ECE 161A: Sampling Theorem

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Continuous Time Fourier Transform

Laplace Transform

$$X_c(s) = \int_{-\infty}^{\infty} x_c(t) e^{-st} dt$$

Continuous Time Fourier Transform

$$\begin{aligned} X_c(j\Omega) &= X_c(s)|_{s=j\Omega} = \int_{-\infty}^{\infty} x_c(t) e^{-j\Omega t} dt \\ x_c(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_c(j\Omega) e^{j\Omega t} d\Omega \end{aligned}$$

$\Omega = 2\pi F$, where Ω is in radians per sec and F is in Hz or cycles per sec.

Fourier Series

$$
\begin{aligned}
x_p(t) &= x_p(t+T_0) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_0 t}, \ \text{ where } \Omega_0 = \frac{2\pi}{T_0} \\
c_k &= \frac{1}{T_0}\int_0^{T_0} x_p(t) e^{-jk\Omega_0 t} dt = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x_p(t) e^{-jk\Omega_0 t} dt
\end{aligned}
$$

What is the Fourier transform of a periodic function? For this we will use the following Fourier transform pair

$$
e^{j\Omega_0 t} \ \leftrightarrow \ 2\pi\delta(\Omega - \Omega_0)
$$

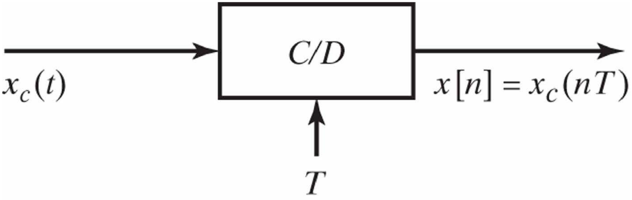
Fourier transform of a periodic function by linearity and Fourier series expansion is

$$
X_p(j\Omega) = 2\pi \sum_{k=-\infty}^{\infty} c_k \delta(\Omega - k\Omega_0)
$$

Some properties of Dirac Delta functions

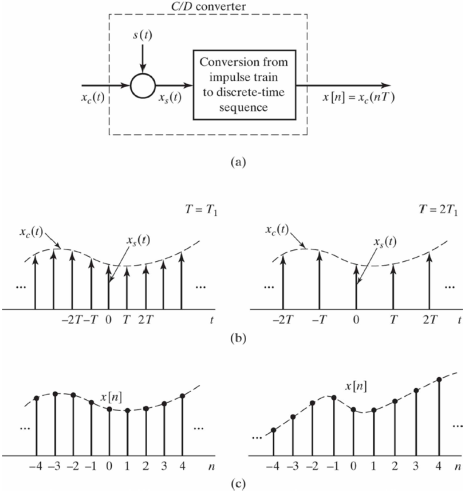
1. $\int_{-\epsilon}^{\epsilon} \delta(t)dt = 1, \ \ \epsilon > 0$
2. $f(t)\delta(t - t_0) = f(t_0)\delta(t - t_0)$
3. $\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0)$
4. $f(t) * \delta(t - t_0) = f(t - t_0)$

Ideal Continuous to Discrete Converter



Sampling with a Pulse Train $s(t) = \sum_n \delta(t - nT)$

Figure 4.2 Sampling with a periodic impulse train, followed by conversion to a discrete-time sequence. (a) Overall system. (b) $x_s(t)$ for two sampling rates. (c) The output sequence for the two different sampling rates.



Mathematical View of Sampling

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

is a periodic function with periodicity T. The fundamental frequency $\Omega_s = \frac{2\pi}{T}$ and the Fourier series expansion is $\sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_s t}$with the Fourier series coefficients given by

$$c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} s(t)e^{-jk\Omega_s t}dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{n=-\infty}^{\infty} \delta(t - nT)e^{-jk\Omega_s t}dt = \frac{1}{T}$$

Fourier Transform of $s(t)$ is given by

$$S(j\Omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

Sampled function (Time Domain View)

$$\begin{aligned} x_s(t) &= x_c(t)s(t) = x_c(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) \\ &= \sum_{n=-\infty}^{\infty} x_c(t)\delta(t-nT) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT) \\ &= \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT), \;\; \text{where} \;\; x[n] = x_c(nT) \end{aligned}$$

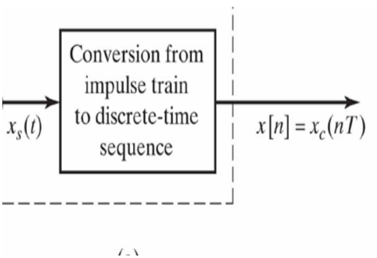
There is a natural connection between the sampled function and the sequence

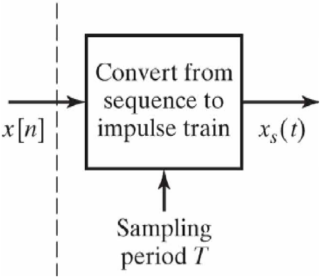
$$x_s(t) \leftrightarrow \{x[n]\}$$

Given $x_s(t)$, one can obtain the sequence $\{x[n]\}$ by stripping away the impulses.
Given $\{x[n]\}$, one can obtain $x_s(t)$ as $x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT)$.

Sampled Function and Corresponding Sequence

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$$





Sampled Function (Frequency Domain View)

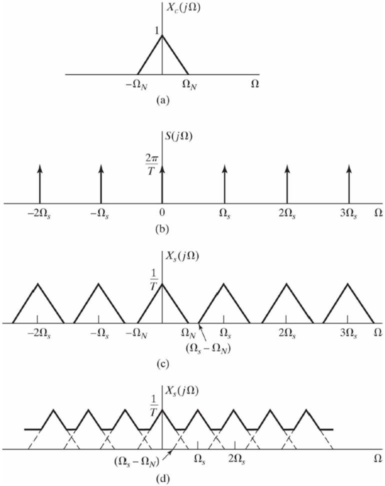
$$x_s(t) = x_c(t)s(t) \leftrightarrow X_s(j\Omega) = \frac{1}{2\pi}\left[X_c(j\Omega) * S(j\Omega)\right]$$

$$\begin{aligned}
X_s(j\Omega) &= \frac{1}{2\pi}\left[X_c(j\Omega) * S(j\Omega)\right] = \frac{1}{2\pi}\left[X_c(j\Omega) * \left(\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\Omega - k\Omega_s)\right)\right] \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty}\left[X_c(j\Omega) * \delta(\Omega - k\Omega_s)\right] = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \\
&= \frac{1}{T}\{\ldots X_c(j(\Omega + \Omega_s)) + X_c(j\Omega) + X_c(j(\Omega - \Omega_s)) + \ldots\}
\end{aligned}$$

Recovery is possible if the shifted copies (images) do not overlap with $X_c(j\Omega)$, i.e. no aliasing

Frequency Domain Representation of Sampling

Figure 4.3 Frequency-domain representation of sampling in the time domain. (a) Spectrum of the original signal. (b) Fourier transform of the sampling function. (c) Fourier transform of the sampled signal with $\Omega_s > 2\Omega_N$. (d) Fourier transform of the sampled signal with $\Omega_s < 2\Omega_N$.



Nyquist-Shannon Sampling Theorem

Let $x_c(t)$ be a bandlimited signal with

$$X_c(j\Omega) = 0, \quad \text{for} \quad |\Omega| \geq \Omega_N.$$

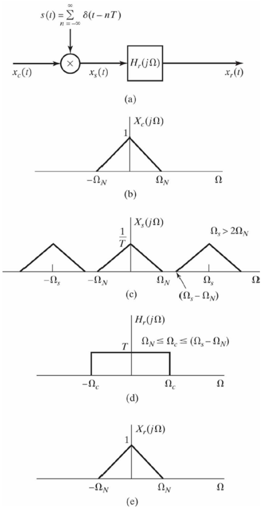
Then $x_c(t)$ is uniquely determined by its samples $x[n] = x_c(nT), n = 0, \pm 1, \pm 2, \ldots$ iff

$$\Omega_s = \frac{2\pi}{T} \geq 2\Omega_N.$$

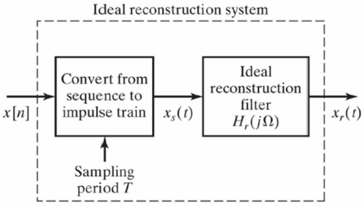
$2\Omega_N$ is referred to as the Nyquist rate.

Recovery

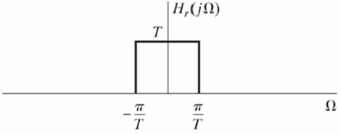
Figure 4.4 Exact recovery of a continuous-time signal from its samples using an ideal lowpass filter.



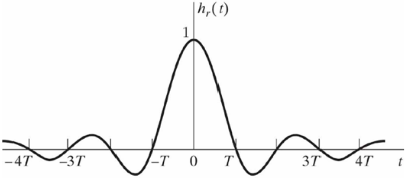
The reconstruction filter is an ideal continuous-time low pass filter


Ideal reconstruction system
Convert from sequence to impulse train
Ideal reconstruction filter $H_r(j\Omega)$
$x[n]$
$x_s(t)$
$x_r(t)$
Sampling period T

(a)


$H_r(j\Omega)$
T
$-\frac{\pi}{T}$
$\frac{\pi}{T}$
Ω

(b)


$h_r(t)$
1
$-4T$
$-3T$
$-T$
0
T
$3T$
$4T$
t

(c)

Reconstruction Filter

$$H_r(j\Omega) = \begin{cases} T & |\Omega| \leq \frac{\Omega_s}{2} \\ 0 & \text{otherwise} \end{cases} \quad \leftrightarrow \quad h_r(t) = \frac{\sin(\frac{\pi t}{T})}{\frac{\pi t}{T}}$$

Assuming no aliasing

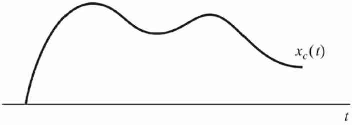
$$X_c(j\Omega) = X_s(j\Omega)H_r(j\Omega)$$

Time domain interpolation

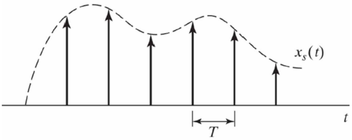
$$x_c(t) = x_s(t) * h_r(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT) * h_r(t) = \sum_{n=-\infty}^{\infty} x[n]h_r(t - nT)$$

Exact reconstruction of the continuous-time signal from the samples.

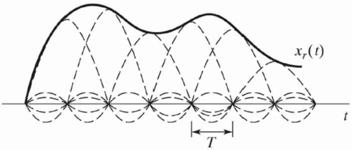
Ideal bandlimited interpolation.



(a)



(b)



(c)

Normalized Frequency (ω) versus Actual Frequencies (Ω)

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT) \leftrightarrow X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega Tn}$$

Now the DTFT of $x[n]$ is defined as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

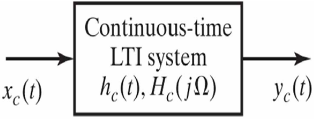
Because $x_s(t) \leftrightarrow \{x[n]\}$, we want $X_s(j\Omega) \leftrightarrow X(e^{j\omega})$ for consistency. This leads to the following relationship between ω and Ω.

$$\omega = \Omega T.$$

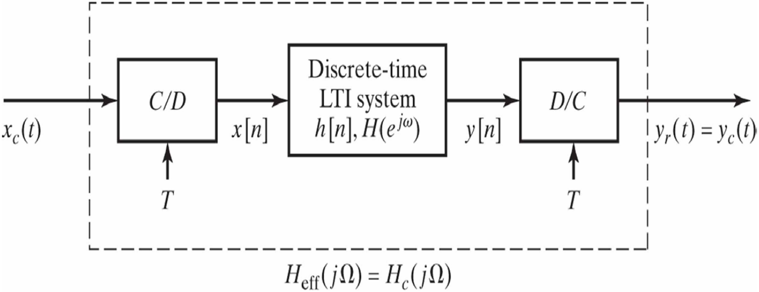
So $\omega = 0$ corresponds to $\Omega = 0$, and $\omega = \pm\pi$ corresponds to $\Omega = \pm\frac{\pi}{T} = \frac{\Omega_s}{2}$.

Example: If the bandwidth of the signal is $4Khz$, and we sample at $10Khz$, there is no aliasing. When we mention $\omega_o = \frac{\pi}{4}$ for the normalized frequency variable, we are referring to a frequency of $5Khz/4$ in the continuous domain.

Digital Systems as substitute for Continuous time Systems

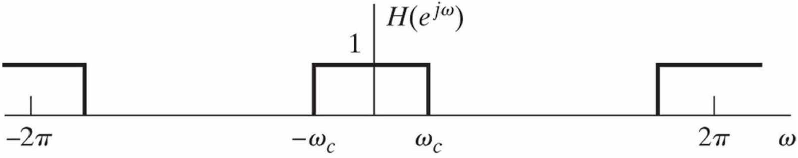


(a)



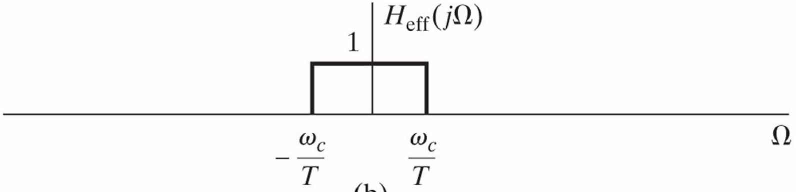
(b)

Example



$H(e^{j\omega})$
1
-2π
$-\omega_c$
ω_c
2π
ω

(a)



$H_{\text{eff}}(j\Omega)$
1
$-\frac{\omega_c}{T}$
$\frac{\omega_c}{T}$
Ω

(b)

Example Design

Problem: $x_c(t)$ is a continuous time signal with bandwidth 8 Khz and we are interested in only retaining the frequencies from 1 KHz to 3 Khz. Design a system to achieve this objective.

Continuous time system design: Design a bandpass filter with passband from 1 to 3 Khz, i.e. $2\pi 1000\ rads/sec \leq |\Omega| \leq 2\pi 3000\ rads/sec$.

Digital system design: Sample the signal at Nyquist rate or above. Let us choose a sampling rate of 20 Khz for the discussion. Design a digital bandpass filter with pass band from $\frac{\pi}{10}$ to $\frac{3\pi}{10}$, i.e. $\frac{\pi}{10} \leq |\omega| \leq \frac{3\pi}{10}$.