Calculus III

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First release, 2024

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1. Introduction

This is the persons i took the course with, they speak mostly spanish but they might help you!

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2. Linear Algebra Fundamentals

In order to understand the concepts present in this module of calculus, a few preliminary concepts in the realm of Linear Algebra are necessary, in order to not leave anybody lost, we'll be reviewing those topics.

2.1 Lines

A line is simply a mathematical object of the form $y = mx + b$ that unites two points in the space we're using, through a straight path, where and b is the cutting point in x=0. and m is tan α, or the slope of this line. We can imagine it as:

$$m = \frac{y_1 - y_0}{x_1 - x_0} \tag{2.1}$$

$$\tag{2.2}$$

another equation of this object is, when working on $\mathbb{R}^2$ is defined as:

$$ax + by = c \tag{2.3}$$

The reason this sort of expression is so useful, is because this **is a general form.** In other words, we can define with this system every single possible line in $\mathbb{R}^{\nvDash}$. For example, let's imagine a line that is perfectly vertical. Such as it can be explained, intuitively, as $x = 1$:

This vector can't be expressed through formula 1,1 or y = mx + b, because it wouldn't cross the '0' axis, and it would have an infinite slope. But if we arrange it through (1,2), we can say:

$$ax + by = c \tag{2.4}$$

$$by = c - ax \tag{2.5}$$

$$y = \frac{c}{b} + \frac{a}{b}x \tag{2.6}$$

We can suppouse $\frac{c}{b}$ as the cutting point with y, and then imagine $\frac{a}{b}$ as 'm', we can then, use an example where we set 'b' to be 0 and define a situation where x = 1.

2.1.1 Lines in $\mathbb{R}^n$

We can define lines in more dimensions than $\mathbb{R}^2$ through different forms. For example, we can imagine it as a parametric equation of the form:

$$p = t\vec{v} \tag{2.7}$$

Example 2.1.1

Find the parametric equation that passes through p = (1,2,3) and is parallel to the vector (1,0,-1)

Solution

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{2.8}$$

With this form, we can imagine that the solution actually is producing 3 different equations, and every single one defines how this object will behave in a different dimension.

therefore:

$$x(t) = 1 + t \tag{2.9}$$
$$y(t) = 2 \tag{2.10}$$
$$z(t) = 3 - t \tag{2.11}$$

Pretty neat, huh?

We can define such an equation in 'n' dimensions that can define a line going through be it a plane or a hyperplane like this. We only need two vectors of the same 'n' dimension. So, taking this form, can we express it in other ways? Well, yeah! ...and we just did. The before introduced equations can be called **the parametric equation of a line.** the equation we generated

From here, we can also imagine that instead of using such a form, we can also equate everything to 't' and from here, we'll start defining a **Symetrical form of the line.**

In the previous example, we can imagine:

$$x - 1 = y - 2 = \frac{z - 3}{-1} = t \tag{2.12}$$

2.2 Planes

The plane can be defined as:

$$ax + by + cz = d \tag{2.13}$$

This plane can be defined with a point and a vector, for example:

Example 2.2.1
given $\vec{PQ} = (x,y,z)$ and $\vec{R} = (a,b,c)$, define a plane.
Solution

$$\begin{pmatrix} x-x_0 \\ y=y_0 \\ z-z_0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{2.14}$$

$$a(x-x_0)+b(y-y_0)+c(z-z_0) = 0 \tag{2.15}$$

$$ax-ax_0+by-by_0+cz-cz_0 = 0 \tag{2.16}$$

$$ax+by+cz = ax_0+by_0+cz_0 \tag{2.17}$$

$$< d = ax_0+by_0+cz_0 > \tag{2.18}$$

$$ax+by+cz = d \tag{2.19}$$

And from there we can define basically every plane in $\mathbb{R}^3$, as you can see, we arrived to the way we defined a plane in this dimension, and we could in theory, decide in a random point, a random vector, and start working from there into defining a plane. But now, can we define a parametric equation of a plane, as we did with a line?

Well, the answer is yes.

the parametric form of a plane is actually pretty simple, as it is supremely similar to the one that defines a line, and is defined as:

$$p+s\vec{v_1}+t\vec{v_2} \tag{2.20}$$

if we generalize this into a system where R = (1,2,3), v_1 = (1,0,1) and v_2 = (1,1,0) we can do a little example, written as:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{2.21}$$

Plane from 3 points in the space

2.3 Vectors

2.3.1 Important operations

Addition and Subtraction

Not much mystery to it, you can add and subtract the components in two different vectors:

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix}$$

Dot Product

We can get a scalar product from the multiplication of the elements two vectors have

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1x_2 + y_1y_2 + z_1z_2$$

This can also be written as:

$$< \vec{u},\vec{v} > \tag{2.22}$$

Properties

- Dot product is commutative: $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
- $\vec{u}\cdot\vec{v} = 0 \impliedby \implies \vec{u}\perp\vec{v}$
- $\vec{u}\cdot\vec{v} = ||\vec{u}||||\vec{v}||\cos\theta$
- Cauchy Schwartz inequality: $|\vec{u}\cdot\vec{v}| \leq ||\vec{u}||||\vec{v}||$
- $\vec{u}\cdot\vec{u} = ||u||^2$

Cross product

Another form to multiply vectors, while getting another vector, is the cross prudict which we define as:

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} x \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_1z_2 - z_1y_2 \\ -(x_1z_2 - z_1x_2) \\ x_1y_2 - y_1x_2 \end{pmatrix}$$

This specific operation cannot be done in dimensions that exceed $\mathbb{R}^3$ and can't be considered formally the multiplication of two vectors when the system is managed in $\mathbb{R}^n|n > 4$. This course, however, is mostly managed for systems at most in 3 dimensions, hyperplanes will probably be tangentially asked about depending on the professor, but i wouldn't count on it.

Properties

Cross product operations have a few specific things that can happen.

- $$\vec{u}x\vec{v} = -\vec{v}x\vec{u}$$
- $$\vec{u}x\vec{v} \perp \vec{u} \wedge \vec{u}x\vec{v} \perp \vec{v}$$
- $$\vec{v}x\vec{u} = 0 \implies \vec{u}//\vec{v}$$
- $$||\vec{u}x\vec{v}|| = Surface$$

Example

Calculate the plane equation that includes the points:

$$\vec{P} = (2,0,0) \tag{2.23}$$
$$\vec{Q} = (0,1,0) \tag{2.24}$$
$$\vec{R} = (0,0,3) \tag{2.25}$$

Solution

First of all, we'll do a normal vector via the cross product. First, we'll need two vectors $\vec{PQ}$ and $\vec{PR}$ via vector subtraction:

$$\vec{PR} = \begin{pmatrix}0\\0\\3\end{pmatrix} - \begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}-2\\0\\3\end{pmatrix} \tag{2.26}$$
$$\vec{PQ} = \begin{pmatrix}0\\1\\0\end{pmatrix} - \begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}-2\\1\\0\end{pmatrix} \tag{2.27}$$

And now, we can generate a cross product for these two mathematical objects, this should give us a vector normal to both:

$$\vec{PQ}x\vec{PR} = \begin{pmatrix}-2\\1\\0\end{pmatrix} x \begin{pmatrix}-2\\0\\3\end{pmatrix} = \tag{2.28}$$

Example 2

Find the Area of the triangle generated by the points:

$$\vec{P} = (2,0,0) \tag{2.29}$$
$$\vec{Q} = (0,1,0) \tag{2.30}$$
$$\vec{R} = (0,0,3) \tag{2.31}$$

For this, we can use the property of cross products being able to calculate a surface area to assume the way we can calculate a triangle from it:

$$A_{\triangle} = \frac{||\vec{p}qx\vec{p}r||}{2} \tag{2.32}$$

As given by our last example, then:

$$A_{\triangle} = \frac{||(3,6,2)||}{2} = \frac{\sqrt{3^2+6^2+2^2}}{2} = \frac{\sqrt{9+36+4}}{2} = \frac{7}{2} \tag{2.33}$$

Projection

A proyection is the vector that can be found from making

$$Proy_{\vec{u}}(\vec{v}) = \alpha\vec{u} \tag{2.34}$$

This has the following properties:

- $\vec{v} - \alpha\vec{u} \perp \vec{u}$
- $\frac{\vec{u}\cdot\vec{v}}{||\vec{u}||^2} = \alpha$

We can also imagine the projection as:

$$Proy_{\vec{u}}(\vec{v}) = (\frac{\vec{u}\cdot\vec{v}}{||\vec{u}||^2})\vec{u} \tag{2.35}$$

Example

Given:

$$\vec{u} = x + y = 3 \tag{2.36}$$
$$\vec{v} = x + y = 1 \tag{2.37}$$

What is the distance between both lines?

Solution

For this we can check for the smallest distance between points, and since they are perpendicular to each other:

$$\min d(\vec{v}, \vec{u}) = d(\vec{v}, \vec{u}) \tag{2.38}$$

lets instantiate the lines in a moment in time where they are both crossing the x axis:

$$P = (0,1) \tag{2.39}$$
$$Q = (0,3) \tag{2.40}$$

2.4 Properties

- two vectors are parallel if one is a multiple of the other, such as:

$$\vec{v} = d\vec{w}; d \in \mathbb{R}$$

- The angle between two vectors can be defined as such:

$$\vec{v}\cdot\vec{w} = ||\vec{v}||||\vec{w}||\cos\theta$$

  If this operation is 0, then the two vectors are orthogonal (parallel) to each other.
- The cross product of two vectors is perpendicular to both vectors that generated it.

3. Non-Linear Fundamentals

3.1 Polar and Cylindrical Coordinates

During the course, we'll be managing non-linear systems and such systems will require managing reference systems that are non-cartesian. This is basically

3.1.1 Polar Coordinates

These can make circles easier to describe.