Assignment-2

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Question-1

$$\dot{x_1} = y - 2x \tag{1}$$
$$\dot{x_2} = \mu + x^2 - y \tag{2}$$

Finding equilibria,

$$eqn1 \implies y = 2x$$
$$eqn2 \implies \mu + x^2 - y = 0$$
$$\implies x^2 - 2x + \mu = 0$$
$$\implies x = \frac{2 \pm \sqrt{4 - 4\mu}}{2}$$
$$\implies x = 1 \pm \sqrt{1 - \mu}$$

We see that when,

- $\mu > 1$: No points of equilibria
- $\mu = 1$: One equilibrium point
- $\mu < 1$: 2 equilibria

The Jacobian is, $\begin{bmatrix} -2 & 1 \\ 2x^* & -1 \end{bmatrix}$
where x^* is the eqbm point of interest.

The characterstic equation will be

$$\begin{vmatrix} -2 - \lambda & 1 \\ 2(1 \pm \sqrt{1-\mu}) & -1 - \lambda \end{vmatrix} = 0$$

$$\implies \lambda^2 + 3\lambda \mp 2\sqrt{1 - \mu} = 0$$

Eqbm i:

$$\lambda^2 + 3\lambda - 2\sqrt{1 - \mu} = 0$$

Notice sum of roots is negative and product of roots is also negative. This means real parts of both roots are negative, therefore it is a stable equilibrium
Eqbm ii:

$$\lambda^2 + 3\lambda + 2\sqrt{1 - \mu} = 0$$

Notice sum of roots is negative and product of roots is positive. This means real parts of one root is negative and other is positive, therefore it is a stable equilibrium.

Question-2

Setting derivatives to zero,

$$x = y \tag{3}$$
$$\mu x + x + sin(x) = 0 \tag{4}$$

Usin Taylor series expansion for sin(x),

$$\begin{aligned} \mu x + x + x - \frac{x^3}{6} &= 0 \\ \implies x &= 0 \text{when} \mu < 0; \\ x &= 0, \pm\sqrt{6(\mu + 2)}\text{otherwise} \end{aligned}$$

Jacobian, $\begin{bmatrix} \mu + cos(x) & 1 \\ 1 & -1 \end{bmatrix}$
Characteristic equation, $\lambda^2 - (\mu + cos(x) - 1)\lambda - \mu - cos(x) - 1 = 0$
For origin,

$$\lambda^2 - (\mu)\lambda - \mu - 2 = 0 \implies D = \mu^2 + 4(\mu + 2) = (\mu + 2)^2 + 4 > 0 \forall \mu$$

Product of roots changes sign at $\mu = -2$ and sum of roots changes sign at $\mu = 0$. We have,

- $\mu < -2$: Sum of roots is negative and product of roots are positive, so we have 2 negative real eigen values. So origin is a stable node.
- $0 > \mu \geq -2$: Sum of roots and product of roots both negative. So one real and one imaginary eigen value. Therefore, origin is a saddle point.
- $\mu \geq 0$: Sum of roots is positive, product of roots is negative. So one real and one imaginary eigen value. Therefore, origin is a saddle point. By the last two cases we infer, it is saddle for $\mu \geq -2$

For the other roots, let's plot the variation of $-\mu - cos(\sqrt{6 + 2\mu}) - 1$ since it dictates the product of eigenvalues and hence the sign of the eigenvalues.

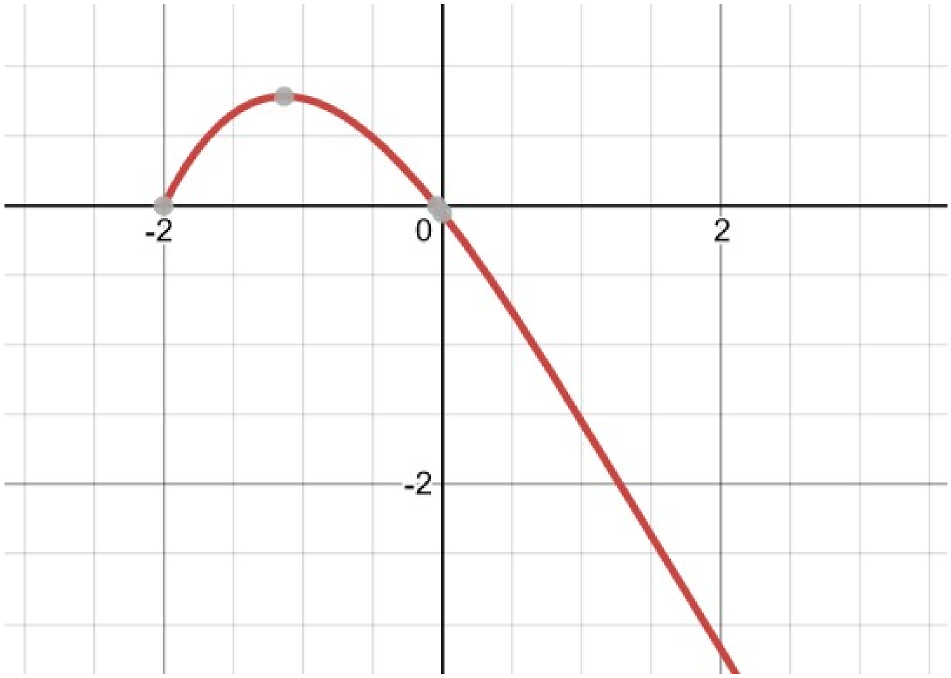

Figure 1: Product of roots

From the figure, we see that it is > 0 between -2 and 0, after which the $-\mu$ term dominates and it is < 0. Let us check the sum of roots $\mu + cos(\sqrt{6 + 2\mu}) - 1$.

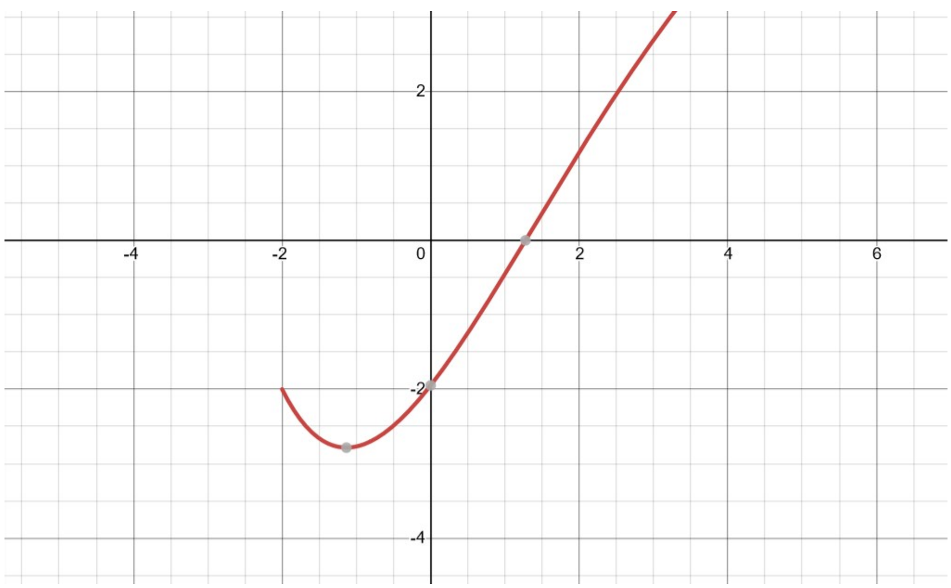

Figure 2: Product of roots

It is < 0 when μ between -2 and 0; > 0 when $\mu > 1.276$. Anyway for $\mu > 0$ product of roots negative so the points are saddle points. For $-2 < \mu < 0$ sum is -ve and product is +ve, leading us to conclude both eigenvalues are negative. Hence, it is stable in that region. We also note that the quadratic has discriminant $> 0 \forall \mu$.

Conclusion: $\mu = -2$ is a critical point and it has supercritical pitchfork bifurcation.

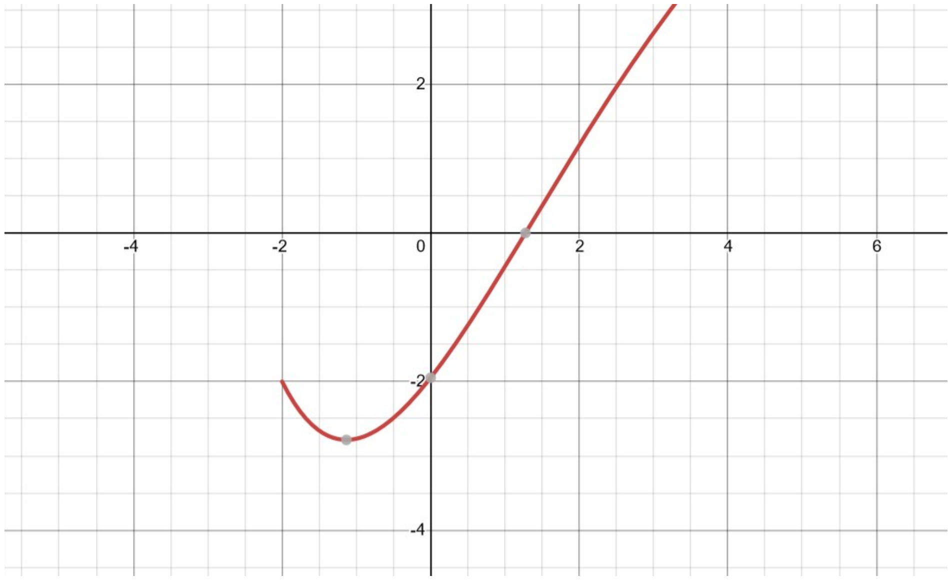

Figure 3: Product of roots

Question-3

a)

Setting derivatives to zero,

$$x_2 = 0$$

$$\implies \mu * x_1 - x_1^3 = 0 \implies x_1 = \pm\sqrt{mu}, 0 \text{when} \mu \geq 0 \text{and} x_1 = 0 \mu < 0$$

Jacobian,

$$\begin{bmatrix} 0 & 1 \\ \mu - 3x_1^2 - 6x_1x_2 & \mu - 1 - 3x_1^2 \end{bmatrix}$$

At each of the equilibrium points, $\begin{bmatrix} 0 & 1 \\ \mu & \mu - 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -2\mu & -2\mu - 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -2\mu & -2\mu - 1 \end{bmatrix}$ The corresponding eigen values,

- $\lambda = -1, \mu$: stable when $\mu < 0$ and saddle when $\mu > 0$
- $\lambda = -1, -2\mu$: stable (because $\mu > 0$ for the root to exist)
- $\lambda = -1, -2\mu$: stable (because $\mu > 0$ for the root to exist)

One root becomes 3 roots after $\mu_{critical} = 0$ and origin transforms from stable to saddle, and other two roots are stable. Hence this is a Supercritical Pitchfork bifurcation.

b)

Once again setting derivatives to zero,

$$x_2 = 0$$

$$\mu - x_1^2 = 0 \implies x_1 \qquad = \pm\sqrt{\mu} \text{when}, \mu > 0; \text{no roots otherwise}$$

Jacobian is, $\begin{bmatrix} 0 & 1 \\ -2(x_1 + x_2) & -1 - 2x_1 \end{bmatrix}$
At each of the equilibrium points,

$$\begin{bmatrix} 0 & 1 \\ \\ -2\sqrt{\mu} & -1-2\sqrt{\mu} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ \\ 2\mu & 2\mu-1 \end{bmatrix}$$

Eigenvalues are, $\lambda = -1, -2\sqrt{\mu}$ and $\lambda = -1, 2\sqrt{\mu}$. So the first point is unstable and other is stable. Therefore, this is a saddle node bifurcation (after critical 2 eqbm points emerge of which one is stable and other is unstable).

c)

Once again setting derivatives to zero,

$$\begin{aligned} x_2 &= 0 \\ \mu x_1 - x_1^3 = 0 \implies x_1 &= 0, \pm\sqrt{\mu}\text{when,}\mu > 0; 0, \text{otherwise} \end{aligned}$$

Jacobian yields, $\begin{bmatrix} 0 & 1 \\ \mu - 3x_1^2 + 6x_1x_2 & \mu - 1 + 3x_1^2 \end{bmatrix}$

Characteristic equation, $\lambda^2 - (\mu - 1 + 3x_1^2)\lambda - (\mu - 3x_1^2) = 0$.

- (0,0), $\lambda = \mu, -1$. Stable for $\mu < 0$ and saddle otherwise.
- ($\pm\sqrt{\mu}$,0):
  Characteristic equation is
  $$\lambda^2 - (4\mu - 1)\lambda + 2\mu = 0$$
  We note that roots change from real to complex at $(4\mu - 1)^2 - 8\mu = 0 \implies \mu = \frac{2\pm\sqrt{3}}{4} = 0.067, 0.933$. Since the discriminant is an upward parabola, it is negative between these roots (so complex eigen values) and positive elsewhere (real eigen values). We note that the real part is +ve when $2\mu > 0$ and $(4\mu - 1) > 0 \implies \mu > 0.25$. Combining these facts we arrive at these conclusions:
  1. $0 < \mu < 0.067$: Stable nodes
  2. $0.067 < \mu < 0.25$: Stable foci
  3. $0.25 < \mu < 0.933$: Unstable foci
  4. $0.933 < \mu$: Unstable nodes

At $\mu = 0$, there is a change from single stable root at origin to saddle at origin and two stable roots elsewhere. So **Supercritical Pitchfork bifurcation at** $\mu = 0$. At $\mu = 0.25$, the two equilibria apart from origin switch from being stable foci to unstable foci. So **Supercritical Hopf bifurcation at** $\mu = 0.25$

Question-4

Question-5

a)

(We used to look for regions such that, $V \leq c$. But here, $V \geq c$, so accordingly we can pose the problem as $-x_2 \leq 0$ to bring to our usual form.)

$$\begin{aligned} V(x) &= -x_2 = 0 \\ \implies \overline{\nabla} V &= -\hat{j} \end{aligned}$$

Checking the dot product of phase vector with normal to the curve,

$$\begin{aligned}
\overline{\nabla} V \cdot (f_1\hat{\imath} + f_2\hat{\jmath}) & \\
&= -\hat{\jmath} \cdot ((a_1 - x_1x_2)\hat{\imath} + (bx_1^2 - cx_2)\hat{\jmath}) && = 0 \\
&= (-bx_1^2 + cx_2)|_{x_2=0} \\
&= -bx_1^2 \\
&\leq 0 \forall x_1
\end{aligned}$$

So trajectories starting in D, stay in D forever.

b)

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = a - x_2 - c$$

We know $c > a$ and $x_2 \geq 0$ in D. So,

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \leq 0$$

always in D. Eg. at $x_2 = 0 \in D$, it has a nonzero value of a - c. So the expression doesn't have a sign change and it is also not identically zero everywhere in D. By Benedixson theorem, no stable orbit exists in D.

Question-6

Question-7

Question-8

Question-9

References

- Students discussed with:
    1. Arvind Ragghav ME18B086
    2. Karthik Srinivasan ME18B149
- Course notes used:
    1. Class notes
    2. PDFs 5.1 and 5.2 given under introductory material on Moodle