Part I: We show that for $\boldsymbol{x}, \boldsymbol{z}$ jointly Gaussian with mean and covariance

$$\begin{bmatrix} \boldsymbol{\mu}_{\boldsymbol{x}} \\ \boldsymbol{\mu}_{\boldsymbol{z}} \end{bmatrix} \qquad\qquad \begin{bmatrix} \boldsymbol{\Sigma}_{\boldsymbol{x}} & \boldsymbol{\Sigma}_{\boldsymbol{xz}} \\ \boldsymbol{\Sigma}_{\boldsymbol{zx}} & \boldsymbol{\Sigma}_{\boldsymbol{z}} \end{bmatrix}$$

$\boldsymbol{x}$ conditioned on $\boldsymbol{z}$ is Gaussian, with conditional mean and conditional covariance

$$\boldsymbol{\mu}_{\boldsymbol{x}}^{+} = \boldsymbol{\mu}_{\boldsymbol{x}} + \boldsymbol{\Sigma}_{\boldsymbol{xz}} \boldsymbol{\Sigma}_{\boldsymbol{z}}^{-1} (\boldsymbol{z} - \boldsymbol{\mu}_{\boldsymbol{z}}) \qquad\qquad \boldsymbol{\Sigma}_{\boldsymbol{x}}^{+} = \boldsymbol{\Sigma}_{\boldsymbol{x}} + \boldsymbol{\Sigma}_{\boldsymbol{xz}} \boldsymbol{\Sigma}_{\boldsymbol{z}}^{-1} \boldsymbol{\Sigma}_{\boldsymbol{zx}}$$

Part II: We recall the update step for sequential Bayesian estimation

$$\underbrace{f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n})}_{\text{Posterior pdf}} \propto \underbrace{f(\boldsymbol{y}_n|\boldsymbol{x}_n)}_{\substack{\text{Likelihood}\\ \text{function}}} \underbrace{f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1})}_{\substack{\text{Predicted}\\ \text{posterior pdf}}}$$

$$= \frac{f(\boldsymbol{x}_n, \boldsymbol{y}_n|\boldsymbol{y}_{1:n-1})}{f(\boldsymbol{y}_n|\boldsymbol{y}_{1:n-1})}$$

and consider that conditioned on $\boldsymbol{y}_{1:n-1}, \boldsymbol{x}_n$ and $\boldsymbol{y}_n$ are jointly Gaussian with mean and covariance

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_{\boldsymbol{x}}^{-} \\ \boldsymbol{H}_n \boldsymbol{\mu}_{\boldsymbol{x}}^{-} \end{bmatrix} \qquad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{\boldsymbol{x}_n}^{-} & \boldsymbol{\Sigma}_{\boldsymbol{x}_n}^{-} \boldsymbol{H}_n^{\mathrm{T}} \\ \boldsymbol{H}_n \boldsymbol{\Sigma}_{\boldsymbol{x}_n}^{-} & \boldsymbol{H}_n \boldsymbol{\Sigma}_{\boldsymbol{x}_n}^{-} \boldsymbol{H}_n^{\mathrm{T}} + \boldsymbol{\Sigma}_{\boldsymbol{v}_n} \end{bmatrix}$$

Now we directly obtain the expressions for the Kalman update step by using the result of Part I

$$\begin{bmatrix} \boldsymbol{I} & -\boldsymbol{\Sigma}_{\boldsymbol{xz}}\boldsymbol{\Sigma}_{\boldsymbol{z}}^{-1} \\ \mathbf{0} & \boldsymbol{I} \end{bmatrix} \boldsymbol{\Sigma} \begin{bmatrix} \boldsymbol{I} & \mathbf{0} \\ -\boldsymbol{\Sigma}_{\boldsymbol{z}}^{-1}\boldsymbol{\Sigma}_{\boldsymbol{xz}}^{\mathrm{T}} & \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Sigma}_{\boldsymbol{x}} - \boldsymbol{\Sigma}_{\boldsymbol{xz}}\boldsymbol{\Sigma}_{\boldsymbol{z}}^{-1}\boldsymbol{\Sigma}_{\boldsymbol{zx}} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_{\boldsymbol{z}} \end{bmatrix}$$

$$\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} \boldsymbol{I} & 0 \\ -\boldsymbol{\Sigma}_{\boldsymbol{z}}^{-1}\boldsymbol{\Sigma}_{\boldsymbol{xz}}^{\mathrm{T}} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \left(\boldsymbol{\Sigma}_{\boldsymbol{x}} - \boldsymbol{\Sigma}_{\boldsymbol{xz}}\boldsymbol{\Sigma}_{\boldsymbol{z}}^{-1}\boldsymbol{\Sigma}_{\boldsymbol{zx}}\right)^{-1} & 0 \\ 0 & \boldsymbol{\Sigma}_{\boldsymbol{z}}^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{\Sigma}_{\boldsymbol{xz}}\boldsymbol{\Sigma}_{\boldsymbol{z}}^{-1} \\ 0 & \boldsymbol{I} \end{bmatrix}$$

$$\det \boldsymbol{\Sigma} = \det\left(\boldsymbol{\Sigma}_{\boldsymbol{x}} - \boldsymbol{\Sigma}_{\boldsymbol{xz}}\boldsymbol{\Sigma}_{\boldsymbol{z}}^{-1}\boldsymbol{\Sigma}_{\boldsymbol{zx}}\right)\det \boldsymbol{\Sigma}_{\boldsymbol{z}}$$

$$
\begin{aligned}
&\left[\boldsymbol{x}^{\mathrm{T}}-\boldsymbol{\mu}_{\boldsymbol{x}}^{\mathrm{T}} \quad \boldsymbol{z}^{\mathrm{T}}-\boldsymbol{\mu}_{\boldsymbol{z}}^{\mathrm{T}}\right] \boldsymbol{\Sigma}^{-1}\left[\boldsymbol{x}^{\mathrm{T}}-\boldsymbol{\mu}_{\boldsymbol{x}}^{\mathrm{T}} \quad \boldsymbol{z}^{\mathrm{T}}-\boldsymbol{\mu}_{\boldsymbol{z}}^{\mathrm{T}}\right]^{\mathrm{T}} \\
&\qquad=\left(\boldsymbol{x}-\boldsymbol{\mu}_{\boldsymbol{x}}^{+}\right)^{\mathrm{T}}\left(\boldsymbol{\Sigma}_{\boldsymbol{x}}^{+}\right)^{-1}\left(\boldsymbol{x}-\boldsymbol{\mu}_{\boldsymbol{x}}^{+}\right)+\left(\boldsymbol{z}-\boldsymbol{\mu}_{\boldsymbol{z}}\right)^{\mathrm{T}} \boldsymbol{\Sigma}_{\boldsymbol{z}}^{-1}\left(\boldsymbol{z}-\boldsymbol{\mu}_{\boldsymbol{z}}\right)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\mu}_{\boldsymbol{x}}^{+} &= \boldsymbol{\mu}_{\boldsymbol{x}}+\boldsymbol{\Sigma}_{\boldsymbol{x}\boldsymbol{z}} \boldsymbol{\Sigma}_{\boldsymbol{z}}^{-1}\left(\boldsymbol{z}-\boldsymbol{\mu}_{\boldsymbol{z}}\right) \\
\boldsymbol{\Sigma}_{\boldsymbol{x}}^{+} &= \boldsymbol{\Sigma}_{\boldsymbol{x}}+\boldsymbol{\Sigma}_{\boldsymbol{x}\boldsymbol{z}} \boldsymbol{\Sigma}_{\boldsymbol{z}}^{-1} \boldsymbol{\Sigma}_{\boldsymbol{z}\boldsymbol{x}}
\end{aligned}
$$

$$
\begin{aligned}
f(\boldsymbol{x} \mid \boldsymbol{z}) &= \frac{f(\boldsymbol{x}, \boldsymbol{z})}{f(\boldsymbol{z})} \\
&= \frac{\sqrt{\det \boldsymbol{\Sigma}_{\boldsymbol{z}}}}{\sqrt{(2 \pi)^{I} \det \boldsymbol{\Sigma}}} \; \frac{\exp \left[-\frac{1}{2}\left[\boldsymbol{x}^{\mathrm{T}}-\boldsymbol{\mu}_{\boldsymbol{x}}^{\mathrm{T}} \quad \boldsymbol{z}^{\mathrm{T}}-\boldsymbol{\mu}_{\boldsymbol{z}}^{\mathrm{T}}\right] \boldsymbol{\Sigma}^{-1}\left[\boldsymbol{x}^{\mathrm{T}}-\boldsymbol{\mu}_{\boldsymbol{x}}^{\mathrm{T}} \quad \boldsymbol{z}^{\mathrm{T}}-\boldsymbol{\mu}_{\boldsymbol{z}}^{\mathrm{T}}\right]^{\mathrm{T}}\right]}{\exp \left[-\frac{1}{2}\left(\boldsymbol{z}-\boldsymbol{\mu}_{\boldsymbol{z}}\right)^{\mathrm{T}} \boldsymbol{\Sigma}_{\boldsymbol{z}}^{-1}\left(\boldsymbol{z}-\boldsymbol{\mu}_{\boldsymbol{z}}\right)\right]} \\
&= \frac{1}{\sqrt{(2 \pi)^{I} \det \boldsymbol{\Sigma}_{\boldsymbol{x}}^{+}}} \exp \left[-\frac{1}{2}\left(\boldsymbol{x}-\boldsymbol{\mu}_{\boldsymbol{x}}^{+}\right)^{\mathrm{T}}\left(\boldsymbol{\Sigma}_{\boldsymbol{x}}^{+}\right)^{-1}\left(\boldsymbol{x}-\boldsymbol{\mu}_{\boldsymbol{x}}^{+}\right)\right]
\end{aligned}
$$