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ISYE 6412 - HW 03.

① From HW-1 we know,

$$R_{\delta_{2,n}}(\theta) = \frac{(\theta-1)^2+n}{(1+n)^2(1+\theta^2)}$$

$$r_\delta = \int_{\forall\theta} \pi(\theta)\, R_\delta(\theta)\, d\theta$$

$$= \int_{-\infty}^{\infty} c_1 (1+\theta^2)\, \phi(\theta-1)\, \frac{(\theta-1)^2+n}{(1+n)^2(1+\theta)^2}\, d\theta$$

$$= \int_{-\infty}^{\infty} c_1\, \phi(\theta-1) \left(\frac{(\theta-1)^2+n}{(1+n)^2}\right) d\theta$$

Replace $\theta - 1 \longrightarrow \alpha$. $\ni d\theta = d\alpha$.

$$\Rightarrow r_\delta = \int_{-\infty}^{\infty} c_1\, \phi(\alpha) \left(\frac{\alpha^2+n}{(1+n)^2}\right) d\alpha.$$

$$= \frac{c_1}{(1+n)^2}\left[\int_{-\infty}^{\infty} \phi(\alpha)\, \alpha^2\, d\alpha + n\int_{-\infty}^{\infty} \phi(\alpha)\, d\alpha\right]$$

$$= \frac{c_1}{(1+n)^2}\left[\, E(\alpha^2) + n\,\right]$$

$\cancel{E}$ "4" where $E(x^2)$, $x \sim N(0,1)$

$\Rightarrow E(x^2) = \text{var}(x) + (E(x))^2$

$= 1$

$\Rightarrow r_\delta = \dfrac{c_1}{1+n}$

For a general procedure $\delta = a + b\bar{Y}_n$

$$R_\delta(\theta) = \frac{(a+(b-1)\theta)^2 + b^2/n}{1+\theta^2}$$

$$r_\delta = \int_{-\infty}^{\infty} c_1(1+\theta^2)\left(\frac{(a+(b-1)\theta)^2 + b^2/n}{(1+\theta^2)}\right)\phi(\theta)\,d\theta$$

$$= c_1 \int_{-\infty}^{\infty}$$

For Bayes procedure, minimise

$$h_\pi^*(y,d) = \int_\Omega L(\theta,d) f_\theta(y)\pi(\theta)\,d\theta$$

for each y

$$= c_1\int_{-\infty}^{\infty} \frac{(\theta-d)^2}{(1+\theta^2)}\left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{(y_i-\theta)^2}{2}\right)}\right)\frac{1}{\sqrt{2\pi}} e^{-\frac{\theta^2}{2}}(1+\theta^2)\,d\theta$$

$$= c_4\int_{-\infty}^{\infty} (\theta^2 - 2\theta d + d^2)(D_0)\left(\exp\left(-\frac{1}{2}\theta^2\left(n + 1\right) + \theta\left(\sum y_i\right)\right)\right)d\theta$$

For Bayes procedure, minimise,

$$r^*_\pi(y,d) = \int_\Omega L(\theta, d) f_\theta(y)\pi(\theta)\, d\theta$$

for all y

$$= \int_{-\infty}^{\infty} \frac{c_1 (1+\theta^2)(\theta-d)^2}{(1+\theta^2)} \left[\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y_i-\theta)^2}{2}} \right] \phi(\theta-1)\, d\theta$$

$$= c_1 \int_{-\infty}^{\infty} (\theta-d)^2 \left[f_\theta(y_1, y_2, \ldots, y_n) \right] \phi(\theta-1)\, d\theta.$$

$\Rightarrow$ minimise $(\theta-d)^2$ for ~~Edist.~~ prior of $\phi(\theta-1)$

We know the solution for $L = (\theta-d)^2$

$\pi(\theta^2) = \phi(\theta-1) = N(1,1)$

& $f = f_\theta(y)$

$$d^* = \frac{\sum y_i/\sigma^2 + \mu/\tau^2}{n/\sigma^2 + 1/\tau^2}$$ where $\pi(\theta) = N(\mu, \tau^2)$ and $f_\theta(x_i) = N(\theta, \sigma^2)$

Here $\sigma = 1$, $\tau = 1$, $\mu = 1$

$$\Rightarrow d^* = \frac{\sum_i y_i/1 + 1}{n+1} = \frac{\sum_i y_i/n + 1/n}{1 + 1/n}.$$

$$\therefore \delta^*_B = \frac{\sum y_i/n + 1/n}{1 + 1/n} = \delta_{2,n}.$$

② a) $m(y) = \int_{0}^{\theta} f_{\theta}(y)\pi(\theta)\, d\theta$

$$= \int_{\beta}^{\infty} \left(\frac{1}{\theta}\right)^{n} \frac{\alpha\beta^{\alpha}}{\theta^{\alpha+1}}\, d\theta$$

($\because$ Y = n i i d uniform RV)

$$= \int_{\beta}^{\infty} \frac{\alpha\beta^{\alpha}}{\theta^{n+\alpha+1}}\, d\theta = \frac{\alpha\beta^{\alpha}}{(n+\alpha)\beta^{n+\alpha}}$$

$$\pi_{\theta|y}(\theta) = \frac{\pi(\theta) f_{\theta}(y)}{m(y)}$$

$$= \frac{\frac{\alpha\beta^{\alpha}}{\theta^{\alpha+1}} \times \left(\frac{1}{\theta}\right)^{n}}{\frac{\alpha\beta^{\alpha}}{(n+\alpha)\beta^{(n+\alpha)}}} \quad \forall\ \theta \geq \beta,\ y_i \leq \theta$$

and 0 otherwise

$$= \frac{(\alpha+n)\beta^{(n+\alpha)}}{\theta^{n+\alpha+1}}$$

$$= \frac{(\alpha+n)\beta^{(\alpha+n)}}{\theta^{\alpha+n+1}} \quad \forall\ \theta \geq \beta,\ y_i \leq \theta$$

(If $\theta < \beta$, $\pi(\theta) = 0$ - defn of Pareto,

if $y_i > \theta$, $f_{\theta}(y) = 0$ - defn of the uniform distribution)

~~case (i) $\beta < \theta$.~~

$$\Rightarrow \pi(\theta|y) = \frac{(\alpha+n)\beta^{(\alpha+n)}}{\theta^{\alpha+n+1}} \quad \forall\, \theta \geq \beta \text{ and } \theta \geq y_i \,\forall i$$

$\theta \geq \beta$ and $\theta \geq y_i \;\forall i$

$$\Rightarrow \theta \geq \beta \text{ and } \theta \geq \max(y_1, \dots, y_n)$$

$$\Rightarrow \theta \geq \beta \text{ and } \theta \geq Y_{(n)}$$

$$\Rightarrow \theta \geq \max(\beta, Y_{(n)})$$

$$\therefore \pi(\theta|y) = \frac{(\alpha+n)\beta^{(\alpha+n)}}{\theta^{\alpha+n+1}} \quad \theta \geq \max(\beta, Y_{(n)})$$

$$\Rightarrow \alpha^* = \alpha+n, \; \beta^* = \cancel{Y_{(n)}} \max(\beta, Y_{(n)})$$

b) $\rho_\pi(y, d) = \int L(\theta, d)\, \pi(\theta|y)\, d\theta.$

~~$\int (\theta - d)^2$~~

We have seen in HW-2, q4 that when $L = (\theta - d)^2$,

$$d^* = \text{mean}(\pi(\theta|y))$$

$$= \cancel{\alpha \, \text{me}} \frac{\alpha^* \beta^*}{\alpha^* - 1}$$

$\delta_{\beta}^{+}(Y)$

$$\Rightarrow \hat{d}^{+} = \frac{(\alpha+n)(\max\{\beta, Y_{(n)}\})}{\alpha+n-1}$$

(c) From HW-2 under $L = |\theta - d|$, the Bayes procedure is the median of $\pi(\theta|y)$

median of Pareto$(\alpha, \beta) = \sqrt{2}\,\beta$

$$\therefore \delta_c^{+}(Y) = \sqrt{2}\,(\ldots)\,\beta^{n}$$

$$= (\sqrt{2})(\max\{\beta, Y_{(n)}\})$$

(3) a) with increase in c we see an increase in assymmetry

b) $E(L(\theta, d)) = \int_{-\infty}^{\infty} L(\theta, d)\,\pi(\theta)\,d\theta$

$$= \int_{-\infty}^{\infty} \left(e^{c(d-\theta)} - c(d-\theta) - 1\right)(\pi(\theta))\,d\theta$$

$$= e^{cd}\int_{-\infty}^{\infty} e^{-\theta c\theta}\,\pi(\theta)\,d\theta - cd\int_{-\infty}^{\infty} \pi(\theta)\,d\theta + c\int_{-\infty}^{\infty} \theta\,\pi(\theta)\,d\theta - \int_{-\infty}^{\infty} \pi(\theta)\,d\theta$$

$$= e^{cd} E(e^{-c\theta}) - cd + c\int_{-\infty}^{\infty} E(\theta) - 1$$

$$\frac{d\, E(L(\theta,d))}{d(d)} = c e^{cd} E(e^{-c\theta}) - c$$

$$\frac{d^2 (E(L(\theta,d)))}{d(d^2)} = c^2 e^{cd} E(e^{-c\theta})$$

$e^{-c\theta} > 0$, ~~ct~~ $\Rightarrow E(e^{-c\theta}) > 0$

$e^{cd} > 0$

$$\therefore \frac{d^2 R(\theta,d)}{d(d^2)} > 0 \quad \forall d$$

If a stationary point exists it must be a minimum

Setting first derivative to 0,

$$c e^{cd} E(e^{-c\theta}) - c = 0$$

$$\Rightarrow e^{cd} * E(e^{-c\theta}) = 1$$

$$\cancel{d \; E(e^{-c\theta})} = \frac{1}{\;}$$

$$\Rightarrow \cancel{E(e^{-c}} \; e^{cd} = \frac{1}{E(e^{-c\theta})}$$

$$\Rightarrow cd^* = -\log E(e^{-c\theta})$$

$$\Rightarrow d^* = -\frac{1}{c} \log\left(E(e^{-c\theta})\right)$$

exists $\because E(e^{-c\theta}) > 0$

& we showed that it is a minimum.

$$\Rightarrow \delta_{Bayes} = \frac{-1}{c} \log\left(E(e^{-c\theta})\right)$$

c) we know that to get Bayes procedure we need to minimise

$$h_\pi(\underline{y}, d) = \int_\Omega L(\theta, d)\, \pi(\theta \mid y)\, d\theta.$$

This is essentially the same qn as previous section with $\pi(\theta \mid y)$ instead of $\pi(\theta)$

$$\Rightarrow h_\pi(\underline{y}, d) = e^{+cd} \int_{-\infty}^{\infty} e^{-c\theta} \pi(\theta \mid y)\, d\theta - cd \int_{-\infty}^{\infty} \pi(\theta \mid y)\, d\theta + c \int_{-\infty}^{\infty} \theta \pi(\theta \mid y)\, d\theta - \int_{-\infty}^{\infty} \pi(\theta \mid y)\, d\theta.$$

$$= e^{cd} E(e^{-c\theta} \mid \underline{Y}) - cd + c E(\theta \mid \underline{Y}) - 1$$

$$\therefore \frac{d(h_*)}{d(d)} = c e^{cd} E(e^{-c\theta} \mid \underline{Y}) - c$$

$$\frac{d^2(h_*)}{d(d^2)} = c^2 e^{cd} E(e^{-c\theta} \mid \underline{Y}) > 0 \quad \forall d$$

By setting derivative to 0 and same argument that stationary point is minima.

$$\cancel{\delta_c(\underline{Y})} = d^* = \frac{-1}{c} \log\left(E\left(e^{-c\theta} \mid \underline{Y}\right)\right)$$

$$\rightarrow \delta_c(\underline{Y}) = \frac{-1}{c} \log\left(E\left(e^{-c\theta} \mid \underline{Y} \right)\right)$$

d) As per hint, we consider a new observation

$$\bar{Y}_n \sim N(\theta, \sigma^2/n)$$

Now. $$m(\bar{y}) = \int_{-\infty}^{\infty} f_\theta(\bar{y}) \pi(\theta) d\theta$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \frac{\sigma^2}{n}}} e^{-\left(\frac{(\bar{y}-\theta)^2}{2\sigma^2/n}\right)} \pi(\theta) d\theta$$

$$\equiv \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \frac{\sigma^2}{n}}} e^{-\left(\frac{(\bar{y}-\theta)^2}{2\sigma^2/n}\right)} \, d\theta$$

$$= 1 \qquad 1 \because \int f_{\bar{y}}(\theta) d\theta = 1$$

$$\therefore \pi(\theta \mid \bar{y}) \equiv \frac{f_\theta(\bar{y}) \times 1}{1}$$

$$= f_\theta(\bar{y})$$

$$\Rightarrow \pi(\theta \mid \bar{y}) \equiv f_\theta(\bar{y})$$

$$\sim N(\bar{y}, \sigma^2/n)$$ makes sense because we assume

this makes sense because we assumed prior is un-informative - we are deciding solely based on obtained data

$$E(e^{-c\theta} \mid \bar{y}) = \int_{-\infty}^{\infty} \frac{e^{-\left(\frac{(\theta-\bar{y})^2}{2\sigma^2/n}\right) - c\theta}}{\sqrt{2\pi \sigma^2/n}} \, d\theta.$$

for exponential term: $\dfrac{(\theta-\bar{y})^2}{2\sigma^2/n} + c\theta$

$$= \frac{\theta^2 - \cancel{2\theta\bar{y}} + \left(-\frac{2\sigma^2}{n}c + 2\bar{y}\right)\theta + \bar{y}^2}{2\sigma^2/n}$$

$$= \left(\left(\theta - \left(\bar{y} - \frac{\sigma^2 c}{n}\right)\right)^2 + \bar{y}^2 - \left(\bar{y} - \frac{\sigma^2 c}{n}\right)^2\right) \frac{1}{\frac{2\sigma^2}{n}}$$

$$= \frac{\left(\theta - \left(\bar{y} - \frac{\sigma^2 c}{n}\right)\right)^2 + \cancel{\bar{y}^2} - \frac{2\bar{y}\sigma^2 c}{n} - \frac{\sigma^4 c^2}{n^2}}{2\sigma^2/n}$$

$\therefore$ ~~$E(e^{-c\theta} \mid \underline{y}) = e^{\left(\bar{y} - \frac{\sigma^2 c}{n} + \frac{2\bar{y}\sigma^2 c}{n} - \frac{\sigma^4 c^2}{n^2}\right)}$~~

$E(e^{-c\theta} \mid \underline{y})$ ~~$=$ constant $\times \phi\left(\frac{y - \left(\bar{y} - \sigma^2 c\right)}{n}\right)$~~

$=$ constant $\times$ integral of Normal dist.

with $\mu = \bar{y} - \frac{\sigma^2 c}{n}$ and var $= \frac{\sigma^2}{n}$

$=$ ~~cons~~ Constant

$$= e^{\left(+\frac{\sigma^4 c^2}{n^2} - \frac{2\bar{y}\sigma^2 c}{n}\right) \times \frac{n}{2\sigma^2}}$$

$$= e^{\left(\frac{c^2\sigma^2}{2n} - \bar{y}c\right)}$$

from part (c),

$$\delta_{\text{Bayes}}(\underline{y}) = \frac{-1}{c} \times \log\left(E(e^{-c\theta} \mid y)\right)$$

$$= \frac{-1}{c}\left(\frac{+c^2\sigma^2}{2n} - \bar{y}c\right)$$

$$\Rightarrow \delta_d(\underline{y}) = \bar{y} - \frac{\sigma^2 c}{2n}$$

e) $h_\pi(y; \delta(y), d) = e^{cd} E(e^{-c\theta} | \underline{y})$

$- cd + c E(\theta | y) - 1$

(from part c)

$= e^{cd} e^{\left(\frac{c^2\sigma^2}{2n^2} - \bar{y}c\right)} - cd + c E(\theta | y) - 1$

(from part d)

$E(\theta | y) = \bar{y} \quad (\because \pi(\theta | y) \sim N(\bar{y}, \frac{\sigma^2}{n}))$

Posterior loss for $\delta_d(y) = \left(\bar{y} - \frac{c\sigma^2}{2n}\right)$

$h_\pi(y, \delta_d(y)) = e^{c\left(\bar{y} - \frac{c\sigma^2}{2n}\right) + \left(\frac{c^2\sigma^2}{2n} - \bar{y}c\right)}$

$- c\left(\bar{y} - \frac{c\sigma^2}{2n}\right) + c\bar{y} - 1$

$= 1 + \frac{c^2\sigma^2}{2n} - 1$

$= \frac{c^2\sigma^2}{2n}.$

Posterior loss for $\delta_e(y) = \left(\bar{y} - \cancel{\frac{c\sigma^2}{2n}}\right)\ \bar{y}$

$h_\pi(y, \delta_e(\bar{y})) = e^{c(\bar{y}) + \frac{c^2\sigma^2}{2n} - \bar{y}c}$

$- c(\bar{y}) + c\bar{y} - 1$

$$= e^{\frac{c^2\sigma^2}{2n}} - 1$$

$e^{\frac{c^2\sigma^2}{2n}}$ ~~α~~ $e^{c\sigma}$

We note that $\frac{c^2\sigma^2}{2n} > 0$

~~as~~ a. $e^x \approx 1 + \frac{x}{1!} + \frac{x^2}{2!} \dots$

$\Rightarrow e^x \geq 1 + x$ (equality holds at $x = 0$)

$\Rightarrow e^x - 1 \geq x$

$\Rightarrow e^{\frac{c^2\sigma^2}{2n}} - 1 \geq \frac{c^2\sigma^2}{2n}$

As expected δ_d performs better (since that is optimal under LINEX)

f) $h_\pi(y, d) = \int_{-\infty}^{+\infty} (\theta - d)^2 \, \pi(\theta \mid y) \, d\theta.$

$$= \int_{-\infty}^{\infty} (\theta^2 - 2d\theta + d^2) \, f_\theta(\bar{y}) \, d\theta$$

(from part d)

$$= E(\theta^2) - 2dE(\theta) + d^2$$

$\theta \sim N(\bar{y}, \frac{\sigma^2}{n})$

$E(\theta^2) = (\bar{y})^2 + \frac{\sigma^2}{n}$, $E(\theta) = \bar{y}$

$\Rightarrow h_\pi(\bar{y}, d) = \left(\bar{y}^2 + \frac{\sigma^2}{n}\right) - 2d\bar{y} + d^2$

For $\delta_d(\bar{y}) = \bar{y} - \left(\frac{c\sigma^2}{2n}\right)$

$$h_\pi(\bar{y}, \delta_d) = \bar{y}^2 + \frac{\sigma^2}{n} - 2\left(\bar{y} - \frac{c\sigma^2}{2n}\right)\bar{y} + \left(\bar{y} - \frac{c\sigma^2}{2n}\right)^2$$

$$= -\bar{y}^2 + \frac{\sigma^2}{n} + \frac{c\bar{y}\sigma^2}{n} + \bar{y}^2 - \frac{2c\sigma^2\bar{y}}{2n} + \frac{c^2\sigma^4}{4n^2}$$

$$= \frac{\sigma^2}{n} + \frac{c^2\sigma^4}{4n^2} = \frac{\sigma^2}{n} + \frac{c^2\sigma^4}{4n^2}$$

For $\delta_e(\bar{y}) = \bar{y}$

$$h_\pi(\bar{y}, \delta_e) = \bar{y}^2 + \frac{\sigma^2}{n} - 2\bar{y}^2 + \bar{y}^2$$

$$= \frac{\sigma^2}{n}.$$

Again since for. $L = (\theta - d)^2$, $\delta_e(\bar{y})$ is optimal we clearly see $\delta_d \neq \delta_e$

$$h_\pi(\bar{y}, \delta_d) > h_\pi(\bar{y}, \delta_e)$$

(4) a) $\pi(\theta|y) = \dfrac{\pi(\theta) f_\theta(y)}{\int_0^\infty \pi(\theta) f_\theta(y)\, d\theta}$.

$$\int_0^\infty \pi(\theta) f_\theta(y)\, d\theta = \int_0^\infty \frac{\theta^q y^{q-1} e^{-\theta y}}{\Gamma(q)} \times \frac{\theta^{\alpha-1} e^{-\theta/\beta}}{\Gamma(\alpha)\beta^\alpha}\, d\theta$$

$$= \frac{y^{q-1}}{\Gamma(q)\Gamma(\alpha)\beta^\alpha} \int_0^\infty \theta^{q+\alpha-1} e^{-\theta\left(y+\frac{1}{\beta}\right)}\, d\theta$$

$\alpha' = q+\alpha$, $\beta' = \dfrac{\beta}{1+\beta y}$

$$= \frac{y^{q-1}\Gamma(\alpha')(\beta')^{\alpha'}}{\Gamma(q)\Gamma(\alpha)\beta^\alpha} \int_0^\infty \frac{\theta^{\alpha'} e^{-\frac{\theta'}{\beta}}\, d\theta}{\Gamma(\alpha')(\beta')^{\alpha'}}$$

$$= \frac{y^{q-1}\Gamma(\alpha')(\beta')^{\alpha'}}{\Gamma(q)\Gamma(\alpha)\beta^\alpha}.$$

$$\therefore \pi(\theta|y) = \frac{\left(\dfrac{\theta^{\alpha-1} e^{-\theta/\beta}}{\Gamma(\alpha)\beta^\alpha}\right)\left(\dfrac{\theta^q y^{q-1} e^{-\theta y}}{\Gamma(q)}\right)}{\dfrac{y^{q-1}\Gamma(\alpha')(\beta')^{\alpha'}}{\Gamma(q)\Gamma(\alpha)\beta^\alpha}}$$

$$= \frac{\theta^{\alpha + q - 1} e^{-\frac{\theta(y\beta + 1)}{\beta}}}{\Gamma(\alpha')(\beta')^{\alpha'}}$$

$$= \frac{\theta^{\alpha' - 1} e^{-\theta/\beta'}}{(\Gamma(\alpha'))(\beta')^{\alpha'}} = \frac{\theta^{\alpha' - 1} e^{-\theta/\beta'}}{\Gamma(\alpha')(\beta')^{\alpha'}}$$

$\Rightarrow \pi(\theta|y) = \text{Gamma}(\alpha', \beta')$

b) We know $\delta_{Bayes}(y)$ with posterior $\pi(\theta|y)$
is $E(\pi(\theta|y))$

$$\Rightarrow \delta_{Bayes}(y) = \text{mean}(\text{Gamma}(\alpha', \beta'))$$

$$= \alpha' \beta'$$

$$= (\alpha + q)\left(\frac{\beta}{1 + \beta y}\right)$$

c) $$h_\pi(y, d) = \int_{\forall \theta} L(\theta, d)\, \pi(\theta|y)\, d\theta$$

$$L^{+}(\theta, d) = \begin{cases} 1 & |\theta - d| > 2 \\ 0 & |\theta - d| < 2 \end{cases}$$

non zero
We have loss only when

$|\tilde{\theta} - d| > 2 \Rightarrow \theta > 2 + d$
and $\theta < d - 2$

$$h_\pi(y, d) = \int_{d+2}^{\infty} \pi(\theta|y)\, d\theta + \int_{-\infty}^{d-2} \pi(\theta|y)\, d\theta$$

$$= \int_{d+2}^{\infty} \pi(\theta|y) d\theta + \int_{d-2}^{d+2} \pi(\theta|y) + \int_{-\infty}^{d-2} \pi(\theta|y) d\theta$$
$$- \int_{d-2}^{d+2} \pi(\theta|y)\, d\theta$$

$$= \int_{-\infty}^{\infty} \pi(\theta|y)\, d\theta - \int_{d-2}^{d+2} \pi(\theta|y)\, d\theta$$

$$= 1 - \int_{d-2}^{d+2} \pi(\theta|y)\, d\theta$$

$$\frac{d(h_\pi(y, d))}{d(d)} = -\left(\pi(d+2|y) - \pi(d-2|y)\right)$$

(Leibniz rule)

$$\frac{d^2(h_\pi(y, d))}{d\, d^2} = -\left(\pi'(d+2|y) - \pi'(d-2|y)\right)$$

Setting derivative $= 0$

$$\Rightarrow \pi(d^* + 2|y) = \pi(d^* - 2|y)$$

~~We~~ ~~want that~~
From ~~the~~ hint, . min d^* is shown to be

$$d^* = \begin{cases} \text{Solution of } \pi(d^*-2|y) = \pi(d^*+2|y) & \text{when } \pi(0|y) < \pi(4|y) \\ 2 & \text{o/w} \end{cases}$$

$\Rightarrow$ ~~δ_{Bayes}~~ $= .\ \pi(0|y) < \pi(4|y)$

$$\Rightarrow \cancel{\theta^{\alpha'}} \cdot 0 < \frac{4^{\alpha'-1} e^{-4/\beta'}}{\Gamma(\alpha')(\beta')^{\alpha'}} \quad \text{when } \alpha' \cancel{\leq 1} > 1$$

If $\alpha' = \alpha + y = 1$,

$$\frac{e^{-y/\beta'}}{\Gamma(\alpha')(\beta')^{\alpha'}} < \frac{4^{\cancel{\alpha'-1}} e^{-y/\beta'}}{\Gamma(\alpha')(\beta')^{\alpha'}}$$

$\Rightarrow 1 < 1$

$\therefore$ Not possible

$\alpha' = \alpha + y = 1 \Rightarrow \pi(0|y) = \pi(4|y)$

If $\alpha' < 1$

$\pi(0|y) \to \infty$ (0 in denominator)

~~$\Rightarrow \delta^*$~~ $= \begin{cases} 2 \\ 2 \end{cases}$

$$\Rightarrow \delta^* = \begin{cases} 2 & \text{if } q+\alpha \le 1 \\ \text{solution } d \text{ of } \pi(d+2|y) = \pi(d-2|y) & q+\alpha > 1 \end{cases}$$

$$= \begin{cases} 2 & q+\alpha \le 1 \\ -2 + \cancel{4}\left(1 - \exp\left(-\frac{4(y+1/\beta)}{\alpha+q-1}\right)\right) & q+\alpha > 1 \end{cases}$$

(as given in qn)

$\therefore$ Bayes confidence interval with fixed width 4

is $[\delta^*(y)-2, \delta^*(y)+2)$.

$$a \triangleq \frac{\alpha \cdot n}{\alpha + \beta + n}, \quad b \triangleq \frac{n}{\alpha + \beta + n}$$

(5) a) $R_\delta(\theta) = E((\theta - d)^2)$

$$= E((\theta - a - b\bar{y}_n)^2)$$

$$= E(\theta^2 + a^2 + b^2 E(\bar{y}_n^2)$$

$$+ 2ab E(\bar{y}_n) - 2a\theta - 2b\theta E(\bar{y}_n))$$

$$E(\bar{y}_n^2) = E\left(\left(\frac{\sum_i y_i}{n}\right)^2\right) = E\left(\frac{\sum_i y_i^2}{n^2} + \frac{2\sum_{i \neq j} y_i y_j}{n^2}\right)$$

$E(\text{cross terms}) = 0$ because of independence

$$\Rightarrow E(\bar{y}_n^2) = \frac{1}{n^2} \, n \, v(\theta) = \left(\frac{\theta(1-\theta)}{n} + \theta^2\right)$$

$$E(\bar{y}_n) = \theta$$

$$\therefore R_\delta(\theta) = \theta^2 + a^2 + b^2\left(\frac{\theta}{n} + \theta^2\right) + 2ab\theta - 2a\theta - 2b\theta^2$$

$$= \theta^2(1 - 2b) + \theta\left(\frac{b^2}{n} + 2ab - 2a\right) + a^2$$

$$\Rightarrow b = \frac{1}{2} \text{ and } \frac{n}{4} + a - 2a = 0$$

$$\Rightarrow a = \frac{n}{4}$$

$$\Rightarrow \frac{\alpha \cdot n}{(\alpha + \beta + n)} = \frac{n}{4} \text{ and } \frac{n}{\alpha + \beta + n} = \frac{n}{2} \Rightarrow \frac{1}{\alpha + \beta + n} = \frac{1}{2}$$

$\cancel{E\left(\bar{y}_n^2\right)}$

$$R_\delta(\theta) = \theta^2 + a^2 + b^2\left(\frac{\theta}{n}\right) + 2ab\theta$$
$$-2a\theta - 2b\theta^2$$
$$= \theta^2(1-2b) + \theta\left(\frac{b^2}{n} + 2ab - 2a\right) + a^2$$

b) eliminating θ terms,

$$b = 1/2$$

$$\cancel{a}_2 \quad \cancel{2}a - 2a + b\frac{1}{4n} = 0 \Rightarrow a = 1/4n$$

$$\Rightarrow \frac{n}{\alpha+\beta+n} = 1/2 \quad \text{and} \quad \frac{\alpha}{\alpha+\beta+n} = \frac{1}{4n}.$$

$$\Rightarrow \frac{\alpha}{n} = \frac{1}{2n} \Rightarrow \alpha^* = 1/2$$

and $\beta^* = n - 1/2$

c) ~~Bayes~~ of $R\delta(\theta)$ =

$$R_f(\theta) = a^2 = \frac{1}{16n^2}$$

$$r_f = \int \frac{1}{16n^2}\pi(\theta)\,d\theta = \frac{1}{16n^2}$$

$\text{var}(\bar{y}_n^{2}) + E(\bar{y}_n)$ $\qquad \frac{n\theta(1-\theta)}{n^2} + \frac{\theta^{2n}}{n^2}$

$\frac{\theta(1-\theta)}{n} + n\theta = \theta\left(\frac{n+1}{n}\right)$

d) Subst $a = 0,\ b = 1,$

$$R_\delta(\theta) = \frac{\theta^2}{n} + E(\bar{y}_n^2) - \frac{2\theta E(\bar{y}_n)}{n}$$

$$= \theta^2 + \frac{\theta}{n} - 2\theta^2 = \frac{\theta}{n} - \theta^2$$

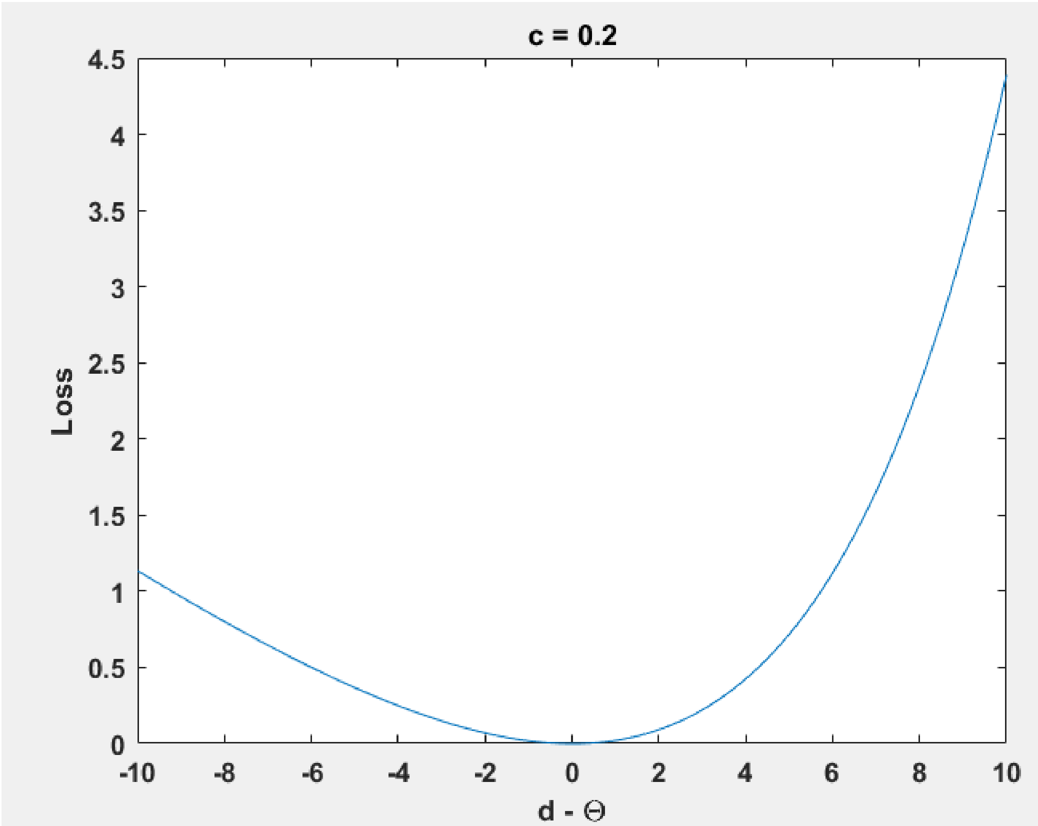
e) $$r_{\delta_{\alpha^2,\beta^2,r}} = a^2 = \left(\frac{\alpha+\beta}{\alpha+\beta+n}\right)^2$$

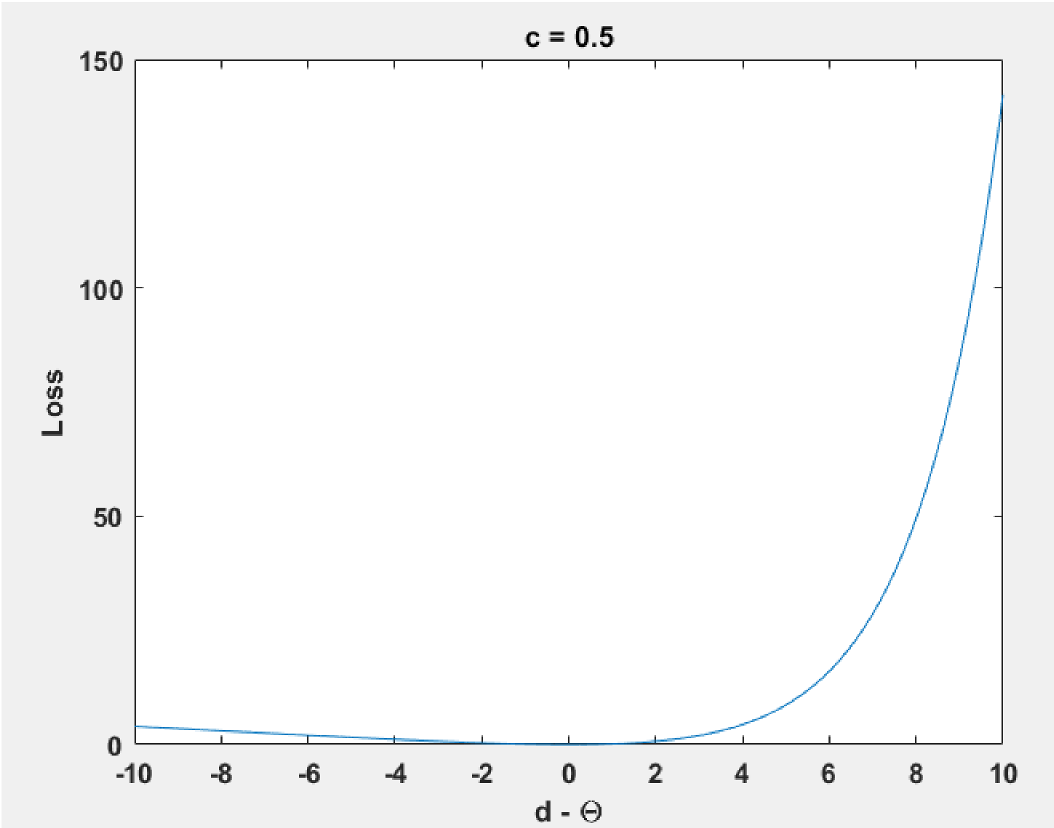
$$r_{\delta_{\text{samplemean}}} = \int\left(\frac{\theta}{n} - \theta^2\right)\pi(\theta)\,d\theta$$

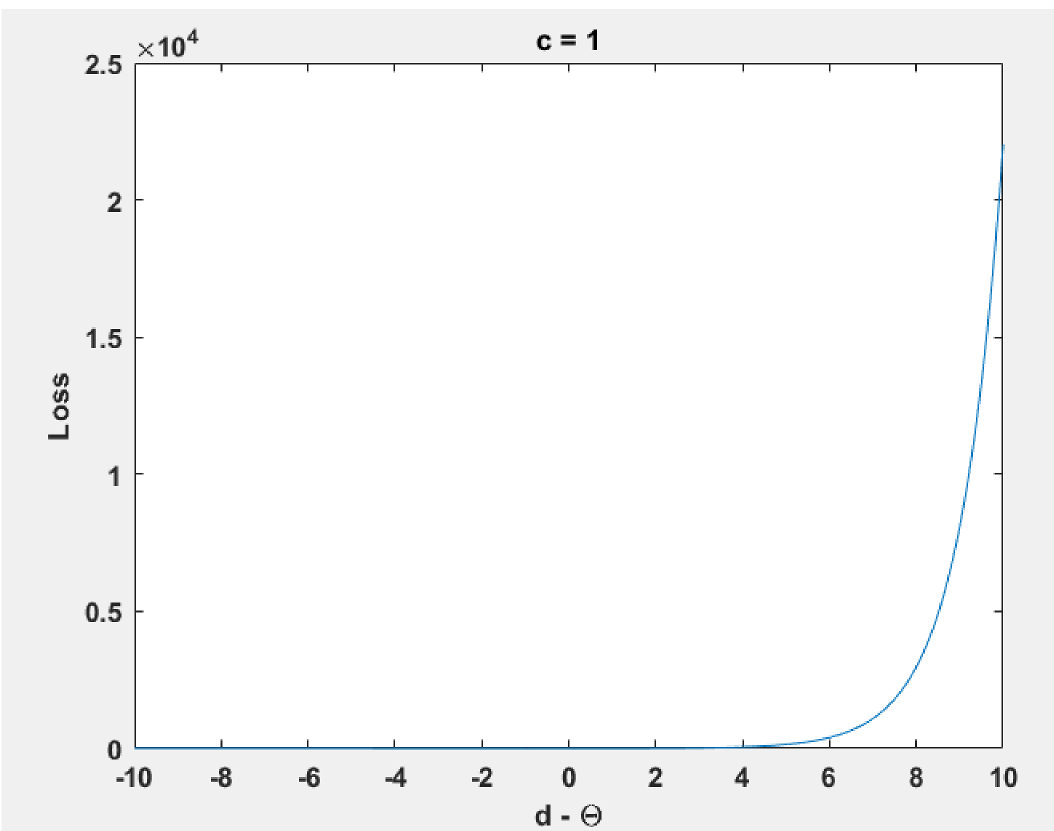
$$= \frac{\alpha}{\alpha+\beta} \cdot \frac{1}{n} - \frac{1}{n}\left(\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} + \frac{\alpha^2}{(\alpha+\beta)^2 n}\right)$$

$$= \frac{1}{n}\left[\frac{\alpha^2}{\alpha+\beta} - \left(\frac{\alpha^2\beta^2}{(\alpha+\beta)^2(\alpha+\beta+1)^2} + \frac{\alpha^2}{(\alpha+\beta)^2 n}\right)\right)$$

Q3a




c = 0.5
Loss
150
100
50
0
-10
-8
-6
-4
-2
0
2
4
6
8
10
d - Θ


c = 1
×10^4
Loss
2.5
2
1.5
1
0.5
0
-10
-8
-6
-4
-2
0
2
4
6
8
10
d - Θ