A BP Method for Track-Before-Detect: Supporting Derivations

Mingchao Liang, Thomas Kropfreiter, and Florian Meyer

This manuscript provides derivations for the letter, "Belief Propagation for Track-Before-Detect" by the same authors [1].

1 Approximation of BP Message $\kappa_{n,j}^{(\ell)}(\mathbf{y}_n)$

In this section, we derive the expression of the mean and covariance of message $\kappa_{n,j}^{(\ell)}(\mathbf{y}_n; \mathbf{z}_j)$, i.e., $\boldsymbol{\mu}_{\kappa,j}^{(\ell)}(\mathbf{y}_n)$ and $\mathbf{C}_{\kappa,j}^{(\ell)}(\mathbf{y}_n)$, that are used in the Gaussian approximation $\tilde{\kappa}_{n,j}^{(\ell)}(\mathbf{y}_n; \mathbf{z}_j) = \mathcal{N}\big(\mathbf{z}_j; \boldsymbol{\mu}_{\kappa,j}^{(\ell)}(\mathbf{y}_n), \mathbf{C}_{\kappa,j}^{(\ell)}(\mathbf{y}_n)\big)$ as discussed in Section III-C. This derivation relies on an interpretation of $\kappa_{n,j}^{(\ell)}(\mathbf{y}_n; \mathbf{z}_j)$ as a probability density function (PDF) of $\mathbf{z}_{k,j}$ conditioned on $\mathbf{y}_n$ and an interpretation of $\beta_{n,j}^{(\ell)}(\mathbf{y}_n)$ as a PDF of $\mathbf{y}_n$. Consequently, we introduce the notation $\tilde{p}^{(\ell)}(\mathbf{z}_j|\mathbf{y}_n) \triangleq \kappa_{n,j}^{(\ell)}(\mathbf{y}_n; \mathbf{z}_j)$ and $\tilde{p}_j^{(\ell)}(\mathbf{y}_n) \triangleq \beta_{n,j}^{(\ell)}(\mathbf{y}_n)$. Based on this new notation, the expression for $\kappa_{n,j}^{(\ell)}(\mathbf{y}_n; \mathbf{z}_j)$ in [1, Eq. (6)], reads

$$\tilde{p}^{(\ell)}(\mathbf{z}_j|\mathbf{y}_n) = \sum_{\mathbf{y}\backslash\mathbf{y}_n} p(\mathbf{z}_j|\mathbf{y}) \prod_{\substack{n'=1\\n'\neq n}}^{N} \tilde{p}_j^{(\ell)}(\mathbf{y}_{n'}). \tag{1}$$

1.1 Derivation of the Mean Vector $\boldsymbol{\mu}_{\kappa,j}^{(\ell)}(\mathbf{y}_n)$

The mean $\boldsymbol{\mu}_{\kappa,j}^{(\ell)}(\mathbf{y}_n)$, can now be computed as the expectation of $\mathbf{z}_j$ with respect to $\tilde{p}^{(\ell)}(\mathbf{z}_j|\mathbf{y}_n)$, i.e.,

$$\begin{aligned}\boldsymbol{\mu}_{\kappa,j}^{(\ell)}(\mathbf{y}_n) &= \int \mathbf{z}_j \bigg(\sum_{\mathbf{y}\backslash\mathbf{y}_n} p(\mathbf{z}_j|\mathbf{y}) \prod_{\substack{\tilde{n}=1\\\tilde{n}\neq n}}^{N} \tilde{p}_j^{(\ell)}(\mathbf{y}_{\tilde{n}}) \bigg) \mathrm{d}\mathbf{z}_j \\ &= \sum_{\mathbf{y}\backslash\mathbf{y}_n} \bigg(\int \mathbf{z}_j p(\mathbf{z}_j|\mathbf{y})\, \mathrm{d}\mathbf{z}_j \bigg) \prod_{\substack{\tilde{n}=1\\\tilde{n}\neq n}}^{N} \tilde{p}_j^{(\ell)}(\mathbf{y}_{\tilde{n}}) \\ &= \sum_{\mathbf{y}\backslash\mathbf{y}_n} \mathrm{E}(\mathbf{z}_j|\mathbf{y}) \prod_{\substack{\tilde{n}=1\\\tilde{n}\neq n}}^{N} \tilde{p}_j^{(\ell)}(\mathbf{y}_{\tilde{n}}).\end{aligned} \tag{2}$$

By further using the measurement model in [1, Eq. (1)], we obtain

$$\begin{aligned}\boldsymbol{\mu}_{\kappa,j}^{(\ell)}(\mathbf{y}_n) &= \sum_{\mathbf{y}\backslash\mathbf{y}_n} \mathrm{E}\bigg(\sum_{n'=1}^{N} r_{n'}\mathbf{h}_{j,n'} + \boldsymbol{\epsilon}_j \;\Big|\; \mathbf{y} \bigg) \prod_{\substack{\tilde{n}=1\\ \tilde{n}\neq n}}^{N} \tilde{p}_j^{(\ell)}(\mathbf{y}_{\tilde{n}}) \\ &= \sum_{\mathbf{y}\backslash\mathbf{y}_n} \bigg(\sum_{n'=1}^{N} r_{n'}\boldsymbol{\mu}_j(\mathbf{x}_{n'}) \bigg) \prod_{\substack{\tilde{n}=1\\ \tilde{n}\neq n}}^{N} \tilde{p}_j^{(\ell)}(\mathbf{y}_{\tilde{n}}) \\ &= \sum_{n'=1}^{N} \sum_{\mathbf{y}\backslash\mathbf{y}_n} r_{n'}\boldsymbol{\mu}_j(\mathbf{x}_{n'}) \prod_{\substack{\tilde{n}=1\\ \tilde{n}\neq n}}^{N} \tilde{p}_j^{(\ell)}(\mathbf{y}_{\tilde{n}}) \end{aligned} \tag{3}$$

where $\boldsymbol{\mu}_j(\mathbf{x}_n)$ is the mean of the Gaussian PDF $p(\mathbf{h}_{j,n}|\mathbf{x}_n)$ introduced in Section II-A.

Since $\tilde{p}_j^{(\ell)}(\mathbf{y}_{\tilde{n}})$ are PDFs that sum and integrate to one, we obtain

$$\sum_{\mathbf{y}\backslash\mathbf{y}_n} r_{n'}\boldsymbol{\mu}_j(\mathbf{x}_{n'}) \prod_{\substack{\tilde{n}=1\\ \tilde{n}\neq n}}^{N} \tilde{p}_j^{(\ell)}(\mathbf{y}_{\tilde{n}}) = \sum_{\mathbf{y}_{n'}} r_{n'}\boldsymbol{\mu}_j(\mathbf{x}_{n'})\tilde{p}_j^{(\ell)}(\mathbf{y}_{n'})$$

for $n' \neq n$, and

$$\sum_{\mathbf{y}\backslash\mathbf{y}_n} r_{n'}\boldsymbol{\mu}_j(\mathbf{x}_{n'}) \prod_{\substack{\tilde{n}=1\\ \tilde{n}\neq n}}^{N} \tilde{p}_j^{(\ell)}(\mathbf{y}_{\tilde{n}}) = r_n\boldsymbol{\mu}_j(\mathbf{x}_n).$$

for $n' = n$. By plugging this result into (3), we obtain

$$\begin{aligned}\boldsymbol{\mu}_{\kappa,j}^{(\ell)}(\mathbf{y}_n) &= r_n\boldsymbol{\mu}_j(\mathbf{x}_n) + \sum_{\substack{n'=1\\ n'\neq n}}^{N} \sum_{\mathbf{y}_{n'}} r_{n'}\boldsymbol{\mu}_j(\mathbf{x}_{n'})\tilde{p}_j^{(\ell)}(\mathbf{y}_{n'}) \\ &= r_n\boldsymbol{\mu}_j(\mathbf{x}_n) + \sum_{\substack{n'=1\\ n'\neq n}}^{N} \sum_{r_{n'}\in\{0,1\}} \int r_{n'}\boldsymbol{\mu}_j(\mathbf{x}_{n'})\tilde{p}_j^{(\ell)}(\mathbf{x}_{n'}, r_{n'})\, \mathrm{d}\mathbf{x}_{n'} \\ &= r_n\boldsymbol{\mu}_j(\mathbf{x}_n) + \sum_{\substack{n'=1\\ n'\neq n}}^{N} \int \boldsymbol{\mu}_j(\mathbf{x}_{n'})\tilde{p}_j^{(\ell)}(\mathbf{x}_{n'}, 1)\, \mathrm{d}\mathbf{x}_{n'}. \end{aligned} \tag{4}$$

Finally, we introduce $\boldsymbol{\mu}_{n,j}^{(\ell)} = \int \boldsymbol{\mu}_j(\mathbf{x}_n)\tilde{p}_j^{(\ell)}(\mathbf{x}_n, 1)\, \mathrm{d}\mathbf{x}_n$ to get the expression

$$\boldsymbol{\mu}_{\kappa,j}^{(\ell)}(\mathbf{y}_n) = r_n\boldsymbol{\mu}_j(\mathbf{x}_n) + \sum_{\substack{n'=1\\ n'\neq n}}^{N} \boldsymbol{\mu}_{n',j}^{(\ell)}. \tag{5}$$

1.2 Derivation of the Covariance Matrix $\mathbf{C}^{(\ell)}_{\kappa,j}(\mathbf{y}_n)$

To get the covariance $\mathbf{C}^{(\ell)}_{\kappa,j}(\mathbf{y}_n)$, we first compute the correlation matrix $\mathbf{R}^{(\ell)}_{\kappa,j}(\mathbf{y}_n)$, as the expectation of $\mathbf{z}_j\mathbf{z}_j^{\mathrm{T}}$ with respect to $\tilde{p}^{(\ell)}(\mathbf{z}_j|\mathbf{y}_n)$

$$\begin{aligned}
\mathbf{R}^{(\ell)}_{\kappa,j}(\mathbf{y}_n) &= \int \mathbf{z}_j\mathbf{z}_j^{\mathrm{T}}\Bigg(\sum_{\mathbf{y}\backslash\mathbf{y}_n} p(\mathbf{z}_j|\mathbf{y}) \prod_{\substack{\tilde{n}=1\\ \tilde{n}\neq n}}^{N} \tilde{p}^{(\ell)}_j(\mathbf{y}_{\tilde{n}})\Bigg)\,\mathrm{d}\mathbf{z}_j \\
&= \sum_{\mathbf{y}\backslash\mathbf{y}_n}\bigg(\int \mathbf{z}_j\mathbf{z}_j^{\mathrm{T}} p(\mathbf{z}_j|\mathbf{y})\,\mathrm{d}\mathbf{z}_j\bigg) \prod_{\substack{\tilde{n}=1\\ \tilde{n}\neq n}}^{N} \tilde{p}^{(\ell)}_j(\mathbf{y}_{\tilde{n}}) \\
&= \sum_{\mathbf{y}\backslash\mathbf{y}_n} \mathrm{E}(\mathbf{z}_j\mathbf{z}_j^{\mathrm{T}}|\mathbf{y}) \prod_{\substack{\tilde{n}=1\\ \tilde{n}\neq n}}^{N} \tilde{p}^{(\ell)}_j(\mathbf{y}_{\tilde{n}}).
\end{aligned} \tag{6}$$

Since r_n is binary, we have $r_n^2 = r_n$. In addition, $\mathrm{E}(\mathbf{h}_{j,n}\mathbf{h}_{j,n}^{\mathrm{T}}|\mathbf{x}) = \mathbf{C}_j(\mathbf{x}_n) + \boldsymbol{\mu}_j(\mathbf{x}_n)\boldsymbol{\mu}_j^{\mathrm{T}}(\mathbf{x}_n)$, where $\mathbf{C}_j(\mathbf{x}_n)$ is the covariance matrix of the Gaussian PDF $p(\mathbf{h}_{j,n}|\mathbf{x}_n)$ introduced in Section II-A. Next, by making use of the measurement model [1, Eq. (6)], we obtain

$$\begin{aligned}
\mathbf{R}^{(\ell)}_{\kappa,j}(\mathbf{y}_n) &= \sum_{\mathbf{y}\backslash\mathbf{y}_n} \mathrm{E}\bigg(\Big(\sum_{n'=1}^{N} r_{n'}\mathbf{h}_{j,n'} + \boldsymbol{\epsilon}_j\Big)\Big(\sum_{n'=1}^{N} r_{n'}\mathbf{h}_{j,n'} + \boldsymbol{\epsilon}_j\Big)^{\mathrm{T}} \,\Big|\, \mathbf{y}\bigg) \prod_{\substack{\tilde{n}=1\\ \tilde{n}\neq n}}^{N} \tilde{p}^{(\ell)}_j(\mathbf{y}_{\tilde{n}}) \\
&= \sum_{\mathbf{y}\backslash\mathbf{y}_n}\bigg(\sum_{n'=1}^{N} r_{n'}\big(\mathbf{C}_j(\mathbf{x}_{n'}) + \boldsymbol{\mu}_j(\mathbf{x}_{n'})\boldsymbol{\mu}_j^{\mathrm{T}}(\mathbf{x}_{n'})\big) + \mathbf{C}_{\boldsymbol{\epsilon}} \\
&\qquad + \sum_{n'=1}^{N}\sum_{\substack{n''=1\\ n''\neq n'}}^{N} r_{n'}r_{n''}\boldsymbol{\mu}_j(\mathbf{x}_{n'})\boldsymbol{\mu}_j^{\mathrm{T}}(\mathbf{x}_{n''})\bigg) \prod_{\substack{\tilde{n}=1\\ \tilde{n}\neq n}}^{N} \tilde{p}^{(\ell)}_j(\mathbf{y}_{\tilde{n}}) \\
&= \mathbf{C}_{\boldsymbol{\epsilon}} + \sum_{n'=1}^{N} \underbrace{\sum_{\mathbf{y}\backslash\mathbf{y}_n} r_{n'}\big(\mathbf{C}_j(\mathbf{x}_{n'}) + \boldsymbol{\mu}_j(\mathbf{x}_{n'})\boldsymbol{\mu}_j^{\mathrm{T}}(\mathbf{x}_{n'})\big) \prod_{\substack{\tilde{n}=1\\ \tilde{n}\neq n}}^{N} \tilde{p}^{(\ell)}_j(\mathbf{y}_{\tilde{n}})}_{\triangleq \mathbf{R}_1} \\
&\qquad + \sum_{n'=1}^{N}\sum_{\substack{n''=1\\ n''\neq n'}}^{N} \underbrace{\sum_{\mathbf{y}\backslash\mathbf{y}_n} r_{n'}r_{n''}\boldsymbol{\mu}_j(\mathbf{x}_{n'})\boldsymbol{\mu}_j^{\mathrm{T}}(\mathbf{x}_{n''}) \prod_{\substack{\tilde{n}=1\\ \tilde{n}\neq n}}^{N} \tilde{p}^{(\ell)}_j(\mathbf{y}_{\tilde{n}})}_{\triangleq \mathbf{R}_2}.
\end{aligned} \tag{7}$$

Next, we discuss the detailed expression of $\mathbf{R}_1$ and $\mathbf{R}_2$. For $\mathbf{R}_1$, we obtain

$$\mathbf{R}_1 = r_{n'}\big(\mathbf{C}_j(\mathbf{x}_{n'}) + \boldsymbol{\mu}_j(\mathbf{x}_{n'})\boldsymbol{\mu}_j^{\mathrm{T}}(\mathbf{x}_{n'})\big)$$

for $n' = n$, and

$$\mathbf{R}_1 = \sum_{\mathbf{y}_{n'}} r_{n'}\big(\mathbf{C}_j(\mathbf{x}_{n'}) + \boldsymbol{\mu}_j(\mathbf{x}_{n'})\boldsymbol{\mu}_j^{\mathrm{T}}(\mathbf{x}_{n'})\big)\tilde{p}^{(\ell)}_j(\mathbf{y}_{n'})$$

otherwise. Similarly, for $\mathbf{R}_2$, we get

$$\mathbf{R}_2 = r_n\boldsymbol{\mu}_j(\mathbf{x}_n)\sum_{\mathbf{y}_{n'}} r_{n'}\boldsymbol{\mu}_j^{\mathrm{T}}(\mathbf{x}_{n'})\tilde{p}^{(\ell)}_j(\mathbf{y}_{n'})$$

if $n' = n$ or $n'' = n$, and

$$\mathbf{R}_2 = \Big(\sum_{\mathbf{y}_{n'}} r_{n'} \boldsymbol{\mu}_j(\mathbf{x}_{n'}) \tilde{p}_j^{(\ell)}(\mathbf{y}_{n'})\Big)\Big(\sum_{\mathbf{y}_{n''}} r_{n''} \boldsymbol{\mu}_j(\mathbf{x}_{n''}) \tilde{p}_j^{(\ell)}(\mathbf{y}_{n''})\Big)^{\mathrm{T}}$$

otherwise. By using Using these results in (7), we obtain

$$\begin{aligned}
\mathbf{R}_{\kappa,j}^{(\ell)}(\mathbf{y}_n) &= \mathbf{C}_{\boldsymbol{\epsilon}} + r_n \mathbf{C}_j(\mathbf{x}_n) + r_n \boldsymbol{\mu}_j(\mathbf{x}_n)\boldsymbol{\mu}_j^{\mathrm{T}}(\mathbf{x}_n) \\
&\quad + \sum_{\substack{n'=1 \\ n'\neq n}}^{N} \sum_{\mathbf{y}_{n'}} r_{n'} \Big(\mathbf{C}_j(\mathbf{x}_{n'}) + \boldsymbol{\mu}_j(\mathbf{x}_{n'})\boldsymbol{\mu}_j^{\mathrm{T}}(\mathbf{x}_{n'})\Big)\tilde{p}_j^{(\ell)}(\mathbf{y}_{n'}) \\
&\quad + 2 r_n \boldsymbol{\mu}_j(\mathbf{x}_n) \sum_{\substack{n'=1 \\ n'\neq n}}^{N} \sum_{\mathbf{y}_{n'}} r_{n'} \boldsymbol{\mu}_j^{\mathrm{T}}(\mathbf{x}_{n'}) \tilde{p}_j^{(\ell)}(\mathbf{y}_{n'}) \\
&\quad + \sum_{\substack{n'=1 \\ n'\neq n}}^{N} \sum_{\substack{n''=1 \\ n''\neq n,n'}}^{N} \Big(\sum_{\mathbf{y}_{n'}} r_{n'} \boldsymbol{\mu}_j(\mathbf{x}_{n'}) \tilde{p}_j^{(\ell)}(\mathbf{y}_{n'})\Big)\Big(\sum_{\mathbf{y}_{k,n''}} r_{n''} \boldsymbol{\mu}_j(\mathbf{x}_{n''}) \tilde{p}_j^{(\ell)}(\mathbf{y}_{n''})\Big)^{\mathrm{T}} \\
&= \mathbf{C}_{\boldsymbol{\epsilon}} + r_n \mathbf{C}_j(\mathbf{x}_n) + r_n \boldsymbol{\mu}_j(\mathbf{x}_n)\boldsymbol{\mu}_j^{\mathrm{T}}(\mathbf{x}_n) + 2 r_n \boldsymbol{\mu}_j(\mathbf{x}_n) \sum_{\substack{n'=1 \\ n'\neq n}}^{N} \boldsymbol{\mu}_{n',j}^{(\ell)\mathrm{T}} \\
&\quad + \sum_{\substack{n'=1 \\ n'\neq n}}^{N} \mathbf{R}_{n',j}^{(\ell)} + \sum_{\substack{n'=1 \\ n'\neq n}}^{N} \sum_{\substack{n''=1 \\ n''\neq n,n'}}^{N} \boldsymbol{\mu}_{n',j}^{(\ell)} \boldsymbol{\mu}_{n'',j}^{(\ell)\mathrm{T}}
\end{aligned} \tag{8}$$

where we have introduced

$$\begin{aligned}
\mathbf{R}_{n,j}^{(\ell)} &= \mathrm{E}_{n,j}^{(\ell)}\Big(\mathbf{C}_j(\mathbf{x}_n) + \boldsymbol{\mu}_j(\mathbf{x}_n)\boldsymbol{\mu}_j^{\mathrm{T}}(\mathbf{x}_n)\Big) \\
&= \int \Big(\mathbf{C}_j(\mathbf{x}_n) + \boldsymbol{\mu}_j(\mathbf{x}_n)\boldsymbol{\mu}_j^{\mathrm{T}}(\mathbf{x}_n)\Big)\tilde{p}_j^{(\ell)}(\mathbf{x}_n, 1)\, \mathrm{d}\mathbf{x}_n.
\end{aligned}$$

The final covariance matrix can now be computed as

$$\begin{aligned}
\mathbf{C}_{\kappa,j}^{(\ell)}(\mathbf{y}_n) &= \mathbf{R}_{\kappa,j}^{(\ell)}(\mathbf{y}_n) - \Big(\boldsymbol{\mu}_{\kappa,j}^{(\ell)}(\mathbf{y}_n)\Big)\Big(\boldsymbol{\mu}_{\kappa,j}^{(\ell)}(\mathbf{y}_n)\Big)^{\mathrm{T}} \\
&= r_n \mathbf{C}_j(\mathbf{x}_n) + \mathbf{C}_{\boldsymbol{\epsilon}} + \sum_{\substack{n'=1 \\ n'\neq n}}^{N} \Big(\mathbf{R}_{n',j}^{(\ell)} - \boldsymbol{\mu}_{n',j}^{(\ell)} \boldsymbol{\mu}_{n',j}^{(\ell)\mathrm{T}}\Big).
\end{aligned} \tag{9}$$

References

[1] M. Liang, T. Kropfreiter, and F. Meyer, “A BP method for track-before-detect,” 2023, submitted.