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ISYE 6412 HW-04

(6)

a) $\pi(\theta) = C(1+\theta^2)k^{-1}\phi(\theta/k)$

where C is such that $\int_{-\infty}^{\infty} \pi(\theta)\,d\theta = 1$

$$h_\pi^+(y,d) = \int_{-\infty}^{\infty} \underbrace{C(1+\theta^2)k^{-1}\phi(\theta/k)}_{\pi(\theta)} \times \underbrace{\frac{(\theta-d)^2}{(1+\theta^2)}}_{L(\theta,d)} \times \underbrace{\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(y_i-\theta)^2}{2\sigma^2}\right)}_{f(y_1,\dots,y_n)} d\theta.$$

$$= \frac{c'}{k}\int_{-\infty}^{\infty} \frac{(\theta-d)^2}{\sqrt{2\pi}}\exp\left(\frac{-\left(\frac{\theta}{k}\right)^2}{2}\right)\exp\left(\sum_{i=1}^{n}\frac{(-(y_i-\theta)^2)}{2\sigma^2}\right)d\theta$$

where c' is all the constants taken out

$(\sqrt{2\pi}, \sigma, k, C)$

$$= c'\int_{-\infty}^{\infty}(\theta-d)^2\exp\left(\frac{-\theta^2}{2k^2}\right)\exp\left(\sum_{i=1}^{n}\frac{-(y_i-\theta)^2}{2\sigma^2}\right)d\theta$$

This is same as h_π^+ we obtained for prior $f_y(y) = N(\theta,\sigma)$,

$\pi(\theta) = N(\mu,\tau)$ and $L(\theta,d) = (\theta-d)^2$

where,

$\therefore$ The Bayes procedure $\delta^+ = \dfrac{\frac{n}{\sigma^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}\bar{y} + \dfrac{\frac{1}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}\mu$

here $\mu = 0, \tau^2 = k^2$

$$\Rightarrow \delta^{*}_{Bayes} = \frac{\frac{n}{\sigma^2}\bar{Y}}{\frac{n}{\sigma^2} + \frac{1}{k^2}}$$

We want the Bayes procedure as $\frac{\sqrt{n}\,\bar{Y}_n}{1+\sqrt{n}}$

$$\frac{n/\sigma^2\,\bar{Y}}{\frac{n}{\sigma^2} + \frac{1}{k^2}} = \frac{\sqrt{n}}{1+\sqrt{n}}\bar{Y}_n$$

$$\Rightarrow \frac{n}{\sigma^2}(1+\sqrt{n}) = (\sqrt{n})\left(\frac{n}{\sigma^2} + \frac{1}{k^2}\right)$$

$$\Rightarrow \frac{n}{\sigma^2} = \frac{\sqrt{n}}{k^2} \Rightarrow k^2 = (\sigma^2/\sqrt{n})$$

$\therefore$ For $k = \sqrt{\frac{\sigma^2}{\sqrt{n}}}$

$\delta_{3,n}$ is Bayes relative to π

b) $R_\delta(\theta, d) = E\left(\frac{(\theta - d)^2}{(1+\theta)^2}\right)$

$$d = \frac{\sqrt{n}\,\bar{Y}_n}{1+\sqrt{n}}$$

$$E\left(\frac{(\theta + a + b\bar{x})^2}{\cancel{4+\theta^2}}\right)$$

$$= \left(E(a + b\bar{x} - \theta)\right)^2 + \mathrm{var}(a + b\bar{x} - \theta))$$

$$= \frac{(a + (b-1)\theta)^2 + b^2\sigma^2/n}{(1+\theta^2)}$$

$$\therefore \text{ when } b = \frac{\sqrt{n}}{\sqrt{n}+1}, \; a = 0$$

$$R_\delta(\theta) = \frac{1}{1+\sqrt{n}} \quad \frac{((b-1)\theta)^2 + \frac{b^2\sigma^2}{n}}{1+\theta^2}$$

$$= \left(\frac{\theta^2}{(1+\sqrt{n})^2} + \frac{\sigma^2}{(1+\sqrt{n})^2}\right)\frac{1}{1+\theta^2}$$

$$= \frac{\theta^2 + \sigma^2}{(1+\sqrt{n})^2(1+\theta^2)}$$

$$= \frac{\theta^2 + 1}{(1+\sqrt{n})^2(1+\theta^2)}$$

$$= \frac{1}{(1+\sqrt{n})^2}$$

$\Rightarrow R_{\delta_{3,n}}(\theta, d)$ is constant with respect to θ and is Bayes with respect to some prior $\pi_a(\theta)$

$\Rightarrow \delta_{3,n}$ is minimax (theorem 3.1)

c) Consider prior,

$$\pi_c = \frac{c}{k}\phi\left(\frac{\theta}{k}\right) \quad \text{where } k = \sqrt{\sigma^2/\sqrt{n}}$$

where c is such that $\int_{-\infty}^{\infty} \pi_c \, d\theta = 1$

$\therefore$ minimise $h_{\pi^*}(y, d) = c' \int_{-\infty}^{\infty} (\theta - d)^2 \frac{1}{k} \exp\left(\frac{-\theta^2}{2k^2}\right) \cdot \exp\left(\frac{\sum -(y_i - \theta)^2}{2\sigma^2}\right) d\theta$

to get the Bayes procedure

solution for this is,

$$\delta^{1}_{Bayes} = \frac{\frac{n}{\sigma^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}\bar{y} + \frac{\frac{1}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}\mu$$

$$\tau^2 = k^2 = \frac{\sigma^2}{\sqrt{n}} ; \quad \mu = 0$$

$$\Rightarrow \delta^{1}_{Bayes} = \frac{\frac{n}{\sigma^2}}{\frac{n}{\sigma^2} + \frac{\sqrt{n}}{\sigma^2}} \cdot \bar{y} = \frac{\sqrt{n}\,\bar{y}}{\sqrt{n} + 1}$$

So $\delta^*_{3,n}$ is Bayes wrt $\pi_c(\theta)$

However,

$$R_{\delta^*_{3,n}}(\theta, d) = E\left(\left(\theta - \frac{d'\sqrt{n}\,\bar{y}}{\sqrt{n}+1}\right)^2\right)$$

$$= E\left(\frac{\left(\theta - \frac{\sqrt{n}\,\bar{y}}{\sqrt{n}+1}\right)}{1+\theta^2}(1+\theta^2)\right)$$

$$= \left(E\left(\frac{\theta - \frac{\sqrt{n}\,\bar{y}}{\sqrt{n}+1}}{1+\theta^2}\right)\right)(1+\theta^2)$$

$$= \frac{(1+\theta^2)}{(1+\sqrt{n})^2} \quad \text{(from hw 1)}$$

$$R_{\delta_{\bar{y}}}(\theta, \bar{y}) = E\left((\theta - \bar{y})^2\right)$$

$$= \frac{\sigma^2}{n} \rightarrow \text{constant}$$

and $\bar{Y}$ is Bayes under the non informative prior $\pi(\theta) = 1$ ~~for $\pi(\theta|y) = f_\theta(\bar{y})$~~

and loss function $(\theta - d)^2$

Because, $\pi(\theta|y) = f_\theta(\bar{y})$

$$\delta_{Bayes} = E(\pi(\theta|y)) = \bar{Y}$$

So $\bar{Y}$ is the new minimax procedure in the new loss function. (thm 3.1)

$R\delta_{3,n}$ is not constant!

~~∴ We can't ta~~

∴ ~~$R\delta_{3,n}$~~ $\delta_{3,n}$ is not the minimax procedure under the new loss function

① ④ $\Theta =$

$$\cancel{h_\pi^+ = \int_0^1 (\theta - d)^2 \pi(\theta) P_\theta (Y = y)\, d\theta.}$$

$$\cancel{= \int_0^{1/2} (\theta - d)^2 \left(\frac{1}{2}\right)}$$

Suppose $\pi(\theta = r) = 1/2$
and $\pi(\theta = 1 - r) = 1/2$ $\quad r \in [0, 1/2]$

$$m(y) = \frac{1}{2} \times (r)^y (1-r)^{1-y} + \frac{1}{2}(1-r)^y r^{1-y}$$

$$\pi(\theta = r \mid y) = \frac{\frac{1}{2} r^y (1-r)^{1-y}}{m(y)}$$

$$\cancel{= r^y (1-r)^{1-y} + (1-r)^y (r}$$

$$= \frac{r^y(1-r)^{1-y}}{r^y(1-r)^{1-y} + r^{1-y}(1-r)^y}$$

Similarly,

$$\pi(\theta = 1-r \mid y) = \frac{r^{1-y}(1-r)^{y}}{r^y(1-r)^{1-y} + r^{1-y}(1-r)^y}$$

$$\left(\because \pi(\theta \mid y) = \frac{\pi(\theta) P_\theta(Y=y)}{m(y)} \right)$$

Under squared error loss,

$$\delta_{Bayes} = E(\pi(\theta \mid y))$$

$$= r\,\pi(\theta = r \mid y) + (1-r)\,\pi(\theta = 1-r \mid y)$$

$$\delta_{Bayes}(Y=0) = \frac{r(1-r)}{(1-r)+r} + \frac{(1-r)\,r}{(1-r)+r} = 2r(1-r)$$

$$\delta_{Bayes}(Y=1) = \frac{r(r)}{(1-r)+r} + \frac{(1-r)(1-r)}{(1-r)+r}$$

$$= r^2 + (1-r)^2 = 2r^2 - 2r + 1$$

Define $u = 2r(1-r)$

$$\Rightarrow \delta_{Bayes}(Y=0) = u, \quad \delta_{Bayes}(Y=1) = 1-u$$

$$R_{\delta_{Bayes}} = \sum \delta_{Bayes}(y) P(Y=y)(\theta - d)^2$$

$$d = \delta_{Bayes}(y)$$

$$= P(Y=0)(\theta - \delta_{Bayes}(0))^2 + P(Y=1)(\theta - \delta_{Bayes}(1))^2$$

$$= (1-\theta)(\theta - u)^2 + \theta(\theta - (1-u))^2$$

$$= (1-\theta)(\theta^2 + u^2 - 2\theta u) + \theta(\theta^2 + u^2 + 1 - 2u - 2\theta + 2u\theta)$$

$$= [\theta^2 + u^2 - 2\theta u - \theta^3 - u^2\theta + 2\theta^2 u + \theta^3 + u^2\theta + \theta - 2u\theta - 2\theta^2 + 2u\theta^2]$$

$$= \theta^2(1 + 2u - 2 + 2u) + \theta(-4u + u^2 + 1 - u^2) + u^2$$

$$= (4u-1)\theta^2 - \theta(4u-1) + u^2$$

When $r = \frac{4}{9}$,

$$u = 2 \times \frac{4}{9} \times \frac{5}{9} = \frac{40}{81}$$

$$\delta_{Bayes}(Y=0) = 2\left(\frac{4}{9}\right)\left(\frac{5}{9}\right) = \frac{40}{81}$$

$$\delta_{Bayes}(Y=1) = 1 - u = \frac{41}{81}$$

$$R_{\delta}^{t}{}_{\text{Bayes}} = \left(\frac{160}{81} - 1\right)\theta^2 - \left(\frac{160}{81} - 1\right)\theta + \left(\frac{40}{81}\right)^2$$

$$= \frac{79}{81}\theta^2 - \frac{79}{81}\theta + \left(\frac{40}{81}\right)^2$$

~~Supremum R_δ^t =~~

~~∵ upward parabola so supremum must be at the ends~~

$$\sup R_\delta^t = \max\left(R_\delta^t\left(\tfrac{4}{9}\right), R_\delta^t\left(\tfrac{5}{9}\right)\right)$$

$$= \cancel{\left(\frac{40}{81}\right)^2 + \max\left(\frac{79}{81}\max\right)}$$

$$\cancel{r_\delta(\pi) = \left(\frac{79}{81}\theta^2 - \frac{79}{81}\theta + \left(\frac{40}{81}\right)^2\right)}$$

$$r_\delta(\pi)_{\text{Bayes}} = \pi\left(\theta = \tfrac{4}{9}\right) R_\delta^t\left(\tfrac{4}{9}\right) + \pi\left(\theta = \tfrac{5}{9}\right) R_\delta^t\left(\tfrac{5}{9}\right)$$

$$= \frac{1}{2}\times\left(\frac{79}{81}\left(-1 + \frac{16}{81} + \frac{25}{81}\right) + 2\times\left(\frac{40}{81}\right)^2\right)$$

$$= \left(\frac{40}{81}\right)^2 + \left(\frac{20}{81}\right)\left(\frac{79}{81}\right)$$

upward parabola so supremum of the rds

$$\sup_{Bayes} R_{\delta}^{*} = \max\left(R_{\delta}\left(\frac{4}{9}\right), R_{\delta}\left(\frac{5}{9}\right)\right)$$

$$= \left(\frac{40}{81}\right)^2 + \frac{79}{81}\max\left(\frac{4}{9}\times\frac{4}{9} - \cancel{\frac{4}{9}}\times\frac{4}{9}, \frac{5}{9}\times\frac{5}{9} - \cancel{\frac{5}{9}}\frac{5}{9}\right)$$

$$= \left(\frac{40}{81}\right)^2 + \frac{79}{81}\max\left(\frac{4}{9}\left(\frac{5}{9}\right), \frac{5}{9}\left(\frac{4}{9}\right)\right)$$

$$= \left(\frac{40}{81}\right)^2 + \frac{79}{81}\times\frac{20}{81}$$

$$= r_{\delta}(\pi) \; !$$

$\therefore$ δ_{Bayes}

$$\delta_{Bayes}(y) = \begin{cases} \frac{40}{81} & \text{when } y=0 \\ \frac{41}{81} & \text{when } y=1 \end{cases}$$

is Bayes wrt $\pi\left(\theta = \frac{4}{9}\right) = \pi\left(\theta = \frac{5}{9}\right) = \frac{1}{2}$

and $r_{\delta_{Bayes}}(\pi) = \sup R_{\delta_{Bayes}}^{*}(\theta)$

$\therefore$ δ_{Bayes} is a minimax procedure.

② a) $r_\delta(\pi) = \pi(\theta) + \dots$

when $r = \frac{1}{9}$

$u = 2 \times \frac{1}{9} \times \frac{8}{9} = \frac{16}{81}$; Ex.

$$\delta_{Bayes} = \begin{cases} \frac{16}{81} & \text{if } y = 0 \\ \frac{65}{81} & \text{if } y = 1 \end{cases}$$

$$R_{\delta_{Bayes}} = \frac{-17}{81}\theta^2 + \frac{17}{81}\theta + \left(\frac{16}{81}\right)^2$$

downward parabola (concave)

→ check for maximum

→ check ends if maximum not in domain

$$\theta_{max} = \frac{-\left(\frac{17}{81}\right)}{2 \times \left(\frac{-17}{81}\right)} = \frac{1}{2}$$

⇒ maxi R_δ

∴ R_δ $\cancel{\frac{1}{2} = 2r(1-r) \Rightarrow r = \frac{1}{2}}$

∴ R_0 which is within $\frac{1}{9} \le \theta \le \frac{8}{9}$

∴ $\sup R_{\delta_{Bayes}} = \frac{-17}{81} \times \frac{1}{4} + \frac{17}{81} \times \frac{1}{2} + a \frac{16}{81}$

② a)

$$= \frac{+17}{81} \times \frac{1}{4} + \left(\frac{16}{81}\right)^2$$

$$= \cancel{\frac{1}{4}} \times \frac{\cancel{3}0\cancel{7}3}{\cancel{32}4} \quad \frac{64}{324} - \frac{17}{324} = \frac{\cancel{4}7}{324}$$

$$r_\delta(\pi) = \frac{1}{2} \times \left(\cancel{\;} \; \frac{17}{81}\left(-\frac{1}{81} + 4 - \frac{64}{81} + 1 \right)\right)$$

$$+ \left(\frac{\cancel{\;}}{81}\right)^{\cancel{4}} \left(\frac{11}{81}\right)^2$$

$$= \frac{17 \times \cancel{1}8}{81 \times 81} + \left(\frac{11}{81}\right)^2$$

$$\to \cancel{\frac{17 \times 8}{81 \times}} \quad \frac{8}{81} < \frac{1}{4} \times \left(\quad \frac{32}{324} < \frac{81}{324} \right)$$

$$\Rightarrow \frac{17}{81}\left(\frac{8}{81}\right) < \frac{1}{4}\left(\frac{1\cancel{8}}{81}\right)$$

~~$\therefore R_\delta < r_\delta(\pi)$~~

$$\Rightarrow r_{\delta_a}(\pi) < \cancel{R_\delta} \sup_{\theta \in \Omega} R_{\delta_a}(\theta)$$

~~thm~~

So we can't say δ_{Bayes} obtained in (1) is ~~Bayes~~ ~~Bayes~~ ~~loss~~. minimax (thm 3.2)

b) $$u = 2\left(\frac{2-\sqrt{2}}{4}\right)\left(\frac{2+\sqrt{2}}{4}\right)$$

$$= \frac{1}{4}$$

b)

$$\delta_{Bayes} = \begin{cases} 1/4 & \text{if } y = 0 \\ 3/4 & \text{if } y = 1 \end{cases}$$

$$R_{\delta^*_{Bayes}}(\theta) = (4u-1)\theta^2 - \theta(4u-1) + u^2$$

$$= (1-1)\theta^2 + -\theta(1-1) + (1)^2$$

$$= 1 \quad \text{a constant}$$

$\cancel{r_\delta(\pi)} = \sup R\delta^*(\theta) = 1$

$$r_{\delta_{Bayes}}(\pi) = 1 + \frac{1}{4} \times \left(\frac{1}{2}\right) + 1 \times \frac{1}{2}$$

$$= 1 \qquad \left(\because \pi\left(\theta = \frac{2-\sqrt{2}}{4}\right) = \pi\left(\theta = \frac{2+\sqrt{2}}{4}\right) = 1/2\right)$$

$$r_{\delta_{Bayes}}(\pi) = \sup R_{\delta^*_{Bayes}}(\theta) = 1$$

$\therefore$ δ_{Bayes} is the minimax procedure

3 a) $m(y) = \sum_{\forall\theta} \pi(\theta) P_\theta(Y=y)$

$\cancel{m(0)} = \pi(0) P_{\theta=0}(Y=1$

$P_\theta(Y=1) = \theta$

$P_\theta(Y=0) = \theta(1-\theta)$

$$m(y=0) = (1-0)(1-r) + \theta\left(\frac{1}{2}\right)r.$$

$$= 1 - r/2$$

$$m(1) = \cancel{(1-\gamma)} \gamma(1-0) + \cancel{(\gamma)}(1-1/2)$$
$$= (1-\gamma)\times 0 + \gamma \times \frac{1}{2}$$
$$= \gamma/2$$

$$\pi(\theta \mid Y=y) = \frac{\pi(\theta)\, f_\theta(Y=y)}{m(y)}$$

	$\theta = 0$	$\theta = 1/2$
$y=0$	$\frac{(1-\gamma)\times 1}{1-\gamma/2}$	$\frac{\cancel{(1-\gamma)\times 0}\ \gamma \times 1/2}{1-\gamma/2}$
$y=1$	$\frac{(1-\gamma)\times 0}{\gamma/2}$	$\frac{\gamma \times 1/2}{\gamma/2}$

$$\Rightarrow \pi(\theta \mid y=0) = \begin{cases} \frac{1-\gamma}{1-\gamma/2} = \frac{2(1-\gamma)}{2-\gamma} & \theta = 0 \\ \frac{\gamma}{2-\gamma} & \theta = 1/2 \\ 0 & \text{ow} \end{cases}$$

$$\pi(\theta \mid y=1) = \begin{cases} 0 & \theta = 0 \\ 1 & \theta = 1/2 \\ 0 & \text{ow.} \end{cases}$$

$$\Rightarrow \pi(\theta/y=0) = \begin{cases} \frac{2(1-\gamma)}{2-\gamma} & \theta = 0 \\ \frac{\gamma}{2-\gamma} & \theta = 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\pi(\theta \mid y=1) = \begin{cases} 0 & \theta = 0 \\ 1 & \theta = 1/2 \\ 0 & \text{otherwise} \end{cases}$$

b) ~~δ~~ $h_\pi^* = \min E\left(L(\theta, d)\, \pi(\theta|y)\right)$

$L = (\theta - d)^2$

$\Rightarrow$ mean of $\pi(\theta|y)$

~~δ~~

$\delta_{Bayes}(Y=0) = \text{mean}\left(\pi(\theta \approx 0 / y = 0)\right)$

$= \frac{1}{2}\left(\frac{r\ \cancel{\alpha}}{\cancel{2}-r}\right) = \frac{r}{2(2-r)}$

$\delta_{Bayes}(Y=1) = \text{mean}\left(\pi(\theta|y=1)\right)$

$= \frac{1}{2} \times 1 = 1/2$

c) $R_{\delta_B}(\theta) = E\left((\theta - d)^2\right)$

$= E\left(\theta - \frac{r}{2(2-r)}\right)$

$= P(Y=0)\left(\theta - \frac{r}{2(2-r)}\right)^2 + P(Y=1)\left(\theta - 1/2\right)^2$

$= \frac{\cancel{1}}{2}(1-\theta)\left(\theta - \frac{r}{2(2-r)}\right)^2 + \theta\left(\theta - 1/2\right)^2$

$R_{\delta_B}(0) = R_{\delta_B}(1/2)$

$\Rightarrow \frac{r^2}{4(2-r)^2} = \frac{1}{2}\left(\frac{1}{2} - \frac{r}{2(2-r)}\right)^2 + \cancel{\frac{1}{2}(\ldots)}$

$\Rightarrow$ Let $\alpha = \dfrac{r}{2(2-1)}$

$\Rightarrow \alpha^2 = \dfrac{1}{2}\left(\dfrac{1}{2} - \alpha\right)^2$

$\Rightarrow 8\alpha^2 = 4\alpha^2 - 4\alpha + 1$

$\Rightarrow 4\alpha^2 + 4\alpha - 1 = 0$

$\Rightarrow \alpha = \dfrac{-4 \pm \sqrt{16+16}}{8}$

But $r/_{2(2-1)} > 0$ since $r > 0$

$\Rightarrow \alpha = \dfrac{-1 + \sqrt{2}}{2}$

$\Rightarrow \dfrac{r}{2(2-r)} = \dfrac{-1+\sqrt{2}}{2}$

$\Rightarrow r = (-1+\sqrt{2})(2-r) \Rightarrow \sqrt{2}\, r = 2(\sqrt{2}-1)$

$\Rightarrow r = 2 - \sqrt{2}$

$\therefore \delta_{Bayes}(1, 4) = \dfrac{r}{2(2-r)} = 1 = \dfrac{\sqrt{2}-1}{2}$

$$\Rightarrow \delta^*(y) = \begin{cases} \frac{\sqrt{2}-1}{2} & y=0 \\ 1/2 & y=1 \end{cases}$$

d) $R_{\delta_B}(\theta) = (1-\theta)\left(\theta - \frac{\sqrt{2}-1}{2}\right)^2 + \theta(\theta - 1/2)^2$

~~Firstly~~ we set $R_{\delta_B}(0) = R_{\delta_B}(1/2)$

So the ~~quadratic~~ sign of the first derivative changes in between

$\Rightarrow$ there is a stationary point in $[0, 1/2]$

$R_{\delta_B}(\theta) = (1-\theta)(\theta^2 - 2\alpha\theta + \alpha^2) + \cancel{\theta(\theta - 1/2)^2}$

where $\alpha = \frac{r}{2(2-1)}$ $\quad + \theta(\theta^2 + \frac{1}{4} - \theta)$

$\cancel{\Rightarrow R_{\delta_B}(\theta) = (1-\alpha)}$. θ^3 term is a. not there

$\cancel{= \theta^2 - 2\alpha\theta + \alpha^2}$

focusing just on θ^2 term,

$\theta^2(1 + 2\alpha - 1)$

coefficient of $\theta^2 = 2\alpha$. , But $\alpha > 0$

$\Rightarrow$ ~~therefore~~ $R_{\delta_B}(\theta)$ is convex, so

the stationary point is a minimum

$\sup R_{\delta_B}(\theta) = \max(R_{\delta_B}(0), R_{\delta_B}(1/2))$

Since, we set them as equal,

$$\sup R_{\delta_B}(\theta) = R_{\delta_B}(0)$$

$$= \alpha^2$$

$$= \left(\frac{\sqrt{2}-1}{2}\right)^2$$

$$\Rightarrow \sup R_{\delta_B}(\theta) = \frac{1}{4}(3-2\sqrt{2})$$

$$r_{\delta_{Bayes}}(\pi) = (1-r) \times \left(R_{\delta_B}(0) \right) + r\left(R_{\delta_B}(1/2)\right)$$

since $R_{\delta_B}(0) = R_{\delta_B}(1)$

$$\Rightarrow r_{\delta_B}(\pi) = R_{\delta_B}(0)$$

$$= \frac{3-2\sqrt{2}}{4} = \sup R_{\delta_B}(\theta)$$

The Bayes procedure has a bayesian risk which equal to the supremum of its risk fnct

$\therefore$ By theorem 3.2,

δ_{Bayes} is minimax

④ b) Consider $\pi(\theta) \sim N(\mu, \tau^2)$

In that case we know

$$\delta_B(y) = \frac{n/\sigma^2}{n/\sigma^2 + 1/\tau^2}\bar{y} + \frac{\frac{1}{\tau^2}}{\frac{n}{\sigma^2} + 1/\tau^2}\mu$$

and $r_{\delta_B}(\pi) = \frac{1}{n/\sigma^2 + \frac{1}{\tau^2}}$

consider sequence of priors

$$\pi_\tau(\theta) \sim N(\mu, \tau^2) \quad \tau \in \{1, 2, \dots\}$$

$$r_\tau = \frac{1}{n/\sigma^2 + \frac{1}{\tau^2}} \to \lim_{\tau\to\infty} r_\tau = \lim_{\tau\to\infty} \frac{1}{n/\sigma^2 + 1/\tau^2}$$

$$\Rightarrow \lim_{\tau\to\infty} r_\tau = \frac{\sigma^2}{n} = r$$

Let $\delta = \bar{y}$

We know, $R_\delta(\theta) = E\left((\theta - \bar{y})^2\right)$

$$= \frac{\sigma^2}{n}$$

$\Rightarrow R_\delta(\theta) = \sigma^2/n$

$\to R_\delta(\theta) \leq r \quad \forall\, \theta \in \Omega$

r_c converges to $r = \frac{\sigma^2}{n}$ & $R_\delta(\theta) <= r \; \forall \theta \in \Omega$

$\therefore$ By part (a), $\delta = \bar{Y}$ is a minimax procedure.

(a) Since δ_k is Bayes under π_k

$$\int_\Omega R_{\delta'}(\theta)\pi_k(\theta)d\theta \geq \int_\Omega R_{\delta_k}(\theta)\pi_k(\theta)d\theta$$

$$= r_k \quad \text{—} ①$$

where δ' is any valid statistical procedure

also $R_{\delta'}(\theta) \leq \sup_\theta R_{\delta'}(\theta)$

$$\Rightarrow \int_\Omega R_{\delta'}(\theta)\pi_k(\theta)d\theta \leq \int \sup_\theta R_{\delta'}(\theta)\pi_k(\theta)d\theta$$

$$= \sup_\theta R_{\delta'}(\theta)\int_\Omega \pi(\theta)d\theta$$

$$\Rightarrow \int_\Omega R_{\delta'}(\theta)\pi_k(\theta)d\theta \leq \sup_\theta R_{\delta'}(\theta)$$

$$\text{—} ②$$

① & ② $\Rightarrow$ $\sup_\theta R_{\delta'}(\theta) \geq r_k \; \forall k$

$$\lim_{k\to\infty} \sup_\theta R_{\delta'}(\theta) \geq \lim_{k\to\infty} r_k$$

$\Rightarrow \sup_\theta R_{\delta'}(\theta) \geq r$ ($\because$ r_k converges to r) — ③

But we are given $R_{\delta^*}(\theta) \leq r \; \forall \theta \in \Omega$

$\Rightarrow \sup_\theta R_{\delta^*}(\theta) = r$ — ④

③ & ④

$\Rightarrow \sup_\theta R_{\delta^*}(\theta) \leq \sup_\theta R_{\delta'}(\theta)$

$\forall$ valid δ'

This is the definition of minimax!

$\therefore$ The procedure δ^* is minimax.

⑤ ⓐ Define $F' = \mathcal{F}_1 - \mathcal{F}_0$ (set difference)

$= \mathcal{F}_1 \cap (\mathcal{F}_0^c)$

$\Rightarrow F' \cup F_0 = \mathcal{F}_1$

Because δ^* is minimax when $F \in \mathcal{F}_0$,

$$\sup_{F \in \mathcal{F}_0} R_{\delta^*}(F) \leq \sup_{F \in \mathcal{F}_0} R_{\delta'}(F) \quad \text{— ①}$$

where δ' is any valid procedure

For any procedure δ',

consider $\sup_{F \in \mathcal{F}_1} R_{\delta'}(F)$

$$= \sup_{F \in (\mathcal{F}_0 \cup \mathcal{F}')} R_{\delta'}(F) \quad (\because \mathcal{F}_0 \cup \mathcal{F}' = \mathcal{F}_1)$$

$$= \max\left(\sup_{F \in \mathcal{F}_0} R_{\delta'}(F), \sup_{F \in \mathcal{F}'} R_{\delta'}(F) \right) \text{ --- } (*)$$

for $\delta' = \delta^*$

$$\sup_{F \in \mathcal{F}_1} R_{\delta^*}(F) = \sup_{F \in \mathcal{F}_0} R_{\delta^*}(F) \text{ —— } (2)$$

$(*)$ sup.

If $\sup_{F \in \mathcal{F}_0} R_{\delta'}(F) \geq \sup_{F \in \mathcal{F}'} R_{\delta'}(F)$ }

$$* \Rightarrow \sup_{F \in \mathcal{F}_1} R_{\delta'}(F) = \sup_{F \in \mathcal{F}_0} R_{\delta'}(F)$$

$$\geq \sup_{F \in \mathcal{F}_0} R_{\delta^*}(F) \quad (\text{from } (1))$$

$$= \sup_{F \in \mathcal{F}_1} R_{\delta^*}(F) \quad (\text{from } (2))$$

$$\Rightarrow \sup_{F \in \mathcal{F}_1} R_{\delta'}(F) \geq \sup_{F \in \mathcal{F}_1} R_{\delta^*}(F) \quad \text{—— (3)}$$

fors all δ' whue

$$\sup_{F \in \mathcal{F}_0} R_{\delta'}(F) \geq \sup_{F \in \mathcal{F}_1} R_{\delta'}(F)$$

If $\sup_{F \in \mathcal{F}_0} R_{\delta'}(F) < \sup_{F \in \mathcal{F}_1} R_{\delta'}(F)$,

$$\sup_{F \in \mathcal{F}_1} R_{\delta'}(F) = \sup_{F \in \mathcal{F}_1} R_{\delta'}(F)$$

$$> \sup_{F \in \mathcal{F}_0} R_{\delta'}(F)$$

$$> \sup_{F \in \mathcal{F}_0} R_{\delta^*}(F) \quad (\text{from } 1)$$

$$= \sup_{F \in \mathcal{F}_1} R_{\delta^*}(F) \quad (\text{from } 2)$$

$$\Rightarrow \sup_{F \in \mathcal{F}_1} R_{\delta'}(F) \geq \sup_{F \in \mathcal{F}_1} R_{\delta^*}(F)$$

$\forall\, \delta'$ whue $\sup_{F \in \mathcal{F}_0} R_{\delta'}(F) < \sup_{F \in \mathcal{F}'} R_{\delta'}(F)$ —— (4)

(3) & (4)

$$\Rightarrow \sup_{F \in \mathcal{F}_1} R_{\delta^*}(F) \leq \sup_{F \in \mathcal{F}_1} R_{\delta'}(F)$$

$\Rightarrow \delta^*(F)$ is minimax when $F \in \mathcal{F}_1$

(5) b) We know, when Y_i are iid Bernoulli(θ) and $L = (\theta - d)^2$, $0 \leq \theta \leq 1$

$$\delta^* = \frac{\sqrt{n}}{1+\sqrt{n}} \bar{Y} + \frac{1}{1+\sqrt{n}}\left(\frac{1}{2}\right)$$

is minimax

So $\mathcal{F}_0$ = Set of all Bernoulli(θ) $0 \leq \theta \leq 1$

[mean (Bernoulli) $= \theta$] },

and δ^* is minimax there.

$$R_{\delta^*}(F) = \frac{1}{4(1+\sqrt{n})^2} \quad \forall F \in \mathcal{F}_0$$

$$\Rightarrow \max_{F \in \mathcal{F}_0} R_{\delta^*}(F) = \frac{1}{4(1+\sqrt{n})^2} \quad — (1)$$

When $0 \leq y_i \leq 1$, $y_i^2 \leq y_i$

$\Rightarrow E(Y_i^2) \leq E(Y_i) = \theta$ (definition of θ)

$$\begin{aligned} var(Y_i) &= E(Y_i^2) - E(Y_i)^2 \\ &= E(Y_i^2) - \theta^2 \\ &\leq E(Y_i^2) - \theta^2 \\ &\leq \theta - \theta^2 \end{aligned}$$

$$\Rightarrow var(Y_i) \leq \theta(1-\theta) \quad — (2)$$

Define $\mathcal{F}_1$ as set of all dist. with $F(0)=0$, $F(1)=1$

i.e. $y_i \in [0,1]$

Now $R_{\delta^*}(F) = E((\theta - \delta^*(y))^2)$

$= (E((\theta - \delta^*(y^*)))^2 + \text{var}(\theta - \delta^*(y))$

~~this depends only on first 2 moments~~

$$= \left(E\left(\theta - \frac{\sqrt{n}\,\bar{y}}{1+\sqrt{n}} - \frac{1}{2}\frac{1}{1+\sqrt{n}}\right)\right)^2$$

$$+ \text{var}\left(\frac{\bar{y}\sqrt{n}}{1+\sqrt{n}}\right) \quad — (3)$$

$$E\left(\theta - \frac{\sqrt{n}\,\bar{y}}{1+\sqrt{n}} - \frac{1}{2}\frac{1}{1+\sqrt{n}}\right)$$

$$= \left(\theta - \frac{\sqrt{n}\,\theta}{1+\sqrt{n}} - \frac{1}{2}\frac{1}{1+\sqrt{n}}\right)^2 = \left(\frac{\theta}{1+\sqrt{n}} - \frac{1}{2}\frac{1}{1+\sqrt{n}}\right)^2$$

$$= \frac{(2\theta-1)^2}{4(1+\sqrt{n})^2}$$

(3) $\Rightarrow$ $R_{\delta^*}(F) = \dfrac{(2\theta-1)^2}{4(1+\sqrt{n})^2} + \text{var}(\bar{y})\dfrac{n}{(1+\sqrt{n})^2}$

(2) $\Rightarrow$ $R_{\delta^*}(F) \le \dfrac{(2\theta-1)^2}{4(1+\sqrt{n})^2} + \dfrac{\theta(1-\theta)}{n}\dfrac{n}{(1+\sqrt{n})^2}$

$\Rightarrow R_{\delta^*}(F) \le \dfrac{\cancel{4\theta^2}\,\cancel{-4\theta}+1}{4(1+\sqrt{n})^2}$ (all θ terms get cancelled)

$\Rightarrow R_{\delta^*}(F) \le \dfrac{1}{4(1+\sqrt{n})^2} \quad \forall F \in \mathcal{F}_1$

$$\Rightarrow \sup_{F \in \mathcal{F}_1} R_{\delta^+}(F) = \frac{1}{4(1+\sqrt{n})^2} = \sup_{F \in \mathcal{F}_0} R_{\delta^+}(F)$$

$$\Rightarrow \sup_{F \in \mathcal{F}_1} R_{\delta^+}(F) = \sup_{F \in \mathcal{F}_0} R_{\delta^+}(F)$$

and δ^+ is minimax in $\mathcal{F}_0$

$\Rightarrow$ δ^+ is minimax in $\mathcal{F}_1$

$\Rightarrow$ δ^+ is minimax procedure ($\because \mathcal{F}_1$ is the set of all distributions)