# ECE 161A: Discrete Time Fourier Transform

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# Discrete Time Fourier Transform

$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\
x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega
\end{aligned}
$$

Will use the notation $X(e^{j\omega}) = \mathcal{F}(x[n])$. $X(e^{j\omega})$ is periodic with periodicity $2\pi$, $X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$.

Some Observations

- $\omega$ is referred to as the normalized frequency.
- Definition consistent with the definition of the continuous time Fourier transform and related through the sampling theorem.
- $x[n]$ can be viewed as the Fourier series coefficients.

# Fourier Series Connection

DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \end{aligned}$$

Fourier Series

$$\begin{aligned} x_c(t) &= x_c(t+T_0) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_0 t}, \text{ where } \Omega_0 = \frac{2\pi}{T_0} \\ c_k &= \frac{1}{T_0}\int_{0}^{T_0} x_c(t)e^{-jk\Omega_0 t}dt \end{aligned}$$

Connection $X(e^{j\omega}) \leftrightarrow x_c(t)$ and $x[n] \leftrightarrow c_k$.

At a variable level we have $t \leftrightarrow \omega$, $T_0 \leftrightarrow 2\pi$, and $\Omega_0 \leftrightarrow 1$.

# Magnitude and Phase

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\measuredangle X(e^{j\omega})} = X_R(e^{j\omega}) + jX_I(e^{j\omega})$$

$$|X(e^{j\omega})| = \sqrt{X_R^2(e^{j\omega}) + X_I^2(e^{j\omega})} \text{ and } \measuredangle X(e^{j\omega}) = \arctan \frac{X_I(e^{j\omega})}{X_R(e^{j\omega})}$$

- Magnitude is usually plotted in db scale, i.e. $20\log_{10}|X(e^{j\omega})|$.
- Phase: there are two popular options
  - $\mathrm{ARG}(X(e^{j\omega}))$ is $\measuredangle X(e^{j\omega})$ limited to the range $[-\pi, \pi]$.
  - $\arg(X(e^{j\omega}))$ is $\measuredangle X(e^{j\omega})$ computed as a continuous function of $\omega$ (unwrapped phase).

**Table 2.3** FOURIER TRANSFORM PAIRS

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| Sequence | Fourier Transform |
|---|---|
| 1. $\delta[n]$ | $1$ |
| 2. $\delta[n-n_0]$ | $e^{-j\omega n_0}$ |
| 3. $1 \quad (-\infty < n < \infty)$ | $\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$ |
| 4. $a^n u[n] \quad (\|a\| < 1)$ | $\frac{1}{1-ae^{-j\omega}}$ |
| 5. $u[n]$ | $\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$ |
| 6. $(n+1)a^n u[n] \quad (\|a\| < 1)$ | $\frac{1}{(1-ae^{-j\omega})^2}$ |
| 7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\|r\| < 1)$ | $\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2 e^{-j2\omega}}$ |
| 8. $\frac{\sin\omega_c n}{\pi n}$ | $X(e^{j\omega}) = \begin{cases} 1, & \|\omega\| < \omega_c, \\ 0, & \omega_c < \|\omega\| \le \pi \end{cases}$ |
| 9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$ | $\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$ |
| 10. $e^{j\omega_0 n}$ | $\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$ |
| 11. $\cos(\omega_0 n+\phi)$ | $\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k)+\pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$ |

# Examples

Delta function:

$$x[n] = \delta[n] \leftrightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = 1.$$

Delayed Delta function:

$$x[n] = \delta[n-n_0] \leftrightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n-n_0] e^{-j\omega n} = e^{-j\omega n_0}.$$

Exponential sequence: $x[n] = a^n u[n], |a| < 1$, has DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

**Figure 2.22** Frequency response for a system with impulse response $h[n] = a^n u[n]$. (a) Real part. $a > 0$; $a = 0.75$ (solid curve) and $a = 0.5$ (dashed curve). (b) Imaginary part. (c) Magnitude. $a > 0$; $a = 0.75$ (solid curve) and $a = 0.5$ (dashed curve). (d) Phase.

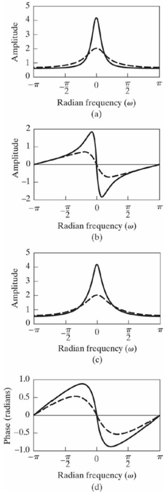

**Table 2.2** FOURIER TRANSFORM THEOREMS

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| Sequence | Fourier Transform |
|---|---|
| $x[n]$ | $X(e^{j\omega})$ |
| $y[n]$ | $Y(e^{j\omega})$ |
| 1. $ax[n]+by[n]$ | $aX(e^{j\omega})+bY(e^{j\omega})$ |
| 2. $x[n-n_d]$ ($n_d$ an integer) | $e^{-j\omega n_d}X(e^{j\omega})$ |
| 3. $e^{j\omega_0 n}x[n]$ | $X(e^{j(\omega-\omega_0)})$ |
| 4. $x[-n]$ | $X(e^{-j\omega})$<br>$X^*(e^{j\omega})$ if $x[n]$ real. |
| 5. $nx[n]$ | $j\dfrac{dX(e^{j\omega})}{d\omega}$ |
| 6. $x[n]*y[n]$ | $X(e^{j\omega})Y(e^{j\omega})$ |
| 7. $x[n]y[n]$ | $\dfrac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$ |

Parseval's theorem:

8. $$\sum_{n=-\infty}^{\infty}|x[n]|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(e^{j\omega})|^2d\omega$$

9. $$\sum_{n=-\infty}^{\infty}x[n]y^*[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

# Time Shifting

Time Shifting: $x[n-n_d] \leftrightarrow e^{-j\omega n_d} X(e^{j\omega})$

Proof:

$$\begin{aligned}
\mathcal{F}(x[n-n_d]) &= \sum_{n=-\infty}^{\infty} x[n-n_d] e^{-j\omega n} \\
&= \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega(m+n_d)} \quad (Change\ of\ variables\ m = n - n_d) \\
&= e^{-j\omega n_d} \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega m} = e^{-j\omega n_d} X(e^{j\omega})
\end{aligned}$$

# Modulation

Modulation: $e^{j\omega_0 n}x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$

Proof:

$$
\begin{aligned}
\mathcal{F}(e^{j\omega_0 n}x[n]) &= \sum_{n=-\infty}^{\infty} e^{j\omega_0 n}x[n]e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega-\omega_0)n} \\
&= X(e^{j(\omega-\omega_0)})
\end{aligned}
$$

# Convolution

$x[n] * y[n] \leftrightarrow X(e^{j\omega})Y(e^{j\omega})$

Proof:

$$\begin{aligned}
\mathcal{F}(x[n] * y[n]) &= \mathcal{F}(\sum_{k=-\infty}^{\infty} x[k]y[n-k]) \\
&= \sum_{k-\infty}^{\infty} x[k]\mathcal{F}(y[n-k]) \;\; (Linearity) \\
&= \sum_{k-\infty}^{\infty} x[k]e^{-j\omega k}Y(e^{j\omega}) \;\; (Time \;\; Shifting) \\
&= X(e^{j\omega})Y(e^{j\omega})
\end{aligned}$$

# LTI Systems and Convolutions

$$y[n] = h[n] * x[n] \leftrightarrow Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$h[n] \leftrightarrow H(e^{j\omega})$ and $H(e^{j\omega})$ is the frequency response (transfer function) of the LTI system.

LTI systems and complex exponential inputs: Consider input $x[n] = e^{j\omega_0 n}$.

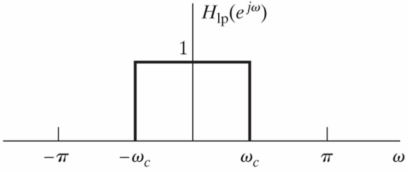
$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega_0(n-k)} \\ &= e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega_0 k} = e^{j\omega_0 n} H(e^{j\omega_0}) \end{aligned}$$

Complex exponential input leads to a scaled complex exponential at the output with the scaling determined by the Transfer function. They are referred as eigenfunctions of LTI systems.

Input: $x[n] = \sum_k c_k e^{j\omega_k n}$. Output: By the linearity property $y[n] = \sum_k c_k H(e^{j\omega_k})e^{j\omega_k n}$

# Parseval's Theorem

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$




$$X(e^{j\omega}) = \begin{cases} 1, & -\omega_c \le \omega \le \omega_c \\ 0, & \text{Otherwise} \end{cases} \quad \text{and } x[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega n} d\omega = \frac{\sin \omega_c n}{\pi n}$$

From Parseval's Theorem

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} \left(\frac{\sin \omega_c n}{\pi n}\right)^2 = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi}$$

**Table 2.1** SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

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| Sequence $x[n]$ | Fourier Transform $X(e^{j\omega})$ |
|---|---|
| 1. $x^*[n]$ | $X^*(e^{-j\omega})$ |
| 2. $x^*[-n]$ | $X^*(e^{j\omega})$ |
| 3. $\mathcal{R}e\{x[n]\}$ | $X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$) |
| 4. $j\mathcal{I}m\{x[n]\}$ | $X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$) |
| 5. $x_e[n]$ (conjugate-symmetric part of $x[n]$) | $X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$ |
| 6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$) | $jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$ |
| *The following properties apply only when $x[n]$ is real:* | |
| 7. Any real $x[n]$ | $X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric) |
| 8. Any real $x[n]$ | $X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even) |
| 9. Any real $x[n]$ | $X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd) |
| 10. Any real $x[n]$ | $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (magnitude is even) |
| 11. Any real $x[n]$ | $\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$ (phase is odd) |
| 12. $x_e[n]$ (even part of $x[n]$) | $X_R(e^{j\omega})$ |
| 13. $x_o[n]$ (odd part of $x[n]$) | $jX_I(e^{j\omega})$ |

Property 1: $x^*[n] \leftrightarrow X^*(e^{-j\omega})$.

$x[n] = x_R[n] + jx_I[n] \leftrightarrow X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega})$, then
$x^*[n] = x_R[n] - jx_I[n] \leftrightarrow X^*(e^{-j\omega}) = X_R(e^{-j\omega}) - jX_I(e^{-j\omega})$.

Note: $x_R[n]$ and $X_R(e^{j\omega})$ are not Fourier transform pairs, i.e.
$x_R[n] \not\leftrightarrow X_R(e^{j\omega})$.

Proof:

$$
\begin{aligned}
\mathcal{F}(x^*[n]) &= \sum_{n=-\infty}^{\infty} x^*[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x^*[n](e^{j\omega n})^* = \sum_{n=-\infty}^{\infty} (x[n]e^{j\omega n})^* \\
&= \left(\sum_{n=-\infty}^{\infty} (x[n]e^{j\omega n})\right)^* = \left(\sum_{n=-\infty}^{\infty} (x[n]e^{-j(-\omega)n})\right)^* \\
&= \left(X(e^{-j\omega})\right)^* = X^*(e^{-j\omega})
\end{aligned}
$$

Property 3: $Re(x[n]) = x_R[n] = \frac{x[n]+x^*[n]}{2} \leftrightarrow X_c(e^{j\omega}) = \frac{X(e^{j\omega})+X^*(e^{-j\omega})}{2}$.
$X_c(e^{j\omega}) = X_c^*(e^{-j\omega})$ (Conjugate-Symmetric part of $X(e^{j\omega})$.)

Property 4: $jIm(x[n]) = jx_I[n] = \frac{x[n]-x^*[n]}{2} \leftrightarrow X_o(e^{j\omega}) = \frac{X(e^{j\omega})-X^*(e^{-j\omega})}{2}$.
$X_o(e^{j\omega}) = -X_o^*(e^{-j\omega})$ (Conjugate-Antisymmetric part of $X(e^{j\omega})$.)

# Real Sequences

For a real sequence $x[n] = x^*[n]$. Hence $X(e^{j\omega}) = X^*(e^{-j\omega})$.

Implications:

$|X(e^{j\omega})| = |X^*(e^{-j\omega})| = |X(e^{-j\omega})|$. The magnitude of the DTFT is an even function. Sufficient to plot $[0, \pi]$.

$\measuredangle X(e^{j\omega}) = \measuredangle X^*(e^{-j\omega}) = -\measuredangle X(e^{-j\omega})|$. The phase of the DTFT is an odd function. Sufficient to plot $[0, \pi]$.

$X(e^{j\omega}) = X^*(e^{-j\omega}) \rightarrow X_R(e^{j\omega}) + jX_I(e^{j\omega}) = X_R(e^{-j\omega}) - jX_I(e^{-j\omega})$.
Hence $X_R(e^{j\omega}) = X_R(e^{-j\omega})$ and $X_I(e^{j\omega}) = -X_I(e^{-j\omega})$.

# Convergence of the Fourier Transform

$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$. Infinite sum and so may not exist for a given $x[n]$.
$X_M(e^{j\omega}) = \sum_{n=-M}^{M} x[n]e^{-j\omega n} \xrightarrow[M\to\infty]{} X(e^{j\omega})$
Complicated because it involves a sequence of functions. Behavior may vary with $\omega$.

Pointwise Convergence: $\lim_{M\to\infty} X_M(e^{j\omega}) = X(e^{j\omega}) \ \forall\omega$.

Uniform Convergence: $\{X_M(e^{j\omega})\}$ converges uniformly to $X(e^{j\omega})$ if given any $\epsilon > 0$, there exists a natural number $N = N(\epsilon)$ such that

$$|X_M(e^{j\omega}) - X(e^{j\omega})| < \epsilon, \text{for every } M > N \text{ and } \forall\omega.$$

Uniform convergence implies pointwise convergence but not the other way around.
The limit of a sequence of continuous functions converging uniformly is also continuous.

# Absolute Summability and Convergence

If the sequence is absolutely summable, the Fourier transform exists.

$$|X(e^{j\omega})| = |\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}| \leq \sum_{n=-\infty}^{\infty} |x[n]e^{-j\omega n}| \leq \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

Can show: Converges uniformly to a continuous function.

For stable LTI systems, the Fourier transform always exists.

Reason: Stable LTI systems have an impulse response that is absolutely summable.

# Convergence Issues with Low Pass Filters

$$H(e^{j\omega}) = \begin{cases} 1, & -\omega_c \leq \omega \leq \omega_c \\ 0, & \text{Otherwise} \end{cases} \quad \text{and } h[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{-j\omega n} d\omega = \frac{\sin \omega_c n}{\pi n}$$

Challenges:

- Non-causal and infinite in duration
- $H_M(e^{j\omega}) = \sum_{n=-M}^{M} h[n]e^{-j\omega n}$ does not converge uniformly. Note that $H_M(e^{j\omega}) = \sum_{n=-M}^{M} h[n]e^{-j\omega n}$ is a continuous function of $\omega$ but the ideal lowpass filter is not
- The ideal impulse response is not absolutely summable

Consequence: Gibbs Phenomenon

# Gibbs Phenomenon

**Figure 2.21** Convergence of the Fourier transform. The oscillatory behavior at $\omega = \omega_c$ is often called the Gibbs phenomenon.