ISyE 6412A **Theoretical Statistics** **Fall 2023**

HW #8 (due at Canvas midnight on Wednesday, Nov 1, ET)

(After you spent at least 30 minutes per question, please take a look at the hints on the second page.)

1. (**Modified from problem 7.37 of our text**). Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution on the interval $(-\theta, 2\theta)$, $\theta > 0$. That is, the X_i's are iid with pdf $f_\theta(x) = \frac{1}{3\theta}\mathbf{1}\{-\theta < x < 2\theta\}$ for $\theta > 0$. Find, if one exists, a best unbiased estimator of θ.

   Answer: Recall that in HW#7, we have shown that

   $$T(\mathbf{X}) = \max\Big(-\min_{1\le i\le n}(X_i), \frac{1}{2}\max_{1\le i\le n}(X_i)\Big) = \max\Big(-X_{(1)}, \frac{1}{2}X_{(n)}\Big)$$

   is a minimal sufficient statistic, and is also complete. Moreover, the distribution of $T(X)$ is

   $$f_T(t) = \begin{cases} \frac{n}{\theta^n}t^{n-1}, & \text{if } 0 \le t \le \theta; \\ 0, & \text{otherwise.} \end{cases}$$

   (we have already shown in class that this family of distributions is complete!!!) Indeed, this is just the distribution of $X_{(n)}$ when $X_1, \cdots X_n$ are iid Uniform$(0, \theta)$. We have also shown in class that $\mathbf{E}_\theta(T) = \frac{n}{n+1}\theta$ or $\mathbf{E}_\theta(\frac{n+1}{n}T) = \theta$. Thus, by Rao-Blackwell Theorem, in this problem the best unbiased estimator of θ is

   $$\delta(\mathbf{X}) = \frac{n+1}{n}T(\mathbf{X}) = \frac{n+1}{n}\max\Big(-\min_{1\le i\le n}(X_i), \frac{1}{2}\max_{1\le i\le n}(X_i)\Big).$$

   $\square$

2. (**Problem 7.55(b) of our text**). Let $X_1, \ldots, X_n$ be a random sample from the pdf $f_\theta(x) = e^{-(x-\theta)}$ for $x > \theta$, where $-\infty < \theta < \infty$. Find the best unbiased estimator of $\phi(\theta) = \theta^r$ for some constant $r \ge 1$ (here r might or might not be an integer).

   Answer: Recall that in HW#7, we have shown that $X_{(1)}$ is a complete sufficient statistic for θ (it is not difficult to show that it is also minimal sufficient), and the density function of $T = X_{(1)}$ is

   $$f_{X_{(1)}}(t) = \begin{cases} ne^{-nt+n\theta}, & \text{if } t > \theta; \\ 0, & \text{otherwise.} \end{cases}$$

   Thus, using the integration by parts, we have

   $$\mathbf{E}_\theta(T^r) = \int_\theta^\infty ne^{n(\theta-t)}t^r dt = -t^r e^{n(\theta-t)}|_\theta^\infty + r\int_\theta^\infty e^{n(\theta-t)}t^{r-1}dt = \theta^r + \frac{r}{n}\mathbf{E}_\theta(T^{r-1}).$$

   Thus $T^r - \frac{r}{n}T^{r-1}$ is an unbiased estimator of θ^r. Since this is the function of complete sufficient statistic, $\delta^* = T^r - \frac{r}{n}T^{r-1}$ with $T = X_{(1)}$ is the best unbiased estimator of θ^r. $\square$

3. (**Problem 7.57 of our text**) Let $X_1, \ldots, X_{n+1}$ be iid Bernoulli(p), and define the function $h(p)$ by $h(p) = \mathbf{P}\Big(\sum_{i=1}^n X_i > X_{n+1}|p\Big)$, the probability that the first n observations exceed the $(n+1)$st.

   (a) Show that

   $$\delta(X_1, \ldots, X_{n+1}) = \begin{cases} 1, & \text{if } \sum_{i=1}^n X_i > X_{n+1}; \\ 0, & \text{otherwise.} \end{cases}$$

   is an unbiased estimator of $h(p)$.

   (b) Find the best unbiased estimator of $h(p)$.

Answer: The proof of part (a) is trivial since δ is Bernoulli and

$$\begin{aligned}\mathbf{E}_p(\delta) &= 0\mathbf{P}_p(\delta=0)+1\mathbf{P}_p(\delta=1)=\mathbf{P}_p(\delta=1)\\ &= \mathbf{P}_p(\sum_{i=1}^{n}X_i>X_{n+1})=h(p),\end{aligned}$$

i.e., δ is indeed an unbiased estimator of $h(p)$.
To solve part (b), first note that $T(\mathbf{X})=\sum_{i=1}^{n+1}X_i$ is a complete sufficient statistic for p (we have proved it in class, haven't we?) Hence, the best unbiased estimator is given by $\delta^*=\mathbf{E}(\delta|T)$, which needs to be calculated explicitly.
By the definition, $\delta(\mathbf{X})$ is either 1 or 0, thus we need to compute

$$\begin{aligned}\delta^*(b) &= \mathbf{E}(\delta|T=b)=\sum_{\mathbf{x}}\delta(\mathbf{x})P_p(\mathbf{X}=\mathbf{x}|T(\mathbf{X})=b)\\ &= \sum_{c}c\mathbf{P}_p(\delta(\mathbf{X})=c|T(\mathbf{X})=b) \qquad \text{(another way to calculate it)}\\ &= \mathbf{P}_p(\delta(\mathbf{X})=1|T(\mathbf{X})=b) \qquad \text{(since } c=0 \text{ or } 1)\\ &= \frac{\mathbf{P}_p(\delta(\mathbf{X})=1 \text{ and } T(\mathbf{X})=b)}{\mathbf{P}_p(T(\mathbf{X})=b)}\\ &= \frac{\mathbf{P}_p(\sum_{i=1}^{n}X_i>X_{n+1} \text{ and } \sum_{i=1}^{n}X_i+X_{n+1}=b)}{\mathbf{P}_p(\sum_{i=1}^{n+1}X_i=b)}.\end{aligned}$$

Now the denominator equals to $\binom{n+1}{b}p^b(1-p)^{n+1-b}$, since $\sum_{i=1}^{n+1}X_i$ has a Binomial$(n+1,p)$ distribution. To find the numerator, it is important to note that X_{n+1} can only take two values 0 or 1, and thus the numerator of the last equation can be written into two parts, depending on whether $X_{n+1}=0$ or 1. Hence, the numerator of the last equation become

$$\mathbf{P}_p(\sum_{i=1}^{n}X_i\geq 1;\sum_{i=1}^{n}X_i=b;\ \text{and } X_{n+1}=0)+\mathbf{P}_p(\sum_{i=1}^{n}X_i\geq 2;\sum_{i=1}^{n}X_i=b-1;\ \text{and } X_{n+1}=1),$$

where the first term is non-zero if and only if $b\geq 1$ and the second term is non-zero if and only if $b-1\geq 2$ (i.e., $b\geq 3$). This suggests that the computation of the numerator needs to consider three different cases of b : (a) when $b=0$; (b) when $b=1$ or 2; and (c) when $b\geq 3$. Therefore,

$$\delta^*(b) = \begin{cases} 0, & \text{if } b=0;\\ \frac{\mathbf{P}_p(\sum_{i=1}^{n}X_i=b \text{ and } X_{n+1}=0)}{\binom{n+1}{b}p^b(1-p)^{n+1-b}} = \frac{\binom{n}{b}p^b(1-p)^{n-b}(1-p)}{\binom{n+1}{b}p^b(1-p)^{n+1-b}} = \frac{\binom{n}{b}}{\binom{n+1}{b}} = \frac{n+1-b}{n+1}, & \text{if } b=1 \text{ or } 2.\\ \frac{\binom{n}{b}p^b(1-p)^{n-b}(1-p)+\binom{n}{b-1}p^{b-1}(1-p)^{n-b+1}p}{\binom{n+1}{b}p^b(1-p)^{n+1-b}} = \frac{\binom{n}{b}+\binom{n}{b-1}}{\binom{n+1}{b}} \equiv 1, & \text{if } 3\leq b(\leq n+1).\end{cases}$$

In other words, the best unbiased estimator of $h(p)$ is $g(\sum_{i=1}^{n+1}X_i)$ with

$$g(b)=\begin{cases}0, & \text{if } b=0\\ \frac{n+1-b}{n+1}, & \text{if } b=1 \text{ or } 2\\ 1, & \text{if } 3\leq b(\leq n+1)\end{cases}$$

Equivalently, the best unbiased estimator of $h(p)$ is

$$\delta^*(X_1,\cdots,X_{n+1})=\begin{cases}0, & \text{if } \sum_{i=1}^{n+1}X_i=0\\ \frac{n+1-\sum_{i=1}^{n+1}X_i}{n+1}=1-\bar{X}_{n+1}, & \text{if } \sum_{i=1}^{n+1}X_i=1 \text{ or } 2\\ 1, & \text{if } 3\leq\sum_{i=1}^{n+1}X_i(\leq n+1)\end{cases}$$

□

4. (**Motivated from Problem 7.59 of our text**). Let $X_1, \ldots, X_n$ be iid $N(\mu, \sigma^2)$, where both μ and σ is unknown, i.e., $\theta = (\mu, \sigma)$. Find the best unbiased estimators of
(a) $\phi(\theta) = \sigma$; (b) $\phi(\theta) = \sigma^2$; and (c) $\phi(\theta) = \sigma^4$.

Answer: We know that $U = (n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$. Then

$$E(U^{p/2}) = \frac{1}{\Gamma(\frac{n-1}{2})2^{\frac{n-1}{2}}} \int_0^\infty u^{\frac{p+n-1}{2}-1} e^{-\frac{u}{2}} du = \frac{\Gamma(\frac{p+n-1}{2})}{\Gamma(\frac{n-1}{2})} 2^{\frac{p}{2}} = C_{p,n}$$

for all $p > 0$. Thus

$$\mathbf{E}\left(\frac{(n-1)S^2}{\sigma^2}\right)^{p/2} = C_{p,n}$$

and

$$\mathbf{E}\left((n-1)^{p/2} S^p / C_{p,n}\right) = \sigma^p.$$

This means that

$$\delta^* = \frac{(n-1)^{p/2}}{C_{p,n}} S^p$$

is an unbiased estimator of σ^p. By Theorem 6.2.25, $(\bar{X}, S^2)$ is a complete sufficient statistic. The unbiased estimator δ^* is a function of the complete sufficient statistic $(\bar{X}, S^2)$. Hence, it is the best unbiased estimator.
In particular, for $p = 1, 2, 4$, by the fact of $\Gamma(u+1) = u\Gamma(u)$, we have

$$\begin{aligned}
C_{1,n} = (U^{1/2}) &= \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} 2^{\frac{1}{2}} \\
C_{2,n} = \mathbf{E}(U) &= \frac{\Gamma(\frac{n-1}{2}+1)}{\Gamma(\frac{n-1}{2})} 2 = \frac{n-1}{2}(2) = n-1 \\
C_{4,n} = \mathbf{E}(U^2) &= \frac{\Gamma(\frac{n-1}{2}+2)}{\Gamma(\frac{n-1}{2})} 2^2 = (\frac{n-1}{2}+1)\frac{n-1}{2}(4) = (n+1)(n-1).
\end{aligned}$$

Hence, the best unbiased estimators for σ, σ^2 and σ^4 are

$$\sqrt{\frac{n-1}{2}} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} S, \qquad \frac{(n-1)S^2}{n-1} = S^2, \qquad \frac{(n-1)^2 S^4}{(n+1)(n-1)} = \frac{n-1}{n+1} S^4,$$

respectively.
For those who are interested in, you can derive more explicit answer for σ, depending on whether $n = 2k+1$ or $n = 2k+2$ for $k \geq 0$. For integer $k \geq 0$, we have

$$\Gamma(k) = (k-1)! \qquad \text{and} \qquad \Gamma(k+\frac{1}{2}) = \frac{(2k)!}{4^k k!}\sqrt{\pi} \qquad \text{and} \qquad \Gamma(k+1) = k!.$$

□

5. (**Modified from 6.31(c) & 7.60**). Let $X_1, \ldots, X_n$ be iid gamma(α, β) with $\alpha > 1$ known. That is, the pdf of X_i is

$$f_\beta(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \qquad 0 \leq x < \infty$$

for $\alpha > 1$ and $\beta > 0$. By Theorems 6.2.10 and 6.2.25, the statistic $T = \sum_{i=1}^n X_i$ is complete sufficient for β, and it is also well known that $T = \sum_{i=1}^n X_i$ has a Gamma$(n\alpha, \beta)$ distribution. To help you find the best unbiased estimator of $\phi(\beta) = 1/\beta$ when $\alpha > 1$ is known, we split it into the following steps.

(a) Show that $\mathbf{E}_\beta\left(\frac{1}{X_1}\right) = \frac{1}{(\alpha-1)\beta}$, and conclude that $\delta = (\alpha-1)/X_1$ is an unbiased estimator of $1/\beta$.

(b) Prove the lemma: if U/V and U are independent random variables, then $\mathbf{E}(\frac{U}{V}) = \mathbf{E}(\frac{1}{V})/\mathbf{E}(\frac{1}{U})$.

(c) Use Basu's Theorem and the lemma in (b) to show that under the setting of this question,

$$\mathbf{E}\Big(\frac{1}{X_1}\Big|T\Big) = \mathbf{E}\Big(\frac{T}{X_1}\frac{1}{T}\Big|T\Big) = \frac{\mathbf{E}\Big(\frac{1}{X_1}\Big)}{\mathbf{E}\Big(\frac{1}{T}\Big)}\frac{1}{T}.$$

(d) Find the best unbiased estimator of $\phi(\beta) = 1/\beta$.

Answer: (a)

$$\mathbf{E}_\beta\Big(\frac{1}{X_1}\Big) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty \frac{1}{x}x^{\alpha-1}e^{-x/\beta}dx = \frac{1}{\Gamma(\alpha)\beta^\alpha}\Big(\Gamma(\alpha-1)\beta^{\alpha-1}\Big) = \frac{1}{(\alpha-1)\beta},$$

as $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$.

(b) Since U/V and $1/U$ are also independent, we have

$$\mathbf{E}(\frac{1}{V}) = \mathbf{E}\Big(\frac{U}{V}\frac{1}{U}\Big) = \mathbf{E}(\frac{U}{V})\mathbf{E}(\frac{1}{U}).$$

(c) Given α, Gamma(α,β) is a scale family in β, and thus $\frac{T}{X_1}$ is ancillary for β. Since T is complete sufficient, by Basu's Theorem, T/X_1 and T are independent. Hence, by the hints and part (b), we have

$$\mathbf{E}\Big(\frac{1}{X_1}\Big|T\Big) = \mathbf{E}\Big(\frac{T}{X_1}\frac{1}{T}\Big|T\Big) = \mathbf{E}\Big(\frac{T}{X_1}\Big)\frac{1}{T} = \frac{\mathbf{E}\Big(\frac{1}{X_1}\Big)}{\mathbf{E}\Big(\frac{1}{T}\Big)}\frac{1}{T}.$$

(d) For the complete sufficient statistic T, it has a Gamma$(n\alpha,\beta)$, and thus the conclusion in (a) applies to T except α being replaced by $n\alpha$. That is, $\mathbf{E}_\beta\Big(\frac{1}{T}\Big) = \frac{1}{(n\alpha-1)\beta}$.

By (a), $\delta = (\alpha-1)/X_1$ is an unbiased estimator of $1/\beta$ and thus by Rao-Blackwell theorem, the best unbiased estimator of $\phi(\beta) = 1/\beta$ is

$$\delta^* = \mathbf{E}(\delta|T) = \mathbf{E}\Big(\frac{\alpha-1}{X_1}|T\Big) = (\alpha-1)\frac{\mathbf{E}\Big(\frac{1}{X_1}\Big)}{\mathbf{E}\Big(\frac{1}{T}\Big)}\frac{1}{T} = (\alpha-1)\frac{\frac{1}{(\alpha-1)\beta}}{\frac{1}{(n\alpha-1)\beta}}\frac{1}{T} = \frac{n\alpha-1}{T} = \frac{n\alpha-1}{\sum_{i=1}^n X_i}.$$

Alternatively, since $\mathbf{E}_\beta\Big(\frac{1}{T}\Big) = \frac{1}{(n\alpha-1)\beta}$, we have $\mathbf{E}_\beta((n\alpha-1)/T) = 1/\beta$. Thus the best unbiased estimator of $\phi(\beta) = 1/\beta$ is

$$\delta^* = (n\alpha-1)/T = \frac{n\alpha-1}{\sum_{i=1}^n X_i}.$$

□

6. (**Motivated from Problem 10.9 of our text**). Suppose that $X_1,\ldots,X_n$ are iid Poisson(θ). Find the best unbiased estimator of

(a) $\phi_1(\theta) = e^{-\theta}$, the probability that $X = 0$.

(b) $\phi_2(\theta) = \theta e^{-\theta}$, the probability that $X = 1$.

(c) A preliminary test of a possible carcinogenic compound can be performed by measuring the mutation rate of microorganisms exposed to the compound. An experimenter places the compound in 15 petri dishes and records the following number of mutant colonies:

10, 7, 8, 13, 8, 9, 5, 7, 6, 8, 3, 6, 6, 3, 5.

Calculate the best unbiased estimators of $e^{-\theta}$, the probability that no mutant colonies emerge, and $\theta e^{-\theta}$, the probability that one mutant colony will emerge.

Answer: First, note that $T(\mathbf{X}) = \sum_{i=1}^n X_i$ is a complete sufficient statistic for θ for Poisson distribution (can you prove it by yourself? Also see problem 7.52 on page 365 of our text). Next, we find an unbiased estimator δ and then derive the best unbiased estimator by Rao-Blackwell Theorem through computing $\delta^* = \mathbf{E}(\delta|T)$.

(a) To find the best unbiased estimator of $h(\theta) = \mathbf{P}_\theta(X=0) = e^{-\theta}$, let

$$\delta = \delta(X_1, \cdots, X_n) = \begin{cases} 1, & \text{if } X_1 = 0; \\ 0, & \text{if } X_1 \neq 0. \end{cases}$$

It is easy to see that $\mathbf{E}_\theta(\delta) = \mathbf{P}_\theta(\delta = 1) = \mathbf{P}_\theta(X_1 = 0) = e^{-\theta}$. So δ is an unbiased estimator of $h(\theta) = \mathbf{P}_\theta(X=0) = e^{-\theta}$. By the Rao-Blackwell theorem, the best unbiased estimator is given by $\delta^* = \mathbf{E}(\delta|T)$.

By the definition, δ is either 1 or 0, and thus, when $T(\mathbf{X}) = b$ for some non-negative integer b,

$$\begin{aligned} \delta^*(b) &= \mathbf{E}(\delta|T=b) \\ &= \sum_c c\mathbf{P}_\theta(\delta(\mathbf{X}) = c|T(\mathbf{X}) = b) \\ &= \mathbf{P}_\theta(\delta(\mathbf{X}) = 1|T(\mathbf{X}) = b) \qquad (\text{since } c = 0 \text{ or } 1) \\ &= \frac{\mathbf{P}_\theta(\delta(\mathbf{X}) = 1 \text{ and } T(\mathbf{X}) = b)}{\mathbf{P}_\theta(T(\mathbf{X}) = b)} \\ &= \frac{\mathbf{P}_\theta(X_1 = 0 \text{ and } \sum_{i=2}^n X_i = b)}{\mathbf{P}_\theta(\sum_{i=1}^n X_i = b)} \end{aligned}$$

Recall that the sum of independent Poisson RVs is still Poisson distributed, and thus $\sum_{i=1}^n X_i$ and $\sum_{i=2}^n X_i$ are Poisson distributed with means $n\theta$ and $(n-1)\theta$, respectively. Thus,

$$\delta^*(b) = \begin{cases} 0, & \text{if } b < 0; \\ \frac{\{e^{-\theta}\}\{[(n-1)\theta]^b e^{-[(n-1)\theta]}/b!)}{(n\theta)^b e^{-(n\theta)}/b!)} = \left(\frac{n-1}{n}\right)^b, & \text{if } b \geq 0. \end{cases}$$

In particular, the possible values of $T = \sum_{i=1}^n X_i$ are $0, 1, 2, \cdots$, and thus for $b = 0, 1, 2, \cdots$, we have

$$\delta^*(b) = \left(\frac{n-1}{n}\right)^b = \left(1 - \frac{1}{n}\right)^b.$$

Replacing b by $T = \sum_{i=1}^n X_i$ will lead to the corresponding best unbiased estimator of $e^{-\theta}$:

$$\delta^* = \left(1 - \frac{1}{n}\right)^T \qquad \text{with} \qquad T = \sum_{i=1}^n X_i.$$

(b) To find the best unbiased estimator of $h(\theta) = \mathbf{P}_\theta(X=1) = \theta e^{-\theta}$, Let

$$\delta = \delta(X_1, \cdots, X_n) = \begin{cases} 1, & \text{if } X_1 = 1; \\ 0, & \text{if } X_1 \neq 1. \end{cases}$$

It is easy to see that $\mathbf{E}_\theta(\delta) = \mathbf{P}_\theta(\delta = 1) = \mathbf{P}_\theta(X_1 = 1) = \theta e^{-\theta}$. So δ is an unbiased estimator of $h(\theta) = \mathbf{P}_\theta(X=1) = \theta e^{-\theta}$. By the Rao-Blackwell theorem, the best unbiased estimator is given by $\delta^* = \mathbf{E}(\delta|T)$.

By the definition, δ is either 1 or 0, and thus, when $T(\mathbf{X}) = b$ for some non-negative integer b,

$$\begin{aligned} \delta^*(b) &= \mathbf{E}(\delta|T=b) \\ &= \sum_c c\mathbf{P}_\theta(\delta(\mathbf{X}) = c|T(\mathbf{X}) = b) \\ &= \mathbf{P}_\theta(\delta(\mathbf{X}) = 1|T(\mathbf{X}) = b) \qquad (\text{since } c = 0 \text{ or } 1) \end{aligned}$$

$$= \frac{\mathbf{P}_\theta(\delta(\mathbf{X}) = 1 \text{ and } T(\mathbf{X}) = b)}{\mathbf{P}_\theta(T(\mathbf{X}) = b)}$$

$$= \frac{\mathbf{P}_\theta(X_1 = 1 \text{ and } \sum_{i=2}^n X_i = b-1)}{\mathbf{P}_\theta(\sum_{i=1}^n X_i = b)}$$

Recall that the sum of independent Poisson RVs is still Poisson distributed, and thus $\sum_{i=1}^n X_i$ and $\sum_{i=2}^n X_i$ are Poisson distributed with means $n\theta$ and $(n-1)\theta$, respectively. Thus,

$$\delta^*(b) = \begin{cases} 0, & \text{if } b = 0; \\ \frac{\{\theta e^{-\theta}\}\{[(n-1)\theta]^{b-1}e^{-[(n-1)\theta]}/(b-1)!)}{(n\theta)^b e^{-(n\theta)}/b!)} = b\frac{(n-1)^{b-1}}{n^b}, & \text{if } b \geq 1. \end{cases}$$

In particular, for $b = 0, 1, 2, \cdots$, we have

$$\delta^*(b) = \frac{b}{n}\Big(1 - \frac{1}{n}\Big)^{b-1}.$$

Replacing b by $T = \sum_{i=1}^n X_i$ will lead to the corresponding best unbiased estimator of $\theta e^{-\theta}$:

$$\delta^* = \frac{T}{n}\Big(1 - \frac{1}{n}\Big)^{T-1} \qquad \text{with} \qquad T = \sum_{i=1}^n X_i.$$

(c) Here $n = 15, T = \sum_{i=1}^n X_i = 104$ and $\bar{X} = T/n = 6.9333$.
For part (a), $\delta_1^* = 0.0007653$.
For part (b), $\delta_2^* = 0.0056850$. □

Hints: If you have already thought about each problem for at least 30 minutes, then please feel free to look at the hints. Otherwise, please try the problem first, as getting help from the hints takes away most of the fun.

Problem 1: you have found the complete sufficient statistic for θ in HW#7, right? Can you use it to find the best unbiased estimator?
Problem 2: Use the complete sufficient statistic T to construct an unbiased estimator. Using integration by parts,

$$\mathbf{E}_\theta(T^r) = \int_\theta^\infty n e^{n(\theta-t)} t^r dt = -t^r e^{n(\theta-t)}|_\theta^\infty + r\int_\theta^\infty e^{n(\theta-t)} t^{r-1} dt = \theta^r + \frac{r}{n}\mathbf{E}_\theta(T^{r-1}).$$

From this relation, can you see $\mathbf{E}_\theta(g(T)) = \theta^r$ and find an unbiased estimator $g(T)$ of θ^r?

Problem 3: Recall that X_{n+1} can only take two possible values: 0 or 1, and you may need to consider different cases of $\sum_{i=1}^{n+1} X_i = b$, depending on whether $b = 0, 1, 2$ or $b \geq 3$.

Problem 4: We have shown in class that $(\bar{X}, S^2)$ is the complete sufficient statistics. Also $U = (n-1)S^2/\sigma^2$ is χ^2_{n-1} distribution, and compute $E(U^{p/2}) = C_{p,n}$, which does not depend on $\theta = (\mu, \sigma)$. Hence, $(n-1)^{p/2}S^p/C_{p,n}$ is an unbiased estimator of σ^p. What happens to $p = 1, 2, 4$?

Problem 5: in part (a), note that

$$\mathbf{E}_\beta\left(\frac{1}{X_1}\right) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty \frac{1}{x} x^{\alpha-1} e^{-x/\beta} dx = \frac{1}{\Gamma(\alpha)\beta^\alpha}\Big(\Gamma(\alpha-1)\beta^{\alpha-1}\Big),$$

In part (b), note that U/V and $1/U$ are also independent.
Part (c) is for the illustration of the use of Basu' Theorem. The key is to prove that $\frac{T}{X_1}$ and T are independent. Please feel free to use the well-known fact on the conditional expectation that if U and W are independent then $\mathbf{E}(Ug(W)|W) = \mathbf{E}(U)g(W)$ for any real-valued function $g(\cdot)$. In particular, $\mathbf{E}(\frac{U}{W}|W) = \mathbf{E}(U)\frac{1}{W}$.
In part (d), you may need to compute the value $\mathbf{E}(1/T)$. Please feel free to use the fact that T has a Gamma$(n\alpha, \beta)$ distribution, and thus the conclusion in (a) applies to $\mathbf{E}_\beta(1/T)$ too except that α is replaced by $n\alpha$.
Also in part (d), you *may* or *may not* use the result in (c), depending on which method you are using: Method 1 (hunting) or Method 2 (Rao-Blackwell). Either approach will be fine for part (d), as long as your final answer is correct.

Problem 6: Use Theorem 6.2.25 to find the complete sufficient statistic T for Poisson distribution. In (a), the indicator variable $\delta = 1(X_1 = 0)$ is unbiased. In (b), the indicator variable $\delta = 1(X_1 = 1)$ is unbiased. Then we can use Method 2 (Rao-Blackwell) to find the best unbiased estimator $\sigma^* = \mathbf{E}(\delta|T)$.