# ECE 161A: DFT Properties

Florian Meyer
University of California, San Diego
Email: flmeyer@ucsd.edu

# Frequency Sampling: Pictorial Depiction

Suppose $x[n]$ is of duration $L$, i.e. $x[n]$ is nonzero for $n = 0, 1, \ldots, L-1$.

The frequency sampling pictorially is shown below.

$$x[n] \leftrightarrow X(e^{j\omega}) = \textstyle\sum_{n=0}^{L-1} x[n] e^{-j\omega n} \underset{?}{\overset{N}{\rightleftarrows}} X_s(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \tilde{X}[k] \delta(\omega - \frac{2\pi}{N} k)$$

Duration L

Only N≥L

ok

Only N≥L

$$\tilde{X}[k] = X(e^{j\omega})|_{\omega = \frac{2\pi}{N} k, k=0,1,..,N-1}$$

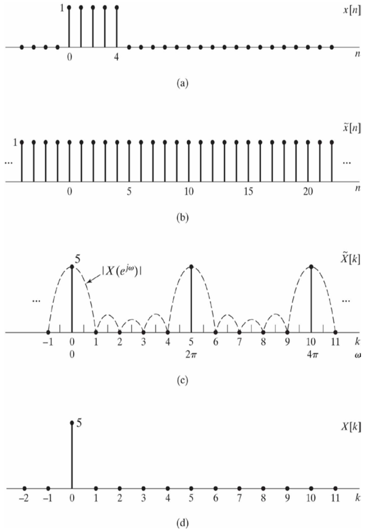
DFS N

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n + rN] = x[((n))_N]$$

Note $X[k] = \tilde{X}[k], 0 \leq k \leq N-1$.

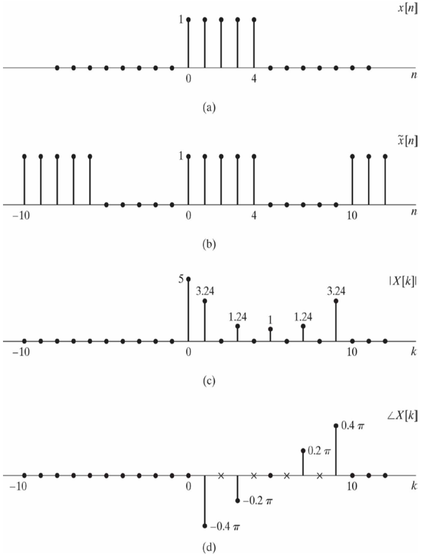
# Example

**Figure 8.10** Illustration of the DFT. (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ formed from $x[n]$ with period $N = 5$. (c) Fourier series coefficients $\tilde{X}[k]$ for $\tilde{x}[n]$. To emphasize that the Fourier series coefficients are samples of the Fourier transform, $|X(e^{j\omega})|$ is also shown. (d) DFT of $x[n]$.



# Example Cont'd

**Figure 8.11** Illustration of the DFT. (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ formed from $x[n]$ with period $N = 10$. (c) DFT magnitude. (d) DFT phase. (x's indicate indeterminate values.)



# Notation and Assumptions

$$x[n] \overset{N}{\longleftrightarrow} X[k]$$

Implied Assumption: Duration $L$ of the signal $x[n]$ is less than or equal to

$N$, i.e. $L \leq N$.

$N$ is a choice of the user.

$x[n]$ has been padded with $N - L$ zeros, i.e. $x[n] = 0, L, .., N - 1$. This is referred to as zero padding.

Notation: $X[k] = \mathcal{DFT}(x[n])$ and $x[n] = \mathcal{IDFT}(X[k])$.

# DFT Properties

**TABLE 8.2** SUMMARY OF PROPERTIES OF THE DFT

| Finite-Length Sequence (Length $N$) | $N$-point DFT (Length $N$) |
| --- | --- |
| 1. $x[n]$ | $X[k]$ |
| 2. $x_1[n], x_2[n]$ | $X_1[k], X_2[k]$ |
| 3. $ax_1[n] + bx_2[n]$ | $aX_1[k] + bX_2[k]$ |
| 4. $X[n]$ | $Nx[((-k))_N]$ |
| 5. $x[((n-m))_N]$ | $W_N^{km} X[k]$ |
| 6. $W_N^{-\ell n} x[n]$ | $X[((k-\ell))_N]$ |
| 7. $\sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$ | $X_1[k]X_2[k]$ |
| 8. $x_1[n]x_2[n]$ | $\frac{1}{N} \sum_{\ell=0}^{N-1} X_1[\ell] X_2[((k-\ell))_N]$ |
| 9. $x^*[n]$ | $X^*[((-k))_N]$ |
| 10. $x^*[((-n))_N]$ | $X^*[k]$ |

# DFT Properties Cont'd

**Table 8.2 (continued)** SUMMARY OF PROPERTIES OF THE DFT

| | |
|---|---|
| 11. $\mathcal{R}e\{x[n]\}$ | $X_{\text{ep}}[k] = \frac{1}{2}\{X[((k))_N] + X^*[((-k))_N]\}$ |
| 12. $j\mathcal{I}m\{x[n]\}$ | $X_{\text{op}}[k] = \frac{1}{2}\{X[((k))_N] - X^*[((-k))_N]\}$ |
| 13. $x_{\text{ep}}[n] = \frac{1}{2}\{x[n] + x^*[((-n))_N]\}$ | $\mathcal{R}e\{X[k]\}$ |
| 14. $x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x^*[((-n))_N]\}$ | $j\mathcal{I}m\{X[k]\}$ |
| Properties 15–17 apply only when $x[n]$ is real. | |
| 15. Symmetry properties | $\begin{cases} X[k] = X^*[((-k))_N] \\ \mathcal{R}e\{X[k]\} = \mathcal{R}e\{X[((-k))_N]\} \\ \mathcal{I}m\{X[k]\} = -\mathcal{I}m\{X[((-k))_N]\} \\ \lvert X[k]\rvert = \lvert X[((-k))_N]\rvert \\ \angle\{X[k]\} = -\angle\{X[((-k))_N]\} \end{cases}$ |
| 16. $x_{\text{ep}}[n] = \frac{1}{2}\{x[n] + x[((-n))_N]\}$ | $\mathcal{R}e\{X[k]\}$ |
| 17. $x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x[((-n))_N]\}$ | $j\mathcal{I}m\{X[k]\}$ |

# DFT Properties: Linearity

$$x_1[n] \overset{N}{\longleftrightarrow} X_1[k] \text{ and } x_2[n] \overset{N}{\longleftrightarrow} X_2[k].$$

Then

$$x[n] = a_1 x_1[n] + a_2 x_2[n] \overset{N}{\longleftrightarrow} X[k] = a_1 X_1[k] + a_2 X_2[k]$$

Note that even if $x_1[n]$ and $x_2[n]$ are of different duration, they are zero padded and made to be of the same length $N$.

Reminder: The DFT length $N$ is a parameter of choice.

# Circular Shift

Linear shift is not meaningful in the context of the DFT. Why?

The DFT and IDFT involve sums from 0 to $N-1$. If we shift a sequence $x[n]$, then some potentially non-zero samples will go outside the interval 0 to $N-1$. This make it difficult to get any meaningful results between the DFT of the original sequence $x[n]$ and the shifted sequence $x[n-m]$.

The circular shift is a linear shift of the periodic signal constructed using $x[n], 0 \leq n \leq N-1$, as one period, i.e. $\tilde{x}[n] = x[((n))_N]$.

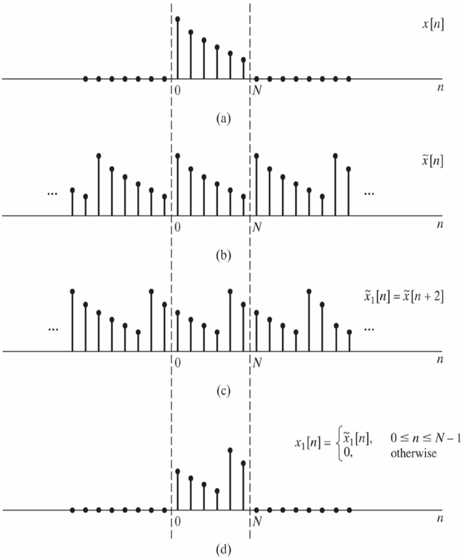
Circular Shift Property:

$$x[((n-m))_N] \stackrel{N}{\longleftrightarrow} W_N^{mk} X[k]$$

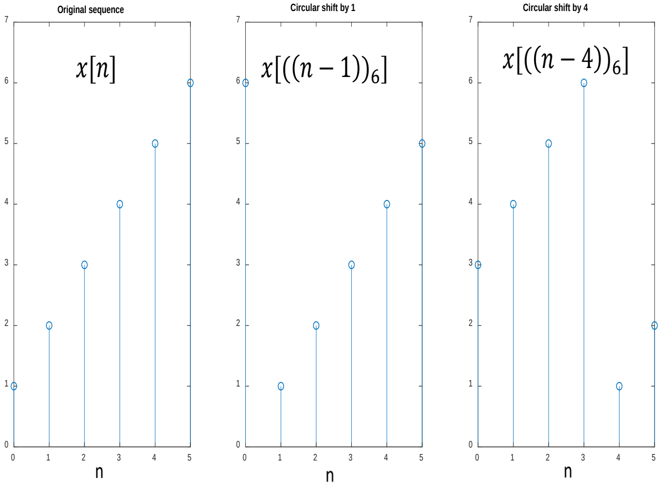
Proof: Follows from the DFS and DFT connection

# Circular Shift = Linear Shift of Periodic extension

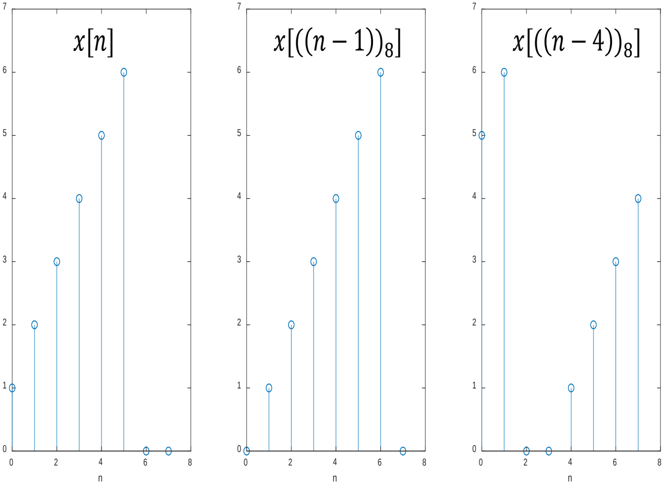
**Figure 8.12** Circular shift of a finite-length sequence; i.e., the effect in the time domain of multiplying the DFT of the sequence by a linear-phase factor.



# Circular Shift: Example 1



# Circular Shift: Example 2



# Circular Convolution

$$x_1[n] \overset{N}{\longleftrightarrow} X_1[k] \text{ and } x_2[n] \overset{N}{\longleftrightarrow} X_2[k].$$

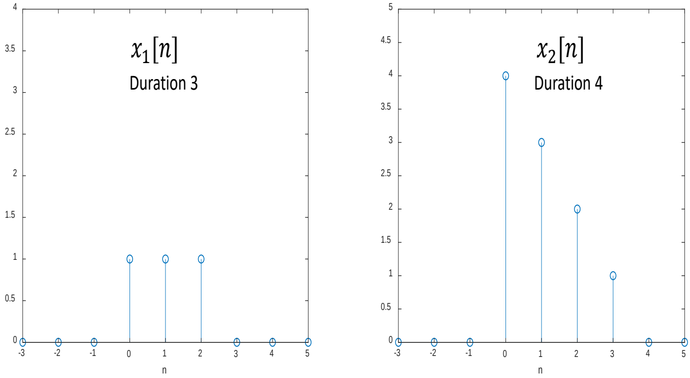
Then

$$x[n] = \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N] = x_1[n] \circledcirc x_2[n] \overset{N}{\longleftrightarrow} X[k] = X_1[k]X_2[k].$$

Proof:

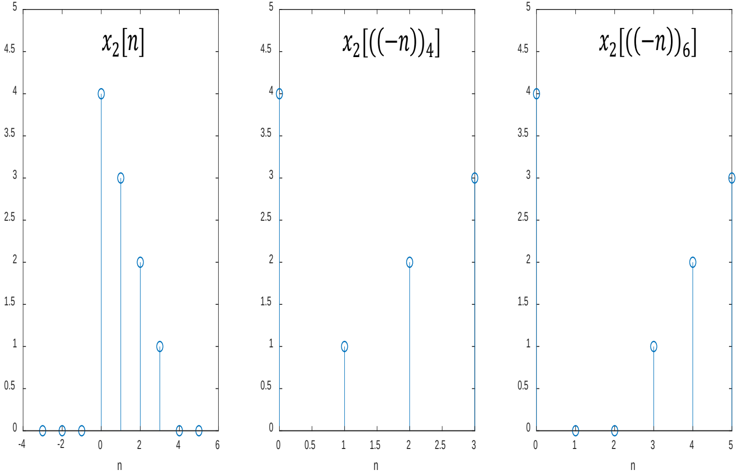
$$\begin{aligned}
X[k] &= \mathcal{DFT}(x[n]) = \mathcal{DFT}\left(\sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]\right) \\
&\overset{Linearity}{=} \sum_{m=0}^{N-1} x_1[m]\mathcal{DFT}(x_2[((n-m))_N]) \overset{Circular\ Shift}{=} \sum_{m=0}^{N-1} x_1[m]W_N^{mk}X_2[k] \\
&= \left(\sum_{m=0}^{N-1} x_1[m]W_N^{mk}\right) X_2[k] = X_1[k]X_2[k]
\end{aligned}$$

# Circular Convolution $x[n] = \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$: Example

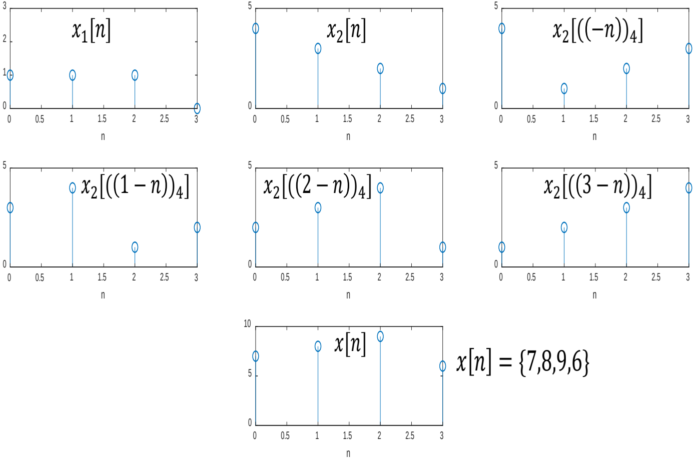



Linear Convolution: $x[n] = \{4, 7, 9, 6, 3, 1, 0, 0..\}$ with $x[n] = 0, n < 0$.

# $x_2[((-n))_N]$ Operation



# Circular Convolution: Example with $N = 4$




$x[n] = \{7,8,9,6\}$

# Circular Convolution: Example with $N = 8$