VECTOR SPACES: DEFINITION OF VECTOR SPACE AND LINEAR INDEPENDENCE.

Mukash Nazgul Kanyshkyzy

Vector spaces

Definition 1. Let V be a non-empty set with two operations:

- Vector Addition: This assigns to any $u, v \in V$ a sum $u + v \in V$.
- Scalar Multiplication: This assigns to any $u \in V$, $\lambda \in \mathbb{K}$, a product $\lambda u \in V$.

Vector spaces

Then V is called a ***vector space*** (over $\mathbb{K}$), whose elements are called ***vectors***, if the following axioms hold for any vectors $u, v, w \in V$ and for any $\lambda, \mu \in \mathbb{K}$:

1. (Commutativity) $u + v = v + u$.
2. (Associativity) $(u + v) + w = u + (v + w)$.

3. (Additive identity) There is a vector in V, denoted by $\mathbf{0}$, and called the ***zero vector***, such that, $u + \mathbf{0} = \mathbf{0} + u = u$.

4. (Additive inverse) There is a vector in V, denoted by $-u$, and called the ***negative*** of u, such that $u + (-u) = (-u) + u = \mathbf{0}$.

5. (Distributivity) $\lambda(u + v) = \lambda u + \lambda v$.

6. (Distributivity) $(\lambda + \mu)u = \lambda u + \mu u$.

7. (Associativity) $(\lambda\mu)u = \lambda(\mu u)$.

8. $1u = u$.

The properties of a vector space.

Theorem 1. Let V be a vector space over a field $\mathbb{K}$.

i. For any scalar $k \in \mathbb{K}$ and $0 \in V$, $k0 = 0$.

ii. For $0 \in \mathbb{K}$ and any vector $u \in V$, $0u = 0$.

iii. If $ku = 0$, where $k \in \mathbb{K}$ and $u \in V$, then $k = 0$ or $u = 0$.

iv. For any $k \in \mathbb{K}$ and any $u \in V$,$(-k)u = k(-u) = -ku$.

Example 1. Let $\mathbb{K} = \mathbb{R}$ and $V = \mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$.

Let $u, v \in \mathbb{R}^2$ and $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Define

$$u + v := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$$

and

$$\lambda u := \begin{pmatrix} \lambda u_1 \\ \lambda u_2 \end{pmatrix}.$$

Then $\mathbb{R}^2$ is a vector space over $\mathbb{R}$ with respect to the defined operations.

Let us show that it is really a vector space over $\mathbb{R}$. For that we need to check all eight axioms in the definition of vector space.

Solution:

1.

$$u + v = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} =$$

$$= \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} = \begin{pmatrix} v_1 + u_1 \\ v_2 + u_2 \end{pmatrix} =$$

$$= \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = v + u.$$

$$2.\ (u+v)+w=\left(\begin{pmatrix}u_1\\u_2\end{pmatrix}+\begin{pmatrix}v_1\\v_2\end{pmatrix}\right)+\begin{pmatrix}w_1\\w_2\end{pmatrix}=$$

$$=\begin{pmatrix}u_1+v_1\\u_2+v_2\end{pmatrix}+\begin{pmatrix}w_1\\w_2\end{pmatrix}=\begin{pmatrix}(u_1+v_1)+w_1\\(u_2+v_2)+w_2\end{pmatrix}=$$

$$=\begin{pmatrix}u_1+(v_1+w_1)\\u_2+(v_2+w_2)\end{pmatrix}=\begin{pmatrix}u_1\\u_2\end{pmatrix}+\begin{pmatrix}v_1+w_1\\v_2+w_2\end{pmatrix}=$$

$$=\begin{pmatrix}u_1\\u_2\end{pmatrix}+\left(\begin{pmatrix}v_1\\v_2\end{pmatrix}+\begin{pmatrix}w_1\\w_2\end{pmatrix}\right)=u+(v+w).$$

3. The zero vector is $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, because

$$u + \mathbf{0} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \mathbf{0} + u.$$

4. $-u = -\binom{u_1}{u_2}$, since $u + (-u) = \mathbf{0}$.

$$5.\ \lambda(u+v) = \lambda\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) =$$

$$= \lambda\begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} = \begin{pmatrix} \lambda(u_1 + v_1) \\ \lambda(u_2 + v_2) \end{pmatrix} =$$

$$= \begin{pmatrix} \lambda u_1 + \lambda v_1 \\ \lambda u_2 + \lambda v_2 \end{pmatrix} = \begin{pmatrix} \lambda u_1 \\ \lambda u_2 \end{pmatrix} + \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix} =$$

$$= \lambda\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \lambda\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda u + \lambda v.$$

$$6.\ (\lambda+\mu)u = (\lambda+\mu)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} (\lambda+\mu)u_1 \\ (\lambda+\mu)u_2 \end{pmatrix} =$$

$$= \begin{pmatrix} \lambda u_1 + \mu u_1 \\ \lambda u_2 + \mu u_2 \end{pmatrix} = \begin{pmatrix} \lambda u_1 \\ \lambda u_2 \end{pmatrix} + \begin{pmatrix} \mu u_1 \\ \mu u_2 \end{pmatrix} =$$

$$= \lambda\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \mu\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \lambda u + \mu u.$$

$$7.\ (\lambda\mu)u = (\lambda\mu)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} (\lambda\mu)u_1 \\ (\lambda\mu)u_2 \end{pmatrix} =$$

$$= \begin{pmatrix} \lambda(\mu u_1) \\ \lambda(\mu u_2) \end{pmatrix} = \lambda\begin{pmatrix} \mu u_1 \\ \mu u_2 \end{pmatrix} =$$

$$= \lambda\left(\mu\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) = \lambda(\mu u).$$

8. $1u = 1\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1u_1 \\ 1u_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u.$

Subspaces and Spanning Sets

One of the examples that led us to introduce the idea of a vector space was the solution set of a homogeneous system. For instance, we've seen such a space that is a planar subset of $\mathbb{R}^3$. There, the vector space $\mathbb{R}^3$ contains inside it another vector space, the plane.

Subspaces and Spanning Sets

Definition. For any vector space, a *subspace* is a subset that is itself a vector space, under the inherited operations.

In other words, a subset W of V is a subspace of V if W satisfies the eight axioms in the definition of vector space. However, if W is a subset of a vector space V, then some of the axioms automatically hold in W, because they already hold in V.

Subspaces and Spanning Sets

Example. The plane from the prior example,

$$P = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} | x + y + z = 0\}$$

is a subspace of $\mathbb{R}^3$.

Subspaces and Spanning Sets

As specified in the definition, the operations are the ones that are inherited from the larger space, that is, vectors add in P as they add in $\mathbb{R}^3$

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

and scalar multiplication is also the same as it is in $\mathbb{R}^3$.

Subspaces and Spanning Sets

To show that P is a subspace, we need only note that it is a subset and then verify that it is a space. Checking that P satisfies the conditions in the definition of a vector space is routine.

Subspaces and Spanning Sets

For instance, for closure under addition, note that if the summands satisfy that $x_1 + y_1 + z_1 = 0$ and $x_2 + y_2 + z_2 = 0$ then the sum satisfies that

$$\begin{aligned} &(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) \\ &= (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) = 0. \end{aligned}$$

Subspaces and Spanning Sets

Example. The x-axis in $\mathbb{R}^2$ is a subspace where the addition and scalar multiplication operations are the inherited ones

$$\begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 0 \end{pmatrix}, r \cdot \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} rx \\ 0 \end{pmatrix}$$

As above, to verify that this is a subspace we simply note that it is a subset and then check that it satisfies the conditions in definition of a vector space.

Subspaces and Spanning Sets

For instance, the two closure conditions are satisfied: (1) adding two vectors with a second component of zero results in a vector with a second component of zero, and (2) multiplying a scalar times a vector with a second component of zero results in a vector with a second component of zero.

Theorem 2. Suppose W is a subset of a vector space V. Then W is a subspace of V if and only if for every $u, v \in W$ and $\lambda \in \mathbb{K}$ the following two conditions hold:

- the sum $u + v \in W$;
- the multiple $\lambda u \in W$.

Example 4. a) Consider the vector space $V = \mathbb{R}^3$. Let W consist of all vectors in $\mathbb{R}^3$ whose entries are equal, that is,

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x = y = z \right\}.$$

For example, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix}, \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$ are vectors in W. We show that W is a subspace of V over $\mathbb{K}$.

Assume $u, v \in W$ and $u = \begin{pmatrix} a \\ a \\ a \end{pmatrix}$ and $v = \begin{pmatrix} b \\ b \\ b \end{pmatrix}$.

Then

$$u + v = \begin{pmatrix} a+b \\ a+b \\ a+b \end{pmatrix} \quad \text{and}$$

$$\lambda u = \begin{pmatrix} \lambda a \\ \lambda a \\ \lambda a \end{pmatrix} \quad \text{for any } \lambda \in \mathbb{K}.$$

So $u + v$ and $\lambda u \in \mathbb{K}$.

Thus, W is a subspace of $\mathbb{R}^3$.

b) Let $\mathrm{W} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + y + z = 1 \right\}$. We show that W is not a subspace of $\mathbb{R}^3$.

Suppose $u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Then $u, v \in W$, but $u + v \notin W$, since $u + v = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, or if $\lambda = 2$, then $2u = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ and $2u \notin W$. Hence, W is not a subspace of $\mathbb{R}^3$.

Subspaces and Spanning Sets

Example. Another subspace of $\mathbb{R}^2$ is its trivial subspace.

$$\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$$

Any vector space has a trivial subspace $\{\vec{0}\}$. At the opposite extreme, any vector space has itself for a subspace. These two are the *improper* subspaces. Other subspaces are *proper*.

Subspaces and Spanning Sets

Example. The definition requires that the addition and scalar multiplication operations must be the ones inherited from the larger space. The set $S = \{1\}$ is a subset of $\mathbb{R}^1$. And, under the operations $1 + 1 = 1$ and $r \cdot 1 = 1$ the set S is a vector space, specifically, a trivial space. However, S is not a subspace of $\mathbb{R}^1$ because those aren't the inherited operations, since of course $\mathbb{R}^1$ has $1 + 1 = 2$.

Subspaces and Spanning Sets

Example. All kinds of vector spaces, not just $\mathbb{R}^n$'s, have subspaces. The vector space of cubic polynomials $\{a + bx + cx^2 + dx^3 | a, b, c, d \in \mathbb{R}\}$ has a subspace comprised of all linear polynomials $\{m + nx | m, n \in \mathbb{R}\}$.

Subspaces and Spanning Sets

Example. Another example of a subspace not taken from an $\mathbb{R}^n$ is one from the examples following the definition of a vector space. The space of all real-valued functions of one real variable $f: \mathbb{R} \to \mathbb{R}$ has a subspace of functions satisfying the restriction $(d^2 f / dx^2) + f = 0$.

Subspaces and Spanning Sets

Example. Being vector spaces themselves, subspaces must satisfy the closure conditions. The set $\mathbb{R}^+$ is not a subspace of the vector space $\mathbb{R}^1$ because with the inherited operations it is not closed under scalar multiplication: if $\vec{v} = 1$ then $-1 \cdot \vec{v} \notin \mathbb{R}^+$.

Subspaces and Spanning Sets

The next result says that Example 2.8 is prototypical. The only way that a subset can fail to be a subspace, if it is nonempty and under the inherited operations, it isn't closed.

Subspaces and Spanning Sets

Lemma. For a nonempty subset S of a vector space, under the inherited operations, the following are equivalent statements.

(1) S is a subspace of that vector space

Subspaces and Spanning Sets

(2) S is closed under linear combinations of pairs of vectors: for any vectors $\overrightarrow{s_1}, \overrightarrow{s_2} \in S$ and scalars r_1, r_2 the vector $\mathrm{r}_1\overrightarrow{s_1} + r_2\overrightarrow{s_2}$ is in S

(3) S is closed under linear combinations of any number of vectors: for any vectors $\overrightarrow{s_1}, \overrightarrow{s_2}, \dots, \overrightarrow{s_n} \in S$ and scalars $r_1, \dots, r_n$ the vector $r_1\overrightarrow{s_1} + \cdots + r_n\overrightarrow{s_n}$ is in S.

Definition 2. A one-element vector space is a ***trivial space***.

Definition 3. Let V be a vector space over $\mathbb{K}$. A vector v in V is a ***linear combination*** of vectors $u_1, u_2, \dots, u_n$ in V if there exist scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ in $\mathbb{K}$ such that

$$v = \lambda_1 u_1 + \cdots + \lambda_n u_n.$$

In general, the question of existence of linear combination $v = \lambda_1 u_1 + \cdots + \lambda_n u_n$ is equivalent to solving a system of linear equations obtained from the linear combination in unknowns $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Example 2. a) Let $V = \mathbb{R}^3$. Show that the vector $v = \begin{pmatrix} 3 \\ 7 \\ -4 \end{pmatrix}$ in $\mathbb{R}^3$ is a linear combination of the vectors $u_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, u_2 = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}, u_3 = \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}$ and write at form $v = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3$.

Solution: a) We have

$$\begin{pmatrix} 3 \\ 7 \\ -4 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} + \lambda_3 \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}.$$

It implies

$$\begin{cases} \lambda_1 + 2\lambda_2 + 3\lambda_3 = 3 \\ 2\lambda_1 + 3\lambda_2 + 5\lambda_3 = 7 \\ 3\lambda_1 + 7\lambda_2 + 6\lambda_3 = -4 \end{cases}.$$

The solution of the system is

$$\lambda_1 = 2, \lambda_2 = -4, \lambda_3 = 3$$

and thus

$$v = 2u_1 - 4u_2 + 3u_3.$$

b) Let $V = P[t]$ and $v = 3t^2 + 5t - 5$. Show that v is a linear combination of vectors

$$p_1 = t^2 + 2t + 1, p_2 = 2t^2 + 5t + 4,$$
$$p_3 = t^2 + 3t + 6$$

and write at form $v = \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3$.

b) We have

$$3t^2 + 5t - 5 = \lambda_1(t^2 + 2t + 1) +$$
$$+\lambda_2(2t^2 + 5t + 4) + \lambda_3(t^2 + 3t + 6).$$

It implies

$$\begin{cases} \lambda_1 + 2\lambda_2 + \lambda_3 = 3 \\ 2\lambda_1 + 5\lambda_2 + 3\lambda_3 = 5 \\ \lambda_1 + 4\lambda_2 + 6\lambda_3 = -5 \end{cases}.$$

The solution of the system is

$$\lambda_1 = 3, \lambda_2 = 1, \lambda_3 = -2$$

and thus

$$v = 3p_1 + p_2 - 2p_3.$$

Example. This is a subspace of the 2×2 matrices $M_{2\times 2}$.

$$L = \{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} | a + b + c = 0\}$$

- To parametrize, express the condition as $a = -b - c$.

$$L = \left\{ \begin{pmatrix} -b - c & 0 \\ b & c \end{pmatrix} \middle| b, c \in \mathbb{R} \right\}$$

$$= \{ b \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} | b, c \in \mathbb{R} \}$$

As above, we've described the subspace as a collection of unrestricted linear combinations. To show it is a subspace, note that a linear combination of vectors from L is a linear combination of linear combinations and so statement (2) is true.

- **Definition 4.** Let V be a vector space over $\mathbb{K}$. Vectors $u_1, \ldots, u_n$ are said to be ***linearly span*** V or to form a ***spanning set*** of V if every $\boldsymbol{v}$ in V is a linear combination of the vectors $u_1, \ldots, u_n$.

The span of the empty subset of a vector space is the trivial subspace.

No notation for the span is completely standard. The square brackets used here are common but so are ‘$span(S)$’ and ‘$sp(S)$’.

Example 3. Let $V = \mathbb{R}^3$. Vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

span $\mathbb{R}^3$. For any $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$ one can have

$v = ae_1 + be_2 + ce_3$.

Example. The span of this set is all of $\mathbb{R}^2$.

$$\left\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix}\right\}$$

To check this we must show that any member of $\mathbb{R}^2$ is a linear combination of these two vectors.

So we ask: for which vectors (with real components x and y) are there scalars c_1 and c_2 such that this holds?

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Gauss's Method

$$\begin{cases} c_1 + c_2 = x, \\ c_1 - c_2 = y, \end{cases} \rightarrow^{-R_1+R_2} \begin{cases} c_1 + c_2 = x, \\ -2c_2 = -x + y, \end{cases}$$

with back substitution gives $c_2 = (x - y)/2$ and $c_1 = (x + y)/2$.

These two equations show that for any x and y there are appropriate coefficients c_1 and c_2 making the above vector equation true. For instance, for $x = 1$ and $y = 2$ the coefficients $c_2 = -1/2$ and $c_1 = 3/2$ will do. That is, we can write any vector in $\mathbb{R}^2$ as a linear combination of the two given vectors.

Since spans are subspaces, and we know that a good way to understand a subspace is to parametrize its description, we can try to understand a set’s span in that way.

Example. Consider, in P_2, the span of the set $\{3x - x^2, 2x\}$. By the definition of span, it is the set of unrestricted linear combinations of the two

$$\{c_1(3x - x^2) + c_2(2x) | c_1, c_2 \in \mathbb{R}\}.$$

Clearly polynomials in this span must have a constant term of zero. Is that necessary condition also sufficient?

We are asking: for which members $a_2x^2 + a_1x + a_0$ of P_2 are there c_1 and c_2 such that

$$a_2x^2 + a_1x + a_0 = c_1(3x - x^2) + c_2(2x)?$$

Since polynomials are equal if and only if their coefficients are equal, we are looking for conditions on a_2, a_1, and a_0 satisfying these.

$$\begin{cases} -c_1 = a_2 \\ 3c_1 + 2c_2 = a_1 \\ 0 = a_0 \end{cases}$$

Gauss's Method gives that $c_1 = -a_2, c_2 = \left(\frac{3}{2}\right) a_2 + \left(\frac{1}{2}\right) a_1$, and $0 = a_0$. Thus the only condition on polynomials in the span is the condition that we knew of – as long as $a_0 = 0$, we can give appropriate coefficients c_1 and c_2 to describe the polynomial $a_0 + a_1 x + a_2 x^2$ as in the span.

For instance, for the polynomial $0 - 4x + 3x^2$, the coefficients $c_1 = -3$ and $c_2 = 5/2$ will do. So the span of the given set is

$$\{a_1 x + a_2 x^2 | a_1, a_2 \in \mathbb{R}\}.$$

This shows, incidentally, that the set $\{x, x^2\}$ also spans this subspace. A space can have more than one spanning set. Two other sets spanning this subspace are $\{x, x^2, -x + 2x^2\}$ and $\{x, x + x^2, x + 2x^2, \ldots\}$.

Naturally, we usually prefer to work with spanning sets that have only a few members.

Definition 6. Let V be a vector space over $\mathbb{K}$. We say that the vector $v_1, \ldots v_n$ in V are ***linearly dependent*** if there exist scalars $\lambda_1, \ldots, \lambda_n$ in $\mathbb{K}$, not all of them 0, such that

$$\lambda_1 v_1 + \ldots + \lambda_n v_n = 0.$$

Otherwise, we say that the vectors are ***linearly independent***.

Example 5. a) Let

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, v_3 = \begin{pmatrix} 4 \\ 9 \\ 5 \end{pmatrix}.$$

We show that they are linearly independent in $\mathbb{R}^3$.

First, consider $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0$.

It implies

$$\lambda_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda_3 \begin{pmatrix} 4 \\ 9 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

From this combination we have

$$\begin{cases} \lambda_1 + \lambda_2 + 4\lambda_3 = 0 \\ \lambda_1 + 3\lambda_2 + 9\lambda_3 = 0 \\ 2\lambda_2 + 5\lambda_3 = 0 \end{cases}.$$

We solve the system by Gaussian Elimination method, and we obtain

$$(\lambda_1, \lambda_2, \lambda_3) = \left(-\frac{2\lambda_3}{2}, -\frac{5\lambda_3}{2}, \lambda_3\right)$$

where λ_3 is any number in $\mathbb{R}$. Set $\lambda_3 = 2$ and have $\lambda_1 = -3, \lambda_2 = -5, \lambda_3 = 2$. So $-3v_1 - 5v_2 + 2v_3 = 0$ and thus v_1, v_2, v_3 are linearly dependent in $\mathbb{R}^3$.

b) Let

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}.$$

We show that they are linearly independent in $\mathbb{R}^3$. First, consider $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0$.

It implies

$$\lambda_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

From this combination we have

$$\begin{cases} \lambda_1 + 2\lambda_2 + \lambda_3 = 0 \\ 2\lambda_1 + 5\lambda_2 + 3\lambda_3 = 0 \\ 3\lambda_1 + 7\lambda_2 + 5\lambda_3 = 0 \end{cases}$$

We solve the system by Gaussian Elimination method, and we obtain that the homogeneous system has a unique solution,

$$\lambda_1 = \lambda_2 = \lambda_3 = 0.$$

Hence, the vectors are linearly independent in $\mathbb{R}^3$.

Exercises for lecture 3

1. Name the zero vector for each of these vector spaces.

(a) The space of degree three polynomials under the natural operations.

(b) The space of 2×4 matrices.

2. For each, list three elements and then show it is a vector space. The set

$$\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \,\middle|\, x + y - z + w = 0 \right\}$$

under the operations inherited from $\mathbb{R}^4$.

3. Show that each of these is not a vector space. (Hint. Check closure by listing two members of each set and trying some operations on them.)

(a) Under the operations inherited from $\mathbb{R}^3$, this set

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, x + y + z = 1 \right\}$$

3. Show that each of these is not a vector space. (Hint. Check closure by listing two members of each set and trying some operations on them.)

(b) Under the operations inherited from $\mathbb{R}^3$, this set

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 + z^2 = 1 \right\}$$

4. Prove or disprove that $\mathbb{R}^3$ is a vector space under these operations.

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad and \quad r\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx \\ ry \\ rz \end{pmatrix}$$