SIO 207A: Fundamentals of Digital Signal Processing
Class 7

Florian Meyer

Scripps Institution of Oceanography
Electrical and Computer Engineering Department
University of California San Diego

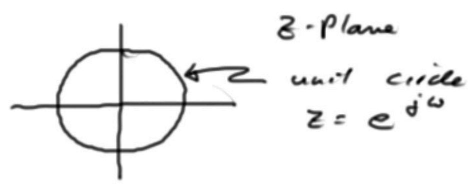
SCRIPPS INSTITUTION OF OCEANOGRAPHY

UC San Diego
JACOBS SCHOOL OF ENGINEERING

Recall Fourier Transform

- Fourier transform is z-transform evaluated on the unit circle

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$



$$X(z)\big|_{z=e^{j\omega}} = X(e^{j\omega})$$

Recall Fourier Transform

- Analysis:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad (1)$$

- Synthesis:

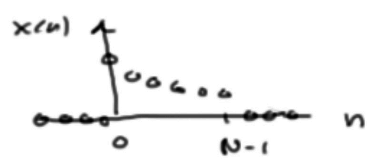
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,\mathrm{d}\omega \qquad (2)$$

Discrete Fourier Series

- **A.** Discrete Fourier Series

Assume $x[n]$ has finite length N,i.e.,

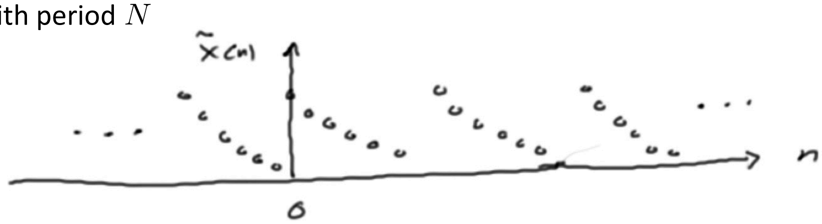
$$x[n] = 0, \quad n < 0 \text{ and } n \geq N$$



Form periodic replication $\tilde{x}[n]$ such that $\tilde{x}[n + lN] = x[n]$

$$n = 0, \ldots, N-1 \quad \text{and } l \in \mathbb{N}$$

$\tilde{x}[n]$ is periodic with period N



Discrete Fourier Series

$\tilde{x}[n]$ can be represented by a (complex exponential) Fourier series

- harmonically related sequences $e^{j\frac{2\pi}{N}kn}$ $\begin{cases} n \text{ is a sample index} \\ k \text{ is a frequency index} \end{cases}$

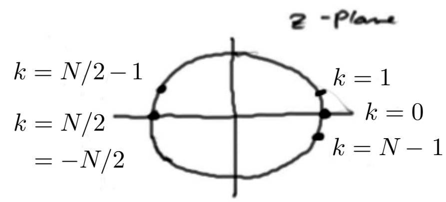
frequency:

$$\omega = \frac{2\pi}{N}k$$

- is periodic in k with period N

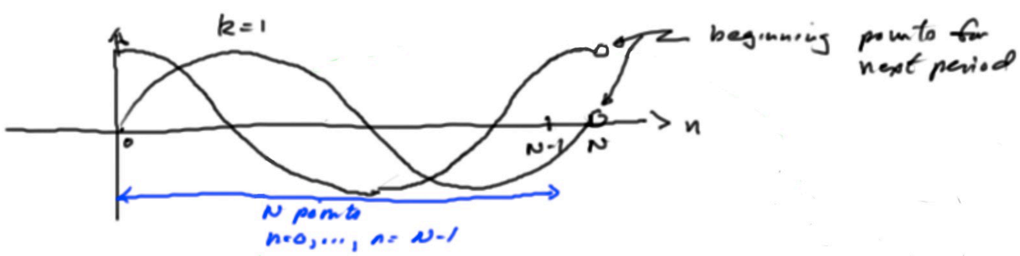
  only sequences for $k = 0, \dots, N-1$ are required

- interpret k as the number of cycles per period



$$k = \{0, 1, \dots, N-1\}$$
$$= \{-N/2, \dots, 0, \dots, N/2 - 1\}$$

Discrete Fourier Series



$$e^{j\frac{2\pi}{N}kn} = \cos\left(\overset{=\omega}{\boxed{\frac{2\pi}{N}k}}n\right) + j\sin\left(\overset{=\omega}{\boxed{\frac{2\pi}{N}k}}n\right)$$

Discrete Fourier Series

- Discrete Fourier Series

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}(k)e^{j\frac{2\pi}{N}kn}$$

- Determine $\tilde{X}(k)$

$$\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi}{N}kn} = \sum_{k'=0}^{N-1}\tilde{X}(k')\underbrace{\boxed{\frac{1}{N}\sum_{n=0}^{N-1}e^{j\frac{2\pi}{N}n(k'-k)}}}_{=\begin{cases}0 \text{ for } k \neq k' \\ 1 \text{ for } k = k'\end{cases}}$$

$$= \tilde{X}(k)$$

Discrete Fourier Series

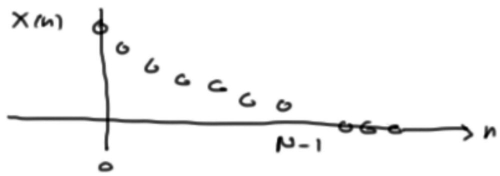
- Analysis and Synthesis Pair:

$$\tilde{X}(k) = \sum_{n=0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi}{N}kn}$$

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}(k)e^{j\frac{2\pi}{N}kn}$$

Discrete Fourier Transform

- Finite Duration Sequence



Fourier transform is z-transform evaluated on unit circles

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$
$$= \sum_{n=0}^{N-1} x[n]\, z^{-n}$$

$$\tilde{X}(k) = X(z)\Big|_{z=e^{j\frac{2\pi}{N}k}} \qquad \omega = \frac{2\pi}{N}k$$

Note: Samples of the Fourier transform lead to a periodic replication of the underlying finite sequence

Discrete Fourier Transform

- Analysis:

$$X(k) = X(e^{j\omega})\Big|_{\omega=\frac{2\pi}{N}k}$$
$$= \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

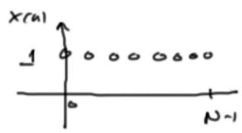
- Synthesis:

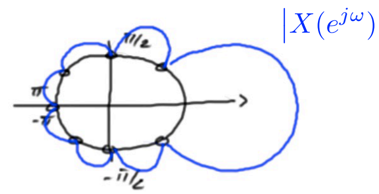
$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn}$$

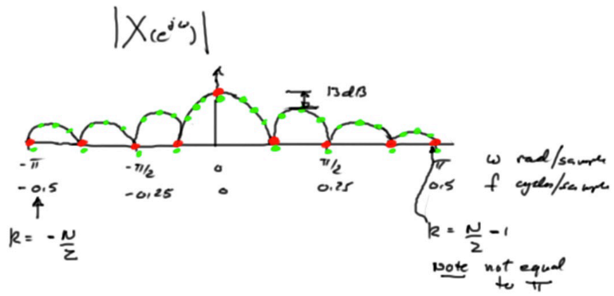
sampling $\Longrightarrow$ periodicity

Zero Padding

- Zero padding does not change the underlying Fourier transform just enables a more dense sampling







$N_{\mathrm{DFT}} = 8$ point DFT

$N_{\mathrm{DFT}} = 32$ point DFT

(zero padding)

DFT / FFT Bins

$$X(k) = \sum_{n=0}^{N-1} x[n]\, e^{-j\underbrace{\boxed{\frac{2\pi}{N}nk}}_{\omega_k n}} \qquad \omega_k = \frac{2\pi}{N}k$$

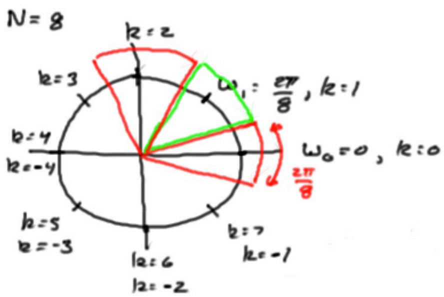
spacing in frequency domain

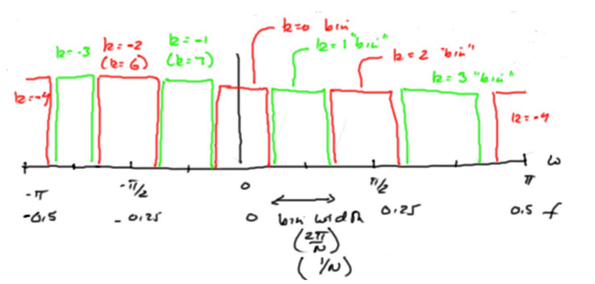
integer k $\{0, \ldots, N-1\}$

"bin index" $\{-N/2, \ldots, 0, \ldots, N/2-1\}$

each "bin" covers a bandwidth equal to $2\pi/N$ rad/samples (or $1/N$ cycles/samples)

edges of bins are $\pm\pi/N$ rad/samples from each center frequency





DFT / FFT Bins

- With "cookie cutter" viewpoint, there is no leakage of signal energy from one bin region with another, i.e., high level signal in bin region $k = 3$ will not show up in bin $k = 1$ after DFT – <u>not</u> true in practice