ISyE 6412A **Theoretical Statistics** **Fall 2023**

HW #1 (due at midnight on Wednesday, Aug 30, ET.)
(There are 2 questions, and please look at both sides. Peer grading will be assigned at 8:00am on Thursday Aug 31)

Problem 1. Suppose that X is a normal random variable with variance 1 and unknown mean θ. It is desired to guess the value of unknown mean θ. Since the experimenter feels the loss is roughly like square error $(d-\theta)^2$ when the true θ is small but is like squared *relative* error $(\theta^{-1}d-1)^2$ when $|\theta|$ is large, he or she chooses loss function $(\theta-d)^2/(1+\theta^2)$ to reflect this behavior.

(a) Specify S, Ω, D, and L (i.e., the sample space, the set of all possible distribution functions, the decision space, and the loss function).

(b) Determine and plot on the same graph the risk function of the 6 procedures δ_i defined by

$$\delta_1(X) = X; \qquad \delta_2(X) = (1+X)/2; \qquad \delta_3(X) = X/2;$$
$$\delta_4(X) = 2X; \qquad \delta_5(X) = 0; \qquad \delta_6(X) = 1;$$

[You can save time by working (e) first but may find it easier to work (b) first. Your calculation will be made simpler if you first compute the risk function of a general procedure of the form $\delta(X) = a + bX$. A check: $R_{\delta_4}(\theta) = (\theta^2 + 4)/(1 + \theta^2)$.]

(c) From these calculations, can you assert that any of these six procedures is inadmissible?

(d) On the basis of the risk functions, if one of these 6 procedures must be used, which procedure would you use, and why? (*Note:* Don't consult any references in answering this. Later you will find out the precise meaning of your present intuition.)

(e) Suppose X is replaced by the vector $(X_1, \ldots, X_n)$ of iid normal $N(\theta, 1)$ random variables. The procedures corresponding to $\delta_1, \delta_2, \delta_3, \delta_6$ are

$$\delta_{1,n}(X_1, \ldots, X_n) = \bar{X}_n; \qquad \delta_{2,n}(X_1, \ldots, X_n) = \frac{\bar{X}_n + n^{-1}}{1 + n^{-1}};$$
$$\delta_{3,n}(X_1, \ldots, X_n) = \frac{\sqrt{n}\, \bar{X}_n}{1 + \sqrt{n}}; \qquad \delta_{6,n}(X_1, \ldots, X_n) = 1.$$

Compute the risk functions of these four procedures, and plot graphs of these four risk functions (or, rather, of $nR_{\delta_i,n}$ to make the results comparable to those of part (b)) for n large (e.g., for $n = 10,000$). [Use the fact that $\bar{X}_n$ is $N(\theta, n^{-1})$ distributed. Again, you may find it is easier first to find $(1+\theta^2)^{-1}\mathbf{E}_\theta(a + b\bar{X}_n - \theta)^2$ for general a, b.]

(f) If n is large, which of the four procedures of part (e) would you use, and why?
(Your answer to this last may differ from the answer to part (d) for the case $n = 1$; does it?)

(g) Suppose the statistician decides to restrict consideration to procedures $\delta_{a,b,n} = a + b\bar{X}_n$ of the form mentioned at the end of (e). He or she is concerned about the behavior of the risk function when $|\theta|$ is large. Show that the risk function approaches 0 as $|\theta| \to \infty$ if and only if $b = 1$. In addition, *among procedures with $b = 1$*, show that the choice $a = 0$ gives uniformly smallest risk function.

[This justification of the procedure $\delta_{1,n} = \bar{X}_n$ under the restriction to procedures of the form $\delta_{a,b,n}$ will seem more sensible to many people than a justification in terms of the "unbiasedness" criterion to be discussed later].

(h) Show that the procedure $\delta_{6,n}$, defined by $\delta_{6,n}(X_1, \ldots, X_n) \equiv 1$, is admissible for each n. [Hints: how can another procedure δ' satisfy $R_{\delta'}(\theta) \le R_{\delta_{6,n}}(\theta)$ when $\theta = 1$?]

Problem 2. Assume that we observe a binomial random variable X with parameter (n, θ), i.e., the probability mass function of X is given by $\mathbf{P}(X = i) = \binom{n}{i}\theta^i(1-\theta)^{n-i}$ for $i = 0, 1, \ldots, n$, where $n \geq 1$ is a known integer and $0 \leq \theta \leq 1$ is unknown. Consider the problem of estimating θ under the so-called "absolute deviation" loss function defined by $L(\theta, d) = |\theta - d|$.

(a) Specify S, Ω, D, and L (i.e., the sample space, the set of all possible distribution functions, the decision space, and the loss function).

(b) When $n = 20$, graph and compare the risk functions of the following three procedures:

$$\delta_1(X) = \frac{X}{n}, \qquad \delta_2(X) = \frac{1}{3}, \quad \text{and} \quad \delta_3(X) = 1.$$

Note that the risk functions may not have simple expressions, and it will be OK to use some computer software to plot the risk functions.

(c) Show that for any given integer $n \geq 1$, the procedure $\delta_2(X) = 1/3$ is admissible. [Hints: how can another procedure δ' satisfy $R_{\delta'}(\theta) \leq R_{\delta_2}(\theta)$ when $\theta = 1/3$?]

(d) Show that when $n = 2$, the procedure $\delta_3(X) = 1$ is admissible.

Remarks: Parts (c) and (d) suggest that an admissible estimator may not be appealing. Of course, it is clear that inadmissible estimators are definitely not desirable.

In Part (b), the following R code can be used to plot the risk functions. For more information about the free statistical software R, please see the website <http://www.r-project.org/>.

```
theta <- seq(0,1,0.0001);
R1 <- 0;
for (i in 0:20){
 R1 <- R1+choose(20,i)*(theta^i)*((1-theta)^(20-i))*abs(i/20 - theta);
}
R2 <- abs(1/3  - theta);
R3 <- abs(1  - theta);
plot(theta, R1,"l", ylab="Risk Function", ylim=c(0,1));
lines(theta, R2, col="red");
lines(theta, R3, col="blue")
```