Lecture 5. Maps Between Spaces: isomorphism and homomorphism.

Definition 1. Let A and B be arbitrary nonempty sets. Suppose to each element $a \in A$ there is assigned a unique element of B so called the ***image*** of a. The collection f of such assignments is called a ***mapping*** (a map) from A to B, and it is denoted by $f : A \to B$.

The set A is called the ***domain*** of the mapping, and B is called ***target set***.

Notations:

- A map f from A to B if and only if $f : A \to B$
- $f(a) = b$ if and only if $f : a \longmapsto b$.

Linear mapping or homomorphism

Definition 2. Let V and U be vector spaces over $\mathbb{K}$. A mapping $F : V \to U$ is called a ***linear mapping*** or ***homomorphism*** if it satisfies the following two conditions

1. for any vectors $v_1, v_2 \in V, F(v_1 + v_2) = F(v_1) + F(v_2)$.
2. for any scalar $\lambda \in \mathbb{K}$ and vector $v \in V$, $F(\lambda v) = \lambda F(v)$.

In other words, F preserves the operations of addition and scalar multiplication.

Note that the conditions above can be written as one condition as follows: for any vectors $v_1, v_2 \in V$, and for any scalars $\lambda, \mu \in \mathbb{K}$

$$F(\lambda v_1 + \mu v_2) = \lambda F(v_1) + \mu F(v_2).$$

Proposition 1. If $F : V \to U$ is a linear mapping, then it sends the zero vector into the zero vector, that is, $F(\mathbf{0}) = \mathbf{0}$.

Example 1. a) Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $F : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix}$. Let $v, v_1, v_2 \in \mathbb{R}^3$ and $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, v_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, v_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ and $\lambda \in \mathbb{R}$. Then

$$F(v_1 + v_2) = F\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) = F\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

$$= F\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + F\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = F(v_1) + F(v_2).$$

$$F(\lambda v) = F\left(\lambda\begin{pmatrix}x\\y\\z\end{pmatrix}\right) = F\begin{pmatrix}\lambda x\\\lambda y\\\lambda z\end{pmatrix} = \begin{pmatrix}\lambda x\\\lambda y\end{pmatrix} = \lambda\begin{pmatrix}x\\y\end{pmatrix} = \lambda F\begin{pmatrix}x\\y\\z\end{pmatrix} = \lambda F(v).$$

Hence F is a linear mapping.

b) Let $F : M_{2,2} \to \mathbb{R}$ given by $F : \begin{pmatrix}a & b\\c & d\end{pmatrix} \to a + d$. Then for $v, v_1, v_2 \in M_{2,2}$ and $v = \begin{pmatrix}a & b\\c & d\end{pmatrix}$, $v_1 = \begin{pmatrix}a_1 & b_1\\c_1 & d_1\end{pmatrix}, v_2 = \begin{pmatrix}a_2 & b_2\\c_2 & d_2\end{pmatrix}$ and $\lambda \in \mathbb{R}$ we have

$$F(v_1 + v_2) = F\left(\begin{pmatrix}a_1 & b_1\\c_1 & d_1\end{pmatrix} + \begin{pmatrix}a_2 & b_2\\c_2 & d_2\end{pmatrix}\right) = F\left(\begin{pmatrix}a_1 + a_2 & b_1 + b_2\\c_1 + c_2 & d_1 + d_2\end{pmatrix}\right) =$$

$$= (a_1 + a_2) + (d_1 + d_2) = (a_1 + d_1) + (a_2 + d_2) =$$

$$= F\begin{pmatrix}a_1 & b_1\\c_1 & d_1\end{pmatrix} + F\begin{pmatrix}a_2 & b_2\\c_2 & d_2\end{pmatrix} = F(v_1) + F(v_2).$$

$$F(\lambda v) = F\left(\lambda\begin{pmatrix}a & b\\c & d\end{pmatrix}\right) = F\begin{pmatrix}\lambda a & \lambda b\\\lambda c & \lambda d\end{pmatrix} = \lambda a + \lambda d =$$

$$= \lambda(a + d) = \lambda F\begin{pmatrix}a & b\\c & d\end{pmatrix} = \lambda F(v).$$

Hence F is a linear mapping.

c) Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F: \begin{pmatrix}x\\y\end{pmatrix} \to \begin{pmatrix}x + 2\\y - 3\end{pmatrix}$. We note that

$$F(\mathbf{0}) = F\left(\begin{pmatrix}0\\0\end{pmatrix}\right) = \begin{pmatrix}0 + 2\\0 - 3\end{pmatrix} = \begin{pmatrix}2\\-3\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix} = \mathbf{0}.$$

So F does not send the zero vector into the zero vector, therefore F is not linear.

d) Let $V = P(t)$ be a space of polynomials in t. Let

$$D: P(t) \to P(t) \quad \text{(the derivative mapping)}$$

defined by

$$D: v \to \frac{dv}{dt}$$

for any $v = v(t) \in P(t)$. Then by properties of derivative mapping in Calculus one can easily show that D is a linear mapping.

e) Again consider a mapping on the space of polynomials

$$J : P(t) \to P(t) \qquad \text{(an integral mapping)}$$

defined by

$$J: v \longmapsto \int_0^1 v dt.$$

Using the properties of integrals, one can show that J is linear.

f) Given two mappings from V to U defined by $F(v) = 0$ and $I(v) = v$ for any $v \in V$. Then it is easy to show that they are linear mappings. F is called zero mapping and I is called identity mapping.

Theorem 1. Let V and U be vector spaces over $\mathbb{K}$. Let $\{v_1, \dots, v_n\}$ be a basis of V and $\{u_1, \dots, u_n\}$ be a basis of U. Then there exists a unique linear mapping $F : V \to U$ such that $F(v_1) = u_1, \dots, F(v_n) = u_n$.

Example 2. Let $F : \mathbb{R} \to \mathbb{R}$ be the linear mapping for which

$$F\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix} \text{ and } F\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\4\end{pmatrix}.$$

Now let us find a formula for $F\begin{pmatrix}a\\b\end{pmatrix}$. By the theorem above it exists and unique if F is linear, and $\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ and $\left\{\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}1\\4\end{pmatrix}\right\}$ are bases of $\mathbb{R}^2$. It is easy to check that they are bases of $\mathbb{R}^2$.

First write $\begin{pmatrix}a\\b\end{pmatrix}$ as a linear combination of $\begin{pmatrix}1\\2\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$.

$$\begin{pmatrix}a\\b\end{pmatrix} = x\begin{pmatrix}1\\2\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}x\\2x+y\end{pmatrix}.$$

Then $a = x$ and $b = 2x + y$. From them we derive $x = a$ and $y = -2a + b$. Then

$$F\begin{pmatrix}a\\b\end{pmatrix} = F\left(x\begin{pmatrix}1\\2\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix}\right) = xF\begin{pmatrix}1\\2\end{pmatrix} + yF\begin{pmatrix}0\\1\end{pmatrix} =$$

$$= a\begin{pmatrix}2\\3\end{pmatrix} + (-2a+b)\begin{pmatrix}1\\4\end{pmatrix} = \begin{pmatrix}b\\-5a+4b\end{pmatrix}.$$

Hence

$$F\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ -5a + 4b \end{pmatrix}.$$

Definition 3. Let $F : V \rightarrow U$ be a linear mapping. The ***kernel of F***, written $KerF$, is the set of vectors in V that map into the zero vector **0** in U, that is,

$$KerF = \{v \in V \,|F(v) = 0\}.$$

The ***image of F***, written ImF, is the set of images of vectors in U, that is,

$$Im(F) = \{F(v) \in U \,|\, v \in V\}.$$

Proposition 2. Let $F : V \rightarrow U$ be a linear mapping. Then $KerF$ is a subspace of V, and ImF is a subspace of U.

Proposition 3. Suppose $v_1, \dots, v_m$ span a vector space V and suppose $F : V \rightarrow U$ is linear. Then $F(v_1), \dots, F(v_m)$ span ImF.

Example 3. a) Let D be the derivative map from $P_3(t)$ to $P_3(t)$. Then

$$D(a_0 + a_1t + a_2t^2 + a_3t^3) = a_1 + a_2t + a_3t^2.$$

We note that $ImD = \{b_0 + b_1t + b_2t^2 | b_0, b_1, b_2 \in \mathbb{R}\} = P_2(t)$ and $KerD = \mathbb{R}$.

b) Let $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}.$$

Then $ImF = \{\begin{pmatrix} a \\ b \\ c \end{pmatrix} | c = 0\} = xy$ −plane and $KerF = \{\begin{pmatrix} a \\ b \\ c \end{pmatrix} | a = b = 0\} = z$ −axis.

Definition 3. Let $F : V \rightarrow U$ be a linear mapping. The ***rank of F*** is defined to be the dimension of its image, and the ***nullity of F*** is defined to be the dimension of its kernel; namely,

$$rank(F) = dim(Im\,F) \qquad \text{and} \qquad nullity(F) = dim(Ker\,F).$$

Theorem 2. Let V be a vector space of finite dimension, and let $F : V \rightarrow U$ be a linear mapping. Then

$$dimV = dim(KerF) + dim(ImF) = nullity(F) + rank(F).$$

Example 4. a) Let D be the derivative map from $P_3(t)$ to $P_3(t)$. We already know that $ImD = \{b_0 + b_1 t + b_2 t^2 | b_0, b_1, b_2 \in \mathbb{R}\} = P_2(t)$ and $Ker\ D = R$.

Then $rank(F) = 3$ and $nullity(F) = 1$. $dimP_3(t) = 3 + 1 = 4$.

b) Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$. We already know that

$ImF = \{\begin{pmatrix} a \\ b \\ c \end{pmatrix} | c = 0\} = xy$ $-$plane and $rank(F) = 2$.

$KerF = \{\begin{pmatrix} a \\ b \\ c \end{pmatrix} | a = b = 0\} = z$ $-$axis and $nullity(F) = 1$. $dim\mathbb{R}^3 = 2 + 1 = 3$.

c) Let $F : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear map: $F\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x - y + z + t \\ 2x - 2y + 3z + 4t \\ 3x - 3y + 4z + 5t \end{pmatrix}$.

We want to find

- a basis and dimension of ImF.
- a basis and dimension of $KerF$.

<u>a basis and dimension ImF:</u> First we find the image of the standard basis vectors of $\mathbb{R}^4$:

$$F\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad F\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}, \quad F\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \quad F\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}.$$

As we know the image vectors span ImF. Now, in order to have a basis of $\mathbb{R}^3$ we need to define independent vectors in the set of image vectors.

To get independent vectors we write the image vectors as rows of matrix (as we did in Lecture 4) and find its echelon matrix.

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix}.$$

So there are 2 independent vectors which span $Im\ F$. Hence

$$\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\}$$

is a basis of ImF and $dim(ImF) = rank(F) = 2$.

a basis and dimension $KerF$: Set $F(v) = 0$ for some $v = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$. From the definition of F we have $F(v) = \begin{pmatrix} x - y + z + t \\ 2x - 2y + 3z + 4t \\ 3x - 3y + 4z + 5t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$$\begin{cases} x - y + z + t = 0 \\ 2x - 2y + 3z + 4t = 0 \\ 3x - 3y + 4z + 5t = 0 \end{cases} \Leftrightarrow \begin{cases} x - y + z + t = 0 \\ z + 2t = 0 \\ z + 2t = 0 \end{cases} \Leftrightarrow \begin{cases} x - y + z + t = 0 \\ z + 2t = 0 \end{cases}$$

The system has free unknowns: y, t.

$$\begin{pmatrix} x \\ u \\ z \\ t \end{pmatrix} = \begin{pmatrix} y + t \\ y \\ -2t \\ t \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix},$$

where y and t any numbers.

Thus,

$$\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \}$$

is a basis of $KerF$ and $dim(KerF) = nullity(F) = 2$. As expected,

$$dim(ImF) + dim(KerF) = 4 = dim\mathbb{R}^4 = 4.$$

We have seen many types of vector spaces so far. In fact, many of them have ”the same” vector space structures. For example, $M_{2,2}$ and $P_3(t)$ are ”the same” as a vector space. They just have different nature of vectors. Another example is $M_{2,2}$ and $\mathbb{K}^4$. To see that we define a new notion so called ***isomorphism*** (”the same structure”) of vector spaces. Furthermore, we see that any n −dimensional vector space V over $\mathbb{K}$ is isomorphic to $\mathbb{K}^n$.

Definition 4. A mapping $F : V \rightarrow U$ is called an ***isomorphism*** if

1. F is linear
2. F is bijective, that is, one-to-one and onto.

If F is an isomorphism, then we say V and U are ***isomorphic*** and write $V \cong U$.

Recall definitions of one-to-one and onto mappings from Discrete Mathematics. A mapping $F : V \to U$ is said to be ***one-to-one***, if $F(v_1) = F(v_2)$, then $v_1 = v_2$. A mapping $F : V \to U$ is said to be ***onto***, if for any $u \in U$ there exists some $v \in V$ such that $F(v) = u$. In other words, a mapping $F : V \to U$ is said to be ***onto***, if $ImF = U$.

Example 5. a) Let $P_2(t)$ be a vector space over $\mathbb{R}$. Define $F : P_2(t) \to \mathbb{R}^3$ as follows

$$a_0 + a_1 t + a_2 t^2 \longmapsto \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}.$$

One can show that F is onto and one-to-one. It is easy to check that it is linear. Hence vector space $P_2(t)$ over $\mathbb{R}$ is isomorphic to $\mathbb{R}^3$ and $P_2(t) \cong \mathbb{R}^3$.

b) Let $P_3(t)$ and $M_{2,2}$ be vector spaces over $\mathbb{R}$. Define $F : M_{2,2} \to P_3(t)$ as follows

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto a + bt + ct^2 + dt^3$$

Then we note that F is onto and one-to-one, and it is linear. Hence vector space $M_{2,2}$ over $\mathbb{R}$ is isomorphic to $P_3(t)$ and $M_{2,2} \cong P_3(t)$.

Theorem 3. Vector spaces are isomorphic if and only if they have the same dimension.

Corollary 1. Any vector space of dimension n over $\mathbb{K}$ is isomorphic to $\mathbb{K}^n$.

The corollary is important when we work with finite dimensional vector spaces.

Let V be a vector space over $\mathbb{K}$ of dimension n. Then $V \cong \mathbb{K}^n$. Suppose $S = \{u_1, \dots, u_n\}$ is a basis of V. Then each vector $v \in V$ can be written as a linear combination of vector in S. Suppose $v = \lambda_1 u_1 + \dots + \lambda_n u_n$ for some $\lambda_1, \dots, \lambda_n \in \mathbb{K}$.

Then $\lambda_1, \dots, \lambda_n$ are called the coordinates of v with respect to S. In this case, we write

$$[v]_S = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}.$$

We note that $[u+v]_S = [u]_S + [v]_S$ and $[\lambda v]_S = \lambda[v]_S$ for all $u, v \in V$ and $\lambda \in \mathbb{K}$. If we consider $\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_n\end{pmatrix}$ as a vector of $\mathbb{K}$, then the correspondence $v \longmapsto \begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_n\end{pmatrix}$ defines an isomorphism between $V \cong \mathbb{K}^n$.

Example 6. Let $V = M_{2,2}$ over $\mathbb{R}$. As a basis of $M_{2,2}$ wetake

$$S = \{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\}.$$

Since

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = a\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + b\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + c\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} + d\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},$$

we have $\left[\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right]_S = \begin{pmatrix}a\\ b\\ c\\ d\end{pmatrix}$.

Then $F : M_{2,2} \to \mathbb{R}^4$ defined as $\begin{pmatrix}a & b\\ c & d\end{pmatrix} \longmapsto \begin{pmatrix}a\\ b\\ c\\ d\end{pmatrix}$. It defines isomorphism and have $M_{2,2} \cong \mathbb{R}^4$.

Let $F : V \to U$ and $G : V \to U$ be linear mappings. We define the sum $F + G$ and the scalar product λF , where $\lambda \in \mathbb{K}$ as follows:

$$(F + G)(v) := F(v) + G(v) \quad \text{and} \quad (\lambda F)(v) := \lambda F(v).$$

Proposition 4. Let V and U be vector spaces over $\mathbb{K}$. $F : V \to U$ and $G: V \to U$ be linear mappings. Then $F + G$ and λF are linear mappings.

Proposition 5. Let V and U be vector spaces over $\mathbb{K}$. Then the collection of all linear mappings from V into U with the above operations forms a vector space over $\mathbb{K}$.

We write $Hom(V, U)$ for space of linear mappings from V into U.

Proposition 6. Suppose $dimV = m$ and $dimU = n$. Then $dim\, Hom(V, U) = mn$.

Glossary

mapping	отображение
image	образ
domain	область

target set	целевой набор
kernel	ядро
nullity	недействительность
one-to-one	взаимно-однозначно

Exercises for lecture 5

1. Show that the following mapping $F: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $F(x, y, z) = (x + 2y - 3z, 4x - 5y + 6z)$ is linear .

2. Show that the following mapping $F: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $F(x, y, z) = (x + 1, y + z)$ is not linear:

3. For the linear mapping $F: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $F(x, y, z) = (x + 2y - 3z, 2x + 5y - 4z, x + 4y + z)$, find a basis and the dimension of the kernel and the image of F.

4. Let V be the vector space of n −square real matrices. Let M be an arbitrary but fixed matrix in V. Let $F: V \longrightarrow V$ be defined by $F(A) = AM + MA$, where A is any matrix in V. Show that F is linear.

5. Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear mapping for which $F(1;\ 2) = (2;\ 3)$ and $F(0;\ 1) = (1;\ 4)$. [Note that $\{(1;\ 2); (0;\ 1)\}$ is a basis of $\mathbb{R}^2$, so such a linear map F exists and is unique.] Find a formula for F; that is, find $F(a; b)$.

6. Suppose a linear mapping $F : V \longrightarrow U$ is one-to-one and onto. Show that the inverse mapping $F^{-1}: U \longrightarrow V$ is also linear.

Homework 5

1. Show that the following mapping $F: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $F(x, y) = (ax + by, cx + dy)$, where a, b, c, d belong to $\mathbb{R}$ is linear:

2. Show that the following mapping $F: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (xy, y)$ is not linear:

3. For the linear mapping $F: \mathbb{R}^4 \to \mathbb{R}^3$ defined by $F(x, y, z, t) = (x + 2y + 3z + 2t, 2x + 4y + 7z + 5t, x + 2y + 6z + 5t)$ find a basis and the dimension of the kernel and the image of F.

4. Let V be the vector space of n −square real matrices. Let M be an arbitrary but fixed matrix in V. Let $F: V \longrightarrow V$ be defined by $F(A) = M + A$, where A is any matrix in V. Show that F is not linear.

5. Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear mapping for which $F(1;\ 2) = (3;\ -1)$ and $F(0;\ 1) = (2;\ 1)$. Find a formula for F; that is, find $F(a; b)$.

6. Give an example of a nonlinear map $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $F^{-1}(0) = \{0\}$ but F is not one-to-one.