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HW-07 ISYE 6412

① a) $f_\theta(x) = \frac{1}{\theta} 1\{\theta < x < 2\theta\} = \frac{1}{\theta} 1(\theta > \frac{x}{2}) 1(\theta < x)$

$\Rightarrow f_\theta(\underline{x}) = \frac{1}{\theta^n}\left(1(\theta > \frac{x_1}{2}) 1(\theta > \frac{x_2}{2}) \dots 1(\theta > \frac{x_n}{2})\right)$
$\times \left(1(\theta < x_1) 1(\theta < x_2) \dots 1(\theta < x_n)\right)$

$\Rightarrow f_\theta(\underline{x}) = \frac{1}{\theta^n} 1(\theta > \max_{\forall i} \frac{x_i}{2}) 1(\theta < \min_{\forall i} x_i)$

$$\frac{f_\theta(\underline{x})}{f_\theta(\underline{y})} = \frac{\frac{1}{\theta^n} 1(\theta > \max_{\forall i} \frac{x_i}{2}) 1(\theta < \min_{\forall i} x_i)}{\frac{1}{\theta^n} 1(\theta > \max \frac{y_i}{2}) 1(\theta < \min_{\forall i} y_i)}$$

$$= \frac{1(\max \frac{x_i}{2} < \theta < \min x_i)}{1(\max \frac{y_i}{2} < \theta < \min y_i)}$$

As we have seen in class and this is independent of θ iff $\max(\frac{x_1}{2}, \frac{x_2}{2}, \frac{x_3}{2}, \dots, \frac{x_n}{2}) = \max(\frac{y_1}{2}, \dots, \frac{y_n}{2})$

& $\min(x_1, x_2, \dots, x_n) = \min(y_1, \dots, y_n)$

$\Rightarrow \max(x_1, \dots, x_n) = \max(y_1, \dots, y_n) = X_{(n)} = Y_{(n)}$

& $\min(x_1, \dots, x_n) = \min(y_1, \dots, y_n) = X_{(1)} = Y_{(1)}$

$\therefore T(X) = \left(\min(x_1, \dots, x_n), \max(x_1, \dots, x_n) \right)$

$= (X_{(1)}, X_{(n)})$ is minimal suff

b) $F(X_{(n)} \leq x) = P(x_1 \leq x) \dots P(x_n \leq x)$

$$= \begin{cases} \left(\frac{x-\theta}{\theta}\right)^n & 2\theta > x > \theta \\ 0 & x \leq \theta \\ 1 & x \geq 2\theta \end{cases}$$

$$\Rightarrow f(X_{(n)} = x) = \begin{cases} \frac{n}{\theta^n}(x-\theta)^{n-1} & 2\theta > x > \theta \\ 0 & \text{otherwise} \end{cases}$$

$$E(X_{(n)}) = \int_{\theta}^{2\theta} \frac{n}{\theta^n} (x-\theta)^{n-1} x \, dx$$

$x - \theta \longrightarrow z$

$$= \frac{n}{\theta^n} \int_0^{\theta} (z+\theta) z^{n-1} dz$$

$$= n\theta \left(\frac{1}{n+1} + \frac{1}{n} \right) = \frac{\theta(2n+1)}{n+1}$$

$F(X_{(1)} \geq x) = P(X_1 \geq x) \dots P(X_n \geq x)$

$= \left(1 - P(X_1 \leq x)\right)^n$

$$F(X_{(1)}) = \begin{cases} 0 & x \le \theta \\ \left(1 - \frac{(x-\theta)}{\theta}\right)^{n} & \theta \le x \le 2\theta \\ 1 & x \ge 2\theta \end{cases}$$

$$f(X_{(1)}) = \begin{cases} \frac{n}{\theta}\left(1 - \left(\frac{x-\theta}{\theta}\right)\right)^{n-1} & \theta \le x \le 2\theta \\ 0 & \text{otherwise} \end{cases}$$

$$E(X_{(1)}) = \int_{\theta}^{2\theta} n\,x \left(1 - \left(\frac{x-\theta}{\theta}\right)\right)^{n-1} dx$$

$$= \int_{\theta}^{2\theta} \frac{n\,x}{\theta^{n}} \left(2\theta - x\right)^{n-1} dx$$

$2\theta - x \to z$

$\Rightarrow x = 2\theta - z$

$$= \int_{0}^{\theta} \frac{n}{\theta^{n}} (2\theta - z) \cdot (z)^{n-1} dz$$

$$= \frac{n\theta}{\;}\left(\frac{2}{n} - \frac{1}{n+1}\right)$$

$$= \frac{\theta\,(n+2)}{(n+1)}$$

- Consider $g(T) = \left(\frac{n+1}{2n+1}\right)(X_{(n)}) - \left(\frac{n+1}{n+2}\right)(X_{(1)})$

$\to E(g(T)) = \theta - \theta = 0$

$\therefore$ T is not a complete ~~statistic~~ statistic

(2) From HW6 q3

$$T(\underline{X}) = \max\left\{1 - \min_{1\le i\le n} \frac{x_i}{i}, \frac{1}{2}\left(\max_{1\le i\le n} \frac{x_i}{i} - 1\right)\right\}$$

is minimal ~~sufficie~~ sufficient for θ

$$P(T(\underline{X}) \le t) = P\left(\max\left\{1 - \min \frac{x_i}{i}, \frac{1}{2} \max \frac{x_i}{i} - 1\right\} \le t\right)$$

$$= P\left(\min \frac{x_i}{i} \ge -1 + t \text{ and } \max \frac{x_i}{i} \le 2t + 1\right)$$

$$= P\left(-1 + t \le \frac{x_i}{i} \le 2t + 1\right)$$

$$= \prod_{i=1}^{n} P\left(-i(1+t) \le x_i \le (2t+1)\, i\right)$$

~~if $t < 0$, $-i(1-t)$~~

$> i$

~~$\rightarrow 1 - t > 1 \rightarrow -i(1+t) < i\theta$~~

~~$(2t+1)\, i < i$~~

~~But $\theta > 0$~~

~~But $\forall \theta > 0$, $-i(2\theta - 1)$~~

if $t < 0$

$\Rightarrow t-1 > 1 \Rightarrow i(t-1) > i = i \times 1 > i \times (2t+1)$
($\because$ ~~let~~ $2t < 0$)

$\Rightarrow i(t-1) > i(2t+1)$

So $P(-i(-1+t) \le x \le (2t+1)i) = 0$ in that case

$t > \theta$

~~$-i(\theta-1) <$~~

$\Rightarrow -i(t-1) \le -i(\theta-1) \le x_i \le (2\theta+1)i < (2t+1)i$

$\therefore$ the probability is 1

$$\therefore \cancel{P(T(X) \le t)} = \begin{cases} 0 & t < 0 \\ \cancel{\frac{3it}{3i\theta} = \frac{t}{i\theta}} & \cancel{0 \le t \le \theta} \\ 1 & t > \theta. \end{cases}$$

~~$\theta < t <$~~

$0 \le t \le \theta$

$$Pr = \prod_{i=1}^{n} \frac{(2t+1)i + i(t-1)}{3i\theta} = \prod_{i=1}^{n} \frac{3it}{3i\theta} = \frac{t^n}{\theta^n}$$

$$\therefore P(T(X) \le t) = \begin{cases} 0 & t < 0 \\ \frac{t^n}{\theta^n} & 0 \le t \le \theta \\ 1 & t > \theta \end{cases}$$

This CDF is same as what we obtained for the min-suff stat ~~for~~ $U[\cancel{\theta}, \theta\cancel{+1}]$ in class.
We showed

$$E(g(t)) = 0 \Rightarrow g(t) = 0$$

$\therefore$ T is ~~that~~ a complete statistic

③ a) $f_\theta(x) = \frac{1}{3\theta} 1(-\theta < x < 2\theta)$

$$= \frac{1}{3\theta} 1(\theta > -x)\, 1\left(\theta > \frac{x}{2}\right)$$

$$= \frac{1}{3\theta} 1\left(\theta > \max\left(\frac{x}{2}, -x\right)\right)$$

Joint density

$$f_\theta(x_1, \dots, x_n) = \frac{1}{(3\theta)^n} 1\left(\theta > \max\left(\frac{x_1}{2}, \frac{x_2}{2}, \dots \frac{x_n}{2}, -x_1, -x_2, \dots, -x_n\right)\right)$$

$$\frac{f_\theta(\underline{x})}{f_\theta(\underline{y})} = \frac{\frac{1}{3\theta^n} 1\left(\theta > \max\left(\frac{x_1}{2}, \dots, \frac{x_n}{2}, -x_1, \dots, -x_n\right)\right)}{\frac{1}{3\theta^n} 1\left(\theta > \max\left(\frac{y_1}{2}, \dots, \frac{y_n}{2}, -y_1, \dots, -y_n\right)\right)}$$

~~These~~ 2 -

This is independent of θ iff

$$\max\left(\frac{x_1}{2}, \dots, \frac{x_n}{2}, -x_1, \dots, -x_n\right) = \max\left(\frac{y_1}{2}, \dots, \frac{y_n}{2}, -y_1, \dots, -y_n\right)$$

$$\therefore T(\underline{X}) = \max\left(\frac{x_1}{2}, \dots, \frac{x_n}{2}, -x_1, \dots, -x_n\right) \text{ is}$$

minimal sufficient

b) $P(T \leq t) = \prod_{i=1}^{n} P\left(\max\left(\frac{-x_i}{2}, x_i\right) \leq t\right)$

$$= \prod_{i=1}^{n} P\left(\frac{-x_i}{2} \leq t\right) P(x_i \geq t)$$

$$= \prod_{i=1}^{n} P(-t \leq x_i \leq 2t)$$

$$= \begin{cases} 0 & t \leq 0 \\ \left(\frac{3t}{3\theta}\right)^n = \frac{t^n}{\theta^n} & 0 < t < \theta \\ 1 & t \geq \theta \end{cases}$$

This is same as that of uniform in the cdf of complete minimal sufficient statistic we got for uniform $[0, \theta]$

$T(x) = \max\left(\frac{x_1}{2}, \dots, \frac{x_n}{2}, -x_1, \dots, -x_n\right)$

is complete sufficient!

(4) b) $f_\theta(x) = \dfrac{\theta}{(1+x)^{1+\theta}}$

$t \triangleq \prod_{i=1}^{n} (1+x_i)$

$f_\theta(x_1, \ldots, x_n) = \prod_{i=1}^{n} \dfrac{\theta}{(1+x_i)^{1+\theta}} = \dfrac{\theta^n}{\left(\prod_{i=1}^{n}(1+x_i)\right)^{1+\theta}}$

$$\frac{f_\theta(\underline{x})}{f_\theta(\underline{y})} = \frac{\dfrac{\theta^n}{\left(\prod_{i=1}^{n}(1+x_i)\right)^{1+\theta}}}{\dfrac{\theta^n}{\left(\prod_{i=1}^{n}(1+y_i)\right)^{1+\theta}}} = \left(\frac{\prod_{i=1}^{n}(1+y_i)}{\prod_{i=1}^{n}(1+x_i)}\right)^{1+\theta}$$

This is independent of θ iff the term inside the brackets is 0 (not possible $\forall\ \underline{y}$) or 1 (possible when $\prod_{i=1}^{n}(1+y_i) = \prod_{i=1}^{n}(1+x_i)$)

$\therefore \dfrac{f_\theta(\underline{x})}{f_\theta(\underline{y})}$ is independent of θ

iff $\prod_{i=1}^{n}(1+x_i) = \prod_{i=1}^{n}(1+y_i)$

$t_x = t_y$

$\therefore T(x)$ is ~~complete~~ minimal suffint

$f_\theta(\underline{x}) = \dfrac{\theta^n}{\cancel{\prod_{i=1}(1+x_i)}} \left(\prod_{i=1}^{n} (1+x_i)\right)^{-(1+\theta)} = \theta^n t^{-(1+\theta)}$

$= \theta^n \exp\left(\cancel{\frac{t\,1}{1+\theta}} -(1+\theta) \log(\cancel{1+}\,t)\right)$

$= \theta^n \exp(-(1+\theta)\, \overset{\log(t)}{\cancel{t}}) = \theta^n \exp\left(-(1+\theta) \sum_{i=1}^{n} \log(1+x_i)\right)$

$w_1(\theta) = -(1+\theta) \Rightarrow \theta < -1$

$\Rightarrow w_1(\theta)$ is an open set in R^k

$h(x) = 1, \quad c(\theta) = \theta^n, \quad t_1(x_i) = \log(1+x_i)$

$\therefore$ By thm. 6.2.25, the statistic is complete

$(T(x) = \sum t_1(x_i) = \sum \log(1+x_i)$

$= \log t$ complete $)$

$\Rightarrow t$ is also sufficient!

(8)

c) $f_\theta(x) = \dfrac{(\log\theta)(\theta^{x})}{(\theta-1)}$

$f_\theta(x_1, \ldots, x_n) = \left(\dfrac{\log\theta}{\theta-1}\right)^n \theta^{\sum_{i=1}^{n} x_i}$

$\dfrac{f_\theta(x_1, \ldots, x_n)}{f_\theta(y_1, \ldots, y_n)} = \theta^{\left(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i\right)}$

possible independent of θ iff $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$

$\therefore T(x) = \sum_{i=1}^{n} x_i$ is the minimal sufficient statistic

$$f_\theta(\underline{x}) = \left(\frac{\log\theta}{\theta-1}\right)^n \theta^{\sum_{i=1}^{n} x_i}$$

$t_1(x_i) = x_i$

$$= \left(\frac{\log\theta}{(\theta-1)}\right)^n \exp\left(\left(\sum_{i=1}^{n} t_1(x_i)\right)\log\theta\right)$$

$w_1(\theta) \triangleq \log\theta \in [0,\infty) \;\forall\, \theta > 1$

$\Rightarrow w_1(\theta)$ is an open set $\forall\, \theta > 1$

$h(x) = 1 \quad , \; C(\theta) = \left(\frac{\log\theta}{(\theta-1)}\right)^n$

$\therefore$ By thm 6.2.25 $T(x)$ is complete sufficient statistic

$(T(x) = \sum t_1(x_i)$

$= \sum x_i = t)$

d) $f_\theta(x) = \prod_{i=1}^{n} f_\theta(x_i)$

$= \cancel{e^{t(x-\theta)}\exp(-e}$

$$= \exp\left(-\sum_{i=1}^{n}(x_i-\theta)\right)\exp\left(-\sum_{i=1}^{n} e^{-(x_i-\theta)}\right)$$

$$= \exp\left(-\sum_{i=1}^{n} x_i + n\theta\right)\exp\left(-\sum_{i=1}^{n}\exp(-x_i+\theta)\right)$$

$$= \exp\left(-\sum_{i=1}^{n} x_i + n\theta\right)\exp\left(e^{\theta}\left(-\sum_{i=1}^{n}\exp(-x_i)\right)\right)$$

$$\frac{f_\theta(\underline{x})}{f_\theta(\underline{y})} = \exp\left(\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i\right) \times \exp\left(e^{\theta}\left(\sum_{i=1}^{n}\exp(-y_i) - \sum_{i=1}^{n}\exp(-x_i)\right)\right)$$

This is indep of θ iff $x_{(i)} = y_{(i)}$
(the 2nd term)

$\therefore$ Order statistics is the minimal sufficient statistic

$T = (X_{(1)}, \cdots, X_{(n)})$

We know for location parameter family, $X_{(n)} - X_{(1)}$ is an ancillary statistic

$\Rightarrow X_{(n)} - X_{(1)}$ doesn't dependent on θ

$\Rightarrow E(X_{(n)} - X_{(1)})$ doesn't dependent on $\theta = c$

$\therefore g(T) = (X_{(n)} - X_{(1)}) - c \Rightarrow E(g(T)) = c - c = 0$

$\Rightarrow g(T) \neq 0$ a.s. $\forall T$

$\therefore T$ is not a complete suff statistic

Q5 b)

(5) a) From H.W.6 we know

$X_{(1)}$ is a minimal sufficient statistic

$P(T \leq t) = \cancel{\prod_{i=1}^{n} P(X_i \geq t)}\; 1 - \prod_{i=1}^{n} P(X_i \leq t)$

$$= \begin{cases} 1 - \left(\int_{\theta}^{t} e^{-(x-\theta)} dx \right)^n & t \geq \theta \\ 0 & t \leq \theta \end{cases}$$

$$= \begin{cases} 1 - e^{-(t-\theta)n} & t > \theta \\ 0 & t \leq \theta \end{cases}$$

$\Rightarrow f(t) = e^{-(t-\theta)n}$

$E(g(t)) = 0 \Rightarrow \int_{-\infty}^{\infty} g(t) f(t)\, dt = 0$

$\Rightarrow \int_{\theta}^{\infty} g(t)\, e^{-n(t-\theta)}\, dt = 0$

Differentiating both sides & using Leibniz Rule

$\Rightarrow -g(\theta) e^{-n(t-\theta)} + \int_{\theta}^{\infty} -n g(t)\, e^{-n(t-\theta)}\, dt = 0$

But $\int_\theta^\infty g(t) e^{-n(t-\theta)} dt = 0 \Rightarrow -n \int_\theta^\infty g(t) e^{-n(t-\theta)} = 0$

$\therefore -g(t) e^{-n(t-\theta)} = 0$

$\Rightarrow g(t) \underbrace{e^{-n(t-\theta)}}_{>0 \ \forall t} = 0 \ \forall \theta$

$\therefore g(t) = 0$ almost surely

$\therefore t$ is a complete sufficient statistic

b) define $Z_i = X_i - \theta$

$F_\theta(X \leq x) = \begin{cases} \int_\theta^x e^{-(x-\theta)} dx = 1 - e^{-(x-\theta)} & x > \theta \\ 0 & ow \end{cases}$

$\Rightarrow F_\theta(X - \theta \leq x) = F(X \leq x + \theta)$

$= \begin{cases} 1 - e^{-x} & x > 0 \\ 0 & ow \end{cases}$

$\Rightarrow F(Z_i \leq z) = \begin{cases} 1 - e^{-z} & z > 0 \\ 0 & ow \end{cases}$

$\therefore$ Distribution of z_i doesnt depend on θ

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x}_n)^2$$

$$= \frac{1}{n-1} \sum_{i=1}^{n} \left(\cancel{z_i} - \frac{\sum_{j=1}^{n} (z_j + \theta)}{n} \right)$$

$z_i + \theta$

$$= \frac{1}{n-1} \sum_{i=1}^{n} \left(z_i - \cancel{z_n} \bar{z}_n + \theta - \frac{n\theta}{n} \right)$$

$$= \frac{1}{n-1} \sum_{i=1}^{n} z_i - \bar{z}_n$$

But distribution of z_i is independent of θ

$\Rightarrow S^2$ is independent of θ

$\Rightarrow S^2$ is an ancillary statistic

$X_{(1)}$ is complete and minimal

S^2 is an ancillary statistic

$\therefore X_{(1)}$ and S^2 are independent of each other

(6) $$f_\theta(\underline{x}) = \frac{1}{\sigma^n} \exp\left(- \frac{\sum_{i=1}^{n} (x_i - \mu)}{\sigma} \right) 1(x_1 > \mu) \ldots 1(x_n > \mu)$$

$$= \frac{1}{\sigma^n} \exp\left(- \frac{\sum_{i=1}^{n} x_i}{\sigma} + \frac{n\mu}{\sigma} \right) 1(\cancel{x_1} \min(x_1, \ldots, x_n) > \mu)$$

$$\frac{f_\theta(\underline{x})}{f_\theta(y)} = \exp\left(\frac{\sum_{i=1}^{n} y_i}{\sigma} - \frac{\sum_{i=1}^{n} x_i}{\sigma}\right) \frac{1(\min(x_1, \ldots, x_n) > \mu)}{1(\min(y_1, \ldots, y_n) > \mu)}$$

is independent of θ iff,

$$\frac{\sum y_i}{\sigma} = \frac{\sum x_i}{\sigma} \text{ and } \min(x_1, \ldots, x_n) = \min(y_1, \ldots, y_n)$$

$$\therefore t_1 = \sum_{i=1}^{n} \frac{x_i}{\sigma}$$

$$t_2 = \min(x_1, x_2, \ldots, x_n) = X_{(1)}$$

$\therefore T(x) = \left(\frac{\sum_{i=1}^{n} x_i}{\cancel{\sigma}}, X_{(1)}\right)$ is minimal sufficient statistic

Define the one to one mapping with $T(x)$,

$$h(T(x)) = \left(\frac{\sum_{i=1}^{n} x_i}{\cancel{\sigma}} - \frac{nX_{(1)}}{\cancel{\sigma}}, X_{(1)}\right) = (T_1, T_2)$$

every $T(X_{n})$ is mapped to a unique (T_1, T_2)

and $T_1 + T_2 = \frac{\sum_{i=1}^{n} x_i}{\cancel{\sigma}}$

$T_2 = X_{(1)}$

thus we can ~~recover~~ invert the mapping (proving one to one)

Consider $\theta = \mu$ & σ is a known constant

~~Then~~ $z_i = x_i - \mu$ is ~~independent~~ ancillary (∵ location parameter distribution)

$$\Rightarrow \cancel{\Sigma} T_1 = \frac{\sum_{i=1}^{n} x_i}{\cancel{\sigma}} - \frac{n x_{(1)}}{\cancel{\sigma}}$$

$$= \frac{\sum_{i=1}^{n}(z_i + \mu)}{\cancel{\sigma}} - \frac{n(z_{(1)} + \mu)}{\cancel{\sigma}}$$

$$= \frac{\sum_{i=1}^{n} z_i}{\cancel{\sigma}} - \frac{n z_{(1)}}{\cancel{\sigma}}$$

is independent of μ

$\Rightarrow T_1$ is ancillary to μ (when σ is known)

T_1 is ancillary and $T_2 = X_{(1)}$ is complete sufficient (when σ is fixed)

∴ By Basu's theorem,

T_1 and T_2 are independent when σ is fixed and known

But this is true for all $\sigma > 0$

$\Rightarrow T_1$ and T_2 are independent even when σ is unknown.