Lecture 1. Linear Systems: solving linear systems.

The system of linear equations play important role in science because many problems of Linear Algebra can be transformed into a system of linear equations and solved algorithmically.

In this lecture, we introduce the first basic concepts of a system of linear equations and give an algorithm, the so-called Gaussian elimination method, to solve them, and then we define the concept of a matrix.

Definition 1. A ***linear equation*** is an equation of the form $a_1x_1 + \cdots + a_px_p = b$ where $a_1, \ldots, a_p, b \in F$ and $x_1, \ldots, x_p$ are ***variables***. The scalars a_j are ***coefficients*** and the scalar b is the ***constant term***.

Definition 2. A ***system of linear equations***, or ***linear system***, is a set of one or more linear equations in the same variables, such as

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1p}x_p &= b_1 \\ a_{21}x_1 + \cdots + a_{2p}x_p &= b_2 \\ &\cdots \\ a_{m1}x_1 + \cdots + a_{mp}x_p &= b_m \end{aligned} \quad . \tag{1}$$

A ***solution*** of the system (1) is a p-tuple $(c_1, \ldots, c_p)$ such that letting $x_j = c_j$ for each j satisfies every equation.

The set of all solutions of the system (1) is called the ***solution set*** or the ***general solution*** of the system. Systems are ***equivalent*** if they have the same solution set. The system (1) is said to be ***homogeneous*** if all the constant terms are zero—that is, if $b_j = 0$ for all j. Otherwise, the system is said to be ***nonhomogeneous***.

The system (1) of linear equations is said to be ***consistent*** if it has one or more solutions, otherwise, it is said to be ***inconsistent***.

Theorem 1. Any system of linear equations has (i) a ***unique solution***, (ii) ***infinitely many solutions***, or (iii) ***no solution***. This situation is pictured in Fig. 1.

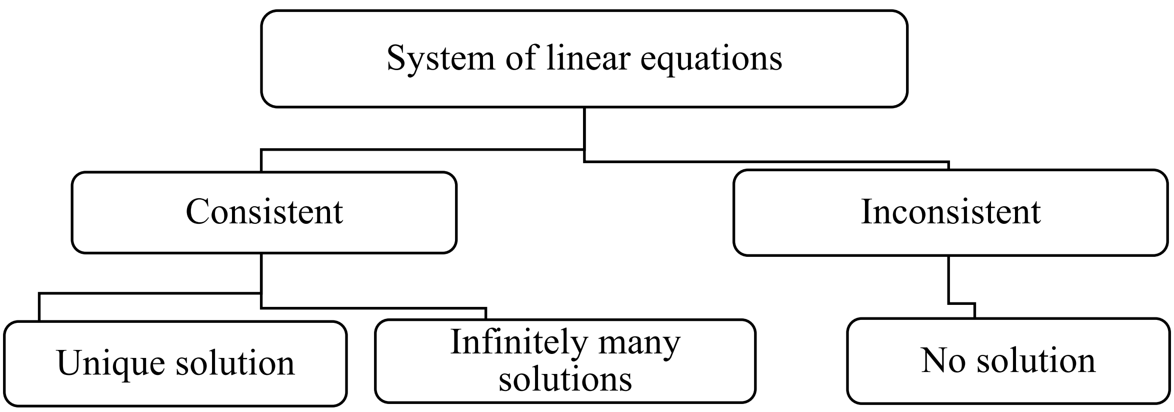

Figure 1

Example 1. Find the solution of the following three systems of linear equations:

$$1)\quad \begin{cases} 3x_1 + 2x_2 = 7 \\ -x_1 + x_2 = 6 \end{cases}, \quad 2)\begin{cases} 3x_1 - 3x_2 = -18 \\ -x_1 + x_2 = 6 \end{cases}, \quad 3)\begin{cases} 3x_1 - 3x_2 = -8 \\ -x_1 + x_2 = 6 \end{cases}.$$

Solution: 1) Let us express from the second equation $x_2 = 6 + x_1$ and substitute in the first equation $3x_1 + 2(6 + x_1) = 7 \Rightarrow 3x_1 + 2x_1 = 7 - 12 \Rightarrow 5x_1 = -5 \Rightarrow x_1 = -1$. Now we find x_2, substituting the found $x_1 = -1$ into the equation $x_2 = 6 + x_1 \Rightarrow x_2 = 6 + (-1) \Rightarrow x_2 = 5$. This system has only one solution, in other words, it has a unique solution.

2) If we multiply the second equation by -3, then we get that the first and second equations are the same. Let us express from the second equation $x_2 = 6 + x_1$ and we can substitute instead of x_1 any numbers. The second system has infinitely many solutions, for example (−1,5), (0,6), (1,7), and so on.

3) If we multiply the second equation by -3, then we get that the first and second equations on the left side are the same, but the answers are different, i.e. this system has no solution.

Elementary operations. To indicate i-th equation in the i-th row of a system of linear equations we write R_i. Now define three operations on linear equations $R_1, R_2, \ldots R_m$ in a system with m linear equations.

- interchanging (or swapping) i-th and j-th equations. We write it shortly as $R_i \leftrightarrow R_j$;
- replacing an i-th equation R_i by a nonzero multiple of itself. We write it shortly as $R_i \rightarrow \lambda R_i$, where $\lambda \neq 0$;
- replacing an j-th equation R_j by the sum of a multiple of i-th equation and itself. We write it shortly as $R_j \rightarrow \lambda R_i + R_j$, where $\lambda \neq 0$.

Example 2. Solve the following linear system using elementary operations

$$1)\begin{cases} x + 2y + z = -1 \\ 2x - y - z = -1 \\ -2x + 2y + 3z = 5 \end{cases}, \qquad 2)\begin{cases} 2x - 2y + 6z = 1 \\ x + y - z = 10 \\ x + z = 5 \end{cases}.$$

Solution: 1) First, try to exclude the unknown x from the second and third equations using the coefficient of x in the first equation. For this reason, we need to perform the elementary operations $R_2 \rightarrow -2R_1 + R_2$ and $R_3 \rightarrow 2R_1 + R_3$ and get

$$\begin{cases} x + 2y + z = -1 \\ -5y - 3z = 1 \\ 6y + 5z = 3 \end{cases}.$$

Repeat the same procedure to exclude y from the third equation using the coefficient of y in the second equation. Use the elementary operation $R_3 \rightarrow 6R_2 + 5R_3$ and have

$$\begin{cases} x + 2y + z = -1 \\ -5y - 3z = 1 \\ 7z = 21 \end{cases}.$$

From the last equation, one can find z. To find y we substitute the value of z into the second equation. Then we substitute the values of y and z into the first equation. Finally, we completely solve the system of linear equations and have $x = 0, y = -2, z = 3$ or write it as $(x, y, z) = (0, -2, 3)$.

2) Interchange the first and second equations, namely, perform $R_1 \leftrightarrow R_2$ and get

$$\begin{cases} x + y - z = 10 \\ 2x - 2y + 6z = 1. \\ x + z = 5 \end{cases}$$

Perform simultaneously $R_2 \rightarrow -2R_1 + R_2$ and $R_3 \rightarrow -R_1 + R_3$, and get

$$\begin{cases} x + y - z = 10 \\ -4y + 8z = -19. \\ -y + 2z = -5 \end{cases}$$

Then perform $R_3 \rightarrow -R_2 + 4R_3$ and get

$$\begin{cases} x + y - z = 10 \\ -4y + 8z = -19. \\ 0 = -1 \end{cases}$$

The last equation gives a false statement. This contradiction implies that the given system of linear equations has no solution.

Definition 3. A system of linear equation is in ***echelon form*** if it has the following form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \cdots + a_{1p}x_p &= b_1 \\ a_{2j_2}x_{j_2} + a_{2j_3}x_{j_3} + \cdots + a_{2p}x_p &= b_2 \\ \cdots \\ a_{mj_m}x_{j_m} + \cdots + a_{mp}x_p &= b_m \end{aligned}, \tag{2}$$

where $1 < j_2 < \cdots < j_r$ and $a_{11}, a_{2j_2}, \ldots, a_{rj_r}$ are not zero. $x_1, x_{j_2}, \ldots, x_{j_m}$ are called ***pivots*** or ***leading unknowns***. The unknows that are not leading in the echelon form are called ***free***.

Example 3. The system $\begin{cases} x - 2y + z = 5 \\ \quad y - 2z = 3 \end{cases}$ is in echelon form. x and y are leading unknowns, z is free variable.

Consider a system of linear equations in echelon form, say with m equations in p unknowns. There are two cases:

i. $m = p$. That is, there are as many equations as unknowns (triangular form). Then the system has a ***unique solution***.
ii. $m < p$. That is, there are more unknowns than equations. Then we can arbitrarily assign values to the $p - m$ free variables and solve uniquely for the m pivot variables, obtaining a solution of the system. Then the system has ***infinitely many solutions***.

Special case of echelon form is triangular form. It is the case when $m = p$.

Definition 4. A system of linear equation is in ***triangular form*** if it has the following form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \cdots + a_{1p}x_p &= b_1 \\ a_{22}x_2 + a_{23}x_3 + \cdots + a_{2p}x_p &= b_2 \\ \cdots \\ a_{pp}x_p &= b_p \end{aligned}, \tag{3}$$

where $a_{11}, a_{22}, \ldots, a_{pp}$ are not zero.

Example. The system $\begin{cases} x - 2y + z = 5 \\ \qquad y - 2z = 3 \\ \qquad\quad 4z = 4 \end{cases}$ is in triangular form. x, y and z are leading variables.

Gaussian Elimination (Gauss method). Gaussian Elimination consists of two parts:

Forward elimination. Step-by-step reduction of the system yielding either a degenerate equation with no solution or an equivalent simpler system in triangular or echelon form.

Backward Elimination. Step-by-step back-substitution to find the solution of the simpler system.

Example 4. Find the solution of the following three systems of linear equations by using Gaussian Elimination:

1) $\begin{cases} x+2y-4z=-4 \\ 2x+5y-9z=-10, \\ 3x-2y+3z=11 \end{cases}$ 2) $\begin{cases} x+2y-3z=-1 \\ -3x+y-2z=-7, \\ 5x+3y-4z=2 \end{cases}$

3) $\begin{cases} x+2y-3z=1 \\ 2x+5y-8z=4 \\ 3x+8y-13z=7 \end{cases}$.

Solution: 1) $\begin{cases} x+2y-4z=-4 \\ 2x+5y-9z=-10 \\ 3x-2y+3z=11 \end{cases} \begin{matrix} R_2 \to -2R_1+R_2 \\ R_3 \to -3R_1+R_3 \end{matrix} \Rightarrow$

$$\begin{cases} x+2y-4z=-4 \\ y-z=-2 \\ -8y+15z=23 \end{cases} \quad R_3 \to 8R_2+R_3 \Rightarrow$$

$$\begin{cases} x+2y-4z=-4 \\ y-z=-2 \\ 7z=7 \end{cases} \quad R_3 \to \frac{1}{7}R_3 \Rightarrow \quad \begin{cases} x=-2y+4z-4 \\ y=z-2 \\ z=1 \end{cases} \Rightarrow$$

$$\begin{cases} x=-2y+4\cdot 1-4 \\ y=1-2 \\ z=1 \end{cases} \Rightarrow \quad \begin{cases} x=-2\cdot(-1) \\ y=-1 \\ z=1 \end{cases} \Rightarrow \quad \begin{cases} x=2 \\ y=-1. \\ z=1 \end{cases}$$

This system has a unique solution $(x,y,z)=(2,-1,1)$.

2) $\begin{cases} x+2y-3z=-1 \\ -3x+y-2z=-7 \\ 5x+3y-4z=2 \end{cases} \begin{matrix} R_2 \to 3R_1+R_2 \\ R_3 \to -5R_1+R_3 \end{matrix} \Rightarrow$

$$\begin{cases} x+2y-3z=-1 \\ 7y-11z=-10 \\ -7y+11z=7 \end{cases} \quad R_3 \to R_2+R_3 \Rightarrow \begin{cases} x+2y-3z=-1 \\ 7y-11z=-10 \\ 0=-3 \end{cases}.$$

The last equation gives a false statement. This system of linear equations has no solution.

3) $\begin{cases} x+2y-3z=1 \\ 2x+5y-8z=4 \\ 3x+8y-13z=7 \end{cases} \begin{matrix} R_2 \to -2R_1+R_2 \\ R_3 \to -3R_1+R_3 \end{matrix} \Rightarrow$

$$\begin{cases} x+2y-3z=1 \\ y-2z=2 \\ 2y-4z=4 \end{cases} \quad R_3 \to -2R_2+R_3 \Longrightarrow$$

$$\begin{cases} x+2y-3z=1 \\ y-2z=2 \\ 0=0 \end{cases} \Longrightarrow \begin{cases} x=-2y+3z+1 \\ y=2z+2 \\ 0=0 \end{cases} \Longrightarrow$$

$$\begin{cases} x=-2(2z+2)+3z+1 \\ y=2z+2 \\ 0=0 \end{cases} \Longrightarrow \begin{cases} x=-4z-4+3z+1 \\ y=2z+2 \\ 0=0 \end{cases} \Longrightarrow$$

$$\begin{cases} x=-z-3 \\ y=2z+2. \\ 0=0 \end{cases}$$

This system has 3 unknowns variables and 2 equations, i.e., it has infinitely many solutions. x and y are pivots variables and z is free variable. The set of solution or general solution of system is $(x,y,z)=(-z-3,2z+2,z)$. We can find a particular solution to this system of linear equations by substituting any rational number instead of a free variable z. If $z=0$, then $(x,y,z)=(-3,2,0)$. If $z=1$, then $(x,y,z)=(-4,\ 4,1)$.

Glossary

variable	переменная
consistent	совместна
general solution	общее решение
equivalent	эквивалентный, равнозначный
homogeneous	однородный
inconsistent	несовместна
unique solution	уникальное решение
infinitely many solutions	бесконечно много решений
indicate	указывать
interchanging	взаимозаменяемость
swapping	замена
exclude	исключить
perform	выполнять
contradiction	противоречие
substitute	подстановка

Exercises for lecture 1

1. Use Gauss's Method to solve each system and conclude "unique solution", "many solutions" or "no solutions".

$$a)\begin{cases} 2x+3y=13 \\ x-y=-1 \end{cases}, \qquad b)\begin{cases} 2x+2y=5 \\ -4x-4y=0 \end{cases}'$$

$$c)\begin{cases} x-3y+z=1 \\ x+y+2z=14 \end{cases}', \qquad d)\begin{cases} 4y+z=20 \\ 2x-2y+z=0 \\ x+z=5 \\ x+y-z=10 \end{cases}.$$

2. For which values of k are there no solutions, many solutions, or a unique solution to this system?

$$\begin{cases} x-y=1 \\ 3x-3y=k \end{cases}$$

3. What conditions must the constants, the b's, satisfy so that of this system has a solution? *Hint.* Apply Gauss's Method and see what happens to the right side.

$$\begin{cases} x-3y=b_1 \\ 3x+y=b_2 \\ x+7y=b_3 \\ 2x+4y=b_4 \end{cases}$$

4. Find the solution of the following system of linear equations by using Gaussian Elimination:

$$\begin{cases} x+y+z=9 \\ 2x+4y-3z=1. \\ 3x+6y-5z=0 \end{cases}$$

5. Find the solution of the following system of linear equations by using Gaussian Elimination:

$$\begin{cases} x+3y=1 \\ 2x+y=-3. \\ 2x+2y=0 \end{cases}$$

6. Find the solution of the following system of linear equations by using Gaussian Elimination:

$$\begin{cases} x + 2y = 4 \\ y - z = 0 \\ x + 2z = 4 \end{cases}.$$

7. Determine the pivot and free variables of the following system:

$$\begin{cases} 2x_1 - 3x_2 - 6x_3 - 5x_4 + 2x_5 = 7 \\ \qquad\qquad x_3 + 3x_4 - 7x_5 = 6 \\ \qquad\qquad\qquad x_4 - 2x_5 = 1 \end{cases}.$$

8. Find the solution of the following system of linear equations by using Gaussian Elimination:

$$\begin{cases} x - 3y + 2z - w + 2t = 2 \\ 3x - 9y + 7z - w + 3t = 7 \\ 2x - 6y + 7z + 4w - 5t = 7 \end{cases}.$$

9. Find those values of a for which the system has a unique solution.

$$\begin{cases} x + 2y + z = 3 \\ ay + 5z = 10 \\ 2x + 7y + az = b \end{cases}.$$

Homework 1

1. Use Gauss's Method to solve each system and conclude "unique solution", "many solutions" or "no solutions".

$$a)\begin{cases} x - z = 0 \\ 3x + y = 1 \\ -x + y + z = 4 \end{cases}, \qquad b)\begin{cases} -x + y = 1 \\ x + y = 2 \end{cases},$$

$$c)\begin{cases} -x - y = 1 \\ -3x - 3y = 2 \end{cases}, \quad d)\begin{cases} 2x + z + w = 5 \\ y - w = -1 \\ 3x - z - w = 0 \\ 4x + y + 2z + w = 9 \end{cases}.$$

2. For which values of k are there no solutions, many solutions, or a unique solution to this system?

$$\begin{cases} x + y = 0 \\ 2x + 2y = k \end{cases}$$

3. What conditions must the constants, the b's, satisfy so that of this system has a solution? *Hint.* Apply Gauss's Method and see what happens to the right side.

$$\begin{cases} x + 2y + 3z = b_1 \\ 2x + 5y + 3z = b_2 \\ x + 8z = b_3 \end{cases}$$

4. Find the solution of the following system of linear equations by using Gaussian Elimination:

$$\begin{cases} x - y = 0 \\ 2x - 2y + z + 2w = 4 \\ y + w = 0 \\ 2z + w = 5 \end{cases}.$$

5. Find the solution of the following system of linear equations by using Gaussian Elimination:

$$\begin{cases} x + y - 2z = 0 \\ x - y = -3 \\ 3x - y - 2z = 0 \end{cases}.$$

6. Find the solution of the following system of linear equations by using Gaussian Elimination:

$$\begin{cases} 2x - 2y = 0 \\ z + 3w = 2 \\ 3x - 3y = 0 \\ x - y + 2z + 6w = 4 \end{cases}.$$

7. Determine the pivot and free variables of the following system:

$$\begin{cases} 2x - 6y + 7z = 1 \\ 4y + 3z = 8. \\ 2z = 4 \end{cases}$$

8. Find the solution of the following system of linear equations by using Gaussian Elimination:

$$\begin{cases} x + 2y - 3z + 4w = 2 \\ 2x + 5y - 2z + w = 1 \\ 5x + 12y - 7z + 6w = 3 \end{cases}.$$

9. Find those pairs of values (a, b) for which the system has more than one solution.

$$\begin{cases} x + 2y + z = 3 \\ ay + 5z = 10 \\ 2x + 7y + az = b \end{cases}.$$