Assignment-5

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Question-1

Part a)

$$d(T(f), T(g)) = max_{x1[0,1]}|T(f) - T(g)| \quad (1)$$
$$= max_{t1[0,1]}|\int_0^t (f(s) - g(s))\, ds| \quad (2)$$
$$\leq max_{t1[0,1]} \int_0^t |f(s) - g(s)|\, ds\text{took mod inside the integral} \quad (3)$$

Let $d(f, g) = max_{x1[0,1]}|f(x) - g(x)| = |f(x1) - g(x1)|$ and since the term inside the integral is always positive, we can do the integration upto its upper limit.

$$\implies d(T(f), T(g)) \leq \int_0^1 |f(x1) - g(x1)|\, ds \quad (4)$$
$$\implies d(T(f), T(g)) \leq d(f, g) * 1 \quad (5)$$
$$\implies d(T(f), T(g)) \leq d(f, g) \quad (6)$$

So a simple example that will hold the equality is f = 1 and g = 0. LHS and RHS both will be equal to 1. Therefore T is not a contraction.

Part b)

In equation 4, if we replace 1 with 0.5,

$$\implies d(T(f), T(g)) \leq \int_0^0 .5|f(x1) - g(x1)|\, ds \quad (7)$$
$$\implies d(T(f), T(g)) \leq 0.5 * d(f, g) \quad (8)$$
$$(9)$$

So, T is a contraction with $\rho = 0.5$

Question-2

Part a)

$$|f(x) - f(y)| = |\frac{1}{1+x} - \frac{1}{1+y}| \quad (10)$$
$$\implies |f(x) - f(y)| \leq |\frac{y - x}{(1+x)(1+y)}| \quad (11)$$
$$\implies |f(x) - f(y)| \leq d(x, y)\frac{1}{(1+x)(1+y)}(\because x, y \geq 0) \quad (12)$$
$$\implies \rho = max_{x,y}\frac{1}{(1+x)(1+y)} \quad (13)$$

We note that we get $\rho = 1$ as the solution. So f is not a contraction.

Part b)

$$f(x^*) = x^* \tag{14}$$
$$\implies \frac{1}{1+x^*} = x^* \tag{15}$$
$$\implies (x^*)^2 + x^* - 1 = 0 \tag{16}$$
$$\implies x^* = \frac{-1 \pm \sqrt{5}}{2} \tag{17}$$

Since, $x^* \in [0, 1]$, we have a unique fixed point, $x^* = \frac{-1+\sqrt{5}}{2}$

Question-3

For locally Lipschitz we just need bounded derivative locally. For globally Lipschitz, the derivative should be bounded, ie shouldnt keep increasing to infinity or decrease to -infinity.

Part a)

Continuously differentiable

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \tag{18}$$

So at $x \neq 0$, $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$ We see that the left hand derivative as well as the right hand derivative do not exist at $\lim_{x \to 0} f'(x)$. So the function is not continuously differentiable.

Locally Lipschitz

At all points other than x = 0, the function is locally Lipschitz, because the derivative is bounded and we can always fix L = max f'(x) $x \in B(x_0, \epsilon)$, so that,

$$\|f(x) - f(y)\| \leq L\|x - y\| \text{ where x,y in } B(x_0, \epsilon)$$

Continuous

Each pieces are continuous functions and at x = 0, $\lim_{x \to 0^+} x^2 \sin \frac{1}{x} = \lim_{x \to 0^-} x^2 \sin \frac{1}{x} = 0 = f(0)$. So the function is continuous.

Globally Lipschitz

We see that the derivative keeps increasing as x keeps increasing so the function is not globally Lipschitz.

Part b)

Continuously differentiable

$$f(x) = \begin{cases} \frac{x^3}{3} + x, & x \geq 0 \\ \frac{x^3}{3} - x, & x \leq 0 \end{cases} \tag{19}$$

So at $x \neq 0$, $f'(x) = \begin{cases} x^2 + 1, & x \geq 0 \\ x^2 - 1, & x \leq 0 \end{cases}$ We see that the left hand derivative and the right hand derivative do not match around x = 0 (-1 and 1). So the function is not continuously differentiable.

Locally Lipschitz

At all points other than x = 0, the function is locally Lipschitz, because the derivative is bounded and we can always fix L = max f'(x) $x \in B(x_0, \epsilon)$, so that,

$$\|f(x) - f(y)\| \leq L\|x - y\| \text{ where x,y in } B(x_0, \epsilon)$$

Continuous

$\frac{x^3}{3}$ and $|x|$ are continuous functions. So sum of continuous functions is also continuous. Therefore, the function is continuous.

Globally Lipschitz

We see that the derivative keeps increasing as x keeps increasing so the function is not globally Lipschitz.

Part c)

Continuously differentiable

$$f(x) = \begin{bmatrix} -x_1 + a|x_2| \\ -(a+b)x_1 + bx_1^2 - x_1x_2 \end{bmatrix} \tag{20}$$

So at $x \neq 0$, $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$ We see that $\frac{\partial f_1}{\partial x_2}$ is not continuously differentiable. So the function is not continuously differentiable.

Locally Lipschitz

At all points other than x = 0, the function is locally Lipschitz, because the derivative is bounded and we can always fix L = max f'(x) $x \in B(x_0, \epsilon)$, so that,

$$\|f(x) - f(y)\| \leq L\|x - y\| \text{ where x,y in } B(x_0, \epsilon)$$

Continuous

f_2 is obviously a continous function. f_1 is also continous, because as shown earlier, $|x|$ is a continous function. So the function f is continuous.

Globally Lipschitz

$$\frac{\partial f_2}{x_1} = 2bx_1 - (a+b) - x_2 \tag{21}$$
$$\frac{\partial f_2}{x_2} = -x_1 \tag{22}$$

We see that the derivatives of f_2 are not globally bounded, so the function is not globally Lipschitz.

Question-4

Holder's inequality:

$$\sum_{i=1}^{n} |a_i||b_i| \leq \left(\sum_{i=1}^{n} |a_i|^r\right)^{\frac{1}{r}} \left(\sum_{i=1}^{n} |b_i|^{\frac{r}{r-1}}\right)^{1-\frac{1}{r}} \tag{23}$$

By applying holder's inequality where b is unit vector and $a \in R^n$, one obtains, the ratio of 2 p-norms as bounded So there exists c_1, c_2, such that,

$$c_1\|x\|_\beta \leq \|x\|_\alpha \leq c_2\|x\|_\beta \tag{24}$$

If f is Lipschitz in $\|\|_\alpha$,

$$\begin{aligned}
&\|f(x) - f(y)\|_\alpha \le L\|x - y\|_\alpha \\
\implies &c_1\|f(x) - f(y)\|_\beta \le L\|x - y\|_\alpha \\
\implies &c_1\|f(x) - f(y)\|_\beta \le Lc_2\|x - y\|_\beta \\
\implies &\|f(x) - f(y)\|_\beta \le \frac{Lc_2}{c_1}\|x - y\|_\beta
\end{aligned}$$

Let $k_1 = \frac{1}{c_2}$ and $k_2 = \frac{1}{c_1}$ If f is Lipschitz in $\|\|_\beta$,

$$\begin{aligned}
&\|f(x) - f(y)\|_\beta \le L\|x - y\|_\beta \\
\implies &k_1\|f(x) - f(y)\|_\alpha \le L\|x - y\|_\beta \\
\implies &k_1\|f(x) - f(y)\|_\alpha \le Lk_2\|x - y\|_\alpha \\
\implies &\|f(x) - f(y)\|_\alpha \le \frac{Lk_2}{k_1}\|x - y\|_\alpha
\end{aligned}$$

So f is Lipschitz in $\|\|_\alpha$ also with $L_\alpha = L_\beta \frac{k_2}{k_1}$ So f is Lipschitz in $\|\|_\alpha$ iff it is Lipschitz in $\|\|_\beta$. Hence proved.

Question-5

Let $z(t) = \int_a^t \beta(s)u(s) \ge u(t) - u(a)\, ds$ and $\alpha(t) = z(t) + u(a) - u(t)$
$\dot{z}(t) = \beta(t)u(t) = \beta(t)\alpha(t) + \beta(t)u(a) - \beta(t)\alpha(t)$

$$\begin{aligned}
&z(a) = 0 \\
&z(t) = \int_a^t exp(\int_s^t \beta(\tau)\, d\tau)(\beta(s) * u(a) - \beta(s)) * \alpha(s)\, ds
\end{aligned}$$

Second term is non negative so,

$$z(t) \le \int_a^t exp(\int_s^t \beta(\tau)\, d\tau)(\beta(s) * u(a))\, ds$$

We note that,

$$\begin{aligned}
\int_a^t \beta(s)exp(\int_s^t \beta(\tau)\, d\tau)\, ds &= -\int_a^t \frac{d}{ds}exp(\int_s^t \beta(\tau)\, d\tau) \\
&= -1 + exp(\int_a^t \beta(\tau)\, d\tau)
\end{aligned}$$

So,

$$u(t) - u(a) \le -u(a) + u(a)exp(\int_a^t \beta(\tau)\, d\tau) \implies u(t) \le u(a)exp(\int_a^t \beta(s)\, ds)$$

Question-6

Part a)

$$\dot{x} \le \|\dot{x}\| \le k_1 + k_2\|x\|$$

$$x \le x_0 + k_1(t - t_0) + \int_{t_0}^t \|x(s)\|\, ds$$

Apply mod and use triangle inequality in rhs

$$\|x\| \le |x_0| + k_1(t - t_0) + \int_{t_0}^t \|x(s)\|\, ds$$

Use Groman Bellman taking into account the constant term for the integral

$$\|x(t)\| \leq \|x_0\| + k_1(t-t_0) + k_2 \int_{t_0}^{t} (\|x_0\| + k_1(s-t_0)) \exp(k_2(t-s))\, ds$$

Integrating by parts and simplifying,

$$\|x(t)\| \leq \|x_0\| exp(k_2(t-t_0)) + \frac{k_1}{k_2}(exp(k_2(t-t_0)) - 1),\ \forall t \geq t_0$$

Part b)

We see that the RHS doesn't have finite escape time and goes to infinity only when $lim_{t\to\infty}$. Since the solution for x(t) is bounded by RHS, it won't be able to blow to ∞ in finite time. So it does not have finite escape time.

Question-7

From question-9 we find that the solution for all $t \geq 0$ is bounded in a compact set (the function in RHS is bounded for all values of t). We also see that function $\dot{x}$ is continuous, and locally Lipschitz in R (because derivative of $\dot{x}$ is continuous). Therefore, the system must have a unique solution for all t ≥ 0 .

Question-8

(From Hassan Khalil Appendix C.2)
Consider $\dot{z} = f(t,z) + \lambda, z(t_0) = u_0$
where $\lambda > 0$. On any compact interval $[t_0, t_1]$ from theorem 3.5, we find that for any $\epsilon > 0$, there exists $\delta > 0$ such that if $\lambda < \delta$ then $z(t,\lambda)$ has a unique solution defined in that interval and,

$$|z(t,\lambda) - u(t)| < \epsilon \tag{25}$$

Claim 1: $v(t) \leq z(t,\lambda) \forall t \in [t_0, t_1]$
If it were not true, there would be times $a, b \in (t_0, t_1]$ such that $v(a) = z(a,\lambda)$ and $v(t) > z(t,\lambda)$ for $a < t \leq b$.
Consequently,

$$\begin{aligned} v(t) - v(a) &> z(t,\lambda) - z(a,\lambda) \\ D^+ v(a) &\leq \dot{z(a,\lambda)} = f(a, z(a,\lambda)) + \lambda > f(a, v(a)) \end{aligned}$$

which contradicts the inequality $D^+ v(a) \leq f(t, v(t))$
Claim 2: $v(t) \leq u(t) \forall t \in [t_0, t_1]$
If the statement is not true, then, there would exist $a \in (t_0, t_1]$ such that $v(a) > u(a)$. Taking $\epsilon = [v(a) - u(a)]/2$ and using eqn 25,

$$v(a) - z(a,\lambda) = v(a) - u(a) + u(a) - z(a,\lambda) \geq \epsilon \tag{26}$$

This contradicts the statement of Claim-1.
Thus, we have shown $v(t) \leq u(t) \forall t \in [t_0, t_1]$. Since this is true on every compact interval we conclude that it holds for all $t \geq t_0$. If it were not the case, let $T < \infty$ be the first time the inequality is violated. We have $v(t) \leq u(t) \forall [t_0, T)$ and, by continuity v(t) = u(T). Consequently, we can extend the inequality to the interval $[T, T + \Delta]$ for some $\Delta > 0$ which contradicts the claim that T is the first time the inequality is violated.

Question-9

By comparing the required statement with the comparison lemma, we can intuitively guess that V should be of the form, $V = \sqrt{x_1^2 + x_2^2}$

$$\dot{V} = \frac{x_1\dot{x_1} + x_2\dot{x_2}}{\sqrt{x_1^2 + x_2^2}}$$
$$\implies \dot{V} = \frac{1}{\sqrt{x_1^2 + x_2^2}}(-x_1^2 - x_2^2 + \frac{2x_1x_2}{1+x_2^2} + \frac{2x_1x_2}{1+x_1^2})$$

Note that

$$\frac{2x_1x_2}{1+x_2^2} \leq 2x_1\frac{|x_2|}{1+x_2^2} \qquad \leq |x_1|$$

So,

$$\dot{V} \leq \frac{1}{\sqrt{x_1^2 + x_2^2}}(-x_1^2 - x_2^2 + |x_1| + |x_2|) \tag{27}$$
$$\implies \dot{V} \leq -V + \sqrt{2} \tag{28}$$

(Use AM-GM after expanding sqrt to get the second term) Define,

$$\dot{u} = -u + \sqrt{2} \tag{29}$$

So u(t) is if $u(0) = \|x(0)\|_2$,

$$u(t) = e^{-t}\|x(0)\|_2 + \sqrt{2}(1 - e^{-t})$$

By using comparison lemma,

$$V(t) \leq u(t)$$
$$V(t) \leq e^{-t}\|x(0)\|_2 + \sqrt{2}(1 - e^{-t})$$

References

- Students discussed with:
  1. Arvind Ragghav ME18B086
  2. Karthik Srinivasan ME18B149
- Course notes used:
  1. Class notes
- Hassan Khalil