ECE 251A: Digital Signal Processing I
Wide Sense Stationary Processes I

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Random/Stochastic Process (RP)

A discrete RP is a infinitely indexed collection of random variables $\{x[n], n \in \mathcal{Z}\}$, where $\mathcal{Z} = \{..., -2, -1, 0, 1, 2, ...\}$, defined over a common probability space.

Formally, the starting point is $(\Omega, \mathcal{F}, P)$, where Ω is the sample space, $\mathcal{F}$ is a field consisting of events, and P is the probability measure.

Random Variable: $X : \Omega \to \mathcal{R}$. Mapping from Ω to the reals. $\zeta \in \Omega \to X[\zeta]$.

Random Process: $x[n] : \Omega \to$ sequences. $\zeta \in \Omega \to x[n, \zeta]$.

Interpretation

- For a given ζ, $x[n, \zeta]$ is a sequence (realization). RP $x[n]$ is a collection (ensemble) of sequences.
- For a fixed n, $x[n, \zeta]$ is a random variable. RP $x[n]$ is an infinite collection of random variables $x[n_1], x[n_2], ...$

Example I

$x[n] = A\cos(\omega_0 n + \phi)$, where ϕ is a random variable uniform between $[-\pi, \pi]$.

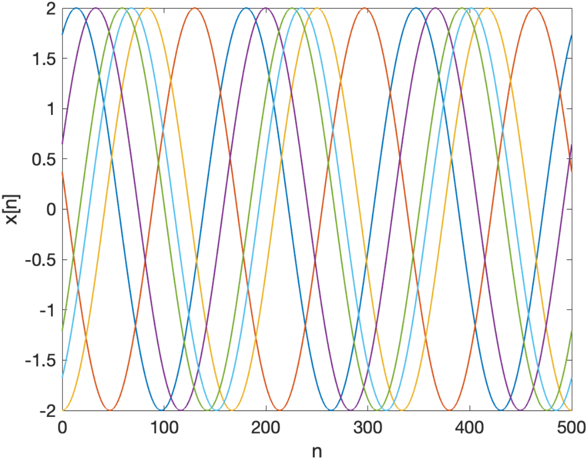

Figure: Six realizations

Examples

1. $x[n] = A\cos(\omega_0 n + \phi)$, where ϕ is a random variable uniform between $[-\pi, \pi]$.
2. $x[n] = w[n]$, where $w[n]$ is a i.i.d. sequence of Gaussian random variables with mean zero and variance 1.
3. $x[n] = ax[n-1] + w[n], |a| < 1$, where $w[n]$ is a i.i.d. sequence of Gaussian random variables with mean zero and variance 1. $x[n]$ is the output of a LTI system $H(z) = \frac{1}{1-az^{-1}}$ with input $w[n]$
4. $x[n] = A$, where A is a random variable uniform between $[-1, 1]$.
5. $x[n] = A\cos(\omega_0 n + \phi)$, where ϕ is a random variable uniform between $[-\pi, \pi]$ and A is a random variable independent of ϕ that is uniform between $[-1, 1]$.

Wide Sense Stationary (WSS) Process

Mean: $E(x[n]) = \mu$. Not a function of time

Autocorrelation: $E(x[n]x^*[n-m]) = r[m]$. Only a function of the time difference

Autocovariance: $E\left((x[n]-\mu)(x[n-m]-\mu)^*\right) = c[m] = r[m] - |\mu|^2$. Only a function of the time difference

Example 1: $x[n] = A\cos(\omega_0 n + \phi)$, where ϕ is a random variable uniform between $[-\pi, \pi]$.

$E(x[n]) = 0$ and $r[m] = \frac{A^2}{2}\cos(\omega_0 m)$. So process is WSS.

Examples Revisited

1. $x[n] = A\cos(\omega_0 n + \phi)$, where ϕ is a random variable uniform between $[-\pi, \pi]$. WSS: Yes ($\mu = 0$, and $r[m] = \frac{A^2}{2}\cos(\omega_0 m)$.)
2. $x[n] = w[n]$, where $w[n]$ is a i.i.d. sequence of Gaussian random variable with mean zero and variance 1. WSS: Yes ($\mu = 0$, and $r[m] = \delta[m]$.)
3. $x[n] = ax[n-1] + w[n], |a| < 1$, where $w[n]$ is a i.i.d. sequence of Gaussian random variable with mean zero and variance 1. $x[n]$ is the output of a LTI system $H(z) = \frac{1}{1-az^{-1}}$ with input $w[n]$. WSS: Yes ($\mu = 0$, and $r[m]$ shaped by $H(z)$)
4. $x[n] = A$, where A is a random variable uniform between $[-1, 1]$. WSS: Yes ($\mu = 0$, and $r[m] = E(A^2) = \frac{1}{3}$.)
5. $x[n] = A\cos(\omega_0 n + \phi)$, where ϕ is a random variable uniform between $[-\pi, \pi]$ and A is a random variable independent of ϕ that is uniform between $[-1, 1]$.WSS: Yes ($\mu = 0$, and $r[m] = \frac{E(A^2)}{2}\cos(\omega_0 m)$.)

Properties of the Autocorrelation Sequence $r[m]$

$r[m] = E(x[n]x^*[n-m])$

1. $r[0] \geqslant |r[m]|$. (Follows from the Cauchy-Schwartz inequality $|E(XY^*)|^2 \leqslant E(|X|^2)E(|Y|^2)$.)
2. $r[m] = r^*[-m]$ (Hermitian Symmetry)
3. $\{r[m]\}$ is a non-negative definite (positive semi-definite) sequence, i.e. $\sum_{l=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} a_l^* a_p r[p-l] \geqslant 0, \forall \mathbf{a}$. Often we have strict inequality (positive definiteness) $\sum_l \sum_p a_l^* a_p r[p-l] > 0$.

Property 1 and 2 are easy to prove and verify for a given sequence. Proof of property 3 is more involved and not so easy to verify for a given sequence. It is equivalent to the power spectrum (Fourier transform of $r[m]$) being positive.

An interesting problem is given only a few autocorrelation values, how do we extend it? Is there only one extension? If there are multiple, how do we choose?

Vector formulation and the Autocorrelation Matrix

$$\mathbf{x}_M[n] = [x[n], x[n-1], \ldots, x[n-M+1]]^T = \begin{bmatrix} x[n] \\ x[n-1] \\ \vdots \\ x[n-M+1] \end{bmatrix} \text{ is a } M \times 1$$

random vector. Then $E(\mathbf{x}_M[n]) = \mu\mathbf{1}$, where $\mathbf{1}$ is a $M \times 1$ vector with all entries equal to one, i.e. $\mathbf{1} = [1, 1, \ldots, 1]^T$. The autocorrelation matrix is given by

$$\begin{aligned}
\mathbf{R}_M &= E(\mathbf{x}_M[n]\mathbf{x}_M^H[n]) = E(\begin{bmatrix} x[n] \\ x[n-1] \\ \vdots \\ x[n-M+1] \end{bmatrix} [x^*[n], x^*[n-1], \ldots, x^*[n-M+1]]) \\
&= \begin{bmatrix} r[0] & r[1] & r[2] & \ldots & r[M-1] \\ r[-1] & r[0] & r[1] & \ldots & r[M-2] \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ r[-(M-1)] & r(-(M-2) & r[-(eM-3)] & \ldots & r[0] \end{bmatrix}
\end{aligned}$$

Note that we could have defined
$\mathbf{x}_M[n] = [x[n+P], x[n-1+P], \ldots, x[n-M+1+P]]^T$ without any impact.

Properties of $\mathbf{R}_M$

$$\mathbf{R}_M = E(\mathbf{x}_M[n]\mathbf{x}_M^H[n]) = \begin{bmatrix} r[0] & r[1] & \dots & r[M-1] \\ r[-1] & r[0] & \dots & r[M-2] \\ \vdots & \vdots & \vdots & \vdots \\ r[-(M-1)] & r(-(M-2) & \dots & r[0] \end{bmatrix}$$

1. $\mathbf{R}_M = \mathbf{R}_M^H$ (Hermitian Symmetry)
2. $\mathbf{R}_M$ is positive semidefinite, $\mathbf{a}^H\mathbf{R}_M\mathbf{a} \geqslant 0, \forall \mathbf{a}$. (usually positive definite)
3. $\mathbf{R}_M$ is Toeplitz, and defined by the first row. The Toeplitz structure allows for low complexity algorithms. Inversion which is usually $O(M^3)$ complexity can be reduced to $O(M^2)$ complexity.

Proof of Positive Semi-Definiteness of $\mathbf{R}_M$

$$\mathbf{a}^H\mathbf{R}_M\mathbf{a} = \mathbf{a}^H E(\mathbf{x}_M[n]\mathbf{x}_M^H[n])\mathbf{a} = E(\mathbf{a}^H\mathbf{x}_M[n]\,\mathbf{x}_M^H[n]\mathbf{a}) = E(y[n]y^*[n]) = E(|y[n]|^2) \geqslant 0.$$

Note that $y[n] = \mathbf{a}^H\mathbf{x}_M[n] = \sum_{l=0}^{M-1} a_l^* x[n-l]$, output of a FIR filter with filter coefficient $\mathbf{a}^* = [a_0^*, a_1^*, ..a_{M-1}^*]^T$ or $A(z) = \sum_{l=0}^{M-1} a_l^* z^{-l}$ and $\mathbf{a}^H\mathbf{R}_M\mathbf{a} = \sum_{l=0}^{M-1}\sum_{p=0}^{M-1} a_p a_l^* r[p-l]$.

If we assume M odd and define $P = \frac{M-1}{2}$ and $\mathbf{x}_M[n] = [x[n+P], \ldots, x[n], \ldots, x[n-P]]^T$, we have a non-causal filter $A(z) = \sum_{l=-P}^{P} a_l^* z^{-l}$ and $\mathbf{a}^H\mathbf{R}_M\mathbf{a} = \sum_{l=-P}^{P}\sum_{p=-P}^{P} a_p a_l^* r[p-l]$.

Note that $\mathbf{a}^H\mathbf{R}_M\mathbf{a}$ as $M \to \infty$ leads to $\sum_{l=-\infty}^{\infty}\sum_{p=-\infty}^{\infty} a_l^* a_p r[p-l]$, the general definition of positive semi-definiteness of an autocorrelation sequence.

Eigendecomposition of $\mathbf{R}_M$

Eigendecomposition: $\mathbf{R}_M = Q\Lambda Q^H$,

where Q is a $M \times M$ matrix containing the orthonormal eigenvectors, i.e. $Q = [q_1, q_2, \dots, q_M]$ and $QQ^H = Q^H Q = \mathbf{I}_M$.

Λ is a diagonal matrix containing the non-negative eigenvalues, i.e. $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, .., \lambda_M)$ with $\lambda_i \geqslant 0$.

Consequences of the Eigendecomposition: $\mathbf{R}_M = Q\Lambda Q^H$.

- $\det\mathbf{R}_M = \det Q \, \det\Lambda \, \det Q^H = \det(QQ^H) \, \det\Lambda = \det\mathbf{I} \, \det\Lambda = \Pi_{i=1}^{M} \lambda_i$
- Trace of $\mathbf{R}_M$:
  $\mathrm{Tr}\mathbf{R}_M = Mr[0] = \mathrm{Tr}(Q\Lambda Q^H) = \mathrm{Tr}(\Lambda Q^H Q) = \mathrm{Tr}(\Lambda) = \sum_{i=1}^{M} \lambda_i$