ISyE 6412A **Theoretical Statistics** **Fall 2023**

HW #7 (due at midnight on Thursday, Oct 26, ET)

(After you spent at least 30 minutes per question, please feel free to take a look at the hints on the second page.)

1. **(Modified from problem 6.23(a))**. Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution on the interval $(\theta, 2\theta)$, $\theta > 0$. That is, the X_i's are iid with pdf $f_\theta(x) = \frac{1}{\theta}\mathbf{1}\{\theta < x < 2\theta\}$ for $\theta > 0$.

   (a) Find a minimal sufficient statistic for θ.

   (b) Is the minimal sufficient statistic in part (a) complete? Justify your answers.

   Answer: (a) Note that the density function of $U(\theta, 2\theta)$ for $\theta > 0$ can be written as

$$f_\theta(x) = \frac{1}{\theta} I(\theta < x < 2\theta) I(\theta > 0).$$

We first use the factorization theorem to find a sufficient statistic, where it is important to write the likelihood function as a function of θ. Next, we show it is minimal sufficient.
First, the joint pdf of the sample $\mathbf{x}$ is

$$\begin{aligned}
f(\mathbf{x}|\theta) &= \prod_{i=1}^{n} [\frac{1}{\theta} I(\theta < x_i < 2\theta)] I(\theta > 0)] \\
&= \frac{1}{\theta^n} \prod_{i=1}^{n} [I(\theta < x_i) I(\theta > \frac{1}{2} x_i) I(\theta > 0)] \\
&= \frac{1}{\theta^n} I(\theta < x_{(1)}) I(\theta > \frac{1}{2} x_{(n)}) I(\theta > 0) \\
&= \frac{1}{\theta^n} I(\frac{1}{2} x_{(n)} < \theta < x_{(1)}) I(\theta > 0) = g(T(\mathbf{x})|\theta) h(\mathbf{x}),
\end{aligned}$$

where $T(\mathbf{x}) = (T_1, T_2) = (x_{(1)}, x_{(n)})$, $g(t_1, t_2|\theta) = \frac{1}{\theta^n} I(\frac{1}{2} t_2 < \theta < t_1) I(\theta > 0)$ and $h(\mathbf{x}) = 1$. Hence, by the Factorization Theorem, $T(\mathbf{X}) = (X_{(1)}, X_{(n)})$ is a sufficient statistic for θ.
Second, $T(\mathbf{X}) = (X_{(1)}, X_{(n)})$ is also minimal sufficient. To see this, for any two sample points $\mathbf{x}$ and $\mathbf{y}$,

$$\frac{f(\mathbf{x}|\theta)}{f(\mathbf{y}|\theta)} = \frac{I(x_{(n)}/2 < \theta < x_{(1)}) I(\theta > 0)}{I(y_{(n)}/2 < \theta < y_{(1)}) I(\theta > 0)},$$

which is a constant of θ only if $x_{(n)}/2 = y_{(n)}/2$ and $x_{(1)} = y_{(1)}$ (why?? how could both be positive for the same value of θ?). On the other hand, if $x_{(1)} = y_{(1)}$ and $x_{(n)} = y_{(n)}$, then this ratio is constant of θ (in fact, it is 1). Thus $T(\mathbf{X}) = (X_{(1)}, X_{(n)})$ is minimal sufficient.
(b) We will show that the minimal sufficient statistic $T(\mathbf{X}) = (X_{(1)}, X_{(n)})$ is not complete.
To see this, note that $U(\theta, 2\theta)$ is from a scale family, and thus we can easily see that $X_{(1)}/X_{(n)}$ is ancillary. In other words, $E_\theta(Z_{(1)}/Z_{(n)}) = C$ is a constant that does not depend on θ. Hence,

$$E_\theta(X_{(1)}/X_{(n)} - C) = 0 \qquad \text{for all } \theta > 0.$$

In other words, let $g(t_1, t_2) = t_1/t_2 - C$, which is a non-zero function of (t_1, t_2) but $E_\theta(g(T_1, T_2)) = E_\theta(g(T(\mathbf{X}))) = 0$ for all θ. Hence $T(\mathbf{X}) = (X_{(1)}, X_{(n)})$ is not complete.
An alternative proof is as follows. let $Z_i = X_i/\theta$, then $Z_1, \ldots, Z_n$ are unobserved random variables and are iid $U(1, 2)$. Denote $C_1 = E(Z_{(1)})$ and $C_2 = E(Z_{(n)})$. Then both C_1 and C_2 are positive constant and do not depend on θ. In fact, we can show that $C_1 = 1 + \frac{1}{n+1} = \frac{n+2}{n+1}$ and $C_2 = 1 + \frac{n}{n+1} = \frac{2n+1}{n+1}$, see problem 1(ii) of HW#2. Thus

$$E_\theta(X_{(1)}) = C_1 \theta \qquad \text{and} \qquad E_\theta(X_{(n)}) = C_2 \theta,$$

and so

$$E_\theta\left(\frac{1}{C_1}X_{(1)} - \frac{1}{C_2}X_{(n)}\right) = 0$$

for all $\theta > 0$. Hence, another alternative choice of non-zero function is $g(t_1, t_2) = t_1/C_1 - t_2/C_2$, then $E_\theta(g(T_1, T_2)) = E_\theta(g(T(\mathbf{X}))) = 0$ for all θ. This again shows that $T(\mathbf{X}) = (X_{(1)}, X_{(n)})$ is not complete. $\square$

2. (**Modified from Ex 6.5 of our text**, also see Problem 4 of HW#6). Assume that $X_1, \cdots, X_n$ are independent random variables with pdfs

$$f(x_i|\theta) = \begin{cases} \frac{1}{3i\theta}, & \text{if } -i(\theta-1) < x_i < i(2\theta+1); \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } i = 1, 2, 3, \ldots$$

where $\theta > 0$. Let $T(\mathbf{X})$ be the (one-dimensional) minimal sufficient statistic for θ you found in HW#6, also see the solution set of Problem 4 of HW#6. Is this minimal sufficient statistic $T(\mathbf{X})$ complete? Justify your answers.

Answer: To check whether $T(\mathbf{X})$ is complete or not, the key step is to find the distribution of $T(X)$! First, note that

$$T(\mathbf{X}) = \max\left\{1 - \min_{1\le i\le n}\frac{x_i}{i}, \quad \frac{1}{2}\left(\max_{1\le i\le n}\frac{x_i}{i} - 1\right)\right\},$$

is always non-negative, no matter whether $\min_{1\le i\le n}\frac{x_i}{i} \le 1$ or > 1 (the latter case implies that $\max_{1\le i\le n}\frac{x_i}{i} \ge 1$.)
Second, for all $t \ge 0$,

$$\begin{aligned}
\mathbf{P}_\theta(T(\mathbf{X}) \le t) &= \mathbf{P}_\theta\Big(\max\left\{1 - \min_{1\le i\le n}\frac{x_i}{i}, \quad \frac{1}{2}\left(\max_{1\le i\le n}\frac{x_i}{i} - 1\right)\right\} \le t\Big) \\
&= \mathbf{P}_\theta\Big(1 - \min_{1\le i\le n}\frac{x_i}{i} \le t \text{ and } \frac{1}{2}\left(\max_{1\le i\le n}\frac{x_i}{i} - 1 \le t\right)\Big) \\
&= \mathbf{P}_\theta\Big(\min_{1\le i\le n}\frac{x_i}{i} \ge 1 - t \text{ and } \max_{1\le i\le n}\frac{x_i}{i} \le 2t + 1\Big) \\
&= \mathbf{P}_\theta\left(1 - t \le \frac{x_i}{i} \le 2t + 1 \text{ for all } i = 1, \cdots, n\right) \\
&= \prod_{i=1}^n \mathbf{P}_\theta\Big(-(t-1) \le \frac{X_i}{i} \le 2t + 1)\Big) \\
&= \prod_{i=1}^n \mathbf{P}_\theta\Big(-i(t-1) \le X_i \le i(2t+1)\Big).
\end{aligned}$$

Now based on the definition of the density function of X_i, we will see that

$$\mathbf{P}_\theta\Big(-i(t-1) \le X_i \le i(2t+1)\Big) = \begin{cases} 0, & \text{if } t < 0; \\ 1, & \text{if } t > \theta. \end{cases}$$

The most interesting case is when $0 \le t < \theta$. In this case, we have

$$\begin{aligned}
\mathbf{P}_\theta\Big(-i(t-1) \le X_i \le i(2t+1)\Big) &= \int_{-i(t-1)}^{i(t+1)} \frac{1}{3i\theta}dx = \frac{i(2t+1) - (-i(t-1))}{3i\theta} = \frac{t}{\theta}, \text{ and} \\
\mathbf{P}_\theta(T(\mathbf{X}) \le t) &= \prod_{i=1}^n \frac{t}{\theta} = \frac{t^n}{\theta^n}.
\end{aligned}$$

Thus the density function of

$$T(\mathbf{X}) = \max\left\{1 - \min_{1\le i\le n}\frac{x_i}{i}, \quad \frac{1}{2}\left(\max_{1\le i\le n}\frac{x_i}{i} - 1\right)\right\},$$

is

$$f_T(t) = \begin{cases} \frac{n}{\theta^n} t^{n-1}, & \text{if } 0 \le t \le \theta; \\ 0, & \text{otherwise.} \end{cases}$$

Now, does this distribution look familiar to you??? In fact, this is just the distribution of $X_{(n)}$ where $X_1, \ldots, X_n$ are iid Uniform$(0, \theta)$. We have shown in class that the family of these distributions is complete. Thus, the desired statistic $T(\mathbf{X})$ is not only minimal sufficient, but also complete! □

3. (**Modified from problem 7.37 of our text**). Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution on the interval $(-\theta, 2\theta)$, $\theta > 0$. That is, the X_i's are iid with pdf $f_\theta(x) = \frac{1}{3\theta}\mathbf{1}\{-\theta < x < 2\theta\}$ for $\theta > 0$.

   (a) Find a minimal sufficient statistic for θ.

   (b) Is the minimal sufficient statistic in part (a) complete? Justify your answers.

Answer: (a) First, the joint probability density function is

$$\begin{aligned} f(\mathbf{x}|\theta) &= \prod_{1 \le i \le n} [\frac{1}{3\theta} I(-\theta < x_i < 2\theta)] \\ &= \frac{1}{(3\theta)^n} I\Big(\theta > -x_i \text{ and } \theta > \frac{1}{2}x_i \text{ for all } i = 1, 2, \cdots, n\Big) \\ &= \frac{1}{(3\theta)^n} I\Big(\theta > \max_{1 \le i \le n}(-x_i) \text{ and } \theta > \max_i(\frac{1}{2}x_i)\Big) \\ &= \frac{1}{(3\theta)^n} I\Big(\theta > \max[\max_{1 \le i \le n}(-x_i), \max_{1 \le i \le n}(\frac{1}{2}x_i)]\Big) \\ &= \frac{1}{(3\theta)^n} I\Big(\theta > T(\mathbf{X})\Big), \end{aligned}$$

where

$$T(\mathbf{X}) = \max[\max_{1 \le i \le n}(-x_i), \max_{1 \le i \le n}(\frac{1}{2}x_i)].$$

is a one-dimensional statistic. If you want, note that we can also write this T in the following equivalent form:

$$T(\mathbf{X}) = \max[-\min_{1 \le i \le n}(x_i), \frac{1}{2}\max_{1 \le i \le n}(x_i)] = \max[-x_{(1)}, \frac{1}{2}x_{(n)}.$$

Thus, by Factorization Theorem, $T(\mathbf{X}) = \max[\max_{1 \le i \le n}(-x_i), \max_{1 \le i \le n}(\frac{1}{2}x_i)]$ is a sufficient statistic for θ.
Next, is it minimal sufficient? Note that the range of X depends on θ and for any two sample points,

$$\frac{f(\mathbf{x}|\theta)}{f(\mathbf{y}|\theta)} = \frac{\frac{1}{(3\theta)^n} I\Big(\theta > T(\mathbf{x})\Big)}{\frac{1}{(3\theta)^n} I\Big(\theta > T(\mathbf{y})\Big)}.$$

Evidently, this ratio is independent of θ if and only if $T(\mathbf{x}) = T(\mathbf{y})$. So

$$T(\mathbf{X}) = \max[\max_{1 \le i \le n}(-x_i), \max_{1 \le i \le n}(\frac{1}{2}x_i)] = \max[-x_{(1)}, \frac{1}{2}x_{(n)}]$$

is a minimal sufficient statistic.
(b) What is the distribution of $T(X) = \max[\max_{1 \le i \le n}(-x_i), \max_{1 \le i \le n}(\frac{1}{2}x_i)] = \max[-x_{(1)}, \frac{1}{2}x_{(n)}]$?

Obviously, it is always non-negative, no matter whether $x_{(1)} \le 0$ or $x_{(1)} > 0$ (in the former case, $T \ge -x_{(1)} \ge 0$, and in the latter case, it implies that $T \ge \frac{1}{2}x_{(n)} \ge \frac{1}{2}x_{(1)} > 0$.) Now for all $t \ge 0$,

$$
\begin{aligned}
P_\theta(T(\mathbf{X}) \le t) &= P_\theta(\max[\max_{1\le i\le n}(-X_i), \max_{1\le i\le n}(\frac{1}{2}X_i)] \le t) \\
&= P_\theta(-X_i \le t \text{ and } \frac{1}{2}X_i \le t \text{ for all } i = 1, \cdots, n) \\
&= P_\theta(-t \le X_i \le 2t \text{ for all } i = 1, \cdots, n) \\
&= \prod_{i=1}^n P_\theta\Big(-t \le X_i \le 2t\Big).
\end{aligned}
$$

Now based on the definition of the density function of X_i, we will see that

$$
\mathbf{P}_\theta\Big(-t \le X_i \le 2t\Big) = \begin{cases} 0, & \text{if } t < 0; \\ 1, & \text{if } t > \theta. \end{cases}
$$

The most interesting case is when $0 \le t < \theta$. In this case, we have

$$
\begin{aligned}
P_\theta\Big(-t \le X_i \le 2t\Big) &= \prod_{i=1}^n \int_{-t}^{2t} \frac{1}{3\theta} dx = \frac{2t - (-t)}{3\theta} = \frac{t}{\theta}, \text{ and thus} \\
P_\theta(T(\mathbf{X}) \le t) &= \prod_{i=1}^n \frac{t}{\theta} = \frac{t^n}{\theta^n}.
\end{aligned}
$$

Thus the density function of $T(\mathbf{X}) = \max[\max_{1\le i\le n}(-x_i), \max_{1\le i\le n}(\frac{1}{2}x_i)] = \max[-x_{(1)}, \frac{1}{2}x_{(n)}]$ is

$$
f_T(t) = \begin{cases} \frac{n}{\theta^n} t^{n-1}, & \text{if } 0 \le t \le \theta; \\ 0, & \text{otherwise.} \end{cases}
$$

Recall that we have already shown in class that this family of distributions is complete!!! Indeed, this is just the distribution of $X_{(n)}$ when $X_1, \cdots X_n$ are iid Uniform$(0, \theta)$. Thus, the minimal sufficient statistic

$$
T(\mathbf{X}) = \max[\max_{1\le i\le n}(-x_i), \max_{1\le i\le n}(\frac{1}{2}x_i)] = \max[-x_{(1)}, \frac{1}{2}x_{(n)}]
$$

is complete for θ.

□

4. **(6.20(b)-(d)).** For each of the following pdfs let $X_1, \ldots X_n$ be iid observations. Find a complete sufficient statistic, or show that one does not exist. For part (b)-(d), please feel free to use Theorem 6.2.25 on page 288 of our text.

   (b) $f_\theta(x) = \frac{\theta}{(1+x)^{1+\theta}}, \qquad 0 < x < \infty,\ \theta > 0$

   (c) $f_\theta(x) = \frac{(\log\theta)\theta^x}{\theta - 1}, \qquad 0 < x < 1,\ \theta > 1$

   (d) $f_\theta(x) = e^{-(x-\theta)}\exp(-e^{-(x-\theta)}), \qquad -\infty < x < \infty,\ -\infty < \theta < \infty$

   Answer: (b) Note that

$$
f_\theta(x) = \frac{\theta}{(1+x)^{1+\theta}} I(x > 0) I(\theta > 0) = I(x > 0)\theta I(\theta > 0) e^{-(1+\theta)\log(1+x)} = h(x)c(\theta)e^{w(\theta)t(x)},
$$

where $h(x) = I(x > 0)$, $c(\theta) = \theta I(\theta > 0)$, $w(\theta) = -(1+\theta)$ and $t(x) = \log(1+x)$. For $\theta > 0$, $w(\theta) = -(1+\theta) \in (-\infty, -1)$, an open set in $\mathcal{R}$. Thus, by Theorem 6.2.25, $T(\mathbf{X}) = \sum_{i=1}^n \log(1 + X_i)$

is complete sufficient for θ.
(c) Observe that

$$f_\theta(x) = \frac{(\log\theta)\theta^x}{\theta-1} I(0<x<1)I(\theta>1) = I(0<x<1)[\frac{\log\theta}{\theta-1}I(\theta>1)]e^{(\log\theta)x} = h(x)c(\theta)e^{w(\theta)t(x)},$$

where $h(x) = I(0<x<1),\ c(\theta) = \frac{\log\theta}{\theta-1}I(\theta>1),\ w(\theta) = \log\theta$ and $t(x) = x$. For $\theta > 1$, $w(\theta) = \log\theta \in (0,\infty)$, an open set in $\mathcal{R}$. Thus, by Theorem 6.2.25, $T(\mathbf{X}) = \sum_{i=1}^n X_i$ is complete sufficient for θ.
(d) Note that

$$f_\theta(x) = e^{-(x-\theta)}\exp(-e^{-(x-\theta)}) = e^{-x}e^{\theta}\exp\Big(-(e^{\theta})e^{-x}\Big) = h(x)c(\theta)e^{w(\theta)t(x)},$$

where $h(x) = e^{-x},\ c(\theta) = e^{\theta},\ w(\theta) = -e^{\theta}$ and $t(x) = e^{-x}$. For $-\infty < \theta < \infty$, $w(\theta) = -e^{-\theta} \in (-\infty, 0)$, an open set in $\mathcal{R}$. Thus, by Theorem 6.2.25, $T(\mathbf{X}) = \sum_{i=1}^n e^{-X_i}$ is complete sufficient for θ. □

5. (**Motivated from problems 6.30 and 7.55(b) of our text**). Let $X_1, \ldots, X_n$ be a random sample from the pdf $f_\theta(x) = e^{-(x-\theta)}$ for $x > \theta$, where $-\infty < \theta < \infty$.

   (a) Show that $X_{(1)} = \min_i X_i$ is a complete sufficient statistic.

   (b) Use Basu's Theorem to show that $X_{(1)}$ and S^2 are independent. Recall that $S^2 = \frac{\sum_{i=1}^n (X_i - \bar{X}_n)^2}{n-1}$.

Answer: (a) First, the joint pdf of the sample $\mathbf{x}$ is

$$f(\mathbf{x}|\theta) = \prod_{i=1}^n [e^{-(x_i-\theta)}I(\theta < x_i)] = e^{-\sum_{i=1}^n x_i} \times e^{n\theta} \times I(\theta < \min(x_i)) = g(T(\mathbf{x})|\theta)h(\mathbf{x}),$$

where $T(\mathbf{x}) = \min x_i = x_{(1)},\ g(t|\theta) = e^{n\theta}I(\theta < t)$ and $h(\mathbf{x}) = e^{-\sum_{i=1}^n x_i}$. Hence, by the Factorization Theorem, $T(\mathbf{X}) = X_{(1)} = \min X_i$ is a sufficient statistic for θ.
Second, we need to find the probability density function of $T = X_{(1)}$. It is easy to see that for $t > \theta$,

$$\begin{aligned} P_\theta(X_i > t) &= \int_t^\infty f(x|\theta)dx = \int_t^\infty e^{-(x-\theta)}dx = e^{-t+\theta} \\ P_\theta(X_{(1)} > t) &= \prod_{i=1}^n P_\theta(X_i > t) = \prod_{i=1}^n e^{-t+\theta} = e^{-nt+n\theta} = e^{n(\theta-t)}. \end{aligned}$$

Thus it is not difficult to show that the density function of $T = X_{(1)}$ is

$$f_{X_{(1)}}(t) = \begin{cases} ne^{n(\theta-t)}, & \text{if } t > \theta; \\ 0, & \text{otherwise.} \end{cases}$$

Third, to prove completeness, suppose that $E_\theta(g(T)) = 0$ for all $\theta \in \mathbb{R}$. Then

$$0 = \int_\theta^\infty [g(t)ne^{n(\theta-t)}]dt = ne^{n\theta}\int_\theta^\infty [g(t)e^{-nt}]dt = 0.$$

Thus $G(\theta) = \int_\theta^\infty [g(t)e^{-nt}]dt = 0$ for all $\theta \in \mathbb{R}$. Applying the result of differentiation of an integral, we obtain that $G'(\theta) = -g(\theta)e^{-n\theta}$ almost everywhere for $\theta \in \mathbb{R}$. Since $G(\theta) = 0$ for all θ, we have $g(\theta)e^{-n\theta} = 0$ almost everywhere for $\theta \in \mathbb{R}$, and, hence, $g(t) = 0$ almost everywhere $t \in \mathbb{R}$. Therefore, $T = X_{(1)}$ is complete and sufficient for $\theta \in \mathbb{R}$.
(b) First, we need to prove that S^2 is ancillary. To see this, let $X_i = \theta + Z_i$. Then $Z_1, \ldots, Z_n$ are iid with density $f_Z(z) = e^{-z}$ for all $z \geq 0$, i.e., the density of Z_i's does not depend on θ. Hence $\bar{X}_n = \theta + \bar{Z}_n$ and

$$S^2 = \frac{\sum_{i=1}^n (X_i - \bar{X}_n)^2}{n-1} = \frac{\sum_{i=1}^n (Z_i - \bar{Z}_n)^2}{n-1} \qquad \text{has a distribution that does not depend on } \theta.$$

Thus, S^2 is ancillary. By Basu's Theorem and part (a), the complete sufficient statistic $X_{(1)}$ is independent of ancillary statistic S^2, i.e., $X_{(1)}$ and S^2 are independent.

(c) By (a), $X_{(1)}$ is a complete sufficient statistic for θ (it is not difficult to show that it is also minimal sufficient), and the density function of $T = X_{(1)}$ is

$$f_{X_{(1)}}(t) = \begin{cases} ne^{-nt+n\theta}, & \text{if } t > \theta; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, using the integration by parts, we have

$$\mathbf{E}_\theta(T^r) = \int_\theta^\infty ne^{n(\theta-t)}t^r dt = -t^r e^{n(\theta-t)}|_\theta^\infty + r\int_\theta^\infty e^{n(\theta-t)}t^{r-1}dt = \theta^r + \frac{r}{n}\mathbf{E}_\theta(T^{r-1}).$$

Thus $T^r - \frac{r}{n}T^{r-1}$ is an unbiased estimator of θ^r. Since this is the function of complete sufficient statistic, $\delta^* = T^r - \frac{r}{n}T^{r-1}$ with $T = X_{(1)}$ is the best unbiased estimator of θ^r. □

6. (**Modified from problem 6.3 of our text**) Let $X_1, \ldots, X_n$ be a random sample from the pdf

$$f(x|\mu,\sigma) = \frac{1}{\sigma}e^{-(x-\mu)/\sigma}, \quad \mu < x < \infty, \quad 0 < \sigma < \infty.$$

Find a (two-dimensional) minimal sufficient statistic $T(\mathbf{X}) = (T_1, T_2)$ for (μ, σ) such that $T_1 = T_1(X_1, \cdots, X_n)$ and $T_2 = T_2(X_1, \cdots, X_n)$ are independent.

Answer: First, the joint pdf of the sample $\mathbf{x}$ is

$$f(\mathbf{x}|\mu,\sigma) = \prod_{i=1}^n [\frac{1}{\sigma}e^{-(x_i-\mu)/\sigma}I(\mu < x_i)] = \frac{1}{\sigma^n}e^{-\sum_{i=1}^n x_i/\sigma + n\mu/\sigma} \times I(\mu < x_{(1)}) = g(T(\mathbf{x})|\mu)h(\mathbf{x}),$$

where $T(\mathbf{x}) = (x_{(1)}, \sum x_i), g(t_1,t_2|\sigma,\mu) = \sigma^n e^{-t_2/\sigma + n\mu/\sigma}I(\mu < t_1)$ and $h(\mathbf{x}) = 1$. Hence, by the Factorization Theorem, $T(\mathbf{X}) = (X_{(1)}, \sum_{i=1}^n X_i)$ is a sufficient statistic for (μ, σ).

Second, for any two samples,$\mathbf{x}$ and $\mathbf{y}$, the (likelihood) ratio

$$\frac{f(\mathbf{x}|\mu,\sigma)}{f(\mathbf{y}|\mu,\sigma)} = \exp\Big\{\frac{1}{\sigma}(\sum_{i=1}^n y_i - \sum_{i=1}^n x_i)\Big\}\frac{I(\mu < x_{(1)})}{I(\mu < y_{(1)})}$$

does not depend on (μ, σ) if and only if $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ and $x_{(1)} = y_{(1)}$. It follows that $\Big(X_{(1)}, \sum_{i=1}^n X_i\Big)$ is a minimal sufficient statistic for (μ, σ).

Third, consider now the statistic $(T_1, T_2) = \Big(X_{(1)}, \sum_{i=1}^n (X_i - X_{(1)})\Big)$. Since there is one-to-one correspondence between this statistic and the previous one, it is also minimal sufficient. In fact, it is no difficult to see that the maximum likelihood estimators of μ and σ are $X_{(1)} = T_1$, and $\sum_{i=1}^n (X_i - X_{(1)})/n = T - 2/n$, respectively. That is, these new T_1 and T_2 can be used to directly construct the estimators of μ and σ.

Finally, suppose for a moment that parameter σ is fixed and known, then we can follow the hints to show that the $Z_i = X_i/\sigma$'s have the distribution in the previous question with parameter $\theta^* = \mu/\sigma$. Then by problem #2, $Z_{(1)}$ is complete sufficient statistic for $\theta^* = \mu/\sigma$ and $\sum_{i=1}^n (Z_i - Z_{(1)}) = \sum_{i=1}^n ((Z_i - \theta^*) - (Z_{(1)} - \theta^*))$ is ancillary statistics. Thus by Basu's theorem, $Z_{(1)}$ and $\sum_{i=1}^n (Z_i - Z_{(1)})$ are independent, and so are $X_{(1)} = Z_{(1)}/\sigma$ and $\sum_{i=1}^n (X_i - X_{(1)}) = (\sum_{i=1}^n (X_i - X_{(1)}))/\sigma$ when σ is known.

Alternatively, without using the result in the previous question, we can directly prove that $X_{(1)}$ is a complete sufficient statistic for μ using the similar arguments in the previous problem. Then let $X_i = \mu + W_i$ and the distribution of W_i's do not depend on μ. Hence, the distribution of statistic $\sum_{i=1}^n (X_i - X_{(1)})$ is the same as the distribution of $\sum_{i=1}^n (W_i - W_{(1)})$ and thus does not depend on μ. It follows that $\sum_{i=1}^n (X_i - X_{(1)})$ is an ancillary statistic for μ. By Basu's theorem we obtain that $X_{(1)}$ and $\sum_{i=1}^n (X_i - X_{(1)})$ are independent when σ is known.

Since this holds for any $\sigma > 0$, we conclude that the statistics $T_1 = X_{(1)}$ and $T_2 = \sum_{i=1}^n (X_i - X_{(1)})$ are independent. □

Hints: If you have already thought about each problem for at least 30 minutes, then please feel free to look at the hints. Otherwise, please try the problem first, as getting help from the hints takes away most of the fun.

Problem 1: Can you find two constants C_1 and C_2 (they might depend on n, but not on θ) such that

$$E_\theta(\frac{1}{C_1}X_{(1)} - \frac{1}{C_2}X_{(n)}) = 0$$

for all θ?

Problem 2: To check completeness of $T(\mathbf{X})$, derive its distribution by noting that

$$\begin{aligned}
\mathbf{P}_\theta(T(\mathbf{X}) \le t) &= \mathbf{P}_\theta\Big(\max\Big\{1 - \min_{1\le i\le n} \frac{x_i}{i}, \quad \frac{1}{2}(\max_{1\le i\le n} \frac{x_i}{i} - 1)\Big\} \le t\Big) \\
&= \mathbf{P}_\theta\Big(1 - \min_{1\le i\le n} \frac{x_i}{i} \le t \text{ and } \frac{1}{2}(\max_{1\le i\le n} \frac{x_i}{i} - 1 \le t\Big) \\
&= \mathbf{P}_\theta\Big(\min_{1\le i\le n} \frac{x_i}{i} \ge 1 - t \text{ and } \max_{1\le i\le n} \frac{x_i}{i} \le 2t + 1\Big) \\
&= \mathbf{P}_\theta(1 - t \le \frac{x_i}{i} \le 2t + 1 \text{ for all } i = 1, \cdots, n) \\
&= \prod_{i=1}^{n} \mathbf{P}_\theta\Big(- (t-1) \le \frac{X_i}{i} \le 2t + 1)\Big) \\
&= \prod_{i=1}^{n} \mathbf{P}_\theta\Big(- i(t-1) \le X_i \le i(2t + 1)\Big)
\end{aligned}$$

for $t > 0$. What happens if $0 \le t \le \theta$? How about if $t < 0$ or if $t > \theta$? Do you see any connections with the problem in which $X_1, \ldots, X_n$ are iid Uniform$(0, \theta)$?

Problem 3: this problem is very different from Problem #1, as the minimal sufficient statistic turns out to be one-dimensional as in Problem #2!!! When proving the completeness, you need to first its probability density function as in problem #2.

Problem 4: For part (b)-(d), please feel free to use Theorem 6.2.25 on page 288 of our text to find the complete sufficient statistic.

Problem 5: In part (a), what is the distribution of $T = X_{(1)}$? In (b), use Basu's theorem.

Problem 6: First, find a minimal sufficient statistic and thus any one-to-one function is also minimal sufficient. Second, you can guess the desired T_1 and T_2 by assuming for a moment that parameter σ is fixed and known, and by finding a complete sufficient statistic and an ancillary statistic for μ (note that they should be the one-to-one function of the minimal sufficient statistic you have found).
Alternatively, T_1 and T_2 can be used to construct reasonable estimates (read: **maximum likelihood estimator**) of μ and σ, respectively.
Third, assume, for a moment, that σ is known, and you can let $Z_i = X_i/\sigma$, and use Basu's theorem to show that your proposed T_1 and T_2 are independent. Since this holds for any $\sigma > 0$, you can conclude that this independence carries over even if σ is unknown, as knowledge of σ has no bearing on the distributions. Also see problem 6.31 of our text for more applications of Basu's theorem.