Lecture 8. Projection, Gram-Schmidt Orthogonalization, projection into a subspace

Inner product in $\mathbb{R}^n$.

Definition 1. Consider arbitrary vectors $u, v \in \mathbb{R}^n$, let

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \quad and \quad v = \begin{pmatrix} v_1 \\ v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

The inner or dot or scalar product of u and v is denoted by $u \cdot v$ and defined as

$$u \cdot v = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n.$$

If $u \cdot v = 0$, then we say u and v are orthogonal. We note that $u \cdot v \in \mathbb{R}$.

Definition 2. The space $\mathbb{R}^n$ with the operations of vector addition, scalar multiplication, and dot product is called Euclidean n −space.

Example 1. Let $u = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}, v = \begin{pmatrix} 5 \\ -6 \\ 1 \end{pmatrix}$ and $w = \begin{pmatrix} -5 \\ -5 \\ -2 \end{pmatrix}$.

Then $u \cdot v = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -6 \\ 1 \end{pmatrix} = 1 \cdot 5 + (-3) \cdot (-6) + 5 \cdot 1 = 5 + 18 + 5 = 28,$

$$u \cdot w = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ -5 \\ -2 \end{pmatrix} = 1 \cdot (-5) + (-3) \cdot (-5) + 5 \cdot (-2) =$$
$$= -5 + 15 - 10 = 0,$$

Hence, u and w are orthogonal.

Basic properties of inner product

Theorem 1. For any vectors $u, v, w \in \mathbb{R}^n$ and any scalar $\lambda \in \mathbb{R}$:

- $(u + v) \cdot w = u \cdot w + v \cdot w$
- $(\lambda u) \cdot v = \lambda(u \cdot v)$
- $u \cdot v = v \cdot u$
- $u \cdot u \geq 0$ and $u \cdot u = 0$ if and only if u is a zero vector.

Norm or Length of a vector

Definition 3. The norm or length of a vector u in $\mathbb{R}^n$, denoted by $||u||$, is defined to be the nonnegative square of $u \cdot u$. In particular, if $u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$, then

$$\left||u|\right| = \sqrt{u \cdot u} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}.$$

A vector u is called a unit if $||u|| = 1$.

Proposition 1. Let $u \in \mathbb{R}^n$. Then $\hat{u} = \frac{u}{||u||}$ is a unit vector. The process of finding $\hat{u}$ from u is $||u||$ called normalizing u.

Example 2. a) Suppose $u = \begin{pmatrix} 1/6 \\ 5/6 \\ -1/6 \\ 1/2 \end{pmatrix}$. Then

$$\left||u|\right| = \sqrt{\frac{1}{6} \cdot \frac{1}{6} + \frac{5}{6} \cdot \frac{5}{6} + \left(-\frac{1}{6}\right) \cdot \left(-\frac{1}{6}\right) + \frac{1}{2} \cdot \frac{1}{2}} = \sqrt{\frac{1}{36} + \frac{25}{36} +\cdot \frac{1}{36} + \frac{1}{4}} = 1.$$

Hence, u is a unit vector.

b) Suppose $v = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$. $\left||u|\right| = \sqrt{1 \cdot 1 + (-3) \cdot (-3) + 5 \cdot 5} = \sqrt{35}.$

Hence, v is not a unit vector. By normalizing v one can have unit vector from v:

$$\hat{u} = \frac{u}{||u||} = \frac{1}{\sqrt{35}} \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{35} \\ -3/\sqrt{35} \\ 5/\sqrt{35} \end{pmatrix}.$$

Inequalities

Theorem (Cauchy-Schwarz inequality). For any vectors $u, v \in \mathbb{R}^n$,

$$|u \cdot v| \leq ||u|| \cdot ||v||.$$

Theorem (Minkowski inequality). For any vectors $u, v \in \mathbb{R}^n$,

$$||u + v|| \leq ||u|| + ||v||.$$

Angle between vectors

Definition 4. The angle θ between nonzero vectors $u, v \in \mathbb{R}^n$ is defined by

$$\cos\theta = \frac{u \cdot v}{||u|| \cdot ||v||}.$$

Example 3. Let $u = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$ and $v = \begin{pmatrix} 5 \\ -6 \\ 1 \end{pmatrix}$.

We have $u \cdot v = 1 \cdot 5 + (-3) \cdot (-6) + 5 \cdot 1 = 5 + 18 + 5 = 28$

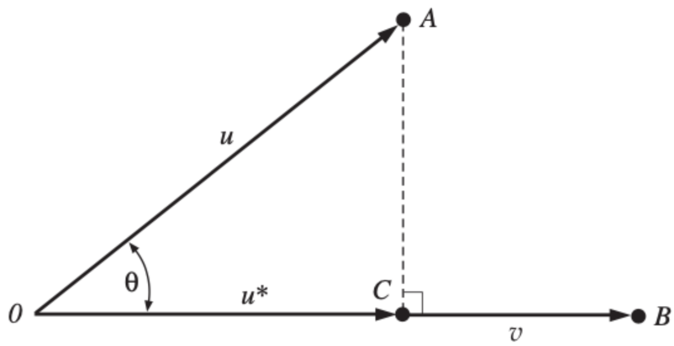
$$||u|| = \sqrt{1^2 + (-3)^2 + 5^2} = \sqrt{35} \quad ||v|| = \sqrt{5^2 + 6^2 + 1^2} = \sqrt{62}.$$

Then

$$\cos\theta = \frac{u \cdot v}{||u|| \cdot ||v||} = \frac{28}{\sqrt{35} \cdot \sqrt{62}} = \frac{28}{\sqrt{2170}}.$$

Projections

Given vectors $u, v \in \mathbb{R}^n$. Consider the projection of u onto the vector v. If we denote the projection of u onto v as u^*, then its length is $||u^*|| = ||u|| \cos\theta$. To obtain u^*, we multiply its length by the unit vector in the direction of v and have



$$u^* = ||u^*|| \frac{v}{||v||} = ||u|| \cos\theta \frac{v}{||v||} = ||u|| \frac{u \cdot v}{||u|| \cdot ||v||} \frac{v}{||v||} = \frac{u \cdot v}{||v||} \frac{v}{||v||}$$
$$= \frac{u \cdot v}{||v||^2} v.$$

Hence,

$$u^* = \frac{u \cdot v}{||v||^2} v.$$

Example 4. Let $u = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$ and $v = \begin{pmatrix} 5 \\ -6 \\ 1 \end{pmatrix}$. We know that $u \cdot v = 28$ and $||v||^2 = 62$. Then the projection u^* of u onto v is

$$u^* = \frac{u \cdot v}{||v||^2} v = \frac{28}{62}\begin{pmatrix} 5 \\ -6 \\ 1 \end{pmatrix} = \frac{14}{31}\begin{pmatrix} 5 \\ -6 \\ 1 \end{pmatrix} = \begin{pmatrix} 70/31 \\ -84/31 \\ 14/31 \end{pmatrix}.$$

Orthogonal vectors

Recall that $u, v \in \mathbb{R}^n$ are said to be orthogonal if $u \cdot v = 0$. Let us find nonzero vector w that is orthogonal to $u = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$ and $v = \begin{pmatrix} 5 \\ -6 \\ 1 \end{pmatrix}$. Let $w = \begin{pmatrix} x \\ y \\ w \end{pmatrix}$. To be orthogonal we have $w \cdot u = 0$ and $w \cdot v = 0$. They imply

$$\begin{cases} x - 3y + 5z = 0 \\ 5x + 6y + z = 0 \end{cases} \Leftrightarrow \begin{cases} x - 3y + 5z = 0 \\ 21y - 24z = 0 \end{cases}$$

Let $z = 7$. Then $y = 8$ and $x = -11$. Thus, $w = \begin{pmatrix} -11 \\ 8 \\ 7 \end{pmatrix}$.

Orthogonal Complement

Definition 5. Let S be a set of vectors $\mathbb{R}^n$. The orthogonal complement of S is defined by

$$S^{\perp} = \{v \in \mathbb{R}^n : u \cdot v = 0 \text{ for every } u \in S\}.$$

Theorem 2. Let W be a subspace of $\mathbb{R}^n$. Then $\mathbb{R}^n$ is the direct of W and $W^{\perp}$, that is $\mathbb{R}^n = W \oplus W^{\perp}$.

Orthogonal basis

Theorem 3. Let $S = \{u_1, u_2, \ldots, u_n\}$ be an orthogonal basis of V. Then for any $v \in \mathbb{R}^n$,

$$v = \frac{v \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{v \cdot u_2}{u_2 \cdot u_2} u_2 + \cdots + \frac{v \cdot u_n}{u_n \cdot u_n} u_n.$$

Example 5. Let $S = \{u_1, u_2, u_3\} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \right\}$. It is easy to check that u_1, u_2, u_3 are basis vectors. We note that $u_1 \cdot u_2 = u_1 \cdot u_3 = u_2 \cdot u_3 = 0$. So, they are orthogonal. Let $v = \begin{pmatrix} 7 \\ 1 \\ 9 \end{pmatrix}$. We write

$$v = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3.$$

Now we will calculate λ_1, λ_2 and λ_3.

$$\lambda_1 = \frac{v \cdot u_1}{u_1 \cdot u_1} = \frac{7+2+9}{1+4+1} = \frac{18}{6} = 3,$$
$$\lambda_2 = \frac{v \cdot u_2}{u_2 \cdot u_2} = \frac{14+1-36}{4+1+16} = \frac{-21}{21} = -1,$$
$$\lambda_3 = \frac{v \cdot u_3}{u_3 \cdot u_3} = \frac{21-2+9}{9+4+1} = \frac{28}{14} = 2.$$

Thus,

$$v = 3u_1 - u_2 + 2u_3.$$

Definition 6. A set of basis vectors $\{v_1, v_2, \dots, v_n\}$ is called orthonormal if $v_i \cdot v_j = 0$ for $i \neq j$ and $v_i \cdot v_i = 1$ for all i. In other words, $\{v_1, v_2, \dots, v_n\}$ is a orthogonal basis such that $||v_i|| = 1$ for all i.
Example 6. Standard basis are orthonormal basis vectors for $\mathbb{R}^n$.

Gram-Schmidt Orthogonalization Process

Suppose $\{v_1, v_2, \dots, v_n\}$ be a basis of $\mathbb{R}^n$. One can obtain orthogonal basis $\{w_1, w_2, \dots, w_n\}$ of $\mathbb{R}^n$ from $\{v_1, v_2, \dots, v_n\}$ using so called Gram-Schmidt Orthogonalization Process.

$$w_1 = v_1$$
$$w_2 = v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1} w_1$$
$$w_3 = v_3 - \frac{v_3 \cdot w_1}{w_1 \cdot w_1} w_1 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2$$
$$\dots$$
$$w_n = v_n - \frac{v_n \cdot w_1}{w_1 \cdot w_1} w_1 - \frac{v_n \cdot w_2}{w_2 \cdot w_2} w_2 - \cdots - \frac{v_n \cdot w_{n-1}}{w_{n-1} \cdot w_{n-1}} w_{n-1}$$

Example 7. Using the Gram-Schmidt orthogonalization process, we orthogonalize the basis

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}:$$

$$w_1 = v_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix},$$

$$w_2 = v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1} w_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} - \frac{1 \cdot (-1) + 2 \cdot 0 + 2 \cdot 2}{1^2 + 2^2 + 2^2} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} - \frac{3}{9} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{4}{3} \\ -\frac{2}{3} \\ \frac{4}{3} \end{pmatrix}.$$

$$w_3 = v_3 - \frac{v_3 \cdot w_1}{w_1 \cdot w_1} w_1 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{9} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - \frac{4}{3 \cdot 4} \begin{pmatrix} -\frac{4}{3} \\ -\frac{2}{3} \\ \frac{4}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{9} \\ -\frac{2}{9} \\ \frac{1}{9} \end{pmatrix}.$$

So we have obtained three orthogonal basis vectors

$$w_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, w_2 = \begin{pmatrix} -\frac{4}{3} \\ -\frac{2}{3} \\ \frac{4}{3} \end{pmatrix}, w_3 = \begin{pmatrix} \frac{2}{9} \\ -\frac{2}{9} \\ \frac{1}{9} \end{pmatrix}.$$

Now we normailize these vectors and have orthonormal basis vectors

$$\widehat{w}_1 = \frac{w_1}{||w_1||} = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \widehat{w}_2 = \frac{w_2}{||w_2||} = \frac{4}{3} \begin{pmatrix} -\frac{4}{3} \\ -\frac{2}{3} \\ \frac{4}{3} \end{pmatrix}, \widehat{w}_3 = \frac{w_3}{||w_3||} = \frac{1}{3} \begin{pmatrix} \frac{2}{9} \\ -\frac{2}{9} \\ \frac{1}{9} \end{pmatrix}.$$

Glossary

norm or length	норма или длина
Cauchy-Schwarz inequality	Неравенство Коши-Шварца
Minkowski inequality	Неравенство Минковского
projection	проекция
orthogonal complement	ортогональное дополнение
Gram-Schmidt Orthogonalization Process	Процесс ортогонализации Грама-Шмидта

Exercises for lecture 8
Projection

1. Normalize the vector $v = (1,2,-2,4)$.
2. Let $u = (1,2,-2)$, $v = (3,-12,4)$ and $k = -3$. Find $||u||, ||v||, ||u + v||, ||ku||$.
3. Find k so that u and v are orthogonal, where:
(a) $u = (3, k, -2), v = (6, -4, -3)$;
(b) $u = (5, k, -4,2), v = (1, -3,2,2k)$.

Gram-Schmidt orthogonalization

4. Let $w = (1, -2, -1,3)$ be a vector in R^4. Find
(a) an orthogonal basis for $w^{\perp}$?;
(b) an orthonormal basis for $w^{\perp}$?
5. Let W be the subspace of R^4 orthogonal to $u_1 = (1,1,2,2)$ and $u_2 = (0,1,2, -1)$. Find
(a) an orthogonal basis for W;
(b) an orthonormal basis for W.
6. Find the Fourier coefficient c and projection cw of v along w, where
(a) $v = (2,3, -5)$ and $w = (1, -5,2)$ in R^3.
7. Let U be the subspace of R^4 spanned by

$$v_1 = (1,1,1,1), \ \ v_2 = (1, -1,2,2), \ \ v_3 = (1,2, -3, -4)$$

(a) Apply the Gram–Schmidt algorithm to find an orthogonal and an orthonormal basis for U.
Find the projection of $v = (1,2, -3,4)$ onto U.

Homework 8
Projection

1. Normalize the vector $v = (2, -1, -4,2)$.
2. Let $u = (1,3, -4)$, $v = (-3,12, -4)$ and $k = -5$. Find $||u||, ||v||, ||u + v||, ||ku||$.
3. Find k so that u and v are orthogonal, where:
(c) $u = (3, k, -2), v = (-6,4,3)$;

(d) $u = (5, k, -4, 2), v = (-1, 3, -2, -2k)$.

Gram-Schmidt orthogonalization

4. Let $w = (1,4,-2,-3)$ be a vector in R^4. Find

(c) an orthogonal basis for $w^{\perp}$?;

(d) an orthonormal basis for $w^{\perp}$?

5. Let W be the subspace of R^4 orthogonal to $u_1 = (1,-1,2,-2)$ and $u_2 = (0,-1,-2,1)$. Find

(c) an orthogonal basis for W;

(d) an orthonormal basis for W.

6. Find the Fourier coefficient c and projection cw of v along w, where

(b) $v = (1,3,1,2)$ and $w = (1,-2,7,4)$ in R^4.

7. Let U be the subspace of R^4 spanned by

$$v_1 = (1,1,1,1), \quad v_2 = (1,-1,-2,-2), \quad v_3 = (1,2,3,4)$$

(b) Apply the Gram–Schmidt algorithm to find an orthogonal and an orthonormal basis for U.

(c) Find the projection of $v = (1,2,-3,4)$ onto U.