

ISYE 6412 - HW 03.

(1) From HW-1 we know,

$$R_{\delta, n}(\theta) = \frac{(\theta-1)^2 + n}{(1+n)^2 (1+\theta^2)}$$

$$\begin{aligned} r_{\delta} &= \int_{-\infty}^{\infty} \pi(\theta) R_{\delta}(\theta) d\theta \\ &= \int_{-\infty}^{\infty} c_1 (1+\theta^2)^{-1} \phi(\theta-1) \frac{(\theta-1)^2 + n}{(1+n)^2 (1+\theta^2)} d\theta \\ &= \int_{-\infty}^{\infty} c_1 \phi(\theta-1) \left(\frac{(\theta-1)^2 + n}{(1+n)^2} \right) d\theta \end{aligned}$$

Replace $\theta-1 \rightarrow \alpha$. $\Rightarrow d\theta = d\alpha$.

$$\begin{aligned} \Rightarrow r_{\delta} &= \int_{-\infty}^{\infty} c_1 \phi(\alpha) \left(\frac{\alpha^2 + n}{(1+n)^2} \right) d\alpha \\ &= \frac{c_1}{(1+n)^2} \left[\int_{-\infty}^{\infty} \phi(\alpha) \alpha^2 d\alpha + n \int_{-\infty}^{\infty} \frac{\phi(\alpha)}{(1+n)} d\alpha \right] \\ &= \frac{c_1}{(1+n)^2} \left[E(\alpha^2) + E(\alpha) n \right] \end{aligned}$$

$$\begin{aligned} \text{E } y \text{ where } \theta(x^2) &\propto N(0, 1) \\ \Rightarrow \text{E}(x^2) &= \text{var}(x) + (\text{E}(x))^2 \\ &= 1 \end{aligned}$$

$$\Rightarrow r_f = \frac{c_1}{1+\eta}$$

For a general procedure $\delta = a + b \bar{y}_n$

$$R_\delta(\theta) = \frac{(a + (b-1)\theta)^2 + b^2/n}{1+\theta^2}$$

$$\begin{aligned} r_f &= \int_{-\infty}^{\infty} c_1 (1+\theta^2) \left(\frac{(a + (b-1)\theta)^2 + b^2/n}{(1+\theta^2)^2} \right) \phi(\theta) d\theta \\ &= c_1 \int_{-\infty}^{\infty} \frac{(a + (b-1)\theta)^2 + b^2/n}{1+\theta^2} \phi(\theta) d\theta \end{aligned}$$

For Bayes procedure, minimise

$$h_{\pi}^{\delta}(y, d) = \int_{\mathcal{R}} L(\theta, d) f_{\theta}(y) \pi(\theta) d\theta$$

$$= c_1 \int_{-\infty}^{\infty} \frac{(0-d)^2}{(1+\theta^2)} \left(\prod_{i=1}^n \frac{1}{\sqrt{2\pi}} e^{-\frac{(y_i-\theta)^2}{2}} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{\theta^2}{2}} d\theta$$

for each y

$$= c_1 \int_{-\infty}^{\infty} (\theta^2 - 2\theta d + d^2) (D_0) \left(\exp \left(-\frac{1}{2} \theta^2 \left(\frac{n}{\sum y_i^2} + \frac{1}{n} \right) \right) \right) d\theta$$

$\theta \left(\sum y_i \right) \text{ and } \left(\sum y_i^2 \right)$

For Bayes procedure, minimise,

$$h_{\pi}^*(y, d) = \int_{\mathcal{R}} L(\theta, d) f_0(y) \pi(\theta) d\theta$$

$$= \int_{-\infty}^{\infty} c_1 \frac{(1+\theta^2)^{-1}}{(1+\theta^2)} \left[\prod_{i=1}^n \frac{1}{\sqrt{2\pi}} e^{-\frac{(y_i - \theta)^2}{2}} \right] \phi(\theta-1) d\theta$$

for all y

$$= c_1 \int_{-\infty}^{\infty} (\theta-d)^2 \left[f_0(y_1, y_2, \dots, y_n) \right] \phi(\theta-1) d\theta$$

\Rightarrow minimise $(\theta-d)^2$ for fixed prior of $\theta \sim \phi(\cdot-1)$

We know the solution for $L = (\theta-d)^2$

$$\pi(\theta) = \phi(\theta-1) = N(1, 1)$$

$$\text{or } \pi, f = f_0(y)$$

$$d^+ = \frac{\sum y_i / \sigma^2 + \mu / \tau^2}{n / \sigma^2 + 1 / \tau^2} \quad \text{where } \pi(\theta) = N(\mu, \tau^2) \text{ and } f_0(x_i) = N(\theta, \sigma^2)$$

$$\text{Here } \sigma^2 = 1, \tau^2 = 1, \mu = 1$$

$$\Rightarrow d^+ = \frac{\sum y_i / 1 + 1}{n + 1} = \frac{\sum y_i / n + 1/n}{1 + 1/n}$$

$$\therefore \delta_B^* = \frac{\sum y_i / n + 1/n}{1 + 1/n} = \delta_{2, n}$$

$$(2) a) m(y) = \int_{\beta}^{\theta} f_{\theta}(y) \pi(\theta) d\theta$$

$$= \int_{\beta}^{\theta} \left(\frac{1}{\theta}\right)^n \frac{\alpha \beta^{\alpha}}{\theta^{\alpha-1}} d\theta \quad (\because y = n \text{ i.i.d. uniform RV})$$

$$= \int_{\beta}^{\theta} \frac{\alpha \beta^{\alpha}}{\theta^{n-\alpha+1}} d\theta = \frac{\alpha \beta^{\alpha}}{(n-\alpha+1) \beta^{n-\alpha+1}}$$

$$\pi(y|\theta) = \frac{\pi(\theta) f_{\theta}(y)}{m(y)}$$

$$= \frac{\pi \frac{\alpha \beta^{\alpha}}{\theta^{\alpha+1}} \times \left(\frac{1}{\theta}\right)^n}{\frac{\alpha \beta^{\alpha}}{(n-\alpha+1) \beta^{n-\alpha+1}}} \quad \forall \theta \geq \beta, y_i \leq \theta \text{ and 0 otherwise}$$

$$= \frac{\pi (\alpha + n) \beta^{(n+\alpha)}}{\theta^{n-\alpha+1}}$$

$$= \frac{(\alpha + n) \beta^{(\alpha+n)}}{\theta^{\alpha+n+1}} \quad \forall \theta \geq \beta, y_i \leq \theta$$

(If $\theta < \beta$, $\pi(\theta) = 0$ - defn of Pareto,

if $y_i > 0$, $f_{\theta}(y) = 0$ - defn of the uniform distribution)

$$\text{case (i)} \quad \beta < \theta.$$

$$\Rightarrow \pi(\theta|y) = \frac{(\alpha+n) \beta^{(\alpha+n)}}{\theta^{\alpha+n+1}} \quad \forall \theta \geq \beta \text{ and } \theta \geq y_i + 1$$

$$\cdot \quad \theta \geq \beta \text{ and } \theta \geq y_i + 1$$

$$\Rightarrow \theta \geq \beta \text{ and } \theta \geq \max(y_1, \dots, y_n)$$

$$\Rightarrow \theta \geq \beta \text{ and } \theta \geq Y(n)$$

$$\Rightarrow \theta \geq \max(\beta, Y(n))$$

$$\therefore \pi(\theta|y) = \frac{(\alpha+n) \beta^{(\alpha+n)}}{\theta^{\alpha+n+1}} \quad \theta \geq \max(\beta, Y(n))$$

$$\Rightarrow \alpha^* = \alpha + n, \quad \beta^* = \max(\beta, Y(n))$$

$$b) \quad h_{\pi}(y, d) = \int L(\theta, d) \pi(\theta|y) d\theta.$$

$$\leftarrow \int (\theta - d)^2$$

We have seen in HW#-2, q4 that when $L = (\theta - d)^2$,

$$d^* = \text{mean}(\pi(\theta|y))$$

$$= \frac{\alpha^* \beta^{*\alpha^*}}{\alpha^* - 1}$$

$$\delta_B^+(Y)$$

$$\Rightarrow \delta_B^+ = \underbrace{(\alpha + n) (\max \{ \beta, Y_{(n)} \})}_{\alpha + n - 1}$$

(c) From HW-2 under $L = | \theta - d |$, the Bayes procedure is the median of $\pi(\theta | y)$

$$\text{median of Pareto}^{(\alpha, \beta)} = \sqrt{2} \beta$$

$$\therefore \delta_C^+(Y) = \sqrt{2} (\alpha + n) \beta^n - (\sqrt{2}) (\max \{ \beta, Y_{(n)} \})$$

(3) a) with increase in c we see an increase in asymmetry

$$b) E(L(\theta, d)) = \int_{-\infty}^{\infty} L(\theta, d) \pi(\theta) d\theta.$$

$$= \int_{-\infty}^{\infty} (e^{c(d-\theta)} - c(d-\theta) - 1) (\pi(\theta)) d\theta.$$

$$= e^{cd} \int_{-\infty}^{\infty} e^{-\theta c} \pi(\theta) d\theta - cd \int_{-\infty}^{\infty} \pi(\theta) d\theta.$$

$$+ c \int_{-\infty}^{\infty} \theta \pi(\theta) d\theta - \int_{-\infty}^{\infty} \pi(\theta) d\theta$$

$$= e^{cd} E(e^{-c\theta}) - cd + c \int_{-\infty}^{\infty} E(\theta) - 1$$

$$\frac{d E(L(0, d))}{d(d)} = c e^{cd} E(e^{-c\theta}) - c$$

$$\frac{d^2 (E(L(0, d)))^2}{d(d^2)} = c^2 e^{cd} E(e^{-c\theta})$$

$$e^{-c\theta} > 0, \text{ and } E(e^{-c\theta}) > 0$$

$$e^{cd} > 0$$

$$\therefore \frac{d^2 R(0, d)}{d(d^2)} > 0 \quad \forall d$$

If a stationary point exists it must be a minimum

Setting first derivative to 0,

$$c e^{cd} E(e^{-c\theta}) - c = 0$$

$$\Rightarrow e^{cd} * E(e^{-c\theta}) = 1 \quad (\text{answer})$$

$$\cancel{E(e^{-c\theta})} = \frac{1}{\cancel{e^{cd}}}$$

$$\Rightarrow \cancel{E(e^{-c\theta})} e^{cd} = \frac{1}{\cancel{E(e^{-c\theta})}}$$

$$\Rightarrow cd^* = -\log E(e^{-c\theta})$$

$$\Rightarrow d^* = -\frac{1}{c} \log (E(e^{-c\theta}))$$

exists $\because E(e^{-c\theta}) > 0$
 & we showed that it is a minimum.

$$\Rightarrow d_{\text{Bayes}} = -\frac{1}{c} \log (E(e^{-c\theta}))$$

c) we know that to get Bayes procedure
 we need to minimise

$$\cdot h_{\pi}(y, d) = \int_{\mathcal{Z}} L(\theta, d) \pi(\theta | y) d\theta.$$

this is essentially the same as previous
 section π with $\pi(\theta | y)$ instead of $\pi(\theta)$

$$\Rightarrow E h_{\pi}(y, d) = e^{+cd} \int_{-\infty}^{\infty} e^{-c\theta} \pi(\theta | y) d\theta \\
= e^{+cd} \int_{-\infty}^{\infty} \pi(\theta | y) d\theta \\
+ c \int_{-\infty}^{\infty} \theta \pi(\theta | y) d\theta - \int_{-\infty}^{\infty} \pi(\theta | y) d\theta.$$

$$= e^{cd} E(e^{-c\theta} | \underline{y}) - cd + c E(\theta | \underline{y}) - 1$$

$$\therefore \frac{d(h_x)}{d(d)} = ce^{cd} E(e^{-c\theta} | \underline{y}) - c$$

$$\frac{d^2(h_x)}{d(d^2)} = c^2 e^{cd} E(e^{-c\theta} | \underline{y}) > 0 \quad \forall d$$

By setting derivative to 0 and same argument that stationary point is min.

$$\delta_c(\underline{y}) = d^* = \frac{-1}{c} \log \left(E(e^{-c\theta} | \underline{y}) \right)$$

$$\rightarrow \delta_c(\underline{y}) = \frac{-1}{c} \log \left(E(e^{-c\theta} | \underline{y}) \right)$$

d) As per hint, we consider a new observation

$$\bar{y}_n \sim N(\theta, \sigma^2/n)$$

$$\text{Now, } m(\theta) = \int_{-\infty}^{\infty} f_{\theta}(\bar{y}) \pi(\theta) d\theta$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2/n}} e^{-\frac{(\bar{y}-\theta)^2}{2\sigma^2/n}} \pi(\theta) d\theta$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2/n}} e^{-\left(\frac{(\bar{y}-\theta)^2}{2\sigma^2/n}\right)} d\theta$$

$$= 1 \quad \because \int_{-\infty}^{\infty} f_{\theta}(\theta) d\theta = 1$$

$$\therefore \pi(\theta|\bar{y}) = \frac{f_{\theta}(\bar{y}) \times 1}{1}$$

$$= f_{\theta}(\bar{y})$$

$$\Rightarrow \pi(\theta|\bar{y}) = f_{\theta}(\bar{y})$$

$$\sim N(\bar{y}, \sigma^2/n)$$

make sense
because we assume

this makes sense because we assumed prior
is non-informative - we are deciding solely based
on observed data

$$E\left(e^{-c\theta/\sigma^2}\right) = \int_{-\infty}^{\infty} \frac{e^{-\left(\frac{(\bar{y}-\theta)^2}{2\sigma^2/n}\right)} e^{-c\theta}}{\sqrt{2\pi\sigma^2/n}} d\theta$$

for exponential form: $\frac{(\theta - \bar{y})^2}{2\sigma^2/n} + c\theta$

$$= \frac{\theta^2 - 2\theta\bar{y} + \left(-\frac{2\sigma^2 c}{n} + 2\bar{y}\right)\theta + \bar{y}^2}{2\sigma^2/n}$$

$$= \left(\theta^2 - \left(\bar{y} - \frac{\sigma^2 c}{n}\right)^2 + \bar{y}^2 - \left(\bar{y} - \frac{\sigma^2 c}{n}\right)^2 \right) \frac{1}{2\sigma^2/n}$$

$$= \frac{\left(\theta - \left(\bar{y} - \frac{r^2 c}{n}\right)\right)^2}{2 \sigma^2/n} + \bar{y} c + \frac{2 \bar{y} r^2 c}{n} - \frac{\sigma^4 c^2}{n^2}$$

$$\therefore E(e^{-c\theta} | Y) = \int_0^\infty e^{-c\theta} \frac{1}{\sigma \sqrt{2\pi/n}} e^{-\frac{\left(\theta - \left(\bar{y} - \frac{r^2 c}{n}\right)\right)^2}{2 \sigma^2/n}} d\theta$$

$$E(e^{-c\theta} | Y) = \text{constant} \times \int_0^\infty e^{-\frac{\left(\theta - \left(\bar{y} - \frac{r^2 c}{n}\right)\right)^2}{2 \sigma^2/n}} d\theta$$

= constant \times integral of Normal dist.

with $\mu = \bar{y} - \frac{r^2 c}{n}$ and var = $\frac{\sigma^2}{n}$

= ~~const~~ constant

$$= e^{\left(\frac{\sigma^4 c^2}{n^2} + \frac{2 \bar{y} r^2 c}{n}\right) \times \frac{n}{2 \sigma^2}} = e^{\left(\frac{c^2 r^2}{2 n \sigma^2} - \bar{y} c\right)}$$

from part (c),

$$\delta_{\text{Bayes}}^*(Y) = \frac{-1}{\bar{y} c} \times \log \left(E(e^{-c\theta} | Y) \right)$$

$$= \frac{-1}{c} \left(\frac{+ c^2 r^2}{2 n \sigma^2} - \bar{y} c \right)$$

$$\Rightarrow \delta_d(Y) = \bar{y} - \frac{r^2 c}{2 n \sigma^2}$$

$$e) \quad h_{\pi}(y; \text{reg}) d) = e^{cd} E(e^{-cd} | Y) \\ = e^{cd + c E(\theta | Y) - 1} \\ \text{(from part c)}$$

$$= e^{cd} e^{\left(\frac{c^2 \sigma^2}{2n} - \bar{y} c\right) - cd + c E(\theta | Y) - 1} \\ \text{(from part d)}$$

$$E(\theta | Y) = \bar{y} \quad (\because \pi(\theta | Y) \sim N(\bar{y}, \frac{\sigma^2}{n}))$$

$$\text{Posterior loss for } d_d(y) = \left(\bar{y} - \frac{c\sigma^2}{2n}\right)$$

$$h_{\pi}(y, d_d(y)) = e^{c(\bar{y} - \frac{c\sigma^2}{2n}) + \left(\frac{c^2 \sigma^2}{2n} - \bar{y} c\right)} \\ = e^{c(\bar{y} - \frac{c\sigma^2}{2n}) + \bar{y} c - 1}$$

$$= \frac{1}{e} \cdot 1 + \frac{c^2 \sigma^2}{2n} = 1$$

$$= \frac{c^2 \sigma^2}{2n}$$

$$\text{Posterior loss for } d_c(y) = \left(\bar{y} - \frac{c\sigma^2}{2n}\right) \bar{y}$$

$$h_{\pi}(y, d_c(y)) = e^{c(\bar{y}) + \frac{c^2 \sigma^2}{2n} - \bar{y} c} \\ = e^{c(\bar{y}) + \bar{y} c - 1}$$

$$= e^{\frac{c^2 \sigma^2}{2n}} - 1$$

Agg of exp

We note that $\frac{c^2 \sigma^2}{2n} > 0$

$$\Rightarrow e^x \approx 1 + x + \frac{x^2}{2!} + \dots$$

$$\Rightarrow e^x \geq 1 + x \quad (\text{equality holds at } x=0)$$

$$\Rightarrow e^x - 1 \geq x$$

$$\Rightarrow e^{\frac{c^2 \sigma^2}{2n}} - 1 \geq \frac{c^2 \sigma^2}{2n}$$

As expected δ_d performs better (since that is optimal under LINEAR)

$$f) h_{\pi}(y, d) = \int_{-\infty}^{\infty} (\theta - d)^2 \pi(\theta | y) d\theta$$

$$= \int_{-\infty}^{\infty} (\theta^2 - 2d\theta + d^2) \pi(\theta | y) d\theta$$

(from part d)

$$= E(\theta^2) - 2dE(\theta) + d^2$$

$$\sim \theta \sim N(\bar{y}, \frac{\sigma^2}{n})$$

$$E(\theta^2) = \left(\bar{y}^2 + \frac{\sigma^2}{n} \right) + E(\theta) = \bar{y}$$

$$\Rightarrow h(\bar{y}, d) = \left(\bar{y}^2 + \frac{\sigma^2}{n} \right) - 2d\bar{y} + d^2$$

$$\text{Find } \delta_d(\bar{y}) = \bar{y} - \left(\frac{c\sigma^2}{2n} \right)$$

$$h(\bar{y}, \delta_d) = \bar{y}^2 + \frac{\sigma^2}{n} - 2 \left(\bar{y} - \frac{c\sigma^2}{2n} \right) \bar{y} + \left(\bar{y} - \frac{c\sigma^2}{2n} \right)^2$$

$$= \bar{y}^2 - \bar{y}^2 + \frac{\sigma^2}{n} + \frac{2c\bar{y}\sigma^2}{n} + \bar{y}^2 - 2\frac{c\sigma^2\bar{y}}{2n} + \frac{c^2\sigma^4}{4n^2}$$

$$= \frac{\sigma^2}{n} + \frac{c^2\sigma^4}{4n^2} = \frac{\sigma^2}{n} + \frac{c^2\sigma^4}{4n^2}$$

$$\text{Find } \delta_e(\bar{y}) = \bar{y}$$

$$h(\bar{y}, \delta_e) = \bar{y}^2 + \frac{\sigma^2}{n} - 2\bar{y}^2 + \bar{y}^2$$

$$= \frac{\sigma^2}{n}$$

Again since for $L = (0 - d)^2$, $\delta_e(\bar{y})$

is optimal we clearly see $\delta_d \neq \delta_e$

$$h(\bar{y}, \delta_d) > h(\bar{y}, \delta_e)$$

$$\textcircled{4} a) \pi(\theta|y) = \frac{\pi(\theta) f_{\theta}(y)}{\int_0^{\infty} \pi(\theta) f_{\theta}(y) d\theta}$$

$$\begin{aligned} \int_0^{\infty} \pi(\theta) f_{\theta}(y) d\theta &= \int_0^{\infty} \frac{\theta^{\nu} y^{\nu-1} e^{-\theta y}}{\Gamma(\nu)} \times \frac{\theta^{\alpha-1} e^{-\theta/\beta}}{\Gamma(\alpha) \beta^{\alpha}} d\theta \\ &= \frac{y^{\nu-1}}{\Gamma(\nu) \Gamma(\alpha) \beta^{\alpha}} \int_0^{\infty} \theta^{\nu+\alpha-1} e^{-\theta(y+\frac{1}{\beta})} d\theta \end{aligned}$$

$$\text{Let } \alpha' = \nu + \alpha, \beta' = \frac{\beta}{1 + \beta y}$$

$$= \frac{y^{\nu-1} \Gamma(\alpha') (\beta')^{\alpha'}}{\Gamma(\nu) \Gamma(\alpha) \beta^{\alpha}} \int_0^{\infty} \frac{\theta^{\alpha'} e^{-\theta/\beta'}}{\Gamma(\alpha') (\beta')^{\alpha'}} d\theta$$

$$= \frac{y^{\nu-1} \Gamma(\alpha') (\beta')^{\alpha'}}{\Gamma(\nu) \Gamma(\alpha) \beta^{\alpha}}$$

$$\therefore \pi(\theta|y) = \frac{\left(\frac{\theta^{\alpha-1} e^{-\theta/\beta}}{\Gamma(\alpha) \beta^{\alpha}} \right) \left(\frac{\theta^{\nu} y^{\nu-1} e^{-\theta y}}{\Gamma(\nu)} \right)}{\frac{y^{\nu-1} \Gamma(\alpha') (\beta')^{\alpha'}}{\Gamma(\nu) \Gamma(\alpha) \beta^{\alpha}}}$$

$$= \frac{\Theta^{\alpha+\eta-1} e^{-\Theta(y\beta+1)}}{\Gamma(\alpha')(\beta')^{\alpha'}}$$

$$= \frac{\Theta^{\alpha'-1} e^{-\Theta/\beta'}}{(\Gamma(\alpha'))(\beta')^{\alpha'}} = \frac{\Theta^{\alpha'-1} e^{-\Theta/\beta'}}{\Gamma(\alpha')(\beta')^{\alpha'}}$$

$$\Rightarrow \pi(\theta|y) = \text{Gamma}(\alpha', \beta')$$

b) We know $\delta_{\text{Bayes}}(y)$ with posterior $\pi(\theta|y)$

$$\Rightarrow \delta \in (\pi(\theta|y))$$

$$\begin{aligned} \Rightarrow \delta_{\text{Bayes}}(y) &= \text{mean}(\text{Gamma}(\alpha', \beta')) \\ &= \alpha' \beta' \\ &= (\alpha + \eta) \left(\frac{\beta}{1 + \beta y} \right) \end{aligned}$$

c) $L^+(y, d) = \int_{\theta \in \Theta} L(\theta, d) \pi(\theta|y) d\theta$

$$L^+(\theta, d) = \begin{cases} 1 & |\theta - d| > 2 \\ 0 & |\theta - d| < 2 \end{cases}$$

non zero

We have less only when

$$|\theta - d| > 2 \Rightarrow \theta > 2 + d$$

$$\text{and } \theta < d - 2$$

$$h_{\pi}(y, d) = \int_{d+2}^{\infty} \pi(\theta|y) d\theta + \int_{-\infty}^{d-2} \pi(\theta|y) d\theta$$

$$= \int_{d+2}^{\infty} \pi(\theta|y) d\theta + \int_{d-2}^{d+2} \pi(\theta|y) d\theta + \int_{-\infty}^{d-2} \pi(\theta|y) d\theta$$

$$- \int_{d-2}^{d+2} \pi(\theta|y) d\theta$$

$$= \int_{-\infty}^{\infty} \pi(\theta|y) d\theta - \int_{d-2}^{d+2} \pi(\theta|y) d\theta$$

$$= 1 - \int_{d-2}^{d+2} \pi(\theta|y) d\theta$$

$$\frac{d(h_{\pi}(y, d))}{d(d)} = -(\pi(d+2|y) - \pi(d-2|y))$$

$$\frac{d^2(h_{\pi}(y, d))}{d^2 d} = -(\pi'(d+2|y) - \pi'(d-2|y)) \quad (\text{Leibniz rule})$$

Setting denominator \rightarrow

$$\Rightarrow \pi(d+2|y) = \pi(d-2|y)$$

We know that

From the hint, $\min d^*$ is shown to be

$$d^* = \begin{cases} \text{solution of } \pi(d^* - 2/y) = \pi(d^* + 2/y) & \text{when } \pi(0/y) < \pi(4/y) \\ 2 & \text{else} \end{cases}$$

$$\Rightarrow \delta_{\text{Bayes}} = \pi(0/y) < \pi(4/y)$$

$$\Rightarrow 0 < \frac{4^{\alpha'-1} e^{-y/\beta'}}{\Gamma(\alpha') (\beta')^{\alpha'}} \quad \text{when } \alpha' \neq 1 > 1$$

$$\nexists \alpha' = \alpha + y \neq 1,$$

$$\frac{e^{-y/\beta'}}{\Gamma(\alpha') (\beta')^{\alpha'}} \leq \frac{e^{2y - y/\beta'}}{4^2 \Gamma(\alpha') (\beta')^{\alpha'}}$$

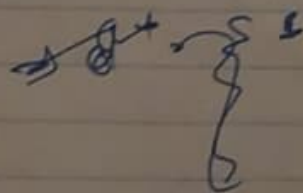
$$\Rightarrow 1 < 1$$

Not possible

$$\alpha' = \alpha + y = 1 \Rightarrow \pi(0/y) = \pi(4/y)$$

$$\nexists \alpha' < 1$$

$$\pi(0/y) \rightarrow \infty \quad (\text{0 in denominator})$$



$$\Rightarrow \delta^+ = \begin{cases} 2 & \text{if } q+x \leq 1 \\ \text{solution of } \pi(d+2/y) & q+x > 1 \\ \quad -q(d-2/y) & \end{cases}$$

$$= \begin{cases} 2 & q+x \leq 1 \\ -2 + 4 \left(1 - \exp \left(- \frac{4(y+1/p)}{x+q-1} \right) \right) & q+x > 1 \end{cases}$$

Bayer

(as given in gn)

\therefore constant interval with fixed width 4

$$\text{as } [\delta^+(y) - 2, \delta^+(y) + 2]$$

$$a \stackrel{\Delta}{=} \frac{\alpha \theta}{\alpha + \beta - n}, \quad b \stackrel{\Delta}{=} \frac{n}{\alpha + \beta - n}$$

$$\textcircled{5} a) R_S(\theta) = E((\theta - d)^2)$$

$$= E((\theta - a - b \bar{y}_n)^2)$$

$$= E(\theta^2 + a^2 + b^2 E(\bar{y}_n^2)$$

$$+ 2ab E(\bar{y}_n) - 2a\theta - 2b\theta E(\bar{y}_n)$$

$$E(\bar{y}_n^2) = E\left(\left(\frac{\sum y_i}{n}\right)^2\right) = E\left(\frac{\sum y_i^2}{n^2} - \frac{12 \sum_{i \neq j} y_i y_j}{n^2}\right)$$

$$E(\text{cross terms}) = 0 \text{ because of independence}$$

$$\Rightarrow E(\bar{y}_n^2) = \frac{1}{n^2} \frac{n \times (\theta)}{\text{var.}} = \left(\frac{\theta}{n}\right)^2 + \theta^2$$

$$E(\bar{y}_n) = \frac{\theta}{n}$$

$$\therefore R_S(\theta) = \theta^2 + a^2 + b^2 \left(\frac{\theta^2}{n} + \theta^2\right) + 2ab \frac{\theta}{n} - 2a\theta - 2b\theta \frac{\theta}{n}$$

$$= \theta^2 \left(1 + \frac{b^2}{n}\right) + a^2 + \frac{2ab\theta}{n} - 2a\theta - \frac{2b\theta^2}{n}$$

$$\Rightarrow b = \frac{1}{2} \text{ and } \frac{b^2}{n} + \frac{2ab}{n} - 2a = 0$$

$$\Rightarrow \frac{1}{4n} + \frac{2a}{n} - 2a = 0 \Rightarrow \frac{1}{4n} + \frac{2a}{n} = 2a \Rightarrow \frac{1}{4n} = \frac{2a}{n} \Rightarrow a = \frac{1}{8}$$

$$\left(\frac{\bar{y}_n^2}{n} \right)$$

$$Rd(\theta) = \theta^2 + a^2 + b^2 \left(\frac{\theta}{h} \right) + 2ab\theta - 2a\theta - 2b\theta^2$$

$$= \theta^2(1-2b) + \theta \left(\frac{b^2}{h} + 2ab - 2a \right) + a^2$$

b) eliminating θ thus,

$$b = 1/2$$

$$a = a - 2a + b \left(\frac{1}{4n} \right) = 0 \Rightarrow a = \frac{1}{4n}$$

$$\Rightarrow \frac{h}{\alpha + \beta + n} = \frac{1}{2} \text{ and } \frac{\alpha}{\alpha + \beta + n} = \frac{1}{4n}$$

$$\Rightarrow \frac{\alpha}{n} = \frac{1}{2n} \Rightarrow \alpha = \frac{1}{2}$$

$$\text{and } \beta = n - \frac{1}{2}$$

c) mean of $RS(\theta) =$

$$Rf(\theta) = a^2 = \frac{1}{16n^2}$$

$$rf = \int \frac{1}{16n^2} \pi(\theta) d\theta = \frac{1}{16n^2}$$

$$\text{var}(\bar{y}_n) = \frac{\sigma^2}{n}$$

$$\frac{n\theta(1-\theta)}{n^2} = \frac{\theta^2}{n}$$

$$\frac{\theta(1-\theta)}{n} + n\theta = \theta\left(\frac{n+1}{n}\right)$$

d)

Subst $a=0, b=1,$

$$R_S(\theta) = \frac{\theta^2}{n} + E(\bar{y}_n^2) - 2\theta E(\bar{y}_n)$$

$$= \theta^2 + \frac{\theta}{n} - 2\theta^2 = \frac{\theta - \theta^2}{n}$$

$$e) \quad r_{\delta_{\text{simplex}}} = a^2 = \left(\frac{\alpha + \beta}{\alpha - (1-\beta)n} \right)^2$$

$$r_{\delta_{\text{simplex}}} = \int \left(\frac{\theta - \theta^2}{n} \right) \pi(\theta) d\theta$$

$$= \frac{\alpha}{\alpha + \beta} \cdot \frac{1}{n} - \frac{1}{n} \left(\frac{\alpha^2}{(\alpha - \beta)^2 (\alpha - (1-\beta)n)^2} + \frac{\alpha}{(\alpha - \beta)^2 (\alpha - (1-\beta)n)} \right)$$

$$= \frac{1}{n} \left[\frac{\alpha}{\alpha + \beta} - \left(\frac{\alpha^2 \beta^2}{(\alpha - \beta)^2 (\alpha - (1-\beta)n)^2} + \frac{\alpha}{(\alpha - \beta)^2 (\alpha - (1-\beta)n)} \right) \right]$$