Lecture 2. Linear Systems: linear geometry, reduced echelon form.

Definition 1. A ***vector (or a column vector)*** is a matrix with a single column. A matrix with a single row is a ***row vector***. The entries of vector are its components.

We use lower-case letters overlined with an arrow such as $\vec{a}, \vec{b}, \ldots$ A column or row vector whose components are all zeros is a zero vector and denoted $\vec{0}$.

Example. Given $\vec{s} = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}$, where $\vec{s}$ is a column vector with components 1, −3, and 0.

Definition 2. The ***vector sum*** of $\vec{u}$ and $\vec{v}$ is the vector defined as

$$\vec{u} + \vec{v} = \begin{pmatrix} u_1 \\ \cdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \cdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ \cdots \\ u_n + v_n \end{pmatrix}.$$

The ***scalar multiplication*** of the real number r and the vector $\vec{v}$ is the vector

$$r \cdot \vec{v} = r \begin{pmatrix} v_1 \\ \cdots \\ v_n \end{pmatrix} = \begin{pmatrix} rv_1 \\ \cdots \\ rv_n \end{pmatrix}.$$

Example. Given vectors $\vec{u} = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 3 \\ 6 \\ -7 \end{pmatrix}$, and scalar $r = 5$. Find the $\vec{u} + \vec{v}$ and $r \cdot \vec{v}$.

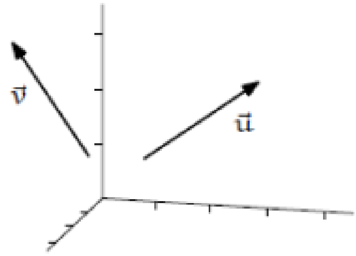
Solution. $\vec{u} + \vec{v} = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 6 \\ -7 \end{pmatrix} = \begin{pmatrix} 1+3 \\ -3+6 \\ 0+(-7) \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ -7 \end{pmatrix}.$

$$r \cdot \vec{v} = 5 \begin{pmatrix} 3 \\ 6 \\ -7 \end{pmatrix} = \begin{pmatrix} 5 \cdot 3 \\ 5 \cdot 6 \\ 5 \cdot (-7) \end{pmatrix} = \begin{pmatrix} 15 \\ 30 \\ -35 \end{pmatrix}.$$

Definition 3. The ***length*** **(or *norm*)** of a vector $\vec{v} \in \mathbb{R}^n$ is the square root of the sum of the squares of its components

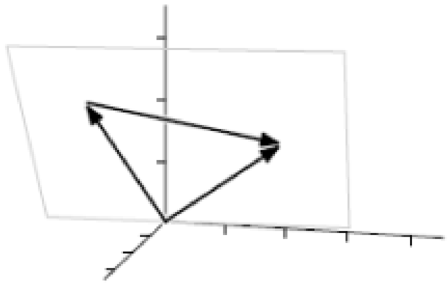
$$||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

We can use that to get a formula for the angle between two vectors. Consider two vectors in $\mathbb{R}^3$ where neither is a multiple of the other



(the special case of multiples will prove below not to be an exception).

They determine a two-dimensional plane – for instance, put them in canonical position and take the plane formed by the origin and the endpoints. In that plane consider the triangle with sides $\vec{u}$, $\vec{v}$ and $\vec{u}-\vec{v}$.



Apply the *Law of Cosines*: $\left|\left|\vec{u}-\vec{v}\right|\right|^2 = \left|\left|\vec{u}\right|\right|^2 + \left|\left|\vec{v}\right|\right|^2 - 2\left|\left|\vec{u}\right|\right| \cdot \left|\left|\vec{v}\right|\right| cos\theta$ where θ is the angle between the vectors. The left side gives

$$(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2$$
$$= (u_1^2 - 2u_1v_1 + v_1^2) + (u_2^2 - 2u_2v_2 + v_2^2) + (u_3^2 - 2u_3v_3 + v_3^2)$$

Canceling squares $u_1^2, \ \ldots, \ v_3^2$ and dividing by 2 gives the formula

$$\theta = \arccos\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{\left|\left|\vec{u}\right|\right|\left|\left|\vec{v}\right|\right|}\right).$$

To give a definition of angle that works in higher dimensions we cannot draw pictures, but we can make the argument analytically.

First, the form of the numerator is clear – it comes from the middle terms of $(u_i - v_i)^2$.

Definition 4. The ***dot product*** (or ***inner product*** or ***scalar product***) of two n-component real vectors is the linear combination of their components

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

Note that the dot product of two vectors is a real number, not a vector, and that the dot product of a vector from $\mathbb{R}^n$ with a vector from $\mathbb{R}^m$ is not defined unless n equals m. Note also this relationship between dot product and length:

$$\vec{u} \cdot \vec{u} = u_1 u_1 + u_2 u_2 + \cdots + u_n u_n = \left|\left|\vec{u}\right|\right|^2.$$

Theorem 1. (Triangle Inequality) For any $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$\left|\left|\vec{u} + \vec{v}\right|\right| \leq \left|\left|\vec{u}\right|\right| + \left|\left|\vec{v}\right|\right|$$

with equality if and only if one of the vectors is a nonnegative scalar multiple of the other one.

This is the source of the familiar saying, "The shortest distance between two points is in a straight line."

Because the Triangle Inequality says that in any $\mathbb{R}^n$ the shortest cut between two endpoints is simply the line segment connecting them, linear surfaces have no bends.

Back to the definition of angle measure. The heart of the Triangle Inequality's proof is the $\vec{u} \cdot \vec{v} \leq \left|\left|\vec{u}\right|\right| \cdot \left|\left|\vec{v}\right|\right|$ line. We might wonder if some pairs of vectors satisfy the inequality in this way: while $\vec{u} \cdot \vec{v}$ is a large number, with absolute value bigger than the right-hand side, it is a negative large number. The next result says that does not happen.

Definition 5. The angle between two nonzero vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ is

$$\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\left|\left|\vec{u}\right|\right| \cdot \left|\left|\vec{v}\right|\right|}\right)$$

(by definition, the angle between the zero vector and any other vector is right).

Corollary 1. Vectors from $\mathbb{R}^n$ are orthogonal, that is, perpendicular, if and only if their dot product is zero.

We know from Lecture 1 that a system of linear equations has three possible solutions: unique solution, infinitely many solutions or no solutions. Now we will consider geometric meaning of these solutions when a system has two or three unknowns.

Given system of two linear equations:

$$\begin{cases} a_1 x_1 + b_1 x_2 = c_1 \\ a_2 x_1 + b_2 x_2 = c_2 \end{cases}.$$

Then the graph of each equation is a line in the plane $\mathbb{R}^2$.

(a) Here two lines intersect in one point. This occurs when the lines have the distinct slopes or equivalently, when the coefficients of x_1 and x_2 are not proportional:

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}.$$

Example. Given the system $\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{cases}$, the solution is $(3,2)$, pictured in Fig.2.

(b) Here the two lines are parallel. This occurs when lines have the same slopes but different y −intercepts, or when

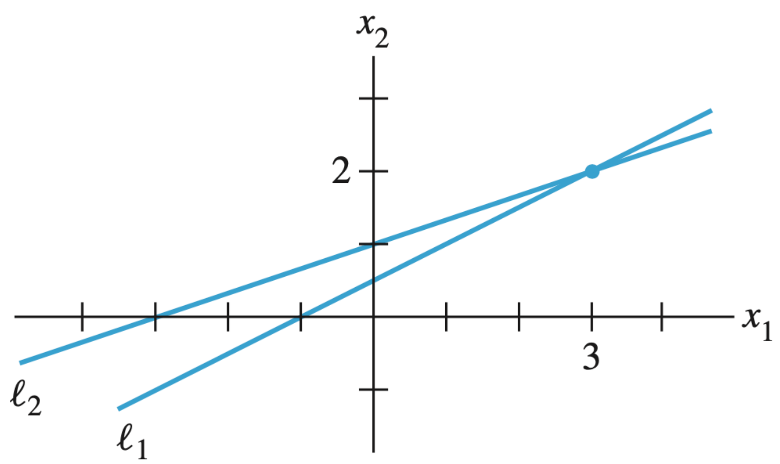
$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}.$$

Example. Given the system $\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 3 \end{cases}$, is has no solution, pictured in Fig.2.

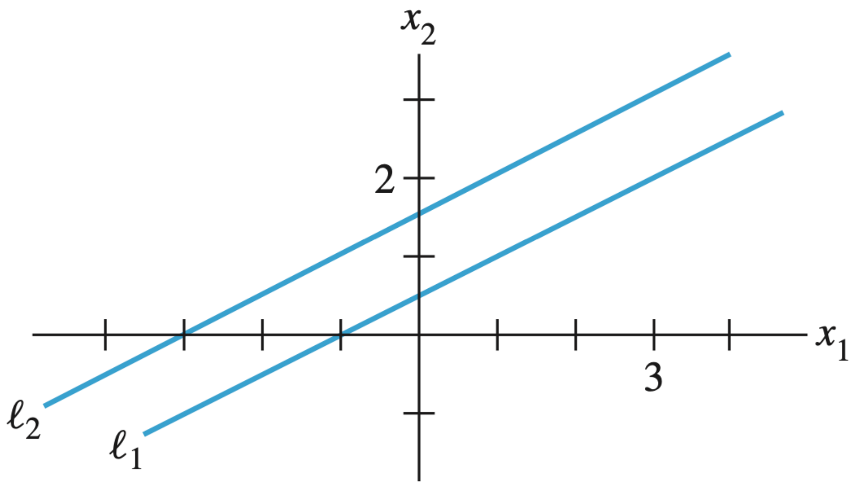
(c) Here the two lines coincide. This occurs when the lines have the same slopes and same y −intercepts, or when the coefficients and constants are proportional.

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

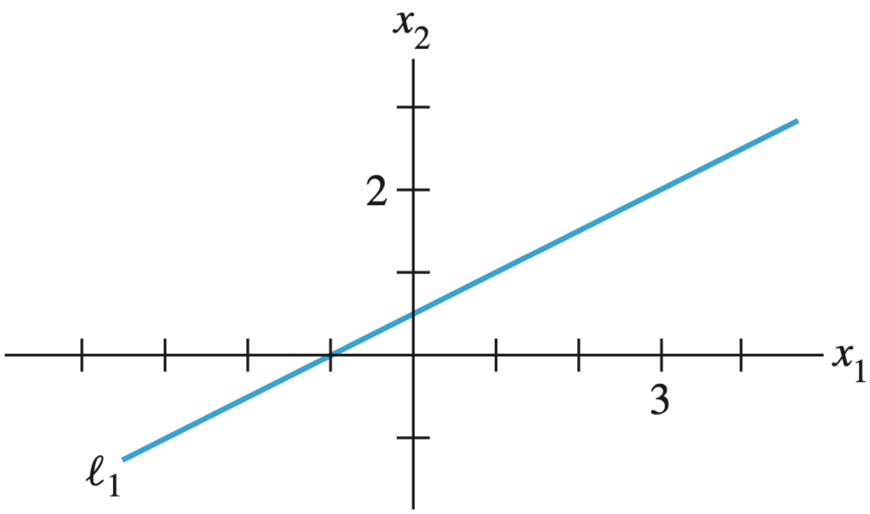
Example. Given the system $\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 1 \end{cases}$, is has infinitely many solutions, pictured in Fig.2.



(a) Unique solution



(b) No solutions



(c) Infinitely many solutions

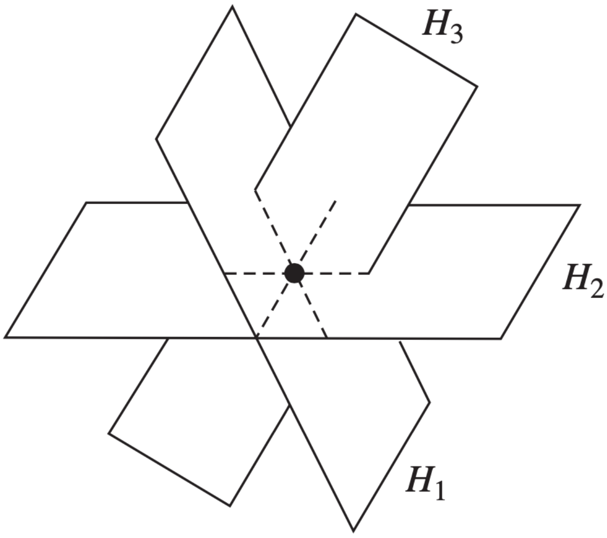
Figure 2.

The system consists of three nondegenerate equations in three unknowns, where the three equations correspond to planes H_1, H_2, H_3 in $\mathbb{R}^3$. That is,

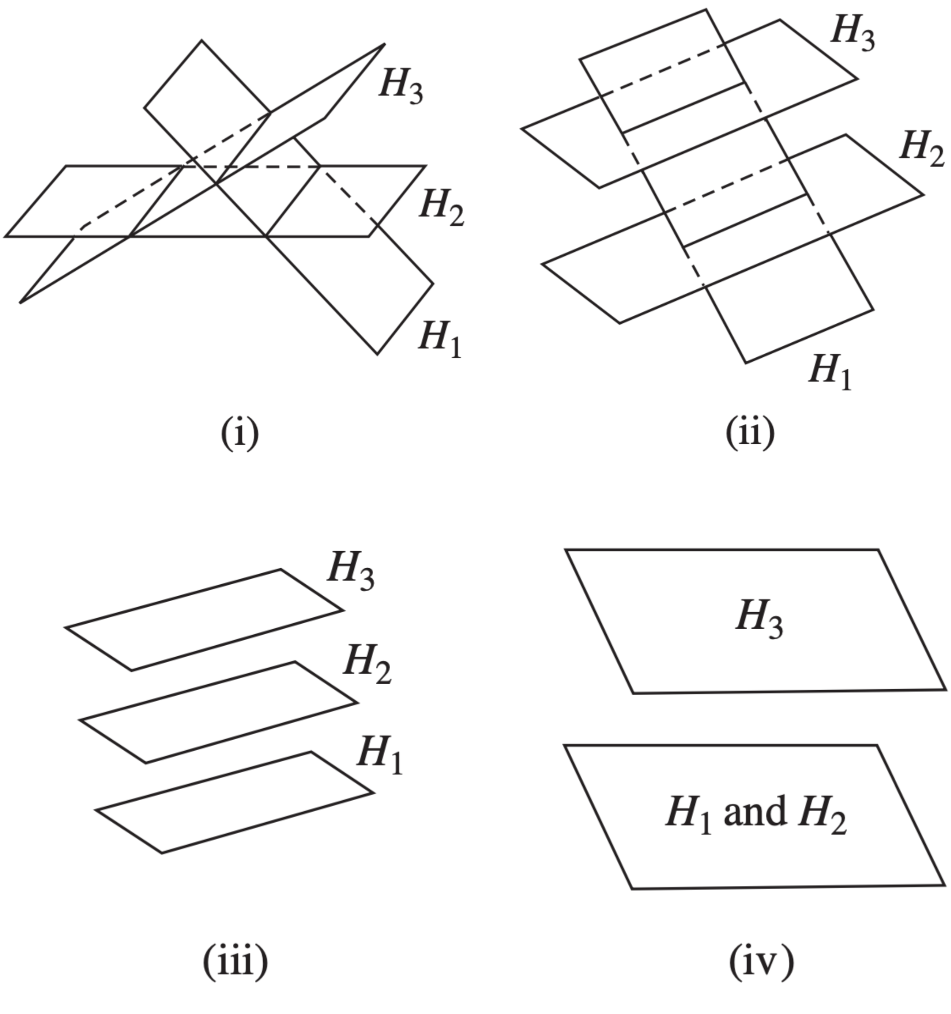
(a) Unique solution: Here the three planes intersect in exactly one point, pictured in Fig.3.

(b) No solution: Here the planes may intersect pairwise but with no common point of intersection, or two of the planes may be parallel, pictured in Fig.4.

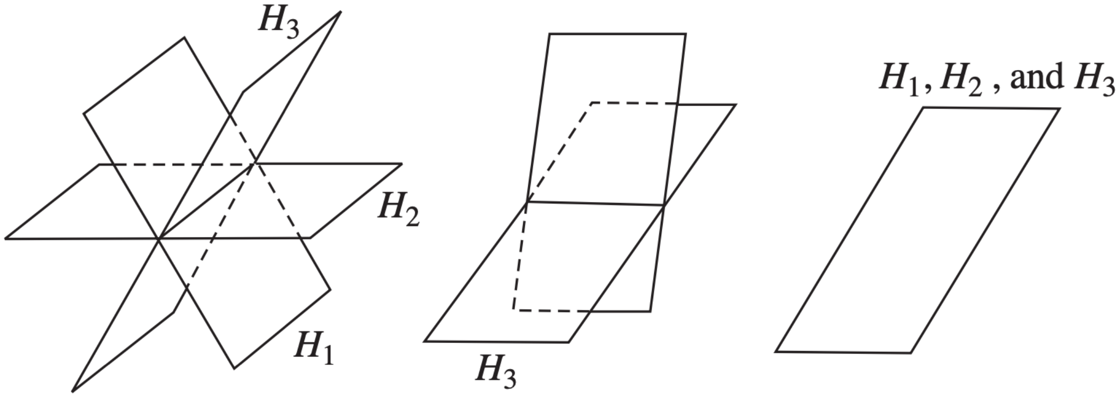
(c) Infinite number of solutions: Here the three planes may intersect in a line (one free variable), or they may coincide (two free variables), pictured in Fig.5.



(a) Unique solution
Figure 3.



(b) No solutions
Figure 4.



(i) (ii) (iii)

(c) Infinite number of solutions

Figure 5.

For the system $\begin{matrix} a_{11}x_1 + \cdots + a_{1p}x_p & = b_1 \\ a_{21}x_1 + \cdots + a_{2p}x_p & = b_2 \\ \cdots & \\ a_{m1}x_1 + \cdots + a_{mp}x_p & = b_m \end{matrix}$, the $m \times p$ matrix $A = \begin{pmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mp} \end{pmatrix}$ is the ***coefficient matrix*** (or ***matrix of coefficients***), $\boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$ is the ***constant vector***, and $\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}$ is the ***unknown vector***. The $m \times (p+1)$ matrix $[A\boldsymbol{b}]$ is the ***augmented matrix*** of the system. It is customary to identify the system of linear equations with the matrix-vector equation $A\boldsymbol{x} = \boldsymbol{b}$. This is valid because a column vector $\boldsymbol{x} = \begin{pmatrix} c_1 \\ \vdots \\ c_p \end{pmatrix}$ satisfies $A\boldsymbol{x} = \boldsymbol{b}$ if and only if $(c_1, \ldots, c_p)$ is a solution of the linear system.

Observe that the coefficients of x_k are stored in column k of A. If $A\boldsymbol{x} = \boldsymbol{b}$ is equivalent to $C\boldsymbol{x} = \boldsymbol{d}$ and column k of C is a pivot column, then x_k is a **basic variable**; otherwise, x_k is a **free variable**.

Example 1. Given the system

$$\begin{cases} x - 2y + z = 0 \\ \quad 2y - 8z = 8 \\ 5x - 5z = 10 \end{cases},$$

with the coefficients of each variable aligned in columns, the coefficient matrix

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{pmatrix},$$

and the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right).$$

To have a unique type of echelon form we introduce ***reduced echelon form matrix*** of a matrix and ***Gauss-Jordan reduction*** which allows to get reduced echelon form matrix. We will observe that Gauss-Jordan reduction is just an extension of Gaussian Elimination method and it has some advantages.

Definition 6. A matrix (a linear system) is in ***reduced echelon form*** (or ***row canonical form***) if, in addition to being in echelon form, each leading entry (coefficient of the leading unknown) is one and is the only nonzero entry in its column (the only nonzero coefficient of the leading unknown).

Example 2. Solve the system from example 2 (1) in Lecture 1

$$\begin{cases} x + 2y + z = -1 \\ 2x - y - z = -1 \\ -2x + 2y + 3z = 5 \end{cases}.$$

Solution. Let's write the system by augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 2 & -1 & -1 & -1 \\ -2 & 2 & 3 & 5 \end{array}\right) \quad \begin{matrix} R_2 \to -2R_1 + R_2 \\ R_3 \to 2R_1 + R_3 \end{matrix} \quad \Rightarrow$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 0 & -5 & -3 & 1 \\ 0 & 6 & 5 & 3 \end{array}\right) \quad R_3 \to 6R_2 + 5R_3 \quad \Rightarrow$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 0 & -5 & -3 & 1 \\ 0 & 0 & 7 & 21 \end{array}\right) \quad R_3 \to \frac{1}{7}R_3 \quad \Rightarrow$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \quad \begin{matrix} R_1 \to -R_3 + R_1 \\ R_2 \to 4R_3 + R_2 \end{matrix} \quad \Rightarrow$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 0 & -4 \\ 0 & -7 & 0 & 14 \\ 0 & 0 & 1 & 3 \end{array}\right) \quad R_2 \to -\frac{1}{7}R_2 \quad \Rightarrow$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 0 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \quad R_1 \to 2R_2 + R_1 \quad \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right).$$

Thus, the obtained matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right),$$

is now in reduced echelon form. From the matrix one can find solution of the given system of linear equations without back-substitution. If we write the matrix as system of linear equations, we have $(x, y, z) = (0, -2, 3)$ or a set of solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}.$$

Example 3. Solve the system

$$\begin{cases} x - y + z = 1 \\ 3x + z = 3 \\ 5x - 2y + 3z = 5 \end{cases}.$$

Solution. Let’s write the system by augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 3 & 0 & 1 & 3 \\ 5 & -2 & 3 & 5 \end{array}\right) \quad \begin{array}{l} R_2 \to -3R_1 + R_2 \\ R_3 \to -5R_1 + R_3 \end{array} \quad \Rightarrow$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -2 & 0 \\ 0 & 3 & -2 & 0 \end{array}\right) \quad R_3 \to -R_2 + R_3 \quad \Rightarrow$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad R_3 \to -\frac{1}{3}R_3 \quad \Rightarrow$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad R_1 \to R_3 + R_1 \quad \Rightarrow$$

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & 1 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Thus, the obtained matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & 1 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

is now in reduced echelon form. From the matrix one can find solution of the given system of linear equations without back-substitution. If we write the matrix as system of linear equations, we have $(x, y, z) = \left(1 - \frac{1}{3}z, \frac{2}{3}z\ , z\right)$ or a set of solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix} z |\ z \in \mathbb{R}\}$$

Definition 7. Let A and B be matrices. If B can be obtained from A by elementary operations, then we say B is ***row equivalent*** to A.

Proposition 1. Row equivalence relation is

- reflexive, that is, any matrix is row equivalent to itself.
- symmetric, that is, if A is row equivalent to B, then B is row equivalent to A.
- transitive, that is, if A is row equivalent to B and B is row equivalent to C, then A is row equivalent to C.

Theorem 2. Each matrix is row equivalent to a unique reduced echelon form matrix.

Example 4. Given the matrices

$$A = \begin{pmatrix} 1 & -3 \\ -2 & 6 \end{pmatrix}, B = \begin{pmatrix} 1 & -3 \\ -2 & 5 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 5 \\ -1 & -4 \end{pmatrix}.$$

Determine which of these matrices are row equivalent?

Solution. The reduced echelon form matrix of

$$A = \begin{pmatrix} 1 & -3 \\ -2 & 6 \end{pmatrix} \text{ is. } \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}.$$

The reduced echelon form matrix of

$$B = \begin{pmatrix} 1 & -3 \\ -2 & 5 \end{pmatrix} \text{ is } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The reduced echelon form matrix of

$$C = \begin{pmatrix} 1 & 5 \\ -1 & -4 \end{pmatrix} \text{ is } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the matrices A and B are not row equivalent, the matrices A and C are not row equivalent and the matrices B and C are row equivalent.

Glossary

column	столбец
row	строка
length	длина
dot product (or inner product or scalar product)	скалярное произведение
plane	плоскость
slope	наклон
intersection	пересечение
augmented matrix	расширенная матрица

Exercises for lecture 2

1. Determine which matrices are in reduced echelon form and which others are only in echelon form.

$$a) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad b) \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$c)\ \begin{pmatrix}1&0&0&0\\0&1&1&0\\0&0&0&0\\0&0&0&1\end{pmatrix},\quad d)\ \begin{pmatrix}1&1&0&1&1\\0&2&0&2&2\\0&0&0&3&3\\0&0&0&0&4\end{pmatrix}$$

2. Use the notation of Example 1 for matrices in echelon form. Suppose each matrix represents the augmented matrix for a system of linear equations. In each case, determine if the system is consistent. If the system is consistent, determine if the solution is unique.

$$a)\ \begin{pmatrix}\blacksquare&*&*&*\\0&\blacksquare&*&*\\0&0&\blacksquare&0\end{pmatrix},\quad b)\ \begin{pmatrix}0&\blacksquare&*&*&*\\0&0&\blacksquare&*&*\\0&0&0&0&\blacksquare\end{pmatrix}$$

3. Determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.

$$\begin{pmatrix}2&3&h\\4&6&7\end{pmatrix}$$

4. Choose h and k such that the system has (a) no solution, (b) a unique solution, and (c) many solutions. Give separate answers for each part.

$$\begin{aligned}x+hy&=2\\4x+8y&=k\end{aligned}$$

5. Decide if the matrices are row equivalent.

$$a)\ \begin{pmatrix}1&2\\4&8\end{pmatrix},\begin{pmatrix}0&1\\1&2\end{pmatrix}\quad b)\ \begin{pmatrix}1&0&2\\3&-1&1\\5&-1&5\end{pmatrix},\begin{pmatrix}1&0&2\\0&2&10\\2&0&4\end{pmatrix}$$

$$c)\ \begin{pmatrix}2&1&-1\\1&1&0\\4&3&-1\end{pmatrix},\begin{pmatrix}1&0&2\\0&2&10\end{pmatrix}$$

6. Find the length of each vector.

$$a)\ \begin{pmatrix}3\\1\end{pmatrix}\quad b)\ \begin{pmatrix}4\\1\\1\end{pmatrix}\quad c)\ \begin{pmatrix}1\\-1\\1\\0\end{pmatrix}$$

7. Use Gauss's Method to solve each system and conclude "unique solution", "many solutions" or "no solutions".

a) $\begin{cases} x + z = 4 \\ x - y + 2z = 5 \\ 4x - y + 5z = 17 \end{cases}$,

b) $\begin{cases} 3x + 6y = 18 \\ x + 2y = 6 \end{cases}$,

8. Solve the following system by using Gaussian elimination:

$$\begin{cases} x - 3y + 2z - t + 2w = 2 \\ 3x - 9y + 7z - t + 3w = 7 \\ 2x - 6y + 7z + 4t - 5w = 7 \end{cases}$$

9. Determine the pivot and free variables in the following system:

$$\begin{cases} 2x - 3y - 6z - 5t + 2w = 7 \\ z + 3t - 7w = 6 \\ t - 2w = 1 \end{cases}$$

Homework 2

1. Determine which matrices are in reduced echelon form and which others are only in echelon form.

$$a) \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad b) \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$c) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad d) \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

2. Use the notation of Example 1 for matrices in echelon form. Suppose each matrix represents the augmented matrix for a system of linear equations. In each case, determine if the system is consistent. If the system is consistent, determine if the solution is unique.

$$a) \begin{pmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & 0 \end{pmatrix}, \quad b) \begin{pmatrix} \blacksquare & * & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare & * \end{pmatrix}$$

3. Determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.

$$\begin{pmatrix} 1 & -3 & -2 \\ 5 & h & -7 \end{pmatrix}$$

4. Choose h and k such that the system has (a) no solution, (b) a unique solution, and (c) many solutions. Give separate answers for each part.

$$\begin{aligned} x + 3y &= 2 \\ 3x + hy &= k \end{aligned}$$

5. Decide if the matrices are row equivalent.

$$a)\ \begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 3 & -1 \\ 2 & 2 & 5 \end{pmatrix} \quad b)\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

6. Find the length of each vector.

$$a)\ \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad b)\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad c)\ \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

7. Use Gauss's Method to solve each system and conclude "unique solution", "many solutions" or "no solutions".

a) $\begin{cases} x + y = 1 \\ x - y = -1 \end{cases}$,

b) $\begin{cases} 2x + y - z = 2 \\ 2x + z = 3 \\ x - y = 0 \end{cases}$.

8. Solve the following system by using Gaussian elimination:

$$\begin{cases} x - 2y - 3z + 4t = 2 \\ 2x + 5y - 2z + t = 1 \\ 5x + 12y - 7z + 6t = 3 \end{cases}$$

9. Determine the pivot and free variables in each of the following systems:

a) $\begin{cases} 2x - 6y + 7z = 1 \\ 4y + 3z = 8 \\ 2z = 4 \end{cases}$

b) $\begin{cases} x - 2y - 3z = 2 \\ 2x + 3y + z = 4 \\ 3x + 4y + 5z = 8 \end{cases}$