ECE 286: Bayesian Machine Perception

Class 2: Bayesian Estimators

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Recap: Bayes Rule

- Recall $f(\boldsymbol{x},\boldsymbol{y}) = f(\boldsymbol{x}|\boldsymbol{y})\,f(\boldsymbol{y}) = f(\boldsymbol{y}|\boldsymbol{x})\,f(\boldsymbol{x})$

- It therefore follows that

$$\underbrace{f(\boldsymbol{x}|\boldsymbol{y})}_{\text{posterior}} = \frac{\overbrace{\boxed{f(\boldsymbol{y}|\boldsymbol{x})}}^{\text{likelihood (}\boldsymbol{y}\text{ is fixed)}}\;\overbrace{\boxed{f(\boldsymbol{x})}}^{\text{prior}}}{\underbrace{\boxed{f(\boldsymbol{y})}}_{\text{evidence}}}$$

Expectation and Covariance

- Expectation of a random variable $\boldsymbol{x}$

discrete case

$$\mathbb{E}\{\boldsymbol{x}\} = \sum_{\boldsymbol{x} \in \mathcal{X}} \boldsymbol{x}\, p(\boldsymbol{x})$$

continuous case

$$\mathbb{E}\{\boldsymbol{x}\} = \int \boldsymbol{x}\, f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$$

- Expectation of transformed random variable $g(\boldsymbol{x})$

$$\mathbb{E}\{g(\boldsymbol{x})\} = \sum_{\boldsymbol{x} \in \mathcal{X}} g(\boldsymbol{x})\, p(\boldsymbol{x}) \qquad \mathbb{E}\{g(\boldsymbol{x})\} = \int g(\boldsymbol{x})\, f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$$

- Covariance of a random variable $\boldsymbol{x}$

$$\mathbb{C}\{\boldsymbol{x}\} = \mathbb{E}\big\{(\boldsymbol{x} - \mathbb{E}\{\boldsymbol{x}\})(\boldsymbol{x} - \mathbb{E}\{\boldsymbol{x}\})^{\mathrm{T}}\big\} = \mathbb{E}\big\{\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}\big\} - \mathbb{E}\{\boldsymbol{x}\}\mathbb{E}\{\boldsymbol{x}\}^{\mathrm{T}}$$

Bayes Risk and Bayesian Estimator

- An estimator is a rule for calculating an estimate $\hat{\boldsymbol{x}}$ of a given quantity $\boldsymbol{x}$ based on measurements $\boldsymbol{y}$

- The cost $C(\boldsymbol{\epsilon})$ is a scalar-valued nonnegative function of the error $\boldsymbol{\epsilon} = \boldsymbol{x} - \hat{\boldsymbol{x}}(\boldsymbol{y})$

- The Bayes risk is defined as the mean cost

$$r \triangleq \mathbb{E}\{C\big(\boldsymbol{x} - \hat{\boldsymbol{x}}(\boldsymbol{y})\big)\} = \int_{\boldsymbol{x}} \int_{\boldsymbol{y}} C\big(\boldsymbol{x} - \hat{\boldsymbol{x}}(\boldsymbol{y})\big) f(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{y}$$

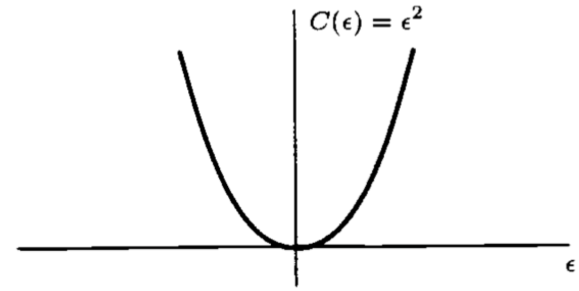
- The Bayesian estimator minimizing the Bayes risk among all possible estimators,

$$\hat{\boldsymbol{x}}^{\mathrm{B}}(\cdot) \triangleq \arg\min_{\hat{\boldsymbol{x}}(\cdot)} r$$

MMSE Estimator

- Quadratic cost:

$$C(\boldsymbol{\epsilon}) = \|\boldsymbol{\epsilon}\|^2$$

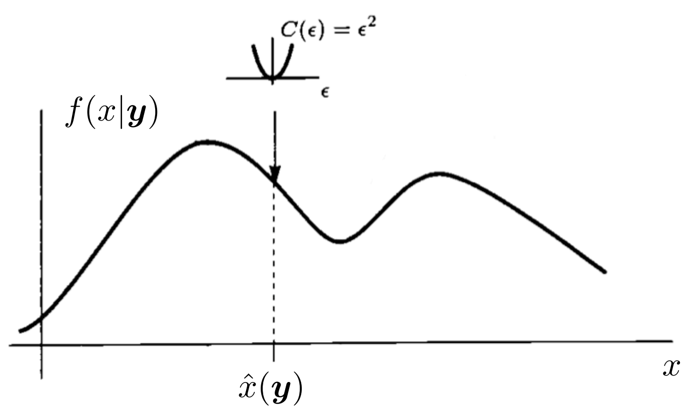


Quadratic error

- Minimum mean-square error (MMSE) estimator

$$\hat{\boldsymbol{x}}(\boldsymbol{y}) = \int \boldsymbol{x} \boxed{f(\boldsymbol{x}|\boldsymbol{y})} \mathrm{d}\boldsymbol{x}$$

posterior

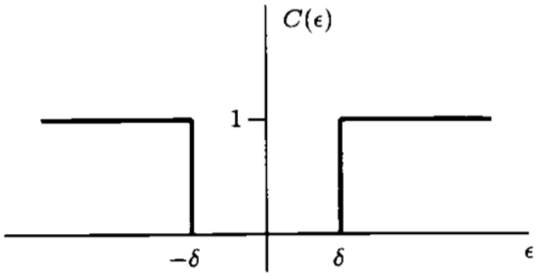


S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory*, Prentice-Hall, 1993.

MAP Estimator

- ``Hit-or-Miss'' cost:

$$C(\boldsymbol{\epsilon}) = \begin{cases} 0 & \text{if } \|\boldsymbol{\epsilon}\| < \delta \quad (\text{for } \delta \to 0) \\ 1 & \text{otherwise} \end{cases}$$

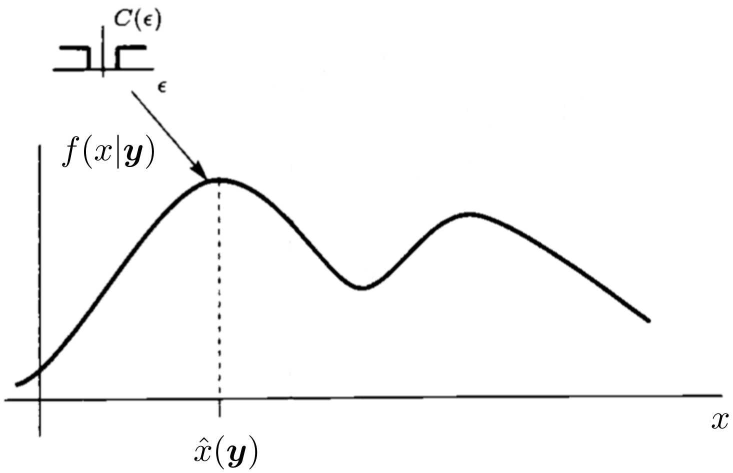


Hit-or-miss error

- Maximum a posteriori (MAP) estimator

$$\hat{\boldsymbol{x}} = \arg\max_{\boldsymbol{x}} \boxed{f(\boldsymbol{x}|\boldsymbol{y})}$$

posterior



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