ISyE 6412A **Theoretical Statistics** **Fall 2023**

HW #8 (due at Canvas midnight on Wednesday, Nov 1, ET)

(After you spent at least 30 minutes per question, please take a look at the hints on the second page.)

1. (**Modified from problem 7.37 of our text**). Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution on the interval $(-\theta, 2\theta)$, $\theta > 0$. That is, the X_i's are iid with pdf $f_\theta(x) = \frac{1}{3\theta}\mathbf{1}\{-\theta < x < 2\theta\}$ for $\theta > 0$. Find, if one exists, a best unbiased estimator of θ.

2. (**Problem 7.55(b) of our text**). Let $X_1, \ldots, X_n$ be a random sample from the pdf $f_\theta(x) = e^{-(x-\theta)}$ for $x > \theta$, where $-\infty < \theta < \infty$. Find the best unbiased estimator of $\phi(\theta) = \theta^r$ for some constant $r \geq 1$ (here r might or might not be an integer).

3. (**Problem 7.57 of our text**) Let $X_1, \ldots, X_{n+1}$ be iid Bernoulli(p), and define the function $h(p)$ by $h(p) = \mathbf{P}\Big(\sum_{i=1}^n X_i > X_{n+1} | p\Big)$, the probability that the first n observations exceed the $(n+1)$st.

   (a) Show that
   $$\delta(X_1, \ldots, X_{n+1}) = \begin{cases} 1, & \text{if } \sum_{i=1}^n X_i > X_{n+1}; \\ 0, & \text{otherwise.} \end{cases}$$
   is an unbiased estimator of $h(p)$.

   (b) Find the best unbiased estimator of $h(p)$.

4. (**Motivated from Problem 7.59 of our text**). Let $X_1, \ldots, X_n$ be iid $N(\mu, \sigma^2)$, where both μ and σ is unknown, i.e., $\theta = (\mu, \sigma)$. Find the best unbiased estimators of
   (a) $\phi(\theta) = \sigma$; (b) $\phi(\theta) = \sigma^2$; and (c) $\phi(\theta) = \sigma^4$.

5. (**Modified from 6.31(c) & 7.60**). Let $X_1, \ldots, X_n$ be iid gamma(α, β) with $\alpha > 1$ known. That is, the pdf of X_i is
   $$f_\beta(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \qquad 0 \leq x < \infty$$
   for $\alpha > 1$ and $\beta > 0$. By Theorems 6.2.10 and 6.2.25, the statistic $T = \sum_{i=1}^n X_i$ is complete sufficient for β, and it is also well known that $T = \sum_{i=1}^n X_i$ has a Gamma$(n\alpha, \beta)$ distribution. To help you find the best unbiased estimator of $\phi(\beta) = 1/\beta$ when $\alpha > 1$ is known, we split it into the following steps.

   (a) Show that $\mathbf{E}_\beta\Big(\frac{1}{X_1}\Big) = \frac{1}{(\alpha-1)\beta}$, and conclude that $\delta = (\alpha - 1)/X_1$ is an unbiased estimator of $1/\beta$.

   (b) Prove the lemma: if U/V and U are independent random variables, then $\mathbf{E}(\frac{U}{V}) = \mathbf{E}(\frac{1}{V})/\mathbf{E}(\frac{1}{U})$.

   (c) Use Basu's Theorem and the lemma in (b) to show that under the setting of this question,
   $$\mathbf{E}\Big(\frac{1}{X_1}\Big|T\Big) = \mathbf{E}\Big(\frac{T}{X_1}\frac{1}{T}\Big|T\Big) = \frac{\mathbf{E}\Big(\frac{1}{X_1}\Big)}{\mathbf{E}\Big(\frac{1}{T}\Big)}\frac{1}{T}.$$

   (d) Find the best unbiased estimator of $\phi(\beta) = 1/\beta$.

6. (**Motivated from Problem 10.9 of our text**). Suppose that $X_1, \ldots, X_n$ are iid Poisson(θ). Find the best unbiased estimator of

   (a) $\phi_1(\theta) = e^{-\theta}$, the probability that $X = 0$.

   (b) $\phi_2(\theta) = \theta e^{-\theta}$, the probability that $X = 1$.

   (c) A preliminary test of a possible carcinogenic compound can be performed by measuring the mutation rate of microorganisms exposed to the compound. An experimenter places the compound in 15 petri dishes and records the following number of mutant colonies:

   10, 7, 8, 13, 8, 9, 5, 7, 6, 8, 3, 6, 6, 3, 5.

   Calculate the best unbiased estimators of $e^{-\theta}$, the probability that no mutant colonies emerge, and $\theta e^{-\theta}$, the probability that one mutant colony will emerge.

Hints: If you have already thought about each problem for at least 30 minutes, then please feel free to look at the hints. Otherwise, please try the problem first, as getting help from the hints takes away most of the fun.

Problem 1: you have found the complete sufficient statistic for θ in HW#7, right? Can you use it to find the best unbiased estimator?
Problem 2: Use the complete sufficient statistic T to construct an unbiased estimator. Using integration by parts,

$$\mathbf{E}_\theta(T^r) = \int_\theta^\infty n e^{n(\theta - t)} t^r dt = -t^r e^{n(\theta-t)}|_\theta^\infty + r\int_\theta^\infty e^{n(\theta-t)} t^{r-1} dt = \theta^r + \frac{r}{n}\mathbf{E}_\theta(T^{r-1}).$$

From this relation, can you see $\mathbf{E}_\theta(g(T)) = \theta^r$ and find an unbiased estimator $g(T)$ of θ^r?

Problem 3: Recall that X_{n+1} can only take two possible values: 0 or 1, and you may need to consider different cases of $\sum_{i=1}^{n+1} X_i = b$, depending on whether $b = 0, 1, 2$ or $b \geq 3$.

Problem 4: We have shown in class that $(\bar{X}, S^2)$ is the complete sufficient statistics. Also $U = (n-1)S^2/\sigma^2$ is χ^2_{n-1} distribution, and compute $E(U^{p/2}) = C_{p,n}$, which does not depend on $\theta = (\mu, \sigma)$. Hence, $(n-1)^{p/2} S^p / C_{p,n}$ is an unbiased estimator of σ^p. What happens to $p = 1, 2, 4$?

Problem 5: in part (a), note that

$$\mathbf{E}_\beta\left(\frac{1}{X_1}\right) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty \frac{1}{x} x^{\alpha-1} e^{-x/\beta} dx = \frac{1}{\Gamma(\alpha)\beta^\alpha}\Big(\Gamma(\alpha-1)\beta^{\alpha-1}\Big),$$

In part (b), note that U/V and $1/U$ are also independent.
Part (c) is for the illustration of the use of Basu' Theorem. The key is to prove that $\frac{T}{X_1}$ and T are independent. Please feel free to use the well-known fact on the conditional expectation that if U and W are independent then $\mathbf{E}(Ug(W)|W) = \mathbf{E}(U)g(W)$ for any real-valued function $g(\cdot)$. In particular, $\mathbf{E}(\frac{U}{W}|W) = \mathbf{E}(U)\frac{1}{W}$.
In part (d), you may need to compute the value $\mathbf{E}(1/T)$. Please feel free to use the fact that T has a Gamma$(n\alpha, \beta)$ distribution, and thus the conclusion in (a) applies to $\mathbf{E}_\beta(1/T)$ too except that α is replaced by $n\alpha$.
Also in part (d), you *may* or *may not* use the result in (c), depending on which method you are using: Method 1 (hunting) or Method 2 (Rao-Blackwell). Either approach will be fine for part (d), as long as your final answer is correct.

Problem 6: Use Theorem 6.2.25 to find the complete sufficient statistic T for Poisson distribution. In (a), the indicator variable $\delta = 1(X_1 = 0)$ is unbiased. In (b), the indicator variable $\delta = 1(X_1 = 1)$ is unbiased. Then we can use Method 2 (Rao-Blackwell) to find the best unbiased estimator $\sigma^* = \mathbf{E}(\delta|T)$.