# SIO 207A: Fundamentals of Digital Signal Processing
# Class 2

Florian Meyer

Scripps Institution of Oceanography
Electrical and Computer Engineering Department
University of California San Diego

SCRIPPS INSTITUTION OF OCEANOGRAPHY

UC San Diego
JACOBS SCHOOL OF ENGINEERING

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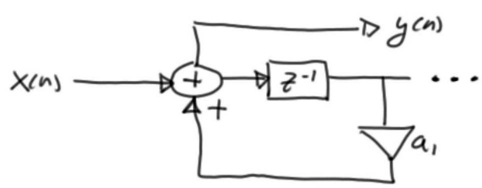
## Infinite Impulse Response (IIR) Filter

- Consider IIR filter with just a single coefficient:




- **A: The feedback creates poles**

  $a_1 = +1,$ system perfectly reinforces the unit step sequence, $u[n] = \begin{cases} 1, & n \geqslant 0 \\ 0, & \text{otherwise} \end{cases}$

  $a_1 = -1,$ system perfectly reinforces the unit alternating sequence

  unstable when $|a_1| \geqslant 1$

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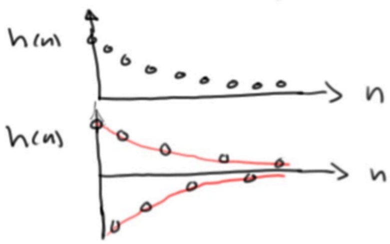
# Infinite Impulse Response (IIR) Filters

- **B: Unit sample response** is a geometric sequence: $h[n] = (a_1)^n$
- Examples

$a_1 = 0.9$

$a_1 = -0.9$




Convolution:

$$y[n] = h[0]x[n] + h[1]x[n-1] + \dots$$
$$= \sum_{k=0}^{\infty} h[k]x[n-k]$$

- **C: Filter Description (input/output)**

Difference Equation:

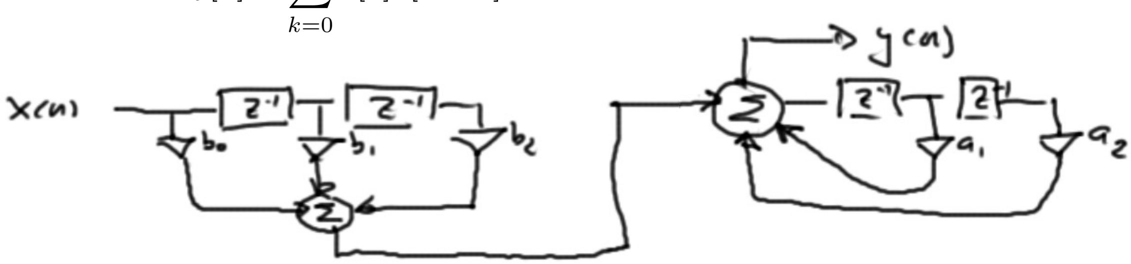
$$y[n] = a_1 y[n-1] + a_2 y[n-2] + \cdots + a_N y[n-N] + x[n]$$
$$= \sum_{k=1}^{N} a_k y[n-k] + x[n]$$





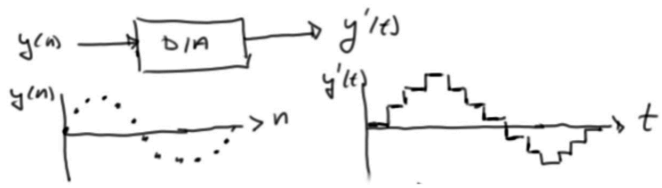
# General Infinite Impulse Response (IIR) Filters

- In general, IIR filters have both feedforward and feedback sections
- **A. Filter description (input/output)**
- Difference Equation: $y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{r=0}^{M} b_r x[n-r]$
- Convolution: $y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$



# Analog System Output

- Analog system output is obtained after digital and analog (D/A) conversion as well as low-pass filtering




- D/A converter
  - "holds" the digital value
  - can be modeled as analog filter with impulse response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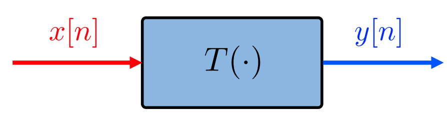
# Linear Shift-Invariant System

Arbitrary Input Signal:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

(representation in terms of delayed unit sample functions)




Transformation $T(\cdot)$

Output Signal:

$$y[n] = T\left(\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]\right)$$

$$= \sum_{k=-\infty}^{\infty} x[k]\,T\big(\delta[n-k]\big) \longleftarrow \text{Linearity}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\,h_k(n)$$

$$= \sum_{k=-\infty}^{\infty} x[k]\,h(n-k) \longleftarrow \text{Shift Invariance}$$

Unit sample response $h[n]$ is a fundamental quantity of interest since it fully describes the linear shift-invariant system

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# Important Class of Linear Shift-Invariant System

- An important class of linear shift-invariant systems consist of those whose input/output relation satisfies a linear constant coefficient difference equation

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{r=0}^{M} b_r x[n-r]$$


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# Stability and Causality

- Stable System: Every bounded input produces a bounded output
- Linear Shift-Invariant system stable iff

$$S \triangleq \sum_{k=-\infty}^{\infty} \left|h[k]\right| < \infty$$ absolutely summable

- Causal system is one in which changes in the output do not precede changes in the input (i.e., non-anticipating)

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# Discrete-Time Convolution

- The output of a linear shift-invariant system is the convolution of the input signal and the unit sample response:

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]\boxed{h(n-m)}$$
$$= x[n] * h[n]$$


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# Frequency Domain Representation

- Often decompositions of the input signal are valuable
- Assume:

$$x[n] = e^{jwn} \qquad -\infty < n < \infty$$

complex exponential sequence

$$= \cos(wn) + j\sin(wn)$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]e^{jw(n-k)}$$
$$= e^{jwn}\sum_{k=-\infty}^{\infty} h[k]e^{-jwk}$$

Define: $H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-jwk}$

$\longrightarrow$ $y[n] = H(e^{j\omega})e^{j\omega n}$

$H(e^{j\omega})$ Describes alternation in phase and magnitude of the complex signal passing through system

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# Frequency Domain Representation

- $H(e^{j\omega})$ is
  - the frequency response of a system with unit sample response $h[n]$
  - is a complex quantity, i.e., $H(e^{j\omega}) = |H(e^{j\omega})| e^{j \arg\{H(e^{jw})\}}$

- **Note:** $x[n] = A\cos\omega n = \dfrac{A}{2}\left[e^{j\omega n} + e^{-j\omega n}\right]$

- **Note:** sampling in time domain $\Longleftrightarrow$ periodicity in frequency domain

  $H(e^{j\omega})$ is in $\omega$ with period $2\pi$



# Frequency Domain Representation

- Since $H(e^{j\omega})$ is periodic it can be represented as a Fourier series (i.e., harmonically related sinusoids)

$$H(e^{j\omega}) \triangleq \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \qquad (1)$$

- Fourier coefficients are the unit sample response $h[n]$ and from the definition of Fourier series

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}\,\mathrm{d}\omega \qquad (2)$$

- Equations (1) and (2) are a Fourier transform pair

# Frequency Domain Representation

- In more general context, we will write the Fourier transform and inverse transform as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad \text{periodic with period } 2\pi$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\,e^{j\omega n}\mathrm{d}\omega$$

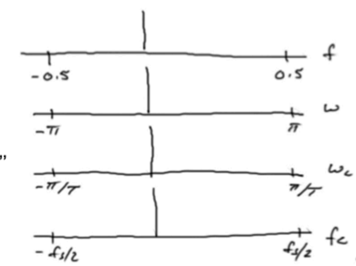
- Four ways to represent frequency domain axis is discrete-time signal processing →

``Normalized Frequency'' $f_c/f_s$ [cycles/sample]

$\omega_c/f_s$ [rad/samples]

``Cont. or Analog Frequency'' [rad/sec]

[cycles/sec] or [Hz]




(Note that $T$ and $f_s = 1/T$ are sampling interval and sampling frequency, respectively.)

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# Input/Output Relation in Frequency Domain

- The response of a linear system due to each complex exponential into which the input $x[n]$ is decomposed is $H(e^{j\omega})$

- Thus, we have

$$y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\,X(e^{j\omega})\,e^{j\omega n}\mathrm{d}\omega$$

$$Y(e^{j\omega}) = H(e^{j\omega})\,X(e^{j\omega})$$

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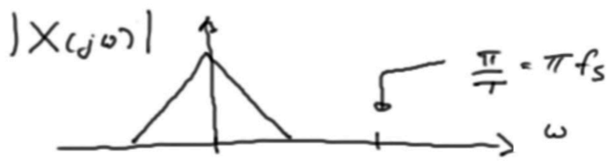
# Sampling Theorem

- Sampling of continuous-time signals
- Recall sampling $\Longleftrightarrow$ periodicity
- <u>Sampling Theorem:</u>

  If the Fourier transform of a continuous-time signal $x(t)$ is zero for all $\omega > 2\pi B$ ("Bandwidth" in Hz: $B$)

  $$X(j\omega) = 0, \quad \omega > 2\pi B$$

  Then $x(t)$ is uniquely determined from its uniformly sampled values if the sampling period is selected to satisfy $T \le 1/2B$, i.e., sampling rate $f_s \ge 2B$

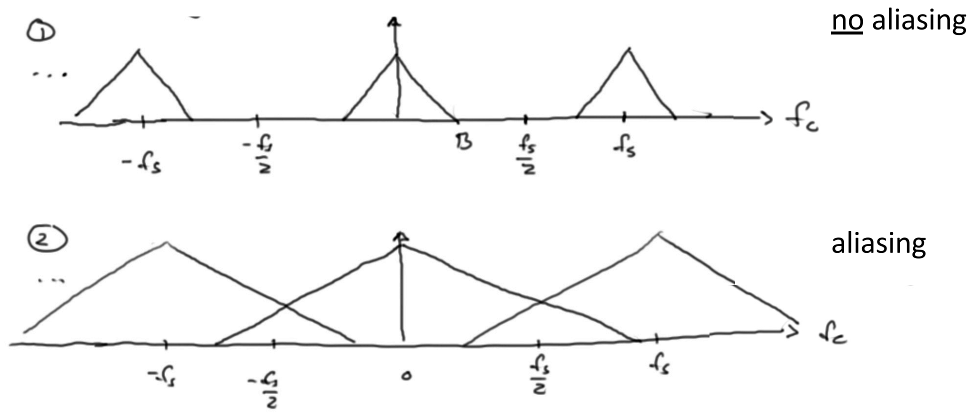







# Aliasing

- Spectral representation after sampling:




- In ① the original analog time-series can be recovered from its samples and in ② it cannot due to aliasing