Lecture 7. Maps Between Spaces: change of basis.

Composition of Linear Mapping

Definition 1. Suppose V, U and W be vector spaces over $\mathbb{K}$ and suppose that $F : V \to U$ and $G : U \to W$ are linear mappings. The composition of linear mapping $G \circ F$ is the mapping from V to W defined by $(G \circ F)(v) = G(F(v))$.

Proposition 1. Let V, U and W be vector spaces over $\mathbb{K}$. If $F : V \to U$ and $G : U \to W$ are linear mappings, then $G \circ F$ is a linear mapping.

Example 1. Let $F, G : \mathbb{R}^2 \to \mathbb{R}^2$ be linear mappings defined by

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix} \text{ and } G\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ x-3y \end{pmatrix}.$$

Then

$$(G \circ F)(v) = G(F(v)) = G\left(F\begin{pmatrix} x \\ y \end{pmatrix}\right) = G\begin{pmatrix} x+y \\ x-y \end{pmatrix} =$$
$$\begin{pmatrix} 2(x+y) \\ (x+y)-3(x-y) \end{pmatrix} = \begin{pmatrix} 2x+3y \\ -2x+4y \end{pmatrix}.$$

$$(F \circ G)(v) = F(G(v)) = F\left(G\begin{pmatrix} x \\ y \end{pmatrix}\right) = F\begin{pmatrix} 2x \\ x-3y \end{pmatrix} =$$
$$\begin{pmatrix} 2x+(x-3y) \\ 2x-(x-3y) \end{pmatrix} = \begin{pmatrix} 3x-3y \\ x+3y \end{pmatrix}.$$

We note that $F \circ G \neq G \circ F$.

Proposition 2. The composition operation for any linear mappings F, G, H is associative, that is,

$$H \circ (G \circ F) = (H \circ G) \circ F.$$

Matrix representation of a linear mapping

Let V be a vector space over $\mathbb{K}$ and $S = \{u_1, \ldots, u_n\}$ be a basis of V. Then for any $v = \lambda_1 u_1 + \ldots + \lambda_n u_n$ we can find its coordinates with respect to S and have

$$[v]_S = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}.$$

Recall that by the correspondence $v \longmapsto [v]_S$ we can construct isomorphism between the spaces V and $\mathbb{R}^n$. Below we see that if $dimV = n$, then there is an isomorphism between $Hom(V, V)$ and $M_{n,n}$ with respect to the basis S.

Definition 2. Let $T \in Hom(V,V)$ over $\mathbb{K}$. Let $S = \{u_1, \ldots, u_n\}$ is a basis of V. Suppose

$$\begin{gathered}
T(u_1) = a_{11}u_1 + a_{21}u_2 + \ldots + a_{n1}u_n, \\
T(u_2) = a_{12}u_1 + a_{22}u_2 + \ldots + a_{n2}u_n, \\
\ldots \\
T(u_n) = a_{1n}u_1 + a_{2n}u_2 + \ldots + a_{nn}u_n,
\end{gathered}$$

Then

$$[T]_S = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

is said to be the matrix representation of T relative to the basis S.

Example 2. a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be linear mappings defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$ and $S = \{e_1, e_2\}$ be the standard basis. Then

$$\begin{gathered}
T(e_1) = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e_1 + e_2 = 1 \cdot e_1 + 1 \cdot e_2, \\
T(e_2) = T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e_1 - e_2 = 1 \cdot e_1 + (-1) \cdot e_2.
\end{gathered}$$

The matrix representation of T in the basis S is

$$[T]_S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

b) Again, consider the linear mapping T in the preceding example $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$ and with new basis $S' = \{f_1, f_2\}$ where $f_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $f_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then

$$T(f_1) = T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \lambda_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 + \lambda_2 \\ \lambda_1 \end{pmatrix}.$$

We obtain $\lambda_1 + \lambda_2 = 2$ and $\lambda_1 = 0$. It implies $\lambda_1 = 0$ and $\lambda_2 = 2$. Thus,

$$T(f_1) = 0\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0f_1 + 2f_2 = 0 \cdot f_1 + 2 \cdot f_2.$$

$$T(f_2) = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mu_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mu_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \mu_1 + \mu_2 \\ \mu_1 \end{pmatrix}.$$

We obtain $\mu_1 + \mu_2 = 1$ and $\mu_1 = 1$. It implies $\mu_1 = 1$ and $\mu_2 = 0$. Thus,

$$T(f_2) = 1\begin{pmatrix}1\\1\end{pmatrix} + 0\begin{pmatrix}1\\0\end{pmatrix} = 1f_1 + 0f_2 = 1 \cdot f_1 + 0 \cdot f_2.$$

The matrix representation of T in the basis S' is

$$[T]_{S'} = \begin{pmatrix}0 & 1\\2 & 0\end{pmatrix}.$$

Properties of Matrix Representations

Theorem 1. Let $T : V \to V$ be a linear mapping and S be a basis of V. Then for any $v \in V$, we have $[T]_S[v]_S = [T(v)]_S$.

Let $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2x+y\\3x-4y\end{pmatrix}$ and $S = \{g_1, g_2\}$ be a basis, where

$$g_1 = \begin{pmatrix}1\\-2\end{pmatrix} \text{ and } g_2 = \begin{pmatrix}1\\-1\end{pmatrix}.$$

Let $v = \begin{pmatrix}2\\3\end{pmatrix}$. Then

$$v = \begin{pmatrix}2\\3\end{pmatrix} = -5\begin{pmatrix}1\\-2\end{pmatrix} + 7\begin{pmatrix}1\\-1\end{pmatrix} = -5g_1 + 7g_2.$$

So

$$[v]_S = \begin{pmatrix}-5\\7\end{pmatrix}.$$

$$T(g_1) = T\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}0\\11\end{pmatrix} = -11\begin{pmatrix}1\\-2\end{pmatrix} + 11\begin{pmatrix}1\\-1\end{pmatrix} = -11g_1 + 11g_2.$$

$$T(g_2) = T\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}1\\7\end{pmatrix} = -8\begin{pmatrix}1\\-2\end{pmatrix} + 9\begin{pmatrix}1\\-1\end{pmatrix} = -8g_1 + 9g_2.$$

Then

$$[T]_S = \begin{pmatrix}-11 & -8\\11 & 9\end{pmatrix}.$$

$$T(v) = T\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}7\\-6\end{pmatrix} = -\begin{pmatrix}1\\-2\end{pmatrix} + 8\begin{pmatrix}1\\-1\end{pmatrix} = -g_1 + 8g_2$$

and so $[T(v)]_S = \begin{pmatrix}-1\\8\end{pmatrix}$.

$$[T]_S[v]_S = \begin{pmatrix}-11 & -8\\11 & 9\end{pmatrix}\begin{pmatrix}-5\\7\end{pmatrix} = \begin{pmatrix}-1\\8\end{pmatrix} = [T(v)]_S.$$

Theorem 2. Let V be an n −dimensional vector space over $\mathbb{K}$, let S be a basis of V. Let $M_{n,n}$ be a vector space of $n \times n$ matrices. Then the mapping

$$\varphi : Hom(V, V) \to M_{n,n}$$

defined by $\varphi(T) = [T]_S$ is a vector space isomorphism. That is, for any $F, G \in Hom(V, V)$ and any $\lambda \in \mathbb{K}$,

1. $\varphi(F + G) = [F]_S + [G]_S$
2. $\varphi(\lambda F) = \lambda[F]_S$
3. φ is one-to-one and onto.

Let V be a vector space over $\mathbb{K}$. Then we select two different bases, say S and S' of V. Then there arises a question: According to these bases how do our representations change?

Change of basis matrix

Definition 3. Let V be a vector space over $\mathbb{K}$ and $S = \{u_1, u_2, \dots, u_n\}$ and $S' = \{v_1, v_2, \dots, v_n\}$ be bases of V. Write basis vectors in S' as a linear combination of basis vectors in S. Suppose

$$\begin{gathered} v_1 = a_{11}u_1 + a_{21}u_2 + \dots + a_{n1}u_n, \\ v_2 = a_{12}u_1 + a_{22}u_2 + \dots + a_{n2}u_n, \\ \dots \\ v_n = a_{1n}u_1 + a_{2n}u_2 + \dots + a_{nn}u_n, \end{gathered}$$

and

$$P = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

Then P is called the change-of-basis matrix or transformation matrix from the ”old” basis S to the ”new” basis S'.

Example 3. Consider $V = \mathbb{R}^2$. Let $S = \{u_1, u_2\} = \{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\}$ and $S' = \{v_1, v_2\} = \{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\5\end{pmatrix}\}$. We write the vectors v_1 and v_2 as linear combinations of u_1 and u_2. Then

$$\begin{pmatrix}1\\2\end{pmatrix} = \lambda_1\begin{pmatrix}1\\1\end{pmatrix} + \lambda_2\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\lambda_1 + \lambda_2\\ \lambda_1\end{pmatrix}. \text{Then } \lambda_1 = 2, \lambda_2 = -1.$$

$$\begin{pmatrix}2\\5\end{pmatrix} = \mu_1\begin{pmatrix}1\\1\end{pmatrix} + \mu_2\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\mu_1 + \mu_2\\ \mu_1\end{pmatrix}. \text{Then } \mu_1 = 5, \mu_2 = -3.$$

Thus,

$$v_1 = 2u_1 - u_2$$
$$v_2 = 5u_1 - 3u_2$$

and hence,

$$P = \begin{pmatrix} 2 & 5 \\ -1 & -3 \end{pmatrix}.$$

Proposition 3. Let P be a change-of-basis matrix from a basis S to a basis S' and Q be a change-of-basis matrix from a basis S' to a basis S. Then $Q = P^{-1}$.

Consider $V = \mathbb{R}^2$. Let $S = \{u_1, u_2\} = \{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$ and $S' = \{v_1, v_2\} = \{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}\}$. We have just seen that the change-of-basis matrix from S to S' is

$$P = \begin{pmatrix} 2 & 5 \\ -1 & -3 \end{pmatrix}.$$

We will see below that the change-of-basis matrix Q from S' to S is P^{-1}.

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} \lambda_1 + 2\lambda_2 \\ 2\lambda_1 + 5\lambda_2 \end{pmatrix}. \text{Then } \lambda_1 = 3, \lambda_2 = -1.$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mu_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \mu_2 \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} \mu_1 + 2\mu_2 \\ 2\mu_1 + 5\mu_2 \end{pmatrix}. \text{Then } \mu_1 = 5, \mu_2 = -2.$$

Thus,

$$u_1 = 3v_1 - v_2$$
$$u_2 = 5v_1 - 2v_2$$

and hence,

$$Q = \begin{pmatrix} 3 & 5 \\ -1 & -2 \end{pmatrix}.$$

Let us find the inverse of P by the formula given in Lecture 6 and have

$$P^{-1} = \frac{1}{2 \cdot (-3) - 5 \cdot (-1)} \begin{pmatrix} -3 & -5 \\ 1 & 2 \end{pmatrix} = -\begin{pmatrix} -3 & -5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ -1 & -2 \end{pmatrix} = Q.$$

Change of coordinates

Theorem 3. Let P be the change-of-basis matrix from a basis S to a basis S' in a vector space V. Then

$$P[v]_{S'} = [v]_S \text{ or } P^{-1}[v]_S = [v]_{S'}.$$

Let $S = \{e_1, e_2, e_3\} = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ and $S' = \{f_1, f_2, f_3\} = \left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$. We know that they are bases of $\mathbb{R}^3$. Let $v = \begin{pmatrix}1\\-2\\3\end{pmatrix}$.

Since $f_1 = e_1 + e_2 + e_3$, $f_2 = e_1 + e_2$, $f_3 = e_1$ we have

$$P = \begin{pmatrix}1 & 1 & 1\\1 & 1 & 0\\1 & 0 & 0\end{pmatrix}.$$

Then it is clear that $v = e_1 - 2e_2 + 3e_3$ and have $[v]_S = \begin{pmatrix}1\\-2\\3\end{pmatrix}$. One can write $v = 3f_1 - 5f_2 + 3f_3$. Consequently, $[v]_{S'} = \begin{pmatrix}3\\-5\\3\end{pmatrix}$. Then

$$P[v]_{S'} = \begin{pmatrix}1 & 1 & 1\\1 & 1 & 0\\1 & 0 & 0\end{pmatrix}\begin{pmatrix}3\\-5\\3\end{pmatrix} = \begin{pmatrix}1\cdot 3 + 1\cdot(-5) + 1\cdot 3\\1\cdot 3 + 1\cdot(-5) + 0\cdot 3\\1\cdot 3 + 0\cdot(-5) + 0\cdot 3\end{pmatrix} = \begin{pmatrix}1\\-2\\3\end{pmatrix} = [v]_S.$$

Two matrix representations

Theorem 4. Let P be he change-of-basis matrix from a basis S to a basis S' vector space V. Then for any linear mapping T on V ($T : V \to V$)

$$[T]_{S'} = P^{-1}[T]_S P.$$

Example 4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be linear mappings defined by $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+y\\x-y\end{pmatrix}$ and $S = \{e_1, e_2\}$ be the standard basis. The matrix representation of T in the basis S is

$$[T]_S = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}.$$

If $S = \{f_1, f_2\}$ where $f_1 = \begin{pmatrix}1\\1\end{pmatrix}$ and $f_2 = \begin{pmatrix}1\\0\end{pmatrix}$,then

$$[T]_{S'} = \begin{pmatrix}0 & 1\\2 & 0\end{pmatrix}.$$

One can easily obtain that

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } P^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then

$$P^{-1}[T]_S P = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} = [T]_{S'}$$

Similarity

Definition 4. Suppose matrices A and B be matrices for which there exists an invertible matrix P such that $B = P^{-1}AP$. Then B is said to be similar to A.

Example 5. If $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$, then B is similar to A, because there exists matrix $P = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ such that

$$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}.$$

Theorem 5. Two matrices represent the same linear mapping if and only if the matrices are similar.

Glossary

matrix representation	матричное представление
similarity	подобие

Exercises for lecture 7

Algebra of Linear Operators

1. Let F and G be the linear operators on $\mathbb{R}^2$ defined by $F(x, y) = (x + y, 0)$ and $G(x, y) = (-y, x)$. Find formulas defining the linear operators:

(a) $F + G$, (b) $5F - 3G$, (c) FG, (d) GF, (e) F^2, (f) G^2.

Matrices and Linear Operators

2. Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x, y) = (4x + 5y, 2x - y)$.

(a) Find the matrix A representing F in the usual basis E.

(b) Find the matrix B representing F in the basis S=$\{u_1, u_2\} = \{(1,4), (2,9)\}$.

(c) Find P such that $B = P^{-1}AP$.
(d) For $v = (a, b)$, find $[v]_S$ and $[F(v)]_S$. Verify that$[F]_S[v]_S = [F(v)]_S$.

3. Find the matrix representing each linear transformation T on $\mathbb{R}^3$ relative to the usual basis of $\mathbb{R}^3$:
(a) $T(x, y, z) = (x, y, 0)$. (b) $T(x, y, z) = (z, y + z, x + y + z)$.
(c) $T(x, y, z) = (2x - 7y - 4z, 3x + y + 4z, 6x - 8y + z)$.

Change of Basis

4. Find the change-of-basis matrix P from the usual basis E of $\mathbb{R}^2$ to a basis S, the change-of-basis matrix Q from S back to E, and the coordinates of $v = (a, b)$ relative to S, for the following bases S:
(a) $S = \{(1,2), (3,5)\}$. (c) $S = \{(2,5), (3,7)\}$.
(b) $S = \{(1, -3), (3, -8)\}$. (d) $S = \{(2,3), (4,5)\}$.

Linear Operators and Change of Basis

5. Consider the linear operator F on $\mathbb{R}^2$ defined by $F(x, y) = (5x + y, 3x - 2y)$ and the following bases of $\mathbb{R}^2$:
$S = \{(1,2), (2,3)\}$ and $S' = \{(1,3), (1,4)\}$
(a) Find the matrix A representing F relative to the basis S.
(b) Find the matrix B representing F relative to the basis S'.
(c) Find the change-of-basis matrix P from S to S'.
(d) How are A and B related?

Homework 7

Algebra of Linear Operators

1. Let F and G be the linear operators on $\mathbb{R}^2$ defined by $F(x, y) = (x + y, 0)$ and $G(x, y) = (-y, x)$. Find formulas defining the linear operators:
(a) GF, (b) F^2, (c) G^2.

Matrices and Linear Operators

2. Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x, y) = (5x + 4y, -x + 2y)$.
(a) Find the matrix A representing F in the usual basis E.
(b) Find the matrix B representing F in the basis S=$\{u_1, u_2\} = \{(4, 1), (3, 4)\}$.
(c) Find P such that $B = P^{-1}AP$.
(d) For $v = (a, b)$, find $[v]_S$ and $[F(v)]_S$. Verify that$[F]_S[v]_S = [F(v)]_S$.

3. Find the matrix representing each linear transformation T on $\mathbb{R}^3$ relative to the usual basis of $\mathbb{R}^3$:
(a) $T(x, y, z) = (0, y, z)$. (b) $T(x, y, z) = (y, x + z, x + y + z)$.

(c) $T(x, y, z) = (2x - 5y - 2z, 3x + y + 6z, 4x - 2y + z)$.

Change of Basis

4. Find the change-of-basis matrix P from the usual basis E of $\mathbb{R}^2$ to a basis S, the change-of-basis matrix Q from S back to E, and the coordinates of $v = (a, b)$ relative to S, for the following bases S:

(a) $S = \{(1,3), (2,5)\}$. (c) $S = \{(3,5), (2,7)\}$.
(b) $S = \{(1, -2), (2, -8)\}$. (d) $S = \{(3,2), (5,4)\}$.

Linear Operators and Change of Basis

5. Consider the linear operator F on $\mathbb{R}^2$ defined by $F(x, y) = (3x + y, 5x - 2y)$ and the following bases of $\mathbb{R}^2$:

$$S = \{(1,3), (3,4)\} \quad \text{and} \quad S' = \{(1,2), (1,4)\}$$

(a) Find the matrix A representing F relative to the basis S.
(b) Find the matrix B representing F relative to the basis S'.
(c) Find the change-of-basis matrix P from S to S'.
(d) How are A and B related?