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ISYE 6412-A , HOMEWORK - 1

(1) a) data - X - normal random variable mean θ variance 1

$$X \sim N(\theta, 1)$$

$S = \mathbb{R}$ (entire real line)

$\Rightarrow X \in (-\infty, \infty)$

$$\Omega \cdot = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\theta}{1}\right)^2} \quad \theta \in \mathbb{R}$$

$$\cancel{\Omega = \{\theta : \theta \in \mathbb{R}\}} = \{\theta : \theta \in \mathbb{R}\}$$

$$D = \{d : d \in \mathbb{R}\}$$

$$L = \frac{(\theta - d)^2}{1+\theta^2}$$

b) $R(\theta, \delta) = E_\theta(L_\theta(\theta, \delta_1))$

$$R(\theta, \delta_1) = E_\theta\left(\frac{(\theta - X)^2}{1+\theta^2}\right)$$

$$= \frac{1}{1+\theta^2} E_\theta\left((\theta - X)^2\right)$$

$$= \frac{\text{Var}(X)}{1+\theta^2} = \boxed{\frac{1}{1+\theta^2}}$$

$$R(\theta, \delta_2) = E\left[\frac{\left(\theta - \left(\frac{1+x}{2}\right)\right)^2}{1+\theta^2}\right]$$

$$= \frac{1}{1+\theta^2} E\left(\left(\theta - \left(\frac{1+x}{2}\right)\right)^2\right)$$

$$= \frac{\cancel{2}\,4\,1}{4(1+\theta^2)} E\left(\left((4\theta - 1) - x\right)^2\right)$$

$$= \frac{1}{4(1+\theta^2)} E\left((4\theta-1)^2 + x^2 - 2(4\theta-1)x\right)$$

$$= \frac{1}{4(1+\theta^2)}\left[(4\theta-1)^2 + E(x^2) - 2(4\theta-1)E(x)\right]$$

$$= \frac{1}{4(1+\theta^2)}\left[(4\theta-1)^2 + E(x^2) - (E(x))^2 + (E(x))^2 - 2(4\theta-1)E(x)\right]$$

$$= \frac{1}{4(1+\theta^2)}\left[(4\theta-1)^2 + 1 + \theta^2 - 2\theta(4\theta-1)\right]$$

$$= \frac{1}{4(1+\theta^2)}\left(16\theta^2 + 1 + 8\theta + 1 + \theta^2 - 8\theta^2 + 2\right)$$

$$= \frac{1}{4(1+\theta^2)}\left(9\theta^2 + 8\theta + 3\right)$$

$$R(\theta, \delta_3) = E\left(\left(\theta - \frac{x}{2}\right)^2\right) \cdot \frac{1}{1+\theta^2}$$

$$= E\left(4\theta^2 + x^2 - 4x\theta\right) \frac{1}{(1+\theta^2)4}$$

$$= \frac{1}{4(1+\theta^2)}\left(4\theta^2 + E(X^2) - \theta^2 + \theta^2 - 4\theta^2\right)$$

$$= \frac{1}{4(1+\theta^2)}\left(4\theta^2 + 1 + -3\theta^2\right)$$

$$= \boxed{\frac{1}{4}}$$

$$R(\theta, \delta_4) = \frac{E\left((\theta - 2x)^2\right)}{(1+\theta^2)} = \frac{E\left(\theta^2 + 4x^2 - 4x\theta\right)}{1+\theta^2}$$

$$= \frac{1}{1+\theta^2}\left(\theta^2 + 4E(x^2) - 4\theta^2 + 4\theta^2 - 4\theta E(x)\right)$$

$$= \frac{1}{1+\theta^2}\left(\theta^2 + 4 + 4\theta^2 - 4\theta^2\right)$$

$$= \boxed{\frac{\theta^2+4}{\theta^2+1}}$$

$$R(\theta, \delta_5) = \frac{E(\theta^2)}{1+\theta^2} = \boxed{\frac{\theta^2}{1+\theta^2}}$$

$$R(\theta, \delta_6) = \frac{E\left((\theta-1)^2\right)}{1+\theta^2} = \boxed{\frac{(\theta-1)^2}{1+\theta^2}}$$

① c) From the graph,

$\delta_1, \delta_2, \delta_3, \delta_5, \delta_6$ are better than δ_4,
i.e.

$$R_{\delta_k}(\theta) \leq R_{\delta_4}(\theta) \quad \forall k \in \{1,2,3,5,6\}$$

and $R_{\delta_k}(\theta_0) < R_{\delta_4}(\theta_0)$ for some θ_0
$\forall k \in \{1,2,3,5,6\}$

$\Rightarrow$ δ_4 is inadmissible

Similarly δ_3 and δ_5 are better than δ_6

$\Rightarrow$ δ_6 is inadmissible

d) δ_3 because it has the least risk function for all θ on an average (is always 0.25)

$\delta_3 = x/2$

e) $R(\theta, \delta) = E_\theta ($

Consider a general $\delta = a + b\bar{X}_n$

$$R(\theta, \delta) = \frac{E_\theta\left((\theta - a - b\bar{X}_n)^2\right)}{1+\theta^2}$$

$$= \frac{E_\theta\left(b^2\bar{X}_n^2 + a^2 + \theta^2 - 2a\theta - 2b\theta\bar{X}_n + 2ab\bar{X}_n\right)}{1+\theta^2} \quad \text{—— (1)}$$

$$\text{var}(\bar{X}_n) = \frac{1}{n} \Rightarrow E(\bar{X}_n^2) - (E(\bar{X}_n))^2 = \frac{1}{n}$$

$$\Rightarrow E(\bar{X}_n^2) = \frac{1}{n} + \theta^2 \quad \text{—— (2)}$$

$$(1) \Rightarrow R = \frac{b^2 E_\theta(\bar{X}_n^2) + a^2 + \theta^2 - 2a\theta - 2b\theta E(\bar{X}_n) + 2abE(\bar{X}_n)}{1+\theta^2}$$

(using (2)) $$= \frac{1}{1+\theta^2}\left(b^2\left(\theta^2 + \frac{1}{n}\right) + a^2 + \theta^2 - 2a\theta - 2b\theta^2 + 2ab\theta\right)$$

$$= \frac{1}{1+\theta^2}\left(\theta^2(b^2 - 2b + 1) + \theta(2ab - 2a) + a^2 + b^2/n\right)$$

e) contd.

$$\Rightarrow R(\theta,\delta) = \frac{1}{1+\theta^2}\left(\theta^2(b-1)^2 + \theta(2a)(b-1) + a^2 + \frac{b^2}{n}\right)$$

$\delta_{1,n}$: $a=0,\ b=1$

$$\Rightarrow R(\theta,\delta_{1,n}) = \cancel{\frac{a^2+b^2/n}{}} \quad \frac{1}{n(\theta^2+1)}$$

$$\Rightarrow R(\theta,\delta_{1,n}) = \frac{1}{n(\theta^2+1)}$$

$$\delta_{2,n} = \frac{\bar{X}_n + \frac{1}{n}}{1+\frac{1}{n}}$$

$$= \left(\frac{n}{n+1}\right)\bar{X}_n + \frac{1}{n+1}$$

$a = \frac{1}{n+1}$, $b = \frac{n}{n+1}$

$$R(\theta,\delta_{2,n}) = \frac{1}{1+\theta^2}\left(\theta^2\left(\frac{n}{n+1}-1\right)^2 + \theta\left(\frac{2}{n+1}\right)\left(\frac{n}{n+1}-1\right) + \frac{1}{(n+1)^2} + \frac{n^2}{(n+1)^2 n}\right)$$

$$= \frac{1}{(1+\theta^2)}\left(\frac{\theta^2}{(n+1)^2} - \frac{2\theta}{(n+1)^2} + \frac{1}{(1+n)}\right)$$

~~$R_\theta(\theta, \delta_{3,n})$~~

$$\delta_{3,n} = \frac{\sqrt{n}\,\bar{X}_n}{1+\sqrt{n}} \; \cancel{\sqrt{n}\bar{X}_n}$$

$$a = 0, \; b = \frac{\sqrt{n}}{1+\sqrt{n}}$$

$$R(\theta, \delta_{3,n}) = \frac{1}{1+\theta^2}\left(\theta^2\left(\frac{\sqrt{n}}{1+\sqrt{n}} - 1\right)^2 + \frac{(\sqrt{n})^2}{(1+\sqrt{n})^2 n}\right)$$

$$= \frac{1}{1+\theta^2}\left(\theta^2\left(\frac{1}{1+\sqrt{n}}\right)^2 + \frac{1}{(1+\sqrt{n})^2}\right)$$

$$= \frac{1}{(1+\theta^2)(1+\sqrt{n})^2}(\theta^2+1) = \frac{1}{(1+\sqrt{n})^2}$$

$$\delta_{6,n} = 1 \Rightarrow a = 1, \; b = 0$$

$$R(\theta, \delta_{6,n}) = \frac{1}{1+\theta^2}\left(\theta^2 - 2\theta + 1\right)$$

$$= \frac{(\theta-1)^2}{1+\theta^2}$$

f) Upon inspection R_{δ_1} and R_{δ_2} are very close with ~~$\sqrt{n}$~~ values.

I would choose δ_1 procedure because it is slightly better.
$\delta_1 = \bar{X}_n$

Answer is different from part (d) [$\delta_3 = X/2$]

g) $R = E\left(\frac{(\cancel{a \sim \theta})\,\theta - a - b\bar{X}_n)^2}{1+\theta^2}\right)$

$$= E\left(\frac{b^2\bar{X}_n^2 + a^2 + \theta^2 - 2a\theta - 2\theta b\bar{X}_n + 2ab\bar{X}_n}{1+\theta^2}\right)$$

$\text{var}(\bar{X}_n) = 1/n \Rightarrow E(\bar{X}_n^2) - \theta^2 = \frac{1}{n}$

$\Rightarrow E(\bar{X}_n^2) = \frac{1}{n} + \theta^2$

$$= \frac{1}{1+\theta^2}\left(b^2\theta^2 + \frac{b^2}{n} + a^2 + \theta^2 - 2a\theta - 2\theta^2 b + 2ab\theta\right)$$

$$\lim_{\theta\to\infty} R = \lim_{\theta\to\infty}\left(\frac{b^2\theta^2 + b^2/n + a^2 + \theta^2 - 2a\theta - 2\theta^2 b + 2ab\theta}{\theta^2+1}\right)$$

$$= \lim_{\theta\to\infty}\left(\frac{1}{\theta^2+1}\right)\left(\theta^2(b^2 - 2b + 1) + \theta(2ab - 2a) + b^2/n + a^2\right)$$

$$= b^2 - 2b + 1 \quad (\theta^2 \text{ is the highest power})$$

$$\lim_{\theta\to\infty} R = (b^2 - 2b + 1)$$

If this is $\lim_{\theta\to\infty} R = 0$

then, $b^2 - 2b + 1 = 0 \Rightarrow b = 1$

If $b = 1$, $\lim_{\theta\to\infty} R = 1 - 2\times1 + 1 = 0$

$\therefore \lim_{\theta\to\infty} R = 0$ iff $b = 1$

② g) contd.

if $b=1$

$$R = \frac{1}{\theta^2+1}\left(\theta^2(b-1)^2 + \theta(2ab-2a) + a^2 + \frac{b^2}{n}\right)$$

$$= \frac{1}{\theta^2+1}\left(a^2 + \frac{1}{n}\right)$$

min R $\quad \frac{dR}{da} = \frac{2a}{\theta^2+1} = 0 \Rightarrow a = 0$

$$\frac{a^2}{\theta^2+1} \geq 0 \quad \because a^2, \theta^2+1 \geq 0$$

$$\Rightarrow \frac{a^2}{\theta^2+1} + \frac{1}{n(\theta^2+1)} \geq \frac{1}{n(\theta^2+1)}$$

$$\Rightarrow \quad \frac{1}{1+\theta^2}\left(a^2 + \frac{1}{n}\right) \geq \frac{1}{(1+\theta^2)\,n}$$

$$\Rightarrow \quad R \geq \frac{1}{n(1+\theta^2)}$$

which happens only when $a = 0$

$$\therefore \quad \min_a R = \frac{1}{n(1+\theta^2)} \quad \text{and} \quad a^*_{min} = 0$$

h) If possible, let δ' ~~be~~ $\delta'_{0,n}$ exists as better procedure than $\delta_{0,n}$ - 1 making $\delta_{0,n}$ inadmissible

Then:

$$R_{\delta'}(\theta) \le R_{\delta_{0,n}}(\theta) = \frac{(\theta - 1)^2}{1+\theta^2} \quad \forall\, \theta \in R \qquad \text{--- (1)}$$

and $R_{\delta'}(\theta_0) < R_{\delta}(\theta_0) = \dfrac{(\theta_0 - 1)^2}{1+\theta^2}$ for at least one θ_0 --- (2)

~~$f_n \dfrac{(\theta-1)^2}{1+\theta^2}$; $\dfrac{\partial A}{\partial \theta} = \phi$~~

, equation (1)

$$R_{\delta'}(\theta) = \int_{y_1=-\infty}^{y_1=\infty} \cdots \int_{y_n=-\infty}^{y_n=\infty} \frac{(\theta - \delta(y_1, \ldots, y_n))^2}{1+\theta^2} f_{y_1}(Y=y_1) \times f_Y(Y=y_2) \cdots f_Y(y=y_n)\, dy_n\, dy_{n-1} \cdots dy_1$$

At $\theta = 1$, $R_{\delta_{0,n}}(1) = 0$

We know that $y_1, y_2, \ldots, y_n$ are independent

and $\int_{-\infty}^{\infty} f(y=k)\, dk = 1$ (property of pdf)

cont

So for $R_{\delta'}(\theta) \leq R_{\delta_{6,n}}(1) = 0$,

only possibility is $R_{\delta'}(\theta) = 0$
$\hookrightarrow \theta = 1$

$\Rightarrow \delta'(y_1, \ldots, y_N) = 1$

But this is the same as $\delta_{6,N}$. So eqn (2) is not satisfied! ($\because \delta' = \delta_{6,N} = 1 \ \forall \theta$)

This is a contradiction

So $\delta_{6,n} = 1$ is an admissible procedure

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a) $S = \{0, 1, 2, 3, \dots, 20\}$

$\Omega = \{ {}^{n}C_i\, \theta^{i} (1-\theta)^{n-i} ;\ \theta \in [0,1] \}$

$D = \{ 0 \leq d \leq 1 \}$ (point estimate of θ)

$L(\theta, d) = |\theta - d|$ (given)

θ is the probability of success

b)

$R(\theta, \delta_1) = E(L(\theta, \delta_1))$

$= E\left(\left| \theta - \frac{x}{n} \right| \right)$

$R(\theta, \delta_2) = E(L(\theta, \delta_2))$

$= E\left(\left| \theta - \frac{1}{3} \right| \right)$

$= |\theta - 1/3|$

$R(\theta, \delta_3) = E(L(\theta, \delta_3))$

$= E(|\theta - 1|)$

$= |\theta - 1|$

To plot $R(\theta, \delta_1) = R_{\delta_1}(\theta)$ ~~we~~ the code performing Monte Carlo simulation ~~shown~~ to estimate $E(|\theta - \frac{x}{n}|)$ for $n = 20$ is used.

Comparison:

(i) Risk functions of δ_2 and δ_3 are similar - ~~just~~ they ~~shifted equations~~

are just shifted along x-axis

(ii) Risk function of δ_1 is reasonable - stays low and is 0 when $\theta = 0$ or $\theta = 1$

Since $R_{\delta_1}(\theta)$ stays reasonably low for all θ compared to other risk functions in the question,

δ_1 is the preferred procedure

$$\left(\delta_1(x) = \frac{x}{n}\right)$$

However it is to be noted that no procedure is ~~better~~ ~~than~~ "better than" ~~or equivalent~~ the other or "equivalent"

- The procedures are "incomparable"

c) ~~R~~

If possibil let δ' be better then $\delta_2(x) = 1/3$.

$\Rightarrow$ (i) $R_{\delta'}(\theta) \leq R_{\delta_2}(\theta) = |\theta - 1/3| \quad \forall \theta \in [0, 1]$ ~~$\{0, \ldots, 1\}$~~

(ii) $R_{\delta'}(\theta_0) \lneq R_{\delta_2}(\theta_0)$ for some θ_0

$$R_{\delta'}(\theta) = \sum_{x \in S} L(\theta, \delta'(x)) P_\theta(X = x)$$

$$= \sum_{x \in S} L(\theta, \delta'(x)) P_\theta(X = x)$$

$$= \sum_{x=0}^{n} |\theta - \delta'(x)| \, {}^nC_x \theta^x (1-\theta)^{1-x}$$

When $\theta = 1/3$,

$R_{\delta_2}(1/3) = 0$

By (i), $R_{\delta'}(\theta = 1/3) \leq 0$

$$\Rightarrow \sum_{x=0}^{n} |1/3 - \delta'(x)| \, {}^nC_x \theta^x (1-\theta)^{1-x} \leq 0$$

$$\Rightarrow \delta'(x) = 1/3 \; \forall x \in \{1, 2, 3, \ldots, 20\}$$

$\Rightarrow \delta' \equiv \delta_3.$

This is a contradiction to our assumption that δ' is better than δ_2

$\therefore \delta_2$ is admissible procedure. $(\forall n \geq 1)$

d) If $\delta_3(x) = 1$ is not admissible, let there exist a better procedure $\delta'(x)$. ~~such that~~ Then,

(i) $R_{\delta'}(\theta) \leq R_{\delta_3}(\theta) = |\theta - 1| \;\forall\; \theta \in [0, 1]$

(ii) $R_{\delta'}(\theta_0) < R_{\delta_3}(\theta_0)$ for some θ_0

$$R_{\delta'}(\theta) = \sum_{x=0}^{2} {}^2C_i (\theta)^i (1-\theta)^{2-i} |\theta - \delta'(x)|$$

$$= |\theta - \delta'(0)|(1-\theta)^2 + 2|\theta - \delta'(1)|(\theta)(1-\theta) + |\theta - \delta'(2)|(\theta^2)$$

If $R_{\delta'}(1) \leq R_{\delta_3}(1) = 0$,

$\Rightarrow |1 - \delta'(2)| \leq 0$

$\Rightarrow \delta'(2) = 1$

Now $\delta'(0)$, and $\delta'(1)$ can be either 0 or 1

4 possibilities:

(i) $\delta'(0) = \delta'(1) = 0$

(ii) $\delta'(0) = 0, \quad \delta'(1) = 1$

(iii) $\delta'(0) = 1, \quad \delta'(1) = 1$

(iv) $\delta'(0) = 1, \delta'(1) = 0$

~~We see that~~

case (iii) $\Rightarrow \delta'(X) \equiv \delta_3^4(X) = 1 \ \forall x$

So contradiction

From the plot we see that the other ~~functions~~ risk values of the procedures are not $\leq R_{\delta_3}(\theta)$

less than the risk value of $R_{\delta_3}(\theta)$ δ_3

i.e. (*) ~~doesn't~~ doesn't ~~seem to hold for~~ all the

$\forall \theta$

i.e. $R_{\delta'}(\theta) \leq R_{\delta_3}(\theta)$ doesn't hold for any δ' we have tried

$\therefore$ this contradicts our assumption δ' is better than δ_3

$\Rightarrow \delta_3(X) = 1$ is admissible for $n=2$!