ECE 286 Class 16: Graph-Based Cooperative Localization II

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Message Representation

- Direct implementation of the SPA algorithm (message and belief calculation rules) is still computationally infeasible
- Two alternative feasible approximations:
  - using a parametric representation for the messages and beliefs
    $\Rightarrow$ Gaussian SPA, ...
  - using a particle representation for the messages and beliefs
    $\Rightarrow$ nonparametric SPA, ...
- For simplicity with start with a static scenario

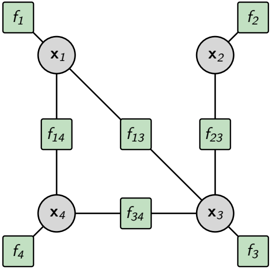
Factor Graph

- Consider K agents with state vectors $\mathbf{x}_k$, $k \in \{1, \ldots, K\}$
- Recall that the posterior pdf $f(\mathbf{x}|\mathbf{y})$ factorizes as

$$f(\mathbf{x}|\mathbf{y}) \propto \left[\prod_{k=1}^{K} f(\mathbf{x}_k)\right] \prod_{(k',l)\in\mathcal{E}} f(\mathbf{y}_{k'l}|\mathbf{x}_{k'}, \mathbf{x}_l)$$

where $\mathbf{y}_{kl} = H(\mathbf{x}_k, \mathbf{x}_l) + \mathbf{v}_{kl}$ are noisy "pairwise" observations

- Representation by factor graph:



$k \in \{1, 2, 3, 4\}$

$\mathcal{E} = \{(1,3), (1,4), (2,3), (3,4)\}$

$f_k \triangleq f(\mathbf{x}_k)$

$f_{kl} \triangleq f(\mathbf{y}_{kl}|\mathbf{x}_k, \mathbf{x}_l)$

Recall SPA for Cooperative Localization

- Recall that $\mathcal{N}_k$ denotes the "neighbor" set of agent $k \in \{1, \ldots, K\}$, which comprises all agents $l \in \{1, \ldots, K\} \backslash \{k\}$ such that $(k, l) \in \mathcal{E}$
- Belief of agent state $\mathbf{x}_k$:

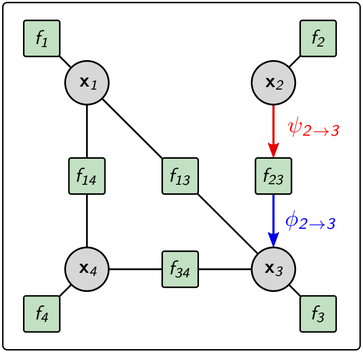
$$b(\mathbf{x}_k) \propto f(\mathbf{x}_k) \prod_{l \in \mathcal{N}_k} \phi_{l \to k}(\mathbf{x}_k)$$

- Extrinsic information:

$$\psi_{k \to l}(\mathbf{x}_k) = f(\mathbf{x}_k) \prod_{l' \in \mathcal{N}_k \backslash \{l\}} \phi_{l' \to k}(\mathbf{x}_k)$$

- "Measurement message" (from function node $f(\mathbf{y}_{kl} | \mathbf{x}_k, \mathbf{x}_l)$ to variable node $\mathbf{x}_k$):

$$\phi_{l \to k}(\mathbf{x}_k) = \int f(\mathbf{y}_{kl} | \mathbf{x}_k, \mathbf{x}_l) \, \psi_{l \to k}(\mathbf{x}_l) \, d\mathbf{x}_l$$



Nonparametric Belief Propagation for Coop. Localization

- Nonparametric belief propagation is an extension of the particle filter to a general factorization structure of the joint posterior distribution
- Let us assume we have a particle-representation $\{\mathbf{x}_k^{(j)}\}_{j=1}^J$ for the prior distribution $f(\mathbf{x}_k)$ of each agent $k = 1, \dots, K$ is available
- A parallel processing of messages is used, where at each iteration $\ell = 1, \dots, L$, each agents calculates extrinsic information based on all currently available messages

Nonparametric Belief Propagation for Coop. Localization

- At iteration $\ell = 1$, we have $\psi^{(1)}_{k\to l}(\mathbf{x}_k) = f(\mathbf{x}_k)$ for all $k = 1, \dots, K$

- We can thus express the "measurement messages" at $\ell = 1$ as follows

$$\phi^{(1)}_{l\to k}(\mathbf{x}_k) = \int f(\mathbf{y}_{kl}|\mathbf{x}_k, \mathbf{x}_l)\, \psi^{(1)}_{l\to k}(\mathbf{x}_l)\, d\mathbf{x}_l$$
$$= \int f(\mathbf{y}_{kl}|\mathbf{x}_k, \mathbf{x}_l)\, f(\mathbf{x}_l)\, d\mathbf{x}_l$$

- A particle-based approximation of the "measurement message" can now be calculated by means of Monte Carlo integration and using the particles $\{\mathbf{x}_l^{(j)}\}_{j=1}^{J} \sim f(\mathbf{x}_l)$, i.e.,

$$\tilde{\phi}^{(1)}_{l\to k}(\mathbf{x}_k) = \frac{1}{J}\sum_{j=1}^{J} f\big(\mathbf{y}_{kl}|\mathbf{x}_k, \mathbf{x}_l^{(j)}\big)$$

Nonparametric Belief Propagation for Coop. Localization

- At iterations $\ell > 1$, an approximation of $\psi^{(\ell)}_{k\to l}(\mathbf{x}_k)$ is given by

$$\tilde{\psi}^{(\ell)}_{k\to l}(\mathbf{x}_k) = f(\mathbf{x}_k) \prod_{l' \in \mathcal{N}_k \setminus \{l\}} \tilde{\phi}^{(\ell-1)}_{l'\to k}(\mathbf{x}_k)$$

- A particle representation $\big\{\big(w^{(\ell,j)}_{k\to l}, \mathbf{x}^{(j)}_k\big)\big\}_{j=1}^{J}$ of $\tilde{\psi}^{(\ell)}_{k\to l}(\mathbf{x}_k)$ is obtained using importance sampling with proposal distribution $f_{\mathrm{p}}(\mathbf{x}_k) = f(\mathbf{x}_k)$ and target distribution $f_{\mathrm{t}}(\mathbf{x}_k) \propto f(\mathbf{x}_k) \prod_{l' \in \mathcal{N}_k \setminus \{l\}} \tilde{\phi}^{(\ell-1)}_{l'\to k}(\mathbf{x}_k)$ by
  - calculating unnormalized weights $\tilde{w}^{(\ell,j)}_{k\to l} = \prod_{l' \in \mathcal{N}_k \setminus \{l\}} \tilde{\phi}^{(\ell-1)}_{l'\to k}(\mathbf{x}^{(j)}_k)$
  - normalizing weights $w^{(\ell,j)}_{k\to l} = \tilde{w}^{(\ell,j)}_{k\to l} \big/ \sum_{j'=1}^{J} \tilde{w}^{(\ell,j')}_{k\to l}$

- Similarly, after $\ell = L$ iterations, a particle representation $\big\{\big(w^{(j)}_k, \mathbf{x}^{(j)}_k\big)\big\}_{j=1}^{J}$ of the belief $b(\mathbf{x}_k) \propto f(\mathbf{x}_k) \prod_{l \in \mathcal{N}_k} \phi^{(L)}_{l\to k}(\mathbf{x}_k)$ can also be calculated by means of importance sampling

Nonparametric Belief Propagation for Coop. Localization

- Resampling may be performed after each calculation of particle representations $\left\{\left(w_{k\to l}^{(\ell,j)}, \mathbf{x}_k^{(j)}\right)\right\}_{j=1}^{J}$ of messages $\tilde{\psi}_{k\to l}^{(\ell)}(\mathbf{x}_k)$

- Note that if no resampling is performed, a particle-based approximation of the "measurement message" is calculated from particles $\left\{\left(w_{l\to k}^{(\ell,j)}, \mathbf{x}_l^{(j)}\right)\right\}_{j=1}^{J}$ of $\tilde{\psi}_{l\to k}^{(\ell)}(\mathbf{x}_l)$, as

$$\tilde{\phi}_{l\to k}^{(\ell)}(\mathbf{x}_k) = \sum_{j=1}^{J} f\left(\mathbf{y}_{kl} | \mathbf{x}_k, \mathbf{x}_l^{(j)}\right) w_{l\to k}^{(\ell,j)}$$

- When particle representations $\left\{\left(w_k^{(j)}, \mathbf{x}_k^{(j)}\right)\right\}_{j=1}^{J}$ for beliefs $b(\mathbf{x}_k)$, $k = 1, \dots, K$ are available, estimates of the agents states $\mathbf{x}_k$ can be calculated as

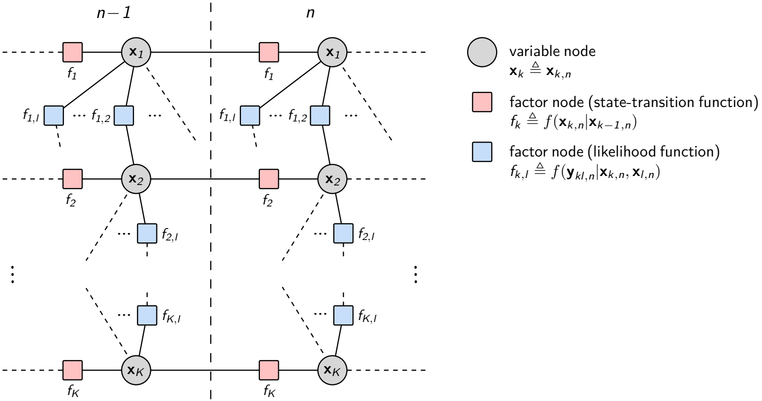
$$\hat{\mathbf{x}}_k = \sum_{j=1}^{J} \mathbf{x}_k^{(j)} w_k^{(j)}$$

The Factor Graph

- Recall factorization of dynamic cooperative localization problem:

$$f(\mathbf{x}_{0:n}|\mathbf{y}_{1:n}) \propto \Big(\prod_{l=1}^{K} f(\mathbf{x}_{l,0})\Big) \prod_{n'=1}^{n} \Big(\prod_{k=1}^{K} f(\mathbf{x}_{k,n'}|\mathbf{x}_{k,n'-1})\Big) \prod_{(k',l')\in\mathcal{E}_{n'}} f(\mathbf{y}_{k'l',n'}|\mathbf{x}_{k',n'},\mathbf{x}_{l',n'})$$

- Representation by factor graph:



Message Passing

Dynamic SPA algorithm:

- "Prediction" message:

$$\phi_{\rightarrow n}(\mathbf{x}_{k,n}) = \int f(\mathbf{x}_{k,n}|\mathbf{x}_{k,n-1})\, b(\mathbf{x}_{k,n-1})\, \mathrm{d}\mathbf{x}_{k,n-1}$$

  Since we send messages only forward in time, we directly use the belief $b(\mathbf{x}_{k,n-1})$ instead of some extrinsic information

- "Measurement" message:

$$\phi_{l\rightarrow k}(\mathbf{x}_{k,n}) = \int f(\mathbf{y}_{kl,n}|\mathbf{x}_{l,n}, \mathbf{x}_{k,n})\, \psi_{l\rightarrow k}(\mathbf{x}_{l,n})\, \mathrm{d}\mathbf{x}_{l,n}$$

- Extrinsic information:

$$\psi_{l\rightarrow k}(\mathbf{x}_{l,n}) = \phi_{\rightarrow n}(\mathbf{x}_{l,n}) \prod_{k'\in\mathcal{N}_{l,n}\setminus\{k\}} \phi_{k'\rightarrow l}(\mathbf{x}_{l,n})$$

- Belief:

$$b(\mathbf{x}_{k,n}) \propto \phi_{\rightarrow n}(\mathbf{x}_{k,n}) \prod_{l\in\mathcal{N}_{k,n}} \phi_{l\rightarrow k}(\mathbf{x}_{k,n})$$

Nonparametric Belief Propagation for Coop. Localization

- In a dynamic scenario with time steps $n = 1, 2, \dots$
  - L iterations are performed for each time step n individually
  - the prediction $\phi_{\rightarrow n}(\mathbf{x}_{k,n}) = \int f(\mathbf{x}_{k,n}|\mathbf{x}_{k,n-1})\, b(\mathbf{x}_{k,n-1})\, \mathrm{d}\mathbf{x}_{k,n-1}$ is used instead of the prior distribution at each step n (see class 15, slide 9)

- For each agent $k = 1, \dots, K$, particles representing $\phi_{\rightarrow n}(\mathbf{x}_{k,n})$ can be calculate from particles representing $b(\mathbf{x}_{k,n-1})$ by means of a conventional particle-based prediction step (see class 4)

- Computational complexity related to the calculation of agent belief $b(\mathbf{x}_{k,n})$ scales as $\mathcal{O}(J^2 L |\mathcal{N}_{k,n}|)$, $k = 1, \dots, K$

Sigma Point Belief Propagation

- Sigma point belief propagation is an extension of the unscented Kalman filter to a general factorization structure of the joint posterior distribution
- Extrinsic information and beliefs are represented by Gaussian distributions
- By reformulating the SPA messages in a higher-dimensional state space, the update step of the unscented Kalman filter can be used to calculate mean and covariance of extrinsic information and beliefs
- For simplicity we start again with a static scenario

Reformulation of SPA Messages

- Recall SPA messages:

$$b(\mathbf{x}_k) \propto f(\mathbf{x}_k) \prod_{l \in \mathcal{N}_k} \phi_{l \to k}(\mathbf{x}_k)$$

$$\psi_{k \to l}(\mathbf{x}_k) = f(\mathbf{x}_k) \prod_{l' \in \mathcal{N}_k \backslash \{l\}} \phi_{l' \to k}(\mathbf{x}_k)$$

$$\phi_{l \to k}(\mathbf{x}_k) = \int f(\mathbf{y}_{kl} | \mathbf{x}_k, \mathbf{x}_l) \psi_{l \to k}(\mathbf{x}_l) \, d\mathbf{x}_l$$

- Equivalently,

$$b(\mathbf{x}_k) \propto \int f(\mathbf{x}_k) \prod_{l \in \mathcal{N}_k} \left[f(\mathbf{y}_{kl} | \mathbf{x}_k, \mathbf{x}_l) \psi_{l \to k}(\mathbf{x}_l) d\mathbf{x}_l \right]$$

$$\psi_{k \to l}(\mathbf{x}_k) = \int f(\mathbf{x}_k) \prod_{l' \in \mathcal{N}_k \backslash \{l\}} \left[f(\mathbf{y}_{kl'} | \mathbf{x}_k, \mathbf{x}_{l'}) \psi_{l' \to k}(\mathbf{x}_{l'}) d\mathbf{x}_{l'} \right.$$

Reformulation of SPA Messages

- Let $\mathcal{N}_k = \{l_1, l_2, \ldots, l_{|\mathcal{N}_k|}\}$, and consider the "composite" vectors

$$\bar{\mathbf{x}}_k \triangleq \left(\mathbf{x}_k^{\mathrm{T}}\ \mathbf{x}_{l_1}^{\mathrm{T}}\ \mathbf{x}_{l_2}^{\mathrm{T}} \cdots \mathbf{x}_{l_{|\mathcal{N}_k|}}^{\mathrm{T}}\right)^{\mathrm{T}} \qquad (\mathbf{x}_k \text{ and its neighbor states})$$

$$\bar{\mathbf{y}}_k \triangleq \left(\mathbf{y}_{kl_1}^{\mathrm{T}}\ \mathbf{y}_{kl_2}^{\mathrm{T}} \cdots \mathbf{y}_{kl_{|\mathcal{N}_k|}}^{\mathrm{T}}\right)^{\mathrm{T}} \quad (\text{all observations involving } \mathbf{x}_k)$$

- We can now write

$$b(\mathbf{x}_k) \propto \int f(\mathbf{x}_k) \prod_{l \in \mathcal{N}_k} \left[f(\mathbf{y}_{kl} | \mathbf{x}_k, \mathbf{x}_l)\, \psi_{l \to k}(\mathbf{x}_l)\, d\mathbf{x}_l \right]$$
$$\propto \int f(\bar{\mathbf{y}}_k | \bar{\mathbf{x}}_k)\, f(\bar{\mathbf{x}}_k)\, d\bar{\mathbf{x}}_{\sim k},$$

with

$$f(\bar{\mathbf{x}}_k) \propto f(\mathbf{x}_k) \prod_{l \in \mathcal{N}_k} \psi_{l \to k}(\mathbf{x}_l) \quad (\text{composite prior})$$

$$f(\bar{\mathbf{y}}_k | \bar{\mathbf{x}}_k) = \prod_{l \in \mathcal{N}_k} f(\mathbf{y}_{kl} | \mathbf{x}_k, \mathbf{x}_l) \qquad (\text{composite likelihood})$$

- Note that $f(\bar{\mathbf{y}}_k | \bar{\mathbf{x}}_k)$ corresponds to the composite observation model

$$\bar{\mathbf{y}}_k = \bar{\mathbf{z}}_k + \bar{\mathbf{v}}_k\,, \quad \text{with } \bar{\mathbf{z}}_k = \bar{H}(\bar{\mathbf{x}}_k)$$

where $\bar{H}(\bar{\mathbf{x}}_k) \triangleq \left(\left(H(\mathbf{x}_k, \mathbf{x}_{l_1})\right)^{\mathrm{T}} \cdots \left(H(\mathbf{x}_k, \mathbf{x}_{l_{|\mathcal{N}_k|}})\right)^{\mathrm{T}} \right)^{\mathrm{T}}$ and $\bar{\mathbf{v}}_k \triangleq \left(\mathbf{v}_{kl_1}^{\mathrm{T}} \cdots \mathbf{v}_{kl_{|\mathcal{N}_k|}}^{\mathrm{T}}\right)^{\mathrm{T}}$

Sigma Point BP

- We can finally express $b(\mathbf{x}_k)$ as the result of a marginalization:
$$b(\mathbf{x}_k) = \int b(\bar{\mathbf{x}}_k)\, d\bar{\mathbf{x}}_{\sim k}$$
with the composite belief
$$b(\bar{\mathbf{x}}_k) \propto f(\bar{\mathbf{y}}_k|\bar{\mathbf{x}}_k)\, f(\bar{\mathbf{x}}_k)$$

- This expression of $b(\bar{\mathbf{x}}_k)$ has the same form as a Bayesian update step $\Rightarrow$ Sigma points, i.e., the unscented transformation, can be used for the calculation of an approximate mean and covariance of $b(\bar{\mathbf{x}}_k)$ and, in turn, of $b(\mathbf{x}_k)$

- More specifically, $\tilde{\boldsymbol{\mu}}_{b(\bar{\mathbf{x}}_k)}$ and $\tilde{\mathbf{C}}_{b(\bar{\mathbf{x}}_k)}$ can be obtained by using the closed-form expressions of the Kalman update step, in which $\boldsymbol{\mu}_{\bar{\mathbf{z}}_k}$, $\mathbf{C}_{\bar{\mathbf{z}}_k}$, and $\mathbf{C}_{\bar{\mathbf{x}}_k\bar{\mathbf{z}}_k}$ are replaced by sigma point based approximations $\tilde{\boldsymbol{\mu}}_{\bar{\mathbf{z}}_k}$, $\tilde{\mathbf{C}}_{\bar{\mathbf{z}}_k}$, and $\tilde{\mathbf{C}}_{\bar{\mathbf{x}}_k\bar{\mathbf{z}}_k}$

Sigma Point BP

Sigma Point BP Algorithm

An approximate calculation of the mean $\boldsymbol{\mu}_{b(\mathbf{x}_k)}$ and covariance matrix $\mathbf{C}_{b(\mathbf{x}_k)}$ of $b(\mathbf{x}_k)$ based on sigma points can be obtained by performing the following two steps:

1. Use the update step based on the unscented transformation to calculate $\tilde{\boldsymbol{\mu}}_{b(\bar{\mathbf{x}}_k)} \approx \boldsymbol{\mu}_{b(\bar{\mathbf{x}}_k)}$ and $\tilde{\mathbf{C}}_{b(\bar{\mathbf{x}}_k)} \approx \mathbf{C}_{b(\bar{\mathbf{x}}_k)}$ representing $b(\bar{\mathbf{x}}_k)$ from $\boldsymbol{\mu}_{\bar{\mathbf{x}}_k}$ and $\mathbf{C}_{\bar{\mathbf{x}}_k}$ representing $f(\bar{\mathbf{x}}_k)$
2. Obtain $\tilde{\boldsymbol{\mu}}_{b(\mathbf{x}_k)}$ and $\tilde{\mathbf{C}}_{b(\mathbf{x}_k)}$ by extracting from $\tilde{\boldsymbol{\mu}}_{b(\bar{\mathbf{x}}_k)}$ and $\tilde{\mathbf{C}}_{b(\bar{\mathbf{x}}_k)}$ the elements related to $\mathbf{x}_k$; this corresponds to the marginalization

$$b(\mathbf{x}_k) = \int b(\bar{\mathbf{x}}_k)\, d\bar{\mathbf{x}}_{\sim k}$$

Sigma Point Belief Propagation

- In a dynamic scenario with time steps $n = 1, 2, \ldots$
  - L iterations are performed for each time step n individually
  - the prediction $\phi_{\rightarrow n}(\mathbf{x}_{k,n}) = \int f(\mathbf{x}_{k,n}|\mathbf{x}_{k,n-1})\, b(\mathbf{x}_{k,n-1}) \mathrm{d}\mathbf{x}_{k,n-1}$ is used instead of the prior distribution at each step n (see class 15, slide 9)

- For each agent $k = 1, \ldots, K$, the mean and covariance representing $\phi_{\rightarrow n}(\mathbf{x}_{k,n})$ can be calculate from the mean and covariance representing $b(\mathbf{x}_{k,n-1})$ by performing the (unscented) Kalman prediction step (see class 4)

- Computational complexity remains constant in number of agents in the network K

- Only means and covariance matrices have to be exchanged among neighboring agents

Summary

- Nonparametric Belief Propagation: Importance sampling and Monte Carlo integration are used to calculate particle-representations of messages and beliefs
- Sigma Point Belief Propagation: By reformulating the SPA messages in a higher-dimensional state space, the update step of the unscented Kalman filter can be used to calculate mean and covariance of beliefs

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