Calculus III

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Contents

1. Introduction

This is the persons i took the course with, they speak mostly spanish but they might help you!

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2. Linear Algebra Fundamentals

In order to understand the concepts present in this module of calculus, a few preliminary concepts in the realm of Linear Algebra are necessary, in order to not leave anybody lost, we'll be reviewing those topics.

2.1 Vectors

2.1.1 Important operations

Addition and Subtraction

Not much mystery to it, you can add and subtract the components in two different vectors:

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

Dot Product

We can get a scalar product from the multiplication of the elements two vectors have

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1x_2 + y_1y_2 + z_1z_2$$

This can also be written as:

$$< \vec{u}, \vec{v} > \tag{2.1}$$

Properties

- Dot product is commutative: $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- $\vec{u} \cdot \vec{v} = 0 \Longleftarrow \Longrightarrow \vec{u} \perp \vec{v}$

- $\vec{u} \cdot \vec{v} = ||\vec{u}||||\vec{v}|| \cos\theta$
- Cauchy Schwartz inequality: $|\vec{u} \cdot \vec{v}| \leq ||\vec{u}||||\vec{v}||$
- $\vec{u} \cdot \vec{u} = ||u||^2$

Cross product

Another form to multiply vectors, while getting another vector, is the cross prudict which we define as:

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} x \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_1z_2 - z_1y_2 \\ -(x_1z_2 - z_1x_2) \\ x_1y_2 - y_1x_2 \end{pmatrix}$$

This specific operation cannot be done in dimensions that exceed $\mathbb{R}^3$ and can't be considered formally the multiplication of two vectors when the system is managed in $\mathbb{R}^n | n > 4$. This course, however, is mostly managed for systems at most in 3 dimensions, hyperplanes will probably be tangentially asked about depending on the professor, but i wouldn't count on it.

Properties

Cross product operations have a few specific things that can happen.

- $$\vec{u}x\vec{v} = -\vec{v}x\vec{u}$$
- $$\vec{u}x\vec{v} \perp \vec{u} \wedge \vec{u}x\vec{v} \perp \vec{v}$$
- $$\vec{v}x\vec{u} = 0 \implies \vec{u}//\vec{v}$$
- $$||\vec{u}x\vec{v}|| = Surface$$

Example

Calculate the plane equation that includes the points:

$$\vec{P} = (2,0,0) \tag{2.2}$$
$$\vec{Q} = (0,1,0) \tag{2.3}$$
$$\vec{R} = (0,0,3) \tag{2.4}$$

Solution

First of all, we'll do a normal vector via the cross product. First, we'll need two vectors $\vec{PQ}$ and $\vec{PR}$ via vector subtraction:

$$\vec{PR} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} \tag{2.5}$$
$$\vec{PQ} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \tag{2.6}$$

And now, we can generate a cross product for these two mathematical objects, this should give us a vector normal to both:

$$\vec{PQ}x\vec{PR} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} x \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} = \tag{2.7}$$

Example 2
Find the Area of the triangle generated by the points:

$$\vec{P} = (2,0,0) \tag{2.8}$$
$$\vec{Q} = (0,1,0) \tag{2.9}$$
$$\vec{R} = (0,0,3) \tag{2.10}$$

For this, we can use the property of cross products being able to calculate a surface area to assume the way we can calculate a triangle from it:

$$A_{\triangle} = \frac{||\vec{p}qx\vec{p}r||}{2} \tag{2.11}$$

As given by our last example, then:

$$A_{\triangle} = \frac{||(3,6,2)||}{2} = \frac{\sqrt{3^2+6^2+2^2}}{2} = \frac{\sqrt{9+36+4}}{2} = \frac{7}{2} \tag{2.12}$$

Projection
A proyection is the vector that can be found from making

$$Proy_{\vec{u}}(\vec{v}) = \alpha\vec{u} \tag{2.13}$$

This has the following properties:
- $\vec{v} - \alpha\vec{u} \perp \vec{u}$
- $\frac{\vec{u}\cdot\vec{v}}{||\vec{u}||^2} = \alpha$

We can also imagine the projection as:

$$Proy_{\vec{u}}(\vec{v}) = (\frac{\vec{u}\cdot\vec{v}}{||\vec{u}||^2})\vec{u} \tag{2.14}$$

Example
Given:

$$\vec{u} = x+y=3 \tag{2.15}$$
$$\vec{v} = x+y=1 \tag{2.16}$$

What is the distance between both lines?

Solution

For this we can check for the smallest distance between points, and since they are perpendicular to each other:

$$\min d(\vec{v},\vec{u}) = d(\vec{v},\vec{u}) \tag{2.17}$$

lets instantiate the lines in a moment in time where they are both crossing the x axis:

$$P = (0,1) \tag{2.18}$$
$$Q = (0,3) \tag{2.19}$$

2.1.2 Properties

- two vectors are parallel if one is a multiple of the other, such as:

$$\vec{v} = d\vec{w}; d \in \mathbb{R}$$

- The angle between two vectors can be defined as such:

$$\vec{v}\cdot\vec{w} = ||\vec{v}||||\vec{w}||\cos\theta$$

  If this operation is 0, then the two vectors are orthogonal (parallel) to each other.
- The cross product of two vectors is perpendicular to both vectors that generated it.

2.2 Lines

A line is simply a mathematical object of the form $y = mx + b$ that unites two points in the space we're using, through a straight path, where and b is the cutting point in x=0. and m is $\tan\alpha$, or the slope of this line. We can imagine it as:

$$m = \frac{y_1 - y_0}{x_1 - x_0} \tag{2.20}$$

$$\tag{2.21}$$

another equation of this object is, when working on $\mathbb{R}^2$ is defined as:

$$ax + by = c \tag{2.22}$$

The reason this sort of expression is so useful, is because this **is a general form.** In other words, we can define with this system every single possible line in $\mathbb{R}^{\nvDash}$. For example, let's imagine a line that is perfectly vertical. Such as it can be explained, intuitively, as $x = 1$:

This vector can't be expressed through formula 1,1 or y = mx + b, because it wouldn't cross the '0' axis, and it would have an infinite slope. But if we arrange it through (1,2), we can say:

$$ax + by = c \tag{2.23}$$
$$by = c - ax \tag{2.24}$$
$$y = \frac{c}{b} + \frac{a}{b}x \tag{2.25}$$

We can suppouse $\frac{c}{b}$ as the cutting point with y, and then imagine $\frac{a}{b}$ as 'm', we can then, use an example where we set 'b' to be 0 and define a situation where x = 1.

2.2.1 Lines in $\mathbb{R}^n$

We can define lines in more dimensions than $\mathbb{R}^2$ through different forms. For example, we can imagine it as a parametric equation of the form:

$$p = t\vec{v} \tag{2.26}$$

Example 2.2.1

Find the parametric equation that passes through p = (1,2,3) and is parallel to the vector (1,0,-1)

Solution

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{2.27}$$

With this form, we can imagine that the solution actually is producing 3 different equations, and every single one defines how this object will behave in a different dimension.

therefore:

$$x(t) = 1 + t \tag{2.28}$$
$$y(t) = 2 \tag{2.29}$$
$$z(t) = 3 - t \tag{2.30}$$

Pretty neat, huh?

We can define such an equation in 'n' dimensions that can define a line going through be it a plane or a hyperplane like this. We only need two vectors of the same 'n' dimension. So, taking this form, can we express it in other ways? Well, yeah! ...and we just did. The before introduced equations can be called **the parametric equation of a line.** the equation we generated

From here, we can also imagine that instead of using such a form, we can also equate everything to 't' and from here, we'll start defining a **Symetrical form of the line.**

In the previous example, we can imagine:

$$x - 1 = y - 2 = \frac{z - 3}{-1} = t \tag{2.31}$$

2.3 Planes

The plane can be defined as:

$$ax + by + cz = d \tag{2.32}$$

This plane can be defined with a point and a vector, for example:

Example 2.3.1
given $\vec{PQ} = (x, y, z)$ and $\vec{R} = (a, b, c)$, define a plane.
Solution

$$\begin{pmatrix} x - x_0 \\ y = y_0 \\ z - z_0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{2.33}$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{2.34}$$

$$ax - ax_0 + by - by_0 + cz - cz_0 = 0 \tag{2.35}$$

$$ax + by + cz = ax_0 + by_0 + cz_0 \tag{2.36}$$

$$< d = ax_0 + by_0 + cz_0 > \tag{2.37}$$

$$ax + by + cz = d \tag{2.38}$$

And from there we can define basically every plane in $\mathbb{R}^3$, as you can see, we arrived to the way we defined a plane in this dimension, and we could in theory, decide in a random point, a random vector, and start working from there into defining a plane. But now, can we define a parametric equation of a plane, as we did with a line?

Well, the answer is yes.

the parametric form of a plane is actually pretty simple, as it is supremely similar to the one that defines a line, and is defined as:

$$p + s\vec{v_1} + t\vec{v_2} \tag{2.39}$$

if we generalize this into a system where R = (1,2,3), v_1 = (1,0,1) and v_2 = (1,1,0) we can do a little example, written as:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{2.40}$$

Example 2.3.2
Determine if the line

$$x = 9 + 5t \tag{2.41}$$

$$y = -3 + t \tag{2.42}$$

$$z = 4t \tag{2.43}$$

Intersects with the plane

$$3x + 10y - 2z + 53 \tag{2.44}$$

If so, determine where.

Solution

We can begin by replacing. Remember the line's equation and how it has x, y and z values. **These values are equivalent to x, y and z in a plane**, and how you can replace it in the plane's equation.

If we can solve the equation for 't', that means we can intersect that line with the plane and that it will intersect in that specific 't' point

$$3(9+5t) + 10(-3+t) - 2(4t) = 53 \tag{2.45}$$
$$27 + 15t - 30 + 10t - 8t = 53 \tag{2.46}$$
$$t = \frac{56}{16} \tag{2.47}$$

Plane from 3 points in the space

Example 2.3.3

Find the line normal to the plane that crosses the points:

$$P = (0, -3, -3) \tag{2.48}$$
$$Q = (-3, 0, -3) \tag{2.49}$$
$$R = (-3, 3, 0) \tag{2.50}$$

Solution We'll begin by getting two vectors from these lines, we can get them by subtracting one point from the other:

$$\vec{PQ} = (-3, 3, 0) \tag{2.51}$$
$$\vec{QR} = (0, -3, 3) \tag{2.52}$$

And now, we'll make a cross product between the two, so we can get a normal vector:

$$\vec{PQ}x\vec{QR} = \begin{pmatrix} -3 \\ 3 \\ 0 \end{pmatrix} x \begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \\ 9 \end{pmatrix} = \vec{n} \tag{2.53}$$

Now, we can get from that normal vector, the direction vector for the line:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix} + t \begin{pmatrix} 9 \\ 9 \\ 9 \end{pmatrix} \tag{2.54}$$

3. Non-Linear Fundamentals

3.1 Polar and Cylindrical Coordinates

During the course, we'll be managing non-linear systems and such systems will require managing reference systems that are non-cartesian. When trying to define parts of a line in algebra, we'll usually be looking at coordinates, be them polar or cartesian. In either case, their information can be converted to the other system through the following formulas.

$$\begin{cases} \rho = \sqrt{x^2 + y^2} \\ \theta = \arctan(\frac{y}{x}) \end{cases} \quad \textit{cartesian to polar} \tag{3.1}$$

$$\begin{cases} \alpha_x = \rho \cos\theta \\ \alpha_y = \rho \sin\theta \end{cases} \quad \textit{polar to cartesian} \tag{3.2}$$

Cartesian coordinates generally translate well to other dimensional spaces, such as would be the case for $\mathbb{R}^3$, however, polar coordinates as we know them usually aren't as translatable in a direct manner, and expressing them in three-dimensional spaces might be better suited to be expressed on a cylindrical or spherical condition.

3.1.1 Cylindrical coordinates

In the case of cylindrical coordinates, the translation is probably the most intuitive, by computing a cylinder with polar coordinates that indicate an (x,y) position, and a 'Z' variable indicating height that allows us to project the vector on a third dimension, this 'z' variable is exactly the same as it would be on a cartesian model. We can express it like such:

$\vec{v} = (\rho, \theta, Z)$

conversion to a cartesian model can be expressed as:

$$\vec{(}\alpha) = \begin{cases} \alpha_x = \rho\cos\theta \\ \alpha_y = \rho\sin\theta & \textit{Cylindrical to cartesian} \\ \alpha_z = Z \end{cases} \tag{3.3}$$

$$\vec{(}\alpha) = \begin{cases} \rho = \sqrt{x^2+y^2} \\ \theta = \arctan(\frac{y}{x}) & \textit{Cartesian to Cylindrical} \\ Z = Z \end{cases} \tag{3.4}$$

These coordinates might be useful to visualize circular functions, such as curves that take a cyllindrical, spring-like form.

3.1.2 Spherical coordinates

A spherical coordinate is formed by a tuple:

$$(\rho,\theta,\phi); \begin{cases} \rho \geq 0 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi \end{cases} \tag{3.5}$$

Where ρ is the magnitude of the vector, θ is the (x,y) coordinates, and ϕ is the (y,z) angle. They must adhere to the following for it to be geometrically coherent:

$$\begin{cases} \rho > 0 \\ 0 \leq \phi \leq \pi \\ 0 \leq \theta \leq 2\pi \end{cases} \tag{3.6}$$

this tuple can generate two vectors:

$$\begin{cases} \rho\sin\phi \\ \rho \end{cases} \tag{3.7}$$

And can be converted to a cartesian model as such:

$$\vec{(}\alpha) = \begin{cases} \alpha_x = \rho\sin\phi\cos\theta \\ \alpha_y = \rho\sin\phi\sin\theta & \textit{Spherical to cartesian} \\ \alpha_z = \rho\cos\phi \end{cases} \tag{3.8}$$

$$\vec{(}\alpha) = \begin{cases} \rho = \sqrt{x^2+y^2+z^2} \\ \theta = \arctan(\frac{y}{x}) & \textit{Cartesian to Spherical} \\ \phi = \arccos(\frac{z}{\sqrt{x^2+y^2+z^2}}) \end{cases} \tag{3.9}$$

These are, of course, pretty useful to visualize spherical objects, but conic objects can also be easily expressed with this reference system.

3.2 Scalar Functions

For this course, there are moments in which we might manage more than a single variable on a single mathematical expression. We can imagine that there are functions can be defined as:

$$f(x_1,\ldots,x_n)\underbrace{U}_{Domain} : \mathbb{R}^n \to \mathbb{R} \tag{3.10}$$

This basically means, they project from a specific dimension to a point in $\mathbb{R}$.
Examples for this sort of function in $\mathbb{R}^2$ include:

$$f(x,y) = x+y \tag{3.11}$$

$$f(x,y) = x^2+y^2 \tag{3.12}$$

$$f(x,y) = e^{x^2y+y} \tag{3.13}$$

We're going to like $\mathbb{R}^2$ expressions on this course because they allow us to think of those functions in terms of a graphical representation on $\mathbb{R}^3$. z will depend on both variables. We write this formally as:

$$\{(x,y,z) \in \mathbb{R}^3 | z = f(x,y)\} \tag{3.14}$$

3.3 Surface Curves

A surface curve, as defined for $\mathbb{R}^3$, can be defined as sets of the form:

$$\{(x,y) \in \mathbb{R}^2 | f(x,y) = constant\} \tag{3.15}$$

A surface curve can be defined with the value 'k' as:

$$\{(x,y) \in \mathbb{R}^2 | f(x,y) = k\} \tag{3.16}$$

generally, we can say for the definition of these level curves:
- If k < 0, there's no solution
- If k = 0, then it's a point
- If k>0, it's an elyptic form

Example 3.3.1

Find the curves of $f(x,y) = 1-x-y$

Solution

$$1-k = x+y \text{ Lines in } \mathbb{R}^{\nvDash} \tag{3.17}$$

Intersection of a Surface with a plane

We can make a surface intersect with a plane by replacing ont of the variables for the plane with

Example

for $f(x,y) = x^2 + y^2$ make an intersection with $x^2 + y^2 = k$

Solution

For this, we can think on the function as circles centered on the origin, whose radius will be 'k'. Now, we can decide that 'x = 0' and we can try and check for values on which this is true for both expressions. This

3.4 Limits and Continuity

3.4.1 Limits

For this course, limits will be a fairly important mathematical operation. We can imagine that limits can be described as:

$$\lim_{x \to x_0} = L \implies \forall_{\varepsilon>0} \exists \delta > 0 \; if \; 0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon \tag{3.18}$$

This means that for every number $\varepsilon > 0$, there exists a $\delta > 0$. this will allow us to define ranges on which a function exists. epsilon and delta are but mathematical definitions of vertical and horizontal values for us to make a limit valid or invalid.

this definition is valid for limits with a single component, for multiple variables, we can assume:

$$\lim_{x \to x_0} = L \implies \forall_{\varepsilon>0} \exists \delta > 0 \; if \; 0 < ||(x,y) - (x_0, y_0)|| < \delta \implies |f(x,y) - L| < \varepsilon \tag{3.19}$$

There are cases in which a limit exists and some others for which it doesn't.

$$\lim f(x,y) \to L \implies (x,y) \to (a,b)$$

A limit doesn't exist if there is two ways of approaching the same point for which the value of the limit is different. This can be seen when

Example

Solve:

$$\frac{xy}{x^2 + y^2}$$

Solution

If we try to separate the values, we might get this

$$\lim_{x \to 0} \frac{x \cdot 0}{x^2 + 0^2} = \lim_{x \to x_0} \frac{0}{x^2} = 0 \tag{3.20}$$

$$\lim_{y \to 0} \frac{y \cdot 0}{0^2 + y^2} = \lim_{y \to y_0} \frac{0}{y^2} = 0 \tag{3.21}$$

It would appear as if the limit existed, but if we take them as a line of the form x=y, then we might get:

$$\lim_{x\to 0}\frac{x\cdot x}{x^2+x^2}=\lim_{x\to 0}\frac{x^2}{2x^2}=\frac{1}{2} \tag{3.22}$$

This doesn't exist, because 0 is different to $\frac{1}{2}$

Given the last example, how can we approach a function on infinite ways? for this, we can imagine a line $y = mx$ that can generalize all the possible lines in the plane. However, this can also be done as explained in section 3.4.1, where we can establish a conversion to polar coordinates that will help our calculations.

Example

Determine if the following limit exists:

$$\lim_{(x,y)\to(0,0)}\frac{y^4}{x^4+3y^4}$$

Solution

$$< Y\ AXIS > \tag{3.23}$$

$$\lim_{y\to 0}\frac{y^4}{3y^4} \tag{3.24}$$

$$\lim_{y\to 0}\frac{1}{3} \tag{3.25}$$

So the limit approaches $\frac{1}{3}$ when the value approaches zero

$$< X\ AXIS > \tag{3.26}$$

$$\lim_{x\to 0}\frac{0}{x^4}=\lim_{x\to 0}0=0 \tag{3.27}$$

You can approach the point from two different places when doing these limits, Therefore the limit DOES NOT EXIST. The limit is going to approach two different points

3.4.2 Polar Coordinates for limits

You can use polar coordinates for making a limit in two variables into one that only uses one. so for example, you could pass a limit as such:

$$\lim_{(x,y)\to(0,0)}F(x,y)=\lim_{\rho\to 0}f(\rho\cos\theta,\rho\sin\phi) \tag{3.28}$$

Then, we can evaluate rho as a single coordinate. This will also allow us to say that a limit exists with certainty.

Example
Determine if the following limit exists:

$$\lim_{(x,y)\to(0,0)} \frac{e^{x^2+y^2}-1}{x^2+y^2}$$

Solution

it doesn't exist, for it doesn't tend to the values we expected. This is a process that would've been a lot more tedious if we did the separation that we applied for our previous example.

Example
evaluate the following limit in its polar form:

$$\frac{x^2}{x^2+y^2}$$

Solution

$$\lim_{r\to 0} \frac{(r^2\cos^2\theta)}{r^2} \tag{3.29}$$
$$\lim_{r\to 0} = \cos^2\theta \tag{3.30}$$

Since it depends on θ, then the limit does not exist. This is because theta might vary to values different from what we want.

The limits of polar coordinates as a method
It is possible that some limits, especially but not uniquely those that have targets different to (0,0), might break when trying to evaluate them through polar coordinates. A limit might break under a bunch of circumstances.

3.4.3 Continuity
a function $\mathbb{R} \to \mathbb{R}$ is continuous if:
- f is defined in x_0
- $\lim_{x\to x_0} f(x)$ exists
- $f(x_0) = \lim_{x\to x_0} f(x)$

And, on two variables this can be seen as:
- f is defined in (x_0, y_0)
- $\lim_{(x,y)\to(X_0,y_0)} f(x,y)$ exists
- $f(x_0,y_0) = \lim_{(x,y)\to(x_0,y_0)} f(x,y)$

You can generalize it to $\mathbb{R}^n$

4. Derivation/Differentiation

a function in a single variable can have every single point on a curve being approximated to a line, called a tangent. This z = (x,y) line variable can be defined by partial derivatives, that can be described as:

$$z_x = \frac{D_z}{D_x} = \frac{D_f}{D_x}(x,y) = \lim_{h \to 0} = \frac{(x+h,y) - f(x,y)}{h} \tag{4.1}$$

$$\frac{d_f}{d_y}(x,y) = \lim_{h \to 0} = \frac{(x,y+h) - f(x,y)}{h} \tag{4.2}$$

Where 'h' is a real number that tends towards zero.

In a more inmediately relevant manner, we can assume an 'f' function is differentiable in a point x_0 when:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \; exists \tag{4.3}$$

this can also be written as:

$$\lim_{h \to 0} \frac{f(x) - f(x_0)}{h} \; exists \tag{4.4}$$

this means that the existence of a limit helps us check for differentiability.

4.1 Partial Derivatives

let:

$$f(x,y)\mathbb{R}^2 \to \mathbb{R} \tag{4.5}$$

Then we can imagine a derivation in respect of a single term, this will mean we'll take any other value as a sort of constant, that we will end up treating as such.

We can notate a derivation of this kind as

$$\frac{\partial f}{\partial x}$$

Example

Generate the partial derivatives of the following function:

$$f(x,y) = \frac{e^{y^2}}{x^2+y^2}$$

4.2 Tangent Planes

The plane tangent to the graph of a function f(x,y) in a point (x_0, y_0) follows the following equation:

$$z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)[x-x_0] + \frac{\partial f}{\partial y}(x_0,y_0)[y-y_0]$$

Example

Find the tangent plane to the graph

$$f(x,y) = \frac{e^{y^2}}{x^2+y^2}$$

in the point (2,1)

Solution

From our example of partial derivatives, we can say:

$$f(2,1) = \frac{e}{5} \tag{4.6}$$

$$\frac{\partial f}{\partial x}(2,1) = \frac{-4e}{25} \tag{4.7}$$

$$\frac{\partial f}{\partial y}(2,1) = \frac{8e}{25} \tag{4.8}$$

Therefore, we can write this plane as:

$$z = \frac{e}{5} + \frac{-4e}{25}(x-2) + \frac{8e}{25}(y-1) \tag{4.9}$$

once expanded, it will translate to a plane when graphed.

Note
The normal vector to a tangent plane is

$$\vec{n} = (\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0), -1)$$

In a similar fashion, the normal vector to a surface in a point is basically the same, defined as:

$$(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0), -1)$$

4.3 Gradients

given

$$f(x,y) : \mathbb{R}^2 \to \mathbb{R}$$

the gradient of f(x,y) is defined as:

$$\nabla f(x,y) = (\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y))$$

and we have to take into account that:

$$\nabla f(x,y) : \mathbb{R}^2 \to \mathbb{R}$$

given all of this, if f(x,y) is differentiable in (x_0, y_0), then we can assume:

$$f(x,y); f(x_0,y_0) + \nabla f(x_0,y_0) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} \tag{4.10}$$

this is an approximation for most situations where this happens, however this can also be written in a more precise way if we assume it as:

If f is differentiable in (x_0, y_0) and

$$\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0)$$

exist. Then

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - (f(x_0,y_0) + \nabla f(x_0,y_0) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix})}{||(x,y) - (x_0,y_0)||} \tag{4.11}$$

4.3.1 Chain Rule

When we have a derivative that is composed of two functions, we can then calculate the derivative such as:

First chain rule case

On our first case, given a composed function, we can say:

$$F(g(x))' = f'(g(x)) \cdot g'(x) \tag{4.12}$$

$$< \text{Also written as: } > \tag{4.13}$$

$$\frac{d(f^o g)}{dx} = \frac{df}{dx}(g(x)) \cdot \frac{dg}{dx} \tag{4.14}$$

we can write also write the chain rule as:

$$\frac{d(f(g(t)))}{dt} = \nabla f(g(t)) \cdot g'(t) \tag{4.15}$$

$$g : \mathbb{R}^2 \to \mathbb{R} \tag{4.16}$$

$$f : \mathbb{R}^2 \to \mathbb{R} \tag{4.17}$$

$$f^o g\mathbb{R} \to \mathbb{R} \tag{4.18}$$

Second chain rule case

given

$$f : \mathbb{R}^2 \to \mathbb{R} | g : \mathbb{R}^2 \to \mathbb{R}$$

we might have a situation where these two functions depend on two other variables u,v. Then:

$$\frac{\partial f(x(u,v),y(u,v))}{\partial u} = \frac{\partial f}{\partial x}(f(x(u,v),y(u,v))) \cdot \frac{\partial f}{\partial u}(u,v) + \frac{\partial f}{\partial y}(f(x(u,v),y(u,v))) \cdot \frac{\partial f}{\partial u}(u,v) \tag{4.19}$$

A consequence of this system is that the gradient of a function is perpendicular to a level curve of the shape we're analyzing. This can be written as:

$$\nabla f(x,y) \perp curve f(x_0,y_0) \tag{4.20}$$

Example 4,3,1

Calculate a line normal to $x^2 + 2y^2 = 3$ in (1,1)

solution

With the gradient of the function, we can then assume it will work like this, given we're assuming that the point they gave us is in the shape:

$$f(x,y) = x^2 + 2y^2 \tag{4.21}$$

$$\nabla f(x,y) = (2x, 4y) \tag{4.22}$$

$$\nabla f(1,1) = (2,4) \tag{4.23}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 4 \end{pmatrix} \tag{4.24}$$

Example 4,3,2
Find the normal line and the tangent plane to:

$$x^2 + y^2 + 2z^2 = 4$$

in (1,1,1)

solution

For the tangent plane:

$$f(x,y,z) = x^2 + y^2 + 2z^2 \tag{4.25}$$
$$\nabla f(x,y,z) = (2x, 2y, 4z) \tag{4.26}$$
$$\nabla f(1,1,1) = (2,2,4) \tag{4.27}$$
$$line = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} \tag{4.28}$$
$$Plane = 2(x-1) + 2(y-1) + 4(z-1) = 0 \tag{4.29}$$
$$2x + 2y + 4z = 8 \tag{4.30}$$

4.4 Superior order derivatives

much like there is a way to do multiple integrals, we can do multiple partial derivatives, such as:

$$\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \tag{4.31}$$
$$f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f \tag{4.32}$$

Theorem
if f(x,y) has second derivatives and they're continuous, then:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} \tag{4.33}$$

4.5 Directional Derivatives

We can talk about a directional derivative for the function 'f' in the direction 'V' as:

$$F_{\vec{v}} f(p) = \frac{\lim_{h \to 0} f(p+hv) - f(p)}{h} \tag{4.34}$$

This is the infinitesimal exchange rate towards v of a function f in p.
Given we have the chain rule, we can rewrite this as:

$$\frac{d}{df}(f(p+tv));t=0 \tag{4.35}$$

A directional derivative is given by:

$$D_{\vec{u}}f(x_0,y_0)=\frac{d}{dt}f(x_0+tu_1,y_0+tu_2);t=0 \tag{4.36}$$

And we say that if a derivative exists, then $||\vec{u}||=1$; f depends on x and y, and x and y both depend on t, this can be seen mathematically as:

$$\frac{df}{dx}f(x_0+tu_1,y_0+tu_2)\underbrace{U_1}_{\frac{dx}{dt}}+\frac{df}{dy}(x_0+tu_1,y_0+tu_2)\underbrace{U_2}_{\frac{dy}{dt}} \tag{4.37}$$

from this we can observe

$$D_{\vec{u}}\ f(x_0,y_0)=\vec{\Delta}f(x_0,y_0)\,\boldsymbol{.}\,\vec{u}=||\vec{\Delta}f(x_0,y_0)||\,\boldsymbol{.}\,||\vec{u}||\,\boldsymbol{.}\cos\theta \tag{4.38}$$

So, given this set of conditions:

$$\begin{cases} D_{\vec{u}}f(x_0,y_0) \text{ is maximal if } \theta \text{ is } 0 \\ D_{\vec{u}}f(x_0,y_0) \text{ is minimal if } \theta \text{ is } \pi \end{cases} \tag{4.39}$$

We can infer then, regarding our direction:

$$\vec{u}=\frac{\vec{\Delta}f(x_0,y_0)}{||\vec{\Delta}f(x_0,y_0)||} \tag{4.40}$$

will make the function grow faster, and:

$$\vec{u}=-\frac{\vec{\Delta}f(x_0,y_0)}{||\vec{\Delta}f(x_0,y_0)||} \tag{4.41}$$

will make it decrease faster.

Gradient Theorem

If we describe formally our previous statements, we can find the gradient theorem. this can be written as:

Given

$$f(x,y,z):\mathbb{R}^3\to\mathbb{R}$$

and a point

$$P\in\mathbb{R}^3$$

, we can say:

- the fastest increasing direction of 'f' in 'P' is

$$\vec{v}=\nabla f(p)$$

- the fastest decreasing direction of 'f' in 'P' is

$$\vec{v}=-\nabla f(p)$$

Note

$$F_{\vec{v}}f(p) = \nabla f(p) \cdot \frac{\nabla f(p)}{||\nabla f(p)||} = ||\nabla f(p)|| \quad (4.42)$$

Example 1

Temperature in a room is measured through the function

$$T(x,y,z) = x^2 + 3y^2 + 2z^3$$

in the point (1,0,1), what is the fastest decreasing option?

Solution

$$\nabla T(x,y,z) = (2x, 6y, 6z^2) \quad (4.43)$$

$$\nabla T(1,0,1) = (2,0,6) \quad (4.44)$$

$$\vec{v} = -\nabla T(1,0,1) = -(2,0,6) \quad (4.45)$$

Optional, normalize the vector

$$\frac{\vec{v}}{||\vec{v}||} = (\frac{-2}{2\sqrt{10}}, 0, \frac{-6}{2\sqrt{10}}) \quad (4.46)$$

Example 1,1

A person moves with a speed of $2\frac{m}{s}$ through the room, what is the variation of temperature the person experiences?

Solution

The minimum rate is:

$$-||\nabla T(1,0,1)|| = -\sqrt{40} = -2\sqrt{10}\frac{^oC}{m} Var_{\frac{^oC}{S}} = Var_{\frac{^oC}{m}} \cdot (2\frac{m}{s}) \quad (4.47)$$

$$(-2\sqrt{10}\frac{^oC}{m}) \cdot (2\frac{m}{s}) \quad (4.48)$$

4.5.1 implicit equations

An implicit surface equation given by $F(x,y,z) = C$ we can define, as a curve:

$$\vec{r}t = < x(t), y(t), z(t) > \quad (4.49)$$

$$\vec{r}t' = < x'(t), y'(t), z'(t) > \quad (4.50)$$

And from there we can assume that this curve is a part of S if:

$$F(x(t), y(y), z(t)) = C; \forall t \quad (4.51)$$

Given this, F depends on x,y, and z and those three variables depend on t, allowing us to affirm:

$$F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_z \frac{dz}{dt} \tag{4.52}$$
$$\vec{\Delta} f \,\text{.}\, \vec{r}t' = 0 \tag{4.53}$$

geometrically this means the gradient is perpendicular to the tangent and therefore, the tangent plane. to S in (x_0, y_0, z_0), which can be expressed vectorially (and therefore, more usefully) as:

$$\vec{\Delta} F(x_0, y_0, z_0) \,\text{.}\, \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = 0 \tag{4.54}$$

The line normal to S in (x_0, y_0, z_0) can be defined as:

$$\vec{n}(t) = \vec{x}_0 + t\vec{\Delta} F(x_0, y_0, z_0) \tag{4.55}$$

$$D_{\vec{v}} f(p) = \nabla f(p) \cdot \vec{v} = ||\nabla f(p)|| \cdot ||\vec{v}|| \cos\theta \tag{4.56}$$

4.6 Theorems

Theorem 8
If f(x,y) is differentiable in (x_0, y_0) then $f(x, y)$ is continuous in (x_0, y_0)

Theorem 9
if $f(x, y)$ has continual partial derivatives in (x_0, y_0), then $f(x, y)$ is differentiable in (x_0, y_0)

5. Optimization

5.1 Global Optimization

5.2 Optimization with equality restrictions

We can restrict the range of a function to find the local minimum and maximum points in a specific set.

For example let a function we're optimizing, that we'll call an objective function:

$$x^2 + y^2 - x + y$$

And we'll have a restriction we call the factible set, such as

$$x^2 + y^2 = 1$$

.

Since $x^2 + y^2 - x + y$ is continuous in the set $x^2 + y^2 = 1$ and the set $x^2 + y^2 = 1$ is closed and bound, for the extreme value theorem the problem has a solution.

5.3 Lagrange Multipliers

If f(x,y); g(x,y) are differentiable, such as we have (x_0, y_0) that are maximum or minimum of f(x,y) when bound to g(x,y) = C, then if $\nabla g(x_0, y_0) \neq \vec{0}$, it means there exists a $\lambda \in \mathbb{R}$

We can write this as:

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \tag{5.1}$$

To make an optimization in such conditions, we must:

- Find critical points inside of our region.
- Find critical points on the border of our region.
- Apply the Lagrange Multiplier method;

■ **Example 5.1** Find maximum and minimum values in

$$f(x,y) = 1 - xy$$

On the region

$$x^2 + \frac{y^2}{4} \leq 1$$

Solution

$$f(x,y) = 1 - xy \tag{5.2}$$

$$g(x,y) = x^2 + \frac{y^2}{4} \leq 1 \tag{5.3}$$

$$\text{<both are continuous and bound; therefore>} \tag{5.4}$$

$$\tag{5.5}$$

■

■ **Example 5.2** let

$$f(x,y) = x^2 + 4x + y^2 + 9$$

bound to:

$$x^2 + y^2 = 9$$

Then:

$$\nabla f(x,y) = (2x+4, 2y) \tag{5.6}$$

$$\nabla g(x,y) = (2x, 2y) \tag{5.7}$$

$$\begin{cases} 2x + 4 = \lambda 2x \\ 2y = \lambda 2y \end{cases} \tag{5.8}$$

■

Imagine the following function:

$$z = x^2 + xy + y^2$$

with every point being part of $\mathbb{R}^2$, if we were to find local minimums and maximums, therefore:

$$f(x,y) = x^2 + xy + y^2 \tag{5.9}$$

$$\partial_x = 2x + y \tag{5.10}$$

$$\partial_y = x + 2y \tag{5.11}$$

$$\begin{cases} 2x + y = 0 \\ x + 2y = 0 \end{cases} \implies \begin{pmatrix} 2 & 1 \| & 0 \\ 1 & 2 \| & 0 \end{pmatrix} \tag{5.12}$$

$$x = 0; y = 0 \tag{5.13}$$

$$(0,0) \tag{5.14}$$

from there, we can imagine a Hessian matrix.

5.4 Implicit function theorem

Let a function 'F(x,y) = 0' that is the implicit equation of a curve in xy; then:

Example

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0 \tag{5.15}$$

$$y=\pm\sqrt{(1-\frac{x^2}{a^2})b^2}=\pm\sqrt{\frac{b^2}{a^2}(a^2-x^2)};x\exists[-a,a] \tag{5.16}$$

This is an example of a situation where fiven F(x,y)=0 and (x_0,y_0), expressed together as $F(x_0,y_0)=0$ we would like to know if we gan obtain

5.4.1 Theorem 1

Let:

- a F(x,y) defined and continuous in a rectangle centered on our point (x_0,y_0); then $D=[x_0-\triangle,x_0+\triangle;y_0-\triangle,y_0+\triangle]$
- $F(x_0,y_0)=0$
- $y\to F(x,y)$ increases (or reduces) in a strictly monotonous fashion.

Then:

- in a vecinity of the point, (*) can determine as a function of $x:y=f(x)$
- $f(x_0)=y_0$
- f is continuous

5.4.2 Theorem 2

We'll assume, besides that we assumed on our first theorem:

- $F_y=(x_0,y_0)\neq 0$
- ∂_x,∂_y exist and are continuous.

Then, including what we concluded on the first iteration, we can conclude that:

- in a vecinity of the point, (*) can determine as a function of $x:y=f(x)$
- $f(x_0)=y_0$
- f is continuous
- f'() exists and is continuous

let's remember a function is differentiable when:

$$F_x(x,y)\Delta x+F_y(x,y)\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y,\varepsilon_1,\varepsilon_2\to 0 \tag{5.17}$$

$$=\Delta x(F_x(x,y)+\varepsilon_1)+\Delta y(F_y(x,y)+\varepsilon_2) \tag{5.18}$$

$$\frac{\Delta y}{\Delta x}=-\frac{F_x+\varepsilon_1}{F_y=\varepsilon_2},\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=-\frac{F_x(x,y)}{F_y(x,y)\neq 0} \tag{5.19}$$

$$f'(x) \tag{5.20}$$

5.4.3 Theorem 3

We assume:

- $F(x_1,\ldots,x_n,y)$ is defined and continuous over $D=[\ldots]$

- $f(x_1^0,\dots,x_n^0,y_0)=0$
- $f_y \neq 0$
- $f_{x1},\dots,f_{xn},f_y$ exist and are continuous

then:

- in a vecinity $(x_1^0,\dots,x_n^0,y_0),(**)$ determines as a function