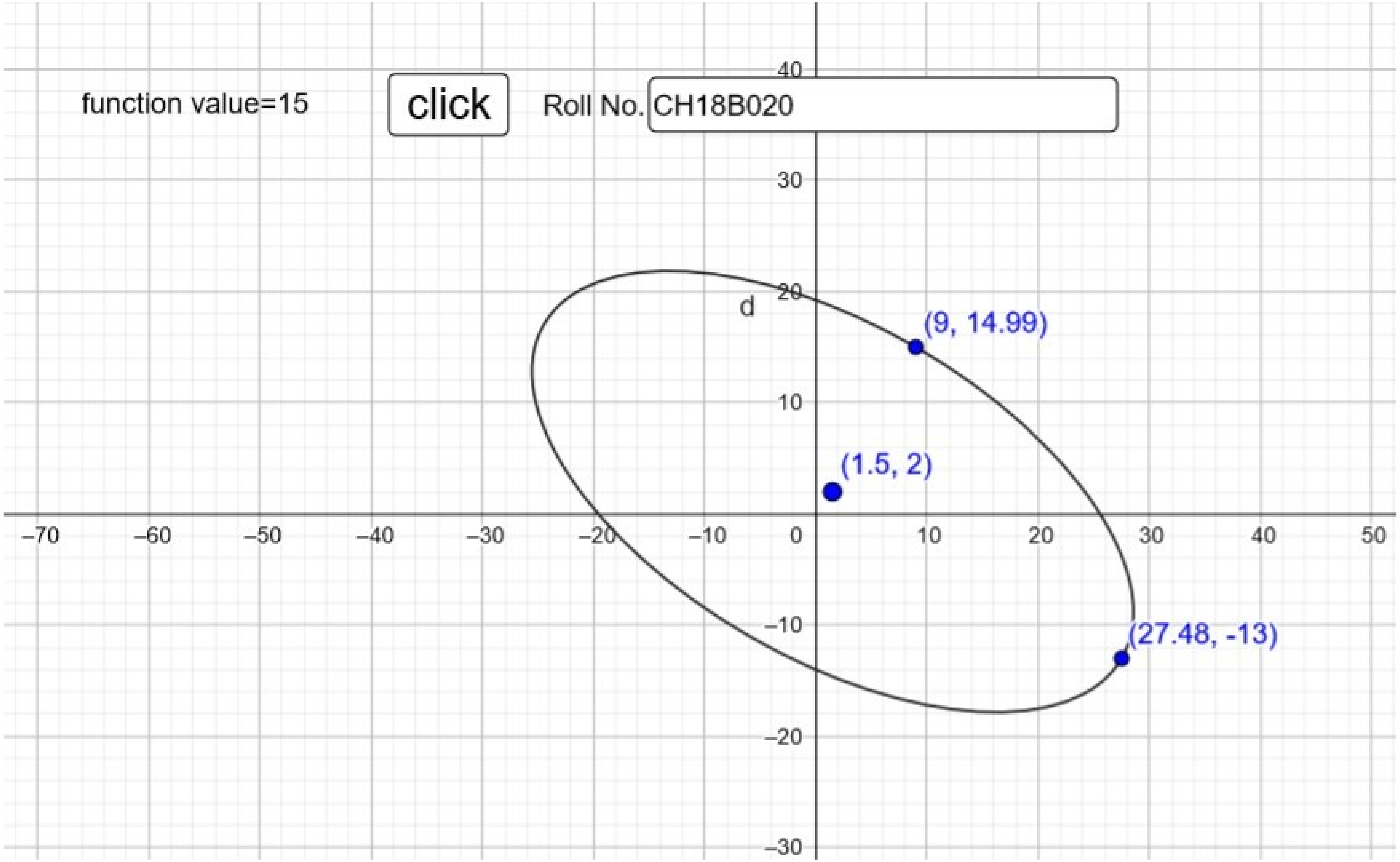

function value=15
click
Roll No. CH18B020
d
(9, 14.99)
(1.5, 2)
(27.48, -13)
40
30
20
10
0
−10
−20
−30
−70
−60
−50
−40
−30
−20
−10
10
20
30
40
50

CH5170 H.W. 2

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CH18B020

① From the graph, we note the following data.

Centre of ellipse $O \equiv (1.5, 2)$

End of major axis $A \equiv (27.48, -13)$

End of minor axis $B \equiv (9, 14.99)$

$f(x)$ at contour $= 15$

We know that the axes of the contour are the eigenvectors of the Hessian Matrix.

Eigen vector along major axis corresponds to the smaller eigen value and eigen vector along minor axis corresponds to the larger eigen value.

Let λ_1 be the larger eigen value & λ_2 be the smaller eigen value.

corresponding eigen vectors: (obtained by subtracting respective coordinates)

V_1 along $\overrightarrow{OB}$: $\begin{bmatrix} 7.5 \\ 12.99 \end{bmatrix}$

V_2 along $\overrightarrow{OA}$: $\begin{bmatrix} 25.98 \\ -15.00 \end{bmatrix}$

We also know,

$$\left(\frac{\lambda_{min}}{\lambda_{max}}\right)^{\cancel{\phi}\,1/2} = \left(\frac{\text{length of minor axis}}{\text{length of major axis}}\right)^2$$

$$= \frac{\left[(9-1.5)^2 + (14.99-2)^2\right]}{\left[(27.48-1.5)^2 + (-13-2)^2\right]}$$

$$= \boxed{0.25}$$

For convenience lets call the ratio β

$$\Rightarrow \lambda_{min} = \beta \lambda_{max} \quad \text{——— (1)}$$

By the eigenvalue decomposition expression, we can write H as:

$$H = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \lambda_{max} & 0 \\ 0 & \lambda_{min} \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}^{-1}$$

$$= d * \left(\begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}^{-1} \right)$$

where $\lambda_{max} = d$

& $\lambda_{min} = d\beta$ (from (1))

We know all values in the matrix.
The matrix multiplication was performed
in MATLAB and the result is

$$H = d \begin{bmatrix} 0.4375 & 0.3248 \\ 0.3248 & 0.8125 \end{bmatrix} \quad — ②$$

General eqn of an ellipse:

$$\frac{1}{2}(x-x^*)^T H (x-x^*) = f(x) - f(x^*)$$

We know $f(x) = 15$ at all points on the ellipse.

and $x^* = [1.5 \quad 2]^T$

$$\Rightarrow (x-x^*)^T H (x-x^*) + 2(f(x^*) - 15) = 0 \quad — ③$$

Comparing this with the standard eqn

$$(x-x^*) P^{-1} (x-x^*) = 1 \quad — ④$$

We find that, $2(f(x^*) - 15) = -1$

$$\Rightarrow f(x^*) = 15 - 1/2$$

$$= \boxed{14.5} \quad — ⑤$$

Substituting this in eqn (4),

$$(x - x^*)^T H (x - x^*) - 1 = 0 \quad — (6)$$

We know any point on ellipse satisfies this eqn.

So substitute the point on major axis as (x_1, x_2)

and also use value of H in eqn 2 in eqn (6)

But $x - x^* = v_1$

$\Rightarrow v_1^T H v_1 - 1 = 0$

$$\Rightarrow d \begin{bmatrix} 7.5 & 12.99 \end{bmatrix} \begin{bmatrix} 0.4375 & 0.325 \\ 0.325 & 0.8125 \end{bmatrix} \begin{bmatrix} 7.5 \\ 12.99 \end{bmatrix} = 1$$

$$\Rightarrow \boxed{d = 0.0044.}$$

$$\therefore \text{required } H = \begin{bmatrix} 0.0019 & 0.0014 \\ 0.0014 & 0.0036 \end{bmatrix}$$

At centre point, $\vec{\nabla} f = 0$

$\Rightarrow H x^* + a = 0 \Rightarrow a = - H x^*$

(Because centre of ellipse is a stationary point)

$$\Rightarrow \boxed{a = \begin{bmatrix} -0.0058 \\ -0.0094 \end{bmatrix}}$$

Also at a point on the ellipse has ~~a~~ function value of

$f(x) = 15$

$\Rightarrow C = 15 - x^T H x - a^T x$

$x = [27.48 \quad -13]^T$

$$\Rightarrow \boxed{C = 14.5137}$$

$$\therefore \boxed{f(x) = 14.514 + [-0.0058 \quad -0.0094]\, x + \frac{1}{2} x^T \begin{bmatrix} 0.0019 & 0.0014 \\ 0.0014 & 0.0036 \end{bmatrix} x}$$

(2) $A \equiv (27.48, -13)$ is a point on the major axis

$$\vec{\nabla} f = Hx + a$$

$$= \begin{bmatrix} 0.0019 & 0.0014 \\ 0.0014 & 0.0036 \end{bmatrix} \begin{bmatrix} 27.48 \\ -13 \end{bmatrix} + \begin{bmatrix} -0.0058 \\ -0.0094 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0289 \\ -0.0167 \end{bmatrix}$$

Steepest descent : $-\vec{\nabla} f = \begin{bmatrix} -0.0289 \\ 0.0167 \end{bmatrix}$

Steepest ascent : $\vec{\nabla} f = \begin{bmatrix} 0.0289 \\ -0.167 \end{bmatrix}$

Direction of no change :

tangential to the $\vec{\nabla} f$.

$$= \begin{bmatrix} -0.0167 \\ -0.0289 \end{bmatrix}$$

(3) We know that the minimum distance along steepest ~~distan~~ descent is given by

$$\alpha^* = \frac{\vec{\nabla} f^T \nabla f}{\nabla f^T H \nabla f} = 899.9604$$

($\because$ after minimum the function starts to increase again)

$$\boxed{\text{distance} = \| \alpha^* p \|_2 = \underline{29.9993}}$$

(4) i) starting point $x = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$$\vec{\nabla} f = \begin{bmatrix} 0.0019 & 0.0014 \\ 0.0014 & 0.0036 \end{bmatrix} \begin{bmatrix} 27.48 \\ -13 \end{bmatrix} + \begin{bmatrix} -0.0058 \\ -0.0094 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9723 \\ 0.7217 \end{bmatrix} \times 10^{-3}$$

Steepest descent $= -\vec{\nabla} f = 10^{-3} \times \begin{bmatrix} -0.9723 \\ -0.7217 \end{bmatrix}$

Steepest ascent $= \vec{\nabla} f = 10^{-3} \times \begin{bmatrix} 0.9723 \\ 0.7217 \end{bmatrix}$

Direction of no change: $10^{-3} \times \begin{bmatrix} 0.7217 \\ -0.9723 \end{bmatrix}$
(tangential to gradient)

ii) Maximum distance we can travel is equivalent to finding the $\min_{\alpha} f(x + \alpha p)$ because function starts to increase only after the minimum.

So we can travel upto $\alpha : \min_{\alpha} f(x + \alpha p)$

$$\alpha^* = \frac{\vec{\nabla} f^T \vec{\nabla} f}{\vec{\nabla} f^T A \vec{\nabla} f} = 255.2110$$ (derived in class)

$\therefore$ Maximum distance $= \| \alpha^{t} p \|_2$

$= \boxed{0.3090}$

(5) If the function is -ve definite the ~~fixed~~ eigen values of the Hessian will become negative

So new $H = -V \Lambda V^{-1}$.

$= - H\text{ old}$

$$\Rightarrow \boxed{H_{new} = \begin{bmatrix} -0.0019 & -0.0014 \\ -0.0014 & -0.0038 \end{bmatrix}}$$

Since centre is a stationary point.

$$\boxed{a = -Hx^* = \begin{bmatrix} 0.0058 \\ 0.0094 \end{bmatrix}}$$

(where $x^* = \begin{pmatrix} 1.5 \\ 2 \end{pmatrix}$)

$a = -\frac{g'}{2}$ Any point on contour has fn value 15.

$\Rightarrow \frac{1}{2} x^t H x + a^t x + c = 15$

Let $x = A$ (point on major axis - ellipse intersection)

$\Rightarrow c = 15 - \frac{1}{2} x^T H x - a^T x$

$= \boxed{15.4863}$

$$\therefore\ \boxed{f(x) = 15.4863 + \begin{bmatrix} 0.0058 \\ 0.0094 \end{bmatrix}^T x + \frac{1}{2} x^T \begin{bmatrix} -0.0019 & -0.0014 \\ -0.0014 & -0.0038 \end{bmatrix} x}$$

Table of Contents


```
clear;
```

Given info

```
minor = [9;14.99];
major = [27.48;-13];
centre = [1.5;2];
c0 = 15;
% minor = [27.99;9.5];
% major = [27.5;-19.65];
% centre = [15;2];
% c0 = 15;
```

Part 1)

Eigen vectors

```
v1 = major-centre;
v2 = minor-centre;
% Eigen values
beta = (norm(v2,2)^2)/(norm(v1,2)^2);
V = [v2 v1];
% Minor axis corresponds to the larger eigen value
H_unscaled = V*[1 0;0 beta]*inv(V);
f_min = c0 - 0.5;
d = 1/(v1'*H_unscaled*v1);
% Get H
H = d*H_unscaled;
% Sanity Check
val1 = (v1'*H*v1)/2 + f_min;
val2 = (v2'*H*v2)/2 + f_min;
% Gradient at any point = Hx + a
a = -H*centre;
% At contour function value is c0
c = -0.5*major'*H*major - a'*major + c0;
```

Part 2) and 3)

```
x = major;
grad = H*x + a;
```

```
% Steepest descent: along -1*gradient
sd = -(H*x + a);
% Steepest ascent: along gradient
sa = -sd;
% No change: tangential (perpendicular to gradient)
nc = [grad(2); -grad(1)];
% Will decrease till centre
alpha = grad'*grad/(grad'*H*grad);
```

Part 4)

```
x = [2;2];
grad = H*x + a;
sd2 = -grad;
sa2 = grad;
nc2 = [grad(2); -grad(1)];
alpha2 = grad'*grad/(grad'*H*grad);
```

Part 5)

```
H_nd = -H;
a_nd = -H_nd*centre;
c_nd = -0.5*major'*H_nd*major - a_nd'*major + c0;
```

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