BY: S. VISHAL
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CS6700 RL ASSIGNMENT - 1

② a) **State Space** (x_k) : Let the state at the kth-stage represent the crop production at that stage.

Action Space (u_k) : The action we perform u_k, is the fraction of crops ~~invested~~ production in the kth stage invested for production in (k+1)th stage

State evolution : $x_{k+1} = x_k + w_k u_k x_k$

where $\{w_k\}$ is iid with $E(w_k) = \bar{w} > 0$

Here we want to ~~optim~~ maximise the total crop production. So,

let cost $g(x_k, u_k, x_{k+1}) = (1-u_k)x_k$ (fraction of crops stored)

& the terminal cost $g_N(x_N) = x_N$

$$\Rightarrow J^* = \max_{u_0, u_1, u_2, \ldots u_{N-1}} E\left(x_N + \sum_{k=0}^{N-1} (1-u_k)x_k\right)$$

Here the decision variables are u_k, $k \in \{0, \ldots, N-1\}$. and the expectation is over the sequence of random variables $\{w_k\}$

Thus by using this MDP, we have set the appropriate objective as desired in the problem as well as obtained an appropriate description.

b) We can use the DP algorithm to solve this problem

$$J_k^*(x_k) = \max_{u_k} E\Big(\underbrace{J_{k+1}^*(x_{k+1})}_{\text{expectation over } w_k} + \underbrace{(1-u_k)x_k}_{\text{transition cost}} \Big)$$

& $J_N^*(x_N) = g_N(x_N) = x_N$

This procedure can be used computationally to arrive at a solution ~~but numerically~~ since we don't know the structure of $J_{k+1}^*(x_{k+1})$, I would like to evaluate the first 2 steps & take a guess and then proceed by induction to generalise the proof

$$J_{N-1}^*(x_{N-1}) = \max_{u_{N-1}} E\Big(\underbrace{(1-u_{N-1})x_{N-1}}_{\text{transition cost } g(\cdot)} + J_N^*(x_N) \Big)$$

$$= \max_{u_{N-1}} \underset{w_{N-1}}{E}\Big((1-u_{N-1})(x_{N-1}) + [x_{N-1} + w_{N-1}u_{N-1}x_{N-1}] \Big)$$

using the state evolution equation

$$= \max_{u_{N-1}} \Big[(1-u_{N-1})(x_{N-1}) + x_{N-1} + \bar{w}\, u_{N-1}x_{N-1} \Big]$$

$(\because E(w_k) = \bar{w})$

Notice the eqn is linear in u_{N-1} so the extrema lie at the ends of the ~~bounds~~ allowed values of u_{N-1}

Sub $u_{N-1} = 0$ & 1

$$\Rightarrow T^*_{N-1}(x_{N-1}) = \max_{u_1} \left\{ 2x_{N-1} , (1+\bar{\omega}) x_{N-1} \right\}$$

$2x_{N-1} > (1+\bar{\omega}) x_{N-1} \Rightarrow \bar{\omega} > 1$ ($\because x$ is +ve)

So, if $\bar{\omega} > 1$, $u^*_{N-1} = 1$

if $0 < \bar{\omega} \leq 1$, $u^*_{N-1} = 0$

& the $J^*_{N-1}(x_{N-1})$ are $(1+\bar{\omega}) x_{N-1}$ & $2x_{N-1}$ correspondingly

transition cost $g(\cdot)$

$$J^*_{N-2} = E\left((1-u_{N-2}) x_{N-2} + J^*_{N-1}(x_{N-1}) \right)$$

If $\bar{\omega} > 1$,

$$J^*_{N-2}(x_{N-2}) = E\left((1-u_{N-2})(x_{N-2}) + (1+\bar{\omega}) x_{N-1} \right)$$

$$= E\left((1-u_{N-2})(x_{N-2}) + (1+\bar{\omega})(x_{N-2} + \omega_{N-2} u_{N-2} x_{N-2}) \right)$$

$$= (1-u_{N-2})(x_{N-2}) + (1+\bar{\omega})(x_{N-2} + u_{N-2} + \bar{\omega} x_{N-2})$$

$$= \max\left((2+\bar{\omega})(x_{N-2}), (x_{N-2})(1+\bar{\omega})^2 \right)$$

since ~~If~~ $\bar{\omega} > 1$,

$$J^*_{N-2} = (1+\bar{w})^2 x_{N-2} \quad \& \quad u^*_{N-2} = 1$$

Claim: If $\bar{\omega} > 1$, $u^*_k = 1 \ \forall k$, $J^*_{N-k}(x_{N-k}) = (1+\bar{\omega})^k x_{N-k}$

We have shown this for $k=1$ & $k=2$.

Now we can try showing it ~~for~~ for $k+1$ if k is true.

$$J^*_{N-k-1} = \max_{u_{N-k-1}} E\left((1-u)_{N-k-1} x_{N-k-1} + J^*_{N-k}(x_{N-k})\right)$$

$$= \max_{u_{N-k-1}} E \, (1-u_{N-k-1}) x_{N-k-1} + (1+\bar{\omega})^k (x_{N-k-1} + x_{N-k-1}\bar{\omega}\, u_{N-k-1})$$

$$= \max \{ x_{N-k-1} (1+(1+\bar{w})^k), \quad x_{N-k-1} (1+\bar{\omega})^{k+1} \}$$

$$= \max \{ x_{N-k-1} (1+(1+\bar{\omega})^k), \ (x_{N-k-1})(1+\bar{\omega})^{k+1} \}$$

$\because (1+\bar{\omega})^k > 1 \Rightarrow 2(1+\bar{\omega})^k > 1+(1+\bar{\omega})^k$

$\Rightarrow (1+\bar{\omega})^{k+1} > 1+(1+\bar{\omega})^k$ $\longrightarrow \because 1+\bar{\omega} > 1+1 = 2$

$$= x_{N-k-1}(1+\bar{\omega})^{k+1}$$

$$\& \quad u^*_{N-k-1} = 1$$

Thus the proposition holds for all k. (by induction)

If $u^*_{N-k} = ~~\text{[illegible]}~~ 1$ if $\bar{w} > 1$

$\left(\& J^*_{N-k}(x_{N-k}) = (1+\bar{w})^k x_{N-k}\right)$

If $\bar{w} < 1$

$$J^*_{N-2}(x_{N-2}) = \max_{u_{N-2}} E\Big((1-u_{N-2})(x_{N-2}) + J^*_N(x_{N-1})\Big)$$

$$= \max_{u_{N-2}} E\Big((1-u_{N-2})(x_{N-2}) + 2x_{N-1}\Big)$$

$$= \max_{u_{N-2}} E\Big((1-u_{N-2})(x_{N-2}) + 2\big(x_{N-2} + u_{N-2} w_{N-2} x_{N-2}\big)\Big)$$

$$= \max\{3x_{N-2}, (2+\bar{w}) x_{N-2}\}$$

$\because \bar{w} < 1$, $3 > 2+\bar{w}$

$\Rightarrow J^*_{N-2}(x_{N-2}) = 2x_{N-2}$.

Claim: $J^*_{N-k}(x_{N-k}) = k x_{N-k}$ if $\bar{w} < 1$

$\& \ u^*_{N-k} = 0$

$$J^*_{N-k-1} = \max_{u_{N-k-1}} E\Big((1-u_{N-k-1}) x_{N-k-1} + k\big(x_{N-k-1} + (u_{N-k-1}) w_{N-k-1} x_{N-k-1}\big)\Big)$$

$$= \max_{u_{N-k-1}} \Big[(1-u_{N-k-1}) x_{N-k-1} + k\big(x_{N-k-1} + u_{N-k-1}\bar{w} x_{N-k-1}\big)\Big]$$

$= \max \left\{ (k+1) \bar{\omega}\, x_{N-k-1},\ x_{N-k-1} (k)(1+\bar{\omega}) \right\}$

$(k+1)\, x_{N-k-1} \ \&> k\, x_{N-k-1}\, (1+\bar{\omega})$

$\Rightarrow (1+\bar{\omega}) < \dfrac{k+1}{k} \Rightarrow \bar{\omega} < \dfrac{1}{k}.$

So if $\bar{\omega} \le \dfrac{1}{N}$, we can complete the induction & the modified claim:

$V^{*}_{N-k}(x_{N-k}) = k\, x_{N-k}$, & $u_{N-k} = 0 \ \forall k$

if $\bar{\omega} \le \dfrac{1}{N}$ holds

However if $\bar{\omega} < 1$ but $\bar{\omega} > \dfrac{1}{k}$,

new claim: choose $u^{*}_{N-j} = 0 \quad \forall j \in \{1, \dots, k\}$

$u^{*}_{N-j} = 1 \quad \forall j > k$

where k is such that

$\dfrac{1}{k+1} \le \bar{\omega} \le \dfrac{1}{k}$

From the previous proof we can infer that

as long as $\bar{\omega} < \dfrac{1}{j}$, $u^{*}_{N-j} = 0$.

We still need to show the second part: $u^{*}_{N-j} = 1$ if $\bar{\omega} > \dfrac{1}{j}$

So consider a point after $N-k$,

say $N-k-j$, $j>0$

we have shown that $u^*_{N-k-1} = 1$

(base case)

Now ~~for it~~ we can try induction

Assume $J^*_{N-k-j} = (k)(1+\bar{\omega})^j x_{N-k-j}$

($\because$ ~~k~~ $u=0$ k times followed by $u=1$ j times)

$$J^*_{N-k-j-1} = \max_{u_{N-k-j-1}} E\left((1-u_{N-k-j-1})x_{N-k-j-1} + J^*_{N-k-j}\right)$$

$$= \max_{u_{N-k-j-1}} E\left((1-u_{N-k-j-1})x_{N-k-j-1} + k(1+\bar{\omega})^j(x_{N-k-j-1} + \omega_{N-k-j-1}u_{N-k-j-1})\right)$$

$$= \max_{u_{N-k-j-1}} \left[(1-u_{N-k-j-1}) + k(1+\bar{\omega})^j(x_{N-k-j-1} + \bar{\omega}\, u_{N-k-j-1})\right]$$

$$= \max\left\{x_{N-k-j}\left(1+k(1+\bar{\omega})^j\right),\; x_{N-k-j}(1+\bar{\omega})^{j+1}k\right\}$$

$u=0$ $\qquad$ $u=1$

$(1+\bar{\omega})^{j+1}k$ vs $(1+k(1+\bar{\omega})^j)$

$\Rightarrow = \max\left(\left[(1+w)^j k\right]\{1+\bar{w}\},\ \left[(1+\bar{w})^j k\right]\left(\frac{1}{k(1+w)}+1\right)\right)$

removing common factors

$\therefore \max\left((1+\bar{w}),\ \frac{1}{k(1+\bar{w})}+1\right)$

$\bar{w} > \frac{1}{k} \Rightarrow \bar{w} > \frac{1}{k(1+\bar{w})}$

$\hookrightarrow$ same factor >1

$\therefore J^*_{N-k-j-1} = (1+w)^{j+1} k \xi$

$\Rightarrow u_{N-k-j-1} = 1$ better.

$\therefore$ By principle of mathematical Induction

our claim is true

<u>Optimal policy</u>

i) If $\bar{w} > 1$, $u^*_0 = u^*_1 = \cdots = u^*_{N-1} = 1$

ii) If $0 < \bar{w} < \frac{1}{N}$, $u^*_0 = u^*_1 = \cdots = u^*_{N-1} = 0$

iii) If $\frac{1}{N} < \bar{w} < 1$, $u^*_0 = u^*_1 = \cdots = u^*_{N-k-1} = 1$

$u^*_{N-k} = u^*_{N-k+1} = \cdots = u^*_{N-1} = 0$

where k is such that $\frac{1}{k+1} < \bar{w} \leq \frac{1}{k}$