ECE 275A: Parameter Estimation I
Sequential Bayesian Estimation

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State-Space Model

- Consider a sequence of states $\mathbf{x}_n$ and a sequence of measurements $\mathbf{y}_n$ at discrete time $n = 1, 2, \ldots$

State-transition model

State $\mathbf{x}_n$ evolves according to

$$\mathbf{x}_n = \mathbf{g}_n(\mathbf{x}_{n-1}) + \underbrace{\mathbf{u}_n}_{\text{Driving noise (white)}}, \quad n = 1, 2, \ldots$$

This determines the state-transition pdf $p(\mathbf{x}_n|\mathbf{x}_{n-1})$

Measurement model

Measurement $\mathbf{y}_n$ depends on state $\mathbf{x}_n$ according to

$$\mathbf{y}_n = \mathbf{h}_n(\mathbf{x}_n) + \underbrace{\mathbf{v}_n}_{\text{Measurement noise (white)}}, \quad n = 1, 2, \ldots$$

This determines the likelihood function $p(\mathbf{y}_n|\mathbf{x}_n)$

Markovian Properties

- Noise sequences $\mathbf{u}_n$ and $\mathbf{v}_n$ are assumed mutually independent and independent of $\mathbf{x}_0$.
- Recall:

$$\boxed{\begin{array}{ll} \mathbf{x}_n = \mathbf{g}_n(\mathbf{x}_{n-1}) + \mathbf{u}_n & \mathbf{u}_n \text{ is white} \\ \mathbf{y}_n = \mathbf{h}_n(\mathbf{x}_n) + \mathbf{v}_n & \mathbf{v}_n \text{ is white} \end{array}}$$

- At time n, the state $\mathbf{x}_n$ summarizes all information about the present and past
- Mathematically expressed by "Markovian properties":

$$\boxed{\begin{array}{c} p(\mathbf{x}_n|\mathbf{x}_{0:n-1}, \mathbf{y}_{1:n-1}) = p(\mathbf{x}_n|\mathbf{x}_{n-1}) \\ p(\mathbf{y}_n|\mathbf{x}_{0:n}, \mathbf{y}_{1:n-1}) = p(\mathbf{y}_n|\mathbf{x}_n) \end{array}} \quad \text{where } \mathbf{y}_{1:n-1} \triangleq \begin{pmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_{n-1} \end{pmatrix}$$

Sequential Bayesian Estimation

- We wish to estimate the current state $\mathbf{x}_n$ from the past and current measurements $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$, i.e., from $\mathbf{y}_{1:n}$, for $n = 1, 2, \ldots$
- MMSE estimator:

$$\hat{\mathbf{x}}_n = \mathrm{E}\{\mathbf{x}_n | \mathbf{y}_{1:n}\} = \int \mathbf{x}_n \, p(\mathbf{x}_n | \mathbf{y}_{1:n}) \, \mathrm{d}\mathbf{x}_n$$

- The posterior pdf $p(\mathbf{x}_n | \mathbf{y}_{1:n})$ can be calculated recursively/sequentially

B. Ristic, S. Arulampalam, and N. Gordon, *Beyond the Kalman Filter: Particle Filters for Tracking Applications*, Artech House, 2004.

Sequential Bayesian Estimation

- Consider joint posterior pdf $p(\mathbf{x}_{0:n}|\mathbf{y}_{1:n})$
- For sequential calculation of the "marginal" posterior pdf $p(\mathbf{x}_n|\mathbf{y}_{1:n})$ we consider **factorization and marginalization** of the joint posterior pdf $p(\mathbf{x}_{0:n}|\mathbf{y}_{1:n})$

Factorization

$$p(\mathbf{x}_{0:n}|\mathbf{y}_{1:n}) \propto p(\mathbf{x}_0) \prod_{n'=1}^{n} p(\mathbf{y}_{n'}|\mathbf{x}_{n'})\, p(\mathbf{x}_{n'}|\mathbf{x}_{n'-1})$$

Marginalization

$$p(\mathbf{x}_n|\mathbf{y}_{1:n}) \propto \int p(\mathbf{x}_0) \left(\prod_{n'=1}^{n} \left[p(\mathbf{y}_{n'}|\mathbf{x}_{n'})\, p(\mathbf{x}_{n'}|\mathbf{x}_{n'-1}) \right) \mathrm{d}\mathbf{x}_{0:n-1}$$

Sequential Bayesian Estimation

- The Markovian properties enable sequential calculation of $p(\mathbf{x}_n|\mathbf{y}_{1:n})$
- As derived in class, one recursion consists of two steps:

Prediction step

$$\underbrace{p(\mathbf{x}_n|\mathbf{y}_{1:n-1})}_{\substack{\text{Predicted}\\\text{posterior pdf}}} = \int \underbrace{p(\mathbf{x}_n|\mathbf{x}_{n-1})}_{\substack{\text{State-transition}\\\text{pdf}}} \underbrace{p(\mathbf{x}_{n-1}|\mathbf{y}_{1:n-1})}_{\substack{\text{Previous}\\\text{posterior pdf}}} \, \mathrm{d}\mathbf{x}_{n-1}$$

Measurement update step

$$\underbrace{p(\mathbf{x}_n|\mathbf{y}_{1:n})}_{\text{Posterior pdf}} \propto \underbrace{p(\mathbf{y}_n|\mathbf{x}_n)}_{\substack{\text{Likelihood}\\\text{function}}} \underbrace{p(\mathbf{x}_n|\mathbf{y}_{1:n-1})}_{\substack{\text{Predicted}\\\text{posterior pdf}}}$$

- In general, the prediction and update step can not be evaluated in closed form, and feasible approximations are needed
- For linear-Gaussian models, the prediction and update step can be evaluated in closed form $\rightarrow$ this is known as Kalman filtering