ISyE 6412A **Theoretical Statistics** **Fall 2023**

HW #4 (due at Canvas midnight on Wednesday, September 20, ET)

(There are 6 questions, and please look at both sides.)

The hints to Problems 1, 2, 4, 5(b), and 6(c) can be found on page #3 of this pdf file.)

1. (**Finding minimax procedures, case I**). Suppose that Y is Bernoulli random variable with $\mathbf{P}_\theta(Y = 1) = \theta$ and $\mathbf{P}_\theta(Y = 0) = 1 - \theta$, and it is desired to guess the value of θ on the basis of X under the squared error loss function $L(\theta, d) = (\theta - d)^2$. Assume that the domain of θ is $\Omega = \{\frac{4}{9} \leq \theta \leq \frac{5}{9}\}$ (since we still want to estimate θ, the decision space D can still be $[0, 1]$). Find the minimax procedure under the squared error loss function when $\Omega = \{\frac{4}{9} \leq \theta \leq \frac{5}{9}\}$.

2. (**Finding minimax procedures, case II**). When finding the minimax procedures, sometimes the desired prior distribution might not necessarily put the point mass at the boundary of Ω. To illustrate this, let us consider the setting of the previous problem with a single Bernoulli random variable Y, but now we assume that the domain of θ is $\Omega = [\frac{1}{9}, \frac{8}{9}]$. Find the minimax estimator of θ under the squared error loss function $L(\theta, d) = (\theta - d)^2$ when $\Omega = [\frac{1}{9}, \frac{8}{9}]$. For that purpose, let us consider two kinds of Bayesian procedures.

   (a) Assume that θ has a prior distribution on two endpoints of Ω with probability mass function $\pi_a(\theta = \frac{1}{9}) = \frac{1}{2}$ and $\pi_a(\theta = \frac{8}{9}) = \frac{1}{2}$. For the corresponding Bayes procedure, denoted by δ_a, show that its Bayes risk $r_{\delta_a}(\pi_a) < \sup_{\theta \in \Omega} R_{\delta_a}(\theta)$, and thus this direction does not work.

   (b) Consider another prior distribution $\pi_b(\theta = \frac{2-\sqrt{2}}{4}) = \frac{1}{2}$ and $\pi_b(\theta = \frac{2+\sqrt{2}}{4}) = \frac{1}{2}$, which is a well-defined prior over $\Omega = [\frac{1}{9}, \frac{8}{9}]$, since $\frac{1}{9} < \frac{2-\sqrt{2}}{4} < \frac{2+\sqrt{2}}{4} < \frac{8}{9}$. Show that the corresponding Bayes procedure, denoted by δ_b, satisfies $r_{\delta_b}(\pi_b) = \sup_{\theta \in \Omega} R_{\delta_b}(\theta)$, and thus we can conclude that δ_b is minimax on $\Omega = [\frac{1}{9}, \frac{8}{9}]$.

3. (**Finding minimax procedures, case III**). When finding the minimax procedures, sometimes the desired prior distribution might not necessarily be symmetric! To see this, under the setting of Problem 1 with a single Bernoulli random variable Y, but we now assume that the domain of θ is $\Omega = [0, \frac{1}{2}]$. Find the minimax estimator of θ under the squared error loss function $L(\theta, d) = (\theta - d)^2$ when $\Omega = [0, \frac{1}{2}]$. To help you find the minimax estimator/procedure when $\Omega = [0, \frac{1}{2}]$, we can split into the following steps.

   (a) Assume that θ has a prior distribution on two endpoints of $\Omega = [0, \frac{1}{2}]$ with probability mass function $\pi(\theta = \frac{1}{2}) = \gamma$ and $\pi(\theta = 0) = 1 - \gamma$. Find the posterior distribution of θ given $Y = y$ for $y = 0$ or 1. [Note that your answer for $\pi(\theta | Y = 0)$ should be different from $\pi(\theta | Y = 1)$, and θ only has two values: 0 and $\frac{1}{2}$].

   (b) Show that the Bayes procedure under the assumption of part (a) is $\delta_B(0) = \frac{\gamma}{4-2\gamma}$ and $\delta_B(1) = \frac{1}{2}$.

   (c) Computing the risk function of δ_B and solving the equation $R_{\delta_B}(\theta = 0) = R_{\delta_B}(\theta = \frac{1}{2})$. Show that $\gamma = 2 - \sqrt{2}$. Thus the Bayes procedure in part (b) becomes

$$\delta^*(Y) = \begin{cases} \frac{\sqrt{2}-1}{2}, & \text{if } Y = 0; \\ \frac{1}{2}, & \text{if } Y = 1. \end{cases}$$

   (d) Show that the procedure δ^* in part (c) is minimax on $\Omega = [0, \frac{1}{2}]$.

4. (**Finding minimax procedures, case IV**). Sometimes the desired prior distribution for the minimax procedure might not exist, but we can use a sequence of priors to find the minimax procedures.

   (a) Let $\pi_k, k = 1, 2, \ldots,$ be a sequence of prior distributions on Ω. Let δ_k denote a Bayes procedure with respect to π_k, and define the Bayes risks of the Bayes procedures

   $$r_k = \int_{\Omega} R_{\delta_k}(\theta)\pi_k(\theta)d\theta.$$

   Show that if the sequence r_k converges to a real-valued number r and if δ_* is a statistical procedure with its risk function $R_{\delta_*}(\theta) \le r$ for all $\theta \in \Omega$, then δ_* is minimax.

   (b) Let $Y_1, \ldots, Y_n$ be iid $N(\theta, \sigma^2)$ with σ known. Consider estimating θ using squared error loss. Show that $\bar{Y} = (Y_1 + \cdots + Y_n)/n$ is a minimax procedure.

5. (**Finding minimax procedures, case V**). Sometimes the minimax properties can be extended from a smaller domain to a larger domain.

   (a) Let $Y_1, \ldots, Y_n$ be iid with distribution F and finite unknown expectation θ, where F belongs to a set $\mathcal{F}_1$ of distributions. Suppose we want to estimate θ under a given loss function $L(\theta, d)$. Show that if δ_* is a minimax procedure when F is restricted to some subset $\mathcal{F}_0$ of $\mathcal{F}_1$, and if $\sup_{F\in\mathcal{F}_0} R_{\delta_*}(F) = \sup_{F\in\mathcal{F}_1} R_{\delta_*}(F)$ (i.e., sup risk of δ_* over $\mathcal{F}_1$ is the same as sup risk over $\mathcal{F}_0$), then δ_* is also minimax when F is permitted to vary over $\mathcal{F}_1$.

   (b) Suppose that $Y_1, \ldots, Y_n$ are iid with unknown mean θ. We further assume that the Y_i's can take any values in the interval $[0, 1]$, i.e., $0 \le Y_i \le 1$, and that $\Omega = \{\theta : 0 \le \theta \le 1\}$ and $D = \{d : 0 \le d \le 1\}$. Show that

   $$\delta_* = \frac{\sqrt{n}}{1+\sqrt{n}}\bar{Y} + \frac{1}{1+\sqrt{n}}\frac{1}{2}$$

   is minimax for estimating θ under the squared error loss function.

6. (**Impact of Loss function on minimax properties**). In HW#1, we assume that $Y_1, \ldots, Y_n$ are iid normal $N(\theta, 1)$ with $\theta \in \Omega = (-\infty, \infty)$ and one of the proposed procedures of estimating θ is

   $$\delta_{3,n}(Y_1, \ldots, Y_n) = \frac{\sqrt{n}\ \bar{Y}_n}{1+\sqrt{n}}.$$

   (a) Under the loss function $L(\theta, d) = (\theta - d)^2/(1 + \theta^2)$, **show** that k can be chosen in the prior density $\pi_a(\theta) = \text{const.} \times (1+\theta^2)k^{-1}\phi(\theta/k)$, where ϕ is the standard normal density, in such a way that $\delta_{3,n}$ is Bayes relative to π_a.

   (b) **Show** that $\delta_{3,n}$ is actually minimax under the loss function $L(\theta, d) = (\theta - d)^2/(1 + \theta^2)$.

   (c) Let us still consider the problem of estimating the normal mean θ, but now under the squared error loss function $L(\theta, d) = (\theta - d)^2$. **Show** that $\delta_{3,n}$ is still Bayes but no longer minimax under the squared error loss function $L(\theta, d) = (\theta - d)^2$.

Hints for problems 1 and 2: you can consider a general case of finding the Bayes procedure when the prior distribution is given by $\pi(\theta = r) = \frac{1}{2}$ and $\pi(\theta = 1 - r) = \frac{1}{2}$ for some $0 \leq r \leq \frac{1}{2}$. The corresponding Bayes procedure is of the form $\delta(0) = u$ and $\delta(1) = 1 - u$, where u depends on r. Next, the risk function of this Bayes procedure is of the form

$$A\theta^2 - A\theta + B,$$

where $A = 4u - 1$ and $B = u^2$. When $r = \frac{4}{9}$, the coefficient $A > 0$, which allows us to prove the minimax properties in Problem 2. However, when $r = \frac{1}{9}$, the coefficient $A < 0$, which is the case in Problem 2(a). Meanwhile, problem 2(b) corresponds to the case of $u = 1/4$ and $A = 0$.

Hints for problem 4: In (a), it is possible that $r_k < r$ but $\lim_{k\to\infty} r_k = r$. In (b) consider the priors $N(\mu, \tau^2)$ for θ. We have shown in class that the Bayes risk of the corresponding Bayes procedure is $\frac{1}{n/\sigma^2 + 1/\tau^2}$, denoted by r_τ. What is $r = \lim_{\tau\to\infty} r_\tau$? What is the risk function of $\delta_* = \bar{Y}$? Does the condition in part (a) hold?

Hints for problem 5 (b): Let the subset $\mathcal{F}_0 =$ the set of Bernoulli distributions, and let $\mathcal{F}_1 =$ the class of all distribution functions F with $F(0) = 0$ and $F(1) = 1$, i.e., $0 \leq Y_i \leq 1$. We have shown in class that δ_* is minimax as F varies over $\mathcal{F}_0$ (i.e., Bayes with constant risk). In order to use part (a), we need to consider the risk function $R_{\delta_*}(F)$ when F varies over $\mathcal{F}_1$. Note that if $0 \leq Y_i \leq 1$, then $Y_i^2 \leq Y_i$ and $\mathbf{E}_F(Y_i^2) \leq \mathbf{E}_F(Y_i) = \theta$, with equality if $F \in \mathcal{F}_0$. Use this to prove that when F varies over $\mathcal{F}_1$, $\mathrm{Var}_F(Y_i) \leq \theta - \theta^2$, and thus $R_{\delta_*}(F) \leq \frac{1}{4(1+\sqrt{n})^2}$, where the right-hand side is the (constant) risk of δ_* over $\mathcal{F}_0$.

Hints for problem 6(c): will it be Bayes relative to the prior density $\pi_b(\theta) = C_1\pi_a(\theta)/(1+\theta^2) = k^{-1}\phi(\theta/k)$ under the squared error loss? What are $\sup_{\theta\in\Omega} R_{\delta_{3,n}}(\theta)$ and $\sup_{\theta\in\Omega} R_{\bar{Y}}(\theta)$ under the squared error loss?]