ECE 286: Bayesian Machine Perception

Class 8: Probabilistic Data Association

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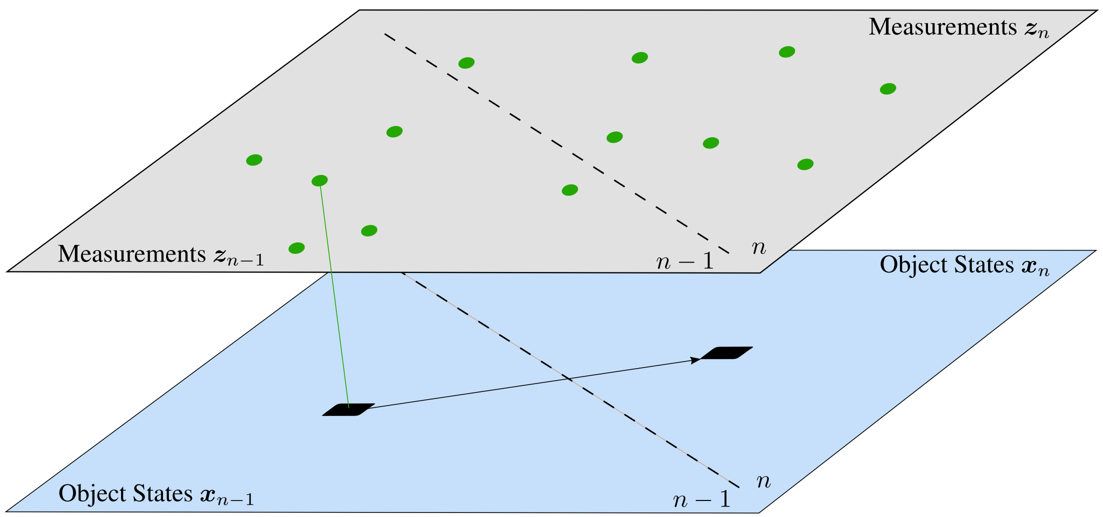
Basic Setting

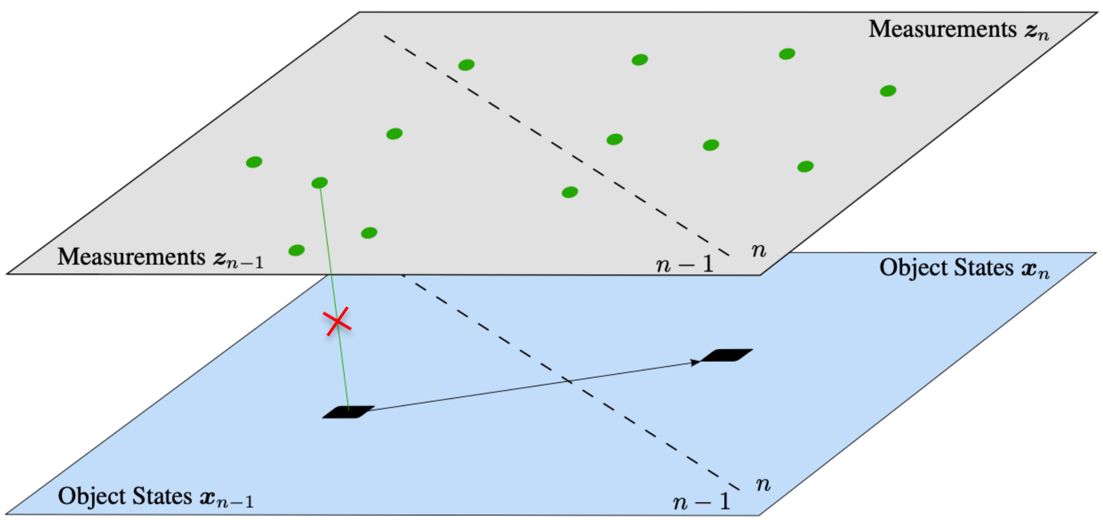
- “Single object tracking in clutter” problem



Basic Setting

- “Single object tracking in clutter” problem

- Measurement-origin uncertainty (MOU), false clutter measurements and missed detections



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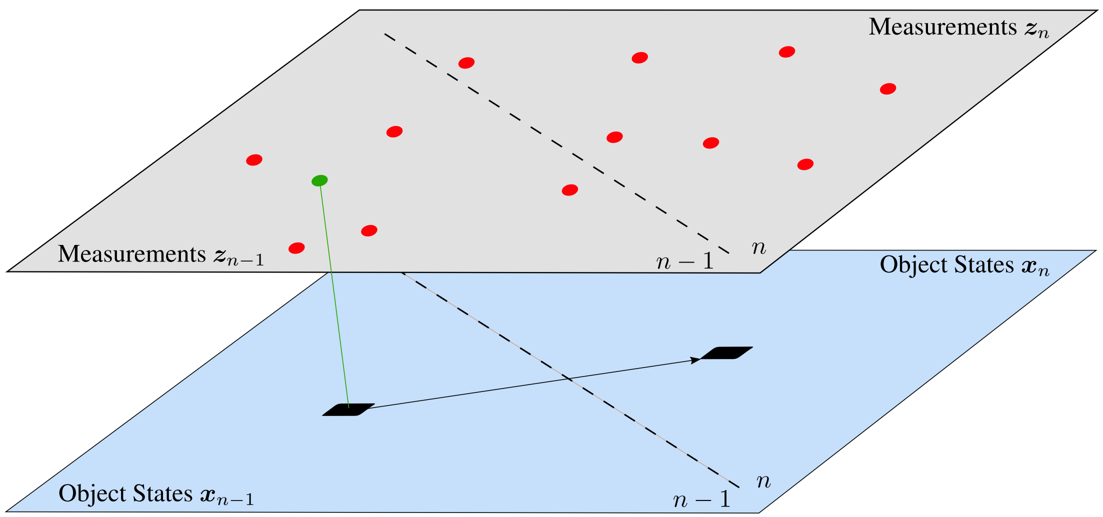

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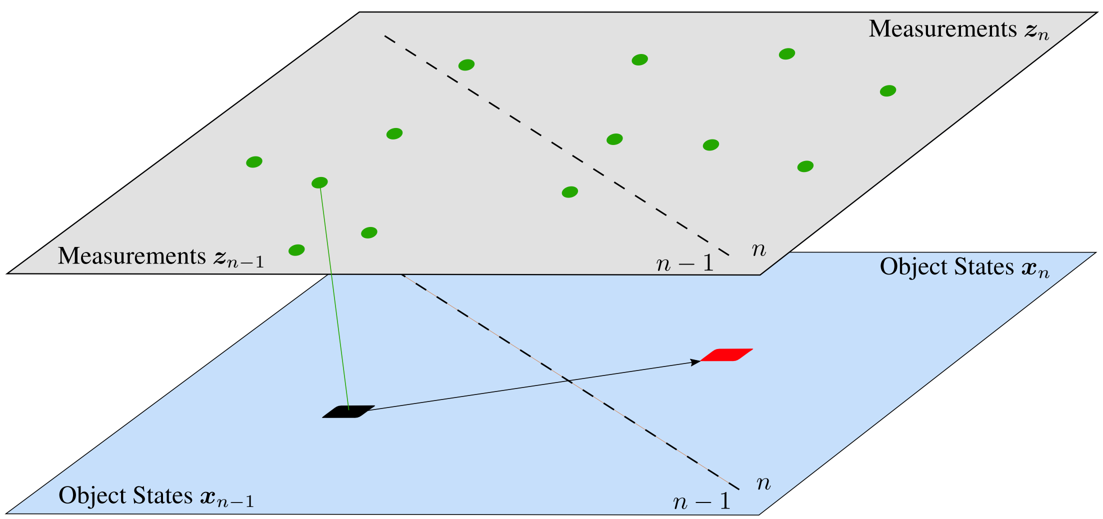
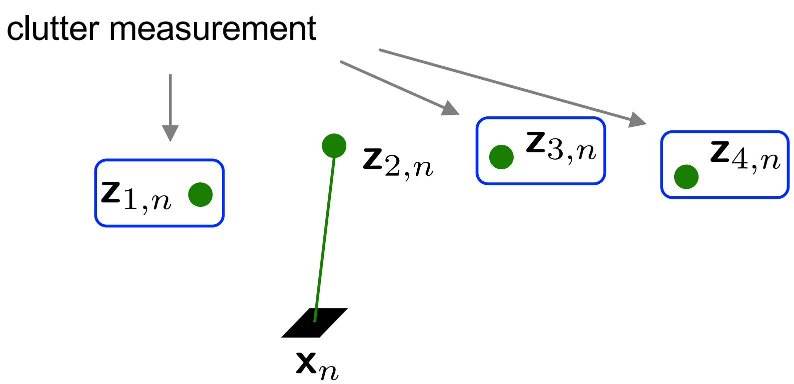


The Association Variable

- Object-oriented association variable $a_n \in \{0, 1, \ldots, \boxed{M_n}\}$ ← number of measurements
  - $a_n = m > 0$: at time n, the object generates the measurement with index m

- Example 1:

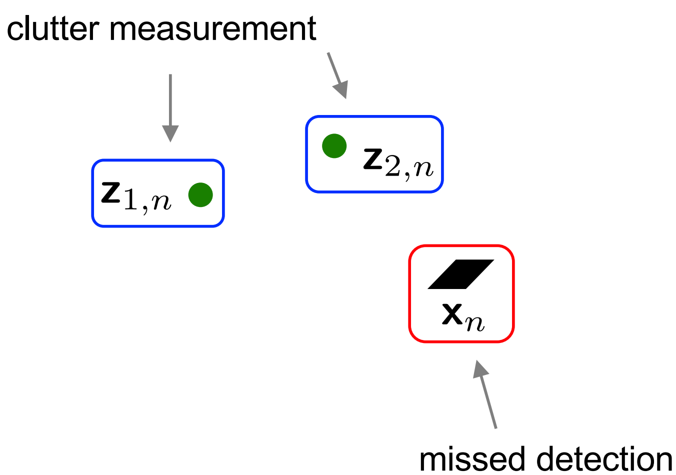


$M_n = 4$ $a_n = 2$

The Association Variable

- Object-oriented association variable $a_n \in \{0, 1, \ldots, M_n\}$
  - $a_n = m > 0$: at time n, the object generates the measurement with index m
  - $a_n = 0$: at time n, the object did not generate a measurement

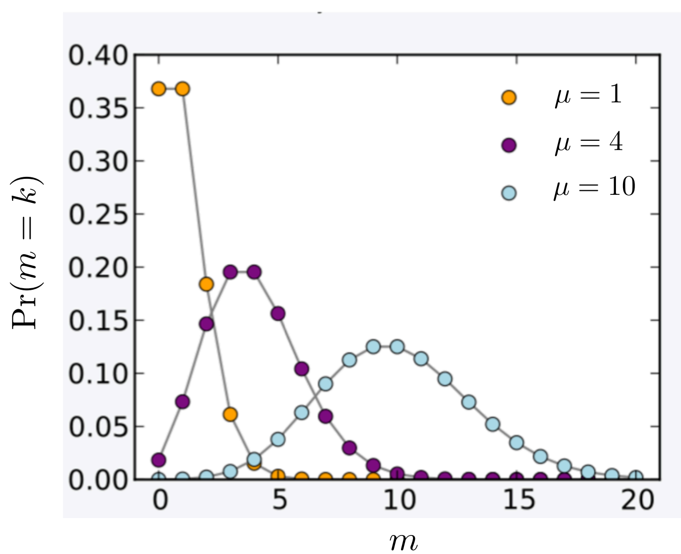
- Example 2:



$M_n = 2$ $\quad$ $a_n = 0$

The Poisson Distribution

- A discrete random variable m is said to have a Poisson distribution with parameter $\mu > 0$, if for $m = 0, 1, 2, \ldots$ the probability mass function is given by



$$p(m) = \frac{\mu^m \, \mathrm{e}^{-\mu}}{m!}$$

The parameter μ is the mean as well as the variance

Single Object Tracking in Clutter

- The state of the object is denoted $\boldsymbol{x}_n \in \mathbb{R}^w$ and the joint measurement is given by $\boldsymbol{z}_n \triangleq [\boldsymbol{z}_{1,n}^{\mathrm{T}}, \boldsymbol{z}_{2,n}^{\mathrm{T}}, \ldots, \boldsymbol{z}_{M_n,n}^{\mathrm{T}}]^{\mathrm{T}}$ with entries $\boldsymbol{z}_{m,n} \in \mathbb{R}^d$

- The association variable a_n is given by
  - $m \in \{1, 2, \ldots, M_n\}$, if measurement $\boldsymbol{z}_{m,n}$ was generated by the object
  - 0, if no measurement was generated by the object

- Association variable a_n and the number of measurements M_n are random variables

Prior Distribution

- Key Assumptions I:
  - Object detection $\theta_n \in \{0, 1\}$ is a Bernoulli trial with success probability $0 < p_{\mathrm{d}} \leqslant 1$
  - The number of clutter measurements L_n is Poisson distributed with mean μ_{c}
  - At most one measurement $\boldsymbol{z}_{m,n}$ is generated by the object

- Joint prior probability mass function (pmf):

$$p(a_n, \theta_n, L_n) = p(a_n|\theta_n, L_n)\, p(\theta_n)\, p(L_n)$$

$$p(\theta_n) = \begin{cases} p_{\mathrm{d}} & \theta_n = 1 \\ 1 - p_{\mathrm{d}} & \theta_n = 0 \end{cases}$$

$$p(a_n|\theta_n = 1, L_n) = \begin{cases} \frac{1}{L_n + 1} & a_n \in \{1, \ldots, M_n\} \\ 0 & a_n = 0 \end{cases}$$

$$p(L_n) = \frac{\mu_{\mathrm{c}}^{L_n}}{L_n!} e^{-\mu_{\mathrm{c}}}$$

$$p(a_n|\theta_n = 0, L_n) = \begin{cases} 0 & a_n \in \{1, \ldots, M_n\} \\ 1 & a_n = 0 \end{cases}$$

Prior Distribution

- After variable transform $L_n + \theta_n \rightarrow M_n$ we obtain

$$p(a_n, M_n) = \begin{cases} p_{\mathrm{d}} \frac{\mu_{\mathrm{c}}^{M_n - 1}}{M_n!} e^{-\mu_{\mathrm{c}}} & a_n \in \{1, \dots, M_n\} \\ (1 - p_{\mathrm{d}}) \frac{\mu_{\mathrm{c}}^{M_n}}{M_n!} e^{-\mu_{\mathrm{c}}} & a_n = 0 \end{cases}$$

- Properties:
  - For all arbitrary $L_n + \theta_n = M_n$ we have $p(a_n, \theta_n, L_n) = p(a_n, M_n)$
  - $p(a_n, M_n)$ is a valid pmf in the sense that $\sum_{M_n=0}^{\infty} \sum_{a_n=0}^{M_n} p(a_n, M_n) = 1$

Likelihood Function

- Key Assumption II:
  - Clutter measurements are independent and identically distributed (iid) according to $f_{\mathrm{c}}(\boldsymbol{z}_{m,n})$
  - Condition on $\boldsymbol{x}_n$, the object-generated measurement $\boldsymbol{z}_{a_n,n}$ is conditionally independent of all the other measurements

- Likelihood function:
  - for $\boldsymbol{z}_n \in \mathbb{R}^{M_n d}$,

measurement model $\boldsymbol{z}_{a_n,n} = h_n(\boldsymbol{x}_n, \boldsymbol{v}_n)$ with noise $\boldsymbol{v}_n$

$$f(\boldsymbol{z}_n|\boldsymbol{x}_n, a_n, M_n) = \begin{cases} \prod_{m=1}^{M_n} f_{\mathrm{c}}(\boldsymbol{z}_{m,n}) & a_n = 0 \\ \frac{\boxed{f(\boldsymbol{z}_{a_n,n}|\boldsymbol{x}_n)}}{f_{\mathrm{c}}(\boldsymbol{z}_{a_n,n})} \prod_{m=1}^{M_n} f_{\mathrm{c}}(\boldsymbol{z}_{m,n}) & a_n \in \{1, \ldots, M_n\} \end{cases}$$

  - For $\boldsymbol{z}_n \notin \mathbb{R}^{M_n d}$,

$$f(\boldsymbol{z}_n|\boldsymbol{x}_n, a_n, M_n) = 0$$

Joint Distributions

- Joint prior for $\boldsymbol{x}_{0:n}$

State transition model $\boldsymbol{x}_n = g_n(\boldsymbol{x}_{n-1}, \boldsymbol{u}_n)$ with noise $\boldsymbol{u}_n$

$$f(\boldsymbol{x}_{0:n}) = f(\boldsymbol{x}_0) \prod_{n'=1}^{n} \boxed{f(\boldsymbol{x}_{n'}|\boldsymbol{x}_{n'-1})}$$

Driving noise independent across time n and independent of $\boldsymbol{x}_0$

- Joint prior for $\boldsymbol{a}_{1:n}$ and $\boldsymbol{M}_{1:n}$

$$p(\boldsymbol{a}_{1:n}, \boldsymbol{M}_{1:n}) = \prod_{n'=1}^{n} p(a_{n'}, M_{n'})$$

Measurement generation independent across time n

- Joint likelihood function

$$f(\boldsymbol{z}_{1:n}|\boldsymbol{x}_{1:n}, \boldsymbol{a}_{1:n}, \boldsymbol{M}_{1:n}) = \prod_{n'=1}^{n} f(\boldsymbol{z}_{n'}|\boldsymbol{x}_{n'}, a_{n'}, M_{n'})$$

Measurement noise and clutter independent across time n

The Joint Posterior Distribution

- The joint posterior distribution ($M_{1:n}$ and $\boldsymbol{z}_{1:n}$ are observed and thus fixed)

$$f(\boldsymbol{x}_{0:n}, \boldsymbol{a}_{1:n} | \boldsymbol{z}_{1:n}) = f(\boldsymbol{x}_{0:n}, \boldsymbol{a}_{1:n}, \boldsymbol{M}_{1:n} | \boldsymbol{z}_{1:n})$$ ⟵ $M_{1:n}$ fixed

Bayes rule ⟶
$$\propto f(\boldsymbol{z}_{1:n} | \boldsymbol{x}_{1:n}, \boldsymbol{a}_{1:n}, \boldsymbol{M}_{1:n}) f(\boldsymbol{x}_{0:n}, \boldsymbol{a}_{1:n}, \boldsymbol{M}_{1:n})$$

$\boldsymbol{x}_{0:n} \perp\!\!\!\perp \boldsymbol{a}_{1:n}, \boldsymbol{M}_{1:n}$ ⟶
$$= f(\boldsymbol{z}_{1:n} | \boldsymbol{x}_{1:n}, \boldsymbol{a}_{1:n}, \boldsymbol{M}_{1:n}) f(\boldsymbol{x}_{0:n}) p(\boldsymbol{a}_{1:n}, \boldsymbol{M}_{1:n})$$

Expressions for joint distributions ⟶
$$= f(\boldsymbol{x}_0) \prod_{n'=1}^{n} f(\boldsymbol{x}_{n'} | \boldsymbol{x}_{n'-1}) f(\boldsymbol{z}_{n'} | \boldsymbol{x}_{n'}, \boldsymbol{a}_{n'}, M_{n'}) p(\boldsymbol{a}_{n'}, M_{n'})$$

Problem Formulation

- Input at time n:
  - All observations up to time $\boldsymbol{z}_{1:n}$
  - ``Markovian'' statistical model

$$f(\boldsymbol{x}_{1:n}, \boldsymbol{a}_{1:n} | \boldsymbol{z}_{1:n}) \propto f(\boldsymbol{x}_0) \prod_{n'=1}^{n} f(\boldsymbol{x}_{n'} | \boldsymbol{x}_{n'-1}) f(\boldsymbol{z}_{n'} | \boldsymbol{x}_{n'}, \boldsymbol{a}_{n'}, M_{n'}) p(\boldsymbol{a}_{n'}, M_{n'})$$

- Output at time n:
  - Estimate of $\hat{\boldsymbol{x}}_n$

- Calculation of an estimate $\hat{\boldsymbol{x}}_n$ is based on the marginal posterior pdf $f(\boldsymbol{x}_n | \boldsymbol{z}_{1:n})$

The Factor Graph

- Recall factorization of the joint posterior distribution:

$$f(\boldsymbol{x}_{1:n}, \boldsymbol{a}_{1:n} | \boxed{\boldsymbol{z}_{1:n}}) \propto f(\boldsymbol{x}_0) \prod_{n'=1}^{n} f(\boldsymbol{x}_{n'} | \boldsymbol{x}_{n'-1}) f(\boxed{\boldsymbol{z}_{n'}} | \boldsymbol{x}_{n'}, a_{n'}, \boxed{M_{n'}}) p(a_{n'}, \boxed{M_{n'}})$$

observed and fixed

The Factor Graph

- **Recall factorization of the joint posterior distribution:**

$$f(\boldsymbol{x}_{1:n}, \boldsymbol{a}_{1:n} | \boldsymbol{z}_{1:n}) \propto f(\boldsymbol{x}_0) \prod_{n'=1}^{n} f(\boldsymbol{x}_{n'} | \boldsymbol{x}_{n'-1}) f(\boldsymbol{z}_{n'} | \boldsymbol{x}_{n'}, a_{n'}, M_{n'}) p(a_{n'}, M_{n'})$$

$$\propto f(\boldsymbol{x}_0) \prod_{n'=1}^{n} f(\boldsymbol{x}_{n'} | \boldsymbol{x}_{n'-1}) g_1(\boldsymbol{x}_{n'}, a_{n'}) g_2(a_{n'})$$

$$f(\boldsymbol{z}_n | \boldsymbol{x}_n, a_n, M_n) = \begin{cases} \frac{f(\boldsymbol{z}_{a_n,n} | \boldsymbol{x}_n)}{f_{\mathrm{c}}(\boldsymbol{z}_{a_n,n})} \prod_{m=1}^{M} f_{\mathrm{c}}(\boldsymbol{z}_{m,n}) & a_n \in \{1, \dots, M_n\} \\ \boxed{\prod_{m=1}^{M} f_{\mathrm{c}}(\boldsymbol{z}_{m,n})} & a_n = 0 \end{cases}$$

constant

$$g_1(\boldsymbol{x}_n, a_n) = \begin{cases} \frac{f(\boldsymbol{z}_{a_n,n} | \boldsymbol{x}_n)}{f_{\mathrm{c}}(\boldsymbol{z}_{a_n,n})} & a_n \in \{1, \dots, M_n\} \\ 1 & a_n = 0 \end{cases}$$

The Factor Graph

- **Recall factorization of the joint posterior distribution:**

$$f(\boldsymbol{x}_{1:n}, \boldsymbol{a}_{1:n} | \boldsymbol{z}_{1:n}) \propto f(\boldsymbol{x}_0) \prod_{n'=1}^{n} f(\boldsymbol{x}_{n'} | \boldsymbol{x}_{n'-1}) f(\boldsymbol{z}_{n'} | \boldsymbol{x}_{n'}, a_{n'}, M_{n'}) p(a_{n'}, M_{n'})$$

$$\propto f(\boldsymbol{x}_0) \prod_{n'=1}^{n} f(\boldsymbol{x}_{n'} | \boldsymbol{x}_{n'-1}) g_1(\boldsymbol{x}_{n'}, a_{n'}) g_2(a_{n'})$$

$$p(a_n, M_n) = \begin{cases} p_\mathrm{d} \frac{\mu_\mathrm{c}^{M_n - 1}}{M_n!} e^{-\mu_\mathrm{c}} & a_n = \{1, \dots, M_n\} \\ (1 - p_\mathrm{d}) \boxed{\frac{\mu_\mathrm{c}^{M_n}}{M_n!} e^{-\mu_\mathrm{c}}} & a_n = 0 \end{cases}$$

constant

$$g_2(a_n) = \begin{cases} \frac{p_\mathrm{d}}{\mu_\mathrm{c}} & a_n \in \{1, \dots, M_n\} \\ (1 - p_\mathrm{d}) & a_n = 0 \end{cases}$$

The Factor Graph

- **Recall factorization of the joint posterior distribution:**

$$
\begin{aligned}
f(\boldsymbol{x}_{1:n}, \boldsymbol{a}_{1:n} | \boldsymbol{z}_{1:n}) &\propto f(\boldsymbol{x}_0) \prod_{n'=1}^{n} f(\boldsymbol{x}_{n'} | \boldsymbol{x}_{n'-1}) f(\boldsymbol{z}_{n'} | \boldsymbol{x}_{n'}, a_{n'}, M_{n'}) p(a_{n'}, M_{n'}) \\
&\propto f(\boldsymbol{x}_0) \prod_{n'=1}^{n} f(\boldsymbol{x}_{n'} | \boldsymbol{x}_{n'-1}) g_1(\boldsymbol{x}_{n'}, a_{n'}) g_2(a_{n'}) \\
&= f(\boldsymbol{x}_0) \prod_{n'=1}^{n} f(\boldsymbol{x}_{n'} | \boldsymbol{x}_{n'-1}) g_{\boldsymbol{z}_n}(\boldsymbol{x}_{n'}, a_{n'})
\end{aligned}
$$

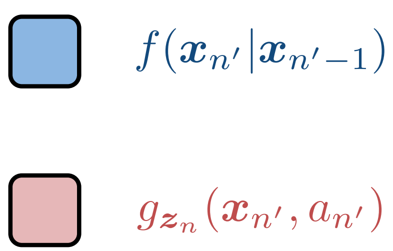
$$
g_{\boldsymbol{z}_n}(\boldsymbol{x}_n, a_n) = g_1(\boldsymbol{x}_n, a_n) g_2(a_n) = \begin{cases} \frac{p_{\mathrm{d}} f(\boldsymbol{z}_{a_n,n} | \boldsymbol{x}_n)}{\mu_{\mathrm{c}} f_{\mathrm{c}}(\boldsymbol{z}_{a_n,n})} & a_n \in \{1, \ldots, M_n\} \\ (1 - p_{\mathrm{d}}) & a_n = 0 \end{cases}
$$

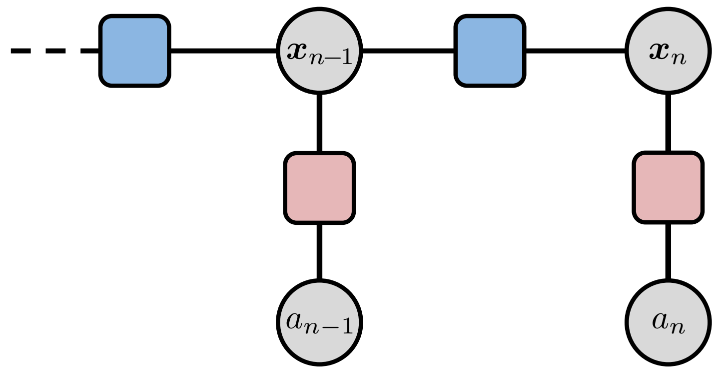
The Factor Graph

- Recall factorization of the joint posterior distribution:

$$f(\boldsymbol{x}_{1:n}, \boldsymbol{a}_{1:n} | \boldsymbol{z}_{1:n}) \propto f(\boldsymbol{x}_0) \prod_{n'=1}^{n} f(\boldsymbol{x}_{n'} | \boldsymbol{x}_{n'-1}) \, g_{\boldsymbol{z}_n}(\boldsymbol{x}_{n'}, a_{n'})$$

- Factor graph for two time steps $n' \in \{n-1, n\}$

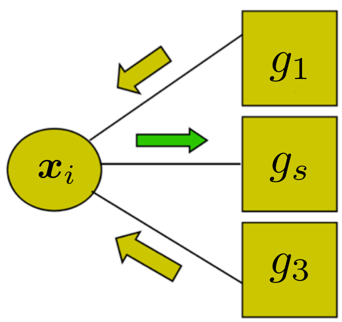


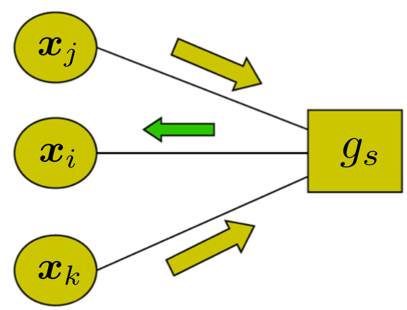


- The factor graph is cycle free $\Rightarrow$ message passing can provide exact marginals

Recall Message Passing Rules

- Message passing protocol: A message to a neighboring node can only be send when it has received messages from all its other neighbors
- Marginal distribution can be calculated as $b(\boldsymbol{x}_i) \propto \prod_{t \in \mathcal{N}(i)} \phi_{ti}(\boldsymbol{x}_i) = \phi_{ti}(\boldsymbol{x}_i)\nu_{it}(\boldsymbol{x}_i)$





$$\nu_{is}(\boldsymbol{x}_i) = \prod_{t \in \mathcal{N}(i) \setminus s} \phi_{ti}(\boldsymbol{x}_i)$$

$$\phi_{si}(\boldsymbol{x}_i) = \int \Big(g_s(\boldsymbol{x}_{\mathcal{N}(s)}) \prod_{j \in \mathcal{N}(s) \setminus i} \nu_{js}(\boldsymbol{x}_j) \Big) \mathrm{d}\boldsymbol{x}_{\mathcal{N}(s) \setminus i}$$

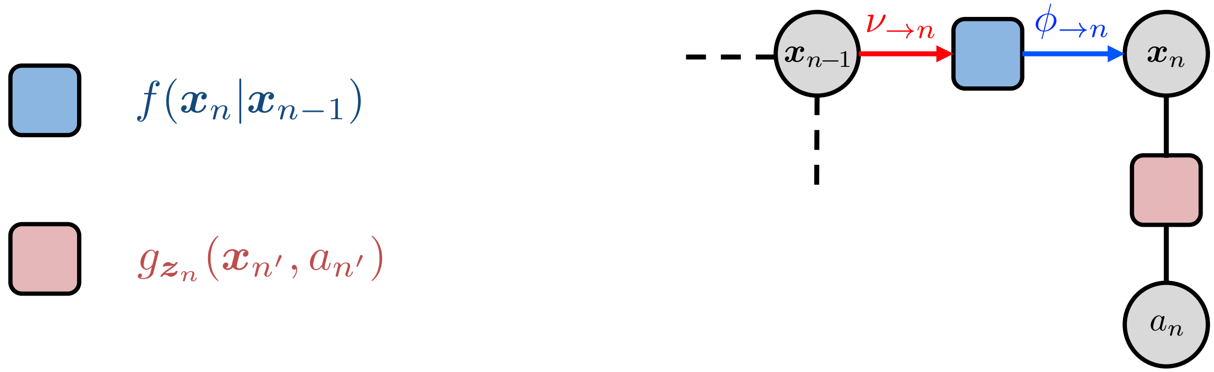
Prediction

- Prediction step:

$$f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1}) = \int f(\boldsymbol{x}_n|\boldsymbol{x}_{n-1}) f(\boldsymbol{x}_{n-1}|\boldsymbol{y}_{1:n-1}) \mathrm{d}\boldsymbol{x}_{n-1}$$

$$\phi_{\rightarrow n}(\boldsymbol{x}_n) = \int f(\boldsymbol{x}_n|\boldsymbol{x}_{n-1})\, \nu_{\rightarrow n}(\boldsymbol{x}_{n-1}) \mathrm{d}\boldsymbol{x}_{n-1}$$

- Factor graph for two time steps $n' \in \{n-1, n\}$



Data Association

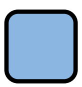
- Data association step:

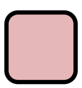
$$\phi_n(\boldsymbol{x}_n) = \sum_{m=0}^{M_n} g_{\boldsymbol{z}_n}(\boldsymbol{x}_n, a_n = m)\, \nu_n(a_n = m)$$

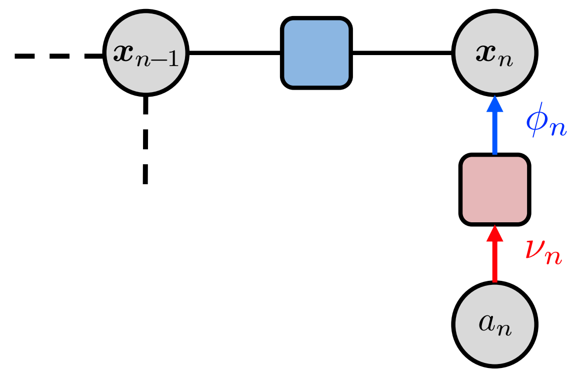
$$\nu_n(a_n) = 1$$

(no other neighbors)

- Factor graph for two time steps $n' \in \{n-1, n\}$

 $f(\boldsymbol{x}_n | \boldsymbol{x}_{n-1})$

 $g_{\boldsymbol{z}_n}(\boldsymbol{x}_{n'}, a_{n'})$



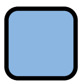
Update

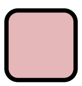
- Update step:

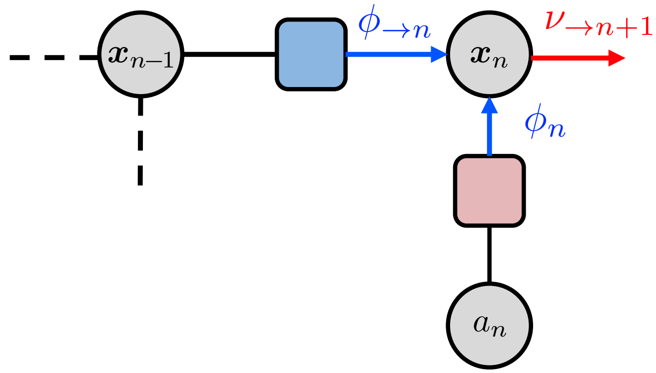
$$f(\boldsymbol{x}_n|\boldsymbol{z}_{1:n}) \propto \phi_n(\boldsymbol{x}_n)\,\phi_{\rightarrow n}(\boldsymbol{x}_n) \qquad f(\boldsymbol{x}_n|\boldsymbol{z}_{1:n}) \propto \phi_n(\boldsymbol{x}_n)\, f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1})$$

$$\nu_{\rightarrow n+1}(\boldsymbol{x}_n) = \phi_n(\boldsymbol{x}_n)\,\phi_{\rightarrow n}(\boldsymbol{x}_n)$$

- Factor graph for two time steps $n' \in \{n-1, n\}$

 $f(\boldsymbol{x}_n|\boldsymbol{x}_{n-1})$

 $g_{\boldsymbol{z}_n}(\boldsymbol{x}_{n'}, a_{n'})$



Particle-Based Update Step (cf. Class 4)

- **Given:** Particles $\{(\boldsymbol{x}_n^{(j)})\}_{j=1}^{J} \simeq f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1})$ representing the predicted posterior PDF

- **Wanted:** Particles $\{(\overline{\boldsymbol{x}}_n^{(j)})\}_{j=1}^{J} \simeq f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n})$ representing the posterior PDF

- Perform importance sampling with proposal distribution $f_{\mathrm{p}}(\boldsymbol{x}_n) = f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1})$ and target distribution $f_{\mathrm{t}}(\boldsymbol{x}_n) \propto \phi_n(\boldsymbol{x}_n) f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1})$
  - calculate unnormalized weights $\tilde{w}_n^{(j)} = \boxed{\sum_{m=0}^{M_n} g_{\boldsymbol{z}_n}(\boldsymbol{x}_n^{(j)}, a_n = m)} \propto f_{\mathrm{t}}(\boldsymbol{x}_n^{(j)})/f_{\mathrm{p}}(\boldsymbol{x}_n^{(j)})$
  - normalize weights $w_n^{(j)} = \tilde{w}_n^{(j)} / \sum_{j'=1}^{J} \tilde{w}_n^{(j')}, \quad j = 1, \ldots, J$

- Perform resampling to get $\{(\overline{\boldsymbol{x}}_n^{(j)})\}_{j=1}^{J} \simeq f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n})$ from $\{(\boldsymbol{x}_n^{(j)}, w_n^{(j)})\}_{j=1}^{J} \simeq f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n})$

Summary

- Single object tracking in clutter
  - possible association events are modelled by discrete random variable
  - data association is performed by summing over all possible association events
  - the sequential estimation problem that can be represented by a cycle free factor graph
  - a particle-based implementation can provide exact estimation results as the number of particles goes to infinity