SIO 209: Signal Processing for Ocean Sciences
Class 6 & 7

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Complex Scalar Random Variables

Ensembles Averages

- Average or mean $\mathrm{E}[x] = \int_{-\infty}^{\infty} x\, p(x)\mathrm{d}x$
- Mean squared $\mathrm{E}[x^2] = \int_{-\infty}^{\infty} x^2\, p(x)\mathrm{d}x$
- Variance $\mathrm{var}[x] = \mathrm{E}\big[(x-\mu)^2\big] = \mathrm{E}\big[x^2\big] - \mu^2 = \sigma^2$
- Autocorrelation sequence

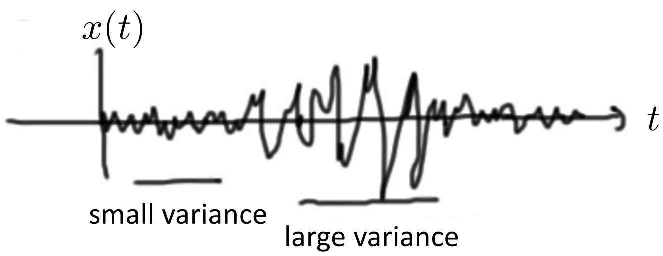
$$\phi_{xx}[n,m] = \mathrm{E}\{x[n]x^*[m]\} \quad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x[n]\, x^*[m]\, p(x[n],x[m])\mathrm{d}x[n]\mathrm{d}x[m]$$

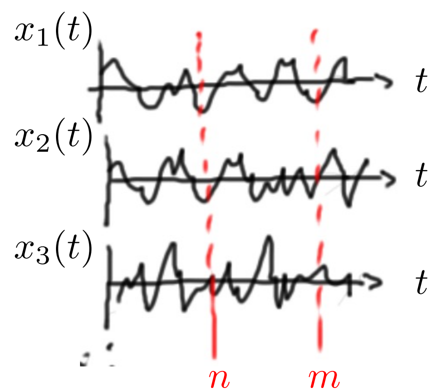
- Cross-correlation sequence

$$\phi_{xy}[n,m] = \mathrm{E}\{x[n]y^*[m]\}$$

Time Dependence of Random Processes

- In general, statistics are a function of time, i.e., $p(x[n])$, $p(x[m])$, and $p(x[n], x[m])$ are a function of time indexes m and n





Time Dependence of Random Processes

Important Properties

- *Wide Sense Stationary (WSS):* mean and autocorrelation sequences (first and second-order moments) are not time-dependent
- *Stationary:* all moments are not time-dependent
- *Ergodic:* temporal averages are the same as ensemble averages
- *Independent and Identically distributed (IID):* all samples follow the same distribution but are statistically independent
- A random process that is stationary is also WSS but not vice versa
- A random process that is IID is ergodic but not vice versa

Sinusoid in Noise

- Signal Model:

amplitude　frequency [rad/samples]　phase

$$x[n] = \underbrace{A\cos(\omega n + \phi)}_{\text{signal}} + \underbrace{w[n]}_{\text{noise (assumed IID)}}$$

- Power of signal is given by $A^2/2$
- Power of noise is given by $\mathrm{E}[w^2]$ or by σ_w^2, in case $w[n]$ is zero-mean
- Signal to noise ratio of sinusoid in zero mean IID noise

$$\mathrm{SNR} = A^2/(2\sigma_w^2)$$

Time Dependence of Random Processes

- For wide-sense stationary sequences, we have

$$\mathrm{E}\{x[n]\} = \mathrm{E}\{x[m]\} \quad \text{and} \quad \mathrm{E}\{x[n]^2\} = \mathrm{E}\{x[m]^2\} \qquad \forall n, m$$

- Furthermore, the autocorrelation sequence can be simplified as follows

$$\phi_{xx}[n,m] \to \phi_{xx}[m]$$

where now m is the separation or lag between the two time samples

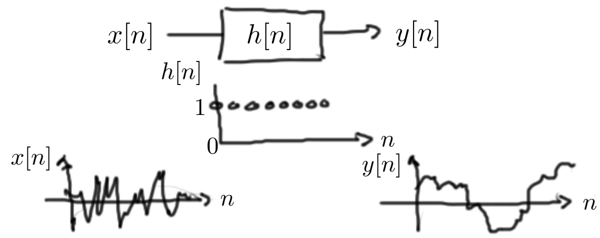
$$\phi_{xx}[m] = \mathrm{E}\{x[n]x^*[n+m]\}$$

(time step n is arbitrary)

- $c_{xx}[m]$ denotes a computational estimation of $\phi_{xx}[m]$ (see homework 3)

Response of Linear System to Random Sequence

- Let is assume we have a WSS process at the input of a linear time-invariant system



$$y[n] = \sum_{k=-\infty}^{\infty} h[n-k]x[k]$$
$$= \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$\mu_y[n] = \mathrm{E}\{y[n]\} = \sum_{k=-\infty}^{\infty} h[k]\,\mathrm{E}\{x[n-k]\} = \mathrm{E}\{x[n]\} \sum_{k=-\infty}^{\infty} h[k]$$ (Eq. 184)

$x[n]$ is WSS

see also Chapter 2.10 and Appendix A.3 in *Oppenheim & Schafer, 2009*

Response of Linear System to Random Sequence

$$\phi_{yy}[m] = \mathrm{E}\{y[n]y^*[n+m]\}$$ (Eq. 187)
$$= \sum_{l=-\infty}^{\infty} \phi_{xx}[m-l] \sum_{k=-\infty}^{\infty} h[k]\,h^*[k+l]$$
$$= \sum_{l=-\infty}^{\infty} \phi_{xx}[m-l]\,c_{hh}[l]$$ where $c_{hh}[l] = \sum_{k=-\infty}^{\infty} h[k]\,h^*[k+l]$

$$\phi_{xy}[m] = \mathrm{E}\{x[n]y^*[n+m]\}$$
$$= \sum_{k=-\infty}^{\infty} h^*[k]\,\phi_{xx}[m-k]$$ (Eq. 195)

Power Spectral Density Estimation

- Let is assume we have a WSS process at the input of a linear time-invariant system

Fourier transform

$$P_{xx}(\omega) = \Upsilon\{\phi_{xx}[m]\} \qquad \text{(Eq. A46)}$$

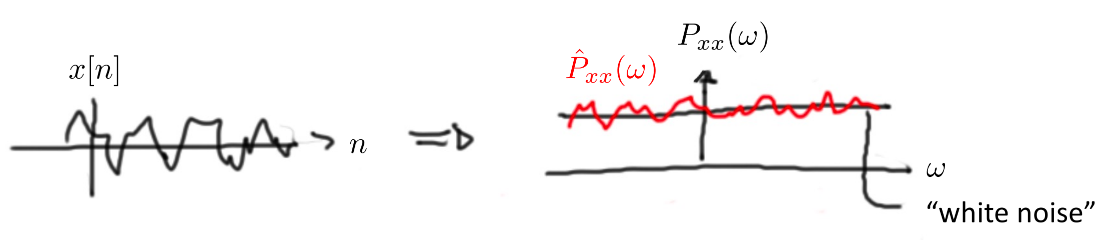
$$\phi_{xx}[m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xx}(\omega)\, e^{j\omega m}\, \mathrm{d}\omega \qquad \text{(Eq. A44b)}$$

- $P_{xx}(\omega)$ is always real valued

$$\phi_{xx}[0] = \mathrm{E}\{x[n]^2\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xx}(\omega)\mathrm{d}\omega$$

Power Spectral Density Estimation

- Let $x[n]$ be an IID random sequence $\Longrightarrow$ power spectral density is "flat"



- $\hat{P}_{xx}(\omega)$ is the estimate of the power spectrum $P_{xx}(\omega)$
- Our goal is to define an estimate of the power spectrum $\hat{P}_{xx}(\omega)$ that is "good"

Cross Spectral Density Function

- Cross-Power Spectral Density Function

$$P_{xy}(\omega) = \Upsilon\{\phi_{xy}(\omega)\} \qquad \text{(Eq. A49)}$$

$$= H^*(e^{j\omega})P_{xx}(\omega) \qquad \text{(Eq. A52)}$$

$$P_{yx}(\omega) = H(e^{j\omega})P_{xx}(\omega)$$

$$P_{xy}(\omega) = P^*_{yx}(\omega) \qquad \text{(Eq. A50)}$$

$$P_{yy}(\omega) = \left|H(e^{j\omega})\right|^2 P_{xx}(\omega)$$

Note that *Oppenheim & Schafer, 2009* uses the alternative definitions

$$\phi_{xx}[m] = \mathrm{E}\{x[n+m]x^*[m]\} \qquad \phi_{xy}[m] = \mathrm{E}\{x[n+m]y^*[m]\}$$

Estimation of Moments

- If $x[n]$ is ergodic, we can calculate an estimate of its mean $\mu_x = \mathrm{E}\{x[n]\}$ and variance $\sigma_x^2 = \mathrm{E}\{(x[n]-\mu_x)^2\}$ using N samples

$$\hat{\mu}_x = \frac{1}{N}\sum_{n=0}^{N-1} x[n] \qquad \hat{\sigma}_x^2 = \frac{1}{N}\sum_{n=0}^{N-1}\left(x[n]-\hat{\mu}_x\right)^2$$

- Are $\hat{\mu}_x$ and $\hat{\sigma}_x^2$ “good” estimates of μ_x and σ_x^2? For $x[n]$ IID, the mean and variance of the estimator read

0 for $N \to \infty$

$$\mathrm{E}\left[\hat{\mu}_x\right] = \frac{1}{N}\sum_{n=0}^{N-1}\mathrm{E}\{x[n]\} = \mu_x \qquad \mathrm{var}[\hat{\mu}_x] = \frac{1}{N}\sigma_x^2$$

(see next slide)

- As the number of samples N increases, the estimates converge to the true moments

Estimation of Moments

$$
\begin{aligned}
\mathrm{var}[\hat{\mu}_x] &= E\big[(\hat{\mu}_x - \mu_x)^2\big] \\
&= E\Big\{\Big(\frac{1}{N}\sum_{n=0}^{N-1} x[n] - \mu_x\Big)^2\Big\} \\
&= \frac{1}{N^2}\,E\Big\{\Big(\sum_{n=0}^{N-1}\big(x[n] - \mu_x\big)\Big)^2\Big\} \\
&= \frac{1}{N^2}\sum_{n=0}^{N-1} E\Big\{\big(x[n] - \mu_x\big)^2\Big\} \quad \longleftarrow \quad \text{(since } x[n] \text{ is IID)} \\
&= \frac{1}{N^2}\sum_{n=0}^{N-1} \sigma_x^2 \\
&= \frac{1}{N}\sigma_x^2
\end{aligned}
$$