ECE 275A Slides 13:
The Unscented Kalman Filter

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The Unscented Kalman Filter

- Consider a random vector $\boldsymbol{x}$ whose mean $\boldsymbol{\mu}_{\boldsymbol{x}}$ and covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{x}}$ are known, and a transformed random vector $\boldsymbol{z} = \boldsymbol{q}(\boldsymbol{x})$
- So called sigma points (SPs) can be used to calculate approximations of the mean $\boldsymbol{\mu}_{\boldsymbol{z}}$, covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{z}}$, and cross-covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{xz}}$
- The resulting approximations are at least as good as those obtained by linearizing $\boldsymbol{q}(\cdot)$ using a first-order Taylor expansion (as done in the extended Kalman filter), since higher-order terms of the expansion are also partly taken into account [Julier, et al., 2000]
- In addition, contrary to an approximation based on the first-order Taylor expansion, an SP-based approximation can also be performed for models that are not differentiable

S. Julier, J. Uhlmann, and H. F. Durrant-Whyte, *A New Method for the Nonlinear Transformation of Means and Covariances in Filters and Estimators*, *IEEE Trans. Automat. Contr.*, 2000

Calculation of Sigma Points

- For a L-dimensional random vector $\boldsymbol{x}$, SPs $\left\{(\boldsymbol{x}^{(j)})\right\}_{j=1}^{2L}$ can be calculated as

$$\boldsymbol{x}^{(j)} = \begin{cases} \boldsymbol{\mu}_{\boldsymbol{x}} + \sqrt{L}\left(\boldsymbol{\Sigma}_{\boldsymbol{x}}^{1/2}\right)_j, & j = 1, \ldots, L \\ \boldsymbol{\mu}_{\boldsymbol{x}} - \sqrt{L}\left(\boldsymbol{\Sigma}_{\boldsymbol{x}}^{1/2}\right)_j, & j = L+1, \ldots, 2L \end{cases}$$

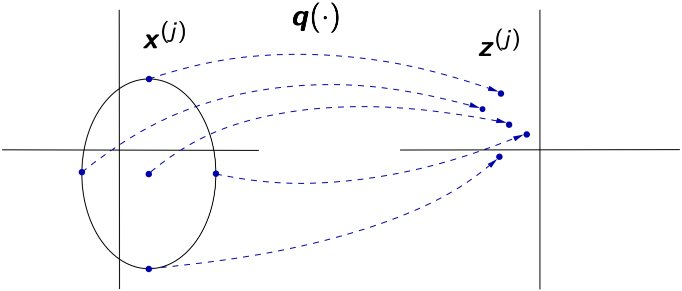
  Here, $\left(\boldsymbol{\Sigma}_{\boldsymbol{x}}^{1/2}\right)_j$ is the jth row or column of the matrix square root of $\boldsymbol{\Sigma}_{\boldsymbol{x}}$

- SPs have the property that the sample mean $\tilde{\boldsymbol{\mu}}_{\boldsymbol{x}} = \frac{1}{2L}\sum_{j=1}^{2L} \boldsymbol{x}^{(j)}$ and sample covariance matrix $\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{x}} = \frac{1}{2L}\sum_{j=1}^{2L}(\boldsymbol{x}^{(j)} - \tilde{\boldsymbol{\mu}}_{\boldsymbol{x}})(\boldsymbol{x}^{(j)} - \tilde{\boldsymbol{\mu}}_{\boldsymbol{x}})^{\mathrm{T}}$ are exactly equal to $\boldsymbol{\mu}_{\boldsymbol{x}}$ and $\boldsymbol{\Sigma}_{\boldsymbol{x}}$, respectively

Unscented Transformation

- SPs $\left\{\boldsymbol{z}^{(j)}\right\}_{j=1}^{2L}$ of $\boldsymbol{z}$ can be obtained by propagating each SP $\boldsymbol{x}^{(j)}$ through $\boldsymbol{q}(\cdot)$:

$$\boldsymbol{z}^{(j)} = \boldsymbol{q}\big(\boldsymbol{x}^{(j)}\big), \qquad j = 1, \ldots, 2L$$



Calculation of Mean and Covariance

- From $\left\{\left(\boldsymbol{x}^{(j)}, \boldsymbol{z}^{(j)}\right\}_{j=1}^{2L}$, one can calculate approximations of $\boldsymbol{\mu}_{\boldsymbol{z}}$, $\boldsymbol{\Sigma}_{\boldsymbol{z}}$ and $\boldsymbol{\Sigma}_{\boldsymbol{xz}}$ as

$$\tilde{\boldsymbol{\mu}}_{\boldsymbol{z}} = \frac{1}{2L}\sum_{j=1}^{2L} \boldsymbol{z}^{(j)}$$

$$\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{z}} = \frac{1}{2L}\sum_{j=1}^{2L} (\boldsymbol{z}^{(j)} - \tilde{\boldsymbol{\mu}}_{\boldsymbol{z}})(\boldsymbol{z}^{(j)} - \tilde{\boldsymbol{\mu}}_{\boldsymbol{z}})^{\mathrm{T}}$$

$$\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{xz}} = \frac{1}{2L}\sum_{j=1}^{2L} (\boldsymbol{x}^{(j)} - \tilde{\boldsymbol{\mu}}_{\boldsymbol{x}})(\boldsymbol{z}^{(j)} - \tilde{\boldsymbol{\mu}}_{\boldsymbol{z}})^{\mathrm{T}}$$

UKF Prediction Step

- We assume the following nonlinear state-transition model
$$\boldsymbol{x}_n = \boldsymbol{w}_n + \boldsymbol{u}_n \quad \text{with} \;\; \boldsymbol{w}_n = \boldsymbol{g}_n(\boldsymbol{x}_n) \qquad (1)$$
Here, the noise $\boldsymbol{u}_n$ is zero-mean and has known covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{u}_n}$; all assumptions of sequential Bayesian estimation discussed previously apply
- As derived in a previous class, the prediction step of sequential Bayesian estimation reads
$$\begin{aligned} f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1}) &\propto \int f(\boldsymbol{x}_n, \boldsymbol{x}_{n-1}|\boldsymbol{y}_{1:n-1})\mathrm{d}\boldsymbol{x}_{n-1} \\ &= \int f(\boldsymbol{x}_n|\boldsymbol{x}_{n-1}, \boldsymbol{y}_{1:n-1})f(\boldsymbol{x}_{n-1}|\boldsymbol{y}_{1:n-1}) \\ &= \int f(\boldsymbol{x}_n|\boldsymbol{x}_{n-1})f(\boldsymbol{x}_{n-1}|\boldsymbol{y}_{1:n-1}) \end{aligned}$$
where $f(\boldsymbol{x}_n|\boldsymbol{x}_{n-1})$ is the state-transition model obtained from (2) and $f(\boldsymbol{x}_{n-1}|\boldsymbol{y}_{1:n-1})$ is the previous posterior pdf
- Even if the previous posterior $f(\boldsymbol{x}_{n-1}|\boldsymbol{y}_{1:n-1})$ is approximated by a Gaussian, a closed-form calculation of $f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1})$ is usually infeasible

UKF Prediction Step

- A feasible special case is when $\boldsymbol{g}_n(\cdot)$ is linear, i.e., $\boldsymbol{w}_n = \boldsymbol{g}_n(\boldsymbol{x}_{n-1}) = \boldsymbol{G}_n \boldsymbol{x}_{n-1}$ and $f(\boldsymbol{x}_{n-1}|\boldsymbol{y}_{1:n-1})$ as well as $f(\boldsymbol{u}_n)$ are Gaussian PDFs. Then $f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1})$ is also Gaussian, and its mean $\boldsymbol{\mu}_{\boldsymbol{x}_n}^-$ and covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{x}_n}^-$ can be calculated based on the Kalman prediction step, i.e.,

$$\boldsymbol{\mu}_{\boldsymbol{x}_n}^- = \boldsymbol{\mu}_{\boldsymbol{w}_n} \qquad \text{with} \qquad \boldsymbol{\mu}_{\boldsymbol{w}_n} = \boldsymbol{G}_n \boldsymbol{\mu}_{\boldsymbol{x}_{n-1}}$$

$$\boldsymbol{\Sigma}_{\boldsymbol{x}_n}^- = \boldsymbol{\Sigma}_{\boldsymbol{w}_n} + \boldsymbol{\Sigma}_{\boldsymbol{u}_n} \qquad \text{with} \qquad \boldsymbol{\Sigma}_{\boldsymbol{w}_n} = \boldsymbol{G}_n \boldsymbol{\Sigma}_{\boldsymbol{x}_{n-1}} \boldsymbol{G}_n^{\mathrm{T}}$$

- If $\boldsymbol{g}_n(\cdot)$ is nonlinear, the UKF prediction step can be applied; here $\boldsymbol{\mu}_{\boldsymbol{w}_n}$ and $\boldsymbol{\Sigma}_{\boldsymbol{w}_n}$ are replaced by the SP approximations $\tilde{\boldsymbol{\mu}}_{\boldsymbol{w}_n}$ and $\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{w}_n}$

UKF Update Step

- We assume the following nonlinear measurement model

$$\boldsymbol{y}_n = \boldsymbol{z}_n + \boldsymbol{v}_n \quad \text{with } \boldsymbol{z}_n = \boldsymbol{h}_n(\boldsymbol{x}_n); \tag{2}$$

  here, the noise $\boldsymbol{v}_n$ is zero-mean and has known covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{v}_n}$; all assumptions of sequential Bayesian estimation discussed previously apply

- As derived in a previous class, the update step of sequential Bayesian estimation reads

$$\begin{aligned} f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n}) &\propto f(\boldsymbol{y}_n, \boldsymbol{x}_n|\boldsymbol{y}_{1:n-1}) \\ &= f(\boldsymbol{y}_n|\boldsymbol{x}_n, \boldsymbol{y}_{1:n-1}) f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1}) \\ &= f(\boldsymbol{y}_n|\boldsymbol{x}_n) f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1}) \end{aligned}$$

  where $f(\boldsymbol{y}_n|\boldsymbol{x}_n)$ is the likelihood function obtained from (2) and $f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1})$ is the predicted posterior pdf

- Even if the predicted posterior $f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1})$ is approximated by a Gaussian, a closed-form calculation of $f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n})$ is infeasible

UKF Update Step

- A feasible special case is when $\boldsymbol{h}_n(\cdot)$ is linear, i.e., $\boldsymbol{z}_n = \boldsymbol{h}_n(\boldsymbol{x}_n) = \boldsymbol{H}_n\boldsymbol{x}_n$ and $f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1})$ as well as $f(\boldsymbol{v}_n)$ are Gaussian PDFs. Then $f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n})$ is also Gaussian, and its mean $\boldsymbol{\mu}_{\boldsymbol{x}_n}$ and covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{x}_n}$ can be calculated based on the Kalman update step, i.e.,

$$\boldsymbol{\mu}_{\boldsymbol{x}_n} = \boldsymbol{\mu}_{\boldsymbol{x}_n}^{-} + \boldsymbol{K}_n(\boldsymbol{y}_n - \boldsymbol{\mu}_{\boldsymbol{y}_n}) \qquad \boldsymbol{\Sigma}_{\boldsymbol{x}_n} = \boldsymbol{\Sigma}_{\boldsymbol{x}_n}^{-} - \boldsymbol{K}_n\boldsymbol{\Sigma}_{\boldsymbol{x}_n\boldsymbol{y}_n}^{\mathrm{T}}$$

  with

$$\begin{aligned}
\boldsymbol{K}_n &= \boldsymbol{\Sigma}_{\boldsymbol{x}_n\boldsymbol{z}_n}(\boldsymbol{\Sigma}_{\boldsymbol{z}_n} + \boldsymbol{\Sigma}_{\boldsymbol{v}_n})^{-1}\\
\boldsymbol{\mu}_{\boldsymbol{y}_n} &= \boldsymbol{\mu}_{\boldsymbol{z}_n} = \boldsymbol{H}_n\boldsymbol{\mu}_{\boldsymbol{x}_n}\\
\boldsymbol{\Sigma}_{\boldsymbol{z}_n} &= \boldsymbol{H}_n\boldsymbol{\Sigma}_{\boldsymbol{x}_n}\boldsymbol{H}_n^{\mathrm{T}}\\
\boldsymbol{\Sigma}_{\boldsymbol{x}_n\boldsymbol{y}_n} &= \boldsymbol{\Sigma}_{\boldsymbol{x}_n\boldsymbol{z}_n} = \boldsymbol{\Sigma}_{\boldsymbol{x}_n}\boldsymbol{H}_n^{\mathrm{T}}
\end{aligned}$$

UKF Update Step

- In the case of nonlinear $\boldsymbol{h}_n(\cdot)$, the SP prediction step can be applied
- This is done by using the closed-form expressions of the linear-Gaussian case, in which $\boldsymbol{\mu}_{\boldsymbol{z}_n}$, $\boldsymbol{\Sigma}_{\boldsymbol{z}_n}$, and $\boldsymbol{\Sigma}_{\boldsymbol{x}_n\boldsymbol{z}_n}$ are replaced by the SP approximations $\tilde{\boldsymbol{\mu}}_{\boldsymbol{z}_n}$, $\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{z}_n}$, and $\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{x}_n\boldsymbol{z}_n}$
- In particular, SP approximations $\tilde{\boldsymbol{\mu}}_{\boldsymbol{x}_n}$ and $\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{x}_n}$ are obtained as

$$\tilde{\boldsymbol{\mu}}_{\boldsymbol{x}_n} = \boldsymbol{\mu}_{\boldsymbol{x}_n}^- + \tilde{\boldsymbol{K}}_n(\boldsymbol{y}_n - \tilde{\boldsymbol{\mu}}_{\boldsymbol{z}_n})\,, \qquad \tilde{\boldsymbol{\Sigma}}_{\boldsymbol{x}_n} = \boldsymbol{\Sigma}_{\boldsymbol{x}_n}^- - \tilde{\boldsymbol{K}}_n \tilde{\boldsymbol{\Sigma}}_{\boldsymbol{x}_n\boldsymbol{z}_n}^{\mathrm{T}},$$

with $\tilde{\boldsymbol{K}}_n = \tilde{\boldsymbol{\Sigma}}_{\boldsymbol{x}_n\boldsymbol{z}_n}(\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{z}_n} + \boldsymbol{\Sigma}_{\boldsymbol{v}_n})^{-1}$