ECE 251A: Digital Signal Processing I
Wide Sense Stationary Processes II

Florian Meyer

Electrical and Computer Engineering Department
Scripps Institution of Oceanography
University of California San Diego

UC San Diego
JACOBS SCHOOL OF ENGINEERING

Power Spectrum

Power Spectral Density (PSD)

$$\begin{aligned} R_x(e^{j\omega}) &= \sum_{m=-\infty}^{\infty} r[m]e^{-j\omega m} \\ r[m] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} R_x(e^{j\omega})e^{j\omega m}d\omega \end{aligned}$$

Notation: $r[m] = r_x[m] = r_{xx}[m]$.

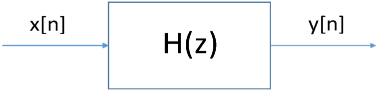
$$R_x(z) = \sum_{m=-\infty}^{\infty} r[m]z^{-m} \ \ \text{ROC} = \{|z| : r_L < |z| < r_R\}$$

The ROC includes the unit circle

Properties of PSD $R_x(e^{j\omega})$

1. PSD is a real function of ω, i.e. $R_x(e^{j\omega}) = R_x^*(e^{j\omega})$. Follows from the Hermitian symmetry of $r[m]$.
2. $R_x(e^{j\omega}) \geqslant 0 \leftrightarrow r[m]$ is a positive semidefinite sequence.
3. $r[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} R_x(e^{j\omega}) d\omega = E(|x[n]|^2) \geqslant 0$.
4. If $x[n]$ is a real RP, then $R_x(e^{j\omega}) = R_x(e^{-j\omega})$.

RP and LTI Systems



Questions of interest: Properties of $y[n]$ if $x[n]$ is WSS and joint properties of $x[n]$ and $y[n]$.

Applications

1. Spectral Shaping (Spectral Factorization): Generating a WSS process with desired autocorrelation sequence. (input a white noise sequence and choosing $H(z)$ appropriately).
2. Filtering/Prediction: Enhancing $x[n]$ in the presence of noise or predicting another RP given $x[n]$.

Example I ($a = 0.5$)

$x[n] = ax[n-1] + w[n]$, where $w[n]$ is a zero mean, variance one, i.i.d. Gaussian sequence. $H(z) = \frac{1}{1-az^{-1}}$.

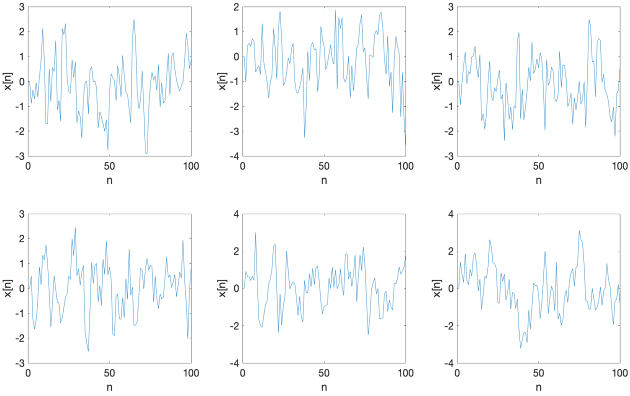


Figure: Six realizations for $a = 0.5$

Example II ($a = -0.9$)

$x[n] = ax[n-1] + w[n]$, where $w[n]$ is a zero mean, variance one, i.i.d. Gaussian sequence. $H(z) = \frac{1}{1-az^{-1}}$.

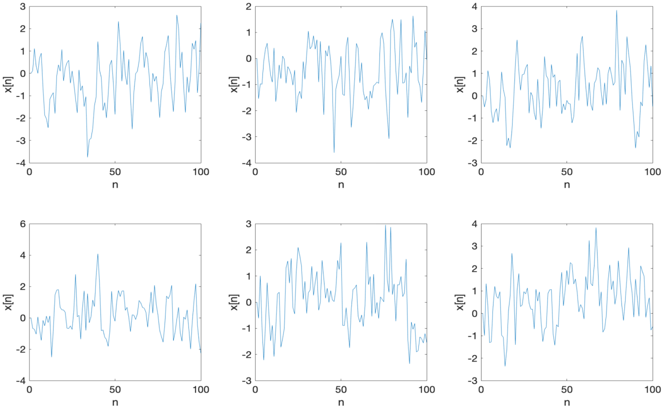


Figure: Six realizations for $a = -0.9$

WSS and LTI Systems

WSS process $x[n]$ is input to a LTI system with transfer function $H(z)$ and output is $y[n]$.

Is $y[n]$ WSS? Answer: Yes

Are $x[n]$ and $y[n]$ jointly WSS? need to check if the cross-correlation $E(x[n+m]y^*[n])$ is also a function of time difference m, i.e. $r_{xy}[m] = E(x[n+m]y^*[n])$. Answer: Yes

Check:
Mean: $E(y[n]) = E(\sum_l h[l]x[n-l]) = \sum_l h[l]E(x[n-l]) = \sum_l h[l]\mu_x = H(e^{j0})\mu_x = H(1)\mu_x$
Cross-Correlation

$$\begin{aligned} E(x[n+m]y^*[n]) &= E(x[n+m]\sum_l h^*[l]x^*[n-l]) \\ &= \sum_l h^*[l]E(x[n+m]x^*[n-l]) = \sum_l h^*[l]r_{xx}[m+l] \\ &= \sum_l h^*[-l]r_{xx}[m-l] = r_{xx}[m] * h^*[-m] \end{aligned}$$

Conclusion: $r_{xy}[m] = r_{xx}[m] * h^*[-m]$. Function of only the time difference.

WSS and LTI Cont'd

Autocorrelation of $y[n]$

$$\begin{aligned} E(y[n+m]y^*[n]) &= E(\sum_l h[l]x[n+m-l]y^*[n]) = \sum_l h[l]r_{xy}[m-l] \\ &= h[m] * r_{xy}[m] = h[m] * h^*[-m] * r_{xx}[m]. \end{aligned}$$

Conclusion: $y[n]$ is WSS with mean $\mu_y = H(1)\mu_x$ and autocorrelation sequence $r_{yy}[m] = h[m] * h^*[-m] * r_{xx}[m]$.

In the z-domain $R_{xy}(z) = \mathcal{Z}(r_{xy}[m]) = R_{xx}(z)H^*(\frac{1}{z^*})$ and simplifies to $R_{xx}(z)H(z^{-1})$ for real systems.

$$R_{yy}(z) = R_{xx}(z)H(z)H^*(\frac{1}{z^*}) \text{ or } R_{yy}(e^{j\omega}) = R_{xx}(e^{j\omega})|H(e^{j\omega})|^2$$

Spectral Factorization: Given $R_{yy}(z)$ and with white noise input, $R_{xx}(z) = 1$, find $H(z)$ or equivalently factor $R_{yy}(z)$ such that $R_{yy}(z) = H(z)H^*(\frac{1}{z^*})$.

Positivity of $R_{xx}(e^{j\omega})$

Proof by Contradiction: Suppose $R_{xx}(e^{j\omega}) < 0$ for $\omega_1 \leqslant \omega \leqslant \omega_2$. Let $H(z)$ be a bandpass filter over this region

$$H(e^{j\omega}) = \begin{cases} 1, & \omega_1 \leqslant \omega \leqslant \omega_2 \\ 0 & \text{otherwise} \end{cases}$$

If $x[n]$ is the input to $H(z)$, then the output $y[n]$ has a $r_{yy}[0]$ given by

$$r_{yy}[0] = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(e^{j\omega})|^2 R_{xx}(e^{j\omega}) d\omega = \frac{1}{2\pi} \int_{\omega_1}^{\omega_2} R_{xx}(e^{j\omega}) d\omega < 0$$

This contradicts the fact that $r_{yy}[0] \geqslant 0$. This suggests the initial assumption of negative PSD is not true.

PSD estimation via power at output of a bandpass filter

$$r_{yy}[0] = \frac{1}{2\pi} \int_{\omega_0 - \delta\omega}^{\omega_0 + \delta\omega} R_{xx}(e^{j\omega}) d\omega \approx \frac{2\delta\omega}{2\pi} R_{xx}(e^{j\omega_0}) \text{ or } R_{xx}(e^{j\omega_0}) \approx \frac{2\pi}{2\delta\omega} r_{yy}[0]$$

Verifying if a Sequence is a Valid Autocorrelation Sequence

Let $\{q[m]\}_{-\infty}^{\infty}$ be a sequence, and we wish to know if it is a valid autocorrelation sequence.

Checking for maximum at the origin and symmetry is easy.

Checking for positive semi-definiteness is difficult.

Compute the DTFT of $q[m]$

$$Q(e^{j\omega}) = \sum_{m=-\infty}^{\infty} q[m]e^{-j\omega m}$$

Main Result: $\{q[m]\}_{-\infty}^{\infty}$ is a valid autocorrelation sequence if and only if $Q(e^{j\omega}) \geqslant 0$.

Proof

- Compute $p[m] = h[m] * h^*[-m] * q[m] = \sum_l \sum_p h[l]h^*[p]q[m+p-l]$ and $P(e^{j\omega}) = |H(e^{j\omega})|^2 Q(e^{j\omega})$
- Let $h[m] = a_m^*$. Then $p[m] = \sum_l \sum_p a_l^* a_p q[m+p-l]$ and $P(e^{j\omega}) = |A(e^{j\omega})|^2 Q(e^{j\omega})$, where $A(e^{j\omega}) = \sum_l a_l^* e^{-j\omega l}$.
- From Fourier theory $p[m] = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(e^{j\omega}) e^{j\omega m} d\omega$
- $p[0] = \sum_l \sum_p a_l^* a_p q[p-l]$ and $p[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(e^{j\omega}) d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} |A(e^{j\omega})|^2 Q(e^{j\omega}) d\omega$,
- $p[0] \geqslant 0$ iff $Q(e^{j\omega}) \geqslant 0$.

Ergodicity

For WSS, one needs to compute the mean $\mu = E(x[n])$ and autocorrelation $r[m] = E(x[n]x^*[n-m])$. This is a challenge because the expectation operation implies an ensemble average, average over a collection of sequences, i.e. $\hat{\mu}_N = \frac{1}{N}\sum_{l=0}^{N-1} x[n, \zeta_l]$ and autocorrelation $\hat{r}_N[m] = \frac{1}{N}\sum_{l=0}^{N-1} x[n, \zeta_l]x^*[n-m, \zeta_l]$.

In practice, we have one realization and may use time averages $\hat{\mu} = \frac{1}{N}\sum_{n=0}^{N-1} x[n, \zeta]$ and $\hat{r}[m] = \frac{1}{N}\sum_{n=0}^{N-1} x[n, \zeta]x^*[n-m, \zeta]$.

Ergodicity relates to when these time averages converge to the ensemble average.

A RP is mean ergodic when the time average $\hat{\mu} = \frac{1}{N}\sum_{n=0}^{N-1} x[n] \to \mu$ in the mean squared sense, i.e. $E(|\hat{\mu} - \mu|^2) \to 0$ as $N \to \infty$.

A RP is correlation ergodic when the time average $\hat{r}[m] \to r[m]$ in the mean squared sense, i.e. $E(|\hat{r}[m] - r[m]|^2) \to 0$ as $N \to \infty$.

Example I

$x[n] = A\cos(\omega_0 n + \phi)$, where ϕ is a random variable uniform between $[-\pi, \pi]$.

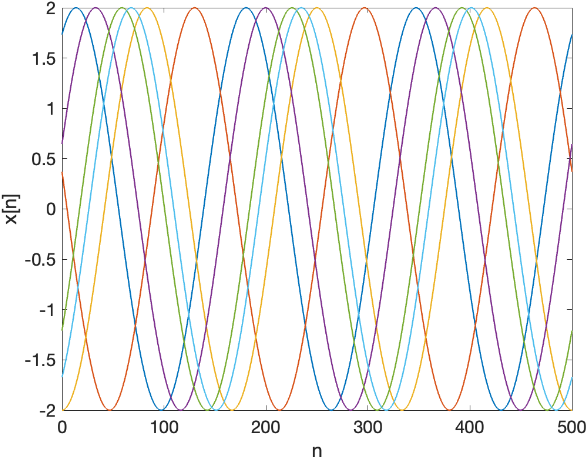


Figure: Six realizations

Examples Revisited

1. $x[n] = A\cos(\omega_0 n + \phi)$, where ϕ is a random variable uniform between $[-\pi, \pi]$. WSS: Yes ($\mu = 0$, and $r[m] = \frac{A^2}{2}\cos(\omega_0 m)$.)
2. $x[n] = w[n]$, where $w[n]$ is a i.i.d. sequence of Gaussian random variable with mean zero and variance 1. WSS: Yes ($\mu = 0$, and $r[m] = \delta[m]$.)
3. $x[n] = ax[n-1] + w[n], |a| < 1$, where $w[n]$ is a i.i.d. sequence of Gaussian random variable with mean zero and variance 1. $x[n]$ is the output of a LTI system $H(z) = \frac{1}{1-az^{-1}}$ with input $w[n]$. WSS: Yes ($\mu = 0$, and $r[m]$ shaped by $H(z)$)
4. $x[n] = A$, where A is a random variable uniform between $[-1, 1]$. WSS: Yes ($\mu = 0$, and $r[m] = E(A^2) = \frac{1}{3}$.)
5. $x[n] = A\cos(\omega_0 n + \phi)$, where ϕ is a random variable uniform between $[-\pi, \pi]$ and A is a random variable independent of ϕ that is uniform between $[-1, 1]$.WSS: Yes ($\mu = 0$, and $r[m] = \frac{E(A^2)}{2}\cos(\omega_0 m)$.)

Examples Revisited

1. $x[n] = A\cos(\omega_0 n + \phi)$, where ϕ is a random variable uniform between $[-\pi, \pi]$. Mean and Correlation Ergodic: Yes
2. $x[n] = w[n]$, where $w[n]$ is a i.i.d. sequence of Gaussian random variables with mean zero and variance 1. Mean and Correlation Ergodic: Yes
3. $x[n] = ax[n-1] + w[n], |a| < 1$, where $w[n]$ is a i.i.d. sequence of Gaussian random variable with mean zero and variance 1. $x[n]$ is the output of a LTI system $H(z) = \frac{1}{1-az^{-1}}$ with input $w[n]$. Mean and Correlation Ergodic: Yes
4. $x[n] = A$, where A is a random variable uniform between $[-1, 1]$. Mean and Correlation Ergodic: No
5. $x[n] = A\cos(\omega_0 n + \phi)$, where ϕ is a random variable uniform between $[-\pi, \pi]$ and A is a random variable independent of ϕ that is uniform between $[-1, 1]$. Mean and Correlation Ergodic: Mean Yes, Correlation No