Calculus III

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Contents

1. Introduction

This is the persons i took the course with, they speak mostly spanish but they might help you!

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2. Linear Algebra Fundamentals

In order to understand the concepts present in this module of calculus, a few preliminary concepts in the realm of Linear Algebra are necessary, in order to not leave anybody lost, we'll be reviewing those topics.

2.1 Vectors

2.1.1 Important operations

Addition and Subtraction

Not much mystery to it, you can add and subtract the components in two different vectors:

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

Dot Product

We can get a scalar product from the multiplication of the elements two vectors have

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1x_2 + y_1y_2 + z_1z_2$$

This can also be written as:

$$< \vec{u}, \vec{v} > \tag{2.1}$$

Properties

- Dot product is commutative: $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- $\vec{u} \cdot \vec{v} = 0 \iff \vec{u} \perp \vec{v}$

- $\vec{u} \cdot \vec{v} = ||\vec{u}||||\vec{v}|| \cos\theta$
- Cauchy Schwartz inequality: $|\vec{u} \cdot \vec{v}| \leq ||\vec{u}||||\vec{v}||$
- $\vec{u} \cdot \vec{u} = ||u||^2$

Cross product

Another form to multiply vectors, while getting another vector, is the cross prudict which we define as:

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} x \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{pmatrix}$$

This specific operation cannot be done in dimensions that exceed $\mathbb{R}^3$ and can't be considered formally the multiplication of two vectors when the system is managed in $\mathbb{R}^n | n > 4$. This course, however, is mostly managed for systems at most in 3 dimensions, hyperplanes will probably be tangentially asked about depending on the professor, but i wouldn't count on it.

Properties

Cross product operations have a few specific things that can happen.

- $$\vec{u} x \vec{v} = -\vec{v} x \vec{u}$$
- $$\vec{u} x \vec{v} \perp \vec{u} \wedge \vec{u} x \vec{v} \perp \vec{v}$$
- $$\vec{v} x \vec{u} = 0 \implies \vec{u} // \vec{v}$$
- $$||\vec{u} x \vec{v}|| = Surface$$

Example

Calculate the plane equation that includes the points:

$$\vec{P} = (2,0,0) \tag{2.2}$$
$$\vec{Q} = (0,1,0) \tag{2.3}$$
$$\vec{R} = (0,0,3) \tag{2.4}$$

Solution

First of all, we'll do a normal vector via the cross product. First, we'll need two vectors $\vec{PQ}$ and $\vec{PR}$ via vector subtraction:

$$\vec{PR} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} \tag{2.5}$$
$$\vec{PQ} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \tag{2.6}$$

And now, we can generate a cross product for these two mathematical objects, this should give us a vector normal to both:

$$\vec{PQ}x\vec{PR} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} x \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} = \tag{2.7}$$

Example 2
Find the Area of the triangle generated by the points:

$$\vec{P} = (2,0,0) \tag{2.8}$$
$$\vec{Q} = (0,1,0) \tag{2.9}$$
$$\vec{R} = (0,0,3) \tag{2.10}$$

For this, we can use the property of cross products being able to calculate a surface area to assume the way we can calculate a triangle from it:

$$A_{\triangle} = \frac{||\vec{p}qx\vec{p}r||}{2} \tag{2.11}$$

As given by our last example, then:

$$A_{\triangle} = \frac{||(3,6,2)||}{2} = \frac{\sqrt{3^2+6^2+2^2}}{2} = \frac{\sqrt{9+36+4}}{2} = \frac{7}{2} \tag{2.12}$$

Projection
A proyection is the vector that can be found from making

$$Proy_{\vec{u}}(\vec{v}) = \alpha\vec{u} \tag{2.13}$$

This has the following properties:
- $\vec{v} - \alpha\vec{u} \perp \vec{u}$
- $\frac{\vec{u}\cdot\vec{v}}{||\vec{u}||^2} = \alpha$

We can also imagine the projection as:

$$Proy_{\vec{u}}(\vec{v}) = (\frac{\vec{u}\cdot\vec{v}}{||\vec{u}||^2})\vec{u} \tag{2.14}$$

Example
Given:

$$\vec{u} = x+y = 3 \tag{2.15}$$
$$\vec{v} = x+y = 1 \tag{2.16}$$

What is the distance between both lines?
Solution
For this we can check for the smallest distance between points, and since they are perpendicular to each other:

$$\min d(\vec{v},\vec{u}) = d(\vec{v},\vec{u}) \tag{2.17}$$

lets instantiate the lines in a moment in time where they are both crossing the x axis:

$$P = (0,1) \tag{2.18}$$
$$Q = (0,3) \tag{2.19}$$

2.1.2 Properties

- two vectors are parallel if one is a multiple of the other, such as:

$$\vec{v} = d\vec{w}; d \in \mathbb{R}$$

- The angle between two vectors can be defined as such:

$$\vec{v} \cdot \vec{w} = ||\vec{v}||||\vec{w}|| \cos\theta$$

  If this operation is 0, then the two vectors are orthogonal (parallel) to each other.
- The cross product of two vectors is perpendicular to both vectors that generated it.

2.2 Lines

A line is simply a mathematical object of the form $y = mx + b$ that unites two points in the space we're using, through a straight path, where and b is the cutting point in x=0. and m is $\tan\alpha$, or the slope of this line. We can imagine it as:

$$m = \frac{y_1 - y_0}{x_1 - x_0} \tag{2.20}$$
$$\tag{2.21}$$

another equation of this object is, when working on $\mathbb{R}^2$ is defined as:

$$ax + by = c \tag{2.22}$$

The reason this sort of expression is so useful, is because this **is a general form.** In other words, we can define with this system every single possible line in $\mathbb{R}^{\nvDash}$. For example, let's imagine a line that is perfectly vertical. Such as it can be explained, intuitively, as $x = 1$:

This vector can't be expressed through formula 1,1 or y = mx + b, because it wouldn't cross the '0' axis, and it would have an infinite slope. But if we arrange it through (1,2), we can say:

$$ax + by = c \tag{2.23}$$
$$by = c - ax \tag{2.24}$$
$$y = \frac{c}{b} + \frac{a}{b}x \tag{2.25}$$

We can suppouse $\frac{c}{b}$ as the cutting point with y, and then imagine $\frac{a}{b}$ as 'm', we can then, use an example where we set 'b' to be 0 and define a situation where x = 1.

2.2.1 Lines in $\mathbb{R}^n$

We can define lines in more dimensions than $\mathbb{R}^2$ through different forms. For example, we can imagine it as a parametric equation of the form:

$$p = t\vec{v} \tag{2.26}$$

Example 2.2.1

Find the parametric equation that passes through p = (1,2,3) and is parallel to the vector (1,0,-1)

Solution

$$\begin{pmatrix}1\\2\\3\end{pmatrix} + t\begin{pmatrix}1\\0\\-1\end{pmatrix} \tag{2.27}$$

With this form, we can imagine that the solution actually is producing 3 different equations, and every single one defines how this object will behave in a different dimension.

therefore:

$$x(t) = 1 + t \tag{2.28}$$
$$y(t) = 2 \tag{2.29}$$
$$z(t) = 3 - t \tag{2.30}$$

Pretty neat, huh?

We can define such an equation in 'n' dimensions that can define a line going through be it a plane or a hyperplane like this. We only need two vectors of the same 'n' dimension. So, taking this form, can we express it in other ways? Well, yeah! ...and we just did. The before introduced equations can be called **the parametric equation of a line.** the equation we generated

From here, we can also imagine that instead of using such a form, we can also equate everything to 't' and from here, we'll start defining a **Symetrical form of the line.**

In the previous example, we can imagine:

$$x - 1 = y - 2 = \frac{z-3}{-1} = t \tag{2.31}$$

2.3 Planes

The plane can be defined as:

$$ax + by + cz = d \tag{2.32}$$

This plane can be defined with a point and a vector, for example:

Example 2.3.1
given $\vec{PQ} = (x, y, z)$ and $\vec{R} = (a, b, c)$, define a plane.
Solution

$$\begin{pmatrix} x - x_0 \\ y = y_0 \\ z - z_0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{2.33}$$
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{2.34}$$
$$ax - ax_0 + by - by_0 + cz - cz_0 = 0 \tag{2.35}$$
$$ax + by + cz = ax_0 + by_0 + cz_0 \tag{2.36}$$
$$< d = ax_0 + by_0 + cz_0 > \tag{2.37}$$
$$ax + by + cz = d \tag{2.38}$$

And from there we can define basically every plane in $\mathbb{R}^3$, as you can see, we arrived to the way we defined a plane in this dimension, and we could in theory, decide in a random point, a random vector, and start working from there into defining a plane. But now, can we define a parametric equation of a plane, as we did with a line?

Well, the answer is yes.

the parametric form of a plane is actually pretty simple, as it is supremely similar to the one that defines a line, and is defined as:

$$p + s\vec{v_1} + t\vec{v_2} \tag{2.39}$$

if we generalize this into a system where R = (1,2,3), v_1 = (1,0,1) and v_2 = (1,1,0) we can do a little example, written as:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{2.40}$$

Example 2.3.2
Determine if the line

$$x = 9 + 5t \tag{2.41}$$
$$y = -3 + t \tag{2.42}$$
$$z = 4t \tag{2.43}$$

Intersects with the plane

$$3x+10y-2z+53 \tag{2.44}$$

If so, determine where.
Solution
We can begin by replacing. Remember the line's equation and how it has x, y and z values. **These values are equivalent to x, y and z in a plane**, and how you can replace it in the plane's equation.
If we can solve the equation for 't', that means we can intersect that line with the plane and that it will intersect in that specific 't' point

$$3(9+5t)+10(-3+t)-2(4t)=53 \tag{2.45}$$
$$27+15t-30+10t-8t=53 \tag{2.46}$$
$$t=\frac{56}{16} \tag{2.47}$$

Plane from 3 points in the space

Example 2.3.3

Find the line normal to the plane that crosses the points:

$$P=(0,-3,-3) \tag{2.48}$$
$$Q=(-3,0,-3) \tag{2.49}$$
$$R=(-3,3,0) \tag{2.50}$$

Solution We'll begin by getting two vectors from these lines, we can get them by subtracting one point from the other:

$$\vec{PQ}=(-3,3,0) \tag{2.51}$$
$$\vec{QR}=(0,-3,3) \tag{2.52}$$

And now, we'll make a cross product between the two, so we can get a normal vector:

$$\vec{PQ}x\vec{QR}=\begin{pmatrix}-3\\3\\0\end{pmatrix}x\begin{pmatrix}0\\-3\\3\end{pmatrix}=\begin{pmatrix}9\\9\\9\end{pmatrix}=\vec{n} \tag{2.53}$$

Now, we can get from that normal vector, the direction vector for the line:

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\-3\\-3\end{pmatrix}+t\begin{pmatrix}9\\9\\9\end{pmatrix} \tag{2.54}$$

3. Non-Linear Fundamentals

3.1 Polar and Cylindrical Coordinates

During the course, we'll be managing non-linear systems and such systems will require managing reference systems that are non-cartesian. When trying to define parts of a line in algebra, we'll usually be looking at coordinates, be them polar or cartesian. In either case, their information can be converted to the other system through the following formulas.

$$\begin{cases} \rho = \sqrt{x^2+y^2} \\ \theta = \arctan(\frac{y}{x}) \end{cases} \quad \textit{cartesian to polar} \tag{3.1}$$

$$\begin{cases} \alpha_x = \rho\cos\theta \\ \alpha_y = \rho\sin\theta \end{cases} \quad \textit{polar to cartesian} \tag{3.2}$$

Cartesian coordinates generally translate well to other dimensional spaces, such as would be the case for $\mathbb{R}^3$, however, polar coordinates as we know them usually aren't as translatable in a direct manner, and expressing them in three-dimensional spaces might be better suited to be expressed on a cylindrical or spherical condition.

3.1.1 Cylindrical coordinates

In the case of cylindrical coordinates, the translation is probably the most intuitive, by computing a cylinder with polar coordinates that indicate an (x,y) position, and a 'Z' variable indicating height that allows us to project the vector on a third dimension, this 'z' variable is exactly the same as it would be on a cartesian model. We can express it like such:

$\vec{v} = (\rho, \theta, Z)$

conversion to a cartesian model can be expressed as:

$$\vec{(}\alpha) = \begin{cases} \alpha_x = \rho\cos\theta \\ \alpha_y = \rho\sin\theta \quad \textit{Cylindrical to cartesian} \\ \alpha_z = Z \end{cases} \tag{3.3}$$

$$\vec{(}\alpha) = \begin{cases} \rho = \sqrt{x^2+y^2} \\ \theta = \arctan(\frac{y}{x}) \quad \textit{Cartesian to Cylindrical} \\ Z = Z \end{cases} \tag{3.4}$$

These coordinates might be useful to visualize circular functions.

3.1.2 Spherical coordinates

A spherical coordinate is formed by a tuple:

$$(\rho,\theta,\phi); \begin{cases} \rho \geq 0 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi \end{cases} \tag{3.5}$$

Where ρ is the magnitude of the vector, θ is the (x,y) coordinates, and ϕ is the (y,z) angle. They must adhere to the following for it to be geometrically coherent:

$$\begin{cases} \rho > 0 \\ 0 \leq \phi \leq \pi \\ 0 \leq \theta \leq 2\pi \end{cases} \tag{3.6}$$

this tuple can generate two vectors:

$$\begin{cases} \rho\sin\phi \\ \rho \end{cases} \tag{3.7}$$

And can be converted to a cartesian model as such:

$$\vec{(}\alpha) = \begin{cases} \alpha_x = \rho\sin\phi\cos\theta \\ \alpha_y = \rho\sin\phi\sin\theta \quad \textit{Spherical to cartesian} \\ \alpha_z = \rho\cos\phi \end{cases} \tag{3.8}$$

$$\vec{(}\alpha) = \begin{cases} \rho = \sqrt{x^2+y^2+z^2} \\ \theta = \arctan(\frac{y}{x}) \qquad\qquad \textit{Cartesian to Spherical} \\ \phi = \arccos(\frac{z}{\sqrt{x^2+y^2+z^2}}) \end{cases} \tag{3.9}$$

These are, of course, pretty useful to visualize spherical objects, but conic objects can also be easily expressed with this reference system.

3.2 Multiple Variable Functions

For this course, there are moments in which we might manage more than a single variable on a single mathematical expression. We can imagine that there are functions can be defined as:

$$f(x) \underbrace{U}_{Domain} \in \mathbb{R}^n \to \mathbb{R}^n \tag{3.10}$$

This basically means, they're not nescessarily defined in an entire dimension, but on a part of it. Examples for this sort of function in $\mathbb{R}^2$ include:

$$f(x,y) = x + y \tag{3.11}$$

$$f(x,y) = x^2 + y^2 \tag{3.12}$$

$$f(x,y) = e^{x^2 y + y} \tag{3.13}$$

We're going to like $\mathbb{R}^2$ expressions on this course because they allow us to think of those functions in terms of a graphical representation on $\mathbb{R}^3$. z will depend on both variables. We write this formally as:

$$\{(x,y,z) \in \mathbb{R}^3 | z = f(x,y)\} \tag{3.14}$$

Example

3.3 Surface Curves

A surface curve, as defined for $\mathbb{R}^3$, can be defined as