellipses around such equilibria, as seen from the phase portrait (at $(\pi, 0)$, $(-\pi, 0)$, $(0, \pi)$ and $(0, -\pi)$).
3. No limit cycle.

Question 3

Part a) $\mu = -1$

From figure 8, we see that origin is an equilibrium point. The trajectory spirals into the origin. Hence, the equilibrium point is a stable focus. Number of equilibrium points = 1. No limit cycle.

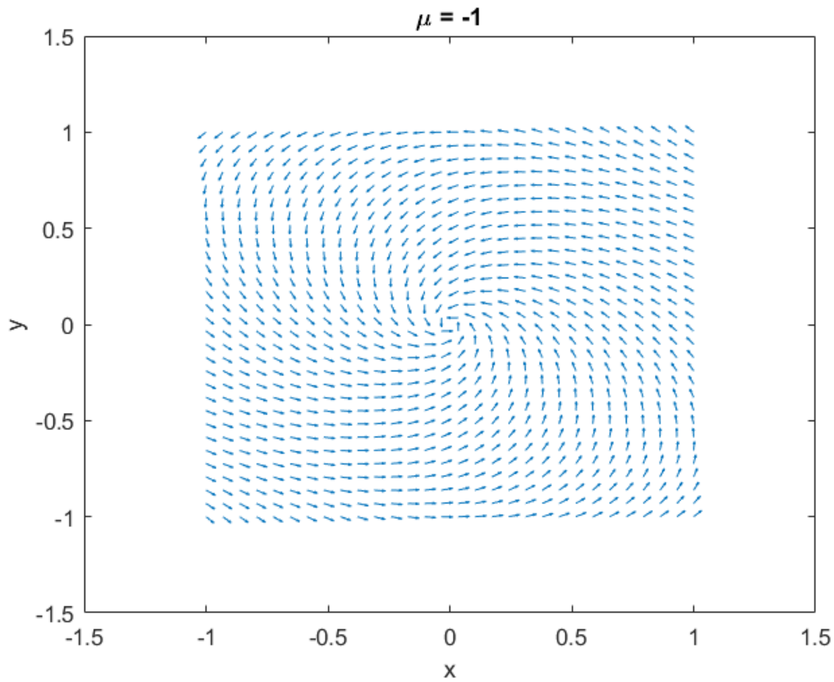

Figure 8: Phase Portrait Q3a

Part b) $\mu = 0$

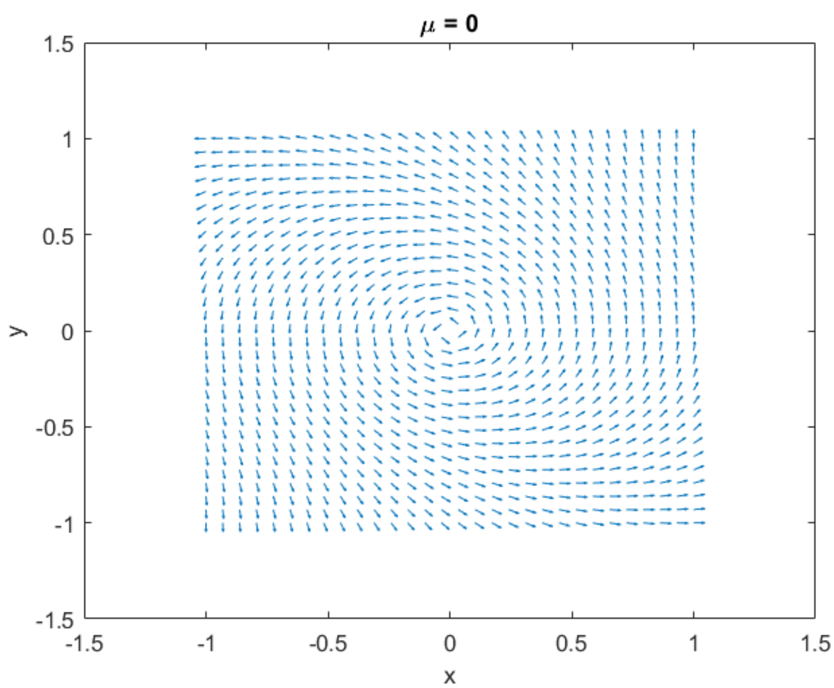

Figure 9: Phase Portrait Q3b

From figure 9, we see that origin is an equilibrium point. We observe that the trajectories are concentric circles around the equilibrium point (1,0). So, the equilibrium point is a center focus. Number of equilibrium points = 1. No limit cycle.

Part c) $\mu = 1$

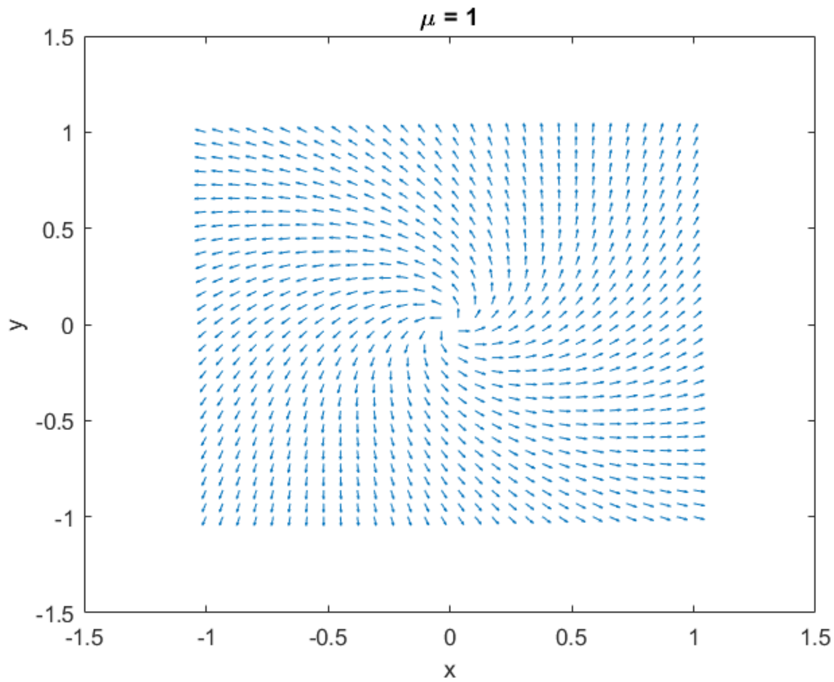

Figure 10: Phase Portrait Q3c

From figure 10, we see that origin is an equilibrium point. A small perturbation from origin will cause the dynamics to evolve such a way that it keeps moving away from the origin. Hence the equilibrium point is an unstable node. Number of equilibrium points = 1. No limit cycle.

Question 4

$$\dot{x} = \mu x + y + sin(x) \tag{5}$$
$$\dot{y} = x - y \tag{6}$$

Setting derivatives to 0, eqn 6 gives

$$x = y \tag{7}$$

$$5 \implies \mu x + x + sin(x) = 0 \tag{8}$$
$$\implies (1 + \mu)x + sin(x) = 0 \tag{9}$$

Part a) $\mu = -2.5$

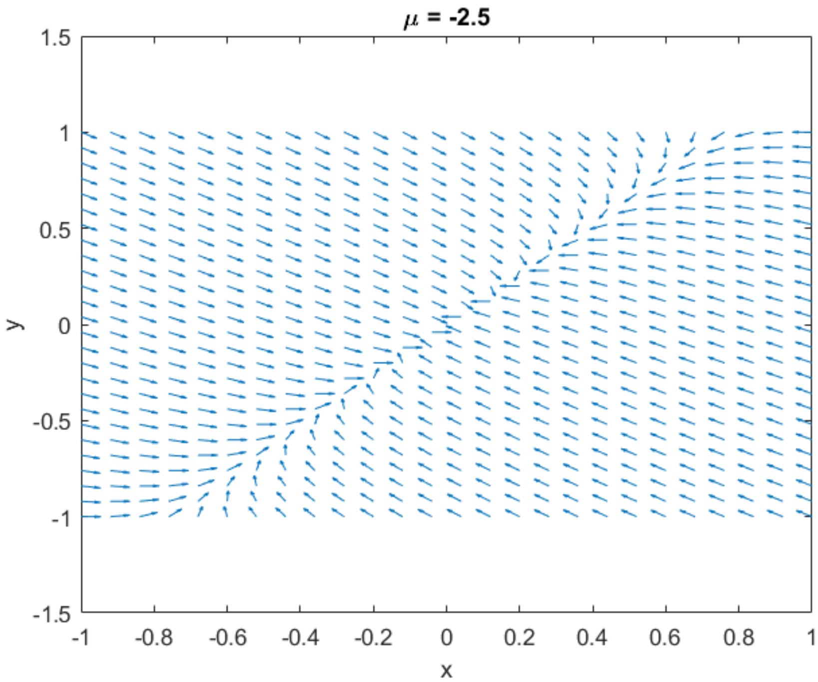

Figure 11: Phase Portrait Q4a

$$9 \implies sin(x) = 1.5x \text{ (at equilibrium point)} \tag{10}$$

$sin(x) \leq 1 \implies x_{eqbm}$(if it exists) < 1. So, we just plot upto x = 1. From figure 11, we see that origin is an equilibrium point. A small perturbation from origin will cause the system to move back to origin restoring the equilibrium. Hence, the equilibrium point is stable. Number of equilibrium points = 1. No limit cycle.

Part b) $\mu = -2$

$$9 \implies sin(x) = x \text{ (at equilibrium point)} \tag{11}$$

Since $sin(x) \leq 1$, $x_{eqbm} < 1$. So, we just plot upto x = 1. From figure 12, we see that origin is an equilibrium point. A small perturbation from origin will cause the system to move back to origin restoring the equilibrium. Hence, the equilibrium point is stable. Number of equilibrium points = 1. No limit cycle.

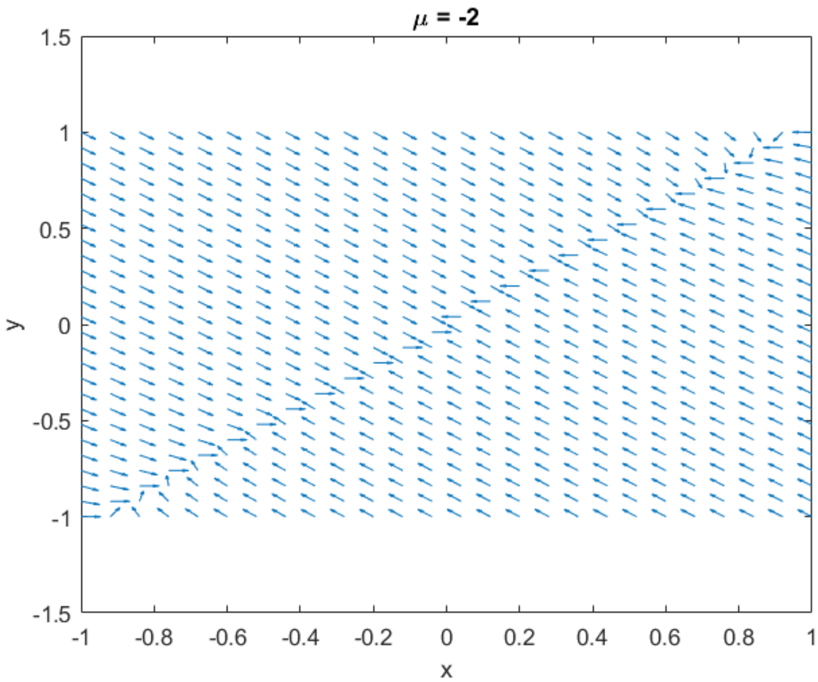

Figure 12: Phase Portrait Q4b

Part c) $\mu = -1.5$

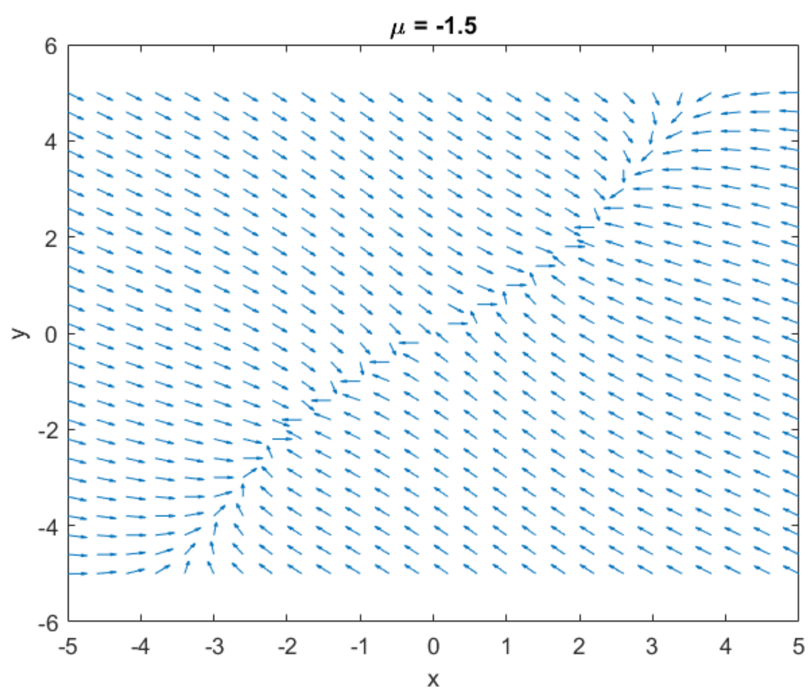

Figure 13: Phase Portrait Q4c

$$9 \implies sin(x) = 0.5x \text{ (at equilibrium point)} \quad (12)$$

Solving the equation, we obtain the solutions as $x = \pm 1.8955, 0$.

The equilibrium points are $(0, 0), (1.8955, 1.8955)$ and $(-1.8955, -1.8955)$.

1. (0,0): Given a small perturbation, in some directions it points towards the equilibrium point (restoration of equilibrium) and in other directions it points away from the point. So, we infer that it is a saddle point.
2. (1.8955,1.8955): A small perturbation from this point will cause the system to move back to origin restoring the equilibrium as seen in figure 13 (look near (2,2)). Hence, the equilibrium point is stable.
3. (-1.8955,-1.8955): A small perturbation from this point will cause the system to move back to origin restoring the equilibrium as seen in figure 13 (look near (-2,-2)). Hence, the equilibrium point is stable.
4. Number of equilibrium points = 3. No limit cycle.

Question 5

Part a)

This is the equation of the system.

$$m\ddot{x} = \frac{-Gm_1m}{x^2} + \frac{Gmm_2}{(a-x)^2} \tag{13}$$

In order to write it in state-space form, we need first-order equation. We break this second-order dynamics to a combination of first-order equations by adding another state, $y = \dot{x}$ (velocity).

$$\dot{x} = y \tag{14}$$

$$\dot{y} = \frac{-Gm_1m}{x^2} + \frac{Gmm_2}{(a-x)^2} \tag{15}$$

where equation 15 is obtained using Newton's Law of Gravitation (equation 13).

Part b)

I am considering only the positive x-axis (i.e. the particle is confined between the 2 masses).

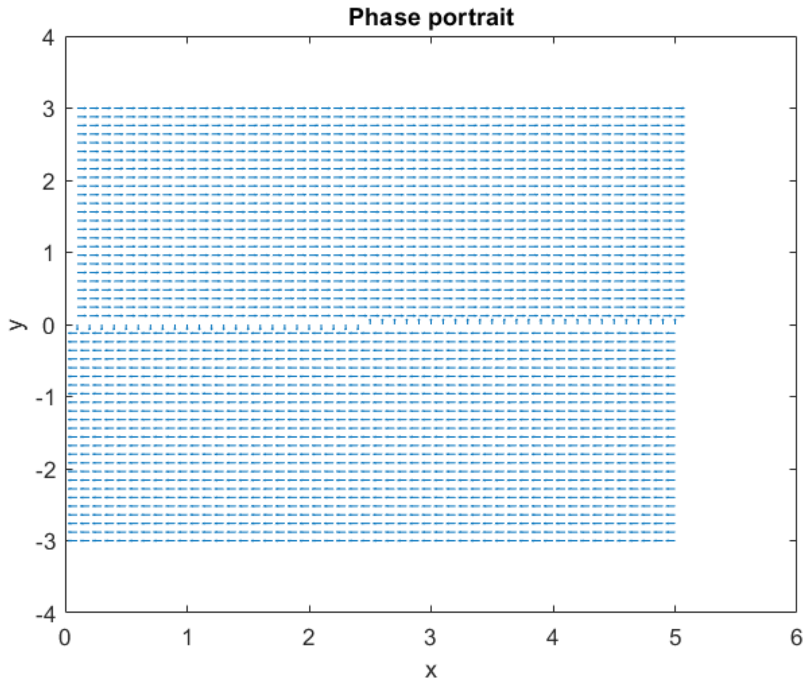

Figure 14: Phase Portrait Q5

From 14, we find that the direction of arrows flip between x=2 and x=3. We also know that, y should be 0 at the equilibrium point by looking at equation 14. Solving eqn 15 by hand, for the equilibrium we find that,

$$Gm\left(\frac{m_2}{(a-x)^2} - \frac{m_1}{x^2}\right) = 0$$
$$\implies \frac{a-x}{x} = \pm\sqrt{\frac{m_2}{m_1}}$$
$$\implies x = \frac{a}{1 \pm \sqrt{\frac{m_2}{m_1}}}$$

Considering only the positive solution we get $x = 2.4025$ units. We find that arrows near the equilibrium point away from it, leading us to conclude that the equilibrium is unstable.

Question 6

At equilibrium

$$-\mu - x^2 = 0$$
$$\implies x^2 = \mu \tag{16}$$

and,

$$y = 0 \tag{17}$$

Part a) $\mu = -2$

From equation 16, we find solution for x-coordinate as: $\sqrt{2}j$. Since this is imaginary, there is no equilibrium point. The phase portrait in figure 15 agrees with the same. Conclusion: Number of equilibrium points = 0. No limit cycle exists.

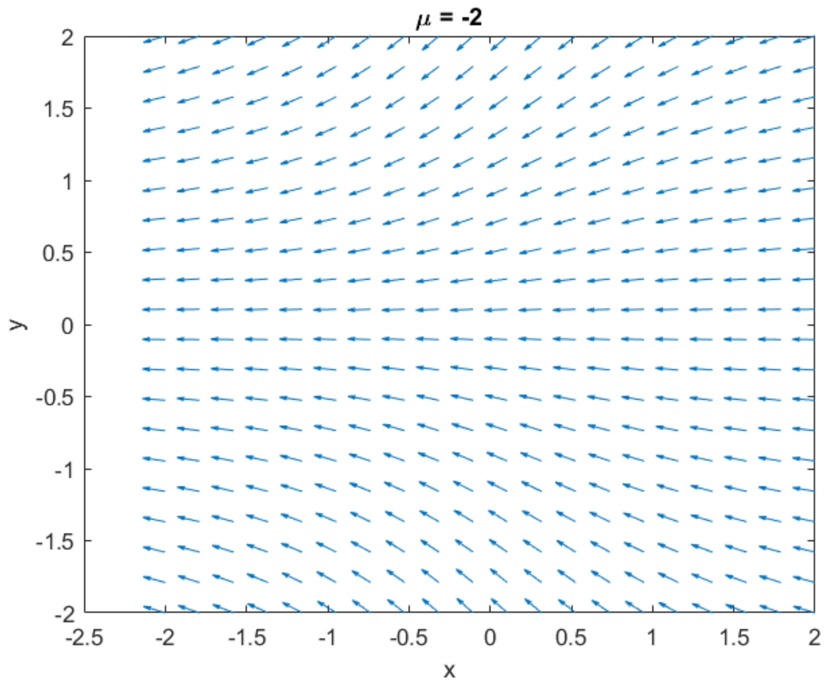

Figure 15: Phase Portrait Q6a

Part b) $\mu = 0$

From equation 16, we find solution for x-coordinate as: $x = 0$. Along with eqn 17, we find the equilibrium point as $(0, 0)$. The phase portrait in figure 16 agrees with the same. When we perturb the system along y axis, equilibrium seems to be restored, however if we perturb along -x axis, the system moves away from the equilibrium point (origin), so the origin is a saddle point.

Conclusion:

1. Number of equilibrium points = 1
2. It is a saddle point
3. Number of limit cycles = 0

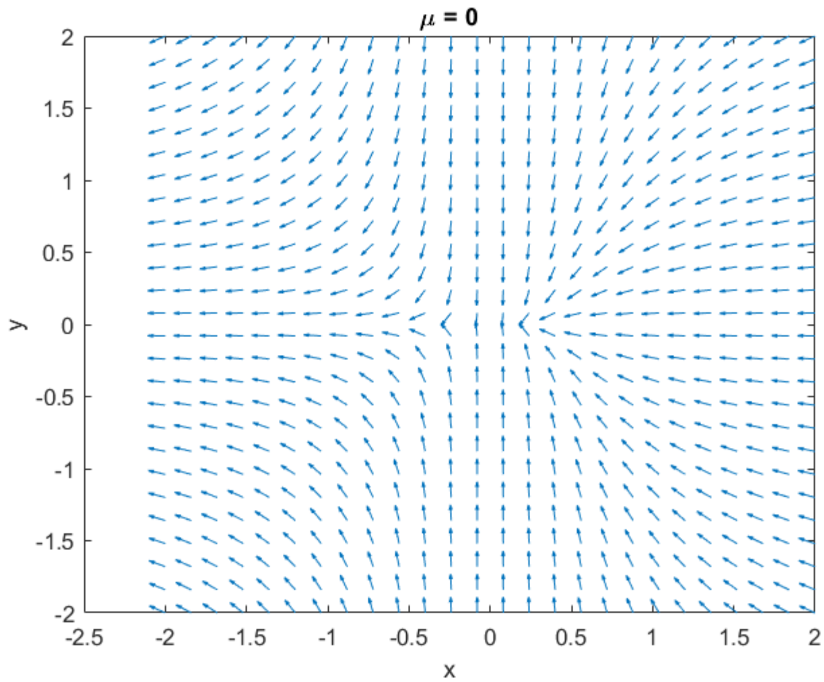

Figure 16: Phase Portrait Q6b

Part c) $\mu = 2$

From equation 16, we find solution for x-coordinate as: $x = \pm\sqrt{2}$. Along with eqn 17, we find that the equilibrium points are $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$. The phase portrait in figure 17 agrees with the same.

We observe that a small perturbation at $(\sqrt{2}, 0)$ will cause the dynamics to proceed such that the equilibrium is restored. So, this equilibrium point is stable.

We observe that a small perturbation at $(-\sqrt{2}, 0)$ along y axis will result in restoration of equilibrium. But perturbation in other directions will cause the dynamics to proceed such that the equilibrium the system moves away from the equilibrium (keeps moving in larger and larger circular orbits). So, this equilibrium point is a saddle point.

Conclusion:

1. Number of equilibrium points = 2
2. $(\sqrt{2}, 0)$ is a stable equilibrium point
3. $(-\sqrt{2}, 0)$ is a saddle point
4. Number of limit cycles = 0

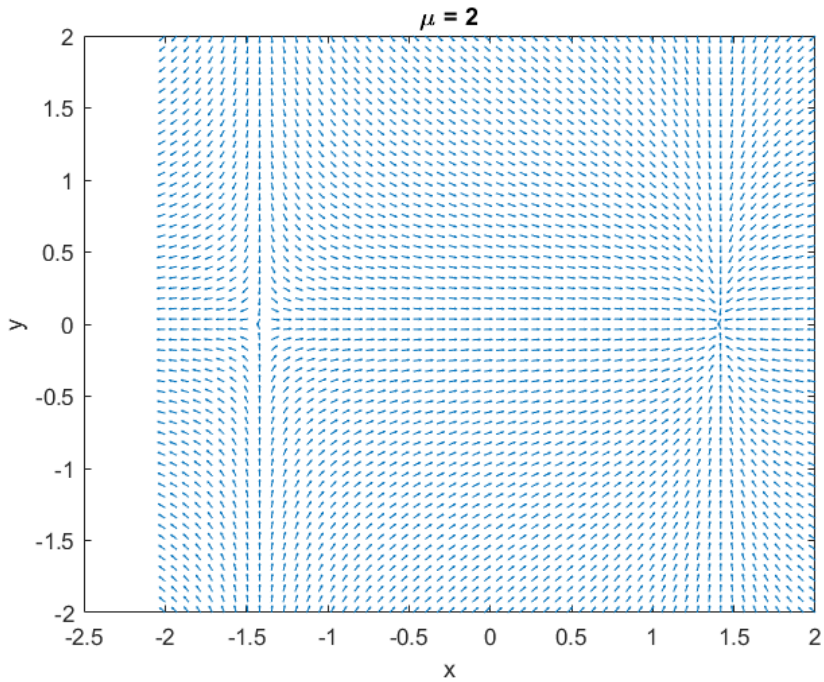

Figure 17: Phase Portrait Q6c

Question 7

Uncontrolled dynamics

We have a second order system, so to represent it in state-space form we can write two first order odes. For the same, define $x = q$ and $y = \dot{q}$. State-space equations:

$$\dot{x} = y \tag{18}$$
$$\dot{y} = -sin(x) - y \tag{19}$$

Setting derivatives to zero we get the solution as: $(n\pi, 0)\forall$ n $\in Z$

From figure 18, we can see that near the origin, the trajectory spirals towards it. Hence, it is a stable focus. This is the same for the points at $x = -2\pi$ and $x = 2\pi$ as well.

At points $x = \pm\pi$, we can see that in some directions (example along $\hat{i} - \hat{j}$ vector) it points towards the equilibrium point (restoration of equilibrium) and in other directions it points away from the point. This is seen clearly in the zoomed in version (figure 19). So, we infer that $x = \pm\pi$ is a saddle point.

Summary:

- Equilibrium points of the form $(2n\pi, 0)$ are stable.
- Equilibrium points of the form $((2n + 1)\pi, 0)$ are saddle points.

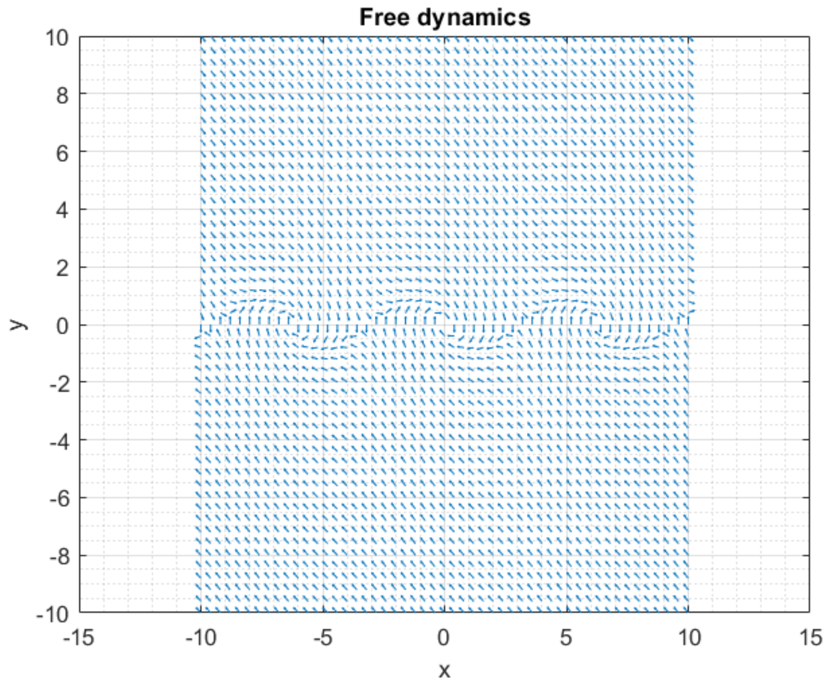

Figure 18: Phase Portrait Q7 - no control

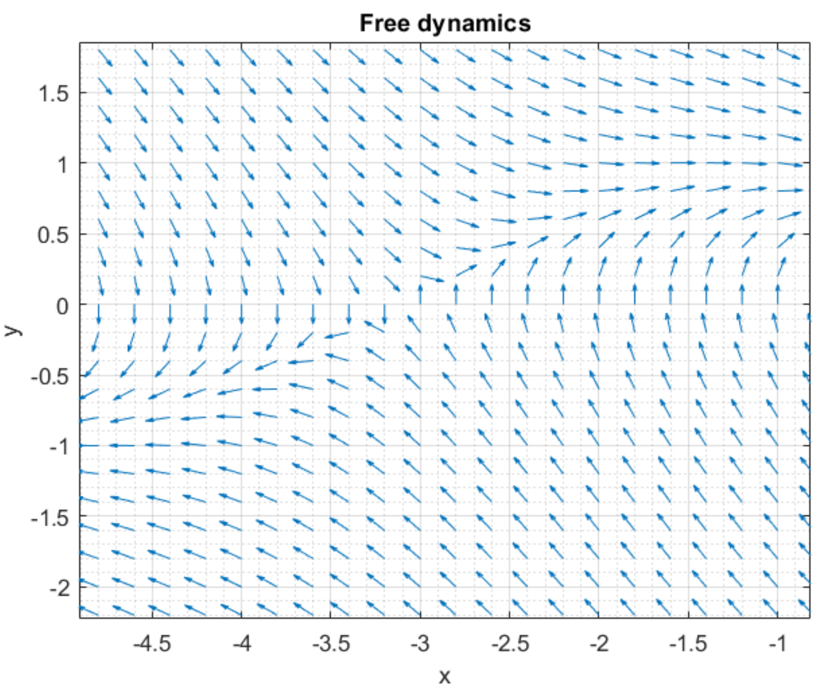

Figure 19: Phase Portrait Q7 - no control - zoomed in at $(-\pi, 0)$

Feedback control

Here we provide an input as shown in equation 20.

$$u = sin(q) - q + 1 \tag{20}$$

$$\ddot{q} + sin(q) + \dot{q} = u \tag{21}$$

Substituting eqn 20 in eqn 21, we get,

$$\ddot{q} + sin(q) + \dot{q} = sin(q) - q + 1$$

$$\ddot{q} + q + \dot{q} - 1 = 0$$

Once again we define $x = q$ and $y = \dot{q}$. State-space equations:

$$\dot{x} = y \tag{22}$$

$$\dot{y} = -x - y + 1 \tag{23}$$

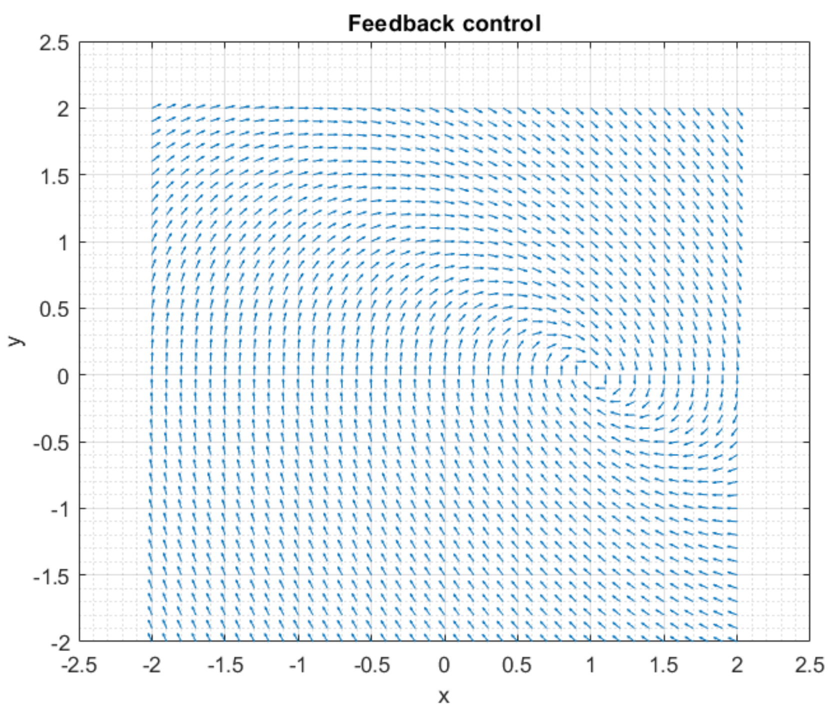

Figure 20: Phase Portrait Q7 - Feedback control

Solving the equations gives $(1, 0)$ as the only equilibrium point and this is reflected in figure 20. We see that trajectories spiral towards the point $(1, 0)$, so it is a stable focus.
$\therefore$ The equilibrium is a stable equilibrium.

References

- Students discussed with:
  1. Arvind Ragghav ME18B086
  2. Karthik Srinivasan ME18B149
- Course notes used:
  1. Class notes
  2. PDFs 5.1 and 5.2 given under introductory material on Moodle