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CH5440 : ~~MULTIVARIATE DATA~~ ANALYSIS ASSIGNMENT - 1

1 a) i) ρ known, b unknown

objective : $\min_{a, b, \hat{u}_i} \sum_i \frac{(y_i - a\hat{u}_i - b)^2}{\sigma_\varepsilon^2} + \frac{(u_i - \hat{u}_i)^2}{\sigma_\delta^2}$

multiply & divide by σ_ε^2 : $\min_{a, b, \hat{u}_i} \frac{1}{\sigma_\varepsilon^2} \sum_{\forall i} (y_i - a\hat{u}_i - b)^2 + \frac{\sigma_\varepsilon^2}{\sigma_\delta^2}(u_i - \hat{u}_i)^2$

this is equivalent to : $\min_{a, b, \hat{u}_i} \sum_{i=1}^{N} (y_i - a\hat{u}_i - b)^2 + \rho (u_i - \hat{u}_i)^2$

$= E$ (say)

$$\frac{\partial E}{\partial a} = 0 \Rightarrow 2\sum_{\forall i} \hat{u}_i (y_i - a\hat{u}_i - b) = 0$$

$$\Rightarrow \sum_{i=1}^{N} \hat{u}_i (y_i - a\hat{u}_i - b) = 0 \quad \text{--- (1)}$$

$$\frac{\partial E}{\partial b} = 0 \Rightarrow 2\sum_{\forall i} (y_i - a\hat{u}_i - b) = 0$$

$$\Rightarrow a\sum_{\forall i} \hat{u}_i + b\sum_{\forall i} 1 = \sum_{\forall i} y_i$$

$$\Rightarrow a \cdot \sum_{\forall i} \hat{u}_i + Nb = \sum_{\forall i} y_i \quad \text{--- (2)}$$

$$\frac{\partial E}{\partial \hat{u}_i} = 0 \ \forall i \Rightarrow \quad a(y_i - a\hat{u}_i - \beta) + \rho(u_i - \hat{u}_i) = 0$$

$$\Rightarrow \boxed{\hat{u}_i = \frac{ay_i - ab + \rho u_i}{a^2 + \rho}} \quad (3)$$

Substitute (3) in (2),

$$Nb = \sum y_i - a \sum \left(\frac{ay_i - ab + \rho u_i}{a^2 + \rho} \right)$$

$$\Rightarrow N(a^2+\rho)b = (a^2+\rho)\sum y_i + \left(-a^2 \sum y_i + a^2 b - a\rho u_i\right)$$

$$\Rightarrow N\rho b = \rho \sum_{\forall i} y_i - \rho a \sum_{\forall i} u_i$$

$$\Rightarrow b = \frac{\sum y_i}{N} - \frac{a \sum u_i}{N}$$

$$\Rightarrow \boxed{b = \bar{y} - a\bar{u}} \quad (4)$$

Substitute (3) in (1)

$$\sum \hat{u}_i \left(y_i - \alpha \left(\frac{\alpha y_i - \alpha b + \rho u_i}{a^2 + \rho} \right) - \beta \right) = 0$$

$$\Rightarrow \sum \hat{u}_i \left(\cancel{a^2 y_i} + \rho y_i - \cancel{a^2 y_i} + \cancel{a^2 b} - a\rho u_i - \cancel{a^2 b} - b\rho \right) = 0$$

$$\Rightarrow \sum \hat{u}_i \left(\rho y_i - (a\rho u_i) - \rho \bar{y} - a\rho \bar{u} \right) = 0$$

(Substituting b value from (4))

$$\cancel{\Rightarrow \sum \hat{u}_i \left(\delta y_i - \alpha \delta u_i \right.}$$

Cancelling δ,

$$\Rightarrow \sum_{\forall i} \hat{u}_i \left((y_i - \bar{y}) - \alpha (u_i - \bar{u}) \right) = 0$$

Substituting value of $\hat{u}_i$ from ③,

$$\sum \left(\frac{a y_i - ab + \delta u_i}{a^2 + \delta} \right) \left((y_i - \bar{y}) - a(u_i - \bar{u}) \right) = 0$$

Substituting b from ④

$$\Rightarrow \sum \left(a y_i - a(\bar{y} - a\bar{u}) + \delta u_i \right) \left((y_i - \bar{y}) - a(u_i - \bar{u}) \right) = 0$$

$$\Rightarrow \sum \left(a(y_i - \bar{y}) + a^2 \bar{u} + \delta u_i \right) \left((y_i - \bar{y}) - a(u_i - \bar{u}) \right) = 0$$

$$\Rightarrow \sum_{\forall i} a(y_i - \bar{y})^2 - \alpha^2 (y_i - \bar{y})(u_i - \bar{u}) + a^2 \bar{u} (y_i - \bar{y}) - a^3 \bar{u} (u_i - \bar{u}) + \delta u_i (y_i - \bar{y}) - \delta a u_i (u_i - \bar{u}) = 0 \quad ⑤$$

a^3 term: $-\sum \bar{u}(u_i - \bar{u}) = \bar{u}\left(\sum u_i - \bar{u} \sum 1\right)$

$= \bar{u}\left(\sum u_i - N\bar{u}\right)$

$= \bar{u}(N\bar{u} - N\bar{u}) = 0$

a^2 term: $\sum \bar{u}(y_i - \bar{y}) - (y_i - \bar{y})(u_i - \bar{u})$

$= \bar{u}\left(\sum y_i - N\bar{y}\right) - \left(\sum y_i u_i + N\bar{y}\bar{u} - \bar{y}\sum u_i - \bar{u}\sum y_i\right)$

$$= -\left(\sum_i y_i u_i + N\bar{y}\bar{u} - N\bar{y}\bar{u} - N\bar{y}\bar{u}\right)$$

$$= -\left(\sum_i y_i u_i - N\bar{y}\bar{u}\right) \qquad \left(\because \frac{\sum_i \alpha_i}{N} = \bar{\alpha}\right)$$

$$= -N S_{yu} \qquad \text{(definition of } S_{yu}) \rightarrow E(YU) - E(Y)E(U)$$

a term: $\sum_i (y_i - \bar{y})^2 - \rho u_i (u_i - \bar{u})$

$$= N S_{yy} - \left[\rho \sum_{\forall i} (u_i^2) - \rho\bar{u}\sum_i u_i\right]$$

$$= N S_{yy} - \rho\left[\sum_i (u_i^2) - N(\bar{u})^2\right]$$

$$= N S_{yy} - \rho N S_{uu} = N(S_{yy} - \rho S_{yu})$$

constant: $\sum_i \rho u_i (y_i - \bar{y}) = \rho\left(\sum_i u_i y_i - \sum_i u_i \bar{y}\right)$

$$= N\rho \cdot \left(\frac{\sum (u_i y_i)}{N} - \bar{u}\bar{y}\right)$$

$$= N\rho S_{yu} \qquad \text{(definition of } S_{yu})$$

$$\downarrow$$

$$E(YU) - E(Y)E(U)$$

So, (5) becomes:

$$-N S_{yu} a^2 + (N S_{yy} - \rho N S_{uu}) a + N\rho S_{yu} = 0$$

$$\Rightarrow S_{yu} a^2 + (\rho S_{uu} - S_{yy}) a - \rho S_{yu} = 0$$

~~$\Rightarrow a = \frac{(S_{yy} - \rho S_{uu}) S_{yy} - \rho S_{uu} + \sqrt{(}}{}$~~

Applying Quadratic Formula,

$$a = \frac{S_{yy} - \rho S_{uu} \pm \sqrt{(S_{yy} - \rho S_{uu})^2 + 4\rho S_{yu}^2}}{2 S_{yu}} \quad \text{—(6)}$$

For minimum 2nd order condition : $\frac{d^2 E}{da^2} > 0$

$$\frac{d^2 E}{da^2} = 2 S_{yu}\, a + (\rho S_{uu} - S_{yy})$$

Subst + root

$$2 S_{yu} \frac{\left(S_{yy} - \rho S_{uu} + \sqrt{(S_{yy} - \rho S_{uu})^2 + 4\rho S_{yu}^2}\right)}{2 S_{yu}} + (\rho S_{uu} - S_{yy})$$

$$= \sqrt{(S_{yy} - \rho S_{uu})^2 + 4\rho S_{yu}^2} > 0 \quad \text{—(7)}$$

other root will give $-\sqrt{(S_{yy} - \rho S_{uu})^2 + 4\rho S_{yu}^2}$

that's not a minimum

$$\therefore \boxed{\hat{a} = \frac{S_{yy} - \rho S_{uu} + \sqrt{(S_{yy} - \rho S_{uu})^2 + 4\rho S_{yu}^2}}{2 S_{yu}} \quad \text{—(8)}}$$

ii) If b is known to be 0.

objective becomes $\min_{a, b, \tilde{u}_i} \sum \frac{(y_i - a\tilde{u}_i)^2}{\sigma_\varepsilon^2} + \frac{(u_i - \tilde{u}_i)^2}{\sigma_\delta^2}$

$$\frac{\partial E}{\partial \hat{u}_i} = 0 \Rightarrow a(y_i - a\hat{u}_i) + \rho(u_i - \hat{u}_i) = 0$$

$$\Rightarrow \boxed{\hat{u}_i = \frac{a y_i + \rho u_i}{a^2 + \rho}} \quad \text{— (9)}$$

$$\frac{\partial E}{\partial a} = 0 \Rightarrow \sum \hat{u}_i (\cancel{a u_i} - a\hat{u}_i + y_i) = 0 \quad \text{— (10)}$$

Subst (9) in (10),

$$\sum \hat{u}_i \left(-a \left(\frac{a y_i + \rho u_i}{a^2 + \rho} \right) + y_i \right) = 0$$

$$\Rightarrow \sum \hat{u}_i (\rho)(y_i - a u_i) = 0$$

Subst from (9) again,

$$\sum \frac{a y_i + \rho u_i}{a^2 + \rho} (y_i - a u_i) = 0$$

$$\Rightarrow \sum a y_i^2 - a y_i u_i + \rho y_i u_i - \cancel{a} a^2 y_i u_i = 0$$

$$\Rightarrow a^2 \left(\sum y_i u_i \right) + \cancel{a} a \left(\rho \sum u_i^2 - \sum y_i^2 \right) - \rho \cancel{y_i u_i} \sum y_i u_i = 0$$

these coeff can be termed as respective variances around zero!

Solving for a ensuring $\frac{d^2 E}{da^2} > 0$ as in previous case,

$$\hat{a} = \frac{\left(\sum y_i^2 - \rho \sum x_i^2 \right) + \sqrt{\left(\sum y_i^2 - a \sum x_i^2 \right)^2 + 4\rho \left(\sum x_i y_i \right)^2}}{2 \sum x_i y_j}$$

b) $$\hat{a} = \frac{S_{yy} - \delta S_{uu} + \sqrt{(S_{yy} - \delta S_{uu})^2 + 4\delta S_{yu}^2}}{2 S_{yu}}$$

$$\lim_{\delta \to 0} \hat{a} = \frac{S_{yy} + \sqrt{(S_{yy} - 0)^2 + 0}}{2 S_{yu}}$$

$$\Rightarrow \hat{a}_{IOLS} = \frac{S_{yy}}{S_{yu}}$$

$$\hat{b}_{IOLS} = \bar{y} - \hat{a}_{IOLS}\bar{u} = \bar{y} - \hat{a}_{IOLS}\bar{u}$$

$$\hat{u}_{IOLS,i} = \frac{\hat{a} y_i - \hat{a}\hat{b}_{IOLS}}{\hat{a}^2} = \frac{y_i - \hat{b}_{OLS}}{\hat{a}_{IOLS}}$$

$$\hat{y}_{i,IOLS} = y_i$$

(Subst $\delta = 0$ in (3))

~~OLS~~

$$\hat{a}_{OLS} = \lim_{\delta \to \infty} \frac{S_{yy} - \delta S_{uu} + \sqrt{(S_{yy} - \delta S_{uu})^2 + 4\delta S_{yu}^2}}{2 S_{yu}}$$

$$= \lim_{\delta \to \infty} \frac{\frac{S_{yy}}{\delta} - S_{uu} + \sqrt{\left(\frac{S_{yy}}{\delta} - S_{uu}\right)^2 + \frac{4 S_{yu}^2}{\delta}}}{\frac{2 S_{yu}}{\delta}}$$

Let $\xi = \frac{1}{\delta}$

$$\Rightarrow \hat{a}_{OLS} = \lim_{\xi \to 0} \frac{\xi S_{yy} - S_{uu} + \sqrt{(\xi S_{yy} - S_{uu})^2 + 4\xi S_{yu}^2}}{2 \xi S_{yu}}$$

OLS :

Recall ~~&~~ the quadratic eqn derived in (a),

$$a^2 S_{yu} + (\delta S_{uu} - S_{yy})a \mp \delta S_{yu} = 0$$

Divide by δ,

$$a^2\left(\frac{S_{yu}}{\delta}\right) + \left(S_{uu} - \frac{S_{yy}}{\delta}\right)a \mp S_{yu} = 0$$

~~lt~~ Let $\xi = \frac{1}{\delta}$, $a^2(\xi S_{yu})$

$\cancel{a^2(\xi\delta)}$ $+ (S_{uu} - \xi S_{yy})a$

$\mp S_{uu} = 0$

$\lim_{\delta\to\infty} \Rightarrow \lim_{\xi\to 0} \Rightarrow \lim_{\xi\to 0} (\xi S_{yu})a^2 + (S_{uu} - \xi S_{yy})a - \cancel{\mp} S_{yu} = 0$

$$\Rightarrow a(S_{uu}) - S_{yu} = 0$$

$$\Rightarrow \hat{a}_{OLS} = \frac{S_{yu}}{S_{uu}}$$

$\hat{b}_{OLS} = \bar{y} - \hat{a}_{OLS}\bar{u}$

$\hat{u}_{i,OLS} = u_i \Rightarrow \hat{u}_{i,OLS} = \hat{u}_i$

~~$\hat{y}_{OLS}$~~ $\hat{y}_{i,OLS} = \hat{a}_{OLS} u_i + \hat{b}_{OLS}$