ECE 251A: Digital Signal Processing I
Gaussian Random Vectors

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Real Gaussian Random Vectors (GRVs)

Real GRVs: $\mathbf{X} \in R^N$. $\mathbf{X} \sim \mathcal{N}(\mu_{\mathbf{x}}, \Sigma_{xx})$

$$f_X(x) = \frac{1}{\sqrt{(2\pi)^N \det \Sigma_{xx}}} e^{-\frac{1}{2}(x-\mu_{\mathbf{x}})^T \Sigma_{xx}^{-1}(x-\mu_{\mathbf{x}})}$$

where the mean $E(\mathbf{X}) = \mu_{\mathbf{x}}$ and the $N \times N$ covariance matrix $\Sigma_{xx} = E[(\mathbf{X} - \mu_{\mathbf{x}})(\mathbf{X} - \mu_{\mathbf{x}})^T]$

Linear Transformation of GRVs

Linear Transformation of GRVs result in GRVs. If $\mathbf{Y} = \mathbf{AX} + \mathbf{d}$, then $\mathbf{Y} \sim \mathcal{N}(\mu_{\mathbf{y}} = \mathbf{A}\mu_{\mathbf{x}} + \mathbf{d}, \Sigma_{yy} = \mathbf{A}\Sigma_{xx}\mathbf{A}^T)$.

In addition, $\mathbf{Y}$ and $\mathbf{X}$ are jointly Gaussian, i.e. the vector $\begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix}$ is a Gaussian Random Vector with mean $\begin{bmatrix} \mu_{\mathbf{x}} \\ \mu_{\mathbf{y}} \end{bmatrix}$ and covariance matrix $\begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix}$, where $\Sigma_{yx} = \Sigma_{xy}^T = \mathbf{A}\Sigma_{xx}$.

Conditional Density

Let $\mathbf{Y}$ and $\mathbf{X}$ be jointly Gaussian, i.e. the vector $\begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix}$ is a Gaussian Random Vector with mean $\begin{bmatrix} \mu_{\mathbf{x}} \\ \mu_{\mathbf{y}} \end{bmatrix}$ and covariance matrix $\begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix}$.

Then the conditional density of $\mathbf{Y}$ given $\mathbf{X} = x$ is also a Gaussian density with conditional mean $\hat{\mathbf{Y}} = \mu_{\mathbf{y}} + \Sigma_{yx}\Sigma_{xx}^{-1}(x - \mu_{\mathbf{x}})$

and conditional covariance matrix $\Sigma_{y|x} = \Sigma_{yy} - \Sigma_{yx}\Sigma_{xx}^{-1}\Sigma_{xy}$.

Expectation of the product of four jointly Gaussian random variables

If X_1, X_2, X_3, X_4 are real, zero mean, jointly Gaussian random variables, then

$$E(X_1X_2X_3X_4) = E(X_1X_2)E(X_3X_4) + E(X_1X_3)E(X_2X_4) + E(X_1X_4)E(X_2X_3)$$

Complex Gaussian Random Vectors

Complex GRV: $\mathbf{X} \in C^N$. $\mathbf{X} = \mathbf{X}_R + j\mathbf{X}_I$ and involves two real jointly Gaussian random vectors $\mathbf{X}_R \in R^N$ and $\mathbf{X}_I \in R^N$.

$\begin{bmatrix} \mathbf{X}_R \\ \mathbf{X}_I \end{bmatrix}$ is a GRV in R^{2N}, with mean $\mu_{\mathbf{x}} = \begin{bmatrix} \mu_{\mathbf{x_R}} \\ \mu_{\mathbf{x_I}} \end{bmatrix}$ and covariance matrix $\begin{bmatrix} \Sigma_{x_R x_R} & \Sigma_{x_R x_I} \\ \Sigma_{x_I x_R} & \Sigma_{x_I x_I} \end{bmatrix}$.

Alternately, one can define this density using the complex quantities

$E(\mathbf{X}) = \mu_{\mathbf{x}} = \mu_{\mathbf{x_R}} + j\mu_{\mathbf{x_I}}$,

$\Sigma_{xx} = E[(\mathbf{X} - \mu_{\mathbf{x}})(\mathbf{X} - \mu_{\mathbf{x}})^H] = \Sigma_{x_R x_R} + \Sigma_{x_I x_I} + j\left(\Sigma_{x_I x_R} - \Sigma_{x_R x_I}\right)$ and

$\mathbf{J}_{xx} = E[(\mathbf{X} - \mu_{\mathbf{x}})(\mathbf{X} - \mu_{\mathbf{x}})^T] = \Sigma_{x_R x_R} - \Sigma_{x_I x_I} + j\left(\Sigma_{x_I x_R} + \Sigma_{x_R x_I}\right)$

Circular GRV

Circular GRV: $\mathbf{X} \in C^N$. $\mathbf{X} = \mathbf{X}_R + j\mathbf{X}_I$ and involves two real, jointly Gaussian random vectors $\mathbf{X}_R \in R^N$ and $\mathbf{X}_I \in R^N$ with $\Sigma_{x_R x_R} = \Sigma_{x_I x_I}$ and $\Sigma_{x_I x_R} + \Sigma_{x_R x_I} = \mathbf{0}$. Alternatively, $\mathbf{J}_{xx} = 0$.

$$f_X(x) = \frac{1}{\pi^N \det\Sigma} e^{-(x-\mu_{\mathbf{x}})^H \Sigma_{xx}^{-1}(x-\mu_{\mathbf{x}})}$$

Notation: $\mathbf{X} \sim \mathcal{CN}(\mu_{\mathbf{x}}, \Sigma_{xx})$.

$\mathbf{X} \sim \mathcal{CN}(0, \mathbf{I})$ refers to a complex GRV $\mathbf{X} = \mathbf{X}_R + j\mathbf{X}_I$, where $E(\mathbf{X}_R) = E(\mathbf{X}_I) = 0$, $E(\mathbf{X}_R\mathbf{X}_R^T) = E(\mathbf{X}_I\mathbf{X}_I^T) = \frac{1}{2}\mathbf{I}$, and $E(\mathbf{X}_R\mathbf{X}_I^T) = 0$

Linear Transformation of circular GRV also results in a circular GRV.

$\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{d}$. If $\mathbf{X} \sim \mathcal{CN}(\mu_x, \Sigma_{xx})$, then $\mathbf{Y} \sim \mathcal{CN}(\mathbf{A}\mu_x + \mathbf{d}, \mathbf{A}\Sigma_{xx}\mathbf{A}^H)$

Conditional Density involving

Let $\mathbf{Y}$ and $\mathbf{X}$ be jointly Circular Gaussian, i.e. the vector $\begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix}$ is a circular Gaussian Random Vector with mean $\begin{bmatrix} \mu_{\mathbf{x}} \\ \mu_{\mathbf{y}} \end{bmatrix}$ and covariance matrix $\begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix}$.

Then the conditional density of $\mathbf{Y}$ given $\mathbf{X} = x$ is also a circular Gaussian density with

conditional mean $\hat{\mathbf{Y}} = \mu_{\mathbf{y}} + \Sigma_{yx}\Sigma_{xx}^{-1}(x - \mu_{\mathbf{x}})$

and conditional covariance matrix $\Sigma_{y|x} = \Sigma_{yy} - \Sigma_{yx}\Sigma_{xx}^{-1}\Sigma_{xy}$.

Expectation of the product of four jointly circular Gaussian random variables

If X_1, X_2, X_3, X_4 are complex, zero mean, jointly circular Gaussian random variables, then

$$E(X_1 X_2^* X_3 X_4^*) = E(X_1 X_2^*) E(X_3 X_4^*) + E(X_1 X_4^*) E(X_3 X_2^*)$$