ECE 161A: The Discrete Fourier Transform (DFT)

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Discrete Time Fourier Transform of Periodic Sequences

$$\tilde{x}[n] \overset{?}{\longleftrightarrow} \tilde{X}(e^{j\omega})$$

We will use the following observation about exponential sequences

$$e^{j\omega_0 n} \longleftrightarrow 2\pi \sum_{r=-\infty}^{\infty} \delta(\omega - \omega_0 + 2\pi r)$$

and so

$$W_N^{-nk} = e^{j\frac{2\pi}{N}kn} \longleftrightarrow 2\pi \sum_{r=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{N}k + 2\pi r)$$

Hence

$$\begin{aligned}
\tilde{x}[n] &= \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] e^{j\frac{2\pi}{N}kn} \overset{linearity}{\longleftrightarrow} \frac{2\pi}{N}\sum_{k=0}^{N-1}\tilde{X}[k] \sum_{r=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{N}k + 2\pi r) \\
&= \frac{2\pi}{N}\sum_{k=-\infty}^{\infty} \tilde{X}[k]\delta(\omega - \frac{2\pi}{N}k)
\end{aligned}$$

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j\frac{2\pi}{N}kn} \overset{linearity}{\longleftrightarrow} = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\tilde{X}[k]\delta(\omega - \frac{2\pi}{N}k)$$

DTFT of periodic sequences

$$\tilde{x}[n] \longleftrightarrow \tilde{X}[k] \longleftrightarrow \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\tilde{X}[k]\delta(\omega - \frac{2\pi}{N}k)$$

A DTFT which has delta functions uniformly spaced implies a periodic time domain sequence.

$$\frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\tilde{X}[k]\delta(\omega - \frac{2\pi}{N}k) \longleftrightarrow \tilde{X}[k] \longleftrightarrow \tilde{x}[n]$$

DFS Summary

With $W_N = e^{-j\frac{2\pi}{N}}$

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-nk} \quad \text{(Synthesis Equation)}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1}\tilde{x}[n]W_N^{nk} \quad \text{(Analysis Equation)}$$

Fourier Transform (DTFT of $\tilde{x}[n]$)

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j\frac{2\pi}{N}kn} \overset{linearity}{\longleftrightarrow} = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\tilde{X}[k]\delta(\omega - \frac{2\pi}{N}k)$$

Procedure for computing DTFT of periodic sequences

$$\tilde{x}[n] \longleftrightarrow \tilde{X}[k] \longleftrightarrow \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\tilde{X}[k]\delta(\omega - \frac{2\pi}{N}k)$$

A DTFT which has delta functions uniformly spaced implies a periodic time domain sequence.

$$\frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\tilde{X}[k]\delta(\omega - \frac{2\pi}{N}k) \longleftrightarrow \tilde{X}[k] \longleftrightarrow \tilde{x}[n]$$

Sampling the DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Sample using a train of impulses in the frequency domain

$$P(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{N}k)$$

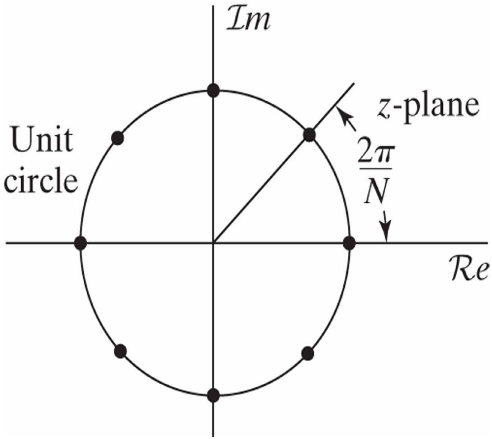
N controls the spacing $\frac{2\pi}{N}$ of the frequency samples.The sampled DTFT is given by

$$\begin{aligned} X_s(e^{j\omega}) &= X(e^{j\omega})P(e^{j\omega}) = X(e^{j\omega})\frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{N}k) \\ &= \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} X(e^{j\frac{2\pi k}{N}})\delta(\omega - \frac{2\pi}{N}k) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \tilde{X}[k]\delta(\omega - \frac{2\pi}{N}k) \end{aligned}$$

where $\tilde{X}[k] = X(e^{j\frac{2\pi k}{N}})$, the samples of the DTFT of $x[n]$ on the unit circle at a spacing of $\frac{2\pi}{N}$.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \xrightarrow{N} \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \tilde{X}[k]\delta(\omega - \frac{2\pi}{N}k) \leftrightarrow \left(N, \tilde{X}[k] = X(e^{j\frac{2\pi k}{N}}), k = 0, 1, \ldots, N-1\right)$$

Figure 8.7 Points on the unit circle at which *X(z)* is sampled to obtain the periodic sequence $\tilde{X}[k]$ *(N* = 8*)*.



Frequency Sampling

$$
\begin{aligned}
x[n] \quad &\leftrightarrow \quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \xrightarrow{N} X_s(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \tilde{X}[k]\delta(\omega - \frac{2\pi}{N}k) \\
&\leftrightarrow \quad \left(N, \tilde{X}[k] = X(e^{j\frac{2\pi k}{N}}), k = 0, 1, \ldots, N-1\right) \leftrightarrow \tilde{x}[n]
\end{aligned}
$$

Important questions

1. What is the relationship $\tilde{x}[n]$ and $x[n]$?
2. Can we recover $x[n]$ from $\tilde{x}[n]$?

Significance: If we can answer yes to the second question, then $\tilde{X}[k]$ has all the information needed to determine $X(e^{j\omega})$.!

Frequency Sampling: Pictorial Depiction

Suppose $x[n]$ is of duration L, i.e. $x[n]$ is nonzero for $n = 0, 1, \ldots, L-1$.

The frequency sampling pictorially is shown below.

$$x[n] \leftrightarrow X(e^{j\omega}) = \sum_{n=0}^{L-1} x[n] e^{-j\omega n} \underset{?}{\overset{N}{\rightleftarrows}} X_s(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \tilde{X}[k] \delta\left(\omega - \frac{2\pi}{N} k\right)$$

Duration L

Only N≥L

$$\tilde{X}[k] = X(e^{j\omega})\big|_{\omega = \frac{2\pi}{N}k, k=0,1,..,N-1}$$

ok

?

Only N≥L

DFS N

$$\tilde{x}[n], n = 0,1,..,N-1$$

Frequency Sampling Theorem (FST)

$$x[n] \leftrightarrow X(e^{j\omega}) \xrightarrow{N} X_s(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \tilde{X}[k]\delta(\omega - \frac{2\pi}{N}k) \leftrightarrow \tilde{X}[k] = X(e^{j\frac{2\pi k}{N}}) \leftrightarrow \tilde{x}[n]$$

Theorem:

1. $\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n + rN]$ (time domain aliasing)
2. If $x[n]$ is of finite duration L, then for $N \geq L$ $x[n]$ can be recovered from $\tilde{x}[n]$ and
$$x[n] = \tilde{x}[n]w[n]$$
where $w[n]$ is rectangular window
$$w[n] = \begin{cases} 1, & 0 \leq k \leq N - 1 \\ 0, & \text{otherwise} \end{cases}$$

Recovery of $x[n]$ from $\tilde{x}[n]$

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n+rN] = \ldots + x[n-N] + x[n] + x[n+N] + \ldots$$

If $x[n]$ is of duration L, then $x[n]$ is nonzero between $0 \le n \le L-1$.
$x[n-N]$ is nonzero between $N \le n \le N+L-1$.
$x[n+N]$ is nonzero between $-N \le n \le -N+L-1$.

No overlap and hence no aliasing for $N \ge L$.

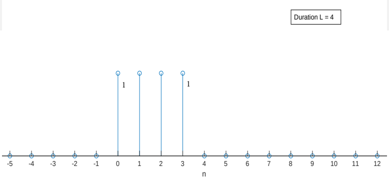
$x[n]$ can be obtained by keeping samples of $\tilde{x}[n], 0 \le n \le N-1$.
Mathematically

$$x[n] = \tilde{x}[n]w[n],$$

where $w[n]$ is rectangular window

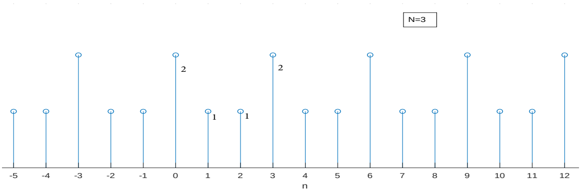
$$w[n] = \begin{cases} 1, & 0 \le k \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Example

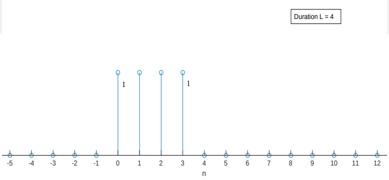


Three samples ($N = 3$)in the frequency domain.

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - r3] = \begin{cases} x[0] + x[3], & n = .., -3, 0, 3, .. \\ x[1], & n = .., -2, 1, 4, .. \\ x[2], & n =, , . -1, 2, 5, .. \end{cases}$$



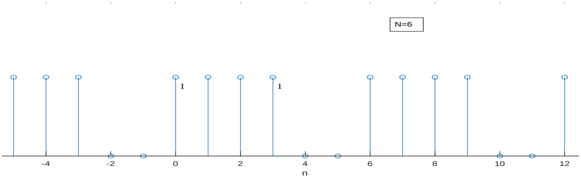
Example Cont'd



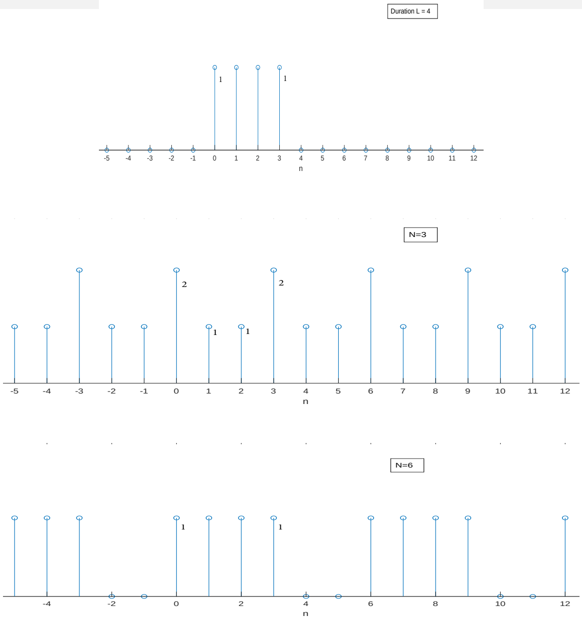
Six samples $(N = 6)$in the frequency domain.

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - r6]$$

A period of $\tilde{x}[n]$ is $x[n]$ padded with 2 $(6 - 4)$ zeros.



Example: Summary



Discrete Time Fourier Transform (DFT)

Define

$$X[k] = \begin{cases} \tilde{X}[k] = X(e^{j\frac{2\pi}{N}k}), & k = 0, 1, \ldots, N-1 \\ 0, & \text{otherwise.} \end{cases}$$

We can now rewrite the equations involving only $x[n]$ and $X[k]$.

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} x[n] W_N^{nk}, k = 0, 1, \ldots, N-1 \quad \text{(DFT)} \\ x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn} = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-nk}, n = 0, 1, \ldots, N-1 \quad \text{(IDFT)} \end{aligned}$$

DFS versus DFT

DFS:

$$\begin{aligned}\tilde{X}[k] &= \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{nk} \\ \tilde{x}[n] &= \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j\frac{2\pi}{N}kn} = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-nk}\end{aligned}$$

DFT:

$$\begin{aligned}X[k] &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} x[n] W_N^{nk}, \ k = 0, 1, \ldots, N-1 \ \text{(DFT)} \\ x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn} = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-nk}, \ n = 0, 1, \ldots, N-1 \ \text{(IDFT)}\end{aligned}$$

1. DFS applies to periodic signals and DFT to finite duration signals.
2. In DFS, N is the periodicity and in DFT $N \geq L$, where L is the duration of the sequence.