SIO 209: Signal Processing for Ocean Sciences
Class 12

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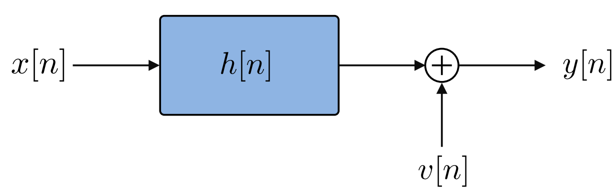
Coherence and Transfer Function Estimation

- **System Model**



- Functions of interest
  - *Transfer Function:* The relationship between $x[n]$ and $y[n]$
  - *Coherence Function:* The degree of causality between $x[n]$ and $y[n]$

P. R. Roth, "Effective Measurements Using Digital Signal Analysis." IEEE Spectrum, 1971

G. C. Carter, "Coherence and Time Delay Estimation." Proc. IEEE, 1987

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Coherence and Transfer Function Estimation

- **Components and Their Definitions**

A. Auto-Power Spectra

conventional power spectral estimation

$$G_{xx}(\omega) \longrightarrow \overline{\hat{G}_{xx}(k)} = \overline{X(k)X^*(k)}$$ where $X(k) = \mathrm{DFT}\{x[n]\}$

$$G_{yy}(\omega) \longrightarrow \overline{\hat{G}_{yy}(k)} = \overline{Y(k)Y^*(k)}$$ where $Y(k) = \mathrm{DFT}\{y[n]\}$

B. Cross-Power Spectra

$$G_{yx}(\omega) \longrightarrow \overline{\hat{G}_{yx}(k)} = \overline{Y(k)X^*(k)}$$

$$\begin{aligned} G_{yx}(\omega) &= H(\omega)G_{xx}(\omega) + G_{vx}(\omega) \\ &= H(\omega)G_{xx}(\omega) \end{aligned}$$

assume $G_{vx}(\omega) = 0$

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Coherence Function Estimation

- The estimate $\overline{\hat{G}_{yx}(k)}$ of the cross-power spectral density $G_{yx}(\omega)$ will not answer question of causality between $x[n]$ and $y[n]$ since $\overline{V(k)X^*(k)} \approx 0$

Magnitude Coherence Squared

$$Y(\omega)^2 = \frac{\left|G_{yx}(\omega)\right|^2}{G_{xx}(\omega)G_{yy}(\omega)} = \frac{\left|H(\omega)\right|^2 G_{xx}(\omega)}{\left|H(\omega)\right|^2 G_{xx}(\omega) + G_{vv}(\omega)}$$

(assuming $G_{vx}(\omega) = 0$)

An estimate of $Y(\omega)^2$ can be computed as

Note that $0 \leqslant Y(\omega)^2 \leqslant 1$

$$\hat{Y}(k)^2 = \frac{\left|\overline{\hat{G}_{yx}(k)}\right|^2}{\overline{\hat{G}_{xx}(k)}\;\overline{\hat{G}_{yy}(k)}}$$

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Coherence Function Estimation

- Interpretation of $Y(\omega)$: The power spectrum at the output, consists of components
  - – caused by system input (i.e., $x[n]$) $Y(\omega)^2 G_{yy}(\omega)$ "coherent output power"
  - – caused by additive noise (i.e., $v[n]$) $\left(1 - Y(\omega)^2\right) G_{yy}(\omega)$

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Coherence Function Estimation: Example 1

Example 6.5. **Physical Illustration of Coherence Measurement.** Consider an airplane flying through a patch of atmospheric turbulence, as illustrated in Figure 6.3. Let the input *x*(*t*) be vertical gust velocity in meters/second (m/s) as measured with a probe extending forward of the airplane, and the output *y*(*t*) be vertical acceleration in *G*'s measured with an accelerometer at the center of gravity of the airplane. The resulting coherence function and autospectra for actual data of this type are presented in Figures 6.4 and 6.5. In this problem, the spectral data were computed over a frequency range from 0.1 to 4.0 Hz with a resolution bandwidth of 0.05 Hz and a record length of about 10 min.

From Figure 6.4, it is seen that the input gust velocity and output airplane acceleration display a relatively strong coherence of 0.8 to 0.9 over the frequency range from about 0.4 to about 2.0 Hz. Below and above this range, however, the coherence function diminishes. At the lower frequencies, the vertical acceleration of the airplane is increasingly due to maneuver loads induced through the control system by the pilot, rather than due to atmospheric turbulence loads. Hence, the loss of coherence at the lower frequencies probably reflects contributions to the output *y*(*t*) from inputs other than the measured input *x*(*t*). At the higher frequencies, the low-pass filtering characteristics of the airplane response plus the decaying nature of the input autospectrum cause the output autospectrum to fall off sharply, as indicated in Figure 6.5. On the other hand, the noise floor for the data acquisition and recording equipment generally does not fall off with increasing frequency. Hence, the diminishing coherence at the higher frequencies probably results from the contributions of extraneous measurement noise. This concludes Example 6.5.

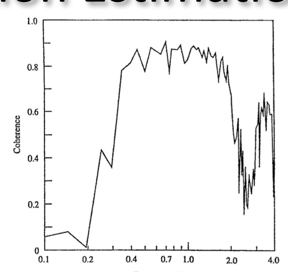

Figure 6.4 Coherence function between gust velocity and response acceleration of airplane. These data resulted from studies funded by the NASA Langley Research Center, Hampton, Virginia, under Contract NAS 1-8538.

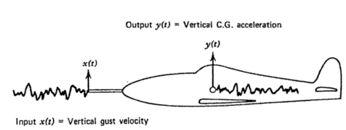

Figure 6.3 Airplane flying through atmospheric turbulence.

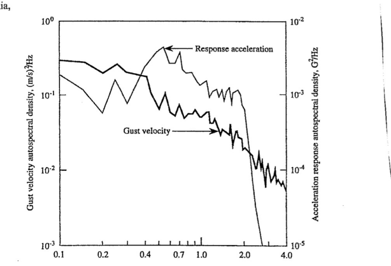

Figure 6.5 Autospectra of gust velocity and response acceleration of airplane. These data resulted from studies funded by the NASA Langley Research Center, Hampton, Virginia, under Contract NAS 1-8538.

J. Bendat and A. Piersol, "Random Data: Analysis and Measurement Procedures." Wiley, 2000

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Coherence Function Estimation: Example 2

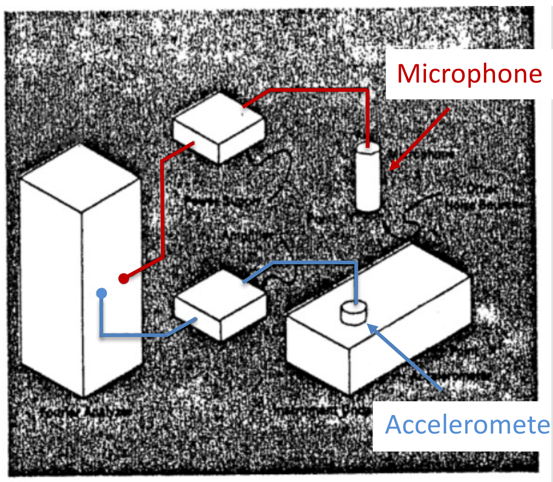

Figure 1—Test setup to determine sources of noise monitored above an electronic instrument case.

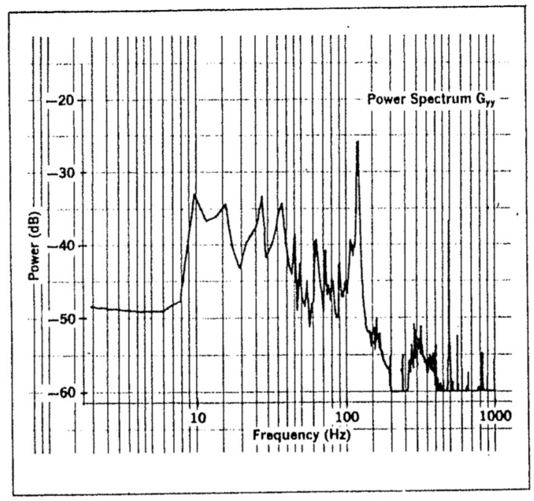

Figure 2—Power spectrum of the noise measured by the microphone shown in Figure 1 (point Y).

P. R. Roth, "How to use the spectrum and coherence function." Sound and Vibration, 1971

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Coherence Function Estimation: Example 2

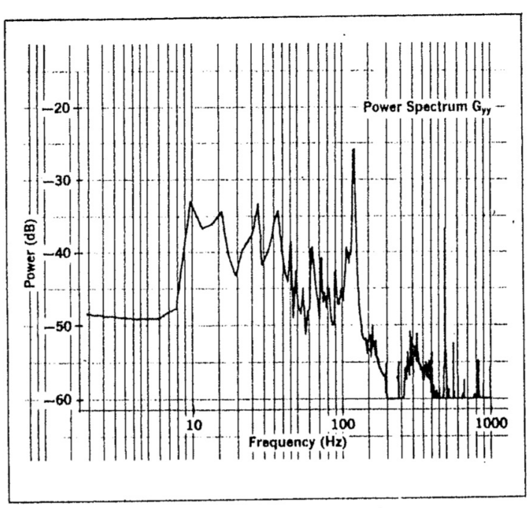

Figure 2—Power spectrum of the noise measured by the microphone shown in Figure 1 (point Y).

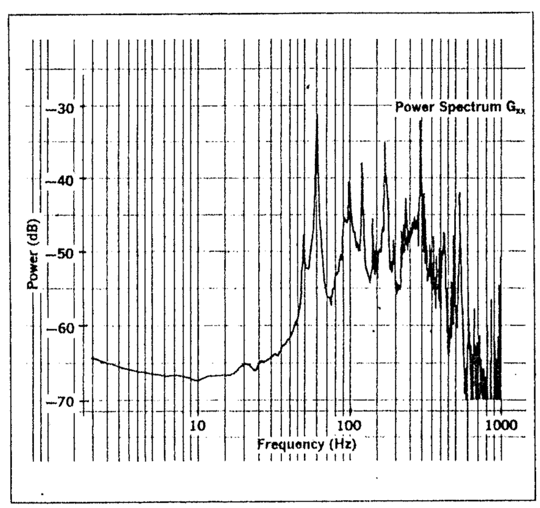

Figure 4—Power spectrum of the vibration measured by the accelerometer shown in Figure 1 (point X).

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Coherence Function Estimation: Example 2

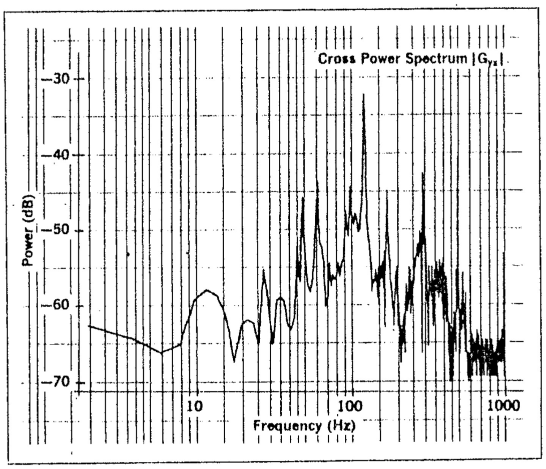

Figure 5—Cross power spectrum of the microphone (Y) and accelerometer (X) signals. The cross spectrum shows the components that are common to X and Y but does not indicate the degree to which the monitored noise is caused by the panel vibration.

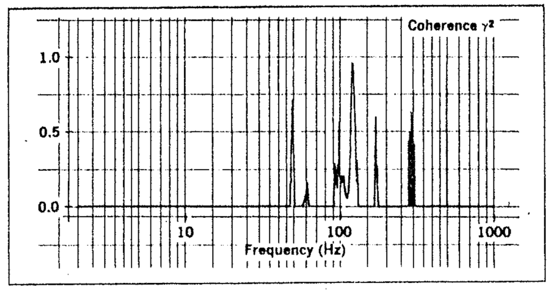

Figure 6—Coherence function of the microphone signal to the accelerometer signal. The coherence function shows the fraction of power at the microphone (Y) which is coherent with the assumed source (X).

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Coherence Function Estimation: Example 2

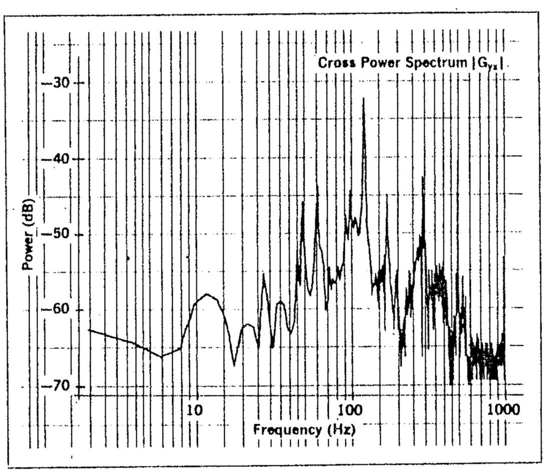

Figure 5—Cross power spectrum of the microphone (Y) and accelerometer (X) signals. The cross spectrum shows the components that are common to X and Y but does not indicate the degree to which the monitored noise is caused by the panel vibration.

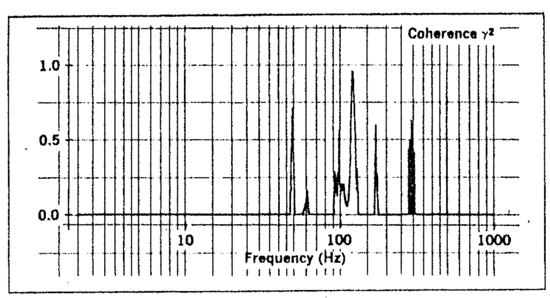

Figure 6—Coherence function of the microphone signal to the accelerometer signal. The coherence function shows the fraction of power at the microphone (Y) which is coherent with the assumed source (X).

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Coherence Function Estimation: Example 2

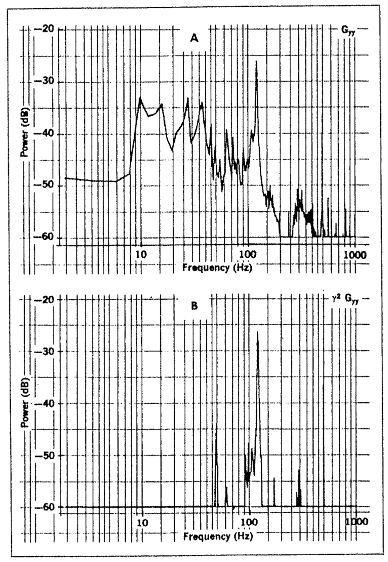

Figure 7—The noise spectrum of the test item is shown above as originally measured (A) and as corrected by the coherence function (B).

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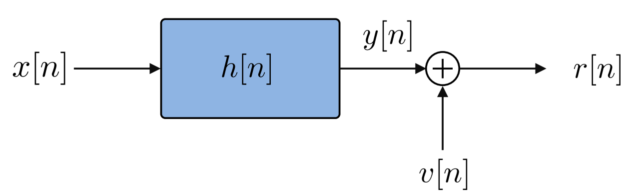
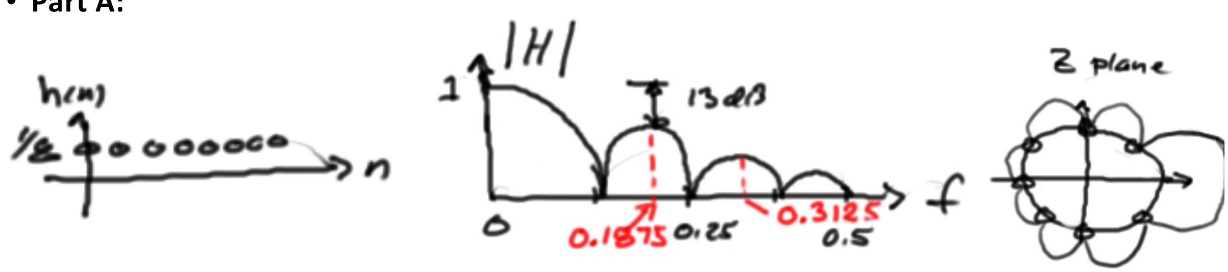
Homework 5 – Transfer and Coherence Function Est.

- Note that in homework 5, $r[n]$ is used to represent the measurement time series and $y[n]$ to represent the output of the linear system



- **Part A:**



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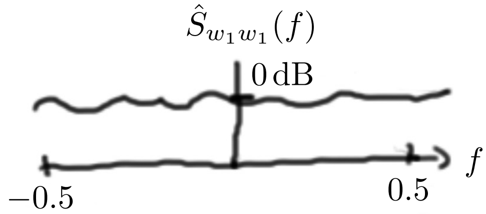
Coherence and Transfer Function Estimation

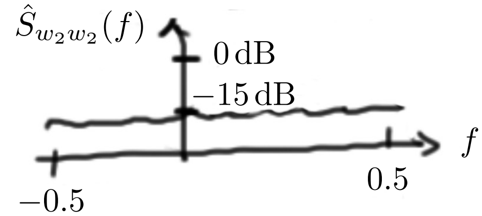
- **Part B: Power Spectra (2-sided)**

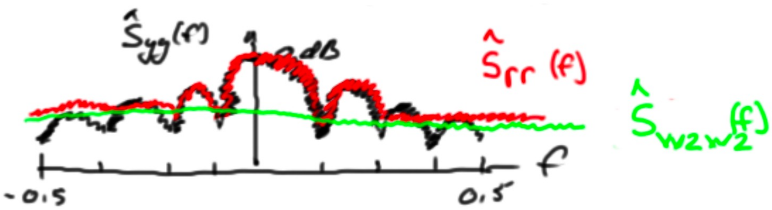
$$\hat{S}_{xx}(f) = \frac{1}{f_s MU}\overline{|X(k)|^2}$$

$$\mathrm{var}(w_1[n]) = 1 = 0\mathrm{dB}$$



$$\mathrm{var}(w_2[n]) = \frac{1}{32} = 2^{-5} = -15\,\mathrm{dB}$$





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