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ISYE 6412 HOMEWORK-02

① a)

$$\text{length}(d) = U - L$$
$$= \bar{Y}_n + c\sigma - (\bar{Y}_n - c\sigma)$$
$$= 2c\sigma \quad — ①$$

$$R_{\delta_c}(\theta) = E(L(\theta, \delta_c))$$
$$= E(r(2c\sigma) - I(\bar{Y}_n - c\sigma \le \theta \le \bar{Y}_n + c\sigma))$$
$$= r(2c\sigma) - P(\bar{Y}_n - c\sigma \le \theta \le \bar{Y}_n + c\sigma)$$
$[\because E(I(A)) = P(A)]$

— ②

$$= r(2c\sigma) - P(\bar{Y}_n \le \theta +$$

$$P(\bar{Y}_n - c\sigma \le \theta \le \bar{Y}_n + c\sigma)$$
$$= P(\bar{Y}_n - c\sigma - \theta \le 0 \le \bar{Y}_n + c\sigma - \theta)$$
$$= P\left(\left(\frac{\bar{Y}_n - \theta}{\sigma/\sqrt{n}}\right) - \frac{c\sigma}{\sigma/\sqrt{n}} \le 0 \le \left(\frac{\bar{Y}_n - \theta}{\sigma/\sqrt{n}}\right) + \frac{c\sigma}{\sigma/\sqrt{n}}\right)$$

— allowed because $\frac{\sigma}{\sqrt{n}} > 0$.

$\{\because \bar{Y}_n \sim N(\theta, \sigma/\sqrt{n})$

$\Rightarrow \left(\frac{\bar{Y}_n - \theta}{\sigma/\sqrt{n}}\right) \sim N(0,1)\}$.

Since $Z \sim N(0,1)$

$$\Rightarrow P(\bar{Y}_n - c\sigma \le \theta \le \bar{Y}_n + c\sigma) = P(Z - c\sqrt{n} \le 0 \le Z + c\sqrt{n})$$
$$= P(-c\sqrt{n} \le Z \le c\sqrt{n}) \quad — ③$$

$(\because z - c\sqrt{n} \le 0 \Rightarrow z \ge -c\sqrt{n}$

$z + c\sqrt{n} \ge 0 \Rightarrow c\sqrt{n} \ge z)$

eqn ② $\Rightarrow R_{\delta_c}(\theta) = \underline{r(2c\sigma) - P(-c\sqrt{n} \le Z \le c\sqrt{n})}$

$= r(2c\sigma) - [P(Z \le c\sqrt{n}) - (P(Z \le -c\sqrt{n}))$

$= r(2c\sigma) - [\Phi(c\sqrt{n}) - \Phi(-c\sqrt{n})]$

$= r(2c\sigma) - [\Phi(c\sqrt{n}) - (1 - \Phi(c\sqrt{n}))]$

(Symmetric property of N dist)

$\Rightarrow R_{\delta_c}(\theta) = \underline{r(2c\sigma) + 1 - 2\Phi(c\sqrt{n})}$

$= 2cr\sigma - 2\Phi(c\sqrt{n}) + 1$

① (b) $R_{\delta_c}(\theta) = r(2c\sigma) + 1 - 2\Phi(c\sqrt{n})$

$$\frac{dR_{\delta_c}(\theta)}{dc} = 2r\sigma - 2\sqrt{n}\,\Phi'(c\sqrt{n})$$

$$\frac{dF}{dx} = f$$

$$\Rightarrow \frac{dR_{\delta_c}(\theta)}{dc} = 2r\sigma - 2\sqrt{n}\,\frac{1}{\sqrt{2\pi}}\,e^{-\frac{(c\sqrt{n})^2}{2}}$$

$$= 2r\sigma - \frac{2\sqrt{n}}{\sqrt{2\pi}}\,e^{-\left(\frac{c^2 n}{2}\right)} \quad \text{—— ④}$$

$2r\sigma$ is a constant wrt c

e^{-c^2} is a decreasing function

$\Rightarrow e^{-\frac{c^2 n}{2}}$ is decreasing ($\because n$ is +ve)

$\Rightarrow \frac{dR_{\delta_c}(\theta)}{dc}$ is increasing

c) we know. max value of $e^{-\frac{c^2 n}{2}} = 1$ $(\forall\, n > 0)$

This means $\frac{dR_{\delta_c}}{dc} \geq 2r\sigma - \frac{2\sqrt{n}}{\sqrt{2\pi}}(1)$ $\forall\, c \geq 0$

$$\Rightarrow \frac{dR_{\delta_c}}{dc} \geq 2r\sigma - \frac{2\sqrt{n}}{\sqrt{2\pi}} \quad — (5)$$

If $r\sigma > \frac{\sqrt{n}}{\sqrt{2\pi}}$

$$\Rightarrow 2r\sigma > \frac{2\sqrt{n}}{\sqrt{2\pi}} \Rightarrow 2r\sigma - \frac{2\sqrt{n}}{\sqrt{2\pi}} > 0 \quad — (6)$$

$$(5) \Rightarrow \frac{dR_{\delta_c}}{dc} > 0 \quad \forall\, c \geq 0 \quad — (7)$$

Since R_{δ_c} is increasing everywhere, minimum is the starting value of $c = 0$

So the "best" "interval estimate" for the given loss function is $c = 0$

$$\Rightarrow \delta_{c^*} = [\bar{Y}_n - 0\sigma, \ \bar{Y}_n + 0\sigma]$$

$$\Rightarrow \delta_{c^*}(Y) = [\bar{Y}_n, \ \bar{Y}_n] \quad — (8)$$

which is nothing but the point estimate of θ.

d) If $r\sigma \leq \sqrt{n}/\sqrt{2\pi}$

$$\frac{dR_{\delta_c}}{dc} = 0 \Rightarrow 2r\sigma - \frac{2\sqrt{n}}{\sqrt{2\pi}} e^{-\frac{c^2 n}{2}} = 0$$

$$\Rightarrow e^{-nc^2/2} = \frac{r\sigma}{\sqrt{n}}\sqrt{2\pi}$$

$$\Rightarrow \frac{-nc^2}{2} = \ln\left(\frac{r\sigma}{\sqrt{n}}\sqrt{2\pi}\right)$$

$$\Rightarrow -nc^2 = 2\ln\left(\frac{r\sigma\sqrt{2\pi}}{\sqrt{n}}\right)$$

$$\Rightarrow c = \sqrt{\frac{\cancel{-}2}{n}\ln\left(\frac{\sqrt{n}}{r\sigma\sqrt{2\pi}}\right)}$$

$$\because \frac{\sqrt{n}}{\sqrt{2\pi}} \geq r\sigma \Rightarrow \frac{\sqrt{n}}{\sqrt{2\pi}\, r\sigma} \geq 1 \quad (\because \sigma > 0)$$

the term inside the square root is always non negative.

$$\therefore c_{opt} = \sqrt{\frac{2}{n}\ln\left(\frac{\sqrt{n}}{r\sigma\sqrt{2\pi}}\right)} \quad \text{— (9)}$$

e) From part (a) we know.

$$R_{\delta_c}(\theta) = r(2c\sigma) - P(-c\sqrt{n} \leq Z \leq c\sqrt{n})$$

Plug in $c = \frac{z_{\alpha/2}}{\sqrt{n}}$

$$\Rightarrow R_{\delta_c}(\theta) = r(2c\sigma) - P(-z_{\frac{\alpha}{2}} \leq Z \leq z_{\alpha/2})$$

$$= 2cr\sigma - 2\Phi(z_{\alpha/2}) + 1$$

From definition of $z_{\alpha/2}$, $\Phi(z_{\alpha/2}) = \cancel{\alpha/2}\left(\frac{1+\alpha}{2}\right)$

$$\Rightarrow R\delta_c(0) = 2cr\sigma - 2 \times \left(\frac{1+\alpha}{2}\right) + 1$$

$$= 2cr\sigma - 1 - \alpha + 1$$

$$= 2cr\sigma - \alpha.$$

$$c_{opt} = \sqrt{\frac{2}{n}\ln\left(\frac{\sqrt{n}}{r\sigma\sqrt{2\pi}}\right)} \qquad \text{(from part d)}$$

$$\Rightarrow \frac{z_{\alpha/2}}{\sqrt{n}} = \sqrt{\frac{2}{n}\ln\left(\frac{\sqrt{n}}{r\sigma\sqrt{2\pi}}\right)}$$

$$\Rightarrow z^2_{\alpha/2} = 2\ln\left(\frac{\sqrt{n}}{r\sigma\sqrt{2\pi}}\right)$$

$$\Rightarrow \frac{r\sigma\sqrt{2\pi}}{\sqrt{n}} = e^{-\frac{z^2_{\alpha/2}}{2}}$$

$$\Rightarrow r = \frac{\sqrt{n}}{\sqrt{2\pi}\,\sigma} e^{-\frac{z^2_{\alpha/2}}{2}}$$

$$\Rightarrow r^* = \frac{\sqrt{n}}{\sqrt{2\pi}\,\sigma} e^{-\frac{z^2_{\alpha/2}}{2}}$$

(2) (a) $S = \{0, 1\}$

$\Omega = \cancel{\{f_1, f_2\}}\ \{\theta : \theta \in \{1/2, 2/3\}\}$

PDF $f_1(y) = \begin{cases} \cancel{y=0 \text{ wp}}\ 1/2, & \text{when } y = 0 \\ \cancel{y=1 \text{ wp}}\ 1/2, & \text{when } y = 1 \end{cases}$

$f_2(y) = \begin{cases} \cancel{y=0} & 2/3 \text{ when } y = 0 \\ & 1/3 \text{ when } y = 1 \end{cases}$

$D = \{d : d_i = \frac{1}{(2+i)} \text{ where } i \in \{0, 1\}\}$

$L_a(\theta, \cancel{\delta}\, d_\delta) = \cancel{\text{if } \theta} \begin{cases} 0 \text{ if } \theta - d = 0 \\ 1 \text{ otherwise} \end{cases}$

b) $R_\delta(\theta) = E_\theta(L(\theta, d))$

$\cancel{= 0 \times P(\theta = d)}$

$= 0 \times P(\theta = \delta(y)) + 1 \times P(\theta \mathrel{!=} \delta(y))$

$= P_\theta(\delta(y) \text{ is the wrong decision})$

$\rightarrow R_\delta(\theta) = P_\theta(\delta \text{ ~~is~~ reaches wrong decision})$

$\underline{\delta_1}$ $R_{\delta_1}(\theta) = \begin{cases} 0 & \text{if } \theta = 1/2 \\ 1 & \text{if } \theta = 1/3 \end{cases}$

$$R_{\delta_2}(\theta) = \begin{cases} 1 & \text{if } \theta = 1/2 \\ 0 & \text{if } \theta = 1/3 \end{cases}$$

$\cancel{R_{\delta_3}(\theta)} = \{$ (Since decisions are fixed, in δ_1 and δ_2, the decision is wrong or right based on θ irrespective of outcome Y)

$\cancel{P(\delta_3 \text{ wrong}) = P(Y=0 \mid \theta = \frac{1}{2}) + P(Y=1 \mid \theta = \frac{1}{3})}$

$\cancel{= \frac{1}{2} +}$ (in other case, the decision will be correct)

$P(\delta_3 \text{ wrong} \mid \theta = 1/2) = P(Y=0 \mid \theta = 1/2)$ (if $Y=1$ we arrive at correct decision)

$= 1/2$

$P(\delta_3 \text{ wrong} \mid \theta = 1/3) = P(Y=1 \mid \theta = 1/3)$ (if $Y=0$, decision is correct)

$= 1/3$

$$\therefore R_{\delta_3}(\theta) = \begin{cases} 1/2 & \text{if } \theta = 1/2 \\ 1/3 & \text{if } \theta = 1/3 \end{cases}$$

$$= \begin{cases} 1/2 & \text{if } \theta = 1/2 \\ 1/3 & \text{if } \theta = 1/3. \end{cases}$$

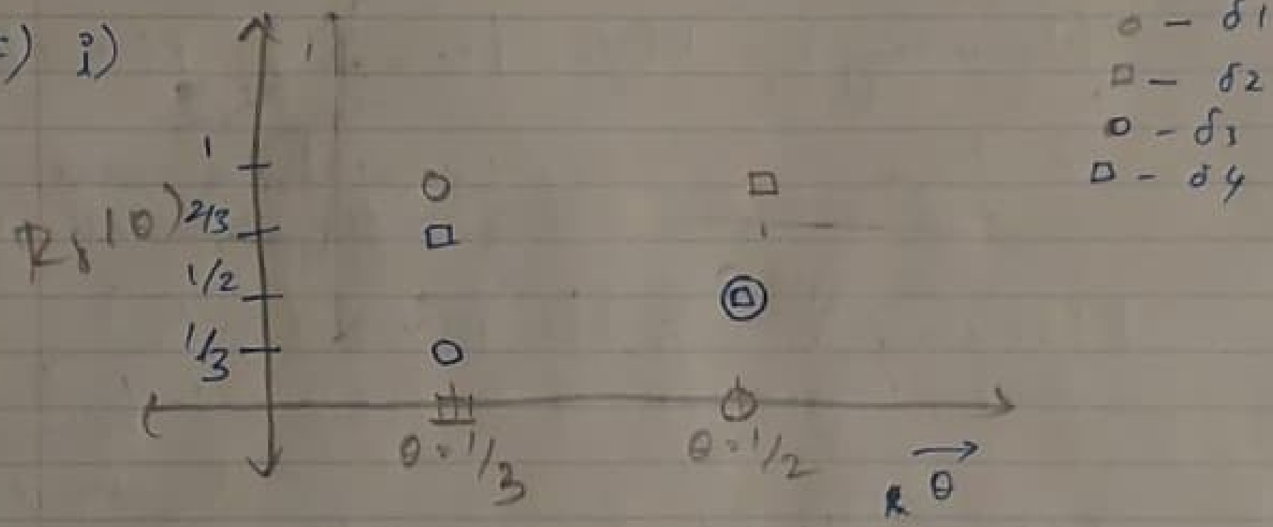
$\underline{\delta_4}$

$P(\delta_4 \text{ wrong} \mid \theta = 1/2) = P(Y = 1 \mid \theta = 1/2)$
$= 1/2$

$P(\delta_4 \text{ wrong} \mid \theta = 1/3) = P(Y = 0 \mid \theta = 1/3)$
$= 1 - 1/3 = 2/3$

$$R_{\delta_4}(\theta) = \begin{cases} 1/2 & \text{when } \theta = 1/2 \\ 2/3 & \text{when } \theta = 1/3 \end{cases}$$

(2) c) i)



We see ~~$R_{\delta_4}(\theta)$~~

δ_4 is inadmissible because δ_3 is better than δ_4

$R_{\delta_4}(1/2) = R_{\delta_3}(1/2) = 1/2$

~~$R_{\delta_4}(1/3) < R_{\delta_3}(1/3)$~~

$R_{\delta_4}(1/3) = 2/3 > R_{\delta_3}(1/3) = 1/3.$

$\Rightarrow$ (a) $R_{\delta_3}(\theta) \leq R_{\delta_4}(\theta) \ \forall\ \theta \in \Omega$

(b) $R_{\delta_3}(\theta_0) < R_{\delta_4}(\theta_0)$ for $\theta_0 = 1/3$

$\therefore$ ~~δ_4~~ is

$\Rightarrow$ δ_3 is better than δ_4
$\Rightarrow$ There exists a procedure better than δ_4
$\Rightarrow$ δ_4 is inadmissible.

Since we have only 4 decision procedures, from the plot we see none of the other procedures (1, 2 and 3) are better than any other. (1, 2, and 3)

admissible procedures : $\delta_1, \delta_2, \delta_3$.

(ii)

$$r_\delta^{\pi} = \sum \pi(\theta_i) R_\delta(\theta_i)$$

$$\pi(\theta = 1/3) = 0.1$$

$$\pi(\theta = 1/2) = 0.9$$

$$r_{\delta_1} = 1 \times 0.1 + 0.9 \times 0 = \boxed{0.1}$$

$$r_{\delta_2} = 0 \times 0.1 + 0.9 \times 1 = 0.9$$

$$r_{\delta 3} = 1/3 \times 0.1 + 1/2 \times 0.9 = \frac{2.9}{6} \simeq 0.483$$

$$r_{\delta 4} = 0.1 \times \frac{2}{3} + 0.9 \times \frac{1}{2} = \frac{3.1}{6} \simeq 0.516$$

Bayes procedure is $\boxed{\delta 1}$

(iii)

$$r_{\delta_1} = 0.6 \times 1 + 0.4 \times 0 = 0.6$$

$$r_{\delta 2} = 0.6 \times 0 + 0.4 \times 1 = \boxed{0.4}$$

$$r_{\delta 3} = 0.6 \times 1/3 + 0.4 \times 1/2 = \boxed{0.4}$$

$$r_{\delta 4} = 0.6 \times 2/3 + 0.4 \times 1/2 = 0.6$$

In (c) ii) Bayes procedure is δ_1

In (c) (iii) Bayes procedures are δ_2 and δ_5.

[Since Bayes procedure is the procedure that minimises r_δ]

d) Yes, multiple distributions including

$$P_\pi(1/3) = 0.05 = 1 - P_\pi(1/2)$$

In general, since δ_1 is calling $\theta = 1/2$, r_{δ_1} ↓ses if $P_\pi(1/2)$ increases chance (less error)

Since δ_3 is better than δ_4 we discard it from our analysis

$$r_{\delta_1} < r_{\delta_2} \Rightarrow 1 \cdot P_\pi(1/3) + 0\,(1 - P_\pi(1/3)) < 0 \cdot P_\pi(1/3) + (1 - P_\pi(1/3))$$

$$\Rightarrow r_{\delta_1} \leq r_{\delta_2} \text{ when } P_\pi(1/3) \leq 1/2 \quad \text{— (1)}$$

$$r_{\delta_1} \leq r_{\delta_3} \Rightarrow P_\pi(1/3) \leq \frac{1}{3} P_\pi(1/3) + \frac{1}{2}(1 - P_\pi(1/3))$$

$$\Rightarrow \frac{7 P_\pi(1/3)}{6} \leq \frac{1}{2}$$

$$\Rightarrow P_\pi(1/3) \leq \frac{3}{7} \quad \text{or}$$

$$\Rightarrow r_{\delta_1} \leq r_{\delta_3} \text{ when } P_\pi(1/3) \leq 3/7 \quad \text{— (2)}$$

So r_{δ_1}, ~~to~~ is minimum (wrt δ) when

prior $P_\pi(1/3) \leq 3/7$ (from eqns ① and ②)

(and so $P_\pi(1/2) = 1 - P(1/3) \geq \frac{4}{7}$)

③ $h^*(y, d) = \sum_{\forall f_i} \pi(\theta_i f_i)\, L(f_i, \delta(y))\, P_{\theta}(Y=y)$

$= \sum_{\forall f_i} \pi(f_i)\, L(f_i, \delta(y))\, f_i(y)$

$= \sum_{\forall f_i} \pi(f_i)\, L(f_i, \delta(y))\, f_i(y)$

$= \sum \pi(f_i)\, L(f_i, \delta(y))\, f_i(y)$

$= \pi(f_0)\, L(\theta f_0, \delta(y))\, f_0(y)$

$+ \pi(f_1)\, L(f_1, \delta(y))\, f_1(y)$ — ①

We know: A procedure $\delta(y)$ is Bayes iff for any $y \in S$ it assigns decision $d = d^*(y)$ that minimises $h(y, d)$

Now, $\delta(y)$ can be d_0 or d_1

If $\delta(y) = d_0$

$h(y, f_0) = \pi(f_1)\, L(f_1, d_0)\, f_1(y)$ $\quad (\because L(f_0, d_0) = 0)$

$= \pi(f_1)\, L\{ w_1\, f_1(y)$ — ②

If $\delta(y) = f_1$

$h(y, f_1) = \pi(f_0)\, L(f_0, d_1)\, f_0(y)$ $\quad (\because L(f_1, d_1) = 0)$

$= \pi(f_0)\, L(f_0, d_1)\, f_0(y)$

$= \pi(f_0)\, w_0\, f_0(y)$ — ③

We need $\min_{d} h(y, d)$

So
case ① If $h(y, f_1) < h(y, f_0)$, d_1 is ~~preferred~~ optimal.

$\Rightarrow \pi(f_0) w_0 f_0(y) < \pi(f_1) w_1 f_1(y)$

$$\Rightarrow \frac{f_0}{f_1} < \frac{\pi(f_1) w_1}{\pi(f_0) w_0}$$

$$\Rightarrow \frac{f_1(y)}{f_0(y)} > \frac{\pi(f_0) w_0}{\pi(f_1) w_1}$$

$\rightarrow \delta^*(y) = d_1$ in above case

case (ii) If $h(y, f_1) = h(y, f_0)$, any of the functions are equally good.

$\pi(f_0) w_0 f_0(y) = \pi(f_1) w_1 f_1(y)$

$$\Rightarrow \frac{f_1(y)}{f_0(y)} = \frac{\pi(f_0) w_0}{\pi(f_1) w_1}$$

$\delta^*(y) = d_0$ or d_1 in this case (both are equally good)

case (iii) d_0 is the optimal decision if $h(y, f_1) > h(y, f_0)$

$\pi(f_0) w_0 f_0(y) > \pi(f_1) w_1 f_1(y)$

$$\Rightarrow \frac{f_1(y)}{f_0(y)} < \frac{\pi(f_0) w_0}{\pi(f_1) w_1}$$

$\delta^*(y) = d_0$ in the above case

If we define $\dfrac{\pi(f_0)\,\cancel{\pi_0}\, w_0}{\pi(f_1)\, w_1}$ as c,

From cases (i), (ii), (iii) we conclude,

$$\delta(y) = \begin{cases} d_1 & , \text{ if } \dfrac{f_1(y)}{f_0(y)} > c. \\ d_1 \text{ or } d_0 & \text{if } \dfrac{f_1(y)}{f_0(y)} = c \\ d_0 & \text{if } \dfrac{f_1(y)}{f_0(y)} < c \end{cases}$$

(4) a) $r_\delta(\pi) = \int_\Omega R_\delta(\theta)\pi(\theta)\,d\theta$

$$= \int_\Omega E_\theta(L(\theta,\delta(Y))\,\pi(\theta)\,d\theta$$

$$= \int_\Omega \left[\int_S L(\theta,\delta(y))\, f_\theta(y)\,dy\right]\pi(\theta)\,d\theta$$

$$= \int_S \left[\int_\Omega L(\theta,\delta(y))\, f_\theta(y)\, \cancel{dy} \,\pi(\theta)\,d\theta\right] dy$$

$$= \int_S \left[\int_\Omega L(\theta,\delta(y)\,\frac{f_\theta(y)\pi(\theta)}{m(y)}\,d\theta\right] m(y)\,dy$$

where $m(y) \triangleq \int f_\theta(\vec{y})\,\pi(\theta)\,d\theta$

(marginal distribution of y)

$$= \int_S \left[\int_\Omega L(\theta,\delta(y))\,\pi(\theta|y)\,d\theta\right] m(y)\,dy$$

$$= \int_S \left[\int_\Omega |\theta - d|^r\,\pi(\theta|y)\,d\theta\right] m(y)\,dy$$

$h_\pi \triangleq \int_\Omega L(\theta,d)\,\pi(\theta|\vec{y})\,d\theta.$

$$= \int_a^b k\,|\theta - d|^r\,\pi(\theta|\vec{y})\,d\theta.$$

~~For~~ ~~$\mathcal{Y}$~~

~~max~~ ~~criterion~~ $\min_{\delta} r_\delta(\pi) \Rightarrow \min_{d} h_\pi(\vec{y}, d)$ for each $\vec{y}$

$\left(\because \; r_\delta(\pi) = \int_{\mathcal{Y}}^{} h_\pi(\vec{y}, d)\, m(y)\, dy \right.,$

$h_\pi(\vec{y}, d) \geq h_\pi(\vec{y}, d^*)$ (where d^* is the optimal decision)

$\Rightarrow h_\pi(\vec{y}, d)\, m(\vec{y}) \geq h_\pi(\vec{y}, d^*)\, m(\vec{y})$

Allowed since $m(\vec{y}) > 0$

$\Rightarrow \int h_\pi(\vec{y}, d)\, m(\vec{y})\, dy \geq \int h_\pi(\vec{y}, d^*)\, m(\vec{y})\, dy$

$\therefore$ optimising for $h_\pi(\vec{y}, d^*)$ gives minimises r_δ $\left.\right)$

$\min_{\delta} r_\delta(\pi)$ gives the Bayes procedure δ^*.

$\therefore$ choose $\delta^*(\vec{y}) = d'$

by minimising $\int_a^b |\theta - d'|^r \pi(\theta | y)\, d\theta$.

to get Bayes procedure $\delta^*(\vec{y})$

b)

$$\min_{d'} \int_a^b (\theta - d^*)^2 \pi(\theta|\vec{y})\, d\theta - h_\pi(y, d') \quad —\ (1)$$

$$\Rightarrow \frac{d\, h_\pi(y, d^*)}{d(d^*)} = \int_a^b \frac{\partial\left((\theta - d)^2 \pi(\theta|y)\right)}{\partial(d)} d\theta$$

(Leibnitz rule)

$$= \int_a^b 2(d - \theta)\pi(\theta|y)\, d\theta \quad —\ (2)$$

Since the

$$\frac{d^2 h_\pi(y, d)}{d(d^2)} = \int_a^b 2\pi(\theta|y)\, d\theta$$

$$= 2 \quad —\ (3)$$ (integral of a pdf $= 1$)

Since 2nd derivative is +ve, if a stationary (3) point exists it should be a minima.

equating derivative to 0 (eqn (2))

$$\int_a^b 2(d - \theta)\pi(\theta|y)\, d\theta = 0$$

$$\Rightarrow d\int_a^b \pi(\theta|y)\, d\theta - \int_a^b \theta\, \pi(\theta|y)\, d\theta = 0 \quad —\ (4)$$

($\because$ d is constant taken out of integral)

$$\int_{\forall\theta} \pi(\theta)\, d\theta = 1$$

$$\int_{\forall\theta} \theta\pi(\theta)\, d\theta = E$$

$$\int_{\forall\theta} \pi(\theta|y)\, d\theta = 1$$

$$\int_{\forall\theta} \theta\,\pi(\theta|y)\, d\theta = E(\theta|y)$$

equ (4) $\Rightarrow$ $2d - E(\theta|y) = 0$

$\Rightarrow d = E(\theta|y)$

= mean of posterior $\pi(\theta|\vec{y})$ of θ

(4) (c) $\min_{d} h_\pi(y, d^*) = \min_{d} \int_a^b |\theta - d|\, \pi(\theta|\vec{y})\, d\theta$

$\Rightarrow d' = \text{median}(\pi(\theta|\vec{y}))$

{ $\because$ for $\min_{c} \int_{a}^{b} |\theta - c|\, g(\theta)\, d\theta \Rightarrow$

$c^* = \text{median}(g(\theta))$

proof :

$$\min_{c} \int_a^b |\theta - c| \, g(\theta)\, d\theta = \int_a^c (c-\theta)\, g(\theta)\, d\theta$$

$$+ \int_c^b (\theta - c)\, g(\theta)\, d\theta$$

$$\Rightarrow \frac{d}{dc}\left(\int_a^b |\theta - c| \, g(\theta)\, d\theta \right)$$

$$= \cancel{\left[(c-c)\, g(\theta) \right]} + \left[\right.$$

$$= \left[(c-c)\, g(\theta) + \int_a^c g(\theta)\, d\theta \right]$$

$$+ \left[-(c-c)\, g(\theta) + \int_c^b -g(\theta)\, d\theta \right]$$

$$= \int_a^c g(\theta)\, d\theta + \int_a^c g(\theta)\, d\theta - \int_a^c g(\theta)\, d\theta$$

(add & subtract $\int_a^c g(\theta)\, d\theta$)

$$- \int_c^b g(\theta)\, d\theta$$

$$= 2 \int_a^c g(\theta)\, d\theta - \int_a^b g(\theta)\, d\theta .$$

~~⇒ equating zero.~~

$$\Rightarrow \frac{d^2}{dc^2}\left(\int_a^b |\theta - c|\, g(\theta)\, d\theta \right) = 2 g(\theta) > 0 \;\forall c$$

So ! stationary point if exists is minimum

equating derivative to 0,

$$2\int_a^c g(\theta)\,d\theta - \underbrace{\int_a^b g(\theta)\,d\theta}_{1} = 0$$

$$\Rightarrow \int_a^c g(\theta)\,d\theta = \frac{1}{2}$$

which is the definition of median!

$\Rightarrow c^* =$ median

alternate proof as per hints:

$$\int_{-\infty}^{\infty} |\theta - c|\, g(\theta)\,d\theta - \int_{-\infty}^{\infty} |\theta - m|\, g(\theta)\,d\theta \geq 0$$

where m is the median

$$\Rightarrow \int_{-\infty}^{\infty} \left(|\theta - c| - |\theta - m| \right) g(\theta)\,d\theta \geq 0$$

~~$\int |\theta - c|$.~~

case (i) : $c > m$,

$$|\theta - c| - |\theta - m| = \begin{cases} c - m & \theta \leq m \\ c + m - 2\theta & m < \theta \leq c \\ m - c & \theta > c \end{cases}$$

$$(c-m)\,\text{sign}(m-\theta) = \begin{cases} c-m & \theta < m \\ 0 & \theta = m \\ m-c & m < \theta < c \\ m-c & \theta > c \end{cases}$$

$m-c = \cancel{m-\theta-c+\theta}$

we can see they are equal when $\theta \geq c$ and $\theta < m$.

when $m < \theta \leq c$,

$$2c \geq 2\theta$$

$$\Rightarrow (m-c)+2c \geq 2\theta + m - c$$

$$\Rightarrow m+c-2\theta \geq m-c$$

$$\therefore |\theta - c| - |\theta - m| \geq (c-m)\,\text{sign}(m-\theta) \quad \forall \theta$$

$$\Rightarrow \int_{-\infty}^{\infty} (|\theta - c| - |\theta - m|)\, g(\theta)\, d\theta \geq \int_{-\infty}^{\infty} (c-m)\,\text{sign}(m-\theta)\, g(\theta)\, d\theta$$

$(\because g(\theta) > 0)$

$$= (c-m) \int_{-\infty}^{\infty} \text{sign}(m-\theta)\, g(\theta)\, d\theta$$

$$= (c-m)\left(\int_{-\infty}^{m} g(\theta)\, d\theta - \int_{m}^{\infty} g(\theta)\, d\theta\right)$$

$$= (c-m)\left(\tfrac{1}{2} - \tfrac{1}{2}\right) \quad \text{(definition of median)}$$

$$= 0$$

$$\Rightarrow \int_{-\infty}^{\infty} (|\theta - c| - |\theta - m|)\, g(\theta)\, d\theta \geq 0$$

when $c > m$.

when $c \leq m$

$$(|\theta - c| - |\theta - m|) = \begin{cases} c - m & \theta \leq c \\ -(\theta - m) \ldots & c < \theta \leq m \\ m - c & \theta \geq m \end{cases}$$

$$= \begin{cases} c - m & \theta < c \\ 2\theta - m - c & c < \theta \leq m \\ m - c & \theta > m \end{cases}$$

$$\theta > c$$

$$\Rightarrow 2\theta > 2c$$

$$\Rightarrow 2\theta - m - c > 2c - m - c$$

$$\Rightarrow 2\theta - m - c > c - m$$

$$(-m+c)\,\text{sign}(m - \theta) = \begin{cases} -m + c & \theta \leq c \\ -m + c & c < \theta \leq m \\ -c + m & \theta \geq m \end{cases}$$

Again equal when $\theta \geq m$ and $\theta < c$

and $|\theta - c| - |\theta - m| > (c - m)\,\text{sign}(m - \theta)$

when $c < \theta \leq m$

$$\therefore |\theta - c| - |\theta - m| \geq (c - m)\,\text{sign}(m - \theta)$$

$$\Rightarrow \int_{-\infty}^{\infty} (|\theta - c| - |\theta - m|)\, g(\theta)\, d\theta \geq \int_{-\infty}^{\infty} (c - m)\,\text{sign}(m - \theta)\, g(\theta)\, d\theta$$

$\because g(\theta) > 0$

$$= \int_{-\infty}^{m} (c-m)\, g(\theta)\, d\theta + \int_{m}^{\infty} (m-c)\, g(\theta)\, d\theta.$$

$$= (c-m)\left(\int_{-\infty}^{m} g(\theta)\, d\theta - \int_{m}^{\infty} g(\theta)\, d\theta\right)$$

$$= (c-m)\left(\frac{1}{2} - \frac{1}{2}\right) \quad \text{(definition of median)}$$

$$= 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \left(|\theta - c| - |\theta - m|\right) g(\theta)\, d\theta \geq 0 \quad \forall\, c \leq m.$$

$$\therefore \int_{-\infty}^{\infty} \left(|\theta - c| - |\theta - m|\right) g(\theta)\, d\theta \geq 0 \quad \forall\, c \neq m.$$

$$\Rightarrow \int_{-\infty}^{\infty} |\theta - c|\, g(\theta)\, d\theta \geq \int_{-\infty}^{\infty} |\theta - m|\, g(\theta)\, d\theta$$

$$\Rightarrow \min_{c} \int_{-\infty}^{\infty} |\theta - c|\, g(\theta)\, d\theta$$

has solution $c^* = m$

completing the required proof for the result we used for (4c)