BY: VISHAL SIVARAMAN

HW-09 ISYE 6412

① $E\left(\bar{X}_n^2 - \frac{1}{n}\right) = \frac{-1}{n} + E(\bar{x}_n^2) = \frac{-1}{n} + \underbrace{(E(\bar{x}_n))^2}_{+var(\bar{x}_n)}$

$= \theta^2 + \frac{1}{n} - \frac{1}{n}$

$= \theta^2$

So $\delta = \bar{x}_n^2 - \frac{1}{n}$; is unbiased.

And $\bar{x}_n = \frac{\sum x_i}{n}$ is a complete sufficient statistic.

$\therefore \delta = \bar{x}_n^2 - \frac{1}{n}$ is the best unbiased estimator

$var\left(\bar{x}_n^2 - \frac{1}{n}\right) = var(\bar{x}_n^2)$

$= E(\bar{Y}_n^4) - (E(\bar{x}_n^2))^2$

$\bar{x}_n \sim N(\theta, \frac{1}{n})$

$\Rightarrow Z = \left(\frac{\bar{x}_n - \theta}{1/\sqrt{n}}\right) \Rightarrow \bar{x}_n = \frac{Z}{\sqrt{n}} + \theta$

$\Rightarrow E(x_n^3) = E\left(\left(\frac{Z}{\sqrt{n}} + \theta\right)^3\right) = E\left(\theta^3 + \frac{Z^3}{n\sqrt{n}} + \frac{3\theta^2 Z}{\sqrt{n}} + \frac{3Z^2\theta}{n}\right)$

$= \theta^3 + \frac{3\theta}{n}$

$E(\bar{x}_n^4) = E\left(\left(\frac{Z}{\sqrt{n}} + \theta\right)^4\right) = E\left(\frac{Z^4}{n^2} + \theta^4 + \frac{4Z\theta^3}{\sqrt{n}} + \frac{4Z^3\theta}{n\sqrt{n}} + \frac{6Z^2\theta^2}{n}\right)$

$$= \frac{3}{n^2} + \theta^4 + \frac{6\theta^2}{n}$$

$$\therefore \operatorname{var}\left(\bar{X}_n^2 - \frac{1}{n}\right) = \theta^4 + \frac{6\theta^2}{n} + \frac{3}{n^2} - \left(\theta^2 + \frac{1}{n}\right)^2$$

$$= \frac{4\theta^2}{n} + \frac{2}{n^2}$$

$$f_\theta(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{(x_i - \cancel{\theta}\,\theta)^2}{2}\right)$$

$$\Rightarrow \frac{\partial}{\partial \theta} \log f_\theta = \sum_{i=1}^{n} (x_i - \theta) = n(\bar{X}_n - \theta)$$

$$I_n = E\left(n^2(\bar{X}_n - \theta)^2\right) = n^2\left[\left(E(\bar{X}_n - \theta)\right)^2 + \operatorname{var}(\bar{X}_n - \theta)\right]$$

$$= n$$

$$(\phi'(\theta))^2 = (2\theta)^2 = 4\theta^2$$

$$\therefore \text{CR lower bound} = \frac{(\phi'(\theta))^2}{I_n} = \frac{4\theta^2}{n}$$

$$\therefore \operatorname{var}\left(\bar{X}_n^2 - \frac{1}{n}\right) = \frac{4\theta^2}{n} + \frac{2}{n^2} > \frac{4\theta^2}{n} = \operatorname{var}\left(\frac{4\theta^2}{n}\right)$$

$\Rightarrow \operatorname{var}\left(\bar{X}_n^2 - \frac{1}{n}\right) >$ ~~than~~ CR lower bound

a) We note that the support of pdf doesn't depend on θ

(2) $f_\theta(x_1, \ldots, x_n) = \prod_{i=1}^{n} \theta x_i^{\theta-1}$

$$\Rightarrow \log f_\theta = \sum_{i=1}^{n} \log \theta + (\theta-1) \sum_{i=1}^{n} \log x_i$$

$$\frac{\partial \log f_\theta}{\partial \theta} = \frac{n}{\theta} + \sum \log x_i$$

$$= \underbrace{-n}_{a(\theta)} \Bigg(\underbrace{\frac{-\sum \log x_i}{n}}_{\delta(x)} - \underbrace{\frac{1}{\theta}}_{g(\theta)} \Bigg)$$

$\therefore$ There exists an unbiased estimator $\delta(x) = \dfrac{-\sum_{i=1}^{n} \log x_i}{n}$

for $g(\theta) = \dfrac{1}{\theta}$

b) $f_\theta(x_1, \ldots, x_n) = \prod_{i=1}^{n} \dfrac{\log \theta}{\theta - 1} \theta^{x_i}$

$$\Rightarrow \log f_\theta(x_1, \ldots, x_n) = n \log\left(\frac{\log \theta}{\theta - 1}\right) + \left(\sum_{i=1}^{n} x_i\right) \log \theta$$

$$= n \log(\log \theta) - n \log(\theta - 1) + (\log \theta) \sum_{i=1}^{n} x_i$$

$$\frac{\partial \log f_\theta}{\partial \theta} = \frac{n}{\log \theta} \cdot \frac{1}{\theta} - \frac{n}{\theta - 1} + \frac{\sum_{i=1}^{n} x_i}{\theta}$$

$$= \frac{n}{\theta}\left(\frac{\sum_{i=1}^{n} x_i}{n} - \left(\left(\frac{\theta}{\theta-1}\right) - \frac{1}{\log(\theta)}\right)\right)$$

$a(\theta)$ $\qquad \delta \qquad g(\theta)$

and we note support of pdf is independent of θ

$\therefore \frac{\sum_{i=1}^{n} x_i}{n}$ is ~~the~~ an unbiased estimator for the ~~test~~

$g(\theta) = \left(\frac{\theta}{\theta-1}\right) - \frac{1}{\log\theta}$ and attains the

Cramer Rao Lower Bound.

(3) a) ~~Ht~~ $f(\underline{x}) = \begin{cases} \prod_{i=1}^{n} \theta_2^{-\theta_1} \theta_1 x_i^{\theta_1 - 1} & x_{(n)} \le \theta_2 \\ 0 & \text{ow} \end{cases}$

$\Rightarrow \log f(\underline{x}) = L(\theta_2) = -n\theta_1 \log\theta_2 + n\log\theta_1 + \sum_{i=1}^{n} (\theta_1 - 1)(\log \sum x_i)$

$\log(L) \propto \frac{1}{\theta_2}$ (θ_2 for $\downarrow$ses $\Rightarrow$ θ_2 ($\uparrow$ses)

Lowest θ_2 possible giving non-zero likelihood $= X_{(n)}$

$\therefore \hat{\theta}_{2,MLE} = X_{(n)}$

b) $\cancel{f(\underline{x})}$ =

$$\log L(\theta_1) = -n\theta_1 \log\theta_2 + n\log\theta_1 + \sum_{i=1}^{n} (\theta_1 - 1)\log x_i$$

Support doesn't depend on θ_1,

$$\frac{\partial}{\partial\theta_1}\log L(\theta) = -n\log\theta_2 + \frac{n}{\theta_1} + \sum_{i=1}^{n}\log(x_i)$$

$$\frac{\partial^2}{\partial\theta_1}\log L(\theta_1) = \frac{-n}{\theta_1^2} > 0 \ \forall\ \theta_1$$

$\therefore$ minima

$$\frac{\partial(\log L)}{\partial\theta_1} = 0 \Rightarrow -n\log\theta_2 + \frac{n}{\theta_1} + \sum_{i=1}^{n}\log(x_i) = 0$$

$$\Rightarrow \hat{\theta}_{1_{MLE}} = n\left(\frac{1}{-\sum_{i=1}^{n}\log(x_i) + n\log(\theta_2)}\right)$$

c) case a):

$\cancel{E(X_{(n)})}$

$$F(X_{(n)} \le x) = \left(\theta_2^{-\theta_1}\frac{\theta_1 x^{\theta_1}}{\theta_1}\right)^n \quad \forall\ x \le \theta_2$$

$$= (\theta_2^{-\theta_1})^n x^{n\theta_1}$$

$$f(X_{(n)}) = (n\theta_1 - 1)(\theta_2^{-\theta_1})^n x^{n\theta_1 - 1}$$

$$\therefore E(X_{(n)}) = \int_0^{\theta_2} (n\theta_1 - 1)\, \theta_2^{-\theta_1}\, x^{n\theta_1}\, dx$$

$$= \left(\frac{n\theta_1 - 1}{n\theta_1 - 1}\right) \theta_2$$

$\therefore$ It is biased

case (b) : we note,

$$\frac{\partial}{\partial \theta_1} \log(L(\theta)) = -n\log(\theta_2) + \frac{n}{\theta_1} + \sum_{i=1}^{n} \log(x_i)$$

$$= \underbrace{n}_{a(\theta)} \Bigg(\underbrace{\frac{1}{\theta_1}}_{g(\theta)} - \underbrace{\left(\frac{n\log(\theta_2) - \sum_{i=1}^{n} \log(x_i)}{n} \right)}_{\delta(x)} \Bigg)$$

and support is independent of θ_1

So $n\log(\theta_2) - \sum_{i=1}^{n} \log(x_i) = \frac{1}{\hat{\theta}_{1,MLE}}$ is

an unbiased estimator of $g(\theta) = \frac{1}{\theta_1}$

d) (i) $f_\theta(\underline{x}) = (\theta_2^{-\theta_1})^n \theta_1^n \prod_{i=1}^{n} x_i^{\theta_1 - 1}$

$= \left(\frac{\theta_2^{-\theta_1}}{\theta_1}\right)^n \cdot \cancel{(\prod x_i)} \left(\prod_{i=1}^{n} x_i\right)^{\theta_1 - 1}$

Let $t \triangleq \prod_{i=1}^{n} x_i$

$\therefore f_\theta(\underline{x}) = \underbrace{\left(\frac{\theta_2^{-\theta_1}}{\theta_1}\right)^n \prod_{k} t^{\theta_1 - 1}}_{g(\theta, t)}$ after

$h(\underline{x}) = 1$

$\Rightarrow t = \prod_{i=1}^{n} x_i$ is a sufficient statistic

(ii) $\log L(\theta_1, \theta_2) = \begin{cases} -n\theta_1 \log \theta_2 + n \log \theta_1 + \sum_{i=1}^{n} (\theta_1 - 1) \log x_i & \text{if } X_{(n)} \leq \theta_2 \\ 0 & \text{ow.} \end{cases}$

As argued earlier,

$L(\theta_1, \theta_2)$ decreases with increase in θ_2

So we find the smallest θ_2 possible to maximise likelihood $\rightarrow \hat{\theta}_{2_{MLE}} = X_{(n)}$

~~Once and for~~ :

~~Further wrt not θ_2 and~~

$\frac{\partial(\log L)}{\partial \theta_1} = 0$

$\Rightarrow \hat{\theta}_{1,MLE} = \frac{n}{-\sum_{i=1}^{n} \log x_i + n\log(X_{(n)})}$

$\frac{\partial^2 \log L}{\partial \theta_1^2} = \frac{-n}{\theta_1^2} > 0 \quad \forall \theta_1, \theta_2$

$\therefore$ It is a ~~...~~ maximum

(ii) $\phi(\theta_1, \theta_2) = P_{\theta_1,\theta_2}(X_1 > 1)$

$= \int^{\theta_2} \theta_2^{-\theta_1} \theta_1 x_1^{\theta_1 - 1} \, dx_1$

$= \theta_2^{-\theta_1} \frac{1}{\cancel{\theta_1}} \cancel{\theta_1} \left[\theta_2^{\theta_1} - 1 \right]$

$= 1 - \frac{1}{\cancel{\theta_2^{\theta}}} \theta_2^{-\theta_1}$

Invariance property of MLE : $\hat{\phi}(\theta_1, \theta_2)_{MLE}$

$= \phi(\hat{\theta}_{1,MLE}, \hat{\theta}_{2,MLE})$

$$= 1 - \left(X_{(n)}\right)^{-\left(\frac{n}{-\sum_{i=1}^{n} \log x_i + n\log(X_{(n)})}\right)}$$

iv)

$\hat{\theta}_{1,MLE} = 12.595$

$\hat{\theta}_{2,MLE} = 25$

$\phi(\hat{\theta}_1, \hat{\theta}_2) = 1 - (\hat{\theta}_2)^{-\hat{\theta}_1}$

$\stackrel{n\to}{\approx} 1$

(4) a) $f_\theta(x) = \prod_{i=1}^{n} \frac{1}{\sigma} \exp\left(-\frac{(x_i - \mu)}{\sigma}\right)$

$$= \frac{1}{\sigma^n} \exp\left(-\left(\frac{\sum_{i=1}^{n} x_i}{\sigma} - \frac{n\mu}{\sigma}\right)\right)$$

$$\Rightarrow \log L(\theta) = \begin{cases} -n\log\sigma - \frac{\sum_{i=1}^{n} x_i}{\sigma} + \frac{n\mu}{\sigma} ; & \mu \le x_{(1)} \\ \text{undefined} & \mu > x_{(1)} \end{cases}$$

$\frac{\partial \log L}{\partial \mu} = \frac{\partial \log L}{\partial t} \neq$

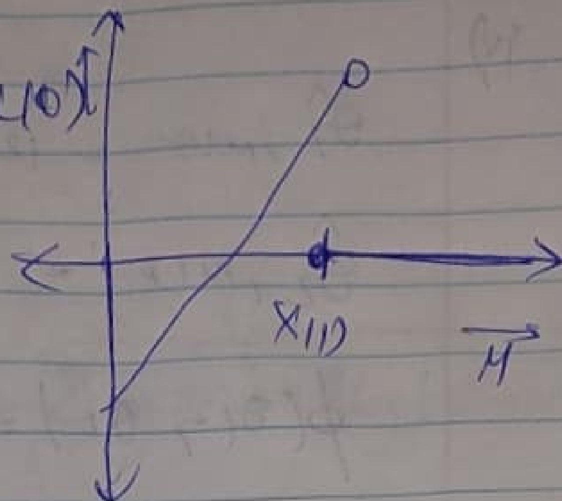
($\because$ we need $x_i \ge \mu \ \forall i$

$\Rightarrow X_{(1)} \ge \mu$)

We observe $L(\mu)$ is linear in μ with a positive slope

So we pick the max value of μ possible

$$\hat{\mu}_{MLE} = X_{(1)}$$



we further observe,

$$\frac{f(\underline{x})}{f(\underline{y})} = \frac{\frac{1}{\sigma}\exp\left(\Sigma \frac{-x_i}{\sigma} + \frac{n\mu}{\sigma}\right) 1(X_{(1)} \geq \mu)}{\frac{1}{\sigma}\exp\left(\Sigma \frac{-y_i}{\sigma} + \frac{n\mu}{\sigma}\right) 1(Y_{(1)} \geq \mu)}$$

is indep of μ iff $X_{(1)} = Y_{(1)}$

$\Rightarrow X_{(1)}$ is a sufficient statistic

$$F_{X_{(1)}}(x) = \max\left(e^{-\frac{(x-\mu)}{\sigma}}\right)^n$$

$$\Rightarrow f_{X_{(1)}}(t) = \frac{n}{\sigma} e^{-\frac{(x-\mu)}{\sigma}(n)}$$

$$\Rightarrow E(X_{(1)}) = \int_{\mu}^{\infty} n\frac{x}{\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)n} dx$$

$$= \mu + \int_{\mu}^{\infty} e^{-\frac{(x-\mu)n}{\sigma}} dx \quad = \mu + \frac{\sigma}{n}$$

$\Rightarrow E\left(X_{(1)} - \frac{\sigma}{n}\right) = \mu$

$\therefore$ $\delta = X_{(1)} - \frac{\sigma}{n}$ is the best unbiased estimator. ($\because$ it is unbiased and fn of T)

b) $$\frac{\partial \log L}{\partial \sigma} = \frac{-n}{\sigma} + \frac{\sum x_i}{\sigma^2} - \frac{n\mu}{\sigma^2}$$

$$\frac{\partial^2 \log L}{\partial \sigma^2} = \underbrace{\frac{n}{\sigma^2}}_{a(\theta)}\left(\underbrace{-\sigma}_{g(\theta)} + \underbrace{\frac{\sum x_i - n\mu}{n}}_{\delta}\right)$$

and support indep of σ

$\therefore$ $\delta = \frac{\sum x_i - n\mu}{n}$ is the best unbiased estimate of $g(\theta) = \sigma$.

$$\frac{\partial \log L}{\partial \sigma} = \frac{-n}{\sigma} + \frac{\sum x_i}{\sigma^2} - \frac{n\mu}{\sigma^2}$$

$$\Rightarrow \frac{\partial^2 \log L}{\partial \sigma^2} = \frac{n}{\sigma^2} - \frac{2\sum x_i}{\sigma^3} + \frac{2n\mu}{\sigma^3}$$

Setting first derivative to 0,

(as $\sigma \neq 0$)

$$\sigma = \frac{\sum_{i=1}^{n} x_i - n\mu}{n} \geq 0 \quad (\because x_i \geq \mu \;\; \forall x_i)$$

2nd derivative : $\frac{1}{\sigma^3}\left(+n\sigma - 2\left(\sum_{i=1}^{n} x_i - n\mu\right)\right)$

$$= \frac{-\left(\sum_{i=1}^{n} x_i - n\mu\right)}{\sigma^3}$$

$\because x_i \geq \mu \Rightarrow -\left(\sum x_i - n\mu\right) \leq 0$

& $\sigma^3 \geq 0$

$\therefore$ 2nd derivative < 0

$\Rightarrow$ it is a maximum!

$$\therefore \hat{\sigma}_{MLE} = \frac{\sum x_i - n\mu}{n}$$

c) We know by setting $\frac{\partial \log L}{\partial \sigma} = 0$

we get a maximum for $\sigma = \frac{\sum x_i - n\mu}{n}$

from part b)

plugging it in expression for $\log L$,

$$\log(L(\theta)) = \begin{cases} -n\log\left(\frac{\sum x_i - n\mu}{n}\right) - \frac{\sum x_i}{\sigma} + \frac{n\mu}{\sigma} ; & \mu \le x_{(1)} \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} -n\log\left(\frac{\sum x_i - n\mu}{n}\right) - n + \frac{n\mu}{\sigma} ; & \mu \le x_{(1)} \\ 0 & \text{otherwise} \end{cases}$$

μ increases $\Rightarrow \frac{n\mu}{\sigma}$ incs

$\Rightarrow a-\mu$ decs $\Rightarrow \log(a-\mu)$ decs $\Rightarrow -n\log(a-\mu)$ incs

$\therefore$ increase in μ still increases likelihood

pick the largest possible $\mu = x_{(1)}$

$\therefore \hat{\mu}_{MLE} = x_{(1)}$

$$\hat{\sigma}_{MLE} = \frac{\sum_{i=1}^{n} x_i - n X_{(1)}}{n}.$$

from part a), we know,

$$E(X_{(1)}) = \mu + \frac{\sigma}{n}$$

$$E(X_i) = \int_{\mu}^{\infty} \frac{x}{\sigma} e^{-\frac{(x-\mu)}{\sigma}} dx.$$

$$= \mu + \int_{\mu}^{\infty} e^{-\left(\frac{x-\mu}{\sigma}\right)} dx.$$

$$= \cancel{\mu\left(1+\frac{n}{\mu}\right)} \; \mu + \sigma$$

$$\therefore E\left(\sum_{i=1}^{n} x_i\right) = n\cancel{\mu}\left(\mu + \frac{\sigma}{\cancel{\mu}}\right)$$

From HW 6 #6 we know $T_1 = X_{(1)}$

$$T_2 = \sum_{i=1}^{n} x_i$$

$$E(T_1) = \mu + \frac{\sigma}{n}.$$

$$E(T_2) = \cancel{n\mu\left(1+\frac{1}{\mu}\right)} \; n\mu + n\sigma$$

$$E\left(-T_1 + \frac{T_2}{n}\right) = \sigma\left(1 - \frac{1}{n}\right)$$

$$\Rightarrow E\left(\left(-T_1 + \frac{T_2}{n}\right)\left(\frac{n}{n-1}\right)\right) = \sigma$$

$$\therefore \delta = \left(\frac{n}{n-1}\right)\left(-X_{(1)} + \frac{\sum_{i=1}^{n} X_i}{n}\right)$$

is an unbiased estimator for σ

But it is a $f(T_1, T_2)$

$$\therefore \delta = \left(\frac{n}{n-1}\right)\left(\bar{X} - X_{(1)}\right)$$ is the best unbiased estimator for σ

$$E\left(\frac{T_2}{n^2}\right) = \frac{\mu}{n} + \frac{\sigma}{n}$$

$$\Rightarrow E\left(T_1 - \frac{T_2}{n^2}\right) = \mu\left(1 - \frac{1}{n}\right)$$

$$\Rightarrow \delta = \left(X_{(1)} - \frac{\sum_{i=1}^{n} X_i}{n^2}\right)\frac{n}{n-1}$$ is an

unbiased estimator of μ.

But it is a function of T_1, T_2

$$\Rightarrow \delta = \left(X_{(1)} - \frac{\bar{X}_n}{n}\right)\left(\frac{n}{n-1}\right)$$ is the

best unbiased estimator for μ

(5) a) $\log L(\theta) = \begin{cases} \log \frac{1}{\theta^n} = -n \log \theta. & \text{if } 0 \le x_i \le \theta \ \forall i \\ 0 & \text{otherwise} \end{cases}$

$\theta \quad = \begin{cases} -n \log \theta & \theta \ge x_i \ \forall i \\ 0 & \text{otherwise} \end{cases}$

$= \begin{cases} -n \log \theta & \theta \ge X_{(n)} \\ 0 & \text{otherwise} \end{cases}$

~~We see~~ $L(\theta)$ ~~r~~ decreases with θ

$\Rightarrow$ choose the smallest θ possible $= X_{(n)}$

$\therefore \hat{\theta}_{MLE} = X_{(n)}$

We have only one unknown here, so we look at only first moment

Set ~~Ef~~ $m_1 = E(X)$

$\Rightarrow \frac{\sum_{i=1}^{n} (X_i)}{n} = \frac{\hat{\theta}_{MOM}}{2}$

$\Rightarrow \hat{\theta}_{MOM} = \frac{2}{n} \sum_{i=1}^{n} X_i$

b) $F_\theta(x) = \frac{x^n}{\theta^n} \Rightarrow \cancel{f_\theta(x) = \begin{cases} \frac{n}{\theta x^{n-1}} \\ \end{cases}}$

$\Rightarrow f_\theta(x) = \begin{cases} \frac{n x^{n-1}}{\theta^n} & 0 \le x \le \theta \\ 0 & ow. \end{cases}$

$$E(X_{(n)}) = \int_0^\theta \frac{n x^n}{\theta^n} dx = \left(\frac{n}{n+1}\right)\theta.$$

$$E(X_{(n)}^2) = \int_0^\theta \frac{n x^{n+1}}{\theta^n} dx = \left(\frac{n}{n+2}\right)\theta^2$$

$$var(X_{(n)}) = n\theta^2\left(\frac{1}{n+2} - \frac{n}{(n+1)^2}\right)$$

$$= n\theta^2\left(\frac{n^2+1+2n-n^2-2n}{(n+1)^2(n+2)}\right)$$

$$= \frac{n\theta^2}{(n+1)^2(n+2)}$$

$$E(\hat{\theta}_{MOM}) = \frac{2}{n} \times \frac{n\theta}{2} = \theta$$

$$var(\hat{\theta}_{MOM}) = \frac{4}{n^2}\sum_{i=1}^{n} var(x_i) = \frac{4}{n^2} \times \frac{\theta^2 n}{12}$$

$$= \frac{\theta^2}{3n}$$

Q) [illegible]

$$g = \frac{n}{(n+1)^2(n+2)} = \frac{1}{(n+1)^2}\left(\frac{n}{n+2}\right)$$

$$\frac{\partial g}{\partial n} = \frac{(n+1)^2(n+2) - 2n(n+1)(n+2) - n(n+1)^2}{\left((n+1)^2(n+2)\right)^2}$$

$< 0 \;\forall\, n$

$\therefore \frac{n}{(n+1)^2(n+2)}$ is a [illegible] fn

at $n=1$, $\frac{1}{(1+1)^2(1+2)} = \frac{1}{3(1+1)^2}$

$\therefore$ MLE: $E(\hat{\theta}_{MLE}) = \frac{n\theta}{n+1}$

$\text{Var}(\hat{\theta}_{MLE}) = \frac{n\theta^2}{(n+1)^2(n+2)}$

Method of Moments: $E(\hat{\theta}_{MOM}) = \theta$

$\text{var}(\hat{\theta}_{MOM}) = \frac{\theta^2}{3n}$

Note that $\frac{n}{(n+1)^2(n+2)} \leq \frac{n}{n^2 \cdot 3} = \frac{1}{3n}$

$\left(\because \begin{matrix} n+1 \geq n \\ n+2 \geq 3 \end{matrix}\right)$

In case of large n,

$$\lim_{n\to\infty} E(\hat{\theta}_{MLE}) = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)\theta = \theta$$

. the bias will be small
and since the variance is lower, we prefer $\hat{\theta}_{MLE}$

In case of small n, ~~we have~~ ~~the~~ we have large bias in MLE but MoM is unbiased so we prefer method of Moments

c) $E(c\hat{\theta}_{MLE}) = \theta$

$$\Rightarrow cE(\hat{\theta}_{MLE}) = \theta$$

$$\Rightarrow c \frac{n\theta}{n+1} = \theta$$

$$\Rightarrow c = \frac{n+1}{n}$$

$$\therefore \delta_c = \left(\frac{n+1}{n}\right) X_{(n)}$$

We note $X_{(n)}$ is a complete sufficient statistic (class notes) and δ_c is unbiased

$\Rightarrow \delta_c$ is the best unbiased estimator

d) $\delta_c = c\hat{\theta}_{MLE}$

$$R_{\delta_c} = E_\theta\left(\left(c X_{(n)} - \theta\right)^2\right)$$

$$= \left(E_\theta\left(c\left(X_{(n)}\right) - \theta\right)\right)^2 + var\left(cX_{(n)} - \theta\right)$$

$$= \left(\frac{cn\theta}{n+1} - \theta\right)^2 + c^2 var\left(X_{(n)}\right)$$

$$= \theta^2\left(\frac{nc}{n+1} - 1\right)^2 + \frac{c^2 n \theta^2}{(n+1)^2(n+2)}$$

$$\frac{dR_{\delta_c}}{dc} = 2\theta^2\left(\frac{nc}{n+1} - 1\right) + \frac{2c n \theta^2}{(n+1)^2(n+2)}$$

$$\frac{d^2R_{\delta_c}}{dc^2} = \theta^2\left(\frac{2n}{n+1} + \frac{2n}{(n+1)^2(n+2)}\right) > 0 \ \forall c$$

$\therefore$ Stationary point is a maximum

$$\frac{dR_{\delta_c}}{dc} = 0 \Rightarrow 2\theta^2\left(\frac{nc}{n+1} - 1\right) + \frac{2cn\theta^2}{(n+1)^2(n+2)} = 0$$

$$\Rightarrow c\left(\frac{n}{n+1} + \frac{n}{(n+1)^2(n+2)}\right) = 1$$

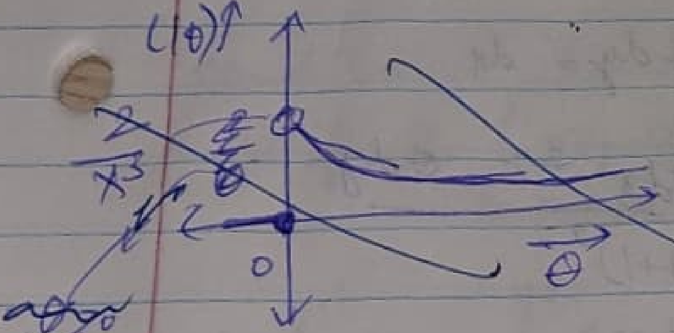
$$\Rightarrow c^* = \frac{(n+1)^2(n+2)}{n\left((n+1)(n+2)+1\right)}$$

$$\therefore \text{reqd. } c = \frac{(n+1)^2(n+2)}{n\left(1+(n+1)(n+2)\right)}$$

(6) a) $L(\theta) = f_\theta(x_1) = \begin{cases} \dfrac{2\theta^2}{(x_1+\theta)^3} & x>0 \\ 0 & x \le 0 \end{cases}$

$\cancel{\dfrac{df}{dx} = \dfrac{-6\theta^2}{(x+\theta)^4}}$ ~~→ no stationary point~~

$< 0 \ \forall \theta$



$\cancel{\dfrac{df}{d\theta} =}$

$$\frac{d}{d\theta}\log L = \log 2 + 2\log\theta - 3\log(x_1+\theta)$$

$$\frac{\partial \log L}{\partial \theta} = \frac{2}{\theta} - \frac{3}{(x_1+\theta)} = 0$$

$$\Rightarrow 2(x_1+\theta) - 3\theta = 0$$

$$\Rightarrow \theta = 2x_1$$

$$\frac{\partial^2 \log L}{\partial \theta^2} = \frac{-2}{\theta^2} + \frac{3}{(x_1+\theta)^2}$$

Subst $\theta = 2x_1$,

$$= -\frac{1}{2x_1^2} + \frac{3}{9x_1^2} = \frac{-1}{2x_1^2} + \frac{1}{3x_1^2} = \frac{-1}{6x_1^2} < 0$$

$$\Rightarrow \left.\frac{\partial^2 \log L}{\partial \theta^2}\right|_{\theta = 2x_1} < 0 \;\forall\, x_1$$

$\therefore \theta = 2x_1$ is a maximum

$$\Rightarrow \hat{\theta}_{MLE} = 2x_1$$

b) $$E(x_1) = \int_0^{\infty} \frac{2\theta^2 x_1\, dx}{(x+\theta)^3} = \int_0^{\infty} \frac{2x\, dx}{\theta\left(\frac{x}{\theta}+1\right)^3}$$

$$y \triangleq \frac{x}{\theta} \Rightarrow \theta\, dy = dx$$

$$\Rightarrow E(x_1) = \int_0^{\infty} \frac{\cancel{2\theta dy}\ 2y\ \theta dy}{\cancel{\theta}(y+1)^3}$$

$$u \triangleq y+1$$

$$\Rightarrow E(x_1) = \theta \int_1^{\infty} \frac{2(u-1)}{u^3}\, du.$$

$$= 2\theta \left(\cancel{\frac{2u}{2u^2}} + \frac{1}{2u^2} - \frac{1}{u} \right)_1^{\infty}$$

$$= \theta$$

$$\Rightarrow E(x_1) = \theta$$

$\therefore E(\hat{\theta}_a) = 2E(x_1) = 2\theta \neq \theta$

$\Rightarrow \hat{\theta}_a$ is not an unbiased estimator

c) $L(\theta, d) = (\theta - d)^2$

$\Rightarrow$ ~~Interm~~ $R_\delta = E(\theta - 2x_1)^2$

$$= \int_0^\infty (\theta^2 + 4x_1^2 - 4x_1\theta) \frac{2\theta^2}{(x_1+\theta)^3} dx$$

$$= 2\theta^2 \left(\int_0^\infty \frac{4x_1^2}{(x_1+\theta)^3} dx - \int_0^\infty \frac{4x_1 \theta}{(x_1+\theta)^3} dx_1 \right.$$

second & third terms $\qquad \left. + \int_0^\infty \frac{\theta^2}{(x_1+\theta)^3} dx \right)$

are finite (proportional to $E(X_1)$ & 1 respectively)

first term: $\int_0^\infty \frac{4x_1^2}{(x_1+\theta)^3} dx = \int_\theta^\infty \frac{4(u-\theta)^2}{u^3} du$

$$= \int_\theta^\infty \frac{4(u^2 - 2\theta u + \theta^2)}{u^3}$$

$$= 4\left(\log(u) + \frac{\theta(4u - \theta)}{2(u^2)}\right)_0^\infty$$

$\longrightarrow \infty$

$\therefore R_{\delta_a} \longrightarrow \infty$

$$R_\delta(\theta) = E\left(\cancel{\ldots}(\theta-17)^2\right)$$

$$= (\theta-17)^2$$

which is a finite value for all finite θ

So the constant estimate is much better than the MLE estimator

d) If $n=2$,

$$L(\theta) = \prod_{i=1}^{n} \begin{cases} \dfrac{2\theta^2}{(x_1+\theta)^3} \times \dfrac{2\theta^2}{(x_2+\theta)^3} & x_1, x_2 > 0 \\ 0 & \text{ow} \end{cases}$$

$$\log L = \log 4 + 4\log\theta - 3\log(x_1+\theta) - 3\log(x_2+\theta)$$

$$\frac{\partial \log L}{\partial\theta} = \frac{4}{\theta} - \frac{3}{x_1+\theta} - \frac{3}{x_2+\theta}$$

$$\frac{\partial^2 \log L}{\partial\theta^2} = \frac{4}{\theta^2} - 3\left(\frac{1}{(x_1+\theta)^2} + \frac{1}{(x_2+\theta)^2}\right)$$

$$\frac{\partial \log L}{\partial \theta} = 4(x_1+\theta)(x_2+\theta) - 3x_2\theta - 3\theta^2 - 3x_1\theta - 3\theta^2$$

$$= 4\theta^2 + 4x_1x_2 + 4x_1\theta + 4x_2\theta - 3x_2\theta - 3x_1\theta - 6\theta^2$$

$$= -2\theta^2 + 4x_1x_2 + \theta(x_1+x_2) = 0$$

$$\Rightarrow 2\theta^2 - (x_1+x_2)\theta - 4x_1x_2 = 0$$

$$\Rightarrow \theta = \frac{1}{4}\left(x_1+x_2 \pm \sqrt{x_1^2+x_2^2+2x_1x_2 + 4\times 4x_2x_1}\right)$$

$$= \frac{1}{4}\left(x_1+x_2 \pm \sqrt{x_1^2+x_2^2+34x_1x_2}\right)$$

$$\frac{\partial^2 \log L}{\partial \theta^2} = \frac{4}{\theta^2} - 3\left(\frac{1}{(x_1+\theta)^2} + \frac{1}{(x_2+\theta)^2}\right)$$

~~we note that $\theta <$~~

~~we note that $\theta < 0 \Rightarrow \frac{1}{x_1+\theta}$~~

Denomenator > 0 always

Numerator

$$= 4(x_1+\theta)^2(x_2+\theta)^2 - 3\theta^2(x_1+\theta)^2 - 3\theta^2(x_2+\theta)^2$$

$$= 4(x_1^2+\theta^2+2x_1\theta)(x_2^2+2x_2\theta+\theta^2) - 3\theta^2(x_1^2+x_2^2 + 2\theta^2 + 2x_1\theta + 2x_2\theta)$$

But $\theta > 0$

$$\Rightarrow \theta = \frac{1}{4}\left(x_1 + x_2 + \sqrt{x_1^2 + 34x_1x_2 + x_2^2}\right)$$

$$\left(\because x_1^2 + 34x_1x_2 + x_2^2 > (x_1+x_2)^2 \; ; x_1, x_2 \geq 0 \right)$$

$$\frac{\partial^2 \log L}{\partial \theta^2} = -\frac{4}{\theta^2} - 3\left(\frac{1}{(x_1+\theta)^2} + \frac{1}{(x_2+\theta)^2}\right).$$

$$\lim_{\theta \to 0} L(\theta) = 0$$

$$\lim_{\theta \to \infty} L(\theta) = 0$$

and we have only one stationary point

Furthermore we note $L(\theta) \geq 0$

$\therefore$ The stationary point is a maximum.

$$\therefore \hat{\theta}_{MLE} = \frac{1}{4}\left[x_1 + x_2 + \sqrt{x_1^2 + 34x_1x_2 + x_2^2}\right]$$

Predicted shape of $L(\theta)$: