IIR Filter Design

Lecture By Prof. Meyer

ECE 161A

Overall Design Steps

Given $\delta_1, \delta_2, \omega_p, \omega_s$

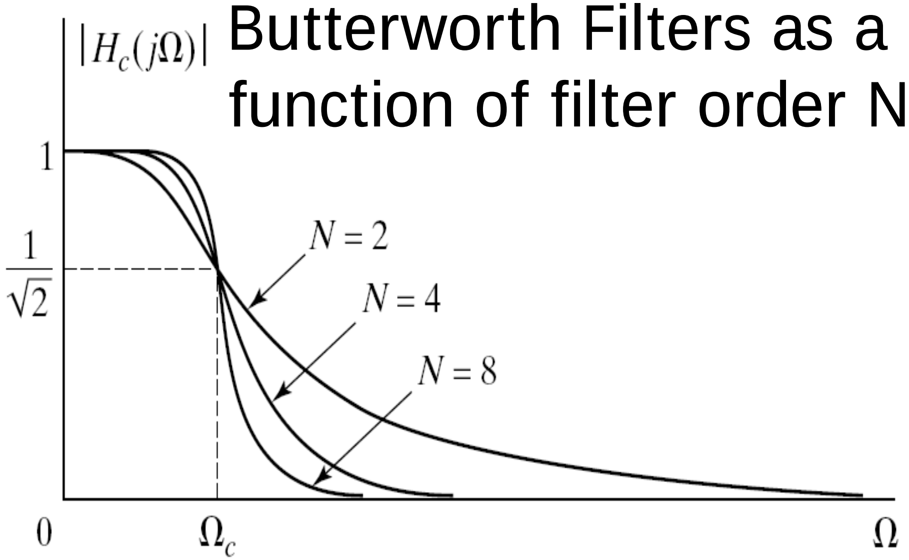
1. Pre-warp to get $\delta_1, \delta_2, \Omega_p, \Omega_s$

where $\Omega_i = \dfrac{2}{T_d}\tan\left(\dfrac{\omega_i}{2}\right)$

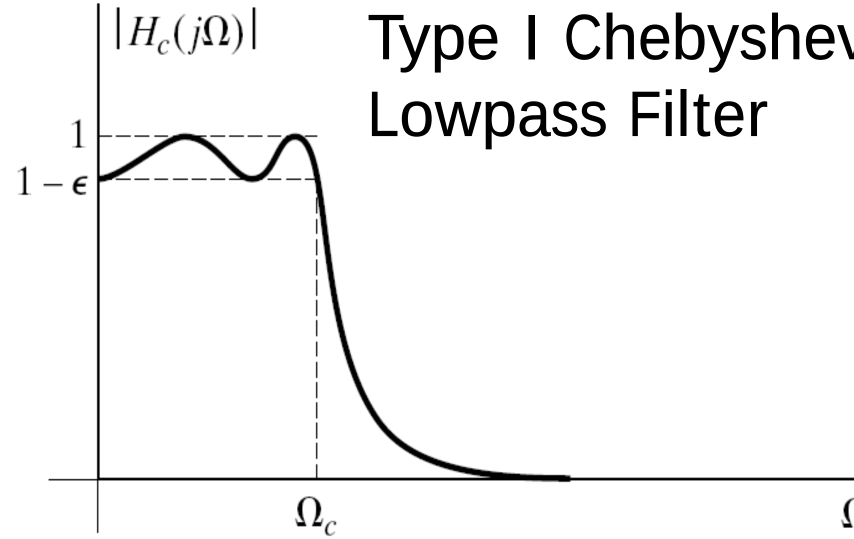
2. Using these specs design Analog Filter $H_c(j\Omega)$ *or* $H_c(s)$

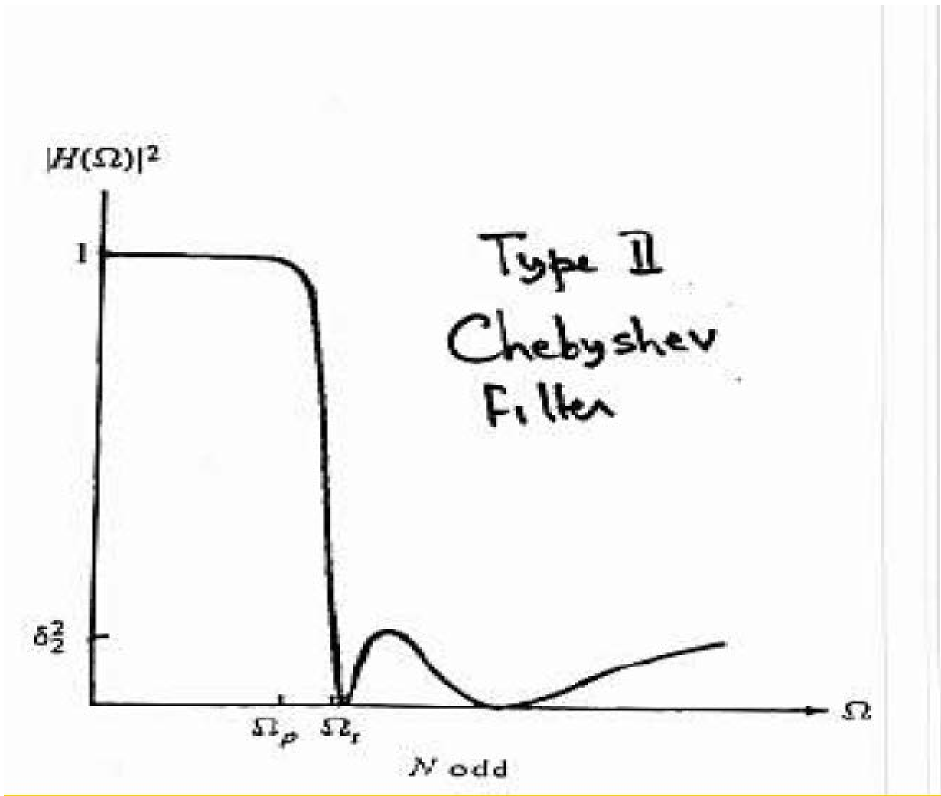
3. $H(z) = H_c\left[\dfrac{2}{T_d}\dfrac{1-z^{-1}}{1+z^{-1}}\right]$

Butterworth Filters as a function of filter order N

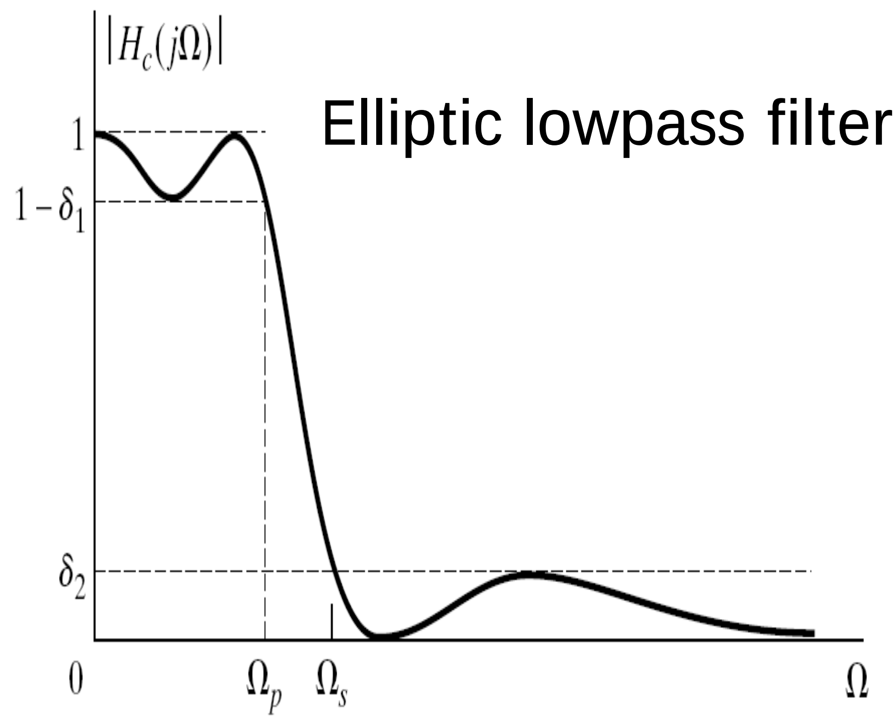


Type I Chebyshev Lowpass Filter



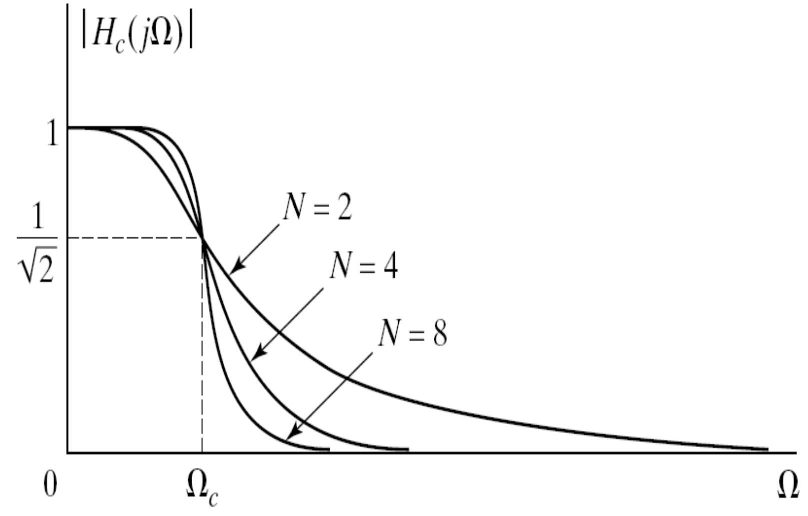


Elliptic lowpass filter



Butterworth Filter

Real coefficient filters



$$|H(j\Omega)|^2 = \frac{1}{1+\left(\dfrac{\Omega}{\Omega_c}\right)^{2N}}$$

$$H(s)H(-s) = \frac{1}{1+\left(\dfrac{s}{j\Omega_c}\right)^{2N}} = \frac{(j\Omega_c)^{2N}}{s^{2N}+(j\Omega_c)^{2N}}$$

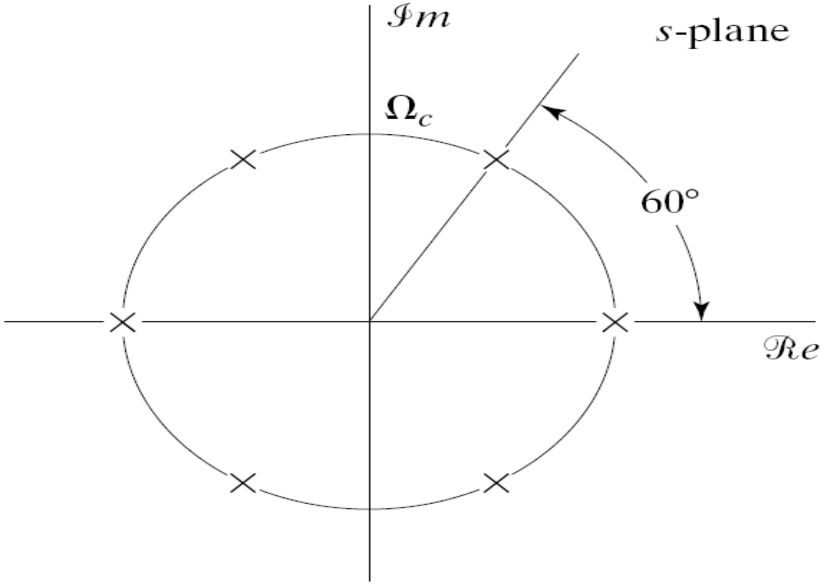
Roots of the Butterworth Filter

$$s^{2N} + (j\Omega_c)^{2N} = 0 \;\text{ or }\; s^{2N} = -(j\Omega_c)^{2N}$$

Now $-1 = e^{j(2k-1)\pi}, k = 0, 1, ..$ allowing us to compute the $2N$ roots as follows

$$\begin{aligned} s_k^{2N} &= e^{j(2k-1)\pi}(j\Omega_c)^{2N} \\ s_k &= e^{j\frac{(2k-1)\pi}{2N}} j\Omega_c = e^{j\frac{(2k-1)\pi}{2N}} e^{j\frac{\pi}{2}} \Omega_c \\ &= e^{j\frac{2k\pi}{2N}} e^{j(\frac{\pi}{2} - \frac{\pi}{2N})} \Omega_c = e^{j(\frac{\pi}{2} - \frac{\pi}{2N})} e^{j\frac{2k\pi}{2N}} \Omega_c \end{aligned}$$

Butterworth Filter Cont'd



N=3

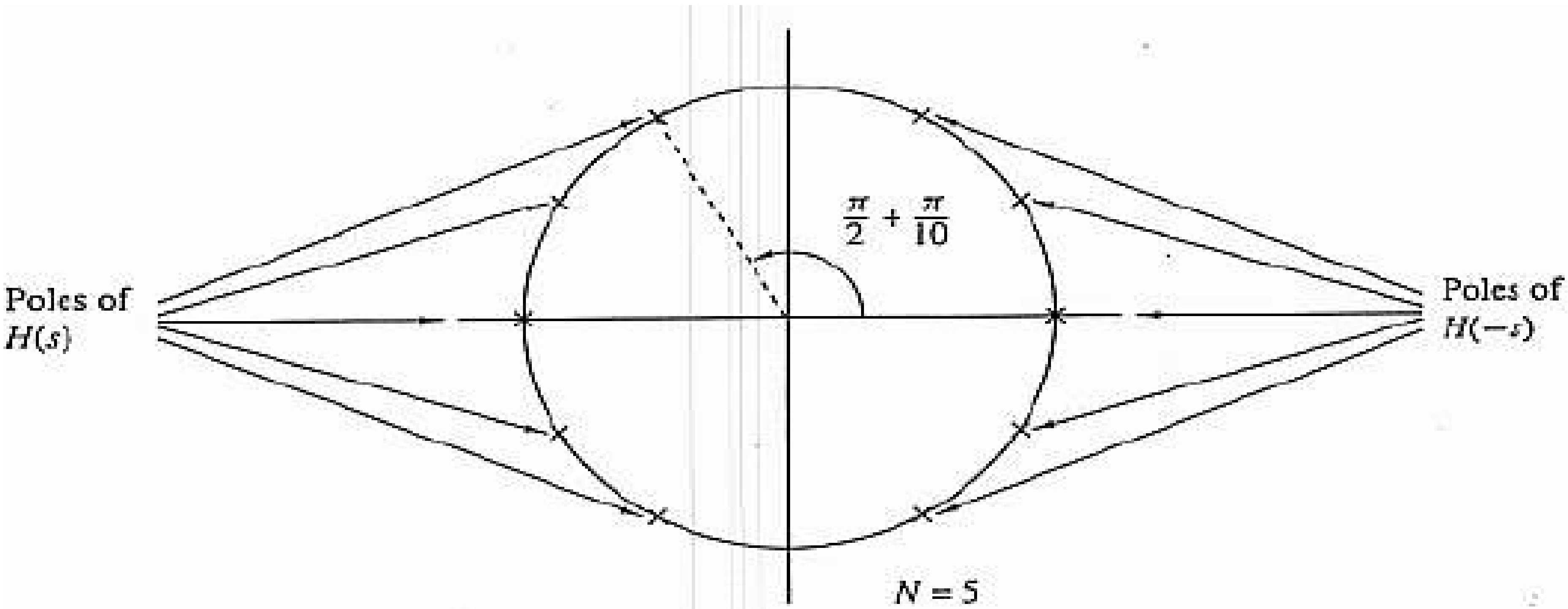
Poles of filter are on a circle of radius Ω_c

$$s_k = \Omega_c e^{j(\frac{\pi}{2} - \frac{\pi}{2N})} e^{\frac{j2\pi k}{2N}} = \Omega_c e^{\frac{j\pi}{2N}(2k+N-1)}, k = 0,1,\ldots,2N-1$$

Butterworth Filters (cont.)

$H_c(s)$ is formed by taking all the poles in the left half plane

$$H_c(s) = \frac{(\Omega_c)^N}{\prod_{k=1}^{N}(s - s_k)}$$



Design Example

- Specification of Digital Filter

$$.89125 \le \left|H(e^{j\omega})\right| \le 1, \quad 0 \le \omega \le .2\pi$$

$$\left|H(e^{j\omega})\right| \le 0.17783, \quad .3\pi \le \omega \le \pi$$

- Specifications of Analog Filter

$$.89125 \le \left|H(j\Omega)\right| \le 1, \quad 0 \le \Omega \le \frac{2}{T_d}\tan\left(\frac{0.2\pi}{2}\right)$$

$$\left|H(j\Omega)\right| \le 0.17783, \quad \frac{2}{T_d}\tan\left(\frac{0.3\pi}{2}\right) \le \Omega \le \infty$$

- Design Constraints $T_d = 1$

$$\left|H_c(j2\tan(.1\pi))\right| \ge .89$$

$$\left|H_c(j2\tan(.15\pi))\right| \le .178$$

Design Example (cont.)

- For the Butterworth filter $|H(j\Omega)|^2 = \dfrac{1}{1+\left(\dfrac{\Omega}{\Omega_c}\right)^{2N}}$ and due to its monotonic response

$$1+\left(\frac{2\tan(.1\pi)}{\Omega_c}\right)^{2N} = \left(\frac{1}{.89}\right)^2$$

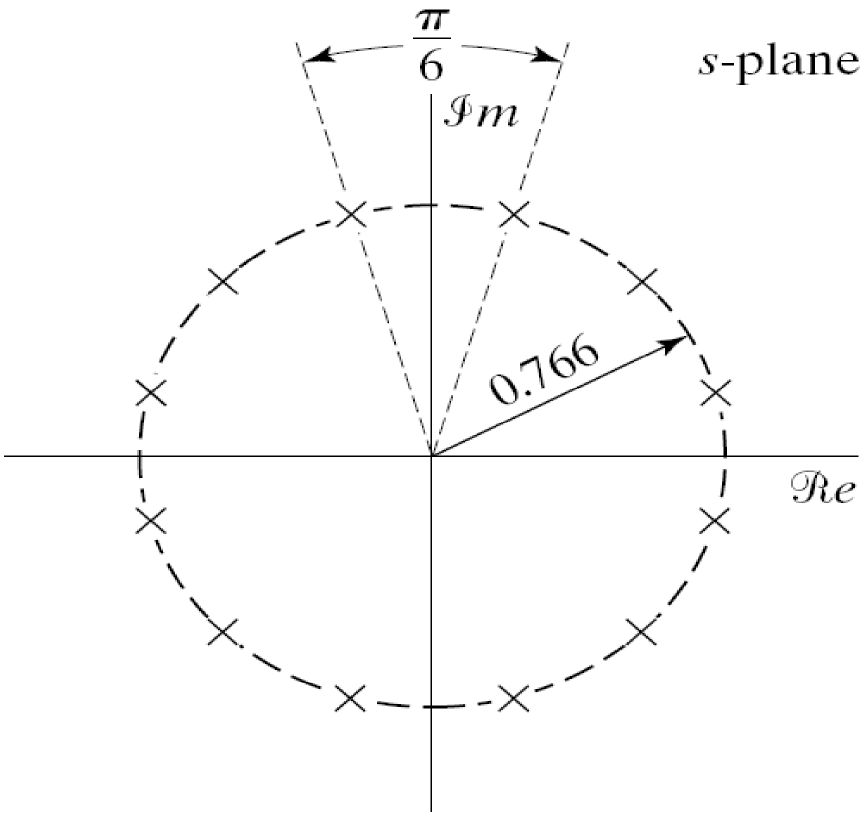
$$1+\left(\frac{2\tan(.15\pi)}{\Omega_c}\right)^{2N} = \left(\frac{1}{.178}\right)^2$$

- Solving for N, yields N= 5.3. Since N has to be an integer it is rounded up and N=6. Solving for Ω_c from the stop-band specs yields Ω_c =.76622

Design Example (Cont. Pg 2)

$$N = 6$$

$$\Omega_c = .766$$



S-Plane locations for pole of $H_c(s)H_c(-s)$ for sixth-order Butterworth Filter in Example 7.4

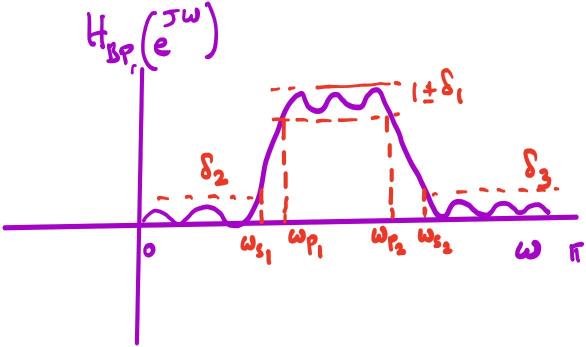
Design Examples (Cont. Pg 3)

$$H_c(s) = \frac{0.20238}{(s^2 + .3996s + 0.59)(s^2 + 1.084s + 0.59)(s^2 + 1.48s + 0.59)}$$

$$H(z) = H_c\left(2\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

$$= \frac{0.0007378(1 + z^{-1})^6}{(1 - 1.27z^{-1} + .71z^{-2})(1 - 1.01z^{-1} + .36z^{-2})(1 - .9z^{-1} + .22z^{-2})}$$

Bandpass Filter Example



Digital Filter Specifications: $\delta_1, \delta_2, \delta_3, \omega_{s_1}, \omega_{p_1}, \omega_{p_2}, \omega_{s_2}$

Analog Filter Specifications

Digital Filter Specifications: $\delta_1, \delta_2, \delta_3, \omega_{s_1}, \omega_{p_1}, \omega_{p_2}, \omega_{s_2}$

The δ's remain the same. Pre-Warp to get the Analog Filter band edges.

$$
\begin{aligned}
\Omega_{s_1} &= \frac{2}{T_d} \tan \frac{\omega_{s_1}}{2} \\
\Omega_{p_1} &= \frac{2}{T_d} \tan \frac{\omega_{p_1}}{2} \\
\Omega_{p_2} &= \frac{2}{T_d} \tan \frac{\omega_{p_2}}{2} \\
\Omega_{s_2} &= \frac{2}{T_d} \tan \frac{\omega_{s_2}}{2}
\end{aligned}
$$

Analog Filter Specifications: $\delta_1, \delta_2, \delta_3, \Omega_{s_1}, \Omega_{p_1}, \Omega_{p_2}, \Omega_{s_2}$

Frequency Transformation

- Change of variable $z^{-1} \longrightarrow G(z^{-1})$ to get desired response (low pass, high pass, band pass, etc.)

$$H(z) \longrightarrow H_d(z) = H(z)\Big|_{z^{-1} \longrightarrow G(z^{-1})}$$

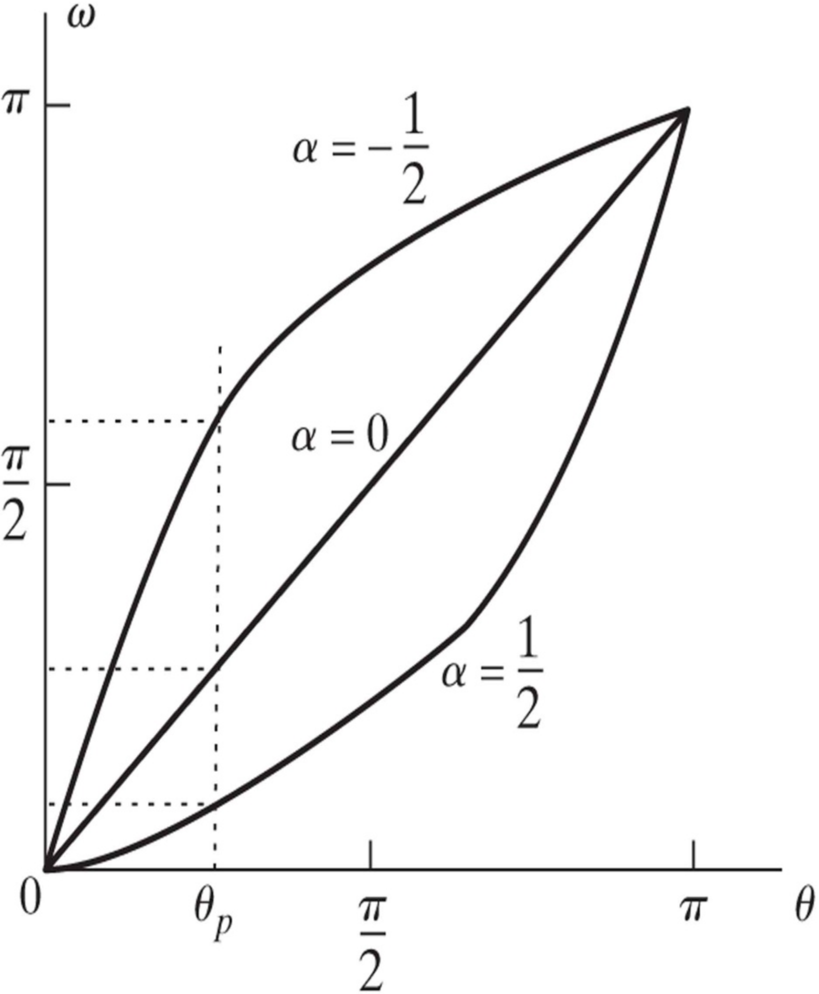
- Alternate Notation

$$Z^{-1} = G(z^{-1})$$

$$H_d(z) = H(Z)\Big|_{Z^{-1} = G(z^{-1})}$$

- Constraints on G(.)
  - Stability be preserved.
  - G(.) is a rational function

Lowpass to Lowpass



$$Z^{-1} = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}, \quad \alpha \ real$$

Also

$$z^{-1} = \frac{Z^{-1} + \alpha}{1 + \alpha Z^{-1}}, \quad \alpha \ real$$

If $Z = e^{j\theta}$ *and* $z = e^{j\omega}$, *then*

$$e^{-j\omega} = \frac{e^{-j\theta} + \alpha}{1 + \alpha e^{-j\theta}}$$

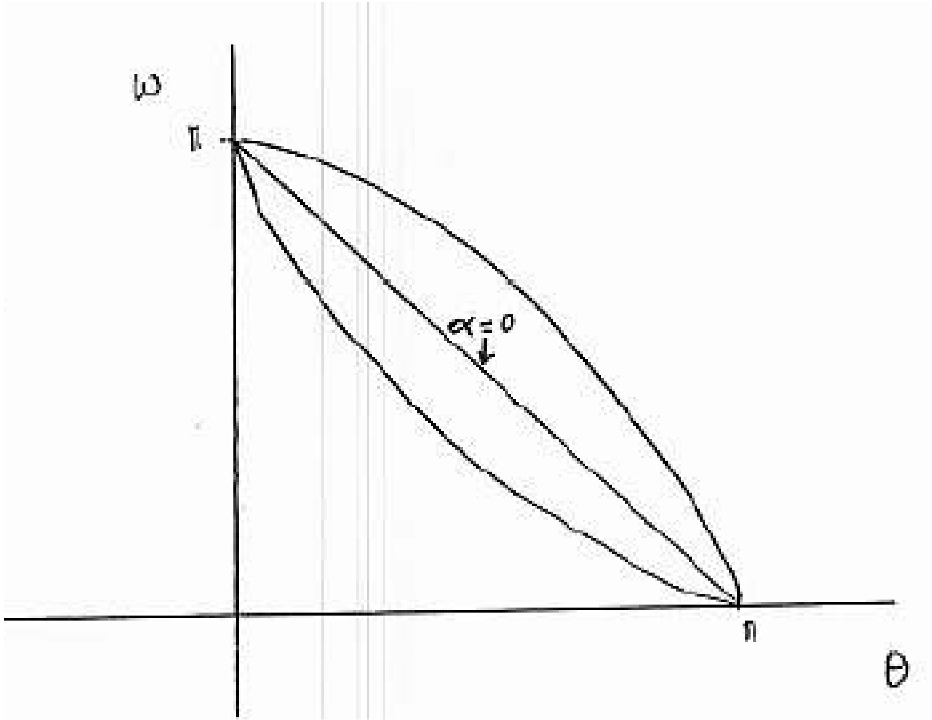
$$\omega = \arctan\left(\frac{(1-\alpha^2)\sin\theta}{2\alpha + (1+\alpha^2)\cos\theta}\right)$$

$$\alpha = \frac{\sin\frac{\theta_p - \omega_p}{2}}{\sin\frac{\theta_p + \omega_p}{2}}$$

Lowpass to Highpass

$$Z^{-1} = -\frac{z^{-1}+\alpha}{1+\alpha z^{-1}}$$

Choose α as follows



$$\alpha = -\frac{\cos\frac{\theta_p+\omega_p}{2}}{\cos\frac{\theta_p-\omega_p}{2}}$$

$$\alpha = 0 \longrightarrow \omega_p = \pi - \theta_p$$

TABLE 7.1 TRANSFORMATIONS FROM A LOWPASS DIGITAL FILTER PROTOTYPE OF CUTOFF FREQUENCY θ_p TO HIGHPASS, BANDPASS, AND BANDSTOP FILTERS

Filter Type	Transformations	Associated Design Formulas
Lowpass	$Z^{-1} = \dfrac{z^{-1} - \alpha}{1 - az^{-1}}$	$\alpha = \dfrac{\sin\left(\frac{\theta_p - \omega_p}{2}\right)}{\sin\left(\frac{\theta_p + \omega_p}{2}\right)}$ $\omega_p =$ desired cutoff frequency
Highpass	$Z^{-1} = -\dfrac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	$\alpha = -\dfrac{\cos\left(\frac{\theta_p + \omega_p}{2}\right)}{\cos\left(\frac{\theta_p - \omega_p}{2}\right)}$ $\omega_p =$ desired cutoff frequency
Bandpass	$Z^{-1} = -\dfrac{z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + 1}$	$\alpha = \dfrac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \cot\left(\dfrac{\omega_{p2} - \omega_{p1}}{2}\right)\tan\left(\dfrac{\theta_p}{2}\right)$ $\omega_{p1} =$ desired lower cutoff frequency $\omega_{p2} =$ desired upper cutoff frequency
Bandstop	$Z^{-1} = \dfrac{z^{-2} - \frac{2\alpha}{1+k} z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k} z^{-2} - \frac{2\alpha}{1+k} z^{-1} + 1}$	$\alpha = \dfrac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \tan\left(\dfrac{\omega_{p2} - \omega_{p1}}{2}\right)\tan\left(\dfrac{\theta_p}{2}\right)$ $\omega_{p1} =$ desired lower cutoff frequency $\omega_{p2} =$ desired upper cutoff frequency