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HW-10 ISYE 6412

① a) $E(X) = \int_{-1}^{1} \frac{x}{2}(1+\theta x)\,dx$

$= \frac{1}{4}$

$= \frac{x^2}{4} + \frac{\theta x^3}{6}\Big|_{-1}^{1} = \frac{\theta}{3}$

$E(X^2) = \int_{-1}^{1} \frac{x^2}{2}(1+\theta x)\,dx = \frac{x^3}{6} + \frac{\theta x^4}{8}\Big|_{-1}^{1}$

$= \frac{1}{3}$

$\therefore \text{var}(X) = E(X^2) - (E(X))^2$

$= \frac{1}{3} - \frac{\theta^2}{9} = \frac{3-\theta^2}{9}$

b) $E(X) = \overline{X}_n$

$\Rightarrow \frac{\theta}{3} = \overline{x}_n$

$\Rightarrow \theta = 3\overline{x}_n$

$\cancel{\hat{\theta}_{MOM} = \frac{\overline{X}_n}{3} = \frac{\sum_{i=1}^{n} x_i}{3n}}$

$$\therefore \hat{\theta}_{MOM} = 3\bar{X}_n = 3\sum_{i=1}^{n}\frac{x_i}{n}$$

(4a) $E(\hat{\theta}_{MoM}) = E\left(\frac{3}{n}\sum_{i=1}^{n} x_i\right) = \frac{3}{n}\sum_{i=1}^{n} E(x_i)$

$$= \frac{3}{n}\frac{n\theta}{3} = \theta$$

Bias $= E(\hat{\theta}_{MoM}) - \theta = \theta - \theta = 0$!

Variance $= var(\hat{\theta}_{MoM}) = var(3\bar{X}_n)$

(4b) $$= var\left(3\frac{\sum_{i=1}^{n} x_i}{n}\right) = \frac{9}{n^2}\sum_{i=1}^{n} var(X_i)$$

$$= \frac{9}{n^2}\, n\frac{(3-\theta^2)}{9} = \frac{3-\theta^2}{n}$$

$$\lim_{n\to\infty} var(\hat{\theta}_{MoM}) = \lim_{n\to\infty}\frac{3-\theta^2}{n} = 0$$

$\therefore var(\hat{\theta}_{MoM}) \to 0$ as $n \to 0$

$\therefore \hat{\theta}_{MoM} = \bar{X}_n$ is a consistent estimator of θ

(2) a) $f_\theta(x_i) = \frac{1}{\sqrt{2\pi\theta}}\left(\exp\left(-\frac{(x_i-\theta)^2}{2\theta}\right)\right)$

$\Rightarrow L(\theta) = \prod_{i=1}^{n}\frac{1}{\sqrt{2\pi\theta}}\exp\left(\frac{-(x_i-\theta)^2}{2\theta}\right)$

$$= (2\pi\theta)^{-n/2}\exp\left(\frac{-1}{2\theta}\sum_{i=1}^{n}(x_i-\theta)^2\right)$$

$$\Rightarrow \log L(\theta) = \frac{-n}{2}\log(2\pi) - \frac{n}{2}\log\theta - \frac{1}{2\theta}\sum_{i=1}^{n}(x_i-\theta)^2$$

$$\Rightarrow \frac{\partial \log L(\theta)}{\partial\theta} = \frac{-n}{2\theta} - \sum_{i=1}^{n}\frac{\left(-2(x_i-\theta)(2\theta) - 2(x_i-\theta)^2\right)}{\cancel{4}\theta^2}$$

$$= \frac{-n}{2\theta} - \sum_{i=1}^{n}\frac{\left(-2x_i\theta + 2\theta^2 + \cancel{2}x_i^2 - \cancel{2}\theta^2 + \cancel{4}x_i\theta\right)}{2\theta^2}$$

$$= \frac{-n}{2\theta} - \sum_{i=1}^{n}\frac{\left(\cancel{\quad} + \cancel{3}\theta^2 - x_i^2\right)}{2\theta^2}$$

$$= \frac{-n}{2\theta} - \frac{\left(n\theta^2 - \sum_{i=1}^{n}x_i^2\right)}{2\theta^2}$$

$$\Rightarrow \frac{\partial^2 \log L}{\partial\theta^2} = \frac{n}{2\cancel{\theta}\theta^2} - \left(\frac{(2n\theta)(2\theta^2) - \cancel{4\theta}\,4\theta\left(n\theta^2 - \sum x_i^2\right)}{4\theta^4}\right)$$

$$= \frac{n}{2\theta^2} \cancel{+} - 4\frac{\sum_{i=1}^{n}x_i^2}{\theta^3}$$

$$= \frac{n\theta \mp 2\theta\sum x_i^2 + n\theta}{2\theta^3} < 0 \quad + \theta > 0$$

$$\theta < 2\frac{\sum x_i^2}{n}$$

equating 1^{st} derivative to 0

$$\frac{-n}{2\theta} - \left(\frac{n\theta^2 - \sum x_i^2}{2\theta^2}\right) = 0$$

$$\Rightarrow -n\theta^2 - n\theta + \sum x_i^2 = 0$$

$$\Rightarrow \theta^2 + \theta - \frac{\sum_{i=1}^{n}x_i^2}{n} = 0$$

So we got the required quadratic eqn.

roots : $\theta = \dfrac{-1 \pm \sqrt{1+4W}}{2}$.

note $\sqrt{1+4W} = \sqrt{1+4\frac{\sum x_i^2}{n}} > 1$

$\therefore$ So one root is +ve & other is -ve

For the pdf to be valid we need $\theta > 0$

$\Rightarrow \theta = \left(-1+\sqrt{1+4W}\right)\frac{1}{2}$

~~And if $\theta > 0$, we noted that~~ $\cancel{\frac{\partial^2 \log L}{\partial \theta^2} > 0}$,

need $\qquad\qquad\qquad\qquad\qquad$ maxm

$-\dfrac{\left(\sqrt{1+4W}-1\right)}{2} + 2W > 0$

$\Leftrightarrow 4W+1 > \sqrt{1+4W}$

$\Leftrightarrow 16W^2 + 8W + 1 > 4W+1$

$\Leftrightarrow 4W^2 \overset{+4W}{} > 0 \Leftrightarrow W(W+1) > 0$

But $W > 0 \;\forall\, x_i \qquad \Leftrightarrow \qquad W > 0$ or $W < -1$

$\Rightarrow \theta = -1+\sqrt{1+4W} < 2\frac{\sum x_i^2}{n}$

$\Rightarrow \dfrac{\partial^2 \log L}{\partial \theta^2} < 0$

$$\therefore \hat{\theta}_{MLE} = \left(-1+\sqrt{1+4\frac{\sum_{i=1}^{n} x_i^2}{n}}\right)\frac{1}{2}$$

$$= \frac{-1+\sqrt{1+4W}}{2}$$

$$\Rightarrow \hat{\theta}_{MLE} = \frac{-1+\sqrt{1+4w}}{2}$$

b) $I_n(\theta) = E\left(-\frac{\partial^2 \log L}{\partial \theta^2}\right)$

$$= E\left(\frac{\sum_{i=1}^n x_i^2}{\theta^3} - \frac{n}{2\theta^2}\right)$$

$E(x_i^2) = \mu^2 + \sigma^2 = \theta^2 + \theta$

$$\Rightarrow I_n(\theta) = \left(\frac{n(\theta^2+\theta)}{\theta^3} - \frac{n}{2\theta^2}\right)$$

$$= \frac{n(2\theta^2 + 2\theta - \theta)}{2\theta^3} = \frac{n}{2\theta^3}(2\theta^2+\theta)$$

$$= \frac{2n\theta + n}{2\theta^2} = \boxed{\frac{n(2\theta+1)}{2\theta^2}}$$

$$I_1(\theta) = E\left(\frac{x_1^2}{\theta^3} - \frac{1}{2\theta^2}\right) = \frac{2\theta^2+2\theta-1}{2\theta^3}$$

$$= \boxed{\frac{2\theta+1}{2\theta^2}}$$

$$\sqrt{n}\left(\phi(\hat{\theta}_{MLE}) - \phi(\theta)\right) \xrightarrow{d} N\left(0, \frac{(\phi'(\theta))^2}{I_1(\theta)}\right)$$

(shown in class)

$$\Rightarrow \text{var}(\phi(\hat{\theta}_{MLE})) = \frac{2\theta^2}{n(2\theta+1)}$$

Here $\phi(\theta) = \theta$

$$\Rightarrow \phi(\hat{\theta}_{MLE}) \xrightarrow{d} N\left(\theta, \frac{2\theta^2}{n(2\theta+1)}\right)$$

c) $\text{avar}(\hat{\theta}_{MLE}) = \dfrac{(\phi'(\theta))^2}{n I_1(\theta)} = \left(\dfrac{\phi'(\theta)}{I_n(\theta)}\right)^2$

$$= \frac{1}{I_n(\theta)} = \frac{2\theta^2}{n(2\theta+1)}$$

$$\approx \left.\frac{2\theta^2}{n(2\theta+1)}\right|_{\theta=\hat{\theta}_{MLE}}$$

$$= \frac{2\left(\dfrac{\sqrt{1+4\frac{\sum_{i=1}^n x_i^2}{n}}-1}{2}\right)^2}{n\left(2\left(\sqrt{1+4\frac{\sum x_i^2}{n}}-1\right)+1\right)}$$

$$= \frac{\left(\sqrt{1+4\frac{\sum_{i=1}^n x_i^2}{n}}-1\right)^2}{2n\left(\sqrt{1+4\frac{\sum_{i=1}^n x_i^2}{n}}\right)}$$

$\Rightarrow$ Approximate variance $= \dfrac{\left(\sqrt{1+4W}-1\right)^2}{2n\left(\sqrt{1+4W}\right)}$

$$= \frac{\left(\sqrt{1+4W}-1\right)^2}{2n\left(\sqrt{1+4W}\right)}$$

d) $\hat{\theta} = \frac{-1 + \sqrt{1+4W}}{2}$

$W = \frac{\sum_{i=1}^{n} x_i^2}{10} = \frac{109.9475}{10} = 10.99475$

~~approx variance =~~

$\therefore \hat{\theta} = \frac{-1 + \sqrt{1+4(10.99475)}}{2}$

$= 2.8533$

Approx variance $= \frac{\left(\sqrt{1+4\times 10.99475} - 1\right)^2}{2\times 10 \times \left(1+\sqrt{1+4\times 10.99475}\right)}$

$= 0.2428$

③ a) $\delta = \begin{cases} 1 & \text{if } X_1 = 1 \\ 0 & \text{ow} \end{cases}$

$E(\delta) = 1 \times P(X_1 = 1) + 0 \times P(X_1 = 0)$

$= \lambda e^{-\lambda}$

$\therefore$ δ is an unbiased estimator of $\phi(\lambda)$

By Rao Blackwell best unbiased estimator is
$E(\delta \mid T) = P(\delta = 1 \mid T)$

$$= P\left(X_1 = 1 \,\middle|\, \sum_{i=1}^{n} x_i = b\right)$$

$$= \frac{P\left(X_1 = 1, \sum_{i=1}^{n} x_i = b\right)}{P\left(\sum_{i=1}^{n} x_i = b\right)}$$

$$= \frac{P\left(X_1 = 1, \sum_{i=2}^{n} x_i = b-1\right)}{P\left(\sum_{i=1}^{n} x_i = b\right)}$$

$$\Rightarrow \delta(b) \overset{\Delta}{=} \begin{cases} 0 & \text{if } b = 0 \\ \dfrac{\lambda e^{-\lambda}\left[((n-1)\lambda)^{b-1} e^{-(n-1)\lambda} / (b-1)!\right]}{\dfrac{(n\lambda)^b e^{-n\lambda}}{b!}} & \text{if } b \geq 1 \end{cases}$$

$$\Rightarrow \delta^{+}(b) = \begin{cases} 0 & \text{if } b = 0 \\ \dfrac{b(n-1)^{b-1}}{n^b} & \text{if } b \geq 1 \\ & \text{where } b = \sum_{i=1}^{n} x_i \end{cases}$$

b) $$f_\theta(\underline{x}) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \quad \forall x \in \{0, 1, 2, \ldots\}$$

$$\Rightarrow \log L = \left(\sum_{i=1}^{n} x_i\right) \log\lambda - n\lambda - \sum_{i=1}^{n} \log(x_i!)$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{\sum_{i=1}^{n} x_i}{\lambda} - n$$

$$\Rightarrow \frac{\partial^2 \log L}{\partial \lambda^2} = \frac{-\sum_{i=1}^{n} x_i}{\lambda^2}$$

$$I_n(\hat{\lambda}) = E\left(-\frac{\partial^2 \log L}{\partial \lambda^2}\right)$$

$$= \frac{\sum_{i=1}^{n} E(x_i)}{\lambda^2} = \frac{n\lambda}{\lambda^2} = \frac{n}{\lambda}$$

$$I_1(\hat{\lambda}) = \frac{1}{\lambda}$$

c) $\phi'(\lambda) = e^{-\lambda} - \lambda e^{-\lambda}$

$\therefore$ CR lower bound $= \dfrac{(\phi'(\lambda))^2}{I_n(\lambda)} = \dfrac{(e^{-\lambda} - \lambda e^{-\lambda})^2}{\frac{n}{\lambda}}$

$$= \left(\frac{\lambda}{n}\right)\left(e^{-\lambda} - \lambda e^{-\lambda}\right)^2$$

d) $\dfrac{\partial \log L}{\partial \lambda} = 0 \Rightarrow \hat{\lambda} = \dfrac{\sum x_i}{n}$.

Note that $\frac{\partial^2 \log L}{\partial L^2}$ ~~at~~ < 0 ($\because x_i \geq 0 \ \forall i$)

~~So $\hat{\lambda}_{MLE}$ is a~~

So the stationary point is a maximum

also note, $\lim_{\lambda \to \infty} L \to 0$ $\lim_{\lambda \to -\infty} L \to -\infty$

$\therefore \hat{\lambda}_{MLE} = \frac{\sum_{i=1}^{n} x_i}{n}$.

$\hat{\phi(\lambda)}_{MLE} = \phi(\hat{\lambda}_{MLE})$

$= \frac{\sum_{i=1}^{n} x_i}{n} \exp\left(-\frac{\sum_{i=1}^{n} x_i}{n}\right)$

$\hat{\phi}(\lambda) - \phi(\lambda) \xrightarrow{} N\left(0, \frac{(\phi'(\lambda))^2}{I_n(\lambda)}\right)$

$\Rightarrow \hat{\phi}(\lambda) \xrightarrow{d} \phi(\lambda) + N\left(0, \frac{(\phi'(\lambda))^2}{I_n(\lambda)}\right)$

$\Rightarrow \hat{\phi}(\lambda) \xrightarrow{d} N\left(\phi(\lambda), \frac{(\phi'(\lambda))^2}{I_n(\lambda)}\right)$ ($\because \phi(\lambda)$ is a constant)

~~$\phi(\lambda)$ and λ~~

Substituting the values,

$$\hat{\phi}(\lambda)_{MLE} \xrightarrow{d} N\left(\lambda e^{-\lambda}, \frac{(e^{-\lambda} - \lambda e^{-\lambda})^2 \lambda}{n}\right)$$

e) Best unbiased estimate :

$b = \sum_{i=1}^{n} x_i$

$\Rightarrow b = 104$

$$\Rightarrow \delta^*(104) = \frac{(104)(15-1)^{103}}{(15)^{104}}$$

$\Rightarrow \hat{\phi(\lambda)}_{\text{Best unbiased}} \simeq 0.0056850$

MLE :

$$\hat{\lambda}_{MLE} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{104}{15} = 6.9333$$

$\Rightarrow \hat{\lambda}_{MLE} = 6.9333$

$\therefore \hat{\phi(\lambda)} = 6.9333\, e^{-6.9333}$

$\simeq 0.0067582$

$$\text{Approximate variance} = \left.\frac{(e^{-\lambda} - \lambda e^{-\lambda})^2 \lambda}{n}\right|_{\lambda = \hat{\lambda}_{MLE} = \frac{104}{15}}$$

$\simeq 0.000015461 \; (= 1.5461 \times 10^{-5})$

(4) a) $f_\theta(\underline{x}) = \begin{cases} \frac{1}{\theta^n} e^{-\frac{\sum_{i=1}^n x_i}{\theta}} & \text{if } x_i > 0 \ \forall i \\ 0 & \text{if } x_i \le 0 \text{ for any } i \end{cases}$

$$\Rightarrow \log L = -n\log\theta + \frac{\sum_{i=1}^n x_i}{\theta_n}$$

$$\Rightarrow \frac{\partial \log L}{\partial\theta} = \frac{-n}{\theta} + \frac{\sum_{i=1}^n x_i}{\theta^2}$$

$$\Rightarrow \frac{\partial^2 \log L}{\partial\theta^2} = \frac{n}{\theta^2} - \frac{2\sum_{i=1}^n x_i}{\theta^3}$$

$$= \frac{1}{\theta^3}\left(n\theta - 2\sum x_i\right)$$

$$\frac{\partial \log L}{\partial\theta} = 0 \Rightarrow \theta^* = \frac{\sum_{i=1}^n x_i}{n}.$$

$$\left.\frac{\partial^2 \log L}{\partial\theta^2}\right|_{\theta=\theta^*} = \frac{1}{\left(\frac{\sum x_i}{n}\right)^3}\left(n\sum x_i - 2\sum x_i\right)$$

$$= \frac{-n^3}{\left(\sum_{i=1}^n x_i\right)^2} < 0 \ \forall \ \underline{x}_i$$

$\therefore \theta^*$ is a maxm sum.

$\Rightarrow \hat{\theta}_{MLE} = \dfrac{\sum_{i=1}^{n} x_i}{n}.$

$\widehat{\phi(\theta)}_{MLE} = \phi(\hat{\theta}_{MLE})$

$= \left(\dfrac{\sum_{i=1}^{n} x_i}{n}\right)^2 = \dfrac{1}{n^2}\left(\sum_{i=1}^{n} x_i\right)^2$

b) $E_\theta(\delta_c - \theta^2)^2 = E\left(c\left(\sum x_i\right)^2 - \theta^2\right)^2$

$= \left(E(cT^2 - \theta^2)\right)^2 + \text{var}(cT^2 - \theta^2)$

$= E(cT^2 - \theta^2)^2$

$= E\left(c^2T^4 + \theta^4 - 2c\theta^2T^2\right)$

Since $T = \sum_{i=1}^{n} x_i$, it is sum of

n iid $\exp\left(\frac{1}{\theta}\right)$ RVs.

$\Rightarrow T \sim$ ~~exp~~ Erlang$\left(n, \frac{1}{\theta}\right)$

$E(T) = n\theta$

$E(T^2) = n\theta^2 + n^2\theta^2 = \theta^2(n)(n+1)$

$MGF_T(s) = \left(1 - s\theta\right)^{-n}$

$$\frac{dg}{ds} = \frac{\theta^{n}}{\lambda}\left(1 - \frac{\theta s}{\cancel{\lambda}}\right)^{-n-1}$$

$$\frac{d^2g}{ds^2} = \frac{n(n+1)\theta^2}{\cancel{\lambda^2}}\left(1 - \frac{\theta s}{\cancel{\lambda}}\right)^{-n-2}$$

$$\frac{d^3g}{ds^3} = \frac{n(n+1)(n+2)\theta^3}{\cancel{\lambda^3}}\left(1 - \frac{s\theta}{\cancel{\lambda}}\right)^{-n-3}$$

$$\frac{d^4g}{ds^4} = \frac{n(n+1)(n+2)(n+3)\theta^4}{\cancel{\lambda^4}}\left(1 - \frac{s\theta}{\cancel{\lambda}}\right)^{-n-4}$$

$$\therefore E(T^3) = \frac{n(n+1)(n+2)\theta^3}{\cancel{\lambda^3}}$$

$$E(T^4) = \frac{n(n+1)(n+2)(n+3)\theta^4}{\cancel{\lambda^4}}$$

$$\Rightarrow E_\theta(\delta_c - \theta^2)^2 = \theta^4 c^2\left(\frac{(n)(n+1)(n+2)(n+3)}{\cancel{\lambda^4}}\right) + \theta^4$$

$$- 2c\theta^2 \frac{n(n+1)\cancel{(n+2)}\theta^2}{\cancel{\lambda^2}}$$

$$= \theta^4\left[c^2(n)(n+1)(n+2)(n+3) - 2cn(n+1) + 1\right]$$

$$\Rightarrow R_{\delta_c}(\theta) = \theta^4\left[c^2 n(n+1)(n+2)(n+3) - 2cn(n+1) + 1\right]$$

c) when $c = \frac{1}{n^2}$ (ie δ_{MLE}^{4}),

$$R_{\delta_{MLE}} = \theta^4 \left(\frac{1}{\theta^4 n^4} n(n+1)(n+2)(n+3) - \frac{2}{n^2} n(n+1) + 1 \right)$$

$$= \frac{\theta^4}{n^3} \left(n^3 + 6n^2 + 11n - 2n^3 - 2n^2 + n^3 \right)$$

$$= \frac{\theta^4}{n^3} \left(4n^2 + 11n + 6 \right)$$

$$= \theta^4 \left(\frac{4n^2 + 11n + 6}{n^3} \right)$$

d) $$\frac{\partial R_{\delta_c}}{\partial c} = 2c\, n(n+1)(n+2)(n+3) - 2n(n+1)$$

$$\Rightarrow \frac{\partial^2 R_{\delta_c}}{\partial c^2} = 2n(n+1)(n+2)(n+3) > 0 \ \forall c$$

$$\frac{\partial R_{\delta_c}}{\partial c} = 2c\, n(n+1)(n+2)(n+3) - 2(n+1)n = 0$$

$$\Rightarrow \boxed{c^* = \frac{1}{(n+2)(n+3)}}$$

It is a minima since $\frac{\partial^2 R_{\delta_c}}{\partial c^2} > 0$

$$\tilde{z}$$

$$R_{\delta_c^4} = \theta^4\left(\left(\frac{1}{(n+2)(n+3)}\right)^2 n(n+1)(n+2)(n+3) - \frac{2}{(n+2)(n+3)}(n)(n+1) + 1\right)$$

$$= \theta^4\left(\frac{n(n+1)}{(n+2)(n+3)} - \frac{2n(n+1)}{(n+2)(n+3)} + 1\right)$$

$$= \theta^4\left(\frac{-n^2-n+n^2+5n+6}{(n+2)(n+3)}\right)$$

$$= \frac{\theta^4(4n+6)}{(n+2)(n+3)}$$

e) $I_n(\theta) = E\left(-\frac{\partial^2 \log L}{\partial\theta^2}\right)$

$$= E\left(-\left(\frac{n}{\theta^2} - \frac{2\sum_{i=1}^{n} x_i}{\theta^3}\right)\right)$$

$$= \frac{-n}{\theta^2} + \frac{2n\theta}{\theta^3} = \frac{n}{\theta^2}$$

$I_1(\theta) = \frac{1}{\theta^2}$

$\phi'(\theta) = 2\theta$

CR lower bound, $= \frac{(\phi'(\theta))^2}{I_n(\theta)}$

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$$= \frac{4\theta^2}{\frac{n}{\theta^2}} = \frac{4\theta^4}{n}$$

Part (a): $\delta_n = \frac{\sum_{i=1}^{n} x_i^2}{n^2} = \frac{1}{n^2} T^2$

$\Rightarrow R\delta_n = \theta^4\left(\frac{4n^2 + 11n + 6}{n^3}\right)$

$$\frac{R\delta_n}{H_n} = \frac{\theta^4\left(\frac{4n^2 + 11n + 6}{n^3}\right)}{4\theta^4/n}$$

$$= \left(4 + \frac{11}{n} + \frac{6}{n^2}\right) \times \frac{1}{4}$$

$$\lim_{n \to \infty} \frac{R_n}{H_n} = \lim_{n \to \infty} \left(4 + \frac{11}{n} + \frac{6}{n^2}\right) \times \frac{1}{4}$$

Sequence-b

$= 1$ ∴ MLE estimators is asymptotically effect

Part (c):

$$R\delta_n = \theta^4 \left(\frac{4n+6}{n(n+1)}\right)$$

$$\lim_{n \to \infty} \frac{R\delta_n}{H_n} = \lim_{n \to \infty} \frac{\theta^4(4n+6)}{(n)(n+1)} \times \frac{n}{4\theta^4}$$

$$= \lim_{n \to \infty} 4\frac{n}{n+1} + \frac{6}{(n+1)4}$$

$= 1$ ∴ Asymptotically effect

Part (d):

$$R\delta_n = \frac{\theta^4(4n+6)}{(n+2)(n+3)}$$

$$\Rightarrow \lim_{n \to \infty} \frac{R\delta_n}{H_n} = \lim_{n \to \infty} \frac{\theta^4(4n+6)}{(n+2)(n+3)} \times \frac{n}{4\theta^4}$$

$$= \lim_{n \to \infty} \frac{4 \cdot n^2}{(n+2)(n+3)} + \frac{6n}{(n+2)(n+3)4}$$

$= 1$

$\therefore$ Post (d) estimator is asymptotically efficient

f) $\int_{-\infty}^{\infty} \pi(\theta)\, d\theta = \int_{-\infty}^{0} 0\, d\theta + \int_{0}^{\infty} \theta^{-2} e^{-1/\theta}\, d\theta$

$u \to 1/\theta \Rightarrow du = \frac{-d\theta}{\theta^2}$

$\Rightarrow \cancel{\int_{-\infty}^{\infty} \pi(\theta)\, d\theta = \int_{-\infty}^{0} -d\theta\, e^{-u}\, du}$

$\Rightarrow \cancel{\int_{-\infty}^{\infty} \pi(\theta)}$

$\Rightarrow \int_{-\infty}^{\infty} \pi(\theta)\, d\theta = \int_{0}^{\infty} e^{-u}\, du$

$\Rightarrow \int_{-\infty}^{\infty} \pi(\theta)\, d\theta = 1$

So $\pi(\theta)$ is a valid pdf

$r_{\delta^*}(\pi) = \min_{\delta} r_\delta(\pi)$

$h_\pi^*(\underline{y}, d) = \int_0^\infty (\theta^2 - d)^2 \prod_{i=1} \frac{1}{\theta^n} \left(e^{-\frac{\sum x_i}{\theta}} \right) \frac{e^{-1/\theta}}{\theta^2} d\theta$

$$\pi(\theta \mid y) = \frac{\theta^{-2} e^{-1/\theta} \frac{1}{\theta^n} e^{-\frac{\sum x_i}{\theta}}}{\int \theta^{-2} e^{-1/\theta} \frac{1}{\theta^n} e^{-\frac{\sum x_i}{\theta}}}$$

$$\cancel{\theta^{-2}} = \int_0^\infty \frac{(\theta^4 + d^2 - 2d\theta^2) e^{-\left(\frac{T+1}{\theta}\right)}}{\theta^{n+2}} d\theta$$

$$= \int_0^\infty \frac{e^{-\left(\frac{T+1}{\theta}\right)}}{\theta^{n-2}} + \frac{d^2 e^{-\frac{(T+1)}{\theta}}}{\theta^{n+2}} - \frac{2d\theta^2 e^{-\frac{(T+1)}{\theta}}}{\theta^n} d\theta$$

$u = 1/\theta \Rightarrow du = \frac{-d\theta}{\theta^2}$

$$= \int_0^\infty \frac{u^{n-4} e^{-(T+1)u}}{\cancel{\theta}} + \frac{u^n d^2 e^{-(T+1)u}}{\cancel{\theta}} - \frac{2d\, u^{n-2} e^{-(T+1)u}}{\cancel{\theta}} du$$

$$= \int_0^\infty (e^{-(T+1)u}) (u^n d^2 - 2d u^{n-2} + u^{n-4}) du$$

$$\Rightarrow h_\pi^*(y, d) = \frac{d^2 \Gamma(n+1)}{(T+1)^{n+1}} - 2d (T+1)^{1-n} \Gamma(n-1) + \frac{\Gamma(n-3)}{(T+1)^{n-3}}$$

$\Rightarrow d^* =$ upward parabola.

$$\Rightarrow d^*_{min} = \frac{+2 (T+1)^{1-n} \Gamma(n-1)}{\frac{2 \Gamma(n+1)}{(T+1)^{n+1}}}$$

$$\Rightarrow \delta^{*} = \frac{(T+1)^2}{n(n-1)} \qquad \left(\because \frac{\Gamma(n-1)}{\Gamma(n+1)} = \frac{1}{n(n-1)} \right)$$

(4) c) $E(T^2) = n(n+1)\,\theta^2$

$$\Rightarrow E\left(\frac{T^2}{n(n+1)}\right) = \theta^2$$

$\Rightarrow \delta = \frac{T^2}{n(n+1)}$ is an unbiased estimate of θ^2

But it is a fn of only T!

$\therefore \delta = \frac{T^2}{n(n+1)}$ is the best unbiased estimated $\Rightarrow c = \frac{1}{n(n+1)}$

$$R_{\delta_0} = \theta^4 \left(\frac{n(n+1)(n+2)(n+3)}{(n)(n+1)^2} - \frac{2}{n(n+1)}(n)(n+1) + 1 \right)$$

$$= \theta^4 \left(\frac{n^2 + 6 + 5n - 2n^2 - 2n + n^2 + n}{n(n+1)} \right)$$

$$= \frac{\theta^4 (4n+6)}{(n(n+1))}$$