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HW-06 ISYE6412

② a) $f(y_i) \sim N(\beta x_i, \sigma^2)$

$(\because y_i = \beta x_i + \varepsilon_i$, βx_i constant and $\varepsilon_i \sim N(0, \sigma^2))$

$$\therefore f_\theta(y) = \prod_{i=1}^{n} f(y_i)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\left(\frac{(y_i - \beta x_i)^2}{2\sigma^2}\right)\right)$$

$$= (2\pi\sigma^2)^{-n/2} \exp\left(-\sum_{i=1}^{n} \frac{(y_i - \beta x_i)^2}{2\sigma^2}\right)$$

$$= (2\pi\sigma^2)^{-n/2} \exp\left(-\sum_{i=1}^{n} \frac{(y_i - \bar{y})^2 + (\bar{y} - \beta x_i)^2}{2\sigma^2}\right)$$

$$= (2\pi\sigma^2)^{-n/2} \exp\left(-\sum \Sigma (y_i - \bar{y}\right.$$

$t_1 \triangleq \bar{y}$, $t_2 \triangleq \sum \frac{(y_i - \bar{y})^2}{n-1}$

$$\therefore f_\theta(y) = \underbrace{(2\pi\sigma^2)^{-n/2} \exp\left(-\left[\frac{(n-1)t_2 + \sum_{i=1}^{n} (t_1 - \beta x_i)^2}{2\sigma^2}\right]\right)}_{g(\theta, \underline{t})}$$

$\theta = (\beta, \sigma^2)$, $h(y) = 1$

$\Rightarrow f_\theta(y) = g(\theta, \underline{t})\, h(y)$

$\therefore$ 2 dimensional sufficient statistic $T = \left(\bar{y}, \frac{\sum(y_i-\bar{y})^2}{n-1}\right)$
(by factorization theorem)

b) $f_\theta(y, x) = m(x)\, f_\theta(y/x)$

$$= m(x_1, \ldots, x_n) \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(y_i-\beta x_i)^2}{2\sigma^2}\right)$$

$$= m(x_1, x_2, \ldots, x_n)\,(2\pi\sigma^2)^{-n/2} \exp\left(-\frac{(n-1)t_2 + \sum_{i=1}^{n}(t_1-\beta x_i)^2}{2\sigma^2}\right)$$

Here x_i is a R.V. so our old $g(\theta, t)$ is not really a g of only θ, t but also has x_i

$$\sum_{i=1}^{n}(t_1-\beta x_i)^2 = \sum_{i=1}^{n} t_1^2 + \beta^2 x_i^2 - 2\beta x_i$$

$$= n t_1^2 + \beta^2 \sum x_i^2 - 2\beta \sum x_i$$

$$= m(x_1, \ldots, x_n)\,(2\pi\sigma^2)^{-n/2} \exp\left(-\frac{\sum_{i=1}^{n} y_i^2 + \beta^2 x_i^2 - 2\beta x_i y_i}{2\sigma^2}\right)$$

Let $t_1 \triangleq \sum_{i=1}^{n} y_i^2$

$t_2 \triangleq \sum_{i=1}^{n} x_i^2$

$t_3 \triangleq \sum_{i=1}^{n} x_i y_i$

$$= m(x_1,\dots,x_n)\,(2\pi\sigma^2)^{-n/2}\exp\left(-\frac{(t_1+\beta^2 t_2-2\beta t_3)}{2\sigma^2}\right)$$

$$g(\theta,t) = (2\pi\sigma^2)^{-n/2}\exp\left(-\frac{(t_1+\beta^2 t_2-2\beta t_3)}{2\sigma^2}\right)$$

$$h(\underline{x}) = m(x_1,\dots,x_n)$$

$$\Rightarrow f_\theta(y,x) = g(\theta,t)\,h(\underline{x})$$

$\therefore$ T is sufficient by factorization thm

(3) $f_\theta(x_i) = \frac{1}{3i\theta}\,1(x_i > -i(\theta-1))\,1(x_i < i(2\theta+1))$

Joint density $= \prod_{i=1}^{n} f_\theta(x_i)$ ($\because$ X_i are indep.)

$$= \prod_{i=1}^{n}\frac{1(x_i > -i(\theta-1))\,1(x_i < i(2\theta+1))}{3i\theta}$$

$$= \prod_{i=1}^{n}\frac{1\left(\frac{x_i}{i} > -(\theta-1)\right)1\left(\frac{x_i}{i} < 2\theta+1\right)}{3i\theta}$$

Let $\prod_{i=1}^{n} 3i = \alpha$ (some constant)

$$= \frac{1}{\alpha\theta^n}\prod_{i=1}^{n}\left(1\left(\theta > 1-\frac{x_i}{i}\right)1\left(\theta > \left(\frac{x_i}{i}-1\right)\frac{1}{2}\right)\right)$$

$$= \frac{1}{\alpha\theta^n}\prod_{i=1}^{n} 1\left(\theta > \max\left(1-\frac{x_i}{i},\ \frac{1}{2}\left(\frac{x_i}{i}-1\right)\right)\right)$$

$$= \frac{1}{\alpha\theta^n} \mathbb{1}\left(\theta > \max\left(\max\left(1-\frac{x_1}{1}, \left(\frac{x_1}{1}-1\right)\frac{1}{2}\right), \ldots, \max\left(1-\frac{x_n}{n}, \left(\frac{x_n}{n}-1\right)\frac{1}{2}\right)\right)\right)$$

$$= \frac{1}{\alpha\theta^n} \mathbb{1}\left(\theta > \max\left(1-\frac{x_1}{1}, 1-\frac{x_2}{2}, \ldots, 1-\frac{x_n}{n}, \left(\frac{x_1}{1}-1\right)\frac{1}{2}, \ldots, \left(\frac{x_n}{n}-1\right)\frac{1}{2}\right)\right)$$

$$= \frac{1}{\alpha\theta^n} \mathbb{1}\left(\theta > \max\left(1-\frac{x_i}{i}\right)\right) \mathbb{1}\left(\theta > \max\left(\frac{x_i}{i}-1\right)\right)$$

$$t_1 = \min\left(\frac{x_i}{i}\right)$$

$$\Rightarrow -t_1 = -\min\left(\frac{x_i}{i}\right) = \max\left(-\frac{x_i}{i}\right)$$

$$\Rightarrow 1-t_1 = \max\left(1-\frac{x_i}{i}\right) \quad \forall\, 1 \le i \le n$$

$$t_2 = \max \frac{x_i}{i}$$

$$\rightarrow \frac{t_2-1}{2} = \frac{1}{2}\left(\max\left(\frac{x_i}{i}\right)-1\right) = \max\left(\frac{x_i}{i}-1\right) \times \frac{1}{2}$$

$$\therefore f_\theta(\underline{x}) = \underbrace{\frac{1}{\alpha\theta^n} \mathbb{1}\left(\theta > (1-t_1)\right) \times \mathbb{1}\left(\theta > \frac{t_2-1}{2}\right)}_{g(\theta, t)}$$

$$h(x) = 1$$

$$\Rightarrow f_\theta(\underline{x}) = h(x) \cdot g(\theta, t)$$

$\therefore T$ is sufficient by factorization thm where $T = (t_1, t_2) = \left(\min\left(\frac{x_i}{i}\right), \max\left(\frac{x_i}{i}\right)\right)$

b) Let $t^+ = \max\left(1-\frac{x_1}{1}, 1-\frac{x_2}{2}, \dots, 1-\frac{x_n}{n}, \left(\frac{x_1}{1}-1\right)\frac{1}{2}, \dots, \left(\frac{x_n}{n}-1\right)\frac{1}{2}\right)$

$= \max\left(1-t_1, \left(\frac{t_2}{2}-1\right)\right)$

$f_\theta(x_i) = \frac{1}{\alpha\theta^n} 1\left(\theta > \max\left(1-\frac{x_1}{1}, 1-\frac{x_2}{2}, \dots, 1-\frac{x_n}{n}, \left(\frac{x_1}{1}-1\right)\frac{1}{2}, \dots, \left(\frac{x_n}{n}-1\right)\frac{1}{2}\right)\right)$

$= \frac{1}{\alpha\theta^n} 1(\theta > t_x^+) \Rightarrow g(\theta, t) = \frac{1}{\alpha\theta^n} 1(\theta > t_x^+),\ h(x) = 1$

$\Rightarrow t_x^+$ is sufficient!

$f_\theta(y_i) = \frac{1}{\alpha\theta^n} 1\left(\theta > \max\left(1-\frac{y_1}{1}, \dots, 1-\frac{y_n}{n}, \left(\frac{y_1}{1}-1\right)\frac{1}{2}, \left(\frac{y_n}{n}-1\right)\frac{1}{2}\right)\right)$

$= \frac{1}{\alpha\theta^n} 1(\theta > t_y^+)$

$$\frac{f_\theta(x_i)}{f_\theta(y_i)} = \frac{\frac{1}{\alpha\theta^n} 1(\theta > t_x^+)}{\frac{1}{\alpha\theta^n} 1(\theta > t_y^+)} = \frac{1(\theta > t_x^+)}{1(\theta > t_y^+)}$$

As shown in class ratio of indicator functions is a constant iff the boundaries are same

i.e. $\frac{1(a<\theta<b)}{1(c<\theta<d)}$ is constant iff $a=c$, $b=d$.

$\Rightarrow \frac{f_\theta(x)}{f_\theta(y)}$ is constant iff $t_x^+ = t_y^+$

$\Rightarrow t^+ = \max\left(-\frac{x_1}{1}+1, \ldots, 1-\frac{x_n}{n}, \left(\frac{x_1}{1}-1\right)\frac{1}{2}, \ldots \left(\frac{x_n}{n}-1\right)\frac{1}{2}\right)$

is a minimal sufficient statistic by Lehman Scheffe thm.

(4) b) $f_\theta(x_i) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-1}{2\mu}(x_i-\mu)^2\right)$

Joint dist

$$f_\theta(\underline{x}) = \prod_{i=1}^{n} f_\theta(x_i).$$

$$= (2\pi\sigma^2)^{-n/2} \exp\left(\sum_{i=1}^{n} \frac{-(x_i-\mu)^2}{2\mu}\right)$$

$$= (2\pi\sigma^2)^{-n/2} \exp\left(-\sum_{i=1}^{n} \frac{x_i^2}{2\mu^2} - \frac{\cancel{2}x_i}{\cancel{2}} + \frac{\mu}{2}\right)$$

$$= (2\pi\sigma^2)^{-n/2} \exp\left(-\sum_{i=1}^{n} \left(\frac{x_i^2}{2\mu^2} + \frac{\mu}{2}\right)\right) \exp\left(\sum_{i=1}^{n} x_i\right)$$

$$t \triangleq \sum_i x_i^2$$

$$g(\theta, t) = (2\pi\sigma^2)^{-n/2} \exp\left(-\sum_{i=1}^{n} \frac{-n\mu}{2} + \frac{t}{2\mu^2}\right)$$

$$h(x) = \exp\left(\sum_{i=1}^{n} x_i\right)$$

$\Rightarrow f_\theta(\underline{x}) = g(\theta, t)\, h(x)$

$\therefore\ t = \sum_{i=1}^{n} x_i^2$ is a sufficient statistic by factorization theorem.

$f_\theta(x_i) = 2\pi$

d) $\cancel{f_\theta(\underline{x}) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{\sum (x_i - \mu)^2}{2\sigma^4}\right)}$

d) $f_\theta(x_i) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)$

$f_\theta(\underline{x}) = \prod_{i=1}^{n} (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)$

$= (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2}\left(\sum_{i=1}^{n} \frac{x_i^2 - 2\mu x_i + \mu^2}{\sigma^2}\right)\right)$

$= (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\left(\frac{\sum_{i=1}^{n} x_i^2 - 2\mu \sum_{i=1}^{n} x_i + n\mu^2}{2\sigma^2}\right)\right)$

$t_1 \triangleq \sum_{i=1}^{n} x_i,\quad t_2 \triangleq \sum_{i=1}^{n} x_i^2 \quad \theta \triangleq (\mu, \sigma^2)$

$= \underbrace{(2\pi\sigma^2)^{-n/2} \exp\left(-\left(\frac{t_2 - 2\mu t_1 + n\mu^2}{2\sigma^2}\right)\right)}_{g(\theta, t)}$

$h(x) = 1$

$\therefore f_\theta(\underline{x}) = g(\theta, t)\, h(x)$

By factorization theorem,

$T(\underline{x}) = \left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i^2\right)$ is a sufficient statistic.

⑤ a) $f(x_i|\theta) = \frac{1}{\sqrt{\pi}} e^{-\frac{(x_i-\theta)^2}{2}}$

$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x_i^2+\theta^2-2x_i\theta)}{2}\right)$

$f_\theta(\underline{x}) = \prod_{i=1}^{n} f_\theta(x_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x_i^2+\theta^2-2x_i\theta)}{2}\right)$

$= (2\pi)^{-n/2} \exp\left(-\frac{\sum_{i=1}^{n}(x_i-\theta)^2}{2\sigma^2}\right)$

$= (2\pi)^{-n/2} \exp\left(-\sum_{i=1}^{n} \frac{(x_i - \bar{x} + \bar{x} - \theta)^2}{2}\right)$

We note $\sum_i (x_i - \bar{x}) = \sum_i x_i - n\bar{x} = 0$

$\bar{x} \triangleq \sum_{i=1}^{n} \frac{x_i}{n}$.

$\Rightarrow f_\theta(\underline{x}) = (2\pi)^{-n/2} \exp\left(-\sum_i \frac{(x_i-\bar{x})^2}{2}\right) \exp\left(\frac{-n(\bar{x}-y)^2}{2}\right)$

$= (2\pi)^{-n/2} \exp\left(\frac{-n(\bar{x}-y)^2}{2}\right) \exp\left(-\frac{\sum_i (x_i-\bar{x})^2}{2}\right)$

$t \hat{=} \bar{x}$

$$g(\theta, t) = (2\pi)^{-n/2} \exp\left(\frac{-n(t-\mu)^2}{2}\right)$$

$$h(x) = \exp\left(-\sum_{i=1}^{n} \frac{(x_i - t)^2}{2}\right)$$

$$f_\theta(\underline{x}) = (2\pi)^{-n/2} \exp\left(\frac{-n(t-\mu)^2}{2}\right) \exp\left(-\sum_{i=1}^{n} \frac{(x_i - t)^2}{2}\right)$$

$$= g(\theta, t)\, h(x)$$

$\therefore$ By factorization theorem, $t = \bar{x}$ is sufficient

$$\frac{f_\theta(\underline{x})}{f_\theta(\underline{y})} = \frac{(2\pi)^{-n/2} \exp\left(\frac{-n(t_x-\theta)^2}{2}\right) \exp\left(-\sum_{i=1}^{n} \frac{(x_i - t_x)^2}{2}\right)}{(2\pi)^{-n/2} \exp\left(\frac{-n(t_y-\theta)^2}{2}\right) \exp\left(-\sum_{i=1}^{n} \frac{(y_i - t_y)^2}{2}\right)}$$

$$t_x = \frac{\sum_{i=1}^{n} x_i}{n}, \quad t_y = \sum_{i=1}^{n} \frac{y_i}{n}$$

$$\Rightarrow \frac{f_\theta(\underline{x})}{f_\theta(\underline{y})} = \exp\left[\frac{-n}{2}\left((t_y-\theta)^2 - (t_x-\theta)^2\right) - \frac{1}{2}\left(\sum_{i=1}^{n}\left((x_i - t_x)^2 + (y_i - t_y)^2\right)\right)\right]$$

This is a constant function of θ iff $(t_y - t_x) = 0 \Rightarrow t_y = t_x$.

$\therefore$ t is a minimal sufficient statistic by Lehmann Scheffe theorem.

b) $f_\theta(x_i) = e^{-(x_i-\theta)} \quad x_i > \theta$

$\cancel{f_\theta(\underline{x}) = \prod_{i=1}^{n} e^{-(x_i-\theta)} = \exp\left(-\sum_{i=1}^{n}(x_i-\theta)\right)}$

$= e^{-(x_i-\theta)} \, 1(x_i > \theta)$

$\Rightarrow f_\theta(\underline{x}) = \prod_{i=1}^{n} \exp(-(x_i-\theta)) \, 1(x_i > \theta)$

$= \exp\left(-\sum_{i=1}^{n}(x_i-\theta)\right) \prod 1(x_1 > \theta) 1(x_2 > \theta) \cdots 1(x_n > \theta)$

$\Rightarrow f_\theta(\cancel{x}) = \left(\exp\left(-\sum_{i=1}^{n} x_i\right)\right) \exp(n\theta) \, 1\left(\theta < \min_{\forall i} x_i\right)$

$= \exp\left(-\sum_{i=1}^{n} x_i\right) \exp(n\theta) \, 1(\theta < \min(x_1, x_2, \ldots, x_n))$

$t \triangleq \min(x_1, x_2, \ldots, x_n)$

$= \left[\exp\left(-\sum_{i=1}^{n} x_i\right)\right] \left[\exp(n\theta) \, 1(\theta < t)\right]$

$g(t;\theta) \triangleq \exp(n\theta) \, 1(\theta < t)$

$h(x) = \exp\left(-\sum_{i=1}^{n} x_i\right)$

$\Rightarrow f_\theta(\underline{x}) = h(x) \, g(t;\theta)$

By Factorization theorem ~~$t(x)$~~ t is a sufficient statistic

$$\frac{f_\theta(\underline{x})}{f_\theta(\underline{y})} = \frac{\exp\left(-\sum_{i=1}^{n} x_i\right) \exp(n\theta)\, 1(\theta < t_x)}{\exp\left(-\sum_{i=1}^{n} y_i\right) \exp(n\theta)\, 1(\theta < t_y)}$$

where,

$t_x = \min(x_1, x_2, \dots, x_n)$; $t_y = \min(y_1, \dots, y_n)$

$$\frac{f_\theta(\underline{x})}{f_\theta(\underline{y})} = \frac{\exp\left(-\sum_{i=1}^{n} x_i\right) 1(\theta < t_x)}{\exp\left(-\sum_{i=1}^{n} y_i\right) 1(\theta < t_y)}$$

which is constant wrt θ iff $t_x = t_y$

$\therefore$ t is a minimal sufficient statistic

d) $f_\theta(x_i; \theta) = \dfrac{1}{\pi(1+(x_i-\theta)^2)}$

$$f_\theta(\underline{x}) = \prod_{i=1}^{n} \frac{1}{\pi(1+(x_i-\theta)^2)}$$

$$= \frac{1}{\pi^n} \cdot \frac{1}{\prod_{i=1}^{n}(1+(x_i-\theta)^2)}$$

$\underline{t} \triangleq (x_{(1)}, x_{(2)}, \dots, x_{(n)})$ (order statistics)

& $\prod_{i=1}^{n}(1+(x_i-\theta)^2) = \prod_{i=1}^{n}(1+(x_{(i)}-\theta)^2)$

[$\because$ it is just rearranging terms]

$$\Rightarrow f(\underline{x}) = \frac{1}{\pi^n} \frac{1}{\prod_{i=1}^{n} (1+(x_{(i)}-\theta)^2)}$$

$$g(\theta, \underline{t}) = \frac{1}{\pi^n} \frac{1}{\prod_{i=1}^{n} (1+(x_{(i)}-\theta)^2)}$$

$h(\underline{x}) = 1$

$\therefore$ $\underline{t}$ = order statistics of $\underset{\sim}{x}$ $x_{(i)}$ is sufficient

Alternate proof:

$P_\theta(\underline{X} = \underline{x} \mid t(\underline{x}) = \underset{\sim}{x_0}) = \frac{1}{n!}$ (has to one of the permutation of $\underset{\sim}{x_0}$)

$\Rightarrow$ independent of θ

$\therefore$ $\underline{t}$ is sufficient

$$\frac{f_\theta(\underline{x})}{f_\theta(\underline{y})} = \frac{\frac{1}{\pi^n} \frac{1}{\prod_{i=1}^{n} (1+(x_{(i)}-\theta)^2)}}{\frac{1}{\pi^n} \frac{1}{\prod_{i=1}^{n} (1+(y_{(i)}-\theta)^2)}}$$

$$= \frac{\prod_{i=1}^{n} (1+(y_{(i)}-\theta)^2)}{\prod_{i=1}^{n} (1+(x_{(i)}-\theta)^2)}$$

Numerator and denominator are product of n quadratic eqns in θ

$\Rightarrow$ order ~~is 2n~~ of polynomial $= 2n$

~~For them to constant be independent~~

For the division of 2 polynomials in θ

to be independent of θ, both must be the same polynomial (or a scaled version of each other but since constant 1 is fixed that is not possible in this case)

$\Leftrightarrow$ all $2n$ roots of Numerator and denominator are same

$\Leftrightarrow \pm j + x_{(i)} = \pm j + y_{(i)}$ (★)

$\Leftrightarrow x_{(i)} = y_{(i)} \ \forall i$

$\frac{f_\theta(x)}{f_\theta(y)} = 1 \Leftrightarrow x_{(i)} = y_{(i)} \ \forall i$

$\Leftrightarrow t_x = t_y$

$\therefore$ x and y have the same order statistic

claim ★: ~~$\pm j + x_{(i)} = \pm$~~

~~proof for ★~~

If all roots are same, then because both are order statistics, the corresponding ith roots are same.

proof: - if possible let $\pm j + x_{(i)} = \pm j + y_{(k)}$

$\Leftrightarrow x_{(i)} = y_{(k)} \quad k \neq i$

case (i): WLOG $k < i$

then by pigeonhole principle $x_{(a)} = y_{(b)}$

But $x_{(a)} < x_{(i)} = y_{(k)} < y_{(b)} \quad a < i, \ b > k$

$\Rightarrow x_{(a)} < y_{(b)}$

$\Rightarrow$ contradiction

Similarly, by symmetry of x and y one can arrive at a contradiction when $k > i$.

$\Rightarrow i = k$

e) $$f_\theta(x_i) = \frac{1}{2} e^{-|x-\theta|}$$

$$f_\theta(x_i) = \begin{cases} \frac{e^{-(x-\theta)}}{2} & x_i > \theta \\ e^{-(\theta - x)} & x_i < \theta \end{cases}$$

$$\frac{e^{-(x_i-\theta)}}{2} 1(x_i > \theta) + \frac{e^{-(\theta - x_i)}}{2} 1(x_i < \theta)$$

$$f_\theta(\underline{x}) = \prod_{i=1}^{n} \frac{e^{-(x_i-\theta)} 1(x_i > \theta) + e^{-(\theta-x_i)} 1(x_i < \theta)}{2}$$

$$= \frac{1}{2^n} \left(e^{-(x_1-\theta)} 1(x_1>\theta) + e^{-(\theta-x_1)} 1(x_1<\theta)\right) \left(e^{-(x_2-\theta)} 1(x_2>\theta) + e^{-(\theta-x_2)} 1(x_2<\theta)\right) \cdots \left(e^{-(x_n-\theta)} 1(x_n>\theta) + e^{-(\theta-x_n)} 1(x_n<\theta)\right)$$

$= \frac{1}{2^n}$ using the fact $1(x_i > \theta) 1(x_i < \theta) = 0$

Once again we note order statistics of $\underline{x} \overset{t}{=} (x_{(1)}, x_{(2)}, \dots, x_{(n)})$

$$\frac{1}{2}\left(e^{-(x_1-\theta)}\mathbb{1}(x_1>\theta) + e^{-(\theta-x_1)}\mathbb{1}(\theta>x_1)\right)\cdots\left(e^{-(x_n-\theta)}\mathbb{1}(x_n>\theta) + e^{-(\theta-x_n)}\mathbb{1}(\theta<x_n)\right)$$

$$= \frac{1}{2}\left(e^{-(x_{(1)}-\theta)}\mathbb{1}(x_{(1)}>\theta) + e^{-(\theta-x_{(1)})}\mathbb{1}(\theta>x_{(1)})\right)\cdots\left(e^{-(x_{(n)}-\theta)}\mathbb{1}(x_{(n)}>\theta) + e^{-(\theta-x_n)}\mathbb{1}(\theta<x_{(n)})\right)$$

$$= g(\theta, t)$$

(By ~~assoc~~ commutativity of multiplication)

$h(x) = 1$

$\therefore t$ = order statistics of $\underline{x}$ is sufficient by factorisation

$$\frac{f_\theta(\underline{x})}{f_\theta(\underline{y})} = \frac{\left(\prod_{i=1}^{n} e^{-(x_i-\theta)}\mathbb{1}(x_i>\theta) + e^{-(\theta-x_i)}\mathbb{1}(x_i<\theta)\right)\times\frac{1}{2^n}}{\frac{1}{2^n}\left(\prod_{i=1}^{n} e^{-(y_i-\theta)}\mathbb{1}(y_i>\theta) + e^{-(\theta-y_i)}(\mathbb{1}(y_i<\theta))\right)}$$

$$= \frac{\prod_{i=1}^{n}\left(e^{-(x_{(i)}-\theta)}\mathbb{1}(x_{(i)}>\theta) + e^{-(\theta-x_{(i)})}\mathbb{1}(x_{(i)}<\theta)\right)}{\prod_{i=1}^{n}\left(e^{-(y_{(i)}-\theta)}\mathbb{1}(y_{(i)}>\theta) + e^{-(\theta-y_{(i)})}\mathbb{1}(y_{(i)}<\theta)\right)}$$

Let $x_{(a)} > \theta$ & $x_{(a-1)} \le \theta$ — case (i)

$y_{(b)} > \theta$ & $y_{(b-1)} \le \theta$ — case (ii)

if such 'a' doesn't exist $\Rightarrow$ $x_{(a)} \le \theta \ \forall a$ or $x_{(a)} \ge \theta \ \forall a$ case (iii)

~~$f_\theta(x)$~~ — e^{-}

case (I)

$$\frac{f(\underline{x})}{f(\underline{y})} = \frac{e^{-\left(-\sum_{i=1}^{a-1} x_{(i)} + (a-1)\theta\right)} e^{-\left(-(n-a)\theta + \sum_{i=a}^{n} x_{(i)}\right)}}{e^{-\left(-\sum_{i=1}^{b-1} y_{(i)} + (b-1)\theta\right)} e^{-\left(-(n-b)\theta + \sum_{i=b}^{n} y_{(i)}\right)}}$$

$$= \exp\left[-\left(\sum_{i=a}^{n} x_{(i)} - \sum_{i=1}^{a-1} x_{(i)} - \sum_{i=b}^{n} y_{(i)} + \sum_{i=1}^{b-1} y_{(i)} + \cancel{2a-2b} + 2(a-b)\theta\right)\right]$$

$a = b$ to eliminate θ

$$= \exp\left[-\left(\sum_{i=a}^{n} (x_{(i)} - y_{(i)}) + \sum_{i=1}^{a-1} (y_{(i)} - x_{(i)})\right)\right]$$

Now a and b are also functions of θ by defintis

to eliminate them we need $x_{(i)} = y_{(i)} \; \forall i$

case (ii)

$$\frac{f(\underline{x})}{f(\underline{y})} = \exp\left(+\sum_{i=1}^{n} x_{(i)} - \sum_{i=b}^{n} y_{(i)} + \sum_{i=1}^{b-1} y_{(i)} + (n-2b-n-1)\theta\right)$$

$\Rightarrow b = \frac{n+1}{2} \Rightarrow y_{(i)} \leq \theta \; \forall i$

$$\Rightarrow \frac{f(x)}{f(y)} = \exp\left(\sum_{i=1}^{n} (x_{(i)} - y_{(i)})\right)$$

$\therefore \; (x_{(i)} < \theta) \Rightarrow (y_{(i)} < \theta) \; \forall i$

case (iii) n

$$\frac{f(\underline{x})}{f(\underline{y})} = \exp\left(\sum_{i=1}^{n} x_i - \sum_{i=b}^{n} y_{(i)} + \sum_{i=1}^{b-1} y_{(i)} \right.$$
$$\left. (-n+2b-n-1)\right)\theta\Big)$$

$\Rightarrow b = \frac{n+1}{2}$

$\Rightarrow y_{(i)} \geq \theta \;\; \forall i$

$\therefore (x_{(i)} \geq \theta) \Rightarrow (y_{(i)} \geq \theta)$

By symmetry of x and y we can also show

(iv) $(y_{(i)} < \theta) \Rightarrow (x_{(i)} < \theta) \;\; \forall i$

(v) $(y_{(i)} > \theta) \Rightarrow (y_{(i)} > \theta) \;\; \forall i.$

Considering (i)-(v), the common solution is $x_{(i)} = y_{(i)}$

And if $x_{(i)} = y_{(i)}$, $\frac{f(\underline{x})}{f(\underline{y})} = 1$ - constant wrt θ.

$\therefore \frac{f(\underline{x})}{f(\underline{y})}$ is constant iff $x_{(i)} = y_{(i)}$

$\Rightarrow$ order statistic of $\underline{x}$ is a minimal sufficient statistic

(b) a) $f_\theta(x \mid N=n) = {}^nC_k \, p^{\sum x_i}$

each trials iid

$\Rightarrow f_\theta(\underline{x} = (x_1, x_2, \ldots, x_n) \mid N=n) = {}^nC_k \, \theta^{\sum x_i} (1-\theta)^{n-\sum x_i}$

where $k = \sum x_i$

$\Rightarrow f_\theta(\underline{x}) = E\left(f_\theta(\underline{x} = (x_1, x_2, \ldots, x_n)) \mid N=n\right)$

$\hookrightarrow$ over N

$$= \sum_{n=1}^{\infty} \left({}^nC_k \, \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{n-\sum_{i=1}^{n} x_i} \right) p_k$$

where $k = \sum x_i$

$\Rightarrow f_\theta(\underline{x}, n) = f(\underline{x} \mid N=n) \, P(N=n)$

$$= {}^nC_x \, \theta^x (1-\theta)^{n-x} \, p_n$$

where $x \triangleq \sum x_i$

$T \triangleq (x, n)$

$\cancel{\dfrac{f_\theta(\underline{y}, n)}{f_\theta(\underline{y}, n)}}$ $\quad g(\theta, (x, n)) = {}^nC_x \, \theta^x (1-\theta)^{n-x} \, p_n$

$h(x_1, x_2, \ldots, x_n) = 1$

$f_\theta(\underline{x}, n) = g(\theta, T) \cdot h(x_1, x_2, \ldots, x_n)$

$\Rightarrow f_\theta(\underline{x}, n)$

$\therefore T = (x, n)$ is a sufficient statistic

$$\frac{f_\theta(\underline{x}, n_1)}{f_\theta(\underline{y}, n_2)} = \frac{{}^{n_1}C_x \theta^x (1-\theta)^{n_1-x} p_{n_1}}{{}^{n_2}C_y \theta^y (1-\theta)^{n_2-y} p_{n_2}}$$

$$x = \sum_{i=1}^{n_1} x_i \; ; \; y = \sum_{i=1}^{n_2} y_i$$

independent $\forall \theta$ iff $\theta^x (1-\theta)^{n_1-x} = \theta^y (1-\theta)^{n_2-y}$

$$\Rightarrow \theta^{x-y} = (1-\theta)^{n_2+x-n_1-y}$$

$$\Rightarrow x-y = n_2+x-n_1-y = 0$$

$$\Rightarrow x=y \text{ and } n_1=n_2$$

$\therefore T = (X, N)$ is minimal sufficient statistic

$$P(N=n) = \begin{cases} p_n & \text{a } \cancel{\text{constant in}} \; n \in \{1,2,3,\dots\} \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow P(N=n)$ ~~is~~ does not depend on θ

$\Rightarrow N$ is an ancillary statistic

(b) $$E\left(\frac{X}{N}\right) = E_N\left(E_X\left(\frac{X}{N} \middle| N=n\right)\right)$$

$$= E\left(E\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)\right) = E\left(\frac{n\theta}{n}\right) = E(\theta)$$

$$= \theta$$

$$\Rightarrow E\left(\frac{X}{N}\right) = \theta$$

$$\text{var}_\theta\left(\frac{X}{N}\right) = E_N\left(\text{var}_X\left(\frac{X}{N}\middle| N=n\right)\right) + \text{var}_N\left(E_X\left(\frac{X}{N}\middle| N=n\right)\right)$$

$$= E\left(\text{var}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)\right) + \text{var}\left(E\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)\right)$$

$$= E\left(\frac{n\theta(1-\theta)}{n^2}\right) + \text{var}\left(\frac{n\theta}{n}\right)$$

$$= \theta(1-\theta)\, E\left(\frac{1}{N}\right) + 0$$

$$\Rightarrow \text{var}_\theta\left(\frac{X}{N}\right) = \theta(1-\theta)\, E\left(\frac{1}{N}\right)$$

① a) $S = [\theta - 1/2,\ \theta + 1/2]$

$\Omega = \{U(\theta - 1/2,\ \theta + 1/2) : -\infty < \theta < \infty\}$

$D = \{d : d \in \mathbb{R}\}$

$L = (\theta - d)^2$

b) $R_{\delta_0}(\theta) = E\left((\theta - d)^2\right) = E\left((\theta - \bar{x}_n)^2\right)$

$$= E\left(\theta^2 - 2\bar{x}_n\theta + \bar{x}_n^2\right)$$

$$= E\,\theta^2 - 2E(\bar{x}_n)\theta + E(\bar{x}_n^2)$$

for uniform β Rvs, $U[\alpha, \beta]$

$E(X) = \frac{\beta + \alpha}{2}$

$var(X)$ ~~$E(X^2)$~~ $= \frac{(\beta - \alpha)^2}{12}$

$\Rightarrow E(\bar{x}_n) = \frac{n\left(\theta - \frac{1}{2} + \theta + \frac{1}{2}\right)}{2} = n\theta$

~~E~~ $var(\bar{x}_n) = \frac{1}{12} \frac{\left(\theta + \frac{1}{2} - \theta + \frac{1}{2}\right)^2}{n}$

$= \frac{1}{12n}$

$E(\bar{x}_n^2) = var(\bar{x}_n) + (E(\bar{x}_n))^2$

$= \theta^2 + \frac{1}{12n}$

$\therefore R_{\delta_0}(\theta) = \theta^2 - 2\theta^2 + \theta^2 + \frac{1}{12n}$

$\Rightarrow R_{\delta_0}(\theta) = \frac{1}{12n}$

c) $f_\theta(x_i) = \frac{1}{\theta + \frac{1}{2} - (\theta - \frac{1}{2})} 1(x_i > \theta - \frac{1}{2}) \, 1(x_i < \theta + \frac{1}{2})$

$= 1(x_i > \theta - \frac{1}{2}) \, 1(x_i < \theta + \frac{1}{2})$

$= 1(\theta < x_i + \frac{1}{2}) \, 1(\theta > x_i - \frac{1}{2})$

$f_\theta(\underline{x}) = \prod_{i=1}^{n} f_\theta(x_i)$

$= \prod_{i=1}^{n} 1(\theta > x_i - \frac{1}{2}) \, 1(\theta < x_i + \frac{1}{2})$

$\Rightarrow f_\theta(\underline{x}) = \mathbb{1}\left(\theta > \max\left(x_1 - \tfrac{1}{2}, x_2 - \tfrac{1}{2}, \ldots, x_n - \tfrac{1}{2}\right)\right)$
$\times\ \mathbb{1}\left(\theta < \min\left(x_1 + \tfrac{1}{2}, x_2 + \tfrac{1}{2}, \ldots, x_n + \tfrac{1}{2}\right)\right)$

$= \mathbb{1}\left(\theta > \max(x_1, x_2, \ldots, x_n) - \tfrac{1}{2}\right) \mathbb{1}\left(\theta < \min(x_1, x_2, \ldots, x_n) + \tfrac{1}{2}\right)$

$x_{(1)} \triangleq \min(x_1, \ldots, x_n)$

$x_{(n)} \triangleq \max(x_1, \ldots, x_n)$

$\Rightarrow f_\theta(\underline{x}) = \mathbb{1}(\theta > x_{(n)} - \tfrac{1}{2})\ \mathbb{1}(\theta < x_{(1)} + \tfrac{1}{2})$

$g(\theta, \underline{t}^T) \triangleq \mathbb{1}(\theta > x_{(n)} - \tfrac{1}{2})\ \mathbb{1}(\theta < x_{(1)} + \tfrac{1}{2})$

$h(\underline{x}) \triangleq 1$

$\rightarrow f_\theta(\underline{x}) = g(\theta, \underline{T})\, h(\underline{x})$

$\therefore$ By factorization theorem, $\underline{T} = (x_{(1)}, x_{(n)})$ is a sufficient statistic for θ

(e) Substituting $b = \frac{1}{2} - \frac{a + n(1-a)}{n+1}$

$$R_{\delta_a}(\theta) = \left[a\frac{1}{n+1} + (1-a)\frac{n}{n+1} + \frac{1}{2} - \frac{a + n(1-a)}{n+1} - \frac{1}{2}\right]$$

$$+ \left[\frac{a^2 n + (1-a)^2 n + 2a(1-a)}{(n+1)^2(n+2)}\right]$$

$$= \frac{a^2 n + (1-a)^2 n + 2a(1-a)}{(n+1)^2 (n+2)}$$

$$\Rightarrow \frac{dR}{da} = \frac{2an - 2(1-a)n + 2 - 4a}{(n+1)^2 (n+2)}$$

$$\Rightarrow \frac{d^2R}{da^2} = a(2n+2n - 4) = 4n - 4 \geq 0 \quad \forall n > 1$$

$\Rightarrow$ 2nd derivative is +ve therefore if a stationary point exists it is a ~~maximum~~ minimum.

$$\frac{dR}{da} = 0 \Rightarrow 2an - 2(1-a)n + 2 - 4a = 0$$

$$\Rightarrow a(2n+2n-4) = 2n-2$$

$$\Rightarrow a^* = 1/2$$

$$b^* = \frac{1}{2} - \frac{\frac{1}{2} + n(1 - 1/2)}{n\frac{1}{2} + 1} = 0$$

$\therefore a^* = 1$, $b^* = 0 \Rightarrow \delta^*(x) = \frac{(x_{(1)} + x_{(n)})}{2}$

gives the uniformly smallest risk function

$$R_{\delta^*} = \frac{\frac{n}{4} + \frac{n}{4} + \frac{2}{4}}{(n+1)^2 (n+2)} = \frac{2n+2}{4(n+1)^2 (n+2)}$$

$\Rightarrow R_{\delta^*} = \dfrac{1}{2(n+1)(n+2)}$

f) $\cancel{\dfrac{1}{2(n+1)(n+2)}}$ $\quad R_{\delta^*} < R_{\delta_0}$

$\Rightarrow \dfrac{1}{2(n+1)(n+2)} < \dfrac{1}{12n}$

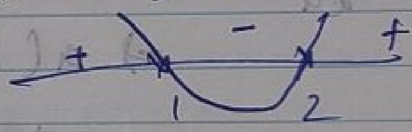
$\Rightarrow 2(n+1)(n+2) > 12n$ (∵ all terms are +ve)

$\Rightarrow 2n^2 + 6n + 2 > 12n$

$\Rightarrow n^2 - 3n + 2 > 0$

$\Rightarrow (n-2)(n-1) > 0$

$\Rightarrow n > 2$ or $n < 1$



$\Rightarrow n \geq 3$.

$\therefore R_{\delta^*} < R_{\delta_0} \ \forall\ n \geq 3,\ \forall\ \theta$

$\Rightarrow \delta^*$ is Better than δ_0

$\Rightarrow \delta_0$ is inadmissible when $n \geq 3$

d) $U_i \stackrel{d}{=} X_i - \theta + 1/2 \Rightarrow U_i =$ uniform $[0, 1]$

$\Rightarrow X_{(1)} = U_{(1)} + \theta - 1/2$

& $X_{(n)} = U_{(n)} + \theta - 1/2$

$$F\left(U_{(1)} \geq u,\ U_{(n)} \leq v\right) = P\left(\min(U_1, U_2, \ldots, U_n) \geq u,\ \max(U_1, U_2, \ldots, U_n) \leq v\right)$$

$$= P(U_1 \geq u, U_2 \geq u, \ldots, U_n \geq u, U_1 \leq v, \ldots, U_n \leq v)$$

$$= P(u \leq U_i \leq v\ \forall i)$$

since i.i.d

$$= \left(P(u \leq U_i \leq v)\right)^n$$

$$= \frac{(v-u)^n}{1} \quad \forall\ 0 \leq u \leq v \leq 1$$

$$\Rightarrow \int_0^v \int_u^v f(u,v)\, du\, dv = (v-u)^n.$$

$$\Rightarrow \int_u^1 f(u,v)\, du = n(v-u)^{n-1}$$

$$\Rightarrow -f(u,v)\, dv = -n(n-1)(v-u)^{n-2}$$

$$\Rightarrow f(u,v) = n(n-1)(v-u)^{n-2}$$

$$P(U_{(1)} \geq u) = \cancel{(1-u)}\ P(U_1, U_2, \geq u, \ldots, U_n \geq u)$$

$$= (1-u)^n \Rightarrow F(u) = 1-(1-u)^n.$$

$$\Rightarrow f_{U_{(1)}}(u) = \begin{cases} n(1-u)^{n-1} & \text{if } 0 \leq u \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$P(U_{(n)} \leq v) = P(U_1 \leq v, \ldots, U_n \leq v)$$

$$= (v-1)^n \Rightarrow F(v) = (v-1)^n$$

$$\Rightarrow f_{U_{(n)}}(v) = \begin{cases} n v^{n-1} & \text{if } 0 \leq v \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

$$E(U_{(1)}) = \int_0^1 u\, n(1-u)^{n-1}\, du$$

$$1-u = \alpha$$

$$= \int_0^1 n(1-\alpha)(\alpha)^{n-1}\, d\alpha$$

$$= n\left[\frac{1}{n} - \frac{1}{n+1}\right] = \frac{1}{n+1}$$

$$E(U_{(n)}) = \int_0^1 v\, n(v)^{n-1}\, dv = \frac{n}{n+1}$$

$$E((U_{(n)})^2) = \int_0^1 v^2\, n(v)^{n-1} = \frac{n}{n+2}$$

$$\Rightarrow \text{var}(U_{(n)}) = \frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2 = \frac{n(n+1)^2 - n^2(n+2)}{(n+1)^2(n+2)}$$

$$= \frac{n^3+2n^2+n-n^3-2n^2}{(n+1)^2(n+2)} = \frac{n}{(n+1)^2(n+2)}$$

$$E((U_{(1)})^2) = \int_0^1 u^2\, n(1-u)^{n-1}\, du$$

$$= \int_0^1 n(\alpha^2+1-2\alpha)(\alpha)^{n-1}\, d\alpha$$

$$= n\left(\frac{1}{n+2} + \frac{1}{n} - \frac{2}{n+1}\right)$$

$$\frac{2}{(n+2)(n+1)}$$

$$\therefore \ \mathrm{var}(U_{(1)}) = E((U_{(1)})^2) - (E(U_{(1)}))^2$$

$$= \frac{2}{(n+1)(n+2)(n+1)} - \left(\frac{1}{(n+1)}\right)^2$$

$$= \cancel{\frac{(n+1)^2 - (n+2)^2}{(n+1)^2(n+2)}}$$

$$= \frac{2(n+1) - (n+2)}{(n+1)^2(n+2)}$$

$$\Rightarrow \mathrm{var}(U_{(1)}) = \frac{n}{(n+1)^2(n+2)}$$

$$\mathrm{cov}(U_{(1)}, U_{(n)}) = E(U_{(1)}U_{(n)}) - E(U_{(1)})E(U_{(n)})$$

$$E(U_{(1)}U_{(n)}) = \int_0^1\int_u^1 uv(n)(n-1)(v-u)^{n-2}\,dv\,du$$

$$v - u = \alpha \Rightarrow v = \alpha + u.$$

$$= n(n-1)\int_0^1 u \int_0^{1-u} (\alpha+u)(\alpha)^{n-2}\,d\alpha.$$

$$= n(n-1)\int_0^1 u\left[\frac{(1-u)^n}{n} + \frac{u(1-u)^{n-1}}{n-1}\right]du.$$

$$= n(n-1)\left(\frac{1}{n^2(n+1)} + \frac{n}{n(n-1)(n+2)}\right).$$

$$\Rightarrow \operatorname{cov}(U_{(1)}, U_{(n)}) = \frac{1}{(n+1)^2(n+2)}$$

$$R_{\delta_{a,b}}(\theta) = E\left((\theta - d)^2\right)$$

$$= \left(E(\theta - \delta_{a,b})\right)^2 + \left(\operatorname{var}(\theta - \delta_{a,b})\right)$$

$$= \left(E\left(\theta - (aX_{(1)} + (1-a)X_{(n)} + b)\right)\right)^2$$

$$+ \operatorname{var}\left(\theta - (aX_{(1)} + (1-a)X_{(n)} + b)\right)$$

$E(X_{(1)}) = E(U_{(1)}) + \theta - 1/2$; $E(X_{(n)}) = E(U_{(n)}) + \theta - 1/2$

$\operatorname{var}(U_{(1)}) = \operatorname{var}(X_{(1)})$;

$\operatorname{var}(U_{(n)}) = \operatorname{var}(X_{(n)})$

$$\operatorname{cov}(X_{(1)}, X_{(n)}) = \operatorname{cov}\left((U_{(1)} + \theta - 1/2)(U_{(n)} + \theta - 1/2)\right)$$

$$= E\left(U_{(1)}U_{(n)} + (\theta - 1/2)^2 + U_{(1)}(\theta - 1/2) + (\theta - 1/2)U_{(n)}\right)$$

$$- E\left(U_{(1)} + \theta - 1/2\right)E\left(U_{(n)}(\theta - 1/2)\right)$$

$$= E(U_{(1)}U_{(n)}) - E(U_{(1)})E(U_{(n)})$$

$$= \operatorname{cov}(U_{(1)}, U_{(n)})$$

$$\Rightarrow \operatorname{cov}(X_{(1)}, X_{(n)}) = \operatorname{cov}(U_{(1)}, U_{(n)})$$

$$\text{var}(a X_{(1)} + (1-a) X_{(n)}) = a^2 \,\text{var}(X_{(1)}) + (1-a)^2 \,\text{var}(X_{(n)}) + 2(a)(1-a)\,\text{cov}(X_{(1)}, X_{(n)})$$

$$\Rightarrow R_{\delta_{a,b}}(\theta) = \left(\theta - \left[\frac{a}{n+1} + \theta - \frac{1}{2} + \frac{n(1-a)}{n+1} + b\right]\right)^2$$

~~$\text{var}(aX_{(1)} + b(1-a)X_{(n)})$~~

$$+ \frac{a^2 n}{(n+1)^2(n+2)} + \frac{(1-a)^2 n}{(n+1)^2(n+2)} + \frac{2(a)(1-a)}{(n+1)^2(n+2)}$$

$$\Rightarrow R_{\delta_{a,b}}(\theta) = \left[\left(\frac{a}{n+1} + \frac{(1-a)n}{n+1} + b - \frac{1}{2}\right)\right]^2 + \frac{a^2 n + (1-a)^2 n + 2a(1-a)}{(n+1)^2(n+2)}$$

For given 'a',

$$\frac{\partial R}{\partial b} = 2\left[\frac{a}{n+1} + \frac{(1-a)n}{n+1} + b - 1/2\right]$$

$$\frac{\partial^2 R}{\partial^2 b} = 2 > 0$$

$\therefore$ stationary point is a minimum

$$\frac{\partial R}{\partial b} = 0 \Rightarrow b^* = \frac{1}{2} - \frac{a + n(1-a)}{n+1}$$