All Pass Filters and Spectral Factorization

Bhaskar D Rao
University of California, San Diego
Email: brao@ucsd.edu

WSS and LTI systems

WSS process $x[n]$ is input to a LTI system with transfer function $H(z)$ and output is $y[n]$.

$y[n]$ is WSS and $x[n]$ and $y[n]$ are jointly WSS.

$y[n]$ is WSS with mean $\mu_y = H(0)\mu_x$ and autocorrelation sequence $r_{yy}[m] = h[m] * h^*[-m] * r_{xx}[m]$.

$x[n]$ and $y[n]$ are jointly WSS and $r_{xy}[m] = E(x[n+m]y^*[n]) = r_{xx}[m] * h^*[-m]$.

In the z-domain $R_{xy}(z) = \mathcal{Z}(r_{xy}[m]) = R_{xx}(z)H^*(\frac{1}{z^*})$ and simplifies to $R_{xx}(z)H(z^{-1})$ for real systems.

$$R_{yy}(z) = R_{xx}(z)H(z)H^*(\frac{1}{z^*}) \text{ or } R_{yy}(e^{j\omega}) = R_{xx}(e^{j\omega})|H(e^{j\omega})|^2$$

Spectral Factorization: Given $R_{yy}(z)$ and with white noise input,

$R_{xx}(z) = 1$, find $H(z)$ or equivalently factor $R_{yy}(z)$ such that $R_{yy}(z) = H(z)H^*(\frac{1}{z^*})$.

All Pass Filters

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}, |a| < 1$$

With $a = re^{j\theta}$, can show $|H_{ap}(e^{j\omega})| = 1$, and

$$\phi(\omega) = \angle H_{ap}(e^{j\omega}) = -\omega - 2\arctan\left[\frac{r\sin(\omega - \theta)}{1 - r\cos(\omega - \theta)}\right]$$

$$\mathrm{grd}(H(e^{j\omega}) = -\frac{d\phi(\omega)}{d\omega} = \frac{1 - r^2}{1 + r^2 - 2r\cos(\omega - \theta)} = \frac{1 - r^2}{|1 - r^{j\theta}e^{-j\omega}|^2} \geq 0$$

A first order all pass filter has a pole at $a = re^{j\theta}$ and a zero at $\frac{1}{a^*} = \frac{1}{r}e^{j\theta}$.

Magnitude and Phase Plots for First Order All Pass Filter

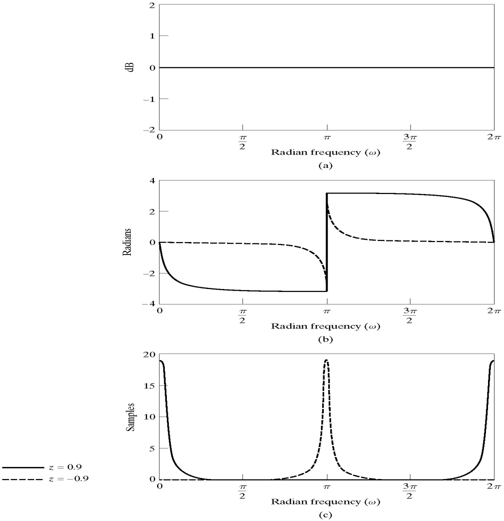
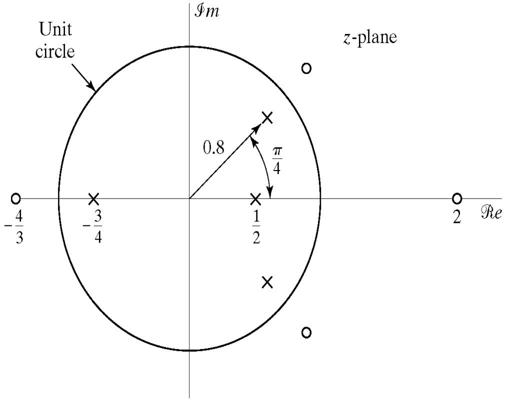


Figure 5.22 Frequency response for all-pass filters with real poles at $z = 0.9$ (solid line) and $z = -0.9$ (dashed line). (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

Poles-Zeros for a General All-Pass Filter

$$H_{ap}(z) = \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \;\; \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k)(z^{-1} - e_k^*)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}, \quad (d_k \;\; \text{real})$$



Spectral-Factorization

Some Notation

$$
\begin{aligned}
H(z) &= h[0] + h[1]z^{-1} + \ldots + h[N-1]z^{-(N-1)} \\
H^*(\frac{1}{z^*}) &= h^*[0] + h^*[1]z^{1} + \ldots + h^*[N-1]z^{(N-1)} = \tilde{H}(z)
\end{aligned}
$$

Will use $\tilde{H}(z) = H^*(\frac{1}{z^*})$ for notational simplicity.

Note: If z_i is a root of $H(z)$, then $\frac{1}{z_i^*}$ is a root of $\tilde{H}(z)$.

Problem: Given $R(e^{j\omega}) \geq 0$, $\forall \omega$, find $H(z)$ or $H(e^{j\omega})$ such that $R(e^{j\omega}) = |H(e^{j\omega}|^2$ or equivalently $R(z) = H(z)\tilde{H}(z)$.

Spectral-Factorization: Rational Systems

We will consider the version where $R(e^{j\omega}) = \frac{P(e^{j\omega})}{Q(e^{j\omega})}$ where $P(e^{j\omega}) \geq 0$ and $Q(e^{j\omega}) > 0$.We will assume no zeros on unit circle and so $P(e^{j\omega}) > 0$

Equivalent z-domain problem is $R(z) = \frac{P(z)}{Q(z)}$, with Region of Convergence (ROC) an annular ring including the unit circle.

Spectral Factorization in this case is equivalent to

$$R(z) = \frac{P(z)}{Q(z)} = \frac{B(z)\tilde{B}(z)}{A(z)\tilde{A}(z)}$$

We will concentrate on $P(z)$ and similar considerations apply to $Q(z)$.

For the rest of the discussion $R(z) = P(z)$ and $H(z) = B(z)$ and we will also restrict $P(z)$ to have finite order

$$R(z) = \sum_{m=-M}^{M} r[m]z^{-m} \overbrace{=}^{?} H(z)\tilde{H}(z)$$

Since $r[m] = r^*[-m]$, we have $R(z) = R^*(\frac{1}{z^*})$.

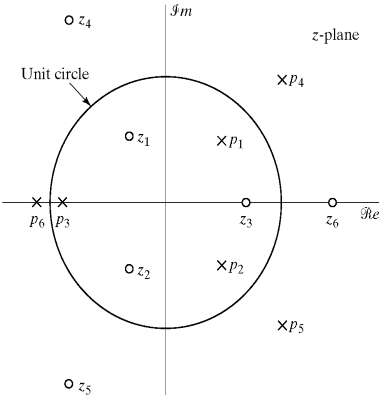
Consequence: if z_i is a zero (root) of $R(z)$, then $\frac{1}{z_i^*}$ is also a zero of the system. If we consider real systems ($r[m] = r^*[-m] = r[-m]$), then z_i^* is also a zero and so is $\frac{1}{z_i}$.

Factorization: If z_i is assigned to $H(z)$, then $\frac{1}{z_i^*}$ is assigned to $\tilde{H}(z)$.

General $R(z) = \frac{P(z)}{Q(z)}$: For the denominator $Q(z)$, because of causality and stability (poles inside unit circle), there is only one useful factorization. For the numerator options exist.

Zeros of $H(z)\tilde{H}(z)$.

$$R(z) = c\ z^M\ \Pi_{i=1}^{M}(1 - z_i z^{-1})(1 - \frac{1}{z_i^*} z^{-1})$$



Minimum-Phase and Maximum-Phase Systems

$$R(z) = d\ z^M\ \Pi_{i=1}^M (1 - z_i z^{-1})(z^{-1} - z_i^*),\ \ |z_i| < 1$$

This factorization is chosen to align with the all-pass numerator. Alternately

$$R(z) = d\ \Pi_{i=1}^M (1 - z_i z^{-1})(1 - z_i^* z) = d\ \Pi_{i=1}^M (z - z_i)(z^{-1} - z_i^*)$$

Also indicates d is real and positive.

Minimum-Phase: $H_{min}(z) = \sqrt{d}\ \ \Pi_{i=1}^M (1 - z_i z^{-1})$. All zeros inside unit circle. The filter is (causally) invertible.

Maximum-Phase: $H_{max}(z) = \sqrt{d}\ \ \Pi_{i=1}^M (z^{-1} - z_i^*)$. All zeros outside unit circle.

$$H_{max}(z) = H_{min}(z) H_{ap}(z),\ \ \text{where}\ \ H_{ap}(z) = \Pi_{i=1}^M \frac{z^{-1} - z_i^*}{1 - z_i z^{-1}}.$$

In general, all factors can be written as $H_{min}(z)H_{ap}(z)$ with an appropriate $H_{ap}(z)$. (Depends on how many zeros have to flipped)

Options for a system with four zeros

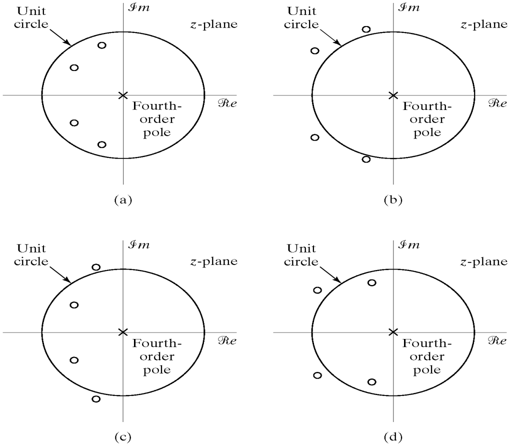


Figure 5.30 Four systems, all having the same frequency-response magnitude. Zeros are at all combinations of $0.9e^{\pm j0.6\pi}$ and $0.8e^{\pm j0.8\pi}$ and their reciprocals.

Impulse response options for a system with four zeros

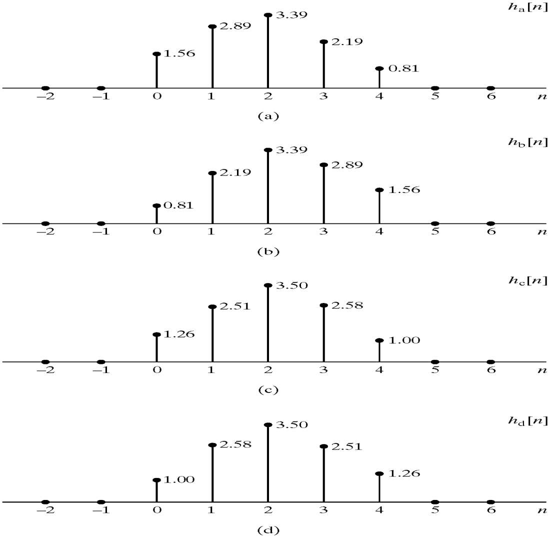


Figure 5.31 Sequences corresponding to the pole–zero plots of Figure 5.30.

Whitening Filter

Consider $x[n] \longrightarrow H(z) \longrightarrow y[n]$ where $x[n]$ is zero mean white noise, i.e. $r_{xx}[m] = \delta[m]$ and $R_{xx}(z) = 1$. So we know from LTI theory that $R_{yy}(e^{j\omega}) = |H(e^{j\omega})|^2$ and $R_{yy}(z) = H(z)\tilde{H}(z)$.

Now we want to white $y[n]$ through linear filtering, i.e.

$$y[n] \longrightarrow G(z) \longrightarrow v[n]$$

Problem (Whitening Operation): Choose $G(z)$ such that $v[n]$ is white, i.e. $r_{vv}[m] = \delta[m]$, and $R_{vv}(z) = 1$.

Answer: $G(z) = \frac{1}{H_{min}(z)}$, where $R_{yy}(z) = H_{min}(z)\tilde{H}_{min}(z)$.

$$R_{vv}(z) = R_{yy}(z)G(z)\tilde{G}(z) = R_{yy}(z)\frac{1}{H_{min}(z)\tilde{H}_{min}(z)} = R_{yy}(z)\frac{1}{R_{yy}(z)} = 1$$

$H_{min}(z)$ is referred as the innovations filter and ensuing model (white noise filtered by $H_{min}(z)$) the innovations representation.