

ISYE 6412-A, HOMEWORK-1

① a) data - X - normal random variable mean 0 variance 1

$$X \sim N(0, 1)$$

$$S = \mathbb{R} \text{ (entire real line)}$$

$$\Rightarrow X \in (-\infty, \infty)$$

$$p = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{x-\theta}{1} \right)^2} \quad \theta \in \mathbb{R}$$

$$\mathcal{R} = \{ \theta : \theta \in \mathbb{R} \} = \{ \theta : \theta \in \mathbb{R} \}$$

$$\mathcal{D} = \{ d : d \in \mathbb{R} \}$$

$$L = \frac{(\theta - d)^2}{1 + \theta^2}$$

$$b) R(\theta, \delta) = E_{\theta}(L_{\theta}(\theta, \delta))$$

$$R(\theta, \delta) = E_{\theta} \left(\frac{(\theta - X)^2}{1 + \theta^2} \right)$$

$$= \frac{1}{1 + \theta^2} E_{\theta}((\theta - X)^2)$$

$$= \frac{\text{var}(X)}{1 + \theta^2} = \boxed{\frac{1}{1 + \theta^2}}$$

$$R(\theta, \delta_2) = E \left[\frac{\left(\theta - \left(\frac{1+X}{2} \right) \right)^2}{1+\theta^2} \right]$$

$$= \frac{1}{1+\theta^2} E \left(\left(\theta - \left(\frac{1+X}{2} \right) \right)^2 \right)$$

$$= \frac{4-1}{4(1+\theta^2)} E \left(\left((4\theta-1) - X \right)^2 \right)$$

$$= \frac{1}{4(1+\theta^2)} E \left((4\theta-1)^2 + X^2 - 2(4\theta-1)X \right)$$

$$= \frac{1}{4(1+\theta^2)} \left[(4\theta-1)^2 + E(X^2) - 2(4\theta-1)E(X) \right]$$

$$= \frac{1}{4(1+\theta^2)} \left[(4\theta-1)^2 + E(X^2) - (E(X))^2 + (E(X))^2 - 2(4\theta-1)E(X) \right]$$

$$= \frac{1}{4(1+\theta^2)} \left[(4\theta-1)^2 + 1 + \theta^2 - 2\theta(4\theta-1) \right]$$

$$= \frac{1}{4(1+\theta^2)} \left(16\theta^2 + 1 + 8\theta + 1 + \theta^2 - 8\theta^2 + 2 \right)$$

$$= \frac{1}{4(1+\theta^2)} \left(9\theta^2 + 8\theta + 3 \right)$$

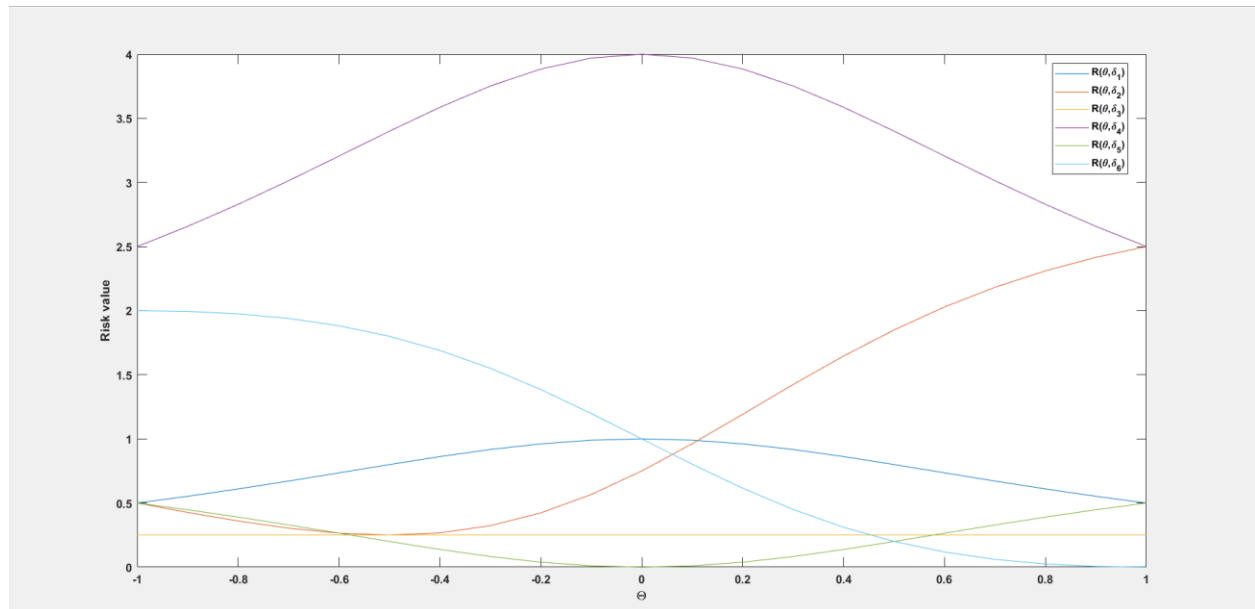
$$\begin{aligned}
 R(\theta, \delta_3) &= E\left(\left(\theta - \frac{x}{2}\right)^2\right) \frac{1}{1+\theta^2} \\
 &= E\left(4\theta^2 + \frac{x^2}{4} - 4x\theta\right) \frac{1}{(1+\theta^2)4} \\
 &= \frac{1}{4(1+\theta^2)} (4\theta^2 + E(x^2) - \theta^2 + \theta^2 - 4\theta^2) \\
 &= \frac{1}{4(1+\theta^2)} (4\theta^2 + 1 - 3\theta^2) \\
 &= \boxed{\frac{1}{4}}
 \end{aligned}$$

$$\begin{aligned}
 R(\theta, \delta_4) &= \frac{E((\theta - 2x)^2)}{(1+\theta^2)} = \frac{E(\theta^2 + 4x^2 - 4x\theta)}{1+\theta^2} \\
 &= \frac{1}{1+\theta^2} (\theta^2 + 4E(x^2) - 4\theta^2 + 4\theta^2 - 4x\theta) \\
 &= \frac{1}{1+\theta^2} (\theta^2 + 4 + 4\theta^2 - 4\theta^2) \\
 &= \boxed{\frac{\theta^2 + 4}{\theta^2 + 1}}
 \end{aligned}$$

$$R(\theta, \delta_5) = \frac{E(\theta^2)}{1+\theta^2} = \boxed{\frac{\theta^2}{1+\theta^2}}$$

$$R(\theta, \delta_6) = \frac{E((\theta - 1)^2)}{1+\theta^2} = \boxed{\frac{(\theta - 1)^2}{1+\theta^2}}$$

Q1 b) plot



① c) From the graph,

$\delta_1, \delta_2, \delta_3, \delta_5, \delta_6$ are better than δ_4 ,
i.e.

$$R_{\delta_k}(\theta) \leq R_{\delta_4}(\theta) \quad \forall k \in \{1, 2, 3, 5, 6\}$$

$$\text{and } R_{\delta_k}(\theta_0) < R_{\delta_4}(\theta_0) \text{ for some } \theta_0 \\ \forall k \in \{1, 2, 3, 5, 6\}$$

$\Rightarrow \delta_4$ is inadmissible

Similarly δ_3 and δ_5 are better than δ_6

$\Rightarrow \delta_6$ is inadmissible

d) δ_3 because it has the least risk function for all θ on an average (it always 0.25)
 $\delta_3 = x/2$

e) $R(\theta, \delta) = E_{\theta}(f(\theta, \delta))$

Consider a general $\delta = a + b\bar{X}_n$

$$R(\theta, \delta) = \frac{E_{\theta}((\theta - a - b\bar{X}_n)^2)}{1 + \theta^2}$$

$$= \frac{E_{\theta}(b^2 \bar{X}_n^2 + a^2 + \theta^2 - 2a\theta - 2b\theta\bar{X}_n + 2ab\bar{X}_n)}{1 + \theta^2} \quad \text{--- ①}$$

$$\text{var}(\bar{X}_n) = \frac{1}{n} \rightarrow E(\bar{X}_n^2) - (E(\bar{X}_n))^2 = \frac{1}{n}$$

$$\rightarrow E(\bar{X}_n^2) = \frac{1}{n} + \theta^2 \quad \text{--- ②}$$

$$\text{①} \Rightarrow R = \frac{b^2 E(\bar{X}_n^2) + a^2 + \theta^2 - 2a\theta - 2b\theta E(\bar{X}_n) + 2ab E(\bar{X}_n)}{1 + \theta^2}$$

$$(\text{using ②}) = \frac{1}{1 + \theta^2} \left(b^2 \left(\theta^2 + \frac{1}{n} \right) + a^2 + \theta^2 - 2a\theta - 2b\theta^2 + 2ab\theta \right)$$

$$= \frac{1}{1 + \theta^2} \left(\theta^2 (b^2 - 2b + 1) + \theta (2ab - 2a) + a^2 + \frac{b^2}{n} \right)$$

e) contd.

$$\Rightarrow R(\theta, \delta) = \frac{1}{1+\theta^2} \left(\theta^2 (b-1)^2 + \theta(2a)(b-1) + a^2 + \frac{b^2}{n} \right)$$

$$\delta_{1,n} : a=0, b=1$$

$$\Rightarrow R(\theta, \delta_{1,n}) = \frac{a^2 + \frac{b^2}{n}}{n(\theta^2 + 1)}$$

$$\Rightarrow R(\theta, \delta_{1,n}) = \frac{1}{n(\theta^2 + 1)}$$

$$\delta_{2,n} \text{ and } a = \frac{\bar{X}_n + \frac{1}{n}}{1 + \frac{1}{n}} \quad \text{and} \quad \text{mm.}$$

$$= \left(\frac{n}{n+1} \right) \bar{X}_n + \frac{1}{n+1}$$

$$a = \frac{1}{n+1}, \quad b = \frac{n}{n+1}$$

$$R(\theta, \delta_{2,n}) = \frac{1}{1+\theta^2} \left(\theta^2 \left(\frac{n}{n+1} - 1 \right)^2 + \theta \left(\frac{2}{n+1} \right) \left(\frac{n}{n+1} - 1 \right) + \frac{1}{(n+1)^2} + \frac{n^2}{(n+1)^2 n} \right)$$

$$= \frac{1}{(1+\theta^2)} \left(\frac{\theta^2}{(n+1)^2} - \frac{2\theta}{(n+1)^2} + \frac{1}{(1+n)^2} \right)$$

$$R_{\theta}(\theta, \delta_{3,n})$$

$$\delta_{3,n} = \frac{\sqrt{n} \bar{X}_n}{1 + \sqrt{n}} \approx \sqrt{n} \bar{X}_n$$

$$a = 0, b = \frac{\sqrt{n}}{1 + \sqrt{n}}$$

$$\begin{aligned} R(\theta, \delta_{3,n}) &= \frac{1}{1 + \theta^2} \left(\theta^2 \left(\frac{\sqrt{n}}{1 + \sqrt{n}} - 1 \right)^2 + \frac{(\sqrt{n})^2}{(1 + \sqrt{n})^2 n} \right) \\ &= \frac{1}{1 + \theta^2} \left(\theta^2 \left(\frac{1}{1 + \sqrt{n}} \right)^2 + \frac{1}{(1 + \sqrt{n})^2} \right) \end{aligned}$$

$$= \frac{1}{(1 + \theta^2)(1 + \sqrt{n})^2} (\theta^2 + 1) = \frac{1}{(1 + \sqrt{n})^2}$$

$$\delta_{6,n} = 1 \Rightarrow a = 1, b = 0$$

$$\begin{aligned} R(\theta, \delta_{6,n}) &= \frac{1}{1 + \theta^2} \left(\theta^2 - 2\theta + \frac{1}{n} \right) \\ &= \frac{(\theta - 1)^2}{1 + \theta^2} \end{aligned}$$

f) Upon inspection R_{δ_1} and R_{δ_2} are very close with δ_1 values.

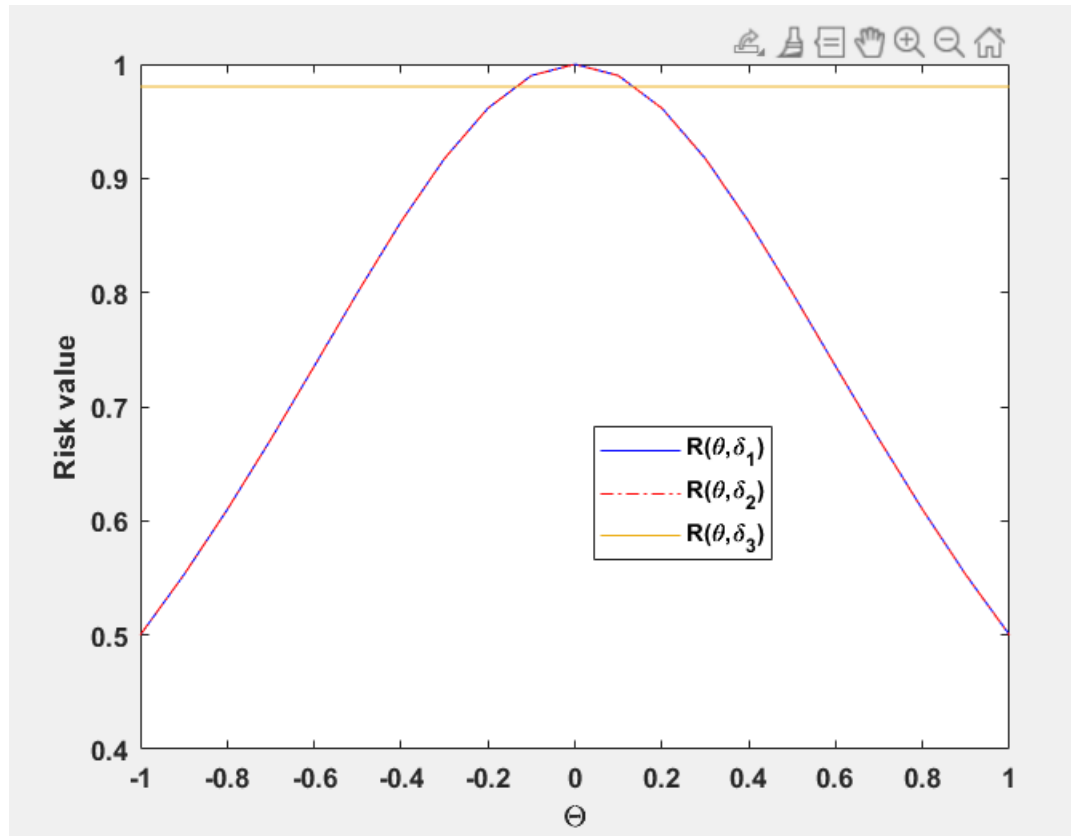
I would choose δ_1 procedure because it is slightly better.

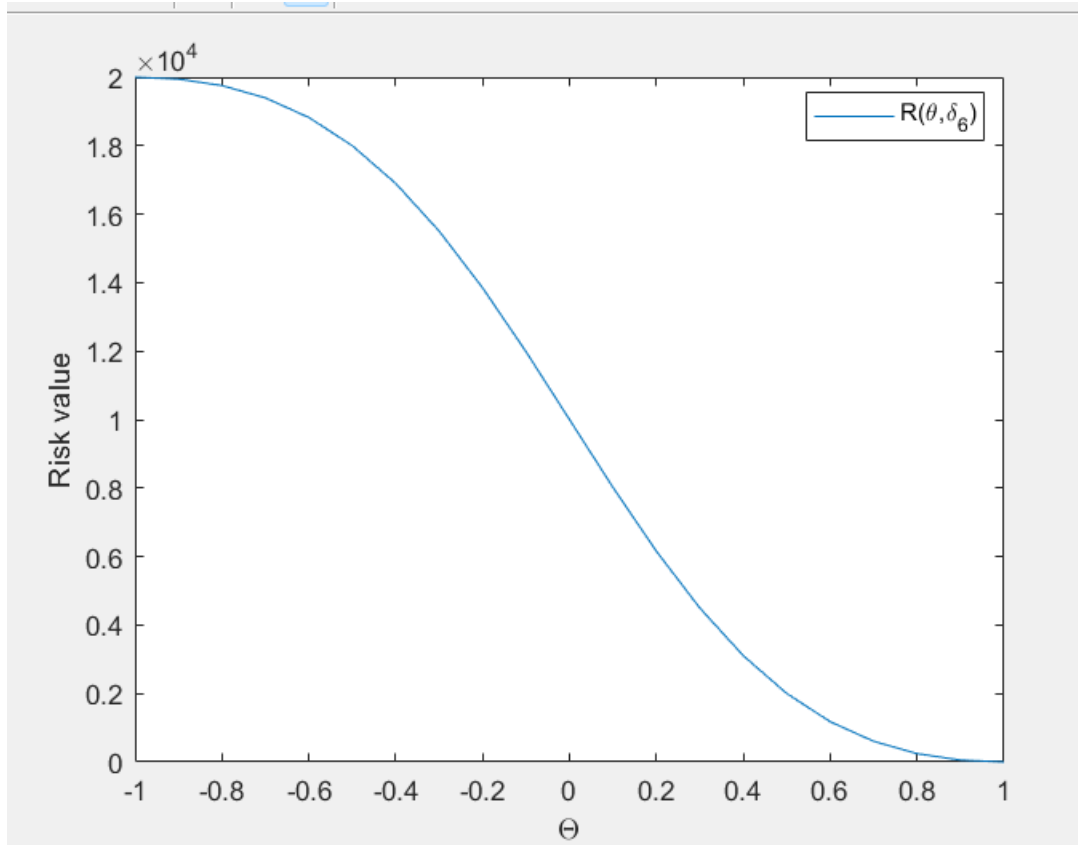
Answer is different from part(d) [$\delta_3 = \bar{X}/2$]

Q1 e) plot

N set as 10000.

All risk values scaled by n





Plotted separately because of high risk value

$$g) R = E \left(\frac{(a - \theta - a - b\bar{X}_n)^2}{1 + \theta^2} \right)$$

$$= E \left(\frac{b^2 \bar{X}_n^2 + a^2 + \theta^2 - 2a\theta - 2\theta b\bar{X}_n + 2ab\bar{X}_n}{1 + \theta^2} \right)$$

$$\text{Var}(\bar{X}_n) = \frac{1}{n} \Rightarrow E(\bar{X}_n^2) - \theta^2 = \frac{1}{n}$$

$$E(\bar{X}_n^2) = \frac{1}{n} + \theta^2$$

$$= \frac{1}{1 + \theta^2} \left(b^2 \theta^2 + \frac{b^2}{n} + a^2 + \theta^2 - 2a\theta - 2\theta^2 b + 2ab\theta \right)$$

$$\lim_{\theta \rightarrow \infty} R = \lim_{\theta \rightarrow \infty} \left(\frac{b^2 \theta^2 + \frac{b^2}{n} + a^2 + \theta^2 - 2a\theta - 2\theta^2 b + 2ab\theta}{\theta^2 + 1} \right)$$

$$= \lim_{\theta \rightarrow \infty} \left(\frac{1}{\theta^2 + 1} \right) (-b^2 \theta^2 (b^2 - 2b + 1) + \theta (2ab - 2a) + \frac{b^2}{n} + a^2)$$

$$= \frac{b^2 - 2b + 1}{1} \quad (\theta^2 \text{ is the highest power})$$

$$\lim_{\theta \rightarrow \infty} R = (b^2 - 2b + 1)$$

$$\text{If this is } \lim_{\theta \rightarrow \infty} R = 0$$

$$\text{then, } b^2 - 2b + 1 = 0 \Rightarrow b = 1$$

$$\text{If } b = 1, \lim_{\theta \rightarrow \infty} R = 1 - 2 \times 1 + 1 = 0$$

$$\therefore \lim_{\theta \rightarrow \infty} R = 0 \text{ iff } b = 1$$

① q) contd.

$$\text{If } b=1$$

$$R = \frac{1}{\theta^2 + 1} \left(\theta^2 (b-1)^2 + \theta (2ab - 2a) + \frac{b^2}{n} \right)$$

$$= \frac{1}{\theta^2 + 1} \left(a^2 + \frac{1}{n} \right)$$

$$\text{min}_\theta R \quad \frac{dR}{da} = \frac{2a}{\theta^2 + 1} - \rho \neq a \cdot 2\phi$$

$$\frac{a^2}{\theta^2 + 1} \geq 0 \quad \therefore a^2 / \theta^2 \geq 0$$

$$\Rightarrow \frac{a^2}{\theta^2 + 1} + \frac{1}{n(\theta^2 + 1)} \geq \frac{1}{n(\theta^2 + 1)}$$

$$\Rightarrow a \frac{1}{1+\theta^2} \left(a^2 + \frac{1}{n} \right) \geq \frac{1}{(1+\theta^2)^n}$$

$$\Rightarrow R \geq \frac{1}{n(1+\theta^2)}$$

which happens only when $a \geq 0$

$$\therefore \min_a R = \frac{1}{n(1+\theta^2)} \quad \text{and} \quad a_{\min}^+ > 0$$

* h) If possible, let δ be $\delta_{b,n}$ exists as better procedure than $\delta_{b,n} = 1$ making $\delta_{b,n}$ inadmissible

Then:

$$R_{\delta'}(\theta) \leq R_{\delta_{b,n}}(\theta) = \frac{(\theta - 1)^2}{1 + \theta^2} \quad \forall \theta \in \mathbb{R} \quad \text{--- (1)}$$

$$\text{and } R_{\delta'}(\theta_0) < R_{\delta}(\theta_0) = \frac{(\theta_0 - 1)^2}{1 + \theta_0^2} \text{ for at least one } \theta_0 \quad \text{--- (2)}$$

$$\frac{d}{d\theta} \frac{(\theta - 1)^2}{1 + \theta^2} = 0$$

equation (1)

$$R_{\delta'}(\theta) = \int_{y_1=-\infty}^{y_1=\infty} \dots \int_{y_N=-\infty}^{y_N=\infty} \frac{(\theta - \delta(y_1, \dots, y_N))^2}{1 + \theta^2} f_{y_1}(y=y_1) \times f_{y_2}(y=y_2) \dots f_{y_N}(y=y_N) dy_N \dots dy_1$$

$$\text{At } \theta = 1, R_{\delta_{b,n}}(1) = 0$$

We know that y_1, y_2, \dots, y_n are independent

$$\text{and } \int_{-\infty}^{\infty} f(y=k) dk = 1 \quad (\text{property of pdf})$$

So for $R_{\delta'}(1) \leq 0 \leq R_{\delta_{6,n}}(1) \geq 0$,

only possibility is $R_{\delta'}(0) \geq 0$
 $\hookrightarrow \theta = 1$

~~But~~ But

$$\Rightarrow \delta'(y_1, \dots, y_N) = 1$$

But this is the same as $\delta_{6,n}$. So
eqn (2) is not satisfied! ($\because \delta' = \delta_{6,n} = 1$
 $\neq 0$)

This is a contradiction

So $\delta_{6,n} = 1$ is an admissible procedure

(2)

a)

$$S = \{0, 1, 2, 3, \dots, 20\}$$

$$\Omega = \{ \text{outcome} \} \sim n C_i \theta^i (1-\theta)^{n-i}; \theta \in [0, 1]$$

$$D = \{0 \leq d \leq 1\} \quad (\text{point estimate of } \theta)$$

$$L(\theta, d) = |\theta - d| \quad (\text{given})$$

θ is the probability of success

b)

$$R(\theta, \delta_1) = E(L(\theta, \delta_1))$$

$$= E(|\theta - \frac{x}{n}|)$$

$$R(\theta, \delta_2) = E(p_{\theta} \cdot L(\theta, \delta_2))$$

$$= E(|\theta - \frac{1}{3}|)$$

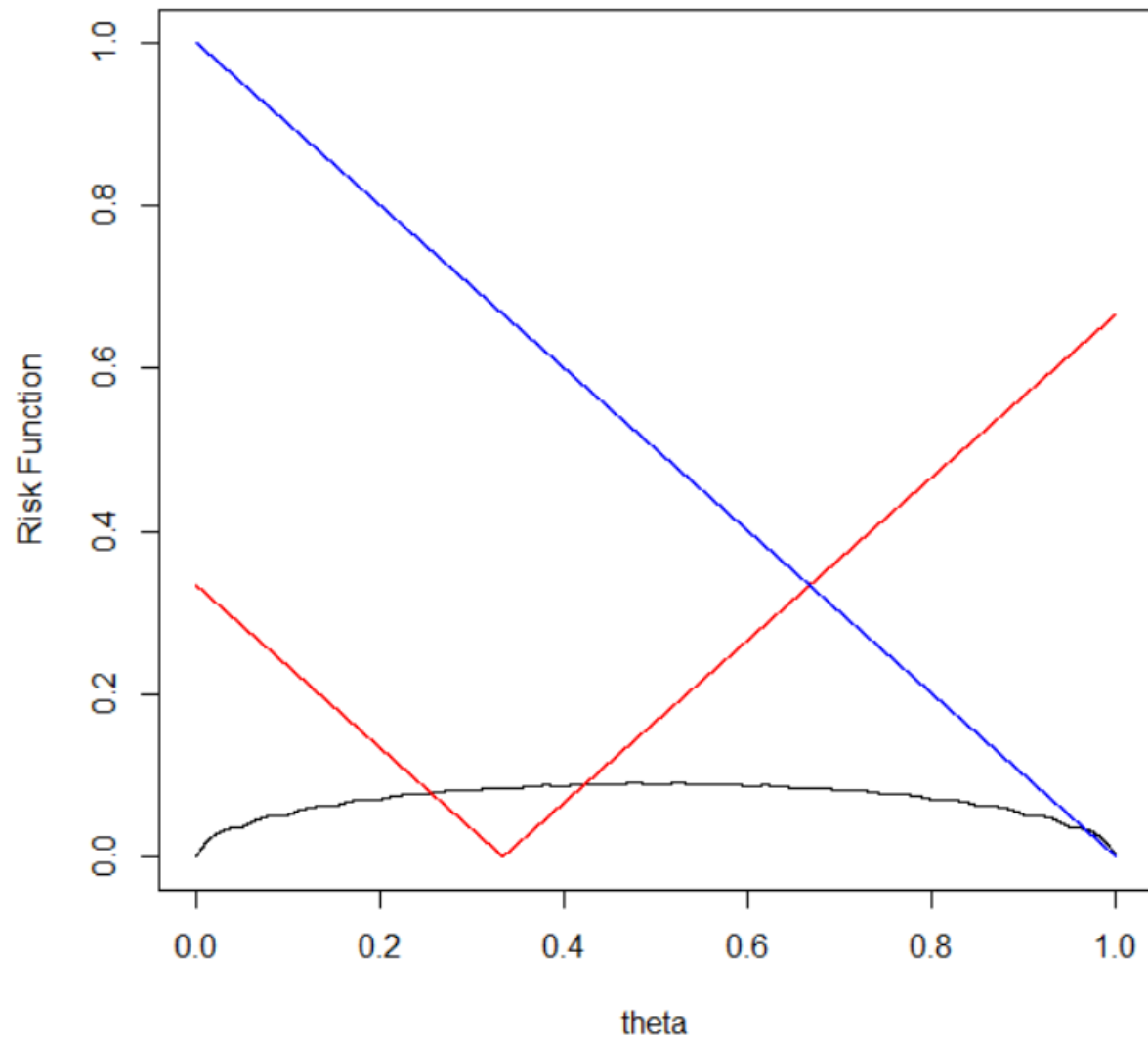
$$= |\theta - \frac{1}{3}|$$

$$R(\theta, \delta_3) = E(L(\theta, \delta_3))$$

$$= E(|\theta - 1|)$$

$$= |\theta - 1|$$

Q2 b) plot



Legend:

- Black – Procedure-1 ($d = X/n$)
- Red – Procedure-2 ($d = 1/3$)
- Blue – Procedure-3 ($d = 1$)

To plot $R(\theta, \delta_1) = R_{\delta_1}(\theta)$ use the code
 performing Monte Carlo simulation ~~show~~ to estimate
 $E\left(\left|\theta - \frac{x}{n}\right|\right)$ for $n=20$ is used.

3 Comparison:

(i) Risk functions of δ_2 and δ_3 are similar - ~~just~~ they
~~shifted equations~~

are just shifted along x -axis

(ii) Risk function of δ_1 is reasonable - stays
 low and is 0 when $\theta = 0$ or $\theta = 1$

Since $R_{\delta_1}(\theta)$ stays reasonably low for all θ compared
 to other ^{risk} functions in the question

δ_1 is the preferred procedure

$$\left(\delta_1(x) = \frac{x}{n}\right)$$

However it is to be noted that no ^{procedure} ~~is~~ is ~~better~~
~~than~~ "better than" ^{or "equivalent"} the other or "equivalent"

The procedures are "uncomparable"

c) R

If possible let δ' be better than $\delta_2(x) = 1/3$.

$$\Rightarrow (i) R_{\delta'}(\theta) \leq R_{\delta_2}(\theta) = |\theta - 1/3| \quad \forall \theta \in [0, 1]$$

$$(ii) R_{\delta'}(\theta) < R_{\delta_2}(\theta) \text{ for some } \theta_0$$

$$R_{\delta'}(\theta) = \sum_{x \in S} L(\theta, \delta'(x)) P_{\theta}(X=x)$$

$$= \sum_{x \in S} L(\theta, \delta'(x)) P_{\theta}(X=x)$$

$$= \sum_{x=0}^{20} |\theta - \delta'(x)|^n (x \theta^n (1-\theta)^{1-x})$$

When $\theta = 1/3$,

$$R_{\delta_2}(1/3) = 0$$

By (i), $R_{\delta'}(\theta = 1/3) \leq 0$

$$\Rightarrow \sum_{x=0}^{20} |1/3 - \delta'(x)|^n (x \theta^n (1-\theta)^{1-x}) \leq 0$$

$$\Rightarrow \delta'(x) = 1/3 \quad \forall x \in \{1, 2, 3, \dots, 20\}$$

$$\Rightarrow \delta' \equiv \delta_3$$

This is a contradiction to our assumption that δ' is better than δ_2

$\therefore \delta_2$ is admissible procedure. ($\forall n \geq 1$)

d) If $\delta_3(x) = 1$ is not admissible, let there exist a better procedure $\delta'(x)$ such that then,

- (i) $R_{\delta'}(\theta) \leq R_{\delta_3}(\theta) = |\theta - 1| \quad \forall \theta \in [0, 1]$
 (ii) $R_{\delta'}(\theta) < R_{\delta_3}(\theta_0)$ for some θ_0

$$R_{\delta'}(\theta) = \sum_{x=0}^{x=2} {}^2C_x (\theta)^x (1-\theta)^{2-x} |\theta - \delta'_x|$$

$$= |\theta - \delta'(0)| (1-\theta)^2 + 2\theta |\theta - \delta'(1)| (\theta)(1-\theta) + |\theta - \delta'(2)| (\theta^2)$$

If $R_{\delta'}(1) \leq R_{\delta_3}(1) = 0$,

$$\Rightarrow |\theta - \delta'(2)| \leq 0$$

$$\Rightarrow \delta'(2) = 1$$

Now $\delta'(0)$, and $\delta'(1)$ can be either 0 or 1

4 possibilities:

- (i) $\delta'(0) = \delta'(1) = 0$
 (ii) $\delta'(0) = 0, \delta'(1) = 1$
 (iii) $\delta'(0) = 1, \delta'(1) = 1$
 (iv) $\delta'(0) = 1, \delta'(1) = 0$

~~We see that~~

$$\text{case (iii)} \Rightarrow \delta'(X) \equiv \delta_3^{\frac{2}{3}}(X) = 1 + X$$

So contradiction

From the plot we see that the other functions
'risk values of the procedures are not $\leq R_{\delta_3}(\theta)$

less than the risk value of $R_{\delta_3}(\theta)$ δ_3

i.e. (i) ~~doesn't~~ doesn't seem to hold for
all the

i.e. $R_{\delta'}(\theta) \leq R_{\delta_3}(\theta)$ $\forall \theta$ doesn't hold
for any δ' we have tried

\therefore this contradicts our assumption δ' is better
than δ_3
 $\Rightarrow \delta_3(X) > 1$ is admissible
for $n=2$!

Q2 d) plot

