SIO 209: Signal Processing for Ocean Sciences
Class 8

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Power Spectral Estimation

- The periodogram is an estimate of the power spectrum $P_{xx}(\omega)$

$$I_N(\omega) = \sum_{m=-(N-1)}^{N-1} c_{xx}[m] e^{-j\omega m} \qquad c_{xx}[m] = \hat{\phi}_{xx}[m]$$

$$= \frac{1}{N} \left| X(e^{j\omega}) \right|^2 \qquad \text{where } X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]\, e^{-j\omega n}$$

see also Sections 10.2 and 10.5 in *Oppenheim & Schafer, 2009*

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Power Spectral Estimation

- The expectation of the periodogram is given by

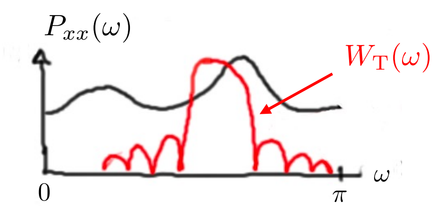
triangular window

$$\mathrm{E}\left[I_N(\omega)\right] = \sum_{m=-(N-1)}^{N-1} \boxed{\frac{N-|m|}{N}}\, \phi_{xx}[m]\, e^{-j\omega m}$$

convolution in time domain is multiplication in frequency domain

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xx}(\theta)\boxed{W_{\mathrm{T}}\left(e^{j(\omega-\theta)}\right)}\mathrm{d}\theta$$

Fourier transform of the triangular window



The periodogram is a biased estimator of the power spectrum $P_{xx}(\omega)$; the bias gets smaller as N increases

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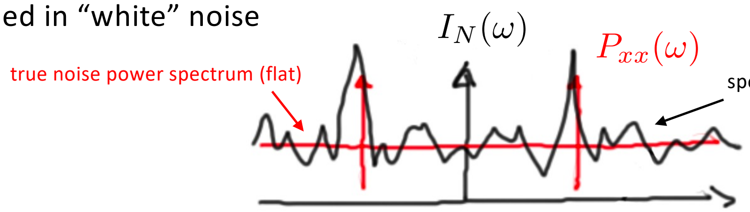
Power Spectral Estimation

- The variance of the periodogram reads

$$\mathrm{var}\left[I_N(\omega)\right] = P_{xx}^2(\omega)\left[1 + \left(\frac{\sin\omega N}{N\sin\omega}\right)^2\right]$$
$$\approx P_{xx}^2(\omega)$$

The variance of the periodogram does not get smaller as N increases

Example: Sinusoid buried in “white” noise



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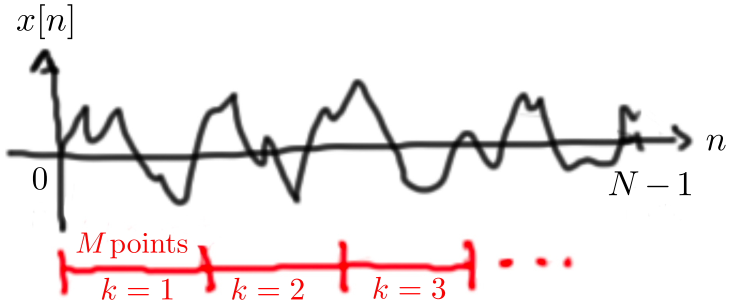
Bartlett’s Procedure of Averaging Periodograms



$$B_{xx}(\omega) = \frac{1}{K}\sum_{k=1}^{K} I_M^{(k)}(\omega)$$

Expectation:

$$\mathrm{E}\big[B_{xx}(\omega)\big] = \frac{1}{K}\sum_{k=1}^{K} \mathrm{E}\big[I_M^{(k)}(\omega)\big] = \mathrm{E}\big[I_M(\omega)\big]$$

Convolution of $P_{xx}(\omega)$ with the Fourier transform of triangular window $(M-|m|)/M$

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Bartlett’s Procedure of Averaging Periodograms

- Following Barlett’s procedure, the variance of the averaged periodogram is obtained as

$$\begin{aligned}\mathrm{var}\big[B_{xx}(\omega)\big] &= \frac{1}{K}\mathrm{var}\big[I_M(\omega)\big] \\ &\approx \boxed{\frac{1}{K}} P_{xx}^2(\omega)\end{aligned}$$

variance reduction resulting from averaging

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Welch's Method of Averaging Modified Periodograms

- Welch's method follows Bartlett's procedure but applies a window function $w[n]$ before computing the estimate of $P_{xx}(\omega)$

$$B_{xx}^{w}(\omega) = \frac{1}{K}\sum_{k=1}^{K} J_M^{(k)}(\omega) \qquad \text{where } J_M^{(k)}(\omega) = \frac{1}{MU}\left|\sum_{n=0}^{M-1} w[n]\,x^{(k)}[n]\,e^{-j\omega n}\right|^2$$

window-dependent normalization constant $\longrightarrow$ $U = \frac{1}{M}\sum_{n=0}^{M-1} w^2[n]$

Note that $J_M^{(k)}(\omega) = I_M^{(k)}(\omega)$ if $w[n]$ is a rectangular window

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Welch's Method of Averaging Modified Periodograms

- Following Welch's procedure, the mean of the averaged periodogram is obtained as

$$\mathrm{E}\left[B_{xx}^{w}(\omega)\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xx}(\theta)\,W\left(e^{j(\omega-\theta)}\right)\mathrm{d}\theta$$

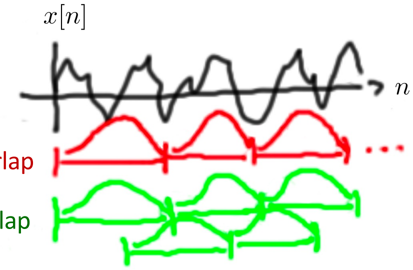
$$W(e^{j\omega}) = \frac{1}{MU}\left|\sum_{n=0}^{M-1} w[n]\,e^{-j\omega n}\right|^2$$

Note that if $w[n]$ is the rectangular window, $W(e^{j\omega})$ is the Fourier transform of the triangular window

- For the variance, we still have

$$\mathrm{var}\left[B_{xx}^{w}(\omega)\right] \approx \frac{1}{K}P_{xx}^{2}(\omega)$$

$x[n]$



0 % overlap

50 % overlap

Typically, overlapping segments are used

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