ECE 251A: Digital Signal Processing I

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Staff

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  - Office Hours: Friday 2:00–3:00 PM (via Zoom)
- TA: Mingchao Liang (m3liang@ucsd.edu)
  Discussion Sessions: Wednesday 9-10 AM and Friday 2-3 PM in EBU1 5101B
- Website: Canvas
- Post all your technical questions on the Canvas discussion board.
- Lectures will be available as a Podcast

Books

- Textbook: Dimitris G. Manolakis, Vinay K. Ingle, and Stephen M. Kogon. *Statistical and Adaptive Signal Processing: Spectral Estimation, Signal Modeling, Adaptive Filtering, and Array Processing*, McGraw-Hill, 2000.

- Additional References:
    - Alan V. Oppenheim and Ronald W. Schafer. *Discrete-Time Signal Processing*. Pearson Education, 2014.
    - Petre Stoica and Randolph L. Moses. *Introduction to Spectral Analysis*. Prentice Hall, 1997.
    - Simon O. Haykin *Adaptive Filter Theory*. Pearson Higher Education, 2013.
    - Ali H. Sayed *Fundamentals of Adaptive Filtering*. John Wiley & Sons, 2003.

Course Overview

- Prerequisite: ECE 161A. Should be familiar with the following topics and concepts: Discete-Time Fourier Transform, Z-Transforms, FIR filters, Discrete Fourier Transform (DFT).

- Prerequisite: ECE 153. Should be familiar with the following topics and concepts: Wide Sense Stationary (WSS) Random Processes, Power Spectrum. Processing using LTI systems.

- Course Content:
  - Short-Time Fourier Transform (STFT)
  - FFT based Power Spectrum Estimation (Periodograms, Modified Periodogram, ..)
  - Model Based Power Spectrum Estimation (Moving Average (MA), AutoRegressive (AR) and ARMA models)
  - AR models and Linear Minimum Mean Squared Estimation (Levinson-Durbin Algorithm and Lattice Filters)
  - Minimum Variance Distortionless Response (MVDR) Power Spectrum Estimation

Grading

- Homework – 30% (Assignments include problems from text and Matlab problems. Matlab problems will have more weighting, Also there will be online Quizzes (mostly multiple choice) on a weekly basis)
  - Usually assigned on Thursday and due the following Thursday (11:59 PM)
  - Must be original work
  - No studying of old solutions
  - Discussion among classmates is ok, but no copying
  - The use of any AI tools is prohibited
  - Please review the academic integrity document at academicintegrity.ucsd.edu
- Midterm – 25%
- Final Exam – 45%

Applications

- Wireless communications
- Speech and audio processing
- Radar signal processing
- Sonar signal processing
- Array signal processing
- Healthcare
- Noise cancellation systems (Headphones, ..)
- Financial markets
- ...

Application sources: Google Scholar or IEEE Signal Processing Magazine

Discrete Time Fourier Transform (DTFT)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Inverse DTFT:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega.$$

Some Observations

- $X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$, periodic with periodicity 2π.
- Sufficient to plot $X(e^{j\omega})$ over the range $[0, 2\pi]$ or $[-\pi, \pi]$.
- ω is referred to as the normalized frequency and is related to true analog frequencies through the sampling theorem, i.e. $\omega = \Omega T$, where Ω is the analog frequency variable (cycles/sec) and T the sampling interval in seconds.
- $[-\pi, \pi]$ correspond to the analog frequency range $[-\Omega_s/2, \Omega_s/2]$, where $\Omega_s = \frac{2\pi}{T}$.

Z-Transform Definition

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Will use the notation $X(z) = \mathcal{Z}(x[n])$.

Definition not complete without specifying the Region of Convergence (ROC)

$$ROC : \{z : \sum_{n=-\infty}^{\infty} |x[n] z^{-n}| = \sum_{n=-\infty}^{\infty} |x[n]| |z|^{-n} < \infty\}$$

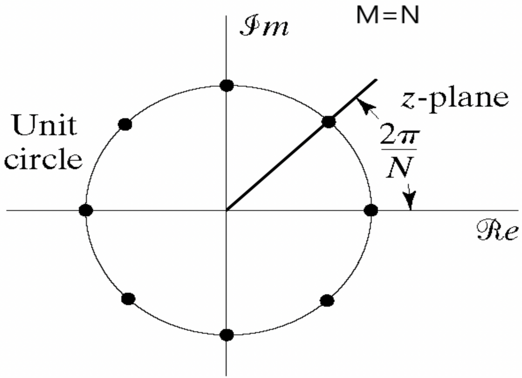
Why Z-transforms?

- More general: Applicable to a larger class of signals
- Easier to manipulate: Complex variable theory can be useful.

Z-Transform versus Discrete Time Fourier Transform(DTFT)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X(z)_{|z=e^{j\omega}}$$

Since $|e^{j\omega}| = 1$, DTFT is the z-transform evaluated on the unit circle.



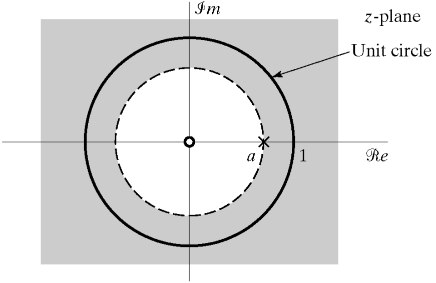
For the Fourier Transform to be defined, ROC must include the unit circle.
Sequence must be absolutely summable.

Example 1: Right-Sided Sequence

Consider $x[n] = a^n u[n]$.

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} \\ &= \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}}, \quad |az^{-1}| < 1 \end{aligned}$$

$ROC : \{z : |az^{-1}| < 1\} = \{z : |z| > |a|\}$

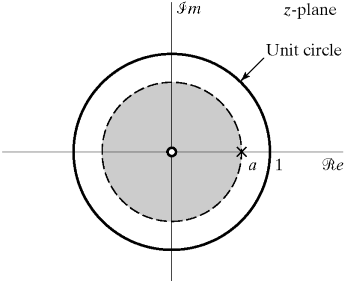


Example 2: Left-Sided Sequence

Consider $x[n] = -a^n u[-n-1]$.

$$
\begin{aligned}
X(z) &= -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} \\
&= -\sum_{n=1}^{\infty} a^{-n} z^n = -\sum_{n=1}^{\infty} (a^{-1}z)^n = -\frac{a^{-1}z}{1-a^{-1}z}, \quad |a^{-1}z| < 1 \\
&= -\frac{az^{-1}}{az^{-1}} \frac{a^{-1}z}{1-a^{-1}z} = -\frac{1}{az^{-1}-1} = \frac{1}{1-az^{-1}}, \quad |a^{-1}z| < 1
\end{aligned}
$$

$ROC : \{z : |a^{-1}z| < 1\} = \{z : |z| < |a|\}$



General Sequences

1. Finite Duration Sequences
2. Right-Sided Sequences
3. Left-Sided Sequences
4. Two-Sided Sequences

Finite Duration Sequences

1. Finite Duration Sequences: Sequences that are non-zero for $-\infty < N_1 \leqslant n \leqslant N_2 < \infty$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=N_1}^{N_2} x[n]z^{-n}$$

ROC: All z except possibly $z = 0$ or $z = \infty$ because z^{-n} is infinity at $z = 0$ for positive values of n and similarly for $z = \infty$ for negative values of n.

Right-Sided Sequences

2. Right-Sided Sequences: Sequences that are zero for $n < N_1 < \infty$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=N_1}^{\infty} x[n]z^{-n}$$

ROC: $\{z : |z| > r_R\}$ except possibly $z = \infty$

$N_1 \geqslant 0$, is a causal sequence

Reasoning: If $\sum_{n=N_1}^{\infty} |x[n]||z|^{-n}$ converges for $|z| = r_R$, it will converge for all $|z| > r_R$ since all terms with positive n get smaller

$$|x[n]r_R^{-n}| = \frac{|x[n]|}{r_R^n} > \frac{|x[n]|}{r^n} = |x[n]r^{-n}|, \quad r > r_R \ \text{ and } \ n > 0$$

Left-Sided Sequences

3. Left-Sided Sequences: Sequences that are zero for $n > N_2 > -\infty$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{N_2} x[n]z^{-n}$$

ROC: $\{z : |z| < r_L\}$ except possibly $z = 0$

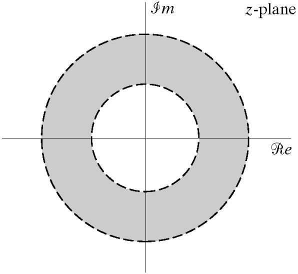
Reasoning: If $\sum_{n=-\infty}^{N_2} |x[n]||z|^{-n}$ converges for $|z| = r_L$, it will converge for all $|z| < r_L$ since all terms with negative n get smaller

Two-Sided Sequences

4. Two-Sided Sequences: Sequences that are neither left-sided or right-sided, i.e. sequence defined over $-\infty < n < \infty$.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

ROC: $\{z : r_R < |z| < r_L\}$ is an annular ring
Reasoning: If $\sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{-1} x[n]z^{-n} + \sum_{n=0}^{\infty} x[n]z^{-n}$, sum of a left-sided sequence and a right-sided sequences. Both sequences must converge, and the ROC is the intersection.



Summary of ROCs

1. Finite Duration Sequences. ROC: All z except possibly $z = 0$ or $z = \infty$
2. Right-Sided Sequences. ROC: $\{z : |z| > r_R\}$ except possibly $z = \infty$
3. Left-Sided Sequences. ROC: $\{z : |z| < r_L\}$ except possibly $z = 0$
4. Two-Sided Sequences. ROC: $\{z : r_R < |z| < r_L\}$ is an annular ring

Examples

Delta function:

$$x[n] = \delta[n] \leftrightarrow X(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = 1.$$

ROC: Entire z-plane

Exponential sequence: $x[n] = a^n u[n]$, has Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}}, \quad |az^{-1}| < 1$$

ROC: $\{z : |az^{-1}| < 1\} = \{z : |z| > |a|\}$

Two Sided Exponential:

$$x[n] = a^{|n|}, |a| < 1 \leftrightarrow X(z) = \frac{1}{1 - az^{-1}} - \frac{1}{1 - a^{-1}z^{-1}}$$

ROC: $\{z : |a| < |z| < \frac{1}{|a|}\}$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n - n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\|R_x$
4	3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Linear Time Invariant Systems and Difference Equations

$y[n] = h[n] * x[n] = \sum_k h[k]x[n-k]$, where $h[n]$ is the impulse response of the system (convolution)

In the z-domain, $Y(z) = H(z)X(z)$ leading to $H(z) = \frac{Y(z)}{X(z)}$, the transfer function.

Difference equation description of a LTI system

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m] \quad \text{or} \quad y[n] = -\sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k] + \sum_{m=0}^{M} \frac{b_m}{a_0} x[n-m]$$

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0}{a_0} \frac{\prod_{m=1}^{M}(1 - z_m z^{-1})}{\prod_{k=1}^{N}(1 - p_k z^{-1})}$$

If $A(z) \neq a_0$, then system is an Infinite Impulse Response (IIR) filter

If $A(z) = 1$, then $H(z) = B(z)$ and the filter is a Finite Impulse response (FIR) filter.

Stability and Causality

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0}{a_0} \frac{\prod_{m=1}^{M}(1 - z_m z^{-1})}{\prod_{k=1}^{N}(1 - p_k z^{-1})}$$

Poles determine ROC and cannot be in the ROC. Zeros do not impact ROC

Causality: $h[n] = 0, n < 0$ is a right sided sequence and so ROC: $\{z : |z| > \max_k |p_k|\}$

Stability: $\sum_k |h[k]| < \infty$ and so Fourier transform defined. ROC: Region between poles, but not including any poles, that contains the unit circle

Causal and Stable: ROC: $\{z : |z| > \max_k |p_k|\}$ and includes unit circle. Need $\max_k |p_k|$ to be less than 1. All poles must be inside the unit circle