ECE 161A: LTI Systems

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Typical Sequences

1. Delta function (unit sample sequence) :

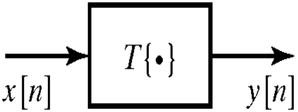
$$x[n] = \delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

2. Delayed delta function : $x[n] = \delta[n - n_0] = \begin{cases} 1 & n = n_0 \\ 0 & n \neq n_0 \end{cases}$
3. Step function : $x[n] = u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$
4. Sines, Cosines, Exponentials: $x[n] = A\cos(\omega_0 n + \phi)$

In general, any signal can be written as a weighted sum of delayed delta functions:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - k].$$

Discrete Time Systems



The input-output relation is given by $y[n] = T\{x[n]\}$

Too general for characterization and design purposes and so constraints are needed.

Special Class of Systems

Memoryless Systems: Output at time n depends only on $x[n]$, i.e input at time n. No memory

Linear Systems: $T\{ax_1[n] + bx_2[n]\} = aT\{x_1[n]\} + bT\{x_2[n]\}$. Additivity and Scaling property

Time-Invariant Systems: If $y[n] = T\{x[n]\}$, then $T\{x[n-n_0]\} = y[n-n_0]$. Delayed input leads to delayed output

Causality: Output $y[n]$ at time $n = n_0$ depends only on the input sequence values for $n \leq n_0$. Output depends only on the present and past

Stable Systems: If $|x[n] \leq B_x < \infty \ \forall n$, then $|y[n]| \leq B_y < \infty \ \forall n$. Must be true for all bounded inputs. Such systems are Bounded-Input Bounded-Output (BIBO) stable

Examples

System 1: $y[n] = x^2[n]$, Memoryless: yes, Linear: no, Time-Invariant: yes, Causal: yes, Stable: yes

System 2: $y[n] = A\cos(\omega_0 n + \phi)x[n]$, Memoryless: yes, Linear: yes, Time-Invariant: No, Causal: yes, Stable: yes

System 3: $y[n] = \frac{1}{2}(x[n] + x[n-1])$, Memoryless: no, Linear: yes, Time-Invariant: yes, Causal: yes, Stable: yes

System 4: $y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n+1])$, Memoryless: no, Linear: yes, Time-Invariant: yes, Causal: no, Stable: yes

Linear Time-Invariant (LTI) Systems

Reason: Simple characterization and design. Response to a unit sample sequence (delta function) is sufficient to define the system

Suppose $h[n] = T\{\delta[n]\}$, then since system is time-invariant $T\{\delta[n-k]\} = h[n-k]$. $h[n]$ is the impulse response of the system.

For LTI systems

$$
\begin{aligned}
y[n] &= T\{x[n]\} = T\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\} \\
&= \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\} \ \ \text{(Linearity)} \\
&= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \ \ \text{(Time-Invariance)} \\
&= x[n] * h[n] \ \ \text{(Convolution)}
\end{aligned}
$$

If we expand the sum

$$y[n] = ... + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + ...$$

Suggests a procedure for carrying out convolution

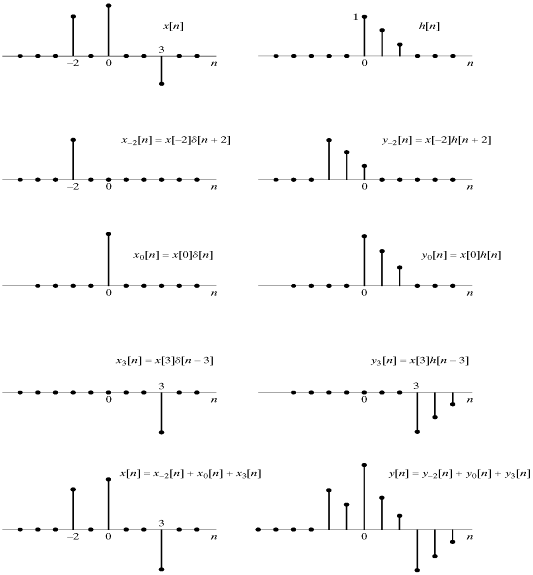


Figure 2.8 Representation of the output of a linear time-invariant system as the superposition of responses to individual samples of the input.

Alternate Approach to Convolution

$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

1: $h_1[k] = h[-k]$ (Flip sequence about origin)

2: Construct $h[n-k]$ for particular n. Amounts to shifting $h_1[k]$ to the right by n for $n > 0$, and to the left by $|n|$ for $n < 0$.

3: Multiply $x[k]$ and $h[n-k]$ term by term to obtain $x[k]h[n-k]$ for each value of k.

4: Sum all the values to get $y[n]$, i.e.$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

5: Repeat above steps (2-4) for all n.

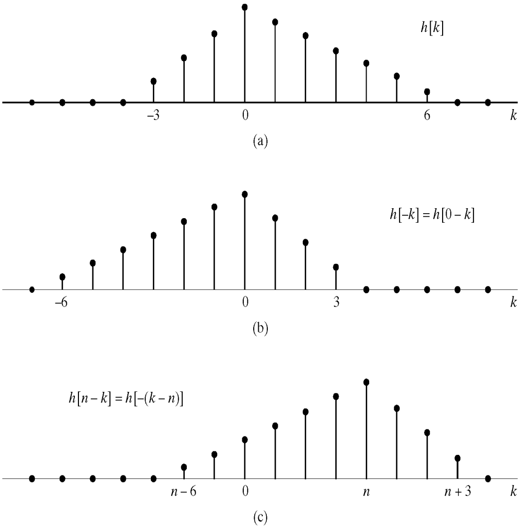


Figure 2.9 Forming the sequence $h[n-k]$. (a) The sequence $h[k]$ as a function of k. (b) The sequence $h[-k]$ as a function of k. (c) The sequence $h[n-k] = h[-(k-n)]$ as a function of k for $n = 4$.

Properties of Convolution

Commutative: $x[n] * h[n] = h[n] * x[n]$. System and input can be interchanged

Distributive: $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$. Leads to the parallel form

Associative: $(x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n])$. leads to cascade forms

Also by the commutative property,
$x[n] * (h_1[n] * h_2[n]) = x[n] * (h_2[n] * h_1[n])$. This allows reordering of systems

Distributive Property

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n].$$

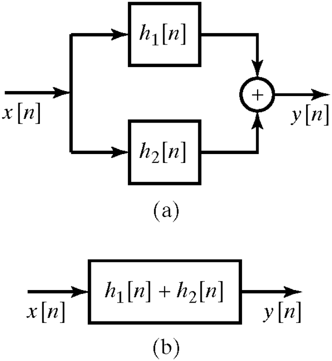
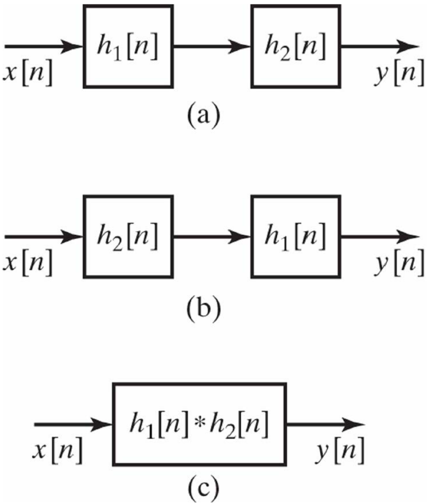


Figure 2.12 (a) Parallel combination of linear time-invariant systems. (b) An equivalent system.

Commutative and Associative Property

$$(x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n]).$$

Figure 2.12 (a) Cascade combination of two LTI systems. (b) Equivalent cascade. (c) Single equivalent system.



Causality and Stability of LTI Systems

LTI systems defined by their impulse response $h[n]$.
Output for any input can be computed as $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

Causality and Stability can be inferred from $h[n]$.

System 1: $h[n] = 2\delta[n] + 3\delta[n-4] + 3\delta[n-5]$

System 2: $h[n] = \delta[n+2] + 2\delta[n] + 3\delta[n-4] + 3\delta[n-5]$

System 3: $h[n] = u[n]$.

Causality LTI Systems

Causality: $h[n] = 0, \forall n < 0$.

Proof:
$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{n} x[k]h[n-k] + \sum_{k=n+1}^{\infty} x[k]h[n-k]$
First term depends on the past values of the input. The second term depends on the future values and should be zero for causal systems. This requires $h[n-k] = 0, k = n+1, n+2, ...$ or $h[n] = 0, n < 0$.

For a causal system

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

In the examples mentioned: System 1 and 3 are causal, System 2 is non-causal.

Stability for LTI Systems

Based on $h[n]$, how do you determine stability?

Main Result: A LTI system is BIBO stable if and only if the impulse response is absolutely summable, i.e $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$.

In the examples mentioned: System 1 and 2 are stable, System 2 is unstable.

Linear Constant-Coefficient Difference Equations

Even general LTI systems are not practical. Why?
$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$ and requires an infinite sum.
This leads to LTI systems described by difference equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

or alternatively

$$y[n] = -\sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k] + \sum_{m=0}^{M} \frac{b_m}{a_0} x[n-m]$$

Can set $a_0 = 1$ without loss of generality (wlog). a_k are referred to as the feedback coefficients and b_m the feedforward coefficients.

No infinite sums and so viable. Design reduces to choosing N, M, a_k's and b_m's.

FIR and IIR Filters

Finite Impulse Response (FIR) filters:

$$y[n] = \sum_{m=0}^{M} b_m x[n-m]$$

Impulse response of a FIR filter: Compare with convolution $y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$.

$h[n] = b_n, 0 \leq n \leq M$, and zero otherwise.
Can be implemented using simple memory, multipliers and adders.

Is it stable?Yes, as long as the b_m are bounded.

IIR filters

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{m=0}^{M} b_m x[n-m]$$

Because of the feedback terms, impulse response is of infinite duration.
Stability depends on the feedback coefficients.

Simple IIR Example

$$y[n] = ay[n-1] + x[n]$$

Assume system at rest, i.e. $y[-1] = 0$. (Clear memory)

Setting $x[n] = \delta[n]$ leads to the output $y[n] = h[n]$.
Can show

$$h[n] = a^n u[n]$$

Stable? Based on absolute summability criteria, Stable if $|a| < 1$.