# Homework 1

ISYE6412 - Theoretical Statistics

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## 1 Problem 1

a)
- $S = \mathbb{R}$
- $\Omega = Normal(\theta, 1)$ where $\theta \in \mathbb{R}$.
- $D$ is the set of all real numbers. Example: $d = 3$ means that our guess for the mean of $X$ is 3.
- $L = \frac{(\theta-d)^2}{1+\theta^2}$

b) Let's calculate the risk function for a general form $\delta(X) = a + bX$.

$$\begin{aligned}\mathbb{E}[L(\theta,\delta(X))] &= \frac{1}{1+\theta^2}\mathbb{E}[(\theta-(a+bX))^2] \\ &= \frac{1}{1+\theta^2}((\mathbb{E}[a+bX]-\theta)^2 + Var(a+bX)) \\ &= \frac{1}{1+\theta^2}(a^2+2a(b-1)\theta+(b-1)^2\theta^2+b^2)\end{aligned}$$

Then, the risk functions are

- $R_{\delta_1}(\theta) = \frac{1}{1+\theta^2}$
- $R_{\delta_2}(\theta) = \frac{0.5(1-\theta+0.5\theta^2)}{1+\theta^2}$
- $R_{\delta_3}(\theta) = \frac{1}{4}$
- $R_{\delta_4}(\theta) = \frac{\theta^2+4}{1+\theta^2}$
- $R_{\delta_5}(\theta) = \frac{\theta^2}{1+\theta^2}$
- $R_{\delta_6}(\theta) = \frac{(\theta-1)^2}{1+\theta^2}$

The graph is as in Figure 1:

c) The procedure $\delta_4(\theta)$ seems inadmisible. At every point the value of the function is higher than the risk of procedure $\delta_1(\theta)$.

d) I would use the procedure number 1 because, even that it does not show the lowest risk value for values near zero, it does for values smaller than -2.5 or higher than 2.5, and since $\theta$ can be any real number, we have to think in all cases, not only near zero.

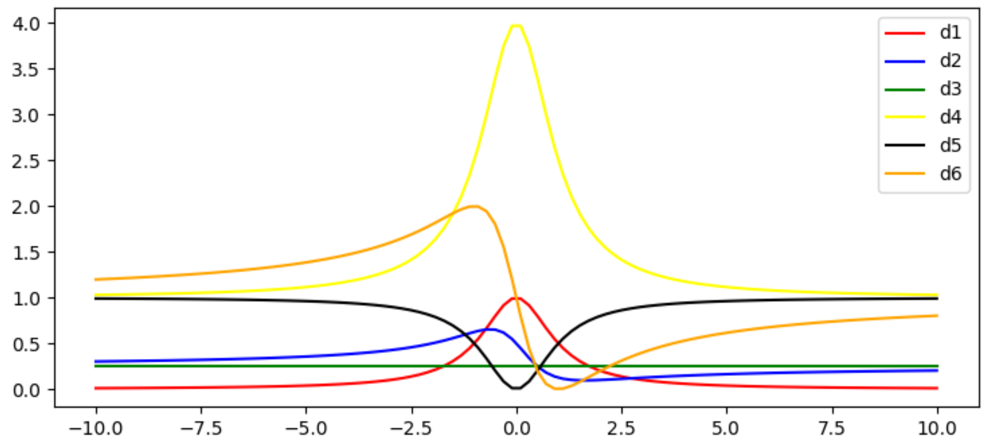



Figure 1: Risk functions of 1.b)

e) Again, let's calculate the risk function for a general form $\delta(\bar{X}_n) = a + b\bar{X}_n$.

$$\begin{aligned}\mathbb{E}[L(\theta, \delta(\bar{X}_n))] &= \frac{1}{1+\theta^2}\mathbb{E}[(\theta - (a + b\bar{X}_n))^2] \\ &= \frac{1}{1+\theta^2}((\mathbb{E}[a + b\bar{X}_n] - \theta)^2 + Var(a + b\bar{X}_n)) \\ &= \frac{1}{1+\theta^2}(a^2 + 2a(b-1)\theta + (b-1)^2\theta^2 + \frac{b^2}{n})\end{aligned}$$

Then, the risk functions are

- $R_{\delta_1}(\theta) = \frac{1}{n(1+\theta^2)}$
- $R_{\delta_2}(\theta) = \frac{n+1-2\theta+\theta^2}{(n+1)^2(1+\theta^2)}$
- $R_{\delta_3}(\theta) = \frac{1}{(1+\sqrt{n})^2}$
- $R_{\delta_6}(\theta) = \frac{(\theta-1)^2}{1+\theta^2}$

The graph is as in Figure 2. Note that they are scaled so they can be compared.

f) In this case, I would choose either procedure 1 or 2, for the same reasons as before. Among procedure 1 or 2, they seem equal.

g) We need to show two directions:

- Let b = 1. Need to show that risk function goes to zero when $|\theta|$ goes to infinity. If b=1, we have

$$\mathbb{E}[L(\theta, \delta(\bar{X}_n))] = \frac{1}{1+\theta^2}(a^2 + \frac{1}{n})$$

  Then, $\lim_{|\theta|\to\infty} \mathbb{E}[L(\theta, \delta(\bar{X}_n))] = 0$.

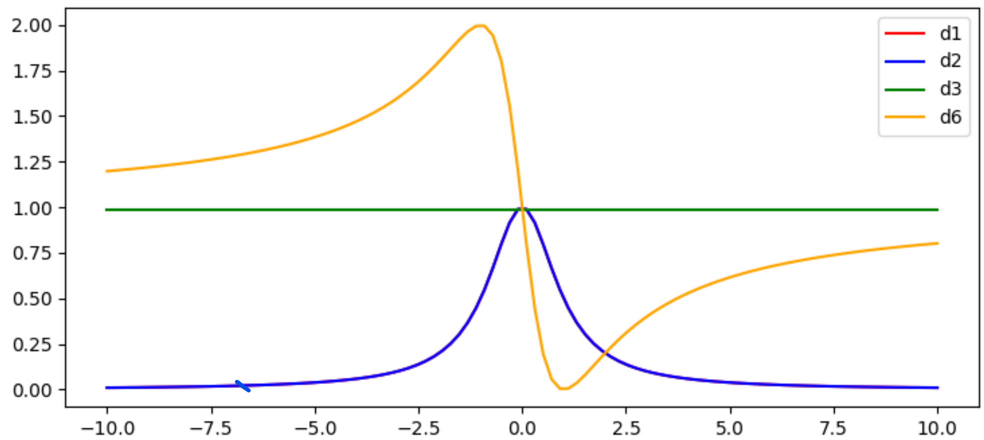



Figure 2: Risk functions of 1.e). The first and second procedures have almost the same graph for high n, that is why the red curve doesn't show, it is behind the blue one.

- Let's show the contrapositive of this direction. So, assume $b \neq 1$. Then,

$$\lim_{|\theta|\to\infty} \mathbb{E}[L(\theta, \delta(\bar{X}_n))] = \lim_{|\theta|\to\infty} \frac{1}{1+\theta^2}(a^2 + 2a(b-1)\theta + (b-1)^2\theta^2 + \frac{b^2}{n})$$
$$= \lim_{|\theta|\to\infty} \frac{(b-1)^2\theta^2}{1+\theta^2} = (b-1)^2,$$

  Which is different from zero since $b \neq 1$.

Now, among procedures with $b = 1$, we take the derivative and make it equal to zero to know the value of $a$ that minimizes the expression.

$$R_\delta = \frac{1}{1+\theta^2}(a^2 + \frac{1}{n})$$
$$\Rightarrow \frac{dR_\delta}{da} := \frac{1}{1+\theta^2}(2a) = 0 \Rightarrow a = 0$$

And we have that second derivative is positive, so it is indeed a minimum. Therefore, $a = 0$ gives the smallest risk function for every $n$.

h) Let $\delta := \delta_{6,n}$. Show by contradiction. Assume not. Then, there exists $\delta'(X)$ such that $R_{\delta'}(\theta) \leq R_\delta(\theta)$ for all $\theta$ and $R_{\delta'}(\theta_0) < R_\delta(\theta_0)$ for at least some $\theta_0$. Then, we have that for all $\theta$,

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{(\theta - \delta'(y_1, ..., y_n))^2}{1+\theta^2} \frac{1}{\sqrt{2\pi}} e^{-0.5(\theta-y_1)^2} \cdots \frac{1}{\sqrt{2\pi}} e^{-0.5(\theta-y_n)^2} dy_1 \dots dy_n \leq \frac{(\theta-1)^2}{1+\theta^2}$$

In particular, this holds for $\theta = 1$. Then, inserting that value:

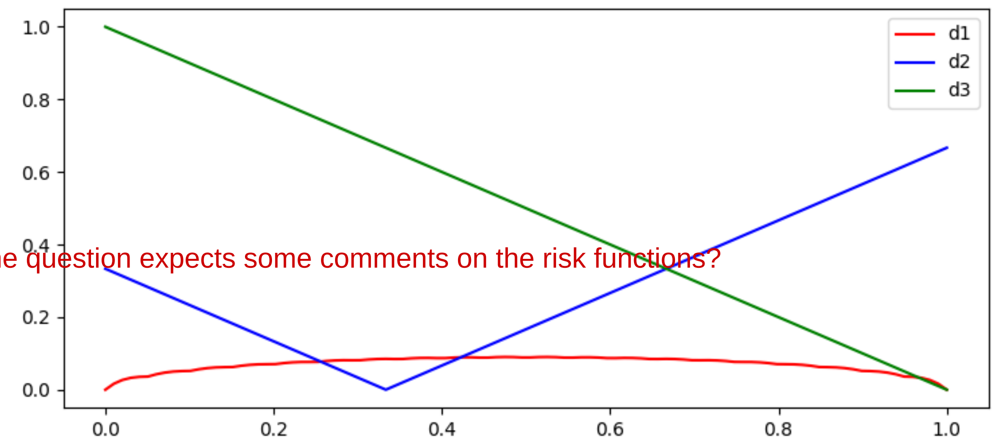
$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{(1 - \delta'(y_1, ..., y_n))^2}{2} \frac{1}{\sqrt{2\pi}} e^{-0.5(1-y_1)^2} \cdots \frac{1}{\sqrt{2\pi}} e^{-0.5(1-y_n)^2} dy_1 \dots dy_n \leq 0$$

We can see that the functions inside the integral is nonnegative. Also, $\frac{1}{\sqrt{2\pi}}$ and the exponential terms are always strictly positive. So, in order to this integral to be less or equal to zero (equal in this case, because everything is nonnegative), the term $(1 - \delta'(y_1, ..., y_n))^2$ must be zero for all $y_1, ..., y_n$. Therefore, $\delta'(y_1, ..., y_n) = 1$ for all $y$. But if this is true, then $R_{\delta'}(\theta_0) < R_\delta(\theta_0)$ is not true for any $\theta_0$, getting the contradiction. Thus, $\delta_{6,n}$ is admissible.

# 2 Problem 2

a)
- $S = \{0, ..., n\}$
- $\Omega = Bin(n, \theta)$ where $\theta \in [0, 1]$.
- $D$ is the set of all real numbers between 0 and 1. Example: $d = 0.6$ means that probability of success of $X$ is 0.6.
- $L = |\theta - \delta|$

b) The graph is as in Figure 3:




(not sure) i think the question expects some comments on the risk functions?

Figure 3: Risk functions of 2.b)

c) Show by contradiction. Assume not. Then, there exists $\delta'(X)$ such that $R_{\delta'}(\theta) \leq R_{\delta_2}(\theta)$ for all $\theta$ and $R_{\delta'}(\theta_0) < R_{\delta_2}(\theta_0)$ for at least some $\theta_0$. Then, we have that for all $\theta$,

$$\sum_{j=0}^{n} |\theta - \delta'| \binom{n}{j} \theta^j (1-\theta)^{n-j} \leq |\theta - \frac{1}{3}|$$

In particular, this holds for $\theta = 1/3$. Then, inserting that value:

$$\sum_{j=0}^{n} |\frac{1}{3} - \delta'(j)| \binom{n}{j} (\frac{1}{3})^j (\frac{2}{3})^{n-j} \leq 0$$

To this inequality to hold, the only possibility is $\delta'(j) = 1/3$ for all $j = 0, ..., n$. But if this is true, then $R_{\delta'}(\theta_0) < R_{\delta}(\theta_0)$ is not true for any $\theta_0$, getting the contradiction. Thus, $\delta_2$ is admissible.

d) Show by contradiction. Assume not. Then, there exists $\delta'(X)$ such that $R_{\delta'}(\theta) \leq R_{\delta_3}(\theta)$ for all $\theta$ and $R_{\delta'}(\theta_0) < R_{\delta_3}(\theta_0)$ for at least some $\theta_0$. Then, we have that for all $\theta \in [0, 1]$,

$$|\theta - \delta'(0)|(1-\theta)^2 + 2|\theta - \delta'(1)|\theta(1-\theta) + |\theta - \delta'(2)|\theta^2 \leq (1 - \theta)$$

In particular, this holds for $\theta = 1$. Then, inserting that value, and observing that we can get rid of the absolute values since $\delta' \leq 1$:

$$(1 - \delta'(2)) \leq 0$$

Which implies $\delta'(2) = 1$. Now, let's divide both sides by $1 - \theta$, with $0 \leq \theta < 1$. Then, using also the value $\delta'(2) = 1$ we have

$$|\theta - \delta'(0)|(1 - \theta) + 2|\theta - \delta'(1)|\theta + \theta^2 - 1 \leq 0$$

The left side is a mix of polynomial with absolute values of $\theta$, which is continuous at $\theta = 1$, so the inequality holds for all $0 \leq \theta \leq 1$. If we let $\theta = 1$, we have

$$2|\theta - \delta'(1)| \leq 0$$

Which implies $\delta'(1) = 1$. Now, let's divide again both sides by $1 - \theta$, with $0 \leq \theta < 1$. Then, using also the value $\delta'(2) = \delta'(1) = 1$ and $\theta^2 - 1 = -(\theta + 1)(1 - \theta)$we have

$$|\theta - \delta'(0)| + 2\theta - \theta - 1 \leq 0$$

The left side is continuous at $\theta = 1$, so the inequality holds for all $0 \leq \theta \leq 1$. If we let $\theta = 1$, we have

$$|\theta - \delta'(0)| + 2 - 2 \leq 0$$

Which implies $\delta'(0) = 1$.

Therefore, $\delta'(X) = 1$, but that would be a contradiction with $R_{\delta'}(\theta_0) < R_{\delta_2}(\theta_0)$ for at least some $\theta_0$. Thus, $\delta_3$ is admissible.