BY: VISHAL SIVARAMAN

ISYE 6412 - HW05

① a) $R_{\delta_{a,b}}(\theta) = E((\theta - a\bar{y} - b)^2)$

$$= \left(E(\theta - a\bar{y} - b)\right)^2 + var(a\bar{y} + b)$$

$$= ((a-1)\theta + b)^2 + \frac{a^2\sigma^2}{n} \quad \text{—— ①}$$

$\left(\because E(\bar{y}) = \theta \text{ and } var(\bar{y}) = \frac{n\sigma^2}{n^2} = \sigma^2/n\right)$

b) i) $a > 1$

consider $a = 1, b = 0, \delta = \bar{y}$

$R_{\bar{y}}(\theta) = \sigma^2/n$

$\because a > 1, \Rightarrow a^2 > 1 \Rightarrow \frac{a^2\sigma^2}{n} > \frac{\sigma^2}{n}.$

$$\Rightarrow \frac{a^2\sigma^2}{n} + \underbrace{((a-1)\theta + b)^2}_{\text{non negative term}} > \frac{\sigma^2}{n}.$$

$\Rightarrow R_{\delta_{a,b}}(\theta) > R_{\delta_{\bar{y}}}(\theta) \quad \forall \theta$

$R_{\bar{Y}} \le R_{\delta_{a,b}} \; \forall \theta$

$R_{\bar{Y}}(\theta_0) < R_{\delta_{a,b}}(\theta_0)$ for at least one θ_0 - in fact θ_0 is all θ

$\therefore$ $\bar{Y}$ is better than $\delta_{a,b}$ when $a > 1$

$\Rightarrow \delta_{a,b}$ is inadmissible

b) ii) $a < 0$

$$R_{\delta_{a,b}}(\theta) = \left[(a-1)\theta + b\right]^2 + \frac{a^2\sigma^2}{n}$$

$$= (a-1)^2\left[\theta + \frac{b}{(a-1)}\right]^2 + a^2\frac{\sigma^2}{n}$$

$$= (a-1)^2\left[\theta - \frac{b}{1-a}\right]^2 + \frac{a^2\sigma^2}{n}$$

We note $a < 0 \Rightarrow 1-a > 1$

$$\Rightarrow (1-a)^2 > 1$$

$$\Rightarrow (1-a)^2\left[\theta - \frac{b}{1-a}\right]^2 > \left[\theta - \frac{b}{1-a}\right]^2 \quad (2)$$

$\Rightarrow R_{\delta_{a,b}}(\theta) > \left(\theta - \frac{b}{1-a}\right)^2$

For any $\delta_{a,b}$ procedure consider $\delta'_{a,b} = \frac{b}{1-a}$.
valid since $a \neq 1$

(constant estimator)

$$R_{\delta'_{a,b}} = E\left(\left(\theta - \frac{b}{1-a}\right)^2\right) = \left(\theta - \frac{b}{1-a}\right)^2$$

$\therefore (2) \Rightarrow R_{\delta_{a,b}}(\theta) > R_{\delta'_{a,b}}(\theta) \; \forall \theta$

$\Rightarrow \delta'_{a,b}$ is better than $\delta_{a,b}$

$\Rightarrow \delta_{a,b}$ is inadmissible

iii) $a=1$; plugging this in eqn ①

$$R_{\delta_{a,b}}(\theta) = b^2 + \frac{\sigma^2}{n} > R_{\bar{Y}} = \frac{\sigma^2}{n} = R_{\bar{Y}}(\theta)$$

$(\because b^2 > 0)$

$R_{\delta_{a,b}}(\theta) > R_{\bar{Y}}(\theta)$

$\bar{Y}$ is better than $\delta_{a,b}$.

$\Rightarrow \delta_{a,b}$ is inadmissible.

② We know that

$$\delta^+ = \frac{n/\sigma^2}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}\bar{Y} + \frac{1/\tau^2}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}\mu$$

is uniquely Bayesian for $Y \sim N(\theta, \sigma^2)$ and $\pi \sim N(\mu, \tau^2)$

$\Rightarrow$ All estimators of the form δ^+ are admissible

Recognising,

$$0 < \frac{n/\sigma^2}{\frac{n}{\sigma^2}+1/\tau^2} < 1 \quad \text{if} \quad 0 < a < 1$$

and comparing coeffts with $a\bar{Y} + b$,

For $0 < a < 1$,

$$\frac{\frac{n}{\sigma^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}} = a \quad \text{and} \quad \frac{\frac{1}{\tau^2}\mu}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}} = b.$$

Given ~~cnth~~ a and b we can get

$$\frac{1}{\tau^2} = \cancel{\text{dum}} \frac{n}{\sigma^2}\left(\frac{1}{a} - 1\right)$$

$$0 \leq a < 1 \Rightarrow \frac{1}{a} > 1 \Rightarrow \frac{1}{a} - 1 > 0$$

So we can always find a $\frac{1}{\tau^2}$

and $\mu = \frac{b}{1-a}$

So for given $0 < a < 1$ and b we can find μ, τ^2 such that under $x \sim N(\mu, \tau^2)$, $\delta_{a,b} = a\bar{y} + b$ is uniquely Bayesian

$\Rightarrow \delta_{a,b}$ is admissible

if $a = 0$, $\delta_{a=0,b} = b$

$$R_{\delta_{a,b}} = (b - \theta)^2.$$

~~If $\delta_{a,b}$ is inad~~

If possible let δ' exist better than $\delta_{a,b}$ making $\delta_{a,b}$ inadmissible.

$\Rightarrow R_{\delta'}(\theta) \leq R_{\delta_{a,b}}(\theta) \quad \forall \theta \in \Omega$

$R_{\delta'}(\theta_0) < R_{\delta_{a,b}}(\theta_0)$ for some θ_0

At $\theta = b$,

$R(\delta_{a,b} \mid \theta = b) = 0$

$\Rightarrow$ $R_{\delta'}(\theta)_{=b} \leq 0 \Rightarrow R_{\delta'}(\theta)_{=b} = 0$ $\left(\because R_\delta(\theta) \geq 0 \; \forall \theta\right)$

$\Rightarrow E\left((\theta - \delta')^2\right) = 0$

$$\Rightarrow \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(b - \delta'(y_1, \ldots, y_n)\right)^2 f_{Y_1}(y) \ldots f_{Y_n}(y)\, dy_1 \ldots dy_n = 0$$

Now $f_{Y_k}(y_k) \geq 0 \; \forall k$ since $f_{Y_k}(y_k)$ is the normal pdf.

$\left(b - \delta'(y_1, \ldots y_n)\right)^2 \geq 0$ since it is a square

So only way for the integral to be zero is for $b - \delta'(y_1, \ldots, y_n) = 0$ identically for all $\underline{y}$

$\Rightarrow \delta'(y_1, \ldots, y_n) = b \; \forall (y_1, \ldots, y_n) \in \mathbb{R}^n$

$\Rightarrow \delta' = \delta_{a=0,b}$

But this is same as $\delta_{a=0,b}$!

$\therefore$ contradiction. There is no δ' better than $\delta_{a=0,b}$.

$\Rightarrow \delta_{a=0,b}$ is admissible

Combining both cases, $\delta_{a,b}$ is admissible $\forall \, 0 \leq a < 1$

(3) $0 \le \bar{Y} \le 1$ $\quad (\because 0 \le Y_i \le 1$
$\Rightarrow 0 \le \sum_{i=1}^{n} Y_i \le n$
$\Rightarrow 0 \le \bar{Y} \le 1)$

case i) If $a \ge 0$,

$0 \le a\bar{Y} \le 1$

$\Rightarrow b \le a\bar{Y} + b \le a+b$

we want this bounds to be within 0 and 1, so that $a\bar{Y}+b$ is in $[0, 1]$

i.e.

$0 \le b \le 1$ — ① and $\underbrace{1+b \ge b}_{\text{trivially true}}$

$0 \le a+b \le 1$ — ②

$\Rightarrow -a \le b \le 1-a$.

$\Rightarrow a \ge -b$ — ③ and $a \le 1-b$ — ④

eqn ③ is unnecessary since ① $\Rightarrow b \ge 0$ and we assumed $a \ge 0$

So when $a \ge 0$: $0 \le b \le 1$ and $a \le 1-b$.

Similarly

case ii) If $a < 0$

$0 \le \bar{Y} \le 1$

$\Rightarrow b \ge a\bar{Y} + b \ge a+b$

Again want the bound b/w 0 and 1

$\Rightarrow 0 \le b \le 1$ — ⑤

$0 \le a+b \le 1$ — ⑥ $\Rightarrow a \ge -b$ — ⑦ and $a \le 1-b$ — ⑧

$\Rightarrow$ when $a < 0$, $a \ge -b$ and $0 \le b \le 1$.

$1-b>0$ from ⑤, & we know $a<0$ already
So eqn ⑧ is trivially true & satisfied.

From cases (i) & (ii)

$0 \geq a$.

$$\left(0 \leq a \leq 1-b \;\&\; 0 \leq b \leq 1\right) \cup \left(-b \leq a \leq 0 \;\&\; 0 \leq b \leq 1\right)$$

$$\Rightarrow -b \leq a \leq 1-b \text{ and } 0 \leq b \leq 1$$

$$\forall\ \bar{Y} \in D = [0,1]$$

④ From qn ①,

$$R_{\delta_{a,b}}(\theta) = \left((a-1)\theta + b\right)^2 + \frac{a^2 \sigma^2}{n}$$

$$= \left((a-1)\theta + b\right)^2 + \frac{a^2 (\theta)(1-\theta)}{n}$$

Since we are given $a<0$, we follow 1 b) ii) trick,

$$R_{\delta_{a,b}}(\theta) = (a-1)^2\left(\theta + \frac{b}{a-1}\right)^2 + \frac{a^2(\theta)(1-\theta)}{n}$$

$$= (a-1)^2\left(\theta - \frac{b}{1-a}\right)^2 + \frac{a^2 (\theta)(\theta,-\theta)}{n}$$

$a<0 \Rightarrow a-1 < -1$

$\Rightarrow (a-1)^2 > 1$

$\Rightarrow (1-a)^2\left(\theta - \frac{b}{1-a}\right)^2 > 1$

$\Rightarrow R_{\delta_{a,b}}(\theta) > \left(\theta - \frac{b}{1-a}\right)^2$

$\left(\because \frac{\theta(1-\theta)a^2}{n} > 0\right)$

For $\delta' = \frac{b}{1-a}$ (say) (defined since $a \neq 1$)

$\Rightarrow R_{\delta'}(\theta) = E\left(\theta - \frac{b}{1-a}\right)^2 = \left(\theta - \frac{b}{1-a}\right)^2$

$\Rightarrow R_{\delta'}(\theta) < R_{\delta_{a,b}}(\theta) \; \forall \theta.$

$\therefore$ δ' is better than $\delta_{a,b}$

$\Rightarrow$ $\delta_{a,b}$ with $0 < b < 1$ and $-b < a < 0$ is inadmissible

(5) For $Y_i \sim$ Bernoulli(θ),

$\frac{\alpha}{\alpha+\beta+n} + \frac{n\bar{Y}}{\alpha+\beta+n}$ is uniquely Bayesian for prior π = Gamma(α, β)

comparing with $a\bar{Y} + b$, we have.

$a = \frac{n}{\alpha+\beta+n}$ and $b = \frac{\alpha}{\alpha+\beta+n}$

$\Rightarrow \frac{b}{a} = \frac{\alpha}{n}$

$\Rightarrow \alpha = \frac{nb}{a}$ assuming $a \neq 0$

Subst. back,

$\frac{nb}{a} + \beta + n = \frac{n}{a}$

$\Rightarrow \beta = \frac{n}{a} - \frac{nb}{a} - n.$

For Gamma(α, β) to be a valid dist.,

$\alpha > 0$ and $\beta > 0$

Since $a, b > 0$, $\alpha > 0$

$\beta > 0 \Rightarrow \frac{n}{a} - \frac{nb}{a} - n > 0$

$\Rightarrow \frac{1-b}{a} - 1 > 0$

$\Rightarrow 1 - b - a > 0$ ($\because a > 0$, Denominator removed without flipping sign)

$\Rightarrow a + b < 1$

$\Rightarrow a < 1 - b$ which is also given.

$\therefore$ when $a \neq 0$, $0 < b < 1$ and $0 \leq a < 1-b$ we can find a prior Gamma(α, β) such that $\delta = a\bar{Y} + b$ is uniquely Bayesian

$\Rightarrow a\bar{Y} + b$ is admissible remaining

corner case : $a=0, \ 0<b<1$

$$R_{\delta^*_{a,b}}(\theta) = (b-\theta)^2 \quad \text{and} \quad \delta_{a=0,b} = b$$

~~If possible let δ' be better than $\delta_{a=0,b}$~~

If possible let δ' be better than $\delta_{a=0,b}$

$\Rightarrow R_{\delta'}(\theta) \leq R_{\delta_{a=0,b}}(\theta) \ \forall \ \theta \in \Omega$

$R_{\delta'}(\theta_0) < R_{\delta_{0,b}}(\theta)$

At $\theta = b$, $\quad R_{\delta_{a=0,b}}(\theta = b) = 0$

$R_{\delta}(\theta) < 0$ not possible

$\Rightarrow R_{\delta'}(\theta) = 0 \ @ \ \theta = b$

$\Rightarrow E\left((\theta - \delta')^2\right) = 0$

$$\Rightarrow \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (b - \delta'(y_1, \dots, y_n))^2 f_{Y_1}(y_1) \cdots f_{Y_n}(y_n) \, dy_1 \cdots dy_n = 0$$

$f_{Y_k}(y_k) > 0 \ \forall \ k$ since $f_{Y_k}(y_k)$ is pdf of Normal distribution.

$\Rightarrow \delta'(y_1, \dots, y_n) = b \ \forall \ (y_1, \dots, y_n) \in \mathbb{R}^n$

$\Rightarrow \delta'(Y) = \delta_{a=0,b}$

Contradiction !

$\Rightarrow$ There is no δ' better than $\delta_{a=0,b}$

$\therefore \delta_{a=0,b}$ is admissible

d) $a = 0$, $0 < b < 1$

From both the cases,

$\delta_{a,b} = a\bar{Y} + b$ is admissible when

$0 \leq a < 1 - b$ and $0 < b < 1$

(6) a) $Y_i \sim N(0, \sigma^2)$

So Y_i equal to σZ_i

$E(Z_i) = 0$, $var(Z_i) = \sigma^2$

~~a~~ Z_i $\sigma Z_i \sim Normal(0,1)$

$\Rightarrow \sigma Z_i \sim N(0, \sigma^2)$

~~Eha~~ $E(Y_i^4) = E(\sigma^4 Z_i^4)$

$= \sigma^4 E(Z_i^4)$

$= 3\sigma^4$ (given)

$E(Y_i^2) = E(\sigma^2 Z_i^2)$

$= \sigma^2 E(Z_i^2)$

$= \sigma^2$

$E(Y_i) = 0$ (given)

$var(Y_i^2) = E(Y_i^4) - (E(Y_i^2))^2 = 2\sigma^4$

$R_{\delta_{a,b}} = E(L(\theta, \delta_{a,b}(Y_1, \ldots, Y_n)))$

$$= E\left((\theta - \delta_{a,b}(y_1, \ldots, y_n))^2\right)$$

$$= \cancel{E\{(\theta - aY}$$

$$= E\left((\sigma^2 - (a s^2 + b)\right)^2$$

$(\because$ given $\theta = \sigma^2$ and

$\delta_{a,b}(y_1, \ldots, y_n = a S^2 + b.$

$$\Rightarrow R_{\delta_{a,b}}(\sigma^2) = \left(E(\sigma^2 - a S^2 - b)\right)^2 + \mathrm{var}(\sigma^2 - a S^2 - b)$$

$$= \left(E(\sigma^2 - a s^2 - b)\right)^2 + a^2 \mathrm{var}(S^2)$$

$$E(S^2) = \frac{E(\Sigma Y_i^2)}{\cancel{n}} = \cancel{\frac{1}{n}} E(\Sigma Y_i^2)$$

Y_i = iid and $E(Y_i^2) = \sigma^2$ (shown)

$$\Rightarrow E(\Sigma Y_i^2) = \Sigma E(Y_i^2) \quad (\because \text{iid})$$

$$= n\sigma^2$$

$$\therefore E(S^2) = a\sigma^2$$

$\cancel{E(S^4)}$.

$$\mathrm{var}(S^2) = \mathrm{var}(\Sigma Y_i^2)$$

y_i are iid

$$= \cancel{\mathrm{var}} \cdot \Sigma(\mathrm{var}(Y_i^2))$$

$$= \sum_{i=1}^{n} 2\sigma^4$$ (computed at the start)

$$= 2n\sigma^4$$

$$\therefore R_{\delta_{a,b}}(\sigma^2) = (\sigma^2 - aE(S^2) - b)^2 + a^2 \operatorname{var}(S^2)$$

$$= (\sigma^2 - na\sigma^2 - b)^2 + a^2 2n\sigma^4.$$

$$\Rightarrow R_{\delta_{a,b}}(\sigma^2) = 2na^2\sigma^4 + ((an-1)\sigma^2 + b)^2$$

b) $R_{\delta_{a=0,b=0}}(\sigma^2) = \sigma^4$

Consider class of estimators $\delta_{a,b=0} = aS^2$

$$R_{\delta_{a,b=0}}(\sigma^2) = 2na^2\sigma^4 + (an-1)^2\sigma^4$$

$$\frac{\partial R}{\partial a} = 4an\sigma^4 + 2an\sigma^4 - 2\sigma^4 = 0$$

$$\Rightarrow a = \frac{1}{3n}$$

Since R_δ is a quadratic in a^2 with +ve coefficient for a^2, the stationary point is a minimum.

Let $\delta' = \cancel{\frac{\sigma^2}{3}} \frac{S^2}{3n}$.

Substituting

$$R_{\delta'}(\sigma^2) = \frac{2}{9n}\sigma^4 + \left(\frac{2}{3}\sigma^2\right)^2$$

$$= \frac{2}{9n}\sigma^4 + \frac{4}{9}\sigma^4$$

$$\frac{2}{9n} \leq \frac{2}{9} \quad \forall n \geq 1$$

$$\Rightarrow R_{\delta'}(\sigma^2) \leq \frac{2}{9}\sigma^4 + \frac{4}{9}\sigma^4$$

$$\leq \frac{2}{3}\sigma^4 < \sigma^4 = R_{\delta_{a=b=0}}(\sigma^2)$$

$$\Rightarrow R_{\delta'}(\sigma^2) < R_{\delta_{a=0,b=0}}(\sigma^2) \quad \forall \sigma^2, n.$$

$\Rightarrow$ δ' is better than $\delta_{a=0,b=0}$

$\Rightarrow$ $\delta_{a=0,b=0}$ is inadmissible.