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CH5440 MVDA ASSIGNMENT - 2 CH18B020

(3) a) $S v_1 = \lambda_1 v_1$

$$\begin{bmatrix} 7 & 21 & 34 \\ 21 & 64 & 102 \\ 34 & 102 & 186 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = \begin{bmatrix} \lambda_1 v_{11} \\ \lambda_1 v_{12} \\ \lambda_1 v_{13} \end{bmatrix}$$

$\Rightarrow (7-\lambda_1) v_{11} + 21 v_{12} + 34 v_{13} = 0$ — (1)

$21 v_{11} + (64-\lambda_1) v_{12} + 102 v_{13} = 0$ — (2)

$34 v_{11} + 102 v_{12} + (186-\lambda_1) v_{13} = 0$ — (3)

$3 \times (1) - (2)$

$-3\lambda_1 v_{11} + (\lambda_1 + 1) v_{12} = 0$

$\Rightarrow v_{12} = \left(\frac{3\lambda_1}{\lambda_1 + 1}\right) v_{11}$ — (4)

$= \frac{280.4}{94.4} v_{11} = 3.012 v_{11}$

$34 \times (2) - 21 \times (3)$

$\left(34(64-\lambda_1) - (21)(102)\right) v_{12}$

$+ \left(102 \times 34 - 21(186-\lambda_1)\right) v_{13} = 0$

$\Rightarrow v_{13} = \frac{34\lambda_1 - 34}{21\lambda_1 - 438} v_{13}$ — (5)

Product of roots = $|S|$

$$= 7\times(64\times186 - (102)^2)$$
$$-21(21\times186 - 34\times102)$$
$$+34(102\times21 - 64\times34)$$

$= 146$

$\Rightarrow \lambda_2\lambda_3 = \dfrac{146}{250.4} = 0.5831$

Solving the quadratic

$$\lambda_2, \lambda_3 = \frac{+6.6 \pm \sqrt{(6.6)^2 - 4\times0.5831}}{2}$$

$= 6.510, \; 0.\cancel{09}0896$

$\therefore \lambda_2 = 6.510, \; \lambda_3 = 0.0896$

Using (4) & (5),

$V_{22} = \left(\dfrac{3\lambda_2}{\lambda_2 - 1.81}\right) V_{21}$ $\quad$ $V_{23} = \left(\dfrac{34\lambda_1 - 434}{21\lambda_1 - 438}\right) V_{22}$

3.544

$\Rightarrow V_{22} = \cancel{2.6}\, V_{21}$ $\quad$ $V_{23} = -2.204\, V_{21}$

Normalising,

$$V_2 = \begin{bmatrix} 0.2330 \\ 0.8258 \\ -0.5155 \end{bmatrix}$$

$\Rightarrow -v_{13} = \dfrac{\cancel{-4820.4}\ -8479.6}{-4820.4} v_{12}$

$\Rightarrow V_{13} = \dfrac{8479.6}{4820.4} \times \dfrac{250.4 \times 3 V_{11}}{249.4}$

$= 5.256\, V_{11}$

Normalising,

$$V_{11} = \frac{1}{\sqrt{1 + (3.012)^2 + (5.256)^2}} = 0.1629$$

$V_{12} = 0.4906$

$V_{13} = 0.8560$

$$\therefore v_1 = \begin{bmatrix} 0.1629 \\ 0.4906 \\ 0.8560 \end{bmatrix}$$

Sum of eigen value = trace (S)

$= 7 + 64 + 186$

$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 71 + 186 = \cancel{250}$

$\Rightarrow \lambda_2 + \lambda_3 = 6.6$

Once again using ④ & ⑤,

$$V_{32} = \left(\frac{3 \times 0.0896}{0.0896 - 1} \right) V_{31} = -0.2952\, V_{31}$$

$$V_{33} = \left(\frac{34 \times 0.0896 - 34}{21 \times 0.0896 - 438} \right) V_{32}$$

$$= -0.0209\, V_{3}$$

Normalising,

$$\therefore\ v_3 = \begin{bmatrix} 0.9589 \\ -0.2831 \\ -0.0200 \end{bmatrix}$$

Summary :

$\lambda_1 = 250.4$, $\lambda_2 = 6.510$, $\lambda_3 = 0.0896$

$$V_1 = \begin{bmatrix} 0.1629 \\ 0.4906 \\ 0.8560 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0.2330 \\ 0.8258 \\ -0.5135 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 0.9589 \\ -0.2831 \\ -0.0200 \end{bmatrix}$$

b) If we take just first component,

$$\%\text{ variance retained} = \frac{250.4}{250.4 + 6.5 + 0.0896} \times 100\%$$

$$= 97.436\% > 95\%$$

$\therefore$ It is enough if we retain just one principal component

c) If there are 2 linear relationships, then we just pick the eigen vectors corresponding to the 2 smallest eigenvalues

such that $v^T \underline{z_s} = 0$ — $\underline{z_s}$ is the shifted sample, v is the eigen vector.

So you get,

$$v_1(z_1 - \bar{z}_1) + v_2(z_2 - \bar{z}_2) + v_3(z_3 - \bar{z}_3) = 0$$

$\bar{z}_1 = 9$, $\bar{z}_2 = 68$, $\bar{z}_3 = 129$.

$$\Rightarrow v_1 z_1 + v_2 z_2 + v_3 z_3 - (v_1 \bar{z}_1 + v_2 \bar{z}_2 + v_3 \bar{z}_3) = 0$$

Substituting each of the eigen vectors,

or. ~~eq.~~ $0.233 Z_1 + 0.8258\, Z_2 - 0.5135\, Z_3$

$+ 7.99 = 0$

$\Rightarrow 0.233 Z_1 + 0.8258 Z_2 - 0.5135 Z_3$
$+ 7.9901 = 0$ — (6)

$0.9589 Z_1 - 0.2831 Z_2 - 0.02 Z_3$
$+ 13.201 = 0$ — (7)

where Z_1 is mass

Z_2 is SVL

Z_3 is HLS

d) we need to project $\begin{bmatrix} 10.1 - 9 \\ 73 - 68 \\ 135.5 - 129 \end{bmatrix}$ along the 3 different axes to get corresponding scores.

(mean shifted sample)

$Z_S = Z - \bar{Z} = \begin{bmatrix} 1.1 \\ 5 \\ 6.5 \end{bmatrix}$

$v_1^T Z_S$

Score along axis - 1 = $v_1^T Z_S = \boxed{8.18962}$

(4)

Score along axis-2 $= v_2^T z_s = \boxed{1.04755}$

Score along axis-3 $= v_3^T z_s = \boxed{-0.49071}$

Since there are 2 linear relationships, we can assume the last 2 scores are just due to noise. The only score that matters for compression is along axis-1 ~~with value~~ which has a value of 8.1962

2) We have 2 linear relationships in 6 & 7 and we need to estimate 2 variables. So we can just solve the equations to get mass and HLS.

Solving 6 & 7 for mass by eliminating HLS,

$(6) \times (+0.5135) + (7) \times (-0.0201)$

$$\Rightarrow (0.9589 \times 0.05135) z_1 + (-0.283 \times 0.5135) z_2 + (-0.0201)(0.233) z_1 + (0.8259)(-0.0201) z_2 = 0$$

$$\Rightarrow z_1 = \frac{-(0.8259 \times (-0.0201)) - (0.283)(0.5135)}{0.9589 \times 0.05135 - 0.0201(0.233)} \times 23$$

~~$\therefore$ mass = 19.2.~~

$0.48771 z_1 - 0.161822 + 6.6422 = 0$

6.6184

$\Rightarrow$ mass $= \dfrac{0.1618 \times 73 - \cancel{6.6422}}{0.4877}$

$= \boxed{10.66g}$

f) Here we have 2 equations and only one variable.

So we can try to minimise the error in both equations

$\min_{\hat{z}} (z - \hat{z})^T (z - \hat{z})$

Eliminating mass from ⑥ & ⑦,

$0.8578 z_2 - 0.48772 z_3 + 4.588 = 0$

2233.

~~0.79185~~

We can set up a TLS problem so that z_2^+, z_3^+ satisfies both ⑥ & ⑦

& obtaining $\hat{z}_1^+$

$$\min_{z_2^+, z_3^+} (z_2 - z_2^+)^2 + (z_3 - z_3^+)^2$$

$$\text{s.t.} \begin{bmatrix} 0.8578 & -0.4877 \end{bmatrix} \begin{bmatrix} z_2^+ \\ z_3^+ \end{bmatrix} = -4.586$$

$$\Rightarrow z_2^+ = \frac{-4.586 + 0.4877 z_3^+}{0.8578}$$

Subst. back we get an unconstrained quadratic problem.

$$\cancel{\min_{z_2^+} (0.8\,73 - z_2^+)^2} + \left(135.5 - \cancel{z_3^+}\right)^2 \quad \left(\cancel{\frac{-4.586 + 0.4877 z_3^+}{0.8578}}\right)$$

$$\min_{z_3^+} \left(73 - \left(\frac{-4.586 + 0.4877 z_3^+}{0.8578}\right)\right)^2 + (135.5 - z_3^+)^2.$$

$$\frac{\partial F}{\partial z_3} = 0 \Rightarrow 214.166 = 1.5685 z_3^+$$

$$\Rightarrow z_3^+ = 136.062$$

$$\therefore z_2^+ = 72.0115$$

$\therefore z_1^* =$ (from eqn 6 or 7)

$$= \boxed{10.343 \ g}$$