ISyE 6412A **Theoretical Statistics** **Fall 2023**

HW #7 (due at midnight on Thursday, Oct 26, ET)

(After you spent at least 30 minutes per question, please feel free to take a look at the hints on the second page.)

1. **(Modified from problem 6.23(a))**. Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution on the interval $(\theta, 2\theta)$, $\theta > 0$. That is, the X_i's are iid with pdf $f_\theta(x) = \frac{1}{\theta}\mathbf{1}\{\theta < x < 2\theta\}$ for $\theta > 0$.

   (a) Find a minimal sufficient statistic for θ.

   (b) Is the minimal sufficient statistic in part (a) complete? Justify your answers.

2. (**Modified from Ex 6.5 of our text**, also see Problem 4 of HW#6). Assume that $X_1, \cdots, X_n$ are independent random variables with pdfs

$$f(x_i|\theta) = \begin{cases} \frac{1}{3i\theta}, & \text{if } -i(\theta-1) < x_i < i(2\theta+1); \\ 0, & \text{otherwise,} \end{cases} \qquad \text{for } i = 1, 2, 3, \ldots$$

   where $\theta > 0$. Let $T(\mathbf{X})$ be the (one-dimensional) minimal sufficient statistic for θ you found in HW#6, also see the solution set of Problem 4 of HW#6. Is this minimal sufficient statistic $T(\mathbf{X})$ complete? Justify your answers.

3. (**Modified from problem 7.37 of our text**). Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution on the interval $(-\theta, 2\theta)$, $\theta > 0$. That is, the X_i's are iid with pdf $f_\theta(x) = \frac{1}{3\theta}\mathbf{1}\{-\theta < x < 2\theta\}$ for $\theta > 0$.

   (a) Find a minimal sufficient statistic for θ.

   (b) Is the minimal sufficient statistic in part (a) complete? Justify your answers.

4. **(6.20(b)-(d)).** For each of the following pdfs let $X_1, \ldots X_n$ be iid observations. Find a complete sufficient statistic, or show that one does not exist. For part (b)-(d), please feel free to use Theorem 6.2.25 on page 288 of our text.

   (b) $f_\theta(x) = \frac{\theta}{(1+x)^{1+\theta}}, \qquad 0 < x < \infty,\ \theta > 0$

   (c) $f_\theta(x) = \frac{(\log\theta)\theta^x}{\theta - 1}, \qquad 0 < x < 1,\ \theta > 1$

   (d) $f_\theta(x) = e^{-(x-\theta)}\exp(-e^{-(x-\theta)}), \qquad -\infty < x < \infty,\ -\infty < \theta < \infty$

5. (**Motivated from problems 6.30 and 7.55(b) of our text**). Let $X_1, \ldots, X_n$ be a random sample from the pdf $f_\theta(x) = e^{-(x-\theta)}$ for $x > \theta$, where $-\infty < \theta < \infty$.

   (a) Show that $X_{(1)} = \min_i X_i$ is a complete sufficient statistic.

   (b) Use Basu's Theorem to show that $X_{(1)}$ and S^2 are independent. Recall that $S^2 = \frac{\sum_{i=1}^n (X_i - \bar{X}_n)^2}{n-1}$.

6. (**Modified from problem 6.3 of our text**) Let $X_1, \ldots, X_n$ be a random sample from the pdf

$$f(x|\mu, \sigma) = \frac{1}{\sigma}e^{-(x-\mu)/\sigma}, \quad \mu < x < \infty, \quad 0 < \sigma < \infty.$$

   Find a (two-dimensional) minimal sufficient statistic $T(\mathbf{X}) = (T_1, T_2)$ for (μ, σ) such that $T_1 = T_1(X_1, \cdots, X_n)$ and $T_2 = T_2(X_1, \cdots, X_n)$ are independent.

Hints: If you have already thought about each problem for at least 30 minutes, then please feel free to look at the hints. Otherwise, please try the problem first, as getting help from the hints takes away most of the fun.

Problem 1: Can you find two constants C_1 and C_2 (they might depend on n, but not on θ) such that

$$E_\theta(\frac{1}{C_1}X_{(1)} - \frac{1}{C_2}X_{(n)}) = 0$$

for all θ?

Problem 2: To check completeness of $T(\mathbf{X})$, derive its distribution by noting that

$$\begin{aligned}
\mathbf{P}_\theta(T(\mathbf{X}) \le t) &= \mathbf{P}_\theta\Big(\max\Big\{1 - \min_{1\le i\le n} \frac{x_i}{i}, \quad \frac{1}{2}(\max_{1\le i\le n} \frac{x_i}{i} - 1)\Big\} \le t\Big) \\
&= \mathbf{P}_\theta\Big(1 - \min_{1\le i\le n} \frac{x_i}{i} \le t \text{ and } \frac{1}{2}(\max_{1\le i\le n} \frac{x_i}{i} - 1 \le t\Big) \\
&= \mathbf{P}_\theta\Big(\min_{1\le i\le n} \frac{x_i}{i} \ge 1 - t \text{ and } \max_{1\le i\le n} \frac{x_i}{i} \le 2t + 1\Big) \\
&= \mathbf{P}_\theta(1 - t \le \frac{x_i}{i} \le 2t + 1 \text{ for all } i = 1, \cdots, n) \\
&= \prod_{i=1}^{n} \mathbf{P}_\theta\Big(- (t-1) \le \frac{X_i}{i} \le 2t + 1)\Big) \\
&= \prod_{i=1}^{n} \mathbf{P}_\theta\Big(- i(t-1) \le X_i \le i(2t + 1)\Big)
\end{aligned}$$

for $t > 0$. What happens if $0 \le t \le \theta$? How about if $t < 0$ or if $t > \theta$? Do you see any connections with the problem in which $X_1, \ldots, X_n$ are iid Uniform$(0, \theta)$?

Problem 3: this problem is very different from Problem #1, as the minimal sufficient statistic turns out to be one-dimensional as in Problem #2!!! When proving the completeness, you need to first its probability density function as in problem #2.

Problem 4: For part (b)-(d), please feel free to use Theorem 6.2.25 on page 288 of our text to find the complete sufficient statistic.

Problem 5: In part (a), what is the distribution of $T = X_{(1)}$? In (b), use Basu's theorem.

Problem 6: First, find a minimal sufficient statistic and thus any one-to-one function is also minimal sufficient. Second, you can guess the desired T_1 and T_2 by assuming for a moment that parameter σ is fixed and known, and by finding a complete sufficient statistic and an ancillary statistic for μ (note that they should be the one-to-one function of the minimal sufficient statistic you have found).
Alternatively, T_1 and T_2 can be used to construct reasonable estimates (read: **maximum likelihood estimator**) of μ and σ, respectively.
Third, assume, for a moment, that σ is known, and you can let $Z_i = X_i/\sigma$, and use Basu's theorem to show that your proposed T_1 and T_2 are independent. Since this holds for any $\sigma > 0$, you can conclude that this independence carries over even if σ is unknown, as knowledge of σ has no bearing on the distributions. Also see problem 6.31 of our text for more applications of Basu's theorem.