ISyE 6412A **Theoretical Statistics** **Fall 2023**

HW #5 (due at Canvas midnight on Wednesday, September 27, ET)

(There are 6 questions. The hints are available on the second page of this pdf file.)

1. Let $Y_1, \ldots, Y_n$ be an iid random sample from a population with unknown mean θ and a finite variance $\sigma^2 > 0$. In the problem of estimating $\theta \in \Omega = \mathbb{R}^1$ (real numbers) with $D = \mathbb{R}^1$ and $L(\theta, d) = (\theta - d)^2$ (squared error loss), consider a general statistical procedure of the form $\delta_{a,b} = a\bar{Y} + b$, where a and b are general constants.

   (a) Calculate the risk function of $\delta_{a,b} = a\bar{Y} + b$.

   (b) Show that $a\bar{Y} + b$ is **inadmissible** whenever (i) $a > 1$; or (ii) $a < 0$; or (iii) $a = 1, b \neq 0$.

2. In Problem 1, assume further that $Y_i \sim N(\theta, \sigma^2)$, where $\sigma^2 > 0$ is known. Show that $a\bar{Y} + b$ is **admissible** if $0 \leq a < 1$.

   Remark: Combining problem 1 and problem 2 studies the admissibility of $a\bar{Y} + b$ for the normal distribution for all cases except $\bar{Y}$ itself which will be shown to be admissible in class. Therefore, under the normality assumption, the procedure $\delta_{a,b} = a\bar{Y} + b$ is **admissible** if and only if (i) $0 \leq a < 1$ or (ii) $a = 1, b = 0$.]

3. In Problem 1, assume further that $Y_i \sim \textit{Bernoulli}(\theta)$, i.e., $\mathbf{P}_\theta(Y_i = 1) = 1 - \mathbf{P}_\theta(Y_i = 0) = \theta$. Assume that the statistician decides to restrict consideration to procedures of the form $\delta_{a,b} = a\bar{Y} + b$, and want to always yield only decisions in $D = [0, 1]$ regardless of the observed value $\mathbf{Y} = (Y_1, \cdots, Y_n)$. Show that the pair (a, b) has to be chosen to satisfy $0 \leq b \leq 1$ and $-b \leq a \leq 1 - b$.

4. In Problem 3 for Bernoulli distribution, when $0 < b < 1$ and $-b < a < 0$, is the procedure $\delta_{a,b} = a\bar{Y} + b$ admissible? Note that the variance $\sigma^2 = \theta(1 - \theta)$ depends on θ here.

5. In Problem 3 for Bernoulli distribution, show that when $0 < b < 1$ and $0 \leq a < 1 - b$, the procedure $\delta_{a,b} = a\bar{Y} + b$ is admissible.

   Remark: For the purpose of completeness, for Bernoulli distribution, it can be shown that $a\bar{Y} + b$ is admissible in the closed triangle $\{(a, b) : a \geq 0, b \geq 0, a + b \leq 1\}$, and it is inadmissible for the remaining values of a and b.

6. Let $Y_1, \cdots, Y_n$ be i.i.d. according to a $N(0, \sigma^2)$ density, and let $S^2 = \sum_{i=1}^n Y_i^2$. We are interested in estimating $\theta = \sigma^2$ under the squared error loss $L(\theta, d) = (\theta - d)^2 = (\sigma^2 - d)^2$ using linear estimator $\delta_{a,b} = aS^2 + b$, where a and b are constants. **Show that**

   (a) The risk of $\delta_{a,b}$ is given by

   $$R_{\delta_{a,b}}(\sigma^2) = \mathbf{E}_\sigma\Big(\sigma^2 - (aS^2 + b)\Big)^2 = 2na^2\sigma^4 + [(an - 1)\sigma^2 + b]^2.$$

   (b) The constant estimator $\delta_{a=0,b=0} = 0$ is inadmissible.

   Remark: this exercise illustrates the fact the constants are not necessarily admissible.

Hints for problem 1 (b) Find a better procedure than $\delta_{a,b} = a\bar{Y} + b$: try $\delta_{1,0} = \bar{Y}$ for case (i) ($a > 1$) or case (iii) ($a = 1$ and $b \neq 0$); and try some constant estimators for case (ii) ($a < 0$). In particular, when $a < 0$, we have $1 - a > 1$, and thus

$$\Big[(a-1)\theta + b\Big]^2 = \Big[(1-a)\theta - b\Big]^2 = (1-a)^2\Big[\theta - \frac{b}{1-a}\Big]^2 \geq \Big[\theta - \frac{b}{1-a}\Big]^2.$$

From this, can you guess the desired constant estimator?

Hints for problem 2 Show that $\delta_{a,b}$ is a Bayes procedure if $0 < a < 1$, and can you find μ, τ^2 (in term of a, b) so that $\delta_{a,b}$ is Bayes with respect to the prior distribution $\theta \sim N(\mu, \tau^2)$? Meanwhile, when $a = 0$, note that $\delta_{a=0,b}$ is the only estimator with zero risk at $\theta = b$, and have we done similar questions in part (h) of HW#1?

Hints for problem 3 it suffices to make sure that $\delta_{a,b} \in [0, 1]$ when $(Y_1, \cdots, Y_n) = (0, \cdots, 0)$ or $(1, \cdots, 1)$, why? Hints: where does the linear function achieve the minimum or maximum values?

Hints for problem 4 Here the variance $\sigma^2 = \theta(1-\theta)$, and it is a special case of problem #1(b).

Hints for problem 5 If $a = 0$, what is risk at $\theta = b$? If $0 < a < 1 - b$, what is the Bayes solution relative to the prior distribution $\pi(\theta) = \text{Beta}(\alpha, \beta)$ with $\alpha > 0$ and $\beta > 0$?

Hints for problem 6 in part (a), let $Y_i = \sigma Z_i$, where $Z_i \sim N(0, 1)$. For the standard normal distributions, we have

$$\mathbf{E}(Z_i) = 0, \mathbf{E}(Z_i^2) = 1, \mathbf{E}(Z_i^3) = 0, \mathbf{E}(Z_i^4) = 3.$$

From this, can you find the mean and variance of $W_i = Y_i^2$? The question can be reduced to the linear estimator of $\delta_{a,b} = a\sum_{i=1}^n W_i + b$ when estimating $\theta = \mathbf{E}_\theta(W_i) = \sigma^2$ under the squared error loss function.
In part (b), let $b = 0$, find a that minimizes the risk function, and such a will yield a better procedure.