Lecture 9. Determinants: definition, properties.

Idea of Determinant of a n × n Matrix

For each $n \times n$ matrix can be assigned a special number called determinant. Below we first consider how it naturally arises in solving of system of linear equations.

Given a system of two linear equations in unknowns x_1 and x_2:

$$\begin{cases} a_{11}x_1 + a_{12}x + 2 = b_1 \\ a_{21}x_1 + a_{22}x + 2 = b_2 \end{cases}$$

In order to exclude x_2 in the second equation, we multiply the first equation by a_{22}, the second equation by $-a_{12}$ and then we add them. Then we have

$$(a_{11}a_{22} - a_{12}a_{21})x_1 = a_{22}b_1 - a_{12}b_2.$$

If $a_{11}a_{22} - a_{12}a_{21} \neq 0$, then one can find x_1

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}.$$

In similar way, one can find x_2

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}.$$

So the expression $a_{11}a_{22} - a_{12}a_{21}$ defines whether the system has a unique solution or not.

If we write coefficients of the system as a matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

then $a_{11}a_{22} - a_{12}a_{21}$ is said to be determinant of matrix A and write it as

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Then following that rule the expressions $a_{22}b_1 - a_{12}b_2$ and $a_{11}b_2 - a_{21}b_1$ will be determinants of matrices, respectively

$$\begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix} \quad and \quad \begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}.$$

So, if $|A| \neq 0$, then

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Definition 1. The determinant of a 2×2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is a

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Example 1. If $A = \begin{pmatrix} 2 & -3 \\ 5 & 6 \end{pmatrix}$, then

$$|A| = \begin{vmatrix} 2 & -3 \\ 5 & 6 \end{vmatrix} = 2 \cdot 6 - (-3) \cdot 5 = 12 + 15 = 27.$$

Determinant of 3×3 matrix

In order to find determinant of 3 × 3 matrix, we consider

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}\, x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}\, x_3 = b_1 \\ a_{31}x_1 + a_{32}x_2 + a_{33}\, x_3 = b_1 \end{cases}$$

Analogously, we can find x_1 and have

$$x_1 = \frac{b_1a_{22}a_{33} - b_1a_{23}a_{32} + b_2a_{32}a_{13} - b_2a_{12}a_{33} + b_3a_{12}a_{23} - b_3a_{22}a_{13}}{a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}}$$

It is a good exercise for students to derive this formula.
The expression

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

defines determinant of any 3 × 3 matrix.

Definition 2. The determinant of a 3×3 matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is a

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

Example 2. $\begin{vmatrix} 2 & 0 & 1 \\ 3 & 1 & 5 \\ 4 & -1 & -2 \end{vmatrix} = 2 \cdot 1 \cdot (-2) + 0 \cdot 5 \cdot 4 + 1 \cdot 3 \cdot (-1) - 1 \cdot 1 \cdot 4 - 0 \cdot 3 \cdot (-2) - 2 \cdot 5 \cdot (-1) = -4 + 0 - 3 - 4 - 0 + 10 = -1.$

We note that the determinant formula of 3 × 3 matrix can be written as a sum of determinants of 2 × 2 matrices:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$
$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Example 3. $\begin{vmatrix} 2 & 0 & 1 \\ 3 & 1 & 5 \\ 4 & -1 & -2 \end{vmatrix} = 2\begin{vmatrix} 1 & 5 \\ -1 & -2 \end{vmatrix} - 0\begin{vmatrix} 3 & 5 \\ 4 & -2 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 4 & -1 \end{vmatrix} = 2 \cdot (1 \cdot (-2) - 5 \cdot (-1)) - 0 \cdot (3 \cdot (-2) - 5 \cdot 4) + 1 \cdot (3 \cdot (-1) - 1 \cdot 4) = -4 + 10 - 0 - 0 - 3 - 4 = -1.$

Cofactor and Minors

Below we give a recursive definition of determinants. Let A be a $n \times n$ matrix. Assume that the determinant of $(n - 1) \times (n - 1)$ matrix is defined. Let M_{ij} denote $(n - 1) \times (n - 1)$ submatrix of A obtained deleting its i −th and j −th column. The determinant $|M_{ij}|$ is called the minor of the element a_{ij} of A, and we define the cofactor of a_{ij}, denoted by A_{ij}:

$$A_{ij} = (-1)^{i+j}|M_{ij}|.$$

Example 4. Let $A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 5 \\ 4 & -1 & -2 \end{pmatrix}$. We find minors $|M_{23}|, |M_{31}|$ and cofactors A_{23}, A_{31}.

$$|M_{23}| = \begin{vmatrix} 2 & 0 \\ 4 & -1 \end{vmatrix} = 2 \cdot (-1) - 0 \cdot 4 = -2,$$
$$A_{23} = (-1)^{2+3}\,|M_{23}| = -1 \cdot (-2) = 2.$$
$$|M_{31}| = \begin{vmatrix} 0 & 1 \\ 1 & 5 \end{vmatrix} = 0 \cdot 5 - 1 \cdot 1 = -1,$$
$$A_{31} = (-1)^{3+1}\,|M_{31}| = 1 \cdot (-1) = -1.$$

Properties of Determinants

Theorem 1. Let A be a square matrix.

1. If A has a row (column) of zeros, then $|A| = 0$.
2. If A has two identical rows (columns), then $|A| = 0$.
3. If A has zeros below the main diagonal, then $|A|$ is the product of diagonal elements. In particular, $|I_n| = 1$ for any n.

Example 5. a) $\begin{vmatrix} 2 & 3 & 2 \\ 3 & 1 & 5 \\ 0 & 0 & 0 \end{vmatrix} = 2 \cdot A_{11} + 3 \cdot A_{12} + 2 \cdot A_{13} = 2|M_{11}| - 3|M_{12}| + 2|M_{13}| = 2\begin{vmatrix} 1 & 5 \\ 0 & 0 \end{vmatrix} - 3\begin{vmatrix} 3 & 5 \\ 0 & 0 \end{vmatrix} + 2\begin{vmatrix} 3 & 1 \\ 0 & 0 \end{vmatrix} = 2 \cdot (1 \cdot 0 - 5 \cdot 0) - 3 \cdot (3 \cdot 0 - 5 \cdot 0) + 2 \cdot (3 \cdot 0 - 1 \cdot 0) = 0 - 0 + 0 = 0.$

b) $\begin{vmatrix} 0 & 3 & 2 \\ 0 & 1 & 5 \\ 0 & -1 & -2 \end{vmatrix} = 0 \cdot A_{11} + 3 \cdot A_{12} + 2 \cdot A_{13} = -3|M_{12}| + 2|M_{13}| = -3\begin{vmatrix} 0 & 5 \\ 0 & -2 \end{vmatrix} + 2\begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix} = -3 \cdot (0 \cdot (-2) - 5 \cdot 0) + 2 \cdot (0 \cdot (-1) - 1 \cdot 0) = 0 + 0 = 0.$

c) $\begin{vmatrix} 2 & 3 & 2 \\ 2 & 3 & 2 \\ 4 & -1 & -2 \end{vmatrix} = 2 \cdot A_{11} + 3 \cdot A_{12} + 2 \cdot A_{13} = 2|M_{11}| - 3|M_{12}| + 2|M_{13}| = 2\begin{vmatrix} 3 & 2 \\ -1 & -2 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ 4 & -2 \end{vmatrix} + 2\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix} = 2 \cdot (3 \cdot (-2) - 2 \cdot (-1)) - 3 \cdot (2 \cdot (-2) - 2 \cdot 4) + 2 \cdot (2 \cdot (-1) - 3 \cdot 4) = -8 + 36 - 28 = 0.$

d) $\begin{vmatrix} 2 & 3 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & -2 \end{vmatrix} = 2 \cdot 1 \cdot (-2) = -4.$

$$\begin{vmatrix} 2 & 3 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & -2 \end{vmatrix} = 2 \cdot A_{11} + 3 \cdot A_{12} + 2 \cdot A_{13} = 2|M_{11}| - 3|M_{12}| + 2|M_{13}|$$

$$= 2\begin{vmatrix} 1 & 5 \\ 0 & -2 \end{vmatrix} - 3\begin{vmatrix} 0 & 5 \\ 0 & -2 \end{vmatrix} + 2\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$= 2 \cdot (1 \cdot (-2) - 5 \cdot 0) - 3 \cdot 0 + 2 \cdot 0 = -4$$

Theorem 2. Suppose B is obtained from A by an elementary row (column) operation.

1. If two rows (columns) of A were interchanged, then $|B| = -|A|$.
2. If a row (column) of A were multiplied by a scalar λ, then $|B| = \lambda|A|$.
3. If a multiple of a row (column) of A were added to another row (column) of A, then $|B| = |A|$.

Example 6. a) We already know that determinant of $\begin{pmatrix} 2 & 3 & 2 \\ 3 & 1 & 5 \\ 4 & -1 & -2 \end{pmatrix}$ is equal to 70. Let B be a matrix obtained from A by interchanging the first and third rows. Then determinant of $B = \begin{pmatrix} 4 & -1 & -2 \\ 3 & 1 & 5 \\ 2 & 3 & 2 \end{pmatrix}$ should be -70.

Checking:

$$|B| = \begin{vmatrix} 4 & -1 & -2 \\ 3 & 1 & 5 \\ 2 & 3 & 2 \end{vmatrix} = 4 \cdot A_{11} + (-1) \cdot A_{12} + (-2) \cdot A_{13}$$

$$= 4|M_{11}| + |M_{12}| - 2|M_{13}| = 4\begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 5 \\ 2 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} =$$

$$= 4 \cdot (1 \cdot 2 - 5 \cdot 3) + (3 \cdot 2 - 5 \cdot 2) - 2 \cdot (3 \cdot 3 - 1 \cdot 2)$$

$$= -52 - 4 - 14 = -70.$$

b) Let $A = \begin{pmatrix} 2 & 3 & 2 \\ 3 & 1 & 5 \\ 4 & -1 & -2 \end{pmatrix}$. Then $|A| = 70$. If $B = \begin{pmatrix} 2 & 3 & 2 \\ 3 & 1 & 5 \\ 8 & -2 & -4 \end{pmatrix}$, then $|B| = 2|A = 140$.

Checking:

$$|B| = \begin{vmatrix} 2 & 3 & 2 \\ 3 & 1 & 5 \\ 8 & -2 & -4 \end{vmatrix} = 2 \cdot A_{11} + 3 \cdot A_{12} + 2 \cdot A_{13}$$

$$= 2|M_{11}| - 3|M_{12}| + 2|M_{13}|$$

$$= 2\begin{vmatrix} 1 & 5 \\ -2 & -3 \end{vmatrix} - 3\begin{vmatrix} 3 & 5 \\ 8 & -4 \end{vmatrix} + 2\begin{vmatrix} 3 & 1 \\ 8 & -2 \end{vmatrix} =$$

$$= 2 \cdot (1 \cdot (-4) - 5 \cdot (-2)) - 3 \cdot (3 \cdot (-4) - 5 \cdot 8) + 2 \cdot (3 \cdot (-2) - 1 \cdot 8) = 12 + 156 - 28 = 140.$$

c) Given matrix $A = \begin{pmatrix} 2 & 3 & 2 \\ 3 & 1 & 5 \\ 4 & -1 & -2 \end{pmatrix}$. Matrix B is obtained from A by elementary operation $R_3 \to -2R_1 + R_3$. Then $|B| = |A| = 70$.

Checking:

$$\begin{vmatrix} 2 & 3 & 2 \\ 3 & 1 & 5 \\ 4 & -1 & -2 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 2 \\ 3 & 1 & 5 \\ 0 & -7 & -6 \end{vmatrix} = 2 \cdot A_{11} + 3 \cdot A_{12} + 2 \cdot A_{13}$$
$$= 2|M_{11}| - 3|M_{12}| + 2|M_{13}|$$
$$= 2\begin{vmatrix} 1 & 5 \\ -7 & -6 \end{vmatrix} - 3\begin{vmatrix} 3 & 5 \\ 0 & -6 \end{vmatrix} + 2\begin{vmatrix} 3 & 1 \\ 0 & -7 \end{vmatrix} =$$
$$= 2 \cdot (1 \cdot (-6) - 5 \cdot (-7)) - 3 \cdot (3 \cdot (-6) - 5 \cdot 0) + 2 \cdot (3 \cdot (-7) - 1 \cdot 0) = 58 + 54 - 42 = 70.$$

Finding Determinants using Properties

$$\begin{vmatrix} 2 & 5 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 2 \\ -1 & -6 & 4 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & -2 & 2 \\ -2 & -3 & 2 & -5 \\ 2 & 5 & -3 & -2 \\ -1 & -6 & 4 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & -2 & 2 \\ 0 & 3 & -2 & -1 \\ 0 & -1 & 1 & -6 \\ 0 & -3 & 2 & 5 \end{vmatrix}$$
$$= \begin{vmatrix} 1 & 3 & -2 & 2 \\ 0 & -1 & 1 & -6 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -2 & 2 \\ 0 & -1 & 1 & -6 \\ 0 & 0 & 1 & -19 \\ 0 & 0 & -1 & 23 \end{vmatrix}$$
$$= \begin{vmatrix} 1 & 3 & -2 & 2 \\ 0 & -1 & 1 & -6 \\ 0 & 0 & 1 & -19 \\ 0 & 0 & 0 & 4 \end{vmatrix} == 1 \cdot (-1) \cdot 1 \cdot 4 = -4.$$

1. $R_1 \leftrightarrow R_3$
2. $R_2 \to 2R_1 + R_2, R_3 \to -2R_1 + R_3, R_4 \to R_1 + R_4$
3. $R_2 \leftrightarrow R_3,$
4. $R_3 \to 3R_2 + R_3, R_4 \to -3R_2 + R_4$
5. $R_4 \to R_3 + R_4$
6. determinant of a diagonal matrix

Theorem 3. $|A^T| = |A|$.

Example 7. $\begin{vmatrix} 1 & 3 & -2 & 2 \\ 0 & -1 & 1 & -6 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 3 & -1 & 1 & -6 \\ -3 & 3 & -2 & -1 \\ 2 & -3 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -1 & -6 & 5 \end{vmatrix} =$

$-\begin{vmatrix} -1 & -6 & 5 \\ -2 & 1 & 2 \\ 3 & -1 & -3 \end{vmatrix} = -\begin{vmatrix} -1 & -6 & 5 \\ 0 & 13 & -8 \\ 0 & -19 & 12 \end{vmatrix} = -\begin{vmatrix} -1 & 0 & 0 \\ -6 & 13 & -8 \\ 5 & -19 & 12 \end{vmatrix} =$

$-(-1)\begin{vmatrix} 13 & -19 \\ -8 & 12 \end{vmatrix} = 13 \cdot 12 - 19 \cdot 8 = 156 - 152 = 4.$

Theorem 4. $|AB| = |A||B|$.

Corollary 1. Let A be a square matrix. If $|A| \neq 0$, then $|A^{-1}| = \frac{1}{|A|}$.

Corollary 2. Let A be a square matrix. Then A^{-1} exists if and only if $|A| \neq 0$.

Glossary

determinant	определитель

Exercises for lecture 9

1. Find $\det(A)$, $adj\ A$, and A^{-1}, where

(a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$

2. Find the inverses of the matrices $\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

3. Use the algorithm to find the inverses of $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

Let A be the corresponding $n \times n$ matrix, and let B be its inverse. Guess the form of B, and then prove that $AB = I$ and $BA = I$.

4. Find A^{-1}, where $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$

5. Let $A = \begin{bmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}$. Find the third column of A^{-1} without computing the other columns.

6. Evaluate: $\begin{vmatrix} 2 & 6 \\ 4 & 1 \end{vmatrix}, \begin{vmatrix} 3 & 5 \\ 4 & -1 \end{vmatrix}$

7. Compute the determinant of each of the following matrices: $\begin{vmatrix} 2 & 1 & 1 \\ 0 & 5 & -2 \\ 1 & -3 & 4 \end{vmatrix}$

8. Find the determinant of the following matrix: $\begin{vmatrix} 2 & -1 & 3 & -4 \\ 2 & 1 & -2 & 1 \\ 3 & 3 & -5 & 4 \\ 5 & 2 & -1 & 4 \end{vmatrix}$

9. Solve the following systems by determinants:

(a) $\begin{cases} 3x + 5y = 8, \\ 4x - 2y = 1. \end{cases}$ (b) $\begin{cases} 2x - 3y = -1, \\ 4x + 7y = -1. \end{cases}$ (c) $\begin{cases} ax - 2by = c, \\ 3ax - 5by = 2c. \end{cases}$ $(ab \neq 0)$

Homework 9

1. Find $\det(A)$, $adj\ A$, and A^{-1}, where

(b) $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

2. Find the inverses of the matrices $\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$

3. Find A^{-1}, where $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

4. Let $A = \begin{bmatrix} -25 & -9 & -27 \\ 546 & 180 & 537 \\ 154 & 50 & 149 \end{bmatrix}$. Find the second and third columns of A^{-1} without computing the first column.

5. Evaluate: $\begin{vmatrix} 2 & -5 \\ 6 & -1 \end{vmatrix}$, $\begin{vmatrix} 3 & 7 \\ 4 & -2 \end{vmatrix}$

6. Compute the determinant of the following matrix $\begin{vmatrix} 3 & -2 & -4 \\ 2 & 5 & -1 \\ 0 & 6 & 1 \end{vmatrix}$

7. Evaluate the following determinants: $a)$ $\begin{vmatrix} 1 & 2 & -1 & 3 & 1 \\ 2 & -1 & 1 & -2 & 3 \\ 3 & 1 & 0 & 2 & -1 \\ 5 & 1 & 2 & -3 & 4 \\ -2 & 3 & -1 & 1 & -2 \end{vmatrix}$

8. Solve the following systems by determinants:

(a) $\begin{cases} 3x + 5y = 8, \\ 2x + 3y = 1. \end{cases}$

(b) $\begin{cases} 2x + 3y = -1, \\ 4x + 7y = -1. \end{cases}$

(c) $\begin{cases} ax - by = c, \\ 3ax - 4by = 2c. \end{cases} \quad (ab \neq 0)$