Lecture 4. Vector Spaces: basis and dimension.

First, we state two equivalent definitions of a basis of a vector space V.

Definition 1. A set of vectors $S = \{v_1, \ldots, v_n\}$ is a ***basis*** of V if

1. S is linearly independent.
2. S spans V.

Definition 2. A set of vectors $S = \{v_1, \ldots, v_n\}$ is a ***basis*** of V if every $v \in V$ can be written uniquely as a linear combination of the basis vectors.

Proposition 1. The Definitions 1 and 2 are equivalent.

Example 1. a) Let $V = \mathbb{R}^3$ over $\mathbb{R}$. Consider the vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We know from Lecture 3 that these vectors span $\mathbb{R}^3$. Furthermore, they are independent. Consider $\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 = 0$. We have

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

One can easily check that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. So e_1, e_2 and e_3 are independent in $\mathbb{R}^3$. Hence $\{e_1, e_2, e_3\}$ is a basis of $\mathbb{R}^3$. This basis is called ***a standard basis*** of $\mathbb{R}^3$.

b) This example generalizes the preceding example. Consider the following n vectors in $\mathbb{R}^n$:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \ldots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

These vectors are linearly independent. Furthermore, any vector

$$v = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix} \in \mathbb{R}^n$$

can be written as a linear combination of $e_1, e_2, \ldots, e_n$. Specifically, $v = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$. Accordingly, the vectors form a basis of $\mathbb{R}^n$ which is called ***standard basis of*** $\mathbb{R}^n$.

c) Let $V = M_{2,2}$ be a vector space of all 2×2 matrices over $\mathbb{R}$. The following four matrices from $M_{2,2}$ form a basis of the vector space $M_{2,2}$ over $\mathbb{R}$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and their independence is evident.

d) Vector space $P_n(t)$ of all polynomials of degree less than or equal to n. The set $S = \{1, t, t^2, \ldots, t^n\}$ is a basis of $P_n(t)$. Any polynomial can be written as

$$a_0 + a_1 t + \ldots + a_n t^n = a_0 1 + a_1 t + \ldots + a_n t^n$$

and their independence is evident.

The following is a fundamental result in linear algebra.

Theorem 1. Let V be a vector space such that one basis has m vectors and another basis has n vectors. Then $m = n$.

Definition 3. A vector space V is said to be of ***finite dimension*** $\boldsymbol{n}$ or $\boldsymbol{n}$ **−dimensional**, written $\dim V = n$ if V has a basis with n vectors. If a vector space V does not have finite basis, then V is said to be of ***infinite dimension*** or to be ***infinite-dimensional***.

The vector space $\{0\}$ is defined to have dimension 0.

Example 2. Prove that

- $\dim \mathbb{R}^n = n$
- $\dim M_{m,n} = mn$
- $\dim P_n(t) = n + 1$

Theorem 2. Let V be a vector space of finite dimension n. Then:

- Any $n + 1$ or more vectors in V are linearly dependent.
- Any linearly independent set $S = \{v_1, \ldots, v_n\}$ with n vectors is a basis of V.
- Any spanning set $T = \{w_1, \ldots, w_n\}$ of V with n elements is a basis of V.

Example 3. a) Let $V = \mathbb{R}^3$. Then $dim\mathbb{R}^3 = 3$. We know the vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

forms a basis of $\mathbb{R}^3$. Then for any nonzero $v \in \mathbb{R}^3$ by the first part of the Theorem above the four vectors e_1, e_2, e_3, v are not linearly independent, consequently $\{e_1, e_2, e_3, v\}$ is not a basis of $\mathbb{R}^3$.

b) Let $V = \mathbb{R}^3$. It is easy to show that

$$v_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

are linearly independent in $\mathbb{R}^3$ and $dim\mathbb{R}^3 = 3$. Then by the second part of the Theorem above $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$.

c) Let $V = \mathbb{R}^3$. We know from Lecture 3 that the vectors

$$f_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, f_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, f_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

span $\mathbb{R}^3$. Taking account that $dim\mathbb{R}^3 = 3$ and the third part of the Theorem above we have $\{f_1, f_2, f_3\}$ is a basis of $\mathbb{R}^3$.

In an echelon form matrix, no nonzero row is a linear combination of the other nonzero rows. The nonzero rows of an echelon form matrix make up a linearly independent set. Namely, rows in an echelon matrix with n columns give us set of linearly independent vectors in $\mathbb{R}^n$.

Example 4. a) Given vectors $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, v_3 = \begin{pmatrix} 4 \\ 9 \\ 5 \end{pmatrix}$. We check whether they are linearly dependent or not by echelon matrix (from Lecture 3 we know that they are linearly dependent). First we write them as rows of matrix and have $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 2 \\ 4 & 9 & 5 \end{pmatrix}$. Perform the following sequence of elementary operations $R_2 \to -R_1 + R_2,\ R_3 \to -4R_1 + R_3,\ R_3 \to -5R_2 + 2R_3$:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 2 \\ 4 & 9 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 4 & 9 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 5 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix}.$$

Finally, we have the echelon matrix with two nonzero rows. It means the third row is a linear combination of the first and second rows, namely, v_3 is a linear combination of v_1 and v_2. Hence, they are linearly dependent in $\mathbb{R}^3$.

b) Given vectors $v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}, v_3 = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$ of $\mathbb{R}^3$. We want to check whether the set $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$. First, we write them as rows of matrix and have

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ -2 & 2 & 3 \end{pmatrix}.$$

After applying the sequence of following elementary operations $R_2 \to -3R_1 + R_2, R_3 \to 2R_1 + R_3, R_3 \to 7R_3, R_3 \to 6R_2 + R_3$ one can obtain the following echelon matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -7 & -4 \\ 0 & 0 & 11 \end{pmatrix}.$$

As we see there are 3 nonzero rows in the echelon matrix and therefore these rows define 3 linearly independent vectors in $\mathbb{R}^3$ obtained from v_1, v_2, v_3. Then v_1, v_2, v_3 are linearly independent vectors in three dimensional vector space. Hence the set $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$.

c) Given set $S = \{v_1, v_2, v_3, v_4\}$, where

$$v_1 = \begin{pmatrix} 1 \\ 3 \\ 1 \\ -2 \\ -3 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 4 \\ 3 \\ -1 \\ -4 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ 3 \\ -4 \\ -7 \\ -3 \end{pmatrix}, v_4 = \begin{pmatrix} 3 \\ 8 \\ 1 \\ -7 \\ -8 \end{pmatrix}.$$

We will extend the set S to a basis of vectors of $\mathbb{R}^5$. First of all, we need to determine whether they are linearly independent or not. If not, we will delete dependent vectors from S.

$$\begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & -3 & -6 & -3 & 3 \\ 0 & -1 & -2 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \end{pmatrix}$$

So we observe that v_1, v_2, v_3, v_4 are linearly dependent and only two vectors are linearly independent. To have a basis of $\mathbb{R}^5$ we need three more vectors so that new five vectors become linearly independent with

$$\{\begin{pmatrix} 1 \\ 3 \\ 1 \\ -2 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \\ -1 \end{pmatrix}\}.$$

We add three more rows

$$\begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and have an echelon matrix with 5 nonzero rows. Now we have five linearly independent vectors.

The set of vectors

$$\{\begin{pmatrix} 1 \\ 3 \\ 1 \\ -2 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\}.$$

is a basis of $\mathbb{R}^5$. We note that this extension of course is not unique. We couldtake other three vectors so that a set of five vectors form a basis of $\mathbb{R}^5$.

Definition 3. The ***rank of a matrix A***, written $rank(A)$, is equal to number of rows in its echelon form matrix.

Example 5. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 2 \\ 4 & 9 & 5 \end{pmatrix}$.

Then rank of A is 2, write $rank(A) = 2$, since its echelon matrix has two nonzero rows

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix}.$$

We consider an application of rank in space of solutions of a homogeneous system of linear equations in n unknowns. Recall that homogeneous systems have either infinitely many solutions or only zero solutions. The set of solutions of systems forms a vector space with respect to the operations in $\mathbb{R}^n$.

Theorem 3. Let W be space of solutions of a homogeneous system of linear equations in n unknowns and A be a matrix of coefficients of unknowns with $rank(A) = r$. Then $dim\, W = n - r$.

Example 6. Consider $\begin{cases} x + y - z = 0 \\ 2x - 3y + z = 0. \\ x - 4y + 2z = 0 \end{cases}$

Then $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 1 \\ 1 & -4 & 2 \end{pmatrix}$. An echelon matrix of A is $\begin{pmatrix} 1 & 1 & -1 \\ 0 & -5 & 3 \end{pmatrix}$ and $rank(A) = 2$. Then $dim\, W = 3 - 2 = 1$, that is, the space of solutions has dimension one. Of course, the system has infinitely many solutions of the form $(x, y, z) = (\frac{2}{5}z, \frac{3}{5}z, z)$, but as a vector space all solutions are linear combination of one vector $\begin{pmatrix} 2/5 \\ 3/5 \\ 1 \end{pmatrix}$.

Definition 4. Let U and W be subsets of a vector space V. The ***sum of U and W***, written $U + W$, consists of all sums $u + w$ where $u \in U$ and $w \in W$.

Theorem 4. Suppose U and W are subspaces of V. Then $U + W$ and $U \cap W$ are subspaces of V.

Theorem 5. Suppose U and W are finite dimensional subspaces of V. Then

$$dim\,(U + W) = dim\, U + dim\, W - dim(U \cap W).$$

Example 7. Given

$$v_1 = \begin{pmatrix} 1 \\ 3 \\ -2 \\ 2 \\ 3 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 4 \\ -3 \\ 4 \\ 2 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ 3 \\ -1 \\ -2 \\ 9 \end{pmatrix}, w_1 = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 2 \\ 1 \end{pmatrix}, w_2 = \begin{pmatrix} 1 \\ 5 \\ -6 \\ 6 \\ 3 \end{pmatrix}, w_3 = \begin{pmatrix} 2 \\ 5 \\ 3 \\ 2 \\ 1 \end{pmatrix}.$$

Let U be a space spanned by u_1, u_2, u_3 and W be a space spanned by w_1, w_2, w_3. Namely, any vector of U and W is a linear combination of u_1, u_2, u_3 and w_1, w_2, w_3, respectively. They are subspaces of $\mathbb{R}^5$.

We will find bases and dimensions of $U, W, U + W$ and $U \cap W$.

To construct a basis of U we need to derive linearly independent vectors from u_1, u_2, u_3. We write them as rows of matrix and find its echelon form matrix.

$$\begin{pmatrix} 1 & 3 & -2 & 2 & 3 \\ 1 & 4 & -3 & 4 & 2 \\ 2 & 3 & -1 & -2 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & -3 & 3 & -6 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \end{pmatrix}.$$

So there are only two nonzero rows or rank of the matrix is two. Thus,

$$u_1 = \begin{pmatrix} 1 \\ 3 \\ -2 \\ 2 \\ 3 \end{pmatrix}, u'_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \\ -1 \end{pmatrix}$$

are linearly independent and they span U, therefore $\{u_1, u_2'\}$ is a basis and $dim\, U = 2$.

In a similar way, we find a basis of W and its dimension.

$$\begin{pmatrix} 1 & 3 & 0 & 2 & 1 \\ 1 & 5 & -6 & 6 & 3 \\ 2 & 5 & 3 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 2 & -6 & 4 & 2 \\ 0 & -1 & 3 & -2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 1 & -3 & 2 & 1 \\ 0 & -1 & 3 & -2 & -1 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 1 & -3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 1 & -3 & 2 & 1 \end{pmatrix}.$$

So there are only two nonzero rows or rank of the matrix is two. Thus,

$$w_1 = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 2 \\ 1 \end{pmatrix}, w'_2 = \begin{pmatrix} 0 \\ 1 \\ -3 \\ 2 \\ 1 \end{pmatrix}$$

are linearly independent and they span W, therefore $\{w_1, w'_2\}$ is a basis and $dim\, W = 2$.

By the definition of $U + W$, the vectors u_1, u_2', w_1, w'_2 span $U + W$.

$$\begin{pmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 1 & 3 & 0 & 2 & 1 \\ 0 & 1 & -3 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & -2 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

So, there are only three nonzero rows or rank of the matrix is three. Thus,

$$u_1 = \begin{pmatrix} 1 \\ 3 \\ -2 \\ 2 \\ 3 \end{pmatrix}, u'_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \\ -1 \end{pmatrix}, w'_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

are linearly independent and they span $U + W$, therefore $\{u_1, u_2', w'_1\}$ is a basis and $dim\ (U + W) = 3$.

To find the dimension of $U \cap W$ we use the formula given above and have $dim(U \cap W) = dim\, U + dim\, W - dim(U + W) = 2 + 2 - 3 = 1$.

Now we find a basis of $U \cap W$ and it consists of one vector.

Let $v = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}$ and $v \in U \cap W$. Then v must be written as a linear combination of their basis vectors.

$$v = \lambda_1 u_1 + \lambda_2 u_2' = \mu_1 w_1 + \mu_2 w'_2 .$$

$$\lambda_1 \begin{pmatrix} 1 \\ 3 \\ -2 \\ 2 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \\ -1 \end{pmatrix} = \mu_1 \begin{pmatrix} 1 \\ 3 \\ 0 \\ 2 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ 1 \\ -3 \\ 2 \\ 1 \end{pmatrix}.$$

We obtain a system of five linear equations in unknowns $\lambda_1, \lambda_2, \mu_1, \mu_2$:

$$\begin{cases} \lambda_1 - \mu_2 = 0 \\ 3\lambda_1 + \lambda_2 - 3\mu_1 - \mu_2 = 0 \\ -2\lambda_1 - \lambda_2 + 3\mu_2 = 0 \\ 2\lambda_1 + 2\lambda_2 - 2\mu_1 - 2\mu_2 = 0 \\ 3\lambda_1 - \lambda_2 - \mu_1 - \mu_2 = 0 \end{cases}$$

The solution is $\{(\lambda_1, \lambda_2, \mu_1, \mu_2) = (\mu_2, \mu_2, \mu_2, \mu_2) | \mu_2 \in R\}$. Let $\mu_2 = 1$ and

$v = u_1 + u_2' = \begin{pmatrix} 1 \\ 4 \\ -3 \\ 4 \\ 2 \end{pmatrix}$ is the vector in $U \cap W$ and span it. Hence it is a basis of $U \cap W$.

Definition 5. The vector space V is said to be the direct sum of its subspaces of U and W, denoted by $V = U \oplus W$ if every $v \in V$ can be written in one and only one way as $v = u + w$ where $u \in U$ and $w \in W$.

Theorem 6. The vector space V is the direct sum of its subspaces of U and W if and only if

- $V = U + W$.
- $U \cap W = \{0\}$.

Example 8. a) Let $V = \mathbb{R}^3$ and $U = \{\begin{pmatrix} a \\ b \\ c \end{pmatrix} | a = c\}$ and $W = \{\begin{pmatrix} a \\ b \\ c \end{pmatrix} | a + b + c = 0\}$. We show that $V = U + W$ but the sum is not direct. Suppose

$v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in V$. Then

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} c \\ a + b - c \\ c \end{pmatrix} + \begin{pmatrix} a - c \\ c - a \\ 0 \end{pmatrix}.$$

Then $V = U + W$. We note that

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} c - 1 \\ a + b - c + 2 \\ c - 1 \end{pmatrix} + \begin{pmatrix} a - c + 1 \\ c - a - 2 \\ 1 \end{pmatrix}.$$

There are two ways of expressing v as a linear combination of vectors U and W. Therefore, the sum is not direct.

b) Let $V = \mathbb{R}^3$ and $U = \{\begin{pmatrix} a \\ b \\ c \end{pmatrix} | a = c\}$ and $W = \{\begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} | c \in \mathbb{R}\}$.

We show that $V = U \oplus W$. Note that

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ a \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ c - a \end{pmatrix}.$$

Then $V = U \oplus W$. Let $v \in U \cap W$. It implies $a = c$ and $a = b = 0$. Then $a = b = c = 0$.

Thus, $v = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $U \cap W = \{0\}$. Hence $V = U \oplus W$.

Glossary

basis	базис, основание
dimension	размерность, измерение
make up	составить
rank	ранг, разряд

Exercises for lecture 4

1. Find a subset of u_1, u_2, u_3, u_4 that gives a basis for $W = span(u_i)$ of $\mathbb{R}^5$, where

(a) $u_1 = (1,1,1,2,3),\ u_2 = (1,2,-1,-2,1),\ u_3 = (3,5,-1,-2,5),\ u_4 = (1,2,1,-1,4)$;

(b) $u_1 = (1,-2,1,3,-1),\ u_2 = (-2,4,-2,-6,2),\ u_3 = (1,-3,1,2,1),$ $u_4 = (3,-7,3,8,-1)$.

2. Consider the subspaces $U = \{(a,b,c,d): b - 2c + d = 0\}$ and $W = \{(a,b,c,d): a = d,\ b = 2c\}$ of $\mathbb{R}^4$. Find a basis and the dimension of (a) U, (b) W.

3. Find a basis and the dimension of the solution space W of each of the following homogeneous systems:

$$\begin{aligned} x + 2y - 2z + 2s - t &= 0 \\ x + 2y - z + 3s - 2t &= 0 \\ 2x + 4y - 7z + s + t &= 0 \end{aligned}$$

4. Find a basis and the dimension of the subspace W of $\boldsymbol{P}(t)$ spanned by

$$u = t^3 + 2t^2 - 2t + 1,\ \ v = t^3 + 3t^2 - 3t + 4, \qquad w = 2t^3 + t^2 - 7t - 7$$

5. Find the dimension of the subspace W of $V = M_{2\times 2}$ spanned by

$$A = \begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -4 \\ -5 & 7 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & -7 \\ -5 & 1 \end{bmatrix}$$

6. Let U_1, U_2, U_3 be the following subspaces of $\mathbb{R}^3$:

$$U_1 = \{(a, b, c): a = c\}, \ U_2 = \{(a; b; c): a + b = c = 0\}, \ U_3 = \{(0; 0; c)\}$$

Show that (a) $\mathbf{R}^3 = U_1 + U_2$, (b) $\mathbf{R}^3 = U_2 + U_3$. When is the sum direct?

Homework 4

1. Find a subset of u_1, u_2, u_3, u_4 that gives a basis for $W = span(u_i)$ of $\mathbb{R}^5$, where

   (a) $u_1 = (1,0,1,0,1), \ u_2 = (1,1,2,1,0), \ u_3 = (2,1,2,1,1), \ u_4 = (1,2,1,1,1)$

   (b) $u_1 = (1,0,1,1,1), \ u_2 = (2,1,2,0,1), \ u_3 = (1,1,2,3,4), \ u_4 = (4,2,5,4,6)$

2. Consider the subspaces $U = \{(a, b, c, d): b - 2c + d = 0\}$ and $W = \{(a, b, c, d): a = d, \ b = 2c\}$ of $\mathbb{R}^4$. Find a basis and the dimension of

   (a) $U \cap W$.

3. Find a basis and the dimension of the solution space W of each of the following homogeneous systems:

$$\text{(a)} \quad \begin{aligned} x + 2y - z + 3s - 4t &= 0 \\ 2x + 4y - 2z - s + 5t &= 0 \\ 2x + 4y - 2z + 4s - 2t &= 0 \end{aligned}$$

4. Find a basis and the dimension of the subspace W of $\boldsymbol{P}(t)$ spanned by

   (a) $u = t^3 + t^2 - 3t + 2, \ v = 2t^3 + t^2 + t - 4, \ w = 4t^3 + 3t^2 - 5t + 2$

5. Let U_1, U_2, U_3 be the following subspaces of $\mathbb{R}^3$:

$$U_1 = \{(a, b, c): a = c\}, \ U_2 = \{(a; b; c): a + b = c = 0\}, \ U_3 = \{(0; 0; c)\}$$

Show that (a) $\mathbf{R}^3 = U_1 + U_3$. When is the sum direct?