SIO 207A: Fundamentals of Digital Signal Processing
Class 3

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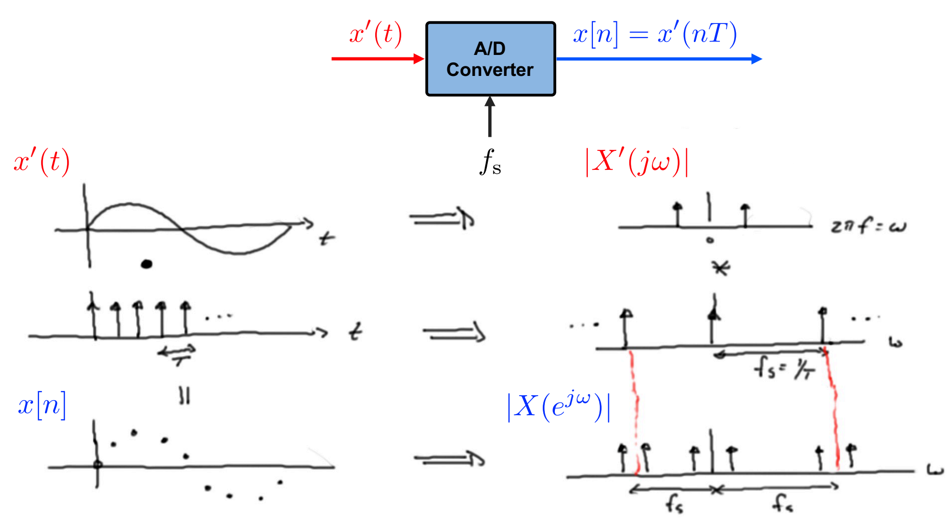
Sampling of Continuous-Time Signals



sampling $\Longrightarrow$ periodicity

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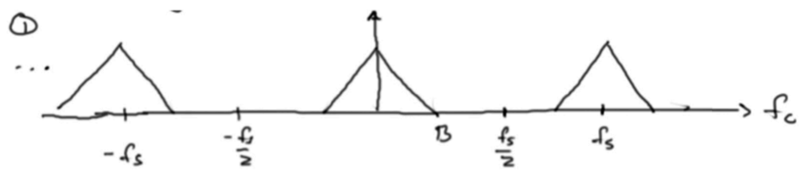
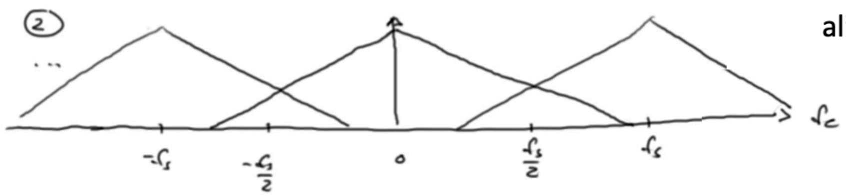
Aliasing

- Spectral representation after sampling:



no aliasing



aliasing

- In ① the original analog time-series can be recovered from its samples and in ② it cannot due to aliasing

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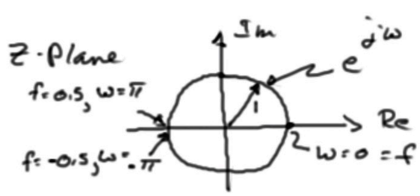
Z-Transform

- The z-transform of discrete time signal $x[n]$ is given by

$$\Gamma\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

where z is an arbitrary complex variable



- The z-transform is linear $\Gamma\{\alpha_1 x_1[n] + \alpha_2 x_2[n]\} = \alpha_1 \Gamma\{x_1[n]\} + \alpha_2 \Gamma\{x_2[n]\}$
- Convolution in time domain is equal to multiplication in z-domain $x[n] * h[n] \iff X(z)H(z)$

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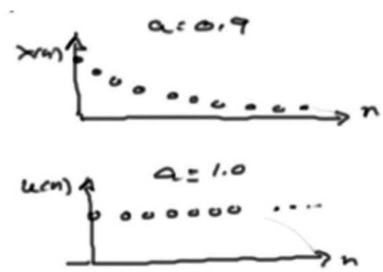
Z-Transform

- Let us again consider a geometric sequence

$$x[n] = \begin{cases} 0, & n = -1, -2, \dots \\ a^n, & n = 0, 1, 2, \dots \end{cases}$$

where a is any constant



- We can now obtain the z-transform by direct substitution

$$X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n} = \sum_{n=0}^{\infty} (a^n) z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

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Convergence

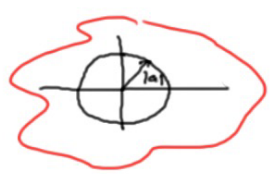
- Let us again consider a geometric sequence

$$S_k = \sum_{n=0}^{k} \alpha^n = \frac{1 - \alpha^{k+1}}{1 - \alpha}, \quad \text{for } \alpha \neq 1$$

- Next, we replace $\alpha = |\alpha| e^{j\phi}$ by az^{-1}, i.e., $X_k(z) = \dfrac{1 - (az^{-1})^{k+1}}{1 - (az^{-1})}$

- We can now calculate

$$X(z) = \lim_{k \to \infty} X_k(z)$$
$$= \begin{cases} \frac{z}{z-a} & |z| \geq |a| \\ \text{unbound} & |z| < |a| \end{cases}$$



region of convergence

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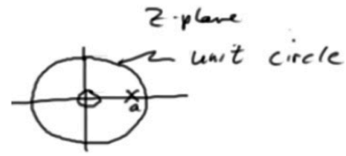
Polynomial Representation

- Let

$$X(z) = \frac{z}{z-a} \quad \text{for} \quad x[n] = a^n u[n]$$

zero at $z = 0$

pole at $z = a$



- Nearly all sequences of interest will have z-transform which can be expressed as the ratio of polynomials in z

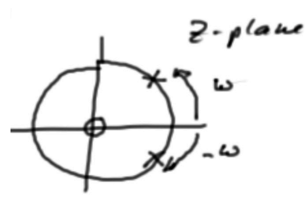
$$\begin{aligned} X(z) &= \frac{b_0 z^M + b_1 z^{M-1} + \cdots + b_M}{z^N + a_1 z^{N-1} + \cdots + a_N} \\ &= \frac{b_0 (z - z_1)(z - z_2)\ldots(z - z_M)}{(z - p_1)(z - p_2)\ldots(z - p_N)} \end{aligned}$$

Z-Transform of Sinusoidal Sequence

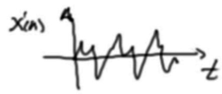
- We consider the sinusoidal sequence $x[n] = (\sin \omega n)u[n]$; note that

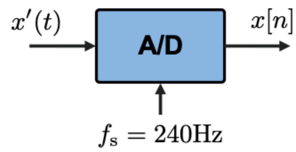
$$\sin \omega n = \left(e^{j\omega n} - e^{-j\omega n}\right)/2j$$

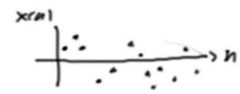
$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} \sin[\omega n] z^{-n} \\ &= \sum_{n=0}^{\infty} \left(\frac{e^{jwn} - e^{-jwn}}{2j}\right) z^{-n} \\ &= \frac{1}{2j} \sum_{n=0}^{\infty} \overset{a}{\boxed{e^{j\omega n}}} z^{-n} - \frac{1}{2j} \sum_{n=0}^{\infty} \boxed{e^{-j\omega n}} z^{-n} \\ &= \frac{1}{2j}\left(\frac{z}{z - e^{j\omega}}\right) - \frac{1}{2j}\left(\frac{z}{z - e^{-j\omega}}\right) \\ &= \frac{z \sin \omega}{(z - e^{j\omega})(z - e^{-j\omega})} \end{aligned}$$



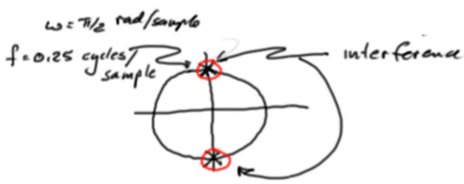
Interference Cancellation



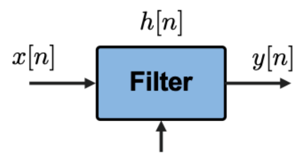




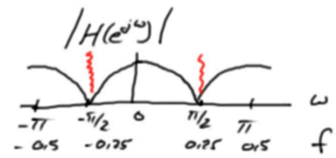
- Problem: $x'(t)$ is contaminated with $f_i = 60\text{Hz}$ interference
- For $f_s = 240\text{Hz}$, 60Hz interference is sampled 4 times per cycle, i.e., $f = 0.25 \frac{\text{cycles}}{\text{samples}}$



$$\omega_i = 2\pi \frac{f_i}{f_s} = 2\pi \frac{60}{240} = \frac{\pi}{2} \frac{\text{rad}}{\text{samples}}$$



place zeros on interference



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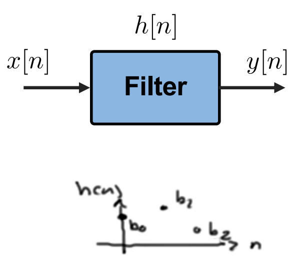
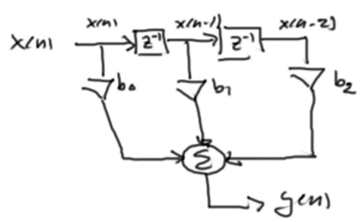
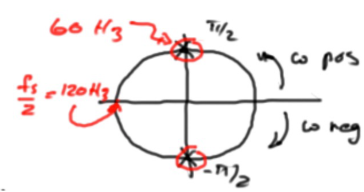
Frequency Domain Representation







O zero locations to cancel interference poles

$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

$$Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z)$$
$$= (b_o + b_1 z^{-1} + b_2 z^{-2}) X(z)$$

$$z_1, z_2 = \frac{-b_1 \pm \sqrt{b_1^2 - 4 b_0 b_2}}{2 b_0}$$

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} = \frac{b_0 z^2 + b_1 z + b_2}{z^2} = \frac{(z - z_1)(z - z_2)}{z^2}$$

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