ECE 161A: The Discrete Fourier Series

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Need for Additional Transforms

Weakness of the Discrete-Time Fourier Transform (DTFT)

$$X(e^{j\omega}) = \sum_{n=-\infty} x[n]e^{-j\omega n} \text{ and } x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Main Problem: Even though the sequence is discrete, the DTFT is a function of a continuous variable.

1. From a computation point of view, there is still the challenge of how many samples of ω in the interval $[-\pi, \pi]$ should we choose?
2. What is the impact of the sampling interval?
3. Can we recover the sequence from the samples of the DTFT?

New Objective

Would like a Transform that is discrete in time and also in the transform domain!

- Discrete Fourier Series (DFS): For periodic sequences
- Discrete Fourier Transform (DFT): For aperiodic sequences
- Fast Fourier Transform (FFT): A very efficient way to compute the DFT.

Discrete Fourier Series (DFS)

$\tilde{x}[n]$ is a periodic sequence with periodicity N, i.e.

$$\tilde{x}[n] = \tilde{x}[n+N] = \tilde{x}[n-N] = \tilde{x}[n+rN],$$

where r is an integer

Let the fundamental frequency be $\frac{2\pi}{N}$ and we can conjecture a Fourier series representation

$$\tilde{x}[n] = \frac{1}{N}\sum_{k} \tilde{X}[k] e^{j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{k} \tilde{X}[k] e_k[n]$$

where $e_k[n] = e^{j\frac{2\pi}{N}kn}$

Need to check this is possible and would prefer finite sums

Properties of the exponential Sequence $e_k[n] = e^{j\frac{2\pi}{N}kn}$

1. They are periodic sequences with periodicity N, $e_k[n] = e_k[n+N]$.

$$e_k[n+N] = e^{j\frac{2\pi}{N}k(n+N)} = e^{j\frac{2\pi}{N}kn}e^{j\frac{2\pi}{N}kN} = e^{j\frac{2\pi}{N}kn}e^{j2\pi k} = e^{j\frac{2\pi}{N}kn} = e_k[n]$$

Good candidates for representing periodic sequences. of periodicity N

2. $e_k[n] = e_{k+N}[n]$, i.e. periodic in the harmonic variable k. Proof is similar.

$$e_{k+N}[n] = e^{j\frac{2\pi}{N}(k+N)n} = e^{j\frac{2\pi}{N}kn}e^{j\frac{2\pi}{N}Nn} = e^{j\frac{2\pi}{N}kn}e^{j2\pi n} = e^{j\frac{2\pi}{N}kn} = e_k[n]$$

This means you do not need to consider values of harmonic variable k outside the range $[0, N-1]$ since they repeat and are the same harmonic sequences!

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j\frac{2\pi}{N}kn}$$

This is a finite sum with finite number of DFS components $\tilde{X}[k], 0 \le k \le N-1$.

Only remaining important step is to show this finite sum is a valid way to represent any sequence.

Orthogonality Property of the Exponential Sequences

3. An important property of the exponential sequences is that they are orthogonal sequences.

$$< e_k[n], e_l[n] > \; = \sum_{n=0}^{N-1} e_k^*[n] e_l[n] = N\delta[k-l]$$

Proof:

$$\sum_{n=0}^{N-1} e_k^*[n] e_l[n] = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}kn} e^{j\frac{2\pi}{N}ln} = \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(l-k)n} = \sum_{n=0}^{N-1} (e^{j\frac{2\pi}{N}(l-k)})^n$$

For $l = k$, $\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(0)n} = \sum_{n=0}^{N-1} 1 = N$.

For $l \neq k$, we have a sum of a geometric series with ratio $r = (e^{j\frac{2\pi}{N}(l-k)})$ and first term 1.

$$\sum_{n=0}^{N-1} (e^{j\frac{2\pi}{N}(l-k)})^n = \frac{1-r^N}{1-r}$$

Note that $r^N = (e^{j\frac{2\pi}{N}(l-k)})^N = e^{j2\pi(l-k)} = 1$ leading to the numerator $1 - r^N = 0$

Validity of the Representation for any periodic sequence with periodicity N

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j\frac{2\pi}{N}kn}$$

We will use vector representation and some linear algebra to do this.

$\tilde{\mathbf{x}} = [\tilde{x}[0], \tilde{x}[1], \ldots, \tilde{x}[N-1]]^T$ and $\mathbf{e}_k = [e_k[0], e_k[1], ..., e_k[N-1]]^T \in C^N$

From the orthogonality of the exponential sequences

$< e_k[n], e_l[n] >= \mathbf{e}_k^H \mathbf{e}_l = N\delta[k-l]$

From linear algebra results, any $\tilde{\mathbf{x}} \in C^N$ can be represented by a basis of N linearly independent vectors.

$\mathbf{e}_k, k = 0, 1, \ldots, N-1$ represent an orthogonal basis for C^N

So any $\tilde{\mathbf{x}} \in C^N$ can be represented a linear combination of the $\mathbf{e}_k, k = 0, 1, \ldots, N-1$, which is the DFS.

Inverting the Expansion

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e_k[n] \;\; \text{(Synthesis Equation)}$$

Finding the DFS coefficients $\tilde{X}[k], k = 0, ...N-1$.

$$\tilde{X}[k] = \sum_{n=0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1}\tilde{x}[n]e_k^*[n] \;\; \text{(Analysis Equation)}$$

Proof:

$$\begin{aligned}
\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi}{N}kn} &= \sum_{n=0}^{N-1}\tilde{x}[n]e_k^*[n] = \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{l=0}^{N-1}\tilde{X}[l]e_l[n]\right)e_k^*[n] \\
&= \frac{1}{N}\sum_{l=0}^{N-1}\tilde{X}[l]\left(\sum_{n=0}^{N-1}e_l[n]e_k^*[n]\right) = \frac{1}{N}\sum_{l=0}^{N-1}\tilde{X}[l]N\delta[k-l] \\
&= \tilde{X}[k]
\end{aligned}$$

Summary

$$\text{Notation: } \tilde{x}[n] \overset{DFS}{\longleftrightarrow} \tilde{X}[k] \text{ or } \tilde{x}[n] \overset{N}{\longleftrightarrow} \tilde{X}[k]$$

Define $W_N = e^{-j\frac{2\pi}{N}}$ and then

$$e_k[n] = e^{j\frac{2\pi}{N}kn} = W_N^{-nk} = W_N^{-(n+N)k} = W_N^{-n(k+N)}.$$

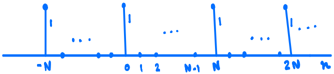
$$\begin{aligned}
\tilde{x}[n] &= \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-nk} \text{ (Synthesis Equation)} \\
\tilde{X}[k] &= \sum_{n=0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1}\tilde{x}[n]W_N^{nk} \\
&= \mathcal{F}_s(\tilde{x}[n]) \text{ (Analysis Equation)}
\end{aligned}$$

Note that even though we only need the DFS coefficients for $k = 0, .., N-1$, the analysis equation can be used for any value of k with the observation that $\tilde{X}[k]$ is periodic, i.e. $\tilde{X}[k] = \tilde{X}[k-N]$.

Example: Periodic Impulse Train

Consider $\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n + rN]$

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n+rN] = \tilde{x}[n+N]$$



$$\begin{aligned}
\tilde{X}[k] &= \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} \sum_{r=-\infty}^{\infty} \delta[n + rN] e^{-j\frac{2\pi}{N}kn} \\
&= \sum_{n=0}^{N-1} \delta[n] e^{-j\frac{2\pi}{N}kn} = e^{-j\frac{2\pi}{N}k0} = 1,\ \ k = 0, 1, \ldots, N-1
\end{aligned}$$

Example: Sum of Exponentials

$$
\begin{aligned}
\tilde{x}[n] &= 3e^{-j\frac{2\pi}{3}n} + 2e^{j\frac{2\pi}{5}n} = 3e^{j\frac{4\pi}{3}n} + 2e^{j\frac{2\pi}{5}n} \\
&= 3e^{j\frac{2\pi}{15}10n} + 2e^{j\frac{2\pi}{15}3n} = \frac{1}{15}\sum_{k=0}^{14} \tilde{X}[k]e^{j\frac{2\pi}{15}kn}
\end{aligned}
$$

$LCM(3,5) = 15$ is the common period for the exponentials.

By inspection $\frac{1}{15}\tilde{X}[10] = 3$, $\frac{1}{15}\tilde{X}[3] = 2$, and the rest are zero. Hence

$$
\tilde{X}[k] = \begin{cases} 30, & k = 3 \\ 45, & k = 10 \\ 0, & k \text{ otherwise} \end{cases}
$$

Note that $0 \leq k \leq 14$.

DFS Properties

TABLE 8.1 SUMMARY OF PROPERTIES OF THE DFS

Periodic Sequence (Period N)	DFS Coefficients (Period N)
1. $\tilde{x}[n]$	$\tilde{X}[k]$ periodic with period N
2. $\tilde{x}_1[n], \tilde{x}_2[n]$	$\tilde{X}_1[k], \tilde{X}_2[k]$ periodic with period N
3. $a\tilde{x}_1[n] + b\tilde{x}_2[n]$	$a\tilde{X}_1[k] + b\tilde{X}_2[k]$
4. $\tilde{X}[n]$	$N\tilde{x}[-k]$
5. $\tilde{x}[n-m]$	$W_N^{km}\tilde{X}[k]$
6. $W_N^{-\ell n}\tilde{x}[n]$	$\tilde{X}[k-\ell]$
7. $\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$ (periodic convolution)	$\tilde{X}_1[k]\tilde{X}_2[k]$
8. $\tilde{x}_1[n]\tilde{x}_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} \tilde{X}_1[\ell]\tilde{X}_2[k-\ell]$ (periodic convolution)
9. $\tilde{x}^*[n]$	$\tilde{X}^*[-k]$
10. $\tilde{x}^*[-n]$	$\tilde{X}^*[k]$

DFS Properties Cont'd

Table 8.1 (continued) SUMMARY OF PROPERTIES OF THE DFS

11. $\mathcal{R}e\{\tilde{x}[n]\}$	$\tilde{X}_e[k] = \frac{1}{2}(\tilde{X}[k] + \tilde{X}^*[-k])$
12. $j\mathcal{I}m\{\tilde{x}[n]\}$	$\tilde{X}_o[k] = \frac{1}{2}(\tilde{X}[k] - \tilde{X}^*[-k])$
13. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}^*[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
14. $\tilde{x}_o[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}^*[-n])$	$j\mathcal{I}m\{\tilde{X}[k]\}$
Properties 15–17 apply only when $x[n]$ is real.	
15. Symmetry properties for $\tilde{x}[n]$ real.	$\begin{cases} \tilde{X}[k] = \tilde{X}^*[-k] \\ \mathcal{R}e\{\tilde{X}[k]\} = \mathcal{R}e\{\tilde{X}[-k]\} \\ \mathcal{I}m\{\tilde{X}[k]\} = -\mathcal{I}m\{\tilde{X}[-k]\} \\ \lvert\tilde{X}[k]\rvert = \lvert\tilde{X}[-k]\rvert \\ \angle\tilde{X}[k] = -\angle\tilde{X}[-k] \end{cases}$
16. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
17. $\tilde{x}_0[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}[-n])$	$j\mathcal{I}m\{\tilde{X}[k]\}$

DFS Properties: Linearity

Note that the sum of two periodic sequences is periodic

$$\tilde{x}_1[n] \stackrel{N}{\longleftrightarrow} \tilde{X}_1[k] \text{ and } \tilde{x}_2[n] \stackrel{N}{\longleftrightarrow} \tilde{X}_2[k].$$

Then

$$\tilde{x}[n] = a_1\tilde{x}_1[n] + a_2\tilde{x}_2[n] \stackrel{N}{\longleftrightarrow} \tilde{X}[k] = a_1\tilde{X}_1[k] + a_2\tilde{X}_2[k]$$

Periodic sequences of different length

$$\tilde{x}_1[n] \stackrel{N_1}{\longleftrightarrow} \tilde{X}_1[k] \text{ and } \tilde{x}_2[n] \stackrel{N_2}{\longleftrightarrow} \tilde{X}_2[k].$$

Then

$$\tilde{x}[n] = a_1\tilde{x}_1[n] + a_2\tilde{x}_2[n] \stackrel{N}{\longleftrightarrow} \tilde{X}[k]$$

where $N = \text{LCM}(N_1, N_2)$.

Note that $\tilde{X}[k] \neq a_1\tilde{X}_1[k] + a_2\tilde{X}_2[k]$, in general

Equality holds if $N_1 = N_2 = N$.

Shift or Delay

$$\tilde{x}[n-m] \stackrel{N}{\longleftrightarrow} W_N^{mk}\tilde{X}[k]$$

Proof: Let $\tilde{y}[n] = \tilde{x}[n-m] = \tilde{y}[n-N]$

$$\begin{aligned}
\tilde{Y}[k] &= \sum_{n=0}^{N-1} \tilde{y}[n] W_N^{nk} = \sum_{n=0}^{N-1} \tilde{x}[n-m] W_N^{nk} = \sum_{l=-m}^{N-m-1} \tilde{x}[l] W_N^{(l+m)k} \\
&= W_N^{mk} \sum_{n=-m}^{N-m-1} \tilde{x}[n] W_N^{nk} = W_N^{mk} \sum_{n=-m}^{N-m-1} \tilde{z}[n], \text{ where } \tilde{z}[n] = \tilde{x}[n] W_N^{nk} \\
&= W_N^{mk} \sum_{n=0}^{N-1} \tilde{z}[n] = W_N^{mk} \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{nk} = W_N^{mk} \tilde{X}[k]
\end{aligned}$$

Periodic Convolution

$$\tilde{x}_1[n] \stackrel{N}{\longleftrightarrow} \tilde{X}_1[k] \text{ and } \tilde{x}_2[n] \stackrel{N}{\longleftrightarrow} \tilde{X}_2[k].$$

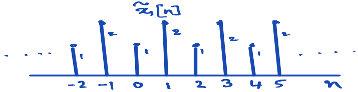
Then

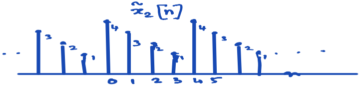
$$\tilde{x}[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \stackrel{N}{\longleftrightarrow} \tilde{X}[k] = \tilde{X}_1[k]\tilde{X}_2[k].$$

Note the sum is over one period and $\tilde{x}[n] = \tilde{x}[n-N]$ (Please verify)
Proof:

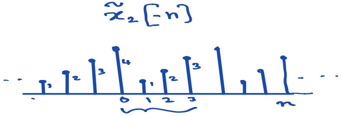
$$\begin{aligned}
\tilde{X}[k] &= \mathcal{F}_s(\tilde{x}[n]) = \mathcal{F}_s\left(\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]\right) \\
&\stackrel{Linearity}{=} \sum_{m=0}^{N-1} \tilde{x}_1[m]\mathcal{F}_s(\tilde{x}_2[n-m]) \stackrel{Shift}{=} \sum_{m=0}^{N-1} \tilde{x}_1[m]W_N^{mk}\tilde{X}_2[k] \\
&= \left(\sum_{m=0}^{N-1} \tilde{x}_1[m]W_N^{mk}\right)\tilde{X}_2[k] = \tilde{X}_1[k]\tilde{X}_2[k]
\end{aligned}$$

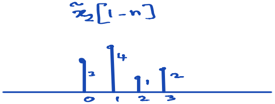
Periodic Convolution: Example



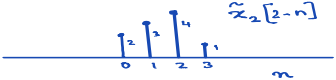


Will use common period of N=4
Only concentrate over index n= 0,1, 2,3.

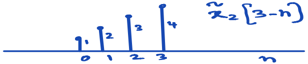




Periodic: So we just repeat.



One period shown



One period shown

$$\tilde{x}[0] = 4\times 1 + 1\times 2 + 2\times 1 + 3\times 2 = 14$$
$$\tilde{x}[n] = \{14, 16, 14, 16\}$$

Complex Conjugate Operation

$$\tilde{x}[n] \stackrel{N}{\longleftrightarrow} \tilde{X}[k] \text{ then } \tilde{x}^*[n] \stackrel{N}{\longleftrightarrow} \tilde{X}^*[-k] = \tilde{X}^*[N-k].$$

Proof:

$$\begin{aligned}
\mathcal{F}_s(\tilde{x}^*[n]) &= \sum_{n=0}^{N-1} \tilde{x}^*[n] W_N^{-nk} = \left(\sum_{n=0}^{N-1} \tilde{x}[n] W_N^{nk} \right)^* \\
&= \left(\sum_{n=0}^{N-1} \tilde{x}[n] W_N^{-n(-k)} \right)^* = \tilde{X}^*[-k]
\end{aligned}$$

Real Sequences

$$\tilde{x}[n] \xleftrightarrow{N} \tilde{X}[k] \text{ then } \tilde{x}^*[n] \xleftrightarrow{N} \tilde{X}^*[-k] = \tilde{X}^*[N-k].$$

For real sequences, $\tilde{x}[n] = \tilde{x}^*[n]$. Hence

$$\tilde{X}[k] = \tilde{X}^*[-k] = \tilde{X}^*[N-k]$$

If $\tilde{X}[k] = |\tilde{X}[k]|e^{j\phi[k]}$, then $\tilde{X}[-k] = |\tilde{X}[-k]|e^{j\phi[-k]}$. Substituting in above equation

$$|\tilde{X}[k]|e^{j\phi[k]} = |\tilde{X}^*[-k]|e^{-j\phi[-k]} = |\tilde{X}[-k]|e^{-j\phi[-k]} = |\tilde{X}[N-k]|e^{-j\phi[N-k]}$$

This implies the following symmetries:

$$|\tilde{X}[k]| = |\tilde{X}[-k]| = |\tilde{X}[N-k]|$$

and

$$\phi[k] = -\phi[-k] = -\phi[N-k]$$

Discrete Time Fourier Transform of Periodic Sequences

$$\tilde{x}[n] \stackrel{?}{\longleftrightarrow} \tilde{X}(e^{j\omega})$$

We will use the following observation about exponential sequences

$$e^{j\omega_0 n} \longleftrightarrow 2\pi \sum_{r=-\infty}^{\infty} \delta(\omega - \omega_0 + 2\pi r)$$

and so

$$W_N^{-nk} = e^{j\frac{2\pi}{N}kn} \longleftrightarrow 2\pi \sum_{r=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{N}k + 2\pi r)$$

Hence

$$\begin{aligned} \tilde{x}[n] &= \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] e^{j\frac{2\pi}{N}kn} \stackrel{linearity}{\longleftrightarrow} \frac{2\pi}{N}\sum_{k=0}^{N-1} \tilde{X}[k] \sum_{r=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{N}k + 2\pi r) \\ &= \frac{2\pi}{N}\sum_{k=-\infty}^{\infty} \tilde{X}[k]\delta(\omega - \frac{2\pi}{N}k) \end{aligned}$$

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j\frac{2\pi}{N}kn} \overset{linearity}{\longleftrightarrow} = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\tilde{X}[k]\delta(\omega - \frac{2\pi}{N}k)$$

DTFT of periodic sequences

$$\tilde{x}[n] \longleftrightarrow \tilde{X}[k] \longleftrightarrow \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\tilde{X}[k]\delta(\omega - \frac{2\pi}{N}k)$$

A DTFT which has delta functions uniformly spaced implies a periodic time domain sequence.

$$\frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\tilde{X}[k]\delta(\omega - \frac{2\pi}{N}k) \longleftrightarrow \tilde{X}[k] \longleftrightarrow \tilde{x}[n]$$