SIO 209: Signal Processing for Ocean Sciences
Class 15

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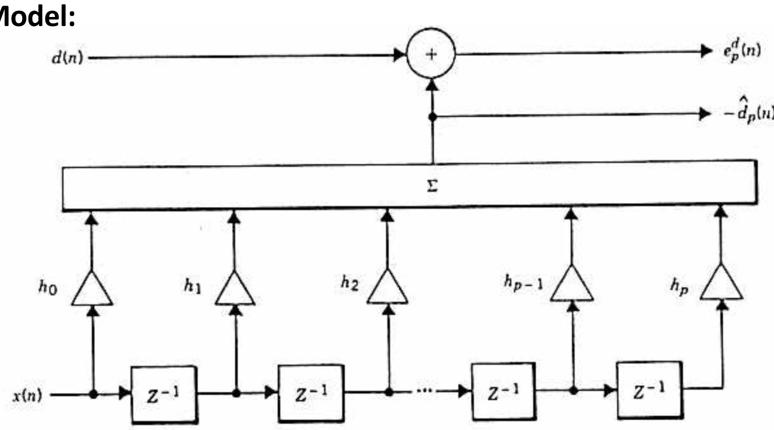
Wiener Filtering and Least Squares Problem

- **General Filter Model:**



- **Goal:** Find FIR filter coefficients $h_0 \dots h_p$ such that the output of the FIR filter, $-\hat{d}_p[n]$, is as similar as possible to $d[n]$

D. DeFatta, J. Lucas, and W. Hodgkiss, "Digital Signal Processing." Wiley, 1988

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Wiener Filtering and Least Squares Problem

- We want to filter the time series $x[n]$ such that it yields an estimate of the time series $d[n]$
- The estimation error is denoted by

$$e^d[n] = d[n] + \sum_{k=0}^{p} h_k x[n-k]$$

- Vector notation of estimation error

$$e^d[n] = d[n] + \boldsymbol{h}^{\mathrm{T}}\boldsymbol{x} \qquad \boldsymbol{h} = [h_0\, h_1 \cdots h_p]^{\mathrm{T}} \qquad \boldsymbol{x} = \big[x[n]\;\, x[n-1]\; \cdots\; x[n-p]\big]^{\mathrm{T}}$$

- The optimality criteria for estimation are given by

$\min\limits_{\boldsymbol{h}} \mathrm{E}\Big\{\big|e^d[n]\big|^2\Big\}$ Wiener Filtering Problem $\qquad \min\limits_{\boldsymbol{h}} \sum\limits_{n} \big|e^d[n]\big|^2$ Least Squares Problem

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Wiener Filtering Problem

- For the Wiener filtering problem, we introduce the scaler ϕ^d_{00}, the vector $\boldsymbol{g}$, and the matrix as Φ

$$\phi^d_{00} = \mathrm{E}\big\{d[n]\, d^*[n]\big\}$$
$$\boldsymbol{g} = [g_0\, g_1 \cdots g_p]^{\mathrm{T}}$$

$$\Phi = \begin{bmatrix} \phi_{00} & \cdots & \phi_{0p} \\ \vdots & & \vdots \\ \phi_{p0} & \cdots & \phi_{pp} \end{bmatrix} = \begin{bmatrix} \phi_{00} & \cdots & \phi_{0p} \\ \vdots & & \vdots \\ \phi^*_{0p} & \cdots & \phi_{pp} \end{bmatrix}$$

where we have used $g_j = \mathrm{E}\big\{d[n]x^*[n-j]\big\}$ and $\phi_{kj} = \mathrm{E}\big\{x[n-k]x^*[n-j]\big\}$

- The Wiener filtering problem follows a stochastic formulation and optimizes the filter in an expected value sense

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Least Squares Problem

- Alternatively, the least-square problem follows a deterministic formulation and optimizes for specific signal sequences
- Here, the statistics of the signals are computed from the data, i.e.,

$$\phi_{00}^{d} = \sum_{n} d[n]\, d^{*}[n] \qquad g_j = \sum_{n} d[n]\, x^{*}[n-j] \qquad \phi_{kj} = \sum_{n} x[n-k]\, x^{*}[n-j]$$

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Optimal Solution and Orthogonality Principle

- To solve for the unknown vector $\boldsymbol{h}$, we write the squared error as follows

$$\left|\, e^{d}[n] \,\right|^{2} = d^{*}[n]\, d[n] + d^{*}[n]\, h^{\mathrm{T}} \boldsymbol{x} + \boldsymbol{h}^{\mathrm{H}} \boldsymbol{x}^{*} d[n] + \boldsymbol{h}^{\mathrm{H}} \boldsymbol{x}^{*} \boldsymbol{h}^{\mathrm{T}} \boldsymbol{x}$$

- By making use of ϕ_{00}^{d}, $\boldsymbol{g}$, and Φ, the mean squared error can be obtained as

$$\mathrm{E}_{p}^{d}(\boldsymbol{h}) = \mathrm{E}\left\{ \left|\, e^{d}[n] \,\right|^{2} \right\} = \phi_{00}^{d} + \boldsymbol{g}^{\mathrm{H}} \boldsymbol{h} + \boldsymbol{h}^{\mathrm{H}} \boldsymbol{g} + \boldsymbol{h}^{\mathrm{H}} \Phi^{\mathrm{T}} \boldsymbol{h}$$

- Minimizing $\mathrm{E}_{p}^{d}(h)$ with respect to h yields

$$0 = \boldsymbol{g} + \Phi^{T} \boldsymbol{h} \qquad \text{or} \qquad \Phi^{T} \boldsymbol{h} = -\boldsymbol{g}$$

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Optimal Solution and Orthogonality Principle

- The optimum coefficient vector of the Wiener filter is thus obtained as $\hat{\boldsymbol{h}} = -\left(\Phi^{\mathrm{T}}\right)^{-1}\boldsymbol{g}$
- By using this solution in the expression of the mean-squared error, we obtain

$$\mathrm{E}_p^d(\hat{\boldsymbol{h}}) = \phi_{00}^d - \boldsymbol{g}^{\mathrm{H}}(\boldsymbol{\Phi}^{\mathrm{T}})^{-1}\boldsymbol{g}$$

- Finally, let's compute the expectation of the product of the error and the complex conjugate of the data, i.e.,

$$\begin{aligned}\mathrm{E}\left\{e^d[n]\boldsymbol{x}^*\right\} &= \mathrm{E}\left\{d[n]\boldsymbol{x}^* + \boldsymbol{h}^{\mathrm{T}}\boldsymbol{x}\boldsymbol{x}^*\right\} \\ &= \boldsymbol{g} + \Phi^{\mathrm{T}}\boldsymbol{h}\end{aligned}$$

- It is straightforward to see that for $\boldsymbol{h} = \hat{\boldsymbol{h}}$, we have $\mathrm{E}\left\{e^d[n]\boldsymbol{x}^*\right\} = 0$

$\longrightarrow$ **Orthogonality Principle of Linear Estimation**: The error of an optimum linear estimator is always orthogonal to the data

Recursive Implementation of the Wiener Filter

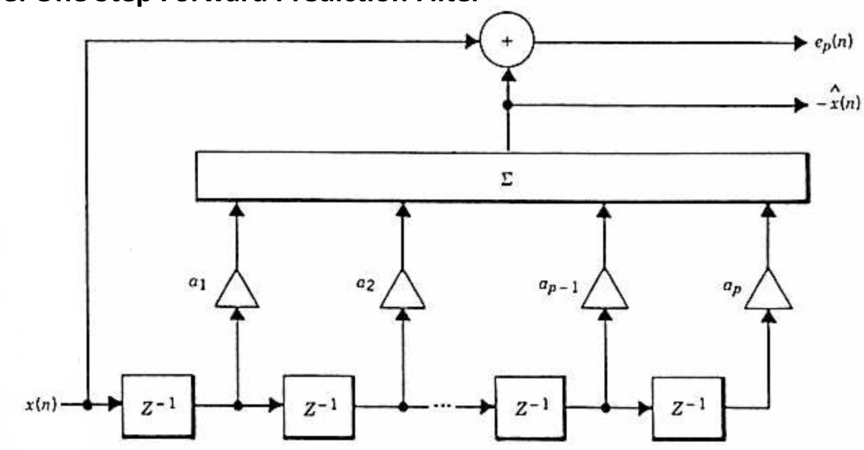
- When the time series $x[n]$ is stationary, the matrix Φ has a Toepliz structure, i.e., the elements on each diagonal are identical
- This property can be expressed mathematically as $\phi_{k,j} = \phi_{k-j}$
- The **Levinson–Durbin algorithm** exploits the structure to avoid direct computation of Φ^{-1}
- In particular, $\hat{\boldsymbol{h}}_p$ is recursively calculated from $\hat{\boldsymbol{h}}_{p-1}$ (see details in Chapter 11 of [*DeFatta et al., 1988*] and Chapter 11 of [*Oppenheim & Schafer, 2009*])

D. DeFatta, J. Lucas, and W. Hodgkiss, "Digital Signal Processing." Wiley, 1988

A. Oppenheim and R. Schafer, "Discrete Time Signal Processing." Prentice Hall, 2009

One Step Prediction Filter

- **Special Case: One Step Forward Prediction Filter**



It is straightforward to see that the one step forward prediction filter used for high resolution spectral analysis is a special case of a Wiener filter

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