ECE 275A Parameter Estimation: The Extended Kalman Filter

Florian Meyer

Electrical and Computer Engineering Department
University of California San Diego

UC San Diego
JACOBS SCHOOL OF ENGINEERING

The Extended Kalman Filter

- Consider again the setting of sequential Bayesian estimation with state-transition and measurement model of the form

$$\boldsymbol{x}_n = \boldsymbol{g}_n(\boldsymbol{x}_{n-1}) + \boldsymbol{u}_n \tag{1}$$

$$\boldsymbol{y}_n = \boldsymbol{h}_n(\boldsymbol{x}_n) + \boldsymbol{v}_n \tag{2}$$

  for $n = 1, 2, \ldots,$ where $\boldsymbol{g}_n(\cdot)$ and $\boldsymbol{h}_n(\cdot)$ are two known nonlinear vector functions and $\boldsymbol{u}_n \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{\boldsymbol{u}_n})$, $\boldsymbol{v}_n \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{\boldsymbol{v}_n})$

- In this nonlinear case, a closed-form sequential implementation of the LMMSE estimator is not available

- However, we can linearize $\boldsymbol{g}_n(\cdot)$ and $\boldsymbol{h}_n(\cdot)$ about suitably chosen points and develop a Kalman filter based on the resulting linear equations; this results in the so-called *extended Kalman filter (EKF)*

- Note that the extended Kalman filter performs well in many applications, but it is not optimal in the LMMSE sense

The Linearized Model

- Consider $f(\boldsymbol{x}_{n-1}|\boldsymbol{y}_{1:n-1}) \approx \mathcal{N}(\tilde{\boldsymbol{\mu}}_{\boldsymbol{x}_{n-1}}, \tilde{\boldsymbol{\Sigma}}_{\boldsymbol{x}_{n-1}})$ and $f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1}) \approx \mathcal{N}(\tilde{\boldsymbol{\mu}}^-_{\boldsymbol{x}_n}, \tilde{\boldsymbol{\Sigma}}^-_{\boldsymbol{x}_n})$
- We first linearize $\boldsymbol{g}_n(\cdot)$ about $\tilde{\boldsymbol{\mu}}_{\boldsymbol{x}_{n-1}}$ as

$$\boldsymbol{g}_n(\boldsymbol{x}_{n-1}) \approx \boldsymbol{g}_n(\tilde{\boldsymbol{\mu}}_{\boldsymbol{x}_{n-1}}) + \boldsymbol{G}_n\big(\boldsymbol{x}_{n-1} - \tilde{\boldsymbol{\mu}}_{\boldsymbol{x}_{n-1}}\big) \tag{3}$$

  where we introduced the Jacobian matrix at $\tilde{\boldsymbol{\mu}}_{\boldsymbol{x}_{n-1}}$, i.e.,

$$\boldsymbol{G}_n = \left.\frac{d\boldsymbol{g}_n(\boldsymbol{x}_{n-1})}{d\boldsymbol{x}_{n-1}}\right|_{\boldsymbol{x}_{n-1}=\tilde{\boldsymbol{\mu}}_{\boldsymbol{x}_{n-1}}}$$

- Similarly, we linearize $\boldsymbol{h}_n(\cdot)$ about $\tilde{\boldsymbol{\mu}}^-_{\boldsymbol{x}_n}$ as

$$\boldsymbol{h}_n(\boldsymbol{x}_n) \approx \boldsymbol{h}_n(\tilde{\boldsymbol{\mu}}^-_{\boldsymbol{x}_n}) + \boldsymbol{H}_n\big(\boldsymbol{x}_n - \tilde{\boldsymbol{\mu}}^-_{\boldsymbol{x}_n}\big) \tag{4}$$

  where we introduced the Jacobian matrix at $\tilde{\boldsymbol{\mu}}^-_{\boldsymbol{x}_n}$, i.e.,

$$\boldsymbol{H}_n = \left.\frac{d\boldsymbol{h}_n(\boldsymbol{x}_n)}{d\boldsymbol{x}_n}\right|_{\boldsymbol{x}_n=\tilde{\boldsymbol{\mu}}^-_{\boldsymbol{x}_n}}$$

The Linearized Model

- By inserting (1) and (2) into (3) and (4), respectively, we obtain the linearized state-transition and measurement model

$$\boldsymbol{x}_n = \boldsymbol{G}_n \boldsymbol{x}_{n-1} + \boldsymbol{u}_n + \left[\boldsymbol{g}_n(\tilde{\boldsymbol{\mu}}_{\boldsymbol{x}_{n-1}}) - \boldsymbol{G}_n \tilde{\boldsymbol{\mu}}_{\boldsymbol{x}_{n-1}}\right]$$

$$\boldsymbol{y}_n = \boldsymbol{H}_n \boldsymbol{x}_n + \boldsymbol{v}_n + \left[\boldsymbol{h}_n(\tilde{\boldsymbol{\mu}}^-_{\boldsymbol{x}_n}) - \boldsymbol{H}_n \tilde{\boldsymbol{\mu}}^-_{\boldsymbol{x}_n}\right]$$

- This differs from the standard linear model used in the Kalman filter in that there are the additional known terms $\boldsymbol{g}_n(\tilde{\boldsymbol{\mu}}_{\boldsymbol{x}_{n-1}}) - \boldsymbol{G}_n \tilde{\boldsymbol{\mu}}_{\boldsymbol{x}_{n-1}}$ and $\boldsymbol{h}_n(\tilde{\boldsymbol{\mu}}^-_{\boldsymbol{x}_n}) - \boldsymbol{H}_n \tilde{\boldsymbol{\mu}}^-_{\boldsymbol{x}_n}$

EKF Prediction Step

- By using the approximation of the previous posterior PDF given by $f(\boldsymbol{x}_{n-1}|\boldsymbol{y}_{1:n-1}) \approx \mathcal{N}(\tilde{\boldsymbol{\mu}}_{\boldsymbol{x}_{n-1}}, \tilde{\boldsymbol{\Sigma}}_{\boldsymbol{x}_{n-1}})$ and performing the same steps as for the derivation of the Kalman prediction step, the prediction step of the EKF is obtained as

$$\tilde{\boldsymbol{\mu}}_{\boldsymbol{x}_n}^{-} = \boldsymbol{g}_n(\tilde{\boldsymbol{\mu}}_{\boldsymbol{x}_{n-1}})$$

$$\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{x}_n}^{-} = \boldsymbol{G}_n \tilde{\boldsymbol{\Sigma}}_{\boldsymbol{x}_{n-1}} \boldsymbol{G}_n^{\mathrm{T}} + \boldsymbol{\Sigma}_{\boldsymbol{u}_n}$$

- The result of the EKF prediction step is a Gaussian approximation of the predicted posterior PDF, i.e., $f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1}) \approx \mathcal{N}\left(\tilde{\boldsymbol{\mu}}_{\boldsymbol{x}_n}^{-}, \tilde{\boldsymbol{\Sigma}}_{\boldsymbol{x}_n}^{-}\right)$

EKF Update Step

- By using the approximation of the predicted posterior PDF given by $f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1}) \approx \mathcal{N}(\tilde{\boldsymbol{\mu}}^{-}_{\boldsymbol{x}_n}, \tilde{\boldsymbol{\Sigma}}^{-}_{\boldsymbol{x}_n})$ and performing the same steps as for the derivation of the Kalman update step, the update step of the EKF is obtained as

$$\boldsymbol{K}_n = \tilde{\boldsymbol{\Sigma}}^{-}_{\boldsymbol{x}_n} \boldsymbol{H}_n^{\mathrm{T}} \left(\boldsymbol{H}_n \tilde{\boldsymbol{\Sigma}}^{-}_{\boldsymbol{x}_n} \boldsymbol{H}_n^{\mathrm{T}} + \boldsymbol{\Sigma}_{\boldsymbol{v}_n} \right)^{-1}$$

$$\tilde{\boldsymbol{\mu}}_{\boldsymbol{x}_n} = \tilde{\boldsymbol{\mu}}^{-}_{\boldsymbol{x}_n} + \boldsymbol{K}_n \left(\boldsymbol{y}_n - \boldsymbol{h}_n(\tilde{\boldsymbol{\mu}}^{-}_{\boldsymbol{x}_n}) \right)$$

$$\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{x}_n} = \tilde{\boldsymbol{\Sigma}}^{-}_{\boldsymbol{x}_n} - \boldsymbol{K}_n \boldsymbol{H}_n \tilde{\boldsymbol{\Sigma}}^{-}_{\boldsymbol{x}_n}$$

- The result of the EKF update step is a Gaussian approximation of the posterior PDF, i.e., $f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n}) \approx \mathcal{N}\left(\tilde{\boldsymbol{\mu}}_{\boldsymbol{x}_n}, \tilde{\boldsymbol{\Sigma}}_{\boldsymbol{x}_n}\right)$

EKF Limitations

- The extended Kalman filter is not optimal (in the LMMSE/MMSE sense) for the original nonlinear problem; its performance will depend on the accuracy of the linearization approximations involved
- The "pseudo error covariances" $\tilde{\boldsymbol{\Sigma}}^{-}_{\boldsymbol{x}_n}$ and $\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{x}_n}$ are not really covariance matrices of the estimation error
- Finally, in contrast to the conventional Kalman algorithm, $\boldsymbol{K}_n$, $\tilde{\boldsymbol{\Sigma}}^{-}_{\boldsymbol{x}_n}$, and $\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{x}_n}$ cannot be precomputed; they must be computed on-line as they depend on the estimates $\tilde{\boldsymbol{\mu}}^{-}_{\boldsymbol{x}_n}$ and $\tilde{\boldsymbol{\mu}}_{\boldsymbol{x}_n}$ and hence on the data $\boldsymbol{y}_n$