SIO 209: Signal Processing for Ocean Sciences
Class 2

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Hilbert Transforms (HTs)

- Constraint of causality of a sequence implies a unique relationship between the real and imaginary parts of the Fourier transform
- Relationships between the real and imaginary parts of complex functions are commonly known as Hilbert transforms
- **Objective:** Investigate relationships of (i) real and imaginary parts and (ii) magnitude and phase

see also Chapter 12 in
Oppenheim & Schafer, 2009

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Hilbert Transforms (HTs)

- **Fourier Transform** $H(e^{j\omega})$ **:**
  - real part of Fourier transform implies imaginary part when $h[n]$ is causal, i.e., $h[n] = 0$ for $n < 0$
  - magnitude of Fourier transform implies phase when "complex cepstrum" $\hat{h}[n]$ is causal, i.e., $\hat{h}[n] = 0$ for $n < 0$
    - the complex cepstrum is defined as the inverse Fourier transform of

$$\hat{H}(e^{j\omega}) = \ln\left|H(e^{j\omega})\right| + j\arg H(e^{j\omega})$$

    - this property is also known as the minimum phase condition

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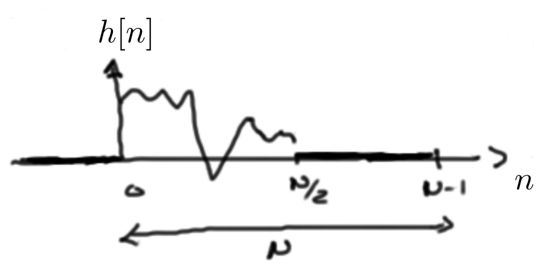
Hilbert Transforms (HTs)

$\omega = \frac{2\pi}{N}k$

k freq index

- **Discrete Fourier Transform** $H(k) = \sum_{n=0}^{N-1} h[n]e^{-j\frac{2\pi}{N}nk}$ **:**
  - real part of discrete Fourier transform implies imaginary part when the following "causality" condition is satisfied
    - $h[n]$ is periodic and $h[n] = 0, \quad N/2 < n < N-1$
    - $h[n]$ is of length N and $h[n] = 0, \quad N/2 < n < N-1$



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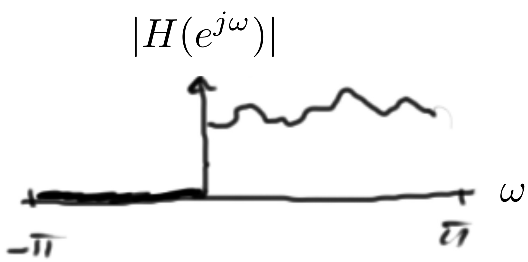
Hilbert Transforms (HTs)

- **Complex Sequence** $h[n]$:
  - the real part of the complex sequence implies the imaginary part when its Fourier transform is "causal", i.e., $-\pi \leq \omega < 0$ for $H(e^{j\omega}) = 0$



HT Relations for Fourier Transform

- Any sequence can be expressed as

$$h[n] = h_{\mathrm{e}}[n] + h_{\mathrm{o}}[n]$$

with even and odd contributions $h_{\mathrm{e}}[n] = \frac{1}{2}\big(h[n] + h[-n]\big)$ and $h_0[n] = \frac{1}{2}\big(h[n] - h[-n]\big)$

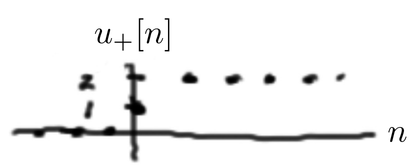
- If $h[n]$ is causal it can be recovered from $h_{\mathrm{e}}[n]$ alone, i.e.,

$$h[n] = \begin{cases} 2h_{\mathrm{e}}[n] & n > 0 \\ h_{\mathrm{e}}[n] & n = 0 \\ 0 & n < 0 \end{cases}$$

$$= h_{\mathrm{e}}[n] u_+[n]$$

Extended Step Function

$$u_+[n] = \begin{cases} 2 & n > 0 \\ 1 & n = 0 \\ 0 & n < 0 \end{cases}$$



HT Relations for Fourier Transform

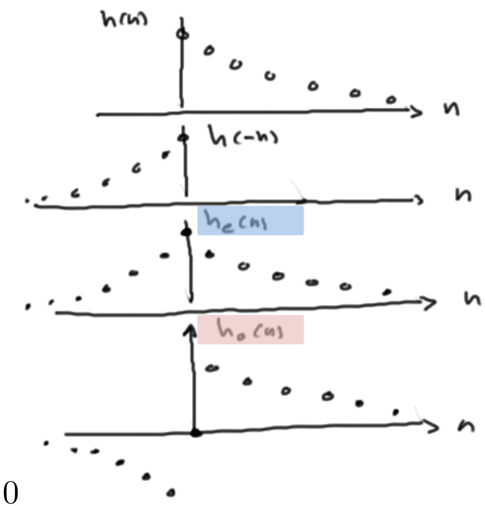
- Consider sequence $h[n]$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} h[n]\,\big(\underbrace{\cos(\omega n)}_{\text{even}} - j\underbrace{\sin(\omega n)}_{\text{odd}}\big)$$

- Note that

$$\sum_{n=-\infty}^{\infty} \overset{\text{even}}{h_{\mathrm{e}}[n]}\,\overset{\text{odd}}{\sin(\omega n)} = 0 \qquad \sum_{n=-\infty}^{\infty} \overset{\text{odd}}{h_{\mathrm{o}}[n]}\,\overset{\text{even}}{\cos(\omega n)} = 0$$



$$h[n] = h_{\mathrm{e}}[n] + h_{\mathrm{o}}[n]$$

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HT Relations for Fourier Transform

- Thus, it follows that for any sequence $h[n]$ the Fourier transform is given

$$H(e^{j\omega}) = H_{\mathrm{R}}(e^{j\omega}) + jH_{\mathrm{I}}(e^{j\omega})$$

Fourier Transform of $h_{\mathrm{e}}[n]$ (purely real)

Fourier Transform of $h_{\mathrm{o}}[n]$ (purely imaginary)

- For causal sequences, we can write

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \big(h_{\mathrm{e}}[n]u_{+}[n]\big)e^{-j\omega n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{\mathrm{R}}(e^{j\theta})\,U_{+}(e^{j(\omega-\theta)})\,\mathrm{d}\theta$$

multi. in time domain is equal to convolution in freq. domain

purely real

real and imaginary

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HT Relations for Fourier Transform

- Finally, the imaginary part of $H(e^{j\omega})$ can be computed as a Hilbert transform of the real part, i.e.,

$$H_{\mathrm{I}}(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{\mathrm{R}}(e^{j\theta})\,\mathrm{Im}\{U_+(e^{j(\omega-\theta)})\}\,\mathrm{d}\theta$$

- Note that the imaginary part of $U_+(e^{j(\omega)})$ is given by

$$\mathrm{Im}\{U_+(e^{j(\omega)})\} = \cot\Big(-\frac{\omega}{2}\Big)$$

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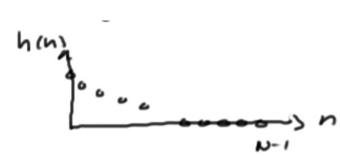
HT Relations for DFT

- Real part of discrete Fourier transform implies imaginary part when the following "causality" condition is satisfied
  - $h[n]$ is periodic and $h[n] = 0, \qquad N/2 < n < N-1$
  - $h[n]$ is of length N and $h[n] = 0, \quad N/2 < n < N-1$



Extended Step Function of Length N

$$h[n] = \begin{cases} 2h_{\mathrm{e}}[n] & n = 1, \dots, \frac{N}{2}-1 \\ h_{\mathrm{e}}[n] & n = 0, \frac{N}{2} \\ 0 & n = \frac{N}{2}+1, \dots, N-1 \end{cases}$$

$$= h_{\mathrm{e}}[n]u_N[n]$$

$$u_N[n] = \begin{cases} 1 & n = 0, \frac{N}{2} \\ 2 & n = 1, \dots, \frac{N}{2}-1 \\ 0 & n = \frac{N}{2}+1, \dots, N-1 \end{cases}$$

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HT Relations for DFT

- Thus, it follows that for any sequence $h[n]$ the Fourier transform is given

$$H(k) = H_{\mathrm{R}}(k) + jH_{\mathrm{I}}(k)$$

DFT of $h_{\mathrm{e}}[n]$ (pure real)

DFT of $h_{\mathrm{o}}[n]$ (pure imaginary)

- For causal sequences, we can write $H(k) = \sum_{n=0}^{N-1} \left(h_{\mathrm{e}}[n]u_N[n]\right) e^{-j\frac{2\pi}{N}nk}$
- Thus, the imaginary part of $H(k)$ can again be computed as an HT of the real part, i.e.,

$$H_{\mathrm{I}}(k) = \frac{2}{N} \sum_{m=0}^{N} H_{\mathrm{R}}(m) \cot\left(\frac{\pi(m-k)}{N}\right)$$

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Minimum Phase Sequences

- We are interested in the conditions under which magnitude and phase are related
- Applications:
  - calculation of the phase response of a stable, causal digital filter when only magnitude information is specified
  - inverse filtering where a phase curve is desired when only autocorrelation function is available
- For future reference, we define the complex logarithm of $H(z)$

$$\hat{H}(z) = \ln H(z) = \ln\left|H(z)\right| + j\arg H(z)$$

see also Section 13 in *Oppenheim & Schafer, 2009*

- Note that if $h[n]$ is real, $\hat{h}[n]$ is also real (if $H(z)$ is conjugate symmetric, $\hat{H}(z)$ is conjugate symmetric as well)

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Minimum Phase Sequences

- It can be shown that $\hat{h}[n]$ causal implies that $h[n]$ is causal and that $H(z)$ has all poles and zeros inside unit circle
- This is all known as the *minimum phase condition*
- Since all poles and zeros of $H(z)$ are inside the unit circle both $h[n]$ and $\hat{h}[n]$ are stable
- Note that
  - $\ln H(z)$ has poles at poles and zeros locations of $H(z)$
  - $\arg H(z)$ is ambiguous due to modulo 2π

see also Sections 5.6 and 13.5 in *Oppenheim & Schafer, 2009*

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Minimum Phase Sequences

- Given that the complex cepstrum $\hat{h}[n]$ is causal, the real part of $\hat{H}(e^{j\omega})$ implies the imaginary part
- It thus follows that $\ln\left|H(e^{j\omega})\right|$ implies $\arg\left[H(e^{j\omega})\right]$
- Note that a sequence or system can be causal but not min-phase
- Since both poles and zeros of $H(z)$ become poles of $\hat{H}(z)$, $\hat{h}[n]$ will be of infinite duration; as consequence, it can only be approximately represented by a DFT where a longer DFT length N results in a more accurate representation

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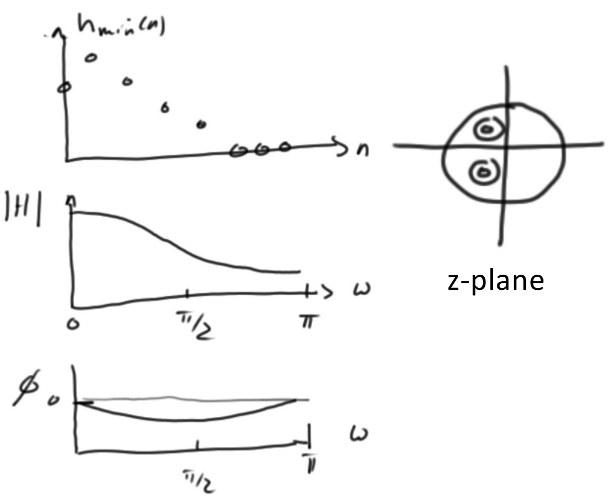
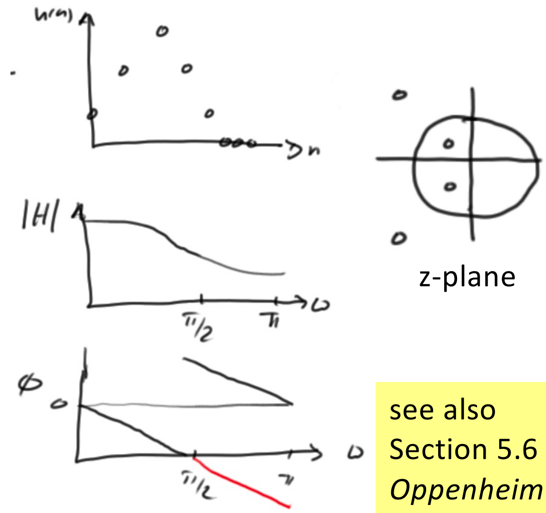
Minimum Phase Sequences



z-plane

minimum phase



z-plane

linear phase

see also Section 5.6 in *Oppenheim & Schafer, 2009*