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ISYE6412 HW-08

① From HW #7,

$$T(x) = \max\left[-x_{(1)}, \frac{1}{2}x_{(n)}\right]$$

is complete sufficient

We also know from HW-7

$$f_T(t) = \begin{cases} \frac{n}{\theta^n} t^{n-1} & \text{if } 0 \le t \le \theta \\ 0 & \text{otherwise.} \end{cases}$$

$$E(T(X)) = \int_0^\theta t \times \frac{n}{\theta^n} t^{n-1} \, dt$$

$$= \frac{n}{\theta^n} \int_0^\theta t^n \, dt = \frac{n\theta^{n+1}}{(n+1)\theta^n}$$

$$= \left(\frac{n}{n+1}\right)\theta.$$

$$\Rightarrow E\left(\left(\frac{n+1}{n}\right)T(x)\right) = \theta$$

$\therefore$ $g(T) = \left(\frac{n+1}{n}\right)T(x)$ is an unbiased estimator of θ

And $g(\cdot)$ is fn of T

$\Rightarrow$ $g(T) = \left(\frac{n+1}{n}\right)T(x)$ is the best unbiased estimator of θ.

(2)

~~Consider $\delta_0 = \begin{cases} 1 & \text{if } X_1 - X_2 = \end{cases}$~~

From HW#7, the ~~usu~~ complete sufficient T is

$T = X_{(1)}$

and $f_T(t) = \begin{cases} n e^{n(\theta - t)} & t > \theta \\ 0 & ow \end{cases}$

Let $g(T)$ be the function which is ~~wa~~ the best unbiased estimator

$$E(g(T)) = \int_{\theta}^{\infty} g(T) n e^{n(\theta - t)} dt = \theta^r$$

$$\Rightarrow e^{n\theta} \int_{\theta}^{\infty} g(t) n e^{-nt} dt = \theta^r$$

Differentiating wrt θ,

$$\Rightarrow e^{n\theta}\left(-g(\theta) n e^{-n\theta}\right) + \cancel{e^{n\theta}} n \left[e^{n\theta} \int_{\theta}^{\infty} g(t) n e^{-nt} dt\right]$$

$$= r\theta^{r-1}$$

$$\Rightarrow -n g(\theta) + n\theta^r = r\theta^{r-1}$$

$$\Rightarrow g(\theta) = \theta^r - \frac{r}{n}\theta^{r-1}$$

$\therefore$ the best unbiased estimator is

$$g(T) = T^r - \frac{r}{n} T^{r-1}$$

where $T = X_{(1)}$

(3)

(a) $E(\delta(X_1, \dots, X_{n+1})) = 1 \times P(\delta = 1) + 0 \times P(\delta = 0)$

$$= P(\delta = 1) = P\left(\sum_{i=1}^{n} X_i > X_{n+1}\right) = h(p)$$

$\therefore$ δ is an unbiased estimator of $h(p)$

(b) $P(\delta = 1) = 1 - P\left(\sum_{i=1}^{n} X_i \le X_{n+1}\right)$

$\sum_{i=1}^{n} X_i \le X_{n+1}$ is possible only in 3 (disjoint) events : (i) $X_1 = X_2 = \dots = X_n = X_{n+1} = 0$

(ii) $X_1 = X_2 = \dots = X_n = 0,\ X_{n+1} = 1$

(iii) One of $X_1, \dots, X_n = 1$ & $X_{n+1} = 1$

otherwise $P(X_i \le X_{n+1}) = 0$

Probability of

case (i) $(1-p)^{n+1}$

(ii) $(1-p)^n p$

(iii) ${}^nC_1\, p(1-p)^{n-1} \times p$

$= {}^nC_1\, p^2 (1-p)^{n-1}$

$\because$ tosses are iid,

$P(x_1 \ldots$ one of $x_1, \ldots, x_n = 1, x_{n+1} = 1)$

$= P(\underbrace{\text{one of } x_1, \ldots, x_n = 1}_{{}^nC_1\, p(1-p)^{n-1}})\, \underbrace{P(x_{n+1}=1)}_{p}$

(ch) Rao Blackwell thm, best unbiased estimator $= E(\delta \mid T=b)$

Pf Case (i) $T=0$ (we note $T \geq 0$ always)

case (ii) $T = 1+0 = 1$

case (iii) $T = 1+1 = 2$

case (iv) $T \geq 3$

$g(T) = E(\delta \mid T=b) = P(\delta = 1 \mid T=b)$

$= P\left(\sum_{i=1}^{n} x_i > x_{n+1} \mid T=b\right)$

$= 1 - P\left(\sum_{i=1}^{n} x_i \leq x_{n+1} \mid T=b\right)$

$P\left(\sum_{i=1}^{n} x_i \leq x_{n+1} \mid T=0\right) = 1$

($\because$ always $\sum x_i = x_{n+1} = 0$ & $\sum x_i = 0$)

$P(\sum x_i \leq x_{n+1} \mid T \geq 3) \Rightarrow$ at least 3 x_i's are $> 1 \Rightarrow$ at least 2 x_i's other than $x_{n+1} \geq 1 \Rightarrow \sum x_i \geq 2$ but $x_{n+1} \leq 1$

$\Rightarrow$ probability is 0

$$P\left(\sum_{i=1}^{n} x_i \leq x_{n+1} \mid T = 1\right)$$

$$= \frac{P\left(\sum_{i=1}^{n} x_i \leq x_{n+1}, \sum_{i=1}^{n+1} x_i = 1\right)}{P\left(\sum_{i=1}^{n+1} x_i = 1\right)}$$

Num. probab. falls under case (ii)

$$= \frac{(1-p)^n p}{{}^{n+1}C_1 (1-p)^n p} = \frac{1}{n+1}$$

$$P\left(\sum_{i=1}^{n} x_i \leq x_{n+1} \mid T = 2\right)$$

$$= \frac{P\left(\sum_{i=1}^{n} x_i \leq x_{n+1}, \sum_{i=1}^{n+1} x_i = 2\right)}{P\left(\sum_{i=1}^{n+1} x_i = 2\right)}$$

$P(\sum x_i \leq x_{n+1}, \sum x_i = 2)$ falls under case (iii)

$$= {}^{n}C_1 p^2 (1-p)^{n-1}$$

$P(\sum x_i = 2) = {}^{n+1}C_2\, p^2 (1-p)^{n-1}$ ($\because$ ~~geometric~~ ~~Binomial~~ Binomial)

$$\Rightarrow P\left(\sum_{i=1}^{n} x_i \leq x_{n+1} \mid T = 2\right)$$

$$= \frac{{}^{n}C_1 p^2 (1-p)^{n-1}}{{}^{n+1}C_2\, p^2 (1-p)^{n-1}}$$

$$= \frac{n}{\frac{(n+1)(n)}{2}} = \frac{2}{n+1}$$

$$\therefore g(T=b) = \begin{cases} 0 & b=0 \\ 1-\frac{1}{n+1} = \frac{n}{n+1} & b=1 \\ 1-\frac{2}{n+1} & b=2 \\ 1-0 & b \geq 3 \end{cases}$$

$$= \begin{cases} 0 & b=0 \\ \frac{n}{n+1} & b=1 \\ \frac{n-1}{n+1} & b=2 \\ 1 & b \geq 3 \end{cases}$$

(4) From class, we showed that

$T(X) = (\bar{X}, S^2)$ is complete and suf

$U = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$ which indep. of (μ, σ^2)

$\Rightarrow E(U^{p/2})$ is indep. of $\mu, \sigma^2 = c_{p,n}$

$$\Rightarrow E\left(\frac{(n-1)^{p/2} S^p}{\sigma^p}\right) = c_{p,n}$$

here S is the R.V. over which exp. is taken, σ^p is just a constant

$$\Rightarrow E\left(\frac{(n-1)^{p/2} S^p}{c_{p,n}}\right) = \sigma^p$$

$\therefore \frac{(n-1)^{p/2} S^p}{c_{p,n}}$ is an unbiased estimator for σ^p

But it is also a function of $T = S^2$.

$\Rightarrow \frac{(n-1)^{p/2} S^p}{c_{p,n}}$ is the best unbiased estimator for σ^p

part. We note, $E(Y) = n$ where $Y \sim \chi^2_n$

$$\Rightarrow \int_0^\infty \frac{y^{n/2} e^{-y/2}}{2^{n/2} \Gamma(n/2)} dy = n$$

$$E(U^{p/2}) = \int_0^\infty \frac{u^{p/2} u^{\frac{n-1}{2}-1} e^{-u/2}}{2^{\frac{(n-1)}{2}} \Gamma\left(\frac{n-1}{2}\right)} du =$$

(a) $p = \cancel{4}\, 1$

$$E\left(U^{1/2}\right) = \int_0^\infty \frac{u^{\frac{n}{2}-1} e^{-u/2}}{2^{\frac{n-1}{2}} \Gamma\left(\frac{n-1}{2}\right)}\, du$$

$$= 2^{1/2}\left(\frac{\Gamma(n/2)}{\Gamma\left(\frac{n-1}{2}\right)}\right) \int_0^\infty f\left(X^2_n\right) du$$

$$= 2^{1/2}\left(\frac{\Gamma(n/2)}{\Gamma\left(\frac{n-1}{2}\right)}\right)$$

$\therefore$ The best unbiased estimator is $\dfrac{S(n-1)^{1/2}}{\sqrt{2}}\left(\dfrac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right)$ for σ

(b) $p = 2$

$$E(U) = \int_0^\infty \frac{u^{\frac{n+1}{2}-1} e^{-u/2}}{2^{\frac{n-1}{2}} \Gamma\left(\frac{n-1}{2}\right)}\, du$$

$$= \frac{2\,\Gamma\left(\frac{n-1}{2}+1\right)}{\Gamma\left(\frac{n-1}{2}\right)} = 2 \times \frac{n-1}{2}$$

$$= n-1$$

$\therefore$ Best estimator is $\dfrac{(n-1)S^2}{(n-1)} = S^2$

for σ^2

c) $p = 4$

$$E(U^2) = \int_0^\infty \frac{u^{\frac{n+3}{2}-1} e^{-u/2}}{2^{\frac{n-1}{2}} \Gamma\left(\frac{n-1}{2}\right)} du.$$

$$= \frac{2^2 \Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} = \frac{2^2 \left(\Gamma\left(\frac{n+1}{2}\right) \left(\frac{n+1}{2}\right)\right)}{\Gamma\left(\frac{n-1}{2}\right)}$$

$$= 2^2 \frac{(n-1)}{2} \frac{(n+1)}{2}$$

$$= (n-1)(n+1)$$

$$\therefore \frac{(n-1)^2 S^4}{(n-1)(n+1)} = \frac{S^4 (n-1)}{(n+1)}$$ is the

best estimator for σ^4.

(5) a) $$E\left(\frac{1}{X_1}\right) = \int_{0^+}^\infty \frac{1}{\Gamma(\alpha) \beta^\alpha} x^{\alpha-1} e^{-x/\beta} \times \frac{1}{x} dx$$

$$= \int_{0^+}^\infty \frac{1}{\Gamma(\alpha) \beta^\alpha} x^{(\alpha-1)-1} e^{-x/\beta} dx$$

$$= \frac{\Gamma(\alpha-1)}{\Gamma(\alpha)} \frac{\beta^{\alpha-1}}{\beta^\alpha} \int_{0^+}^\infty \frac{1}{\Gamma(\alpha-1) \beta^{\alpha-1}} x^{(\alpha-1)-1} e^{-x/\beta} dx$$

Integral is integral of pdf of $f_\beta(x)$ to = 1

$$E\left(\frac{1}{X_1}\right) = \frac{\beta^{\alpha}}{\Gamma\alpha} \frac{1}{(\alpha-1)\beta}$$

$\left(\because \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = \alpha\right)$, $\because \alpha$ is known constant

$\cancel{\Rightarrow E(X_1(\alpha-1)) = \frac{1}{\beta}}$ $\Rightarrow E\left(\frac{\alpha-1}{X_1}\right) = \frac{1}{\beta}$ $\therefore$ $\boxed{\frac{\alpha-1}{X_1}}$ is an unbiased estimator of $\frac{1}{\beta}$

b) $E\left(\frac{U}{V}\right) = \sum_{\forall x} x P\left(\frac{U}{V} = x\right)$

$$= \sum_{\forall x} x \sum_{\forall y} P\left(\frac{U}{V} = x \,\middle|\, \frac{1}{U} = y\right) P\left(\frac{1}{U} = y\right)$$

$E\left(\frac{U}{V} \times \frac{1}{U}\right)$

$$E\left(\frac{1}{V}\right) = E\left(\frac{U}{V} \times \frac{1}{U}\right)$$

we note that $\frac{U}{V}$ and V are independent

$\Rightarrow$ $\frac{U}{V}$ and $\frac{1}{U}$ are independent!

$E(XY) = E(X)E(Y)$ for independent r.v. X, Y

$$\Rightarrow E\left(\frac{1}{V}\right) = E\left(\frac{U}{V}\right) E\left(\frac{1}{U}\right)$$

$$\Rightarrow E\left(\frac{U}{V}\right) = \frac{E(1/V)}{E\left(\frac{1}{U}\right)}$$

c) $\frac{T}{X_1} = \frac{\sum X_i}{X_1}$

Since β is scale parameter, consider $z_i = \frac{X_i}{\beta}$

$\Rightarrow f(z) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}$ indep of β.

$\therefore \frac{\sum X_i}{X_1} = \frac{\sum Z_i}{Z_1} =$ ~~ctd.~~

$\frac{X_i}{X_1} = \frac{X_i/\beta}{X_1/\beta} = Z_i/Z_1$

$\Rightarrow \frac{\sum_{i=1}^{n} X_i}{X_1} = \frac{\sum_{i=1}^{n} Z_i}{Z_1}$ which is independent of β!

$\therefore \frac{T}{X_1}$ is an ancillary statistic ; T is comp. sufficient

$\Rightarrow$ By Basu's thm,

$\frac{T}{X_1}$ is independent of T

$E\left(\frac{1}{X_1} \middle| T\right) = E\left(\frac{T}{X_1} \frac{1}{T} \middle| T\right) =$ ~~$\frac{E\left(\frac{1}{X_1}\right)}{E\left(\frac{1}{T}\right)}$~~

$\cancel{E}\left(\frac{1}{T} \mid T\right) = \frac{1}{T}$

$\Rightarrow E\left(\frac{1}{X_1} \mid T\right) = E\left(\frac{T}{X_1}\right) \cdot \left(\frac{1}{T}\right)$ (independence + as per hint $V = \frac{T}{X_1}$ & $g(W) = \frac{1}{T}$ here)

$$= \frac{E\left(\frac{1}{X_1}\right)}{E\left(\frac{1}{T}\right)} \frac{1}{T}$$

$$\Rightarrow E\left(\frac{1}{X_1} \mid T\right) = \frac{E\left(\frac{1}{X_1}\right)}{E\left(\frac{1}{T}\right)} \frac{1}{T}$$

d) Rao Blackwell thm: unbiased estimator $= \frac{(\alpha-1)}{X_1}$

$\Rightarrow$ Best unbiased estimator $= E\left(\frac{\alpha-1}{X_1} \mid T\right)$

$$= \frac{E\left(\frac{\alpha-1}{X_1}\right)}{E\left(\frac{1}{T}\right)} \frac{1}{T}$$

we are given $T \sim \text{Gamma}(n\alpha, \beta)$

$$\therefore E\left(\frac{1}{X_1} \mid T\right) = \frac{\frac{\alpha-1}{(\alpha-1)\beta}}{\frac{1}{(n\alpha-1)\beta}} \frac{1}{T} \quad \text{(using part (a))}$$

$$= \frac{(n\alpha-1)}{\cancel{}} \frac{1}{T}$$

$\therefore \left(\frac{1}{T}\right)\left(\frac{n\alpha-1}{\cancel{}}\right)$ is the best unbiased estimator of $\frac{1}{\beta}$

⑥ (a) ~~Since~~

~~f(x|θ)~~

$$P(X=x|\theta) = \frac{\theta^x e^{-\theta}}{x!} = e^{x \log\theta} e^{-\theta}$$

$$\Rightarrow h(x) = \frac{1}{x!} \qquad w_1(\theta) = \log\theta$$

$$c(\theta) = e^{-\theta} \qquad t_1(x) = x$$

$\theta \in (0, \infty)$ - open set in $\mathbb{R}^1$

$\therefore$ By thm 6.2.25, $\sum_{i=1}^{n} t_1(x_i) = \sum_{i=1}^{n} x_i$ is a complete sufficient statistic

~~(a) $\phi_1(\theta) = e^{-\theta}$~~

(b) $\phi_2 = \theta e^{-\theta}$

$$\delta = \begin{cases} 1 & \text{if } X_1 = 1 \\ 0 & \text{otherwise} \end{cases}$$

$E(\delta) = P(\delta = 1) = P(X_1 = 1) = \theta e^{-\theta}$

δ is an unbiased estimate

Rao Blackwell thm $\Rightarrow$ $E(\delta|T)$ is the best unbiased

$$E(\delta|T) = P(\delta = 1|T)$$

$$= P\left(X_1 = 1 \,\middle|\, \sum_{i=1}^{n} x_i = b\right)$$

$$= \frac{P\left(X_1 = 1, \sum_{i=1}^{n} x_i = b\right)}{P\left(\sum_{i=1}^{n} x_i = b\right)}$$

Sum of n iid Poisson of $(\theta) \sim$ Poisson$(n\theta)$

$$= \frac{P(X_1 = 1, \sum_{i=2}^{n} x_i = b-1)}{P(\sum_{i=1}^{n} x_i = b)}$$

$$= \frac{P(X_1 = 1)\, P(\sum_{i=2}^{n} x_i = b-1)}{P(\sum_{i=1}^{n} x_i = b)}$$ ($\because x_1$ and $x_2, \ldots, x_n$ are independent)

$$= \frac{\frac{\theta e^{-\theta}}{1!} \frac{((n-1)\theta)^{b-1} e^{-(n-1)\theta}}{(b-1)!}}{\frac{(n\theta)^b e^{-n\theta}}{b!}}$$

$$= b \left(\frac{(n-1)^{b-1}}{n^b} \right) \quad \forall\, b \geq 0$$

$\therefore g(T=b) = b \left(\frac{(n-1)^{b-1}}{n^b} \right)$ is the best estimator for the $\theta e^{-\theta}$

b) $\delta_2 = \begin{cases} 1 & \text{if } X_1 = 0 \\ 0 & \text{ow} \end{cases}$

$E(\delta_2) = P(X_1 = 0) = e^{-\theta}$

$\Rightarrow F_1$ is unbiased estimator of ϕ_2!

Rao Blackwell thm ~~says~~ $E(\delta_1 | T)$ is the best unbiased estimator.

$$E(\delta | T) = P(\delta = 1 | \sum X_i = b)$$

$$= P\left(X_1 = 0 \,\middle|\, \sum_{i=1}^{n} X_i = b\right)$$

$$= \frac{P\left(X_1 = 0, \sum_{i=1}^{n} X_i = b\right)}{P\left(\sum_{i=1}^{n} X_i = b\right)}$$

$$= \frac{P\left(X_1 = 0, \sum_{i=2}^{n} X_i = b\right)}{P\left(\sum_{i=1}^{n} X_i = b\right)}$$

$$= \frac{e^{-\theta}\left(\frac{((n-1)\theta)^{b} e^{-(n-1)\theta}}{b!}\right)}{\frac{n^b \theta^b e^{-(n\theta)}}{b!}}$$

$$= \frac{(n-1)^b}{n^b} = \left(\frac{n-1}{n}\right)^b$$

$\therefore \left(\frac{n-1}{n}\right)^b$ is the best estimate for $\phi(\theta) = e^{-\theta}$

c) (i) $e^{-\theta}$: best unbiased estimate $= \left(\frac{n-1}{n}\right)^{b}$

$= \left(\frac{n-1}{n}\right)^{\sum x_i}$

$b = T = \sum x_i = (0+1) + 8 + 13 + 81 \ldots + 3 + 15$

$= 104$

$n = 15$

$\therefore P(\text{no mutant colony}) = \left(\frac{14}{15}\right)^{104} \approx 7.653 \times 10^{-4}$

$= 7.653 \times 10^{-4}$

(ii) Best estimate : $b\left(\frac{(n-1)^{b-1}}{n^{b}}\right)$

$\therefore P(1 \text{ mutant colony}) = 5.685 \times 10^{-3}$