ECE 275A: Parameter Estimation I

The Kalman Filter

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Recap: State-Space Model

- Consider a sequence of states $\boldsymbol{x}_n$ and a sequence of measurements $\boldsymbol{y}_n$

State-Transition Model:

State $\boldsymbol{x}_n$ evolves according to

$$\boldsymbol{x}_n = g_n(\boldsymbol{x}_{n-1}) + \underbrace{\boldsymbol{u}_n}_{\text{driving noise (white)}}$$

This determines the joint prior

$$f(\boldsymbol{x}_{0:n}) = f(\boldsymbol{x}_0) \prod_{n'=1}^{n} f(\boldsymbol{x}_{n'}|\boldsymbol{x}_{n'-1})$$

Measurement Model:

Measurement $\boldsymbol{y}_n$ is generated as

$$\boldsymbol{y}_n = h_n(\boldsymbol{x}_n) + \underbrace{\boldsymbol{v}_n}_{\text{measurement noise (white)}}$$

This determines the joint likelihood

$$f(\boldsymbol{y}_{1:n}|\boldsymbol{x}_{1:n}) = \prod_{n'=1}^{n} f(\boldsymbol{y}_{n'}|\boldsymbol{x}_{n'})$$

- By using Bayes' rule we obtain the marginal posterior pdf (for $\boldsymbol{y}_{1:n}$ fixed)

$$f(\boldsymbol{x}_{0:n}|\boldsymbol{y}_{1:n}) \propto f(\boldsymbol{x}_{0:n}) f(\boldsymbol{y}_{1:n}|\boldsymbol{x}_{1:n}) = f(\boldsymbol{x}_0) \prod_{n'=1}^{n} f(\boldsymbol{x}_{n'}|\boldsymbol{x}_{n'-1}) f(\boldsymbol{y}_{n'}|\boldsymbol{x}_{n'})$$

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Linear-Gaussian State-Space Model

- Consider a sequence of states $\boldsymbol{x}_n$ and a sequence of measurements $\boldsymbol{y}_n$

State-Transition Model:

State $\boldsymbol{x}_n$ evolves according to

$$\boldsymbol{x}_n = \boldsymbol{G}_n \boldsymbol{x}_{n-1} + \underbrace{\boldsymbol{u}_n}_{\text{driving noise (white)}}$$

with Gaussian driving noise

$$\boldsymbol{u}_n \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}_{\boldsymbol{u}_n})$$

Measurement Model:

Measurement $\boldsymbol{y}_n$ is generated as

$$\boldsymbol{y}_n = \boldsymbol{H}_n \boldsymbol{x}_n + \underbrace{\boldsymbol{v}_n}_{\text{measurement noise (white)}}$$

with Gaussian measurement noise

$$\boldsymbol{v}_n \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}_{\boldsymbol{v}_n})$$

- Prior PDF at $n = 0$, $\boldsymbol{x}_0 \sim \mathcal{N}(\boldsymbol{\mu}_{\boldsymbol{x}_0}, \boldsymbol{\Sigma}_{\boldsymbol{x}_0})$

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Kalman Prediction Step

- Recall prediction step of sequential Bayesian estimation

$$\underbrace{f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1})}_{\text{Predicted posterior pdf}} = \int \underbrace{f(\boldsymbol{x}_n|\boldsymbol{x}_{n-1})}_{\text{State-transition pdf}} \underbrace{f(\boldsymbol{x}_{n-1}|\boldsymbol{y}_{1:n-1})}_{\text{Previous posterior pdf}} \mathrm{d}\boldsymbol{x}_{n-1}$$

- If $f(\boldsymbol{x}_{n-1}|\boldsymbol{y}_{1:n-1}) = \mathcal{N}(\boldsymbol{\mu}_{\boldsymbol{x}_{n-1}}, \boldsymbol{\Sigma}_{\boldsymbol{x}_{n-1}})$ and $f(\boldsymbol{x}_n|\boldsymbol{x}_{n-1}) = \mathcal{N}(\mathbf{G}_n \boldsymbol{x}_{n-1}; \boldsymbol{\Sigma}_{\boldsymbol{u}_n})$

- Then $f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1})$ is Gaussian with mean $\boldsymbol{\mu}^-_{\boldsymbol{x}_n}$ and covariance $\boldsymbol{\Sigma}^-_{\boldsymbol{x}_n}$ given as

$$\boldsymbol{\mu}^-_{\boldsymbol{x}_n} = \boldsymbol{G}_n \boldsymbol{\mu}_{\boldsymbol{x}_{n-1}} \qquad \boldsymbol{\Sigma}^-_{\boldsymbol{x}_n} = \boldsymbol{G}_n \boldsymbol{\Sigma}_{\boldsymbol{x}_{n-1}} \boldsymbol{G}_n^{\mathrm{T}} + \boldsymbol{\Sigma}_{\boldsymbol{u}_n}$$

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Kalman Update Step

- Recall measurement update step of sequential Bayesian estimation

$$\underbrace{f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n})}_{\text{Posterior pdf}} \propto \underbrace{f(\boldsymbol{y}_n|\boldsymbol{x}_n)}_{\substack{\text{Likelihood}\\ \text{function}}} \underbrace{f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1})}_{\substack{\text{Predicted}\\ \text{posterior pdf}}}$$

- If $f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1}) = \mathcal{N}(\boldsymbol{\mu}^-_{\boldsymbol{x}_n}, \boldsymbol{\Sigma}^-_{\boldsymbol{x}_n})$ and $f(\boldsymbol{z}_n|\boldsymbol{x}_n) = \mathcal{N}(\mathbf{H}_n\boldsymbol{x}_n; \boldsymbol{\Sigma}_{\boldsymbol{v}_n})$

- Then $f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n})$ is Gaussian with mean $\boldsymbol{\mu}_{\boldsymbol{x}_n}$ and covariance $\boldsymbol{\Sigma}_{\boldsymbol{x}_n}$ (both can be calculated in closed form)

Kalman Update Step

- Kalman gain

$$\boldsymbol{K}_n = \boldsymbol{\Sigma}^-_{\boldsymbol{x}_n}\boldsymbol{H}_n^{\mathrm{T}}\big(\boldsymbol{H}_n\boldsymbol{\Sigma}^-_{\boldsymbol{x}_n}\boldsymbol{H}_n^{\mathrm{T}} + \boldsymbol{\Sigma}_{\boldsymbol{v}_n}\big)^{-1}$$

- Mean and covariance update

$$\begin{aligned}\boldsymbol{\mu}_{\boldsymbol{x}_n} &= \boldsymbol{\mu}^-_{\boldsymbol{x}_n} + \boldsymbol{K}_n\big(\boldsymbol{y}_n - \boldsymbol{H}_n\boldsymbol{\mu}^-_{\boldsymbol{x}_n}\big)\\ \boldsymbol{\Sigma}_{\boldsymbol{x}_n} &= \boldsymbol{\Sigma}^-_{\boldsymbol{x}_n} - \boldsymbol{K}_n\boldsymbol{H}_n\boldsymbol{\Sigma}^-_{\boldsymbol{x}_n}\end{aligned}$$

Kalman Filter Properties (i)

- Kalman filter provides a procedure for calculating the entire posterior PDF $f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n})$ of linear-Gaussian sequential Bayesian estimation problems

- The covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{x}_n}$ does not depend on the measurements $\boldsymbol{y}_{1:n}$ and can thus be calculated offline

- Since for Gaussian distributions the mean is equal to the maximum (or mode), $\boldsymbol{\mu}_{\boldsymbol{x}_n}$ is the optimum MMSE estimate and MAP estimate, i.e.,

$$\hat{\boldsymbol{x}}_n^{\mathrm{MMSE}} = \hat{\boldsymbol{x}}_n^{\mathrm{MAP}} = \boldsymbol{\mu}_{\boldsymbol{x}_n}$$

and $\boldsymbol{\Sigma}_{\boldsymbol{x}_n}$ is the error covariance matrix of the estimate

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Kalman Filter Properties (ii)

- In case $f(\boldsymbol{u}_{n'})$ and $f(\boldsymbol{v}_{n'}), n' = 0, 1, \ldots, n$ or $f(\boldsymbol{x}_0)$ are not Gaussian distributions, it can be shown that the Kalman filter is the best linear MMSE estimator for $\boldsymbol{x}_n$

- The complexity of the Kalman filter is quite moderate, for I the dimension of the state space and L the dimension of the measurement space
  - it scales as $\mathcal{O}(I^2)$ due matrix multiplication in $\boldsymbol{\Sigma}_{\boldsymbol{x}_n} = (\boldsymbol{I} - \boldsymbol{K}_n\boldsymbol{H}_n)\boldsymbol{\Sigma}_{\boldsymbol{x}_n}^{-}$
  - it scales as $\mathcal{O}(L^{2.4})$ due matrix inversion in $\boldsymbol{K}_n = \boldsymbol{\Sigma}_{\boldsymbol{x}_n}^{-}\boldsymbol{H}_n^{\mathrm{T}}\left(\boldsymbol{\Sigma}_{\boldsymbol{z}_n}\right)^{-1}$

J. Anderson and B. Moore, *Optimal Filtering,* Prentice-Hall, 1979.

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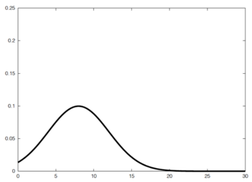
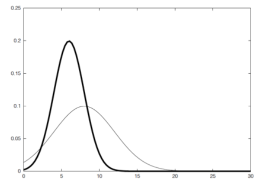
Kalman Filter Illustration



$f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1})$



$f(\boldsymbol{y}_n|\boldsymbol{x}_n)$

S. Thrun, W. Burgard, and D. Fox, *Probabilistic Robotics*, MIT Press, 2006.

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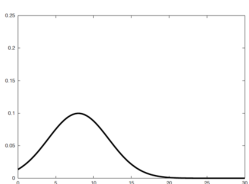
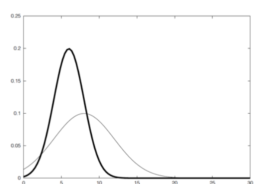
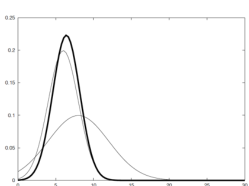
Kalman Filter Illustration



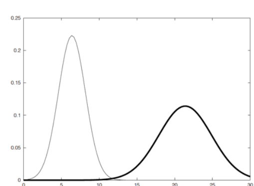
$f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1})$



$f(\boldsymbol{y}_n|\boldsymbol{x}_n)$



$f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n})$



$f(\boldsymbol{x}_{n+1}|\boldsymbol{y}_{1:n})$

S. Thrun, W. Burgard, and D. Fox, *Probabilistic Robotics*, MIT Press, 2006.

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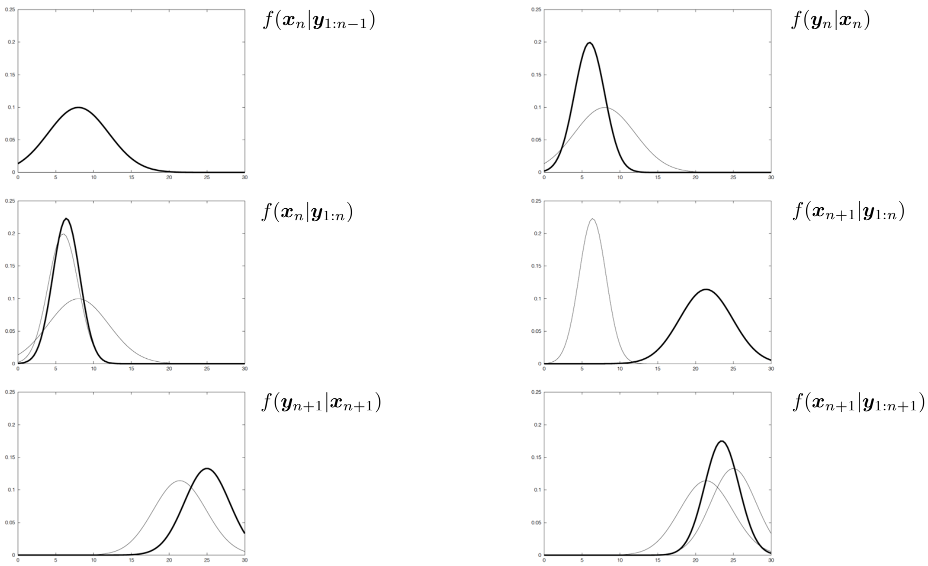
Kalman Filter Illustration



S. Thrun, W. Burgard, and D. Fox, *Probabilistic Robotics*, MIT Press, 2006.

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