ECE 286: Bayesian Machine Perception

Class 5: Undirected Graphical Models

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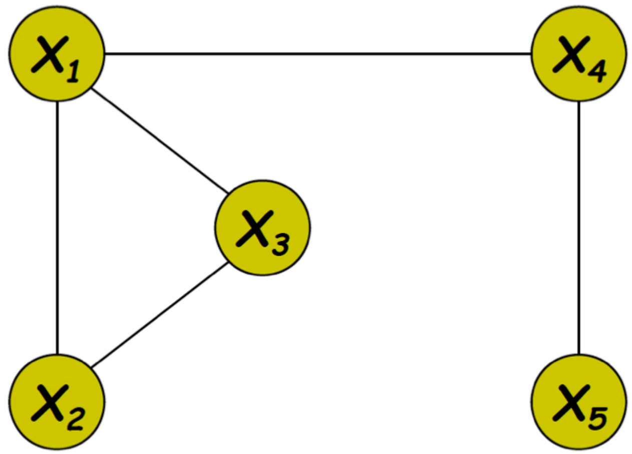
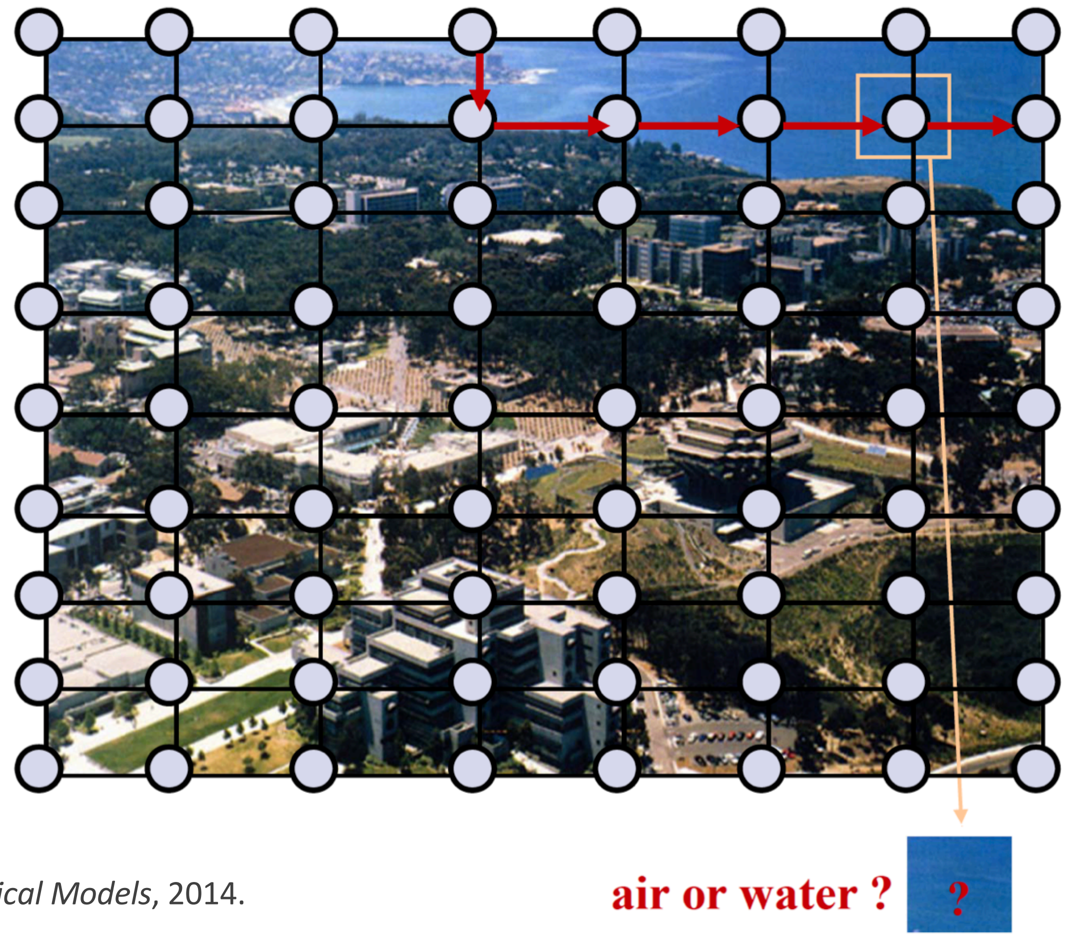
Undirected Graphical Models



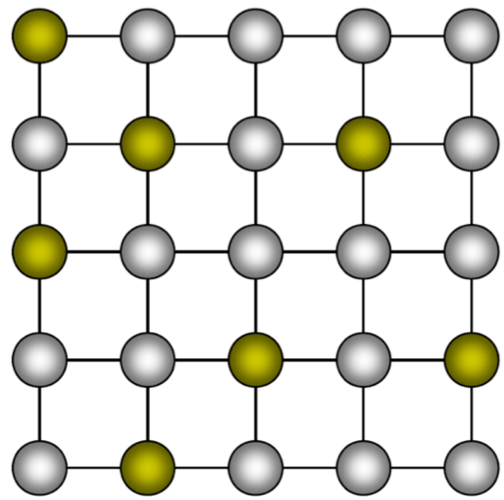
- Model relationships of random variables
- Interactions between *nodes* (or random variables) are *symmetric* in nature
- Rules to extract conditional independencies of variable
- Rules to get numerical explicit joint distributions

Example: Understanding Complex Scenes



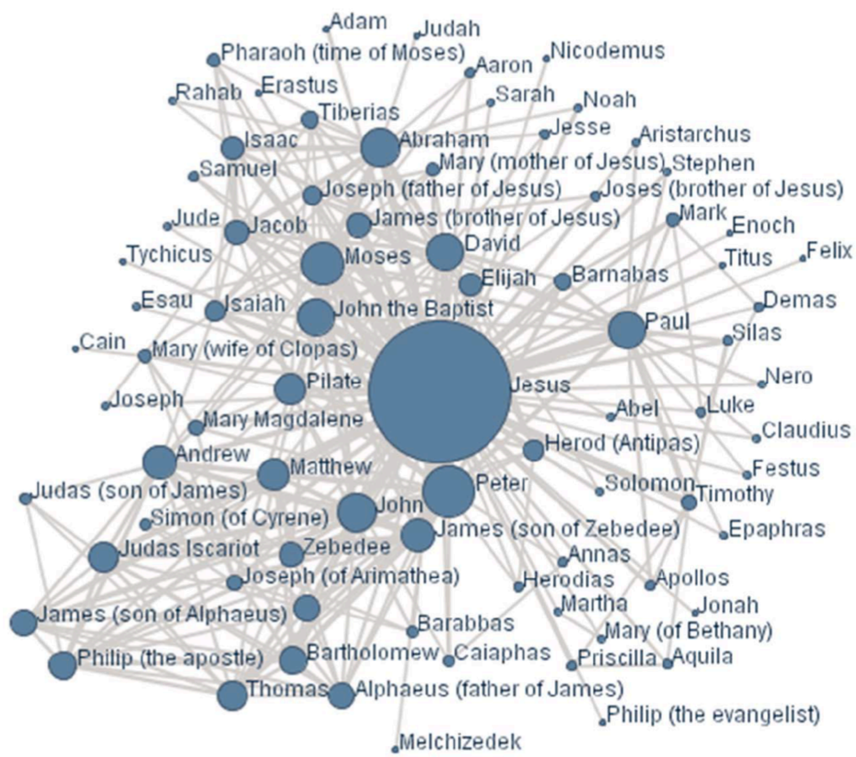
E. Xing, *Probabilistic Graphical Models*, 2014.

Example: The Grid Model



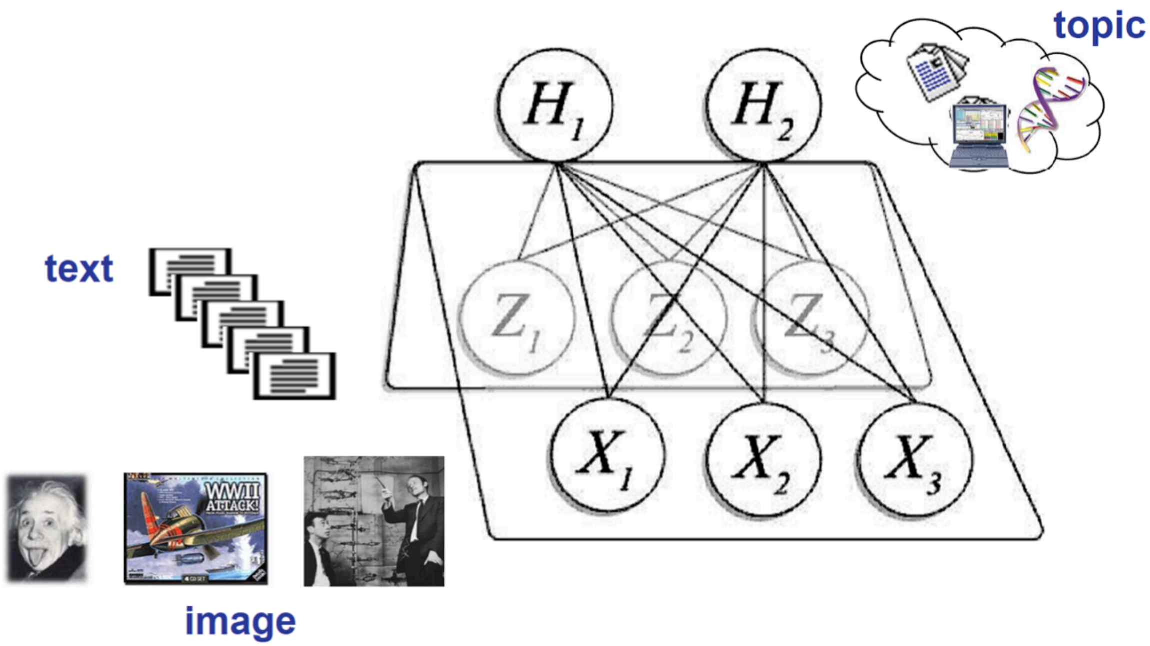
- Naturally arises in image processing, physics, …
- Each node may represent a single ``pixel’’ or an atom
  - States of adjacent or nearby nodes are ``coupled’’ due to pattern continuity or electro-magnetic force, etc.
  - Infer most likely joint configuration from noisy/partial observations

Example: Social Networks



The New Testament

Example: Information Retrieval

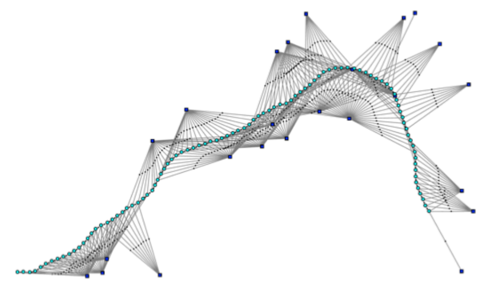


Example: Machine Perception

- **Data:** Stream of measurements from heterogeneous sensors



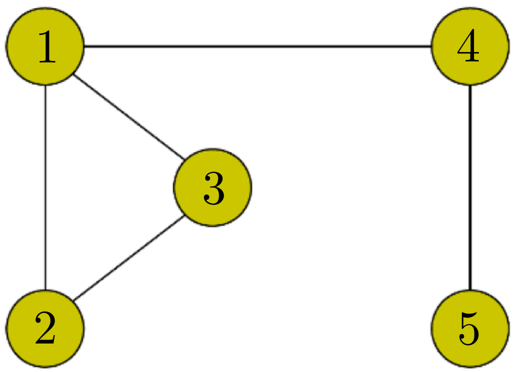

- **Goal:** Infer navigation information, objects in the environment, the floorplan, …



- Graphical representations are useful for
  - modeling large-scale Bayesian problems
  - perform inference
  - establish new scalable algorithms tailored to machine perception applications

Undirected Graphs and Markov Random Fields

- Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected graph with nodes $\mathcal{V}$ and edges $\mathcal{E}$
- *Example:*



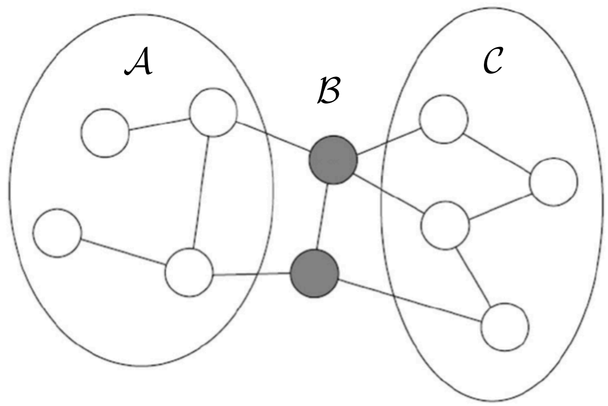
$$\mathcal{V} = \{1, 2, 3, 4, 5\}$$

$$\mathcal{E} = \{(1,2), (1,3), (1,4), (4,5)\}$$

- A set of random variables $\mathcal{X}_{\mathcal{G}} = \{\boldsymbol{x}_v\}_{v \in \mathcal{V}}$ form a Markov random field $(\mathcal{X}_{\mathcal{G}}, \mathcal{G})$ if they satisfy the following Markov properties

Global Markov Property

- Let $\mathcal{G}$ be an undirected graph with subgraphs $\mathcal{A}, \mathcal{B}, \mathcal{C}$



- $\mathcal{B}$ separates $\mathcal{A}$ and $\mathcal{C}$ if every path from a node in $\mathcal{A}$ to a node in $\mathcal{C}$ passes through a node in $\mathcal{B}$:

$$\mathrm{sep}_{\mathcal{G}}(\mathcal{A}, \mathcal{C} | \mathcal{B})$$

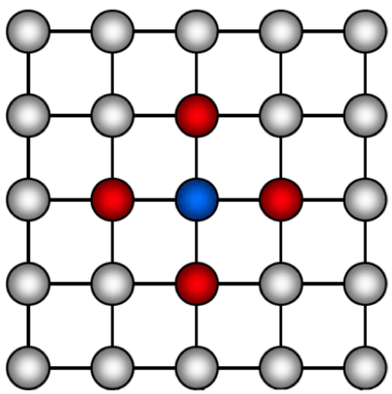
- The global Markov property is satisfied if for any disjoint $\mathcal{A}, \mathcal{B}, \mathcal{C}$ such that $\mathcal{B}$ separates $\mathcal{A}$ and $\mathcal{C}$, $\boldsymbol{x}_{\mathcal{A}}$ is independent of $\boldsymbol{x}_{\mathcal{C}}$ given $\boldsymbol{x}_{\mathcal{B}}$: $\mathrm{sep}_{\mathcal{G}}(\mathcal{A}, \mathcal{C} | \mathcal{B}) \implies \boldsymbol{x}_{\mathcal{A}} \perp\!\!\!\perp \boldsymbol{x}_{\mathcal{C}} \,|\, \boldsymbol{x}_{\mathcal{B}}$

Local Markov Property

- For each node $\boldsymbol{x}_v \in \mathcal{X}$, there is a unique Markov blanket denoted $\mathrm{mb}_{\boldsymbol{x}_v}$, which is the set of random variable corresponding to neighbors of v in $\mathcal{G}$ (nodes that share an edge with v)

- Definition: The local Markov independencies associated with $\mathcal{G}$ are given by

$$\boldsymbol{x}_v \perp\!\!\!\perp \mathcal{X}_{\mathcal{G}} \setminus \{\boldsymbol{x}_v\} \setminus \mathrm{mb}_{\boldsymbol{x}_v} \Big| \mathrm{mb}_{\boldsymbol{x}_v}, \quad \forall \boldsymbol{x}_v \in \mathcal{X}_{\mathcal{G}}$$



Pairwise Markov Property

- Any two variables $\boldsymbol{x}_v, \boldsymbol{x}_u$ corresponding to non-adjacent nodes v, u in graph $\mathcal{G}$ are conditionally independent given all the other variables, i.e.,

$$\boldsymbol{x}_v \perp\!\!\!\perp \boldsymbol{x}_u \,\big|\, \mathcal{X}_{\mathcal{G}} \backslash \{\boldsymbol{x}_v, \boldsymbol{x}_u\}, \quad (u, v) \notin \mathcal{E}$$

- Example: $\boldsymbol{x}_1 \perp\!\!\!\perp \boldsymbol{x}_5 \,\big|\, \{\boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4\}$

$$\mathcal{V} = \{1, 2, 3, 4, 5\}$$

$$\mathcal{E} = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$$

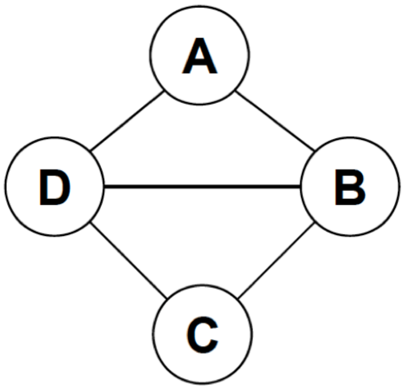
Cliques

- For $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a complete subgraph or clique is a subgraph $\mathcal{G}_1 = (\mathcal{V}_1 \subseteq \mathcal{V}, \mathcal{E}_1 \subseteq \mathcal{E})$ with fully interconnected nodes $\mathcal{V}_1$

- A maximal clique is complete subgraph $\mathcal{G}_1$ of $\mathcal{G}$ where any graph $\mathcal{G}_2 = (\mathcal{V}_1 \subset \mathcal{V}_2 \subseteq \mathcal{V}, \mathcal{E}_1 \subset \mathcal{E}_2 \subseteq \mathcal{E})$ is not complete

- A sub-clique is a clique that is not maximal

- Example:
  - maximal cliques: $\{A, B, D\}$ and $\{B, C, D\}$
  - sub-cliques: $\{A, B\}$, $\{B, C\}$, ...



Factorization

- The distribution $p(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{|\mathcal{V}|})$ factorizes according to graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ if its density can be written in the form

$$p(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{|\mathcal{V}|}) = \frac{1}{Z} \prod_{c \in \mathcal{C}} \psi_c(\boldsymbol{x}^{(c)})$$

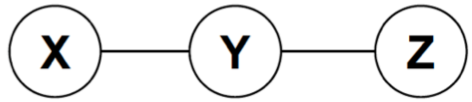
where the potentials $\psi_c(\boldsymbol{x}^{(c)})$ are non-negative functions associated with cliques $\mathcal{C}$ of $\mathcal{G}$ and Z is the partition function

$$Z = \sum_{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{|\mathcal{V}|}} \prod_{c \in \mathcal{C}} \psi_c(\boldsymbol{x}^{(c)})$$

- The potential functions $\psi_c(\boldsymbol{x}^{(c)})$ can be understood as contingency functions of its arguments or local building blocks -> often no probabilistic interpretation

- Factorization according to $\mathcal{G}$ implies the global Markov property

Interpretation of Clique Potentials



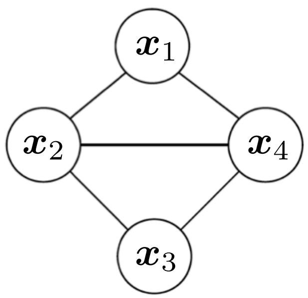
- This model implies that $\boldsymbol{x} \perp\!\!\!\perp \boldsymbol{z} \,|\, \boldsymbol{y}$
- Thus, the joint distribution must factorize as

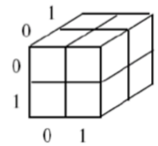
$$p(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = p(\boldsymbol{y})p(\boldsymbol{x}|\boldsymbol{y})p(\boldsymbol{z}|\boldsymbol{y})$$

- We can also write this as $p(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = p(\boldsymbol{y}, \boldsymbol{x})p(\boldsymbol{z}|\boldsymbol{y})$ or $p(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = p(\boldsymbol{z}, \boldsymbol{y})p(\boldsymbol{x}|\boldsymbol{y})$
  - cannot have all potentials to be marginals
  - cannot have all potentials to be conditionals

- Non-negative clique potentials can be thought of as general ``compatibility'', ``goodness'' or ``happiness'' functions over their variables, but not as probability distributions

Example: Factorization Using Maximal Cliques





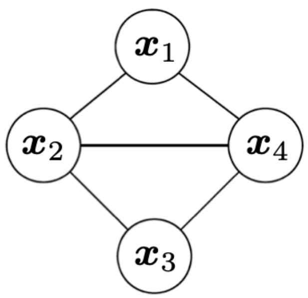
$\psi_c(\cdot,\cdot,\cdot)$

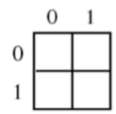
$$p(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4) = \frac{1}{Z}\,\psi_{124}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_4)\,\psi_{234}(\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4)$$

$$Z = \sum_{\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4} \psi_{124}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_4)\,\psi_{234}(\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4)$$

- $p(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4)$ can be represented as two 3-D tables instead of one 4-D table
- The factorization using maximal cliques
  - can always be used without loss of generality
  - often obscures structure that is present in the original set of potentials

Example: Factorization Using Sub-Cliques





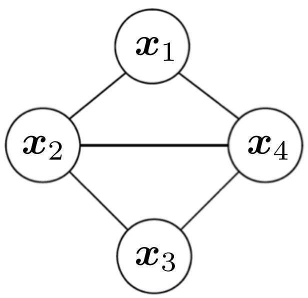
$\psi_c(\cdot, \cdot)$

$$p'(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4) = \frac{1}{Z}\, \psi_{12}(\boldsymbol{x}_1, \boldsymbol{x}_2)\, \psi_{14}(\boldsymbol{x}_1, \boldsymbol{x}_4) \psi_{23}(\boldsymbol{x}_2, \boldsymbol{x}_3) \times \psi_{24}(\boldsymbol{x}_2, \boldsymbol{x}_4) \psi_{34}(\boldsymbol{x}_3, \boldsymbol{x}_4)$$

$$Z = \sum_{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4} \psi_{12}(\boldsymbol{x}_1, \boldsymbol{x}_2)\, \psi_{14}(\boldsymbol{x}_1, \boldsymbol{x}_4)\, \psi_{23}(\boldsymbol{x}_2, \boldsymbol{x}_3) \times \psi_{24}(\boldsymbol{x}_2, \boldsymbol{x}_4)\, \psi_{34}(\boldsymbol{x}_3, \boldsymbol{x}_4)$$

- $p'(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4)$ can be represented as five 2-D tables instead of one 4-D table
- Markov random fields with pairwise interactions is a widely used special case

Example: Canonical Factorization



$$
\begin{aligned}
p''(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4) = \frac{1}{Z}\, & \psi_{124}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_4)\, \psi_{234}(\boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4) \\
& \times \psi_{12}(\boldsymbol{x}_1, \boldsymbol{x}_2)\psi_{14}(\boldsymbol{x}_1, \boldsymbol{x}_4)\psi_{23}(\boldsymbol{x}_2, \boldsymbol{x}_3)\psi_{24}(\boldsymbol{x}_2, \boldsymbol{x}_4)\psi_{34}(\boldsymbol{x}_3, \boldsymbol{x}_4) \\
& \times \psi_1(\boldsymbol{x}_1)\psi_2(\boldsymbol{x}_2)\psi_3(\boldsymbol{x}_3)\psi_4(\boldsymbol{x}_4)
\end{aligned}
$$

$$
Z = \sum_{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4} \psi_{123}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_4)\, \psi_{234}(\boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4) \ldots
$$

- Most general factorization that subsumes any other factorization according to $\mathcal{G}$ as special case

Hammersley-Clifford Theorem

- A positive distribution $p(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{|\mathcal{V}|})$ satisfies the pairwise Markov property with respect to graph $\mathcal{G}$ if and only if it factorizes according to

$$p(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{|\mathcal{V}|}) = \frac{1}{Z} \prod_{c \in \mathcal{C}} \psi_c(\boldsymbol{x}^{(c)})$$

  where $\mathcal{C}$ are cliques of $\mathcal{G}$ and Z is the partition function

- The Hammersley-Clifford theorem
  - identifies weak assumptions on distributions so that equivalence holds between Markov properties and factorization
  - is the central results of the theory of undirected graphical models

Markov Properties and Factorization

- Factorization $(\boldsymbol{F})$ of a distribution $p(\boldsymbol{x}_1, \dots, \boldsymbol{x}_{|\mathcal{V}|})$ according to $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ implies global $(\boldsymbol{G})$, local $(\boldsymbol{L})$, and pairwise $(\boldsymbol{P})$ Markov properties, i.e., $(\boldsymbol{F}) \Rightarrow (\boldsymbol{G}) \Rightarrow (\boldsymbol{L}) \Rightarrow (\boldsymbol{P})$

- However, in general $(\boldsymbol{P}) \not\Rightarrow (\boldsymbol{L}) \not\Rightarrow (\boldsymbol{G}) \not\Rightarrow (\boldsymbol{F})$

- Hammersley and Clifford showed in their celebrated theorem that for (strictly) positive density functions $(\boldsymbol{P}) \Rightarrow (\boldsymbol{F})$ and thus $(\boldsymbol{P}) \Leftrightarrow (\boldsymbol{L}) \Leftrightarrow (\boldsymbol{G}) \Leftrightarrow (\boldsymbol{F})$

Summary: Undirected Graphical Models

- Markov Properties: Conditional independency statements can be extracted from the graph
- Factorization: Local contingency functions (potentials) for each cliques in the graph completely determine the joint distribution
- Hammersley-Clifford theorem: For strictly positive distributions equivalence holds between Markov properties and factorization