CH5170 Assignment-1

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1 Problem Details

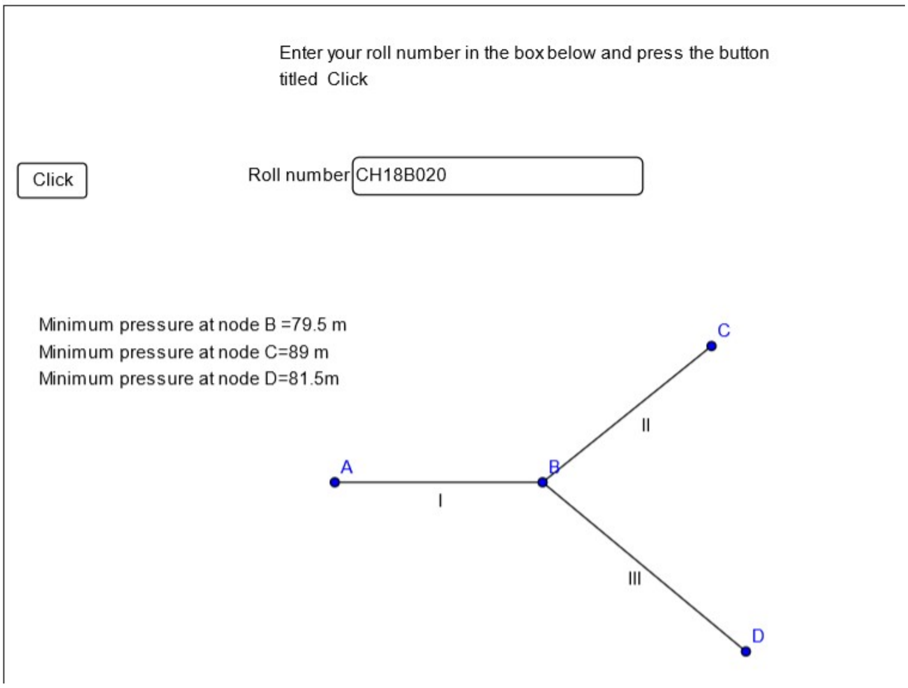

Figure 1: Pressure Values obtained from Geogebra

Other data as obtained from the Problem Sheet are:

1. H_A = 100 m
2. Lengths of the links L_1 = 300 m, L_2 = 500 m, L_1 = 400 m.
3. Flow rates in the links are Q_1 = 9 m^3/min, Q_2 = 3 m^3/min and Q_3 = 2 m^3/min respectively.

Finally, cost of a pipe can be estimated using:

$$c = 1.2654LD^{1.327} \tag{1.1}$$

where L is the length of the pipe (in m), and D is the diameter of the pipe (in mm)

Exercise 1: Head Losses

We make use of the following expression for head loss:

$$\Delta H = 4.457 \cdot 10^8 \cdot \frac{LQ^{1.85}}{D^{4.87}} \tag{2.1}$$

We can first write the equation at node B as:

$$H_A - H_B = 4.457 \cdot 10^8 \cdot \frac{L_1 Q_1^{1.85}}{D_1^{4.87}} \implies H_B = H_A - 4.457 \cdot 10^8 \cdot \frac{L_1 Q_1^{1.85}}{D_1^{4.87}} \tag{2.2}$$

For node C,

$$H_B - H_C = 4.457 \cdot 10^8 \cdot \frac{L_2 Q_2^{1.85}}{D_2^{4.87}} \tag{2.3}$$

Substituting from 2.2 we get H_C as:

$$H_C = H_A - 4.457 \cdot 10^8 \cdot \frac{L_1 Q_1^{1.85}}{D_1^{4.87}} - 4.457 \cdot 10^8 \cdot \frac{L_2 Q_2^{1.85}}{D_2^{4.87}} \tag{2.4}$$

Proceeding similarly for node D:

$$H_B - H_D = 4.457 \cdot 10^8 \cdot \frac{L_3 Q_3^{1.85}}{D_3^{4.87}} \tag{2.5}$$

$$\implies H_D = H_A - 4.457 \cdot 10^8 \cdot \frac{L_1 Q_1^{1.85}}{D_1^{4.87}} - 4.457 \cdot 10^8 \cdot \frac{L_3 Q_3^{1.85}}{D_3^{4.87}} \tag{2.6}$$

Thus we have 3 equations (2.2, 2.4 and 2.6) which relate the Head losses to some known parameters and the pipe diameters(D_1, D_2 and D_3).

Exercise 2: Total Cost

We can simply use cost equation (eqn 2.1) for all the three pipes:

Cost of link 1 (diameter D_1 and length L_1) is $c_1 L_1 = 1.2654 L_1 D_1^{1.327}$

Cost of link 2 (diameter D_2 and length L_2) is $c_2 L_2 = 1.2654 L_2 D_2^{1.327}$

Cost of link 3 (diameter D_3 and length L_3) is $c_3 L_3 = 1.2654 L_3 D_3^{1.327}$

Adding them up we get,

$$Cost = 1.2654 L_1 D_1^{1.327} + 1.2654 L_2 D_2^{1.327} + 1.2654 L_3 D_3^{1.327} \tag{3.1}$$

Exercise 3: Cost in terms of Pressures

Rearranging the Head Loss equation, and obtaining the diameter, we have from eqn 2.1

$$D = (4.457 \cdot 10^8 \cdot \frac{LQ^{1.85}}{\Delta H})^{\frac{1}{4.87}}$$

Proceeding same way for all 3 pipes we obtain:

$$D_1 = (4.457 \cdot 10^8 \cdot \frac{L_1 Q_1^{1.85}}{H_A - H_B})^{\frac{1}{4.87}} \tag{4.1}$$

$$D_2 = (4.457 \cdot 10^8 \cdot \frac{L_2 Q_2^{1.85}}{H_B - H_C})^{\frac{1}{4.87}} \tag{4.2}$$

$$D_3 = (4.457 \cdot 10^8 \cdot \frac{L_3 Q_3^{1.85}}{H_B - H_D})^{\frac{1}{4.87}} \tag{4.3}$$

Substituting diameter values from equations 4.1,4.2,4.3 into the cost equation obtained previously (eqn 3.1), we obtain:

$$\begin{aligned} Totalcost = 1.2654 \cdot L_1 \cdot (4.457 \cdot 10^8 \cdot \frac{L_1 Q_1^{1.85}}{H_A - H_B})^{\frac{1.327}{4.87}} + 1.2654 \cdot L_2 \cdot (4.457 \cdot 10^8 \cdot \frac{L_2 Q_2^{1.85}}{H_B - H_C})^{\frac{1.327}{4.87}} \\ +1.2654 \cdot L_3 \cdot (4.457 \cdot 10^8 \cdot \frac{L_3 Q_3^{1.85}}{H_B - H_D})^{\frac{1.327}{4.87}} \end{aligned} \tag{4.4}$$

Exercise 4: Optimization Formulation

Our objective is to minimize the cost function which we derived above. The objective function:

$$\begin{aligned} Totalcost = 1.2654 \cdot L_1 \cdot (4.457 \cdot 10^8 \cdot \frac{L_1 Q_1^{1.85}}{H_A - H_B})^{\frac{1.327}{4.87}} + 1.2654 \cdot L_2 \cdot (4.457 \cdot 10^8 \cdot \frac{L_2 Q_2^{1.85}}{H_B - H_C})^{\frac{1.327}{4.87}} \\ +1.2654 \cdot L_3 \cdot (4.457 \cdot 10^8 \cdot \frac{L_3 Q_3^{1.85}}{H_B - H_D})^{\frac{1.327}{4.87}} \end{aligned} \tag{5.1}$$

such that (Inequality constraints)

$$\begin{aligned} H_B &\geq 79.5m \\ H_C &\geq 89m \\ H_D &\geq 81.5m \end{aligned} \tag{5.2}$$

And the diameters so obtained should be positive. (bound constraint; which upon substituting in the diameter in terms of head-loss equation 2.2, 2.3 and 2.5 implies $H_A > H_B, H_B > H_C$ and $H_B > H_D$)

Exercise 5: Optimum Pressures

From the inequality and equality constraints we can infer 2 things:

1. **Inequality constraint-1 is redundant:**
For water to flow to tanks C and D from B, we know that $H_B > H_C$ and $H_B > H_D$.
But $H_C > 89 and H_D > 81.5$ which are higher than the limit 79.5 m imposed on H_A as we can see from eqn 5.2. Therefore, Inequality constraint-1 is redundant. Imposing $H_C > 89m$ itself ensures $H_B > 89m$
2. **Optimal values of H_C and H_D:**
Consider the equation 2.3

$$H_B - H_C = 4.457 \cdot 10^8 \cdot \frac{L_2 Q_2^{1.85}}{D_2^{4.87}}$$

Let H_C be some 100 m. If I keep decreasing D_2, the pressure head also keeps falling. Lower the diameter, lower the cost. So we keep lowering the diameter until we hit the minimum value for H_C. At this point, we can't reduce pipe size and hence the pipe cost for link II can't be reduced further.
Therefore, H_C = 89m is the optimal Pressure at node C.
By a similar argument, H_D = 81.5m is the optimal Pressure at node D.
As a result of the above inferences, we can remove all the 3 inequality constraints (automatically satisfied). We can then substitute values of H_C and H_D in equation 6.1 as 89m and 81.5m.
Now the objective is

$$Totalcost = 1.2654 \cdot L_1 \cdot (4.457 \cdot 10^8 \cdot \frac{L_1 Q_1^{1.85}}{H_A - H_B})^{\frac{1.327}{4.87}} + 1.2654 \cdot L_2 \cdot (4.457 \cdot 10^8 \cdot \frac{L_2 Q_2^{1.85}}{H_B - H_C})^{\frac{1.327}{4.87}} + 1.2654 \cdot L_3 \cdot (4.457 \cdot 10^8 \cdot \frac{L_3 Q_3^{1.85}}{H_B - H_D})^{\frac{1.327}{4.87}} \tag{6.1}$$

Where H_B is the only unknown and there are no constraints.

Exercise 6: Unconstrained univariate optimisation

The above optimisation problem was solved in MATLAB and the solution was found to be $H_B = 95.204m$
And of course, as mentioned earlier, $H_C = 89m$ and $H_D = 81.5m$.
The corresponding pipe diameters obtained by substituting Pressure Heads in the equations from part one.
Diameter values:

$$D_1 = 321.5947mm$$

$$D_2 = 223.1900mm$$

$$D_3 = 155.3124mm$$

Cost = Rs. $2.1096 \cdot 10^6$
MATLAB CODE:

```
clear;
% minimise costs
```

```
sol = fmincon(@cost,90,-1,80);
cost_sol = cost(sol);
L1 = 300; L2 = 500; L3 = 400;
Ho = 100; beta = 89; gamma = 81.5;
Q1  = 9; Q2 = 3; Q3 = 2;
D1 = (4.457*10^8)^(1/4.87)*(L1*Q1^1.85/(Ho-sol))^(1/4.87);
D2 = (4.457*10^8)^(1/4.87)*(L2*Q2^1.85/(sol-beta))^(1/4.87);
D3 = (4.457*10^8)^(1/4.87)*(L3*Q3^1.85/(sol-gamma))^(1/4.87);
function f = cost(HA)
    L1 = 300; L2 = 500; L3 = 450;
    Ho = 100; beta = 89; gamma = 81.5;
    Q1  = 9; Q2 = 3; Q3 = 2;
    f = 1.2654*L1*(4.457*10^8)^(1.327/4.87)*((L1*Q1^1.85/(Ho-HA)))^(1.327/4.87);
    f = f + 1.2654*L2*(4.457*10^8)^(1.327/4.87)*((L2*Q2^1.85/(HA-beta)))^(1.327/4.87);
    f = f + 1.2654*L3*(4.457*10^8)^(1.327/4.87)*((L3*Q3^1.85/(HA-gamma)))^(1.327/4.87);
end
```

Exercise 7: Optimisation for the case of Discrete Diameters

Pipes of only a certain set of diameters are available in the market. Since we can use two pipes per link, I am going to use two closest diameter pipes in series as mentioned in the *'problem set 1.pdf'*. This helps us get as close to a single pipe cost as possible.(because larger pipe is costlier than the optimal, and smaller pipe is cheaper than optimal[but can't maintain constraint]) Also, we need to ensure that the pressure constraints are satisfied.

Let the first link have two pipes of lengths l_{11} and l_{12}, and the corresponding diameters be d_{11} and d_{12}. Similarly, we define l_{21}, l_{22}, l_{31}, l_{32}, d_{21}, d_{22}, d_{31}, d_{32}.

For link 1 the diameters are:$d_{11} = 300mm$ $d_{12} = 350mm$.
For link 2 the diameters are:$d_{21} = 200mm$ $d_{22} = 250mm$.
For link 3 the diameters are:$d_{31} = 150mm$ $d_{32} = 200mm$.

Since, we are now using two pipes per link the total cost equation (eq. 3.1) is suitably modified as:

$$\begin{aligned} Cost = {} & 1.2654 l_{11} d_{11}^{1.327} + 1.2654 l_{12} d_{12}^{1.327} \\ & +1.2654 l_{21} d_{21}^{1.327} + 1.2654 l_{22} d_{22}^{1.327} \\ & +1.2654 l_{31} d_{31}^{1.327} + 1.2654 l_{32} d_{32}^{1.327} \end{aligned} \tag{8.1}$$

Further we have the following equality constraints:

$$\begin{aligned} L_1 &= l_{11} + l_{12} \implies l_{12} = L_1 - l_{11} \\ L_2 &= l_{21} + l_{22} \implies l_{22} = L_2 - l_{21} \\ L_3 &= l_{31} + l_{32} \implies l_{32} = L_3 - l_{31} \end{aligned} \tag{8.2}$$

So we can simplify the cost (eq. 8.1) to

$$\begin{aligned} Cost = {} & 1.2654 l_{11} d_{11}^{1.327} + 1.2654 (L_1 - l_{11}) d_{12}^{1.327} \\ & +1.2654 l_{21} d_{21}^{1.327} + 1.2654 (L_2 - l_{21}) d_{22}^{1.327} \\ & +1.2654 l_{31} d_{31}^{1.327} + 1.2654 (L_3 - l_{31}) d_{32}^{1.327} \end{aligned} \tag{8.3}$$

We can write equations relating pressure and diameter similar to those in 2.2, 2.4, 2.6. They come out to be:

$$H_B = H_A - 4.457 \cdot 10^8 \cdot (\frac{l_{11} Q_1^{1.85}}{d_{11}^{4.87}} + \frac{l_{12} Q_1^{1.85}}{d_{12}^{4.87}}) \tag{8.4}$$

$$H_C = H_A - 4.457 \cdot 10^8 \cdot (\frac{l_{11} Q_1^{1.85}}{d_{11}^{4.87}} + \frac{l_{12} Q_1^{1.85}}{d_{12}^{4.87}} + \frac{l_{21} Q_2^{1.85}}{d_{21}^{4.87}} + \frac{l_{22} Q_2^{1.85}}{d_{22}^{4.87}}) \tag{8.5}$$

$$H_D = H_A - 4.457 \cdot 10^8 \cdot (\frac{l_{11} Q_1^{1.85}}{d_{11}^{4.87}} + \frac{l_{12} Q_1^{1.85}}{d_{12}^{4.87}} + \frac{l_{31} Q_2^{1.85}}{d_{31}^{4.87}} + \frac{l_{32} Q_2^{1.85}}{d_{32}^{4.87}}) \tag{8.6}$$

Then substituting the equality constraints in 8.2 and after some simplification, we can express the inequalities in 5.2 in terms of l_{11}, l_{21}, and l_{31}. Notice that the equations are linear in the lengths, so we can express them in $Ax \leq B$ form.

$$(4.457 \cdot 10^8) \cdot \begin{bmatrix} Q_1^{1.85}/d_{11}^{4.87} - Q_1^{1.85}/d_{12}^{4.87} & 0 & 0 \\ Q_1^{1.85}/d_{11}^{4.87} - Q_1^{1.85}/d_{12}^{4.87} & Q_2^{1.85}/d_{21}^{4.87} - Q_2^{1.85}/d_{22}^{4.87} & 0 \\ Q_1^{1.85}/d_{11}^{4.87} - Q_1^{1.85}/d_{12}^{4.87} & 0 & Q_3^{1.85}/d_{31}^{4.87} - Q_3^{1.85}/d_{32}^{4.87} \end{bmatrix} \cdot \begin{bmatrix} l_{11} \\ l_{21} \\ l_{31} \end{bmatrix}$$
$$= \begin{bmatrix} (H_A - 79.5) - L_1 \cdot Q_1^{1.85}/d_{12}^{4.87} \cdot (4.457 \cdot 10^8) \\ (H_A - 89) - L_1 \cdot Q_1^{1.85}/d_{12}^{4.87} \cdot (4.457 \cdot 10^8) - L_2 \cdot Q_2^{1.85}/d_{22}^{4.87} \cdot (4.457 \cdot 10^8) \\ (H_A - 81.5) - L_1 \cdot Q_1^{1.85}/d_{12}^{4.87} \cdot (4.457 \cdot 10^8) - L_3 \cdot Q_3^{1.85}/d_{32}^{4.87} \cdot (4.457 \cdot 10^8) \end{bmatrix} \tag{8.7}$$

We further impose that the lengths should be physically meaningful, that is

$$0 \leq l_{k1} \leq L_k \tag{8.8}$$

for $k \in \{1, 2, 3\}$

In summary we have the cost function (eqn 8.3) and the inequality constraints in (eqns 8.7 and 8.8). This system was optimised in MATLAB and the solutions came out to be:

$l_{11} = 0m$ $l_{12} = 300m$
$l_{21} = 303.206m$ $l_{22} = 196.7940m$
$l_{31} = 370.223m$ $l_{32} = 29.7771m$

and the head losses were
$H_B = 96.8238m$
$H_C = 89m$
$H_D = 81.5m$

Expected Cost = Rs. $2.1908 \cdot 10^6$

Inferences

1. We notice that H_C and H_D are the same as in the previous part! This is because, the arguments made earlier regarding the head losses are still valid here, and hence the 2 pressures are at their respective limiting values.

2. Also, as expected, the cost is more in this case. If it wasn't then we would've obtained that solution in the previous case. So through proof by contradiction we have the cost always to be more in this case(we have imposed the additional constraint of restricting diameters to certain values).

MATLAB CODE:

```
clear;
L1 = 300; L2 = 500; L3 = 400;
Ho = 100; beta = 89; gamma = 81.5; alpha = 79.5;
Q1  = 9; Q2 = 3; Q3 = 2;
d11 = 300; d12=350;
d21 = 200;d22 = 250;
d31 = 150; d32 = 200;
% LHS of the linear inequality constraint
A = zeros(9,3);
A(1,:) = [Q1^1.85/d11^4.87-Q1^1.85/d12^4.87 0 0]*(4.457*10^8);
A(2,:) = [Q1^1.85/d11^4.87-Q1^1.85/d12^4.87 Q2^1.85/d21^4.87-Q2^1.85/d22^4.87 0]*(4.457*10^8)
A(3,:) = [Q1^1.85/d11^4.87-Q1^1.85/d12^4.87 0 Q3^1.85/d31^4.87-Q3^1.85/d32^4.87]*(4.457*10^8)
A(4:6,:) = eye(3);
A(7:9,:) = -1*eye(3);
% RHS
B = zeros(9,1);
B(1) = (Ho-alpha) - L1*Q1^1.85/d12^4.87*(4.457*10^8);
B(2) = (Ho-beta) - L1*Q1^1.85/d12^4.87*(4.457*10^8) - L2*Q2^1.85/d22^4.87*(4.457*10^8);
B(3) = (Ho-gamma) - L1*Q1^1.85/d12^4.87*(4.457*10^8) - L3*Q3^1.85/d32^4.87*(4.457*10^8);
B(4) = L1; B(5) = L2; B(6) = L3;

% minimise costs
sol = fmincon(@cost,[0;300;400],A,B);
cost_sol = cost(sol);
% verification of solution
HA = Ho - 4.457*10^8*Q1^1.85*(sol(1)/d11^4.87 + (L1-sol(1))/d12^4.87);
HB = HA - 4.457*10^8*(Q2^1.85*(sol(2)/d21^4.87 + (L2-sol(2))/d22^4.87));
HC = HA - 4.457*10^8*(Q3^1.85*(sol(3)/d31^4.87 + (L3-sol(3))/d32^4.87));
function f = cost(x)
    L1 = 300; L2 = 500; L3 = 450;
    d11 = 300; d12=350;
    d21 = 200;d22 = 250;
    d31 = 150; d32 = 200;
    f = 1.2654*(x(1)*d11^1.327+(L1-x(1))*d12^1.327);
    f = f+ 1.2654*(x(2)*d21^1.327+(L2-x(2))*d22^1.327);
    f = f+ 1.2654*(x(3)*d31^1.327+(L3-x(3))*d32^1.327);
end
```