ECE 286: Bayesian Machine Perception

Class 4: The Particle Filter

Florian Meyer

Electrical and Computer Engineering Department
University of California San Diego

UC San Diego
JACOBS SCHOOL OF ENGINEERING

Estimation of Expectation and Covariance

- Samples $\boldsymbol{x}^{(j)} \sim f(\boldsymbol{x}),\; j = 1, \ldots, J$ of continuous random vector $\boldsymbol{x}$
- Expectation of a random vector $\boldsymbol{x}$

$$\mathbb{E}\{\boldsymbol{x}\} = \int \boldsymbol{x}\, f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \approx \frac{1}{J} \sum_{j=1}^{J} \boldsymbol{x}^{(j)} = \tilde{\boldsymbol{\mu}}_{\boldsymbol{x}}$$

- Expectation of transformed random vector $g(\boldsymbol{x})$

$$\mathbb{E}\{g(\boldsymbol{x})\} = \int g(\boldsymbol{x})\, f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \approx \frac{1}{J} \sum_{j=1}^{J} g(\boldsymbol{x}^{(j)})$$

- Covariance of a random vector $\boldsymbol{x}$

$$\mathbb{C}\{\boldsymbol{x}\} = \mathbb{E}\{\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}\} - \mathbb{E}\{\boldsymbol{x}\}\mathbb{E}\{\boldsymbol{x}\}^{\mathrm{T}} \approx \frac{1}{J} \sum_{j=1}^{J} \boldsymbol{x}^{(j)} \boldsymbol{x}^{(j)\mathrm{T}} - \tilde{\boldsymbol{\mu}}_{\boldsymbol{x}} \tilde{\boldsymbol{\mu}}_{\boldsymbol{x}}^{\mathrm{T}}$$

- Approximations are arbitrarily good for J sufficiently large

Importance Sampling (i)

- Recap expectation of transformed random vector

$$\mathbb{E}\{g(\boldsymbol{x})\} = \int g(\boldsymbol{x})\, f_{\mathrm{t}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int g(\boldsymbol{x})\, \frac{f_{\mathrm{t}}(\boldsymbol{x})}{f_{\mathrm{p}}(\boldsymbol{x})} f_{\mathrm{p}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$

- Draw samples proposal pdf $\boldsymbol{x}^{(j)} \sim f_{\mathrm{p}}(\boldsymbol{x}),\ j = 1, \ldots, J$

- Calculate approximate expectation

Note that $\sum_{j=1}^{J} w^{(j)} = 1$ for $J \rightarrow \infty$

$$\mathbb{E}\{g(\boldsymbol{x})\} \approx \frac{1}{J} \sum_{j=1}^{J} g(\boldsymbol{x}^{(j)})\, \frac{f_{\mathrm{t}}(\boldsymbol{x}^{(j)})}{f_{\mathrm{p}}(\boldsymbol{x}^{(j)})} = \sum_{j=1}^{J} g(\boldsymbol{x}^{(j)})\, w^{(j)}$$

where $w^{(j)} = \dfrac{1}{J} \dfrac{f_{\mathrm{t}}(\boldsymbol{x}^{(j)})}{f_{\mathrm{p}}(\boldsymbol{x}^{(j)})}$

- This motivates a particle representation $\left\{\boldsymbol{x}^{(j)}, w^{(j)}\right\}_{j=1}^{J} \simeq f_{\mathrm{t}}(\boldsymbol{x})$

Importance Sampling (ii)

- **Goal:** Calculate

$$\{\boldsymbol{x}^{(j)}, w^{(j)}\}_{j=1}^{J} \simeq f_{\mathrm{t}}(\boldsymbol{x})$$

- **Step 1:** Sample

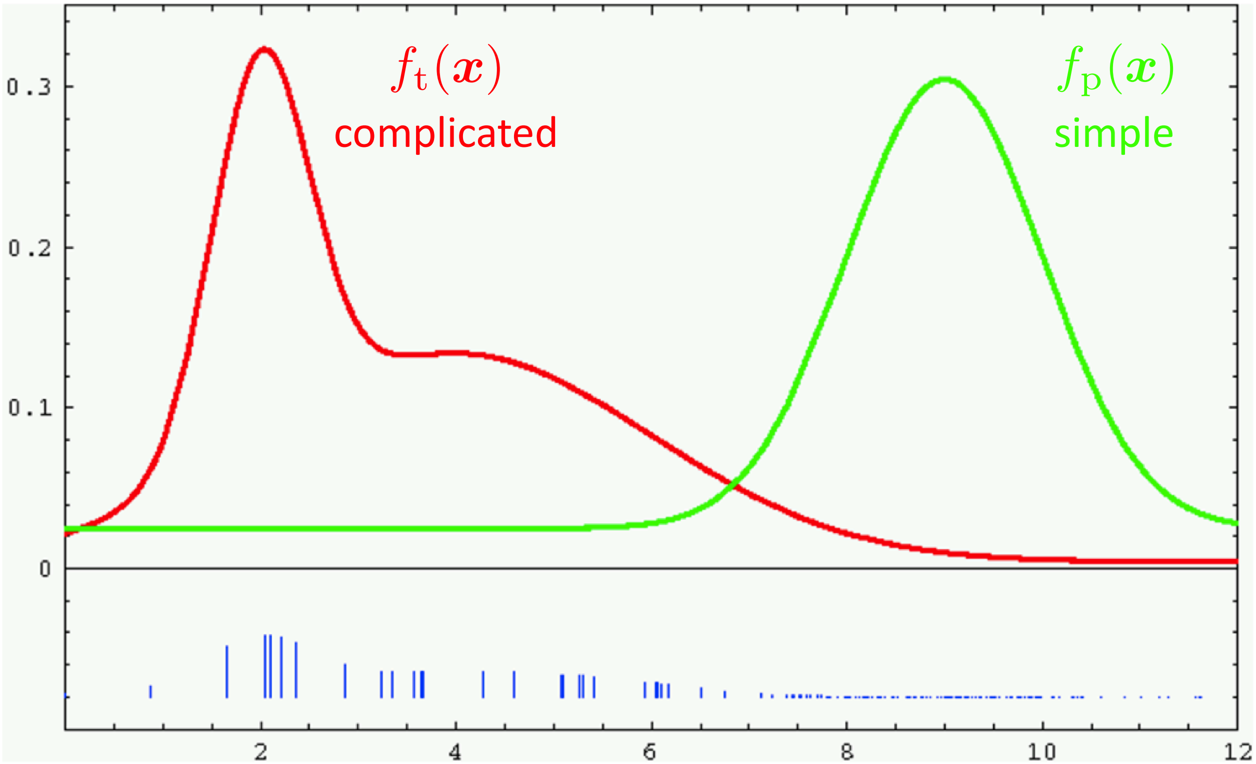
$$\boldsymbol{x}^{(j)} \sim f_{\mathrm{p}}(\boldsymbol{x})$$

$$j \in \{1 \dots J\}$$

- **Step 2:** Calculate weights

$$w^{(j)} = \frac{1}{J} \frac{f_{\mathrm{t}}\big(\boldsymbol{x}^{(j)}\big)}{f_{\mathrm{p}}\big(\boldsymbol{x}^{(j)}\big)}$$

$$j \in \{1 \dots J\}$$



Estimation of Expectation and Covariance

- Particles and weights $\{(\boldsymbol{x}^{(j)}, w^{(j)})\}_{j=1}^{J} \simeq f(\boldsymbol{x})$ representing the PDF of random vector $\boldsymbol{x}$
- Expectation of a random vector $\boldsymbol{x}$

$$\mathbb{E}\{\boldsymbol{x}\} \approx \sum_{j=1}^{J} w^{(j)} \boldsymbol{x}^{(j)} = \tilde{\boldsymbol{\mu}}_{\boldsymbol{x}}$$

- Expectation of transformed random vector $g(\boldsymbol{x})$

$$\mathbb{E}\{g(\boldsymbol{x})\} \approx \sum_{j=1}^{J} g(\boldsymbol{x}^{(j)})\, w^{(j)}$$

- Covariance of a random vector $\boldsymbol{x}$

$$\mathbb{C}\{\boldsymbol{x}\} \approx \sum_{j=1}^{J} w^{(j)} \boldsymbol{x}^{(j)} \boldsymbol{x}^{(j)\mathrm{T}} - \tilde{\boldsymbol{\mu}}_{\boldsymbol{x}} \tilde{\boldsymbol{\mu}}_{\boldsymbol{x}}^{\mathrm{T}}$$

- Approximations are arbitrarily good for J sufficiently large

Example: Fusion of Range Measurements (i)

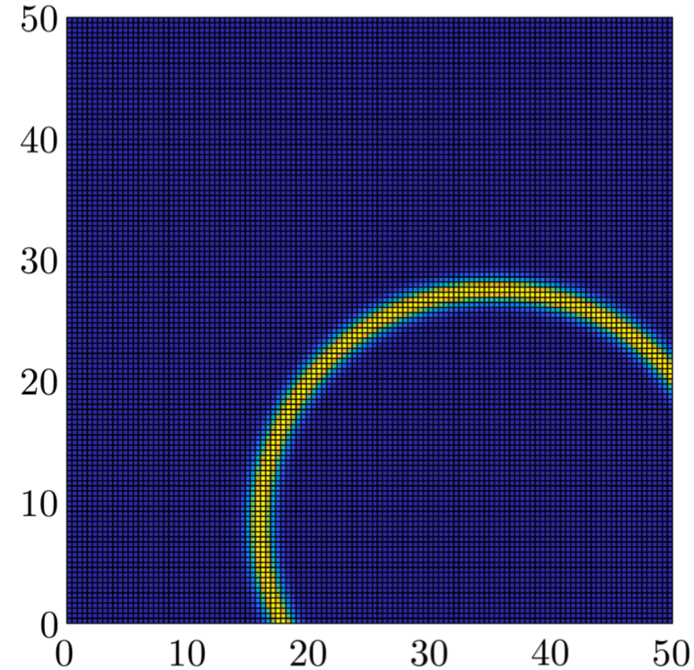
- $\boldsymbol{x}$ is the unknown position of an agent

- Measurement $\boldsymbol{y} = [y_1 \;\, y_2 \;\, y_3]^{\mathrm{T}}$ is modeled as

$$y_i = \|\boldsymbol{x} - \boldsymbol{p}_i\| + v_i$$

  where $\boldsymbol{p}_i$ is the position of a known reference and $v_i \sim \mathcal{N}(0;\sigma_v)$ is noise (independent across i)

- Each likelihood function $f(y_i|\boldsymbol{x})$ describes potential agent positions on a ring



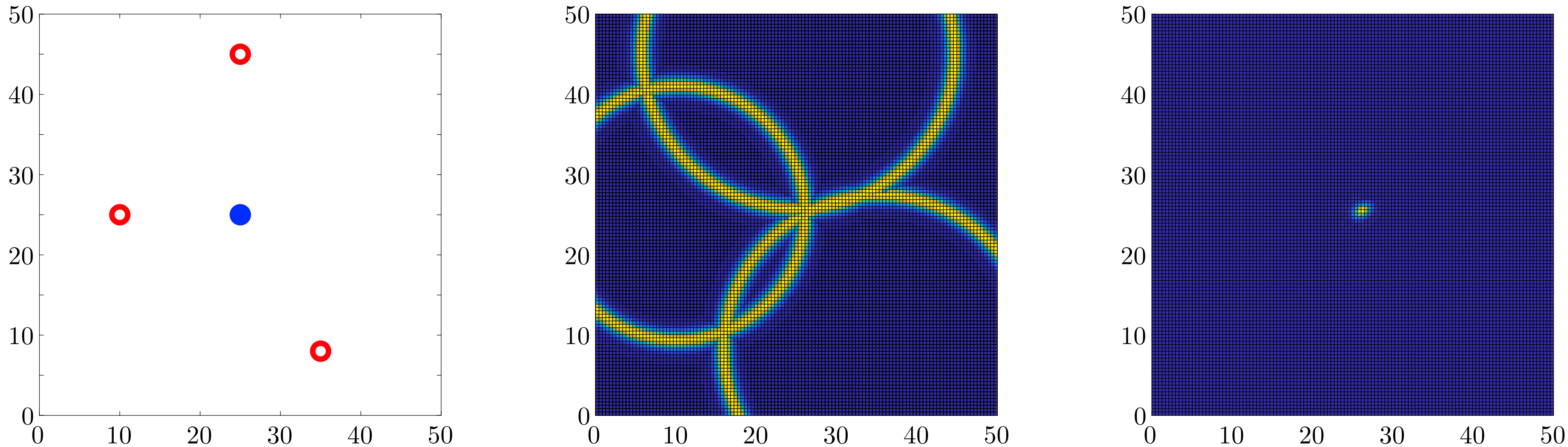
$f(y_1|\boldsymbol{x}),\; \boldsymbol{p}_1 = [35\,8]^{\mathrm{T}}$

Example: Fusion of Range Measurements (ii)

- Marginal posterior PDF ($\boldsymbol{y}$ is observed)

$$f(\boldsymbol{x}|\boldsymbol{y}) \propto f(\boldsymbol{y}|\boldsymbol{x})f(\boldsymbol{x}) = \prod_{i=1}^{3} f(y_i|\boldsymbol{x})f(\boldsymbol{x})$$

- Example scenario with uniform prior $f(\boldsymbol{x})$

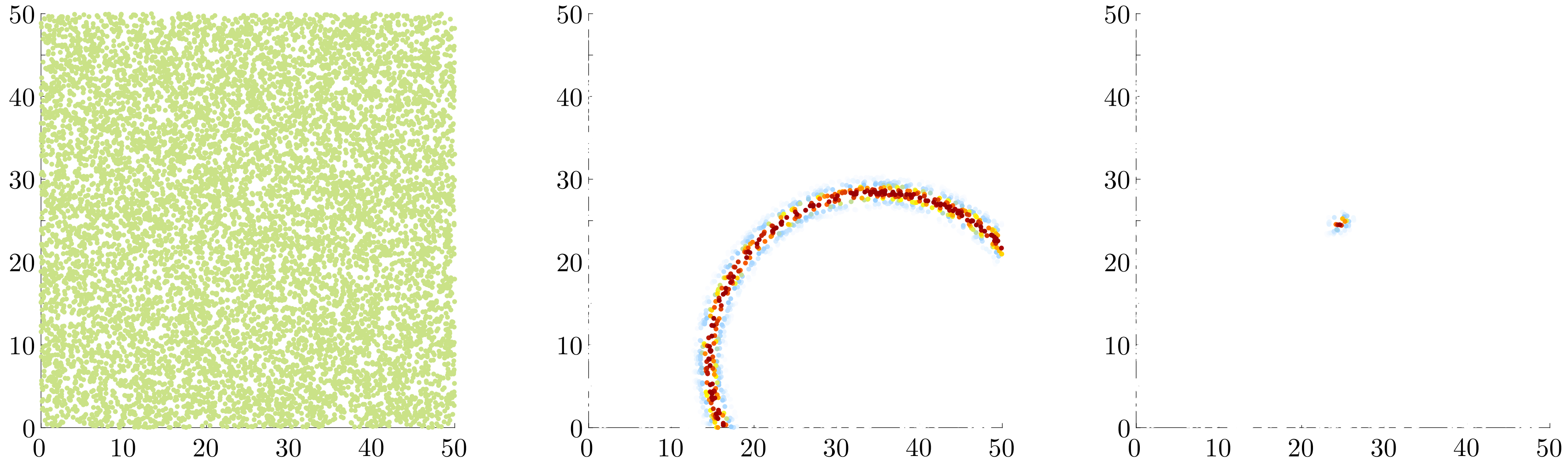


Topology

$\sum_{i=1}^{3} f(y_i|\boldsymbol{x})$

$f(\boldsymbol{x}|\boldsymbol{y})$

Example: Fusion of Range Measurements (iii)



$\{\boldsymbol{x}^j\}_{j=1}^J \sim f(\boldsymbol{x})$ $\{\boldsymbol{x}_1^j, w_1^{(j)}\}_{j=1}^J \simeq f(\boldsymbol{x}|y_1)$ $\{\boldsymbol{x}^j, w^{(j)}\}_{j=1}^J \simeq f(\boldsymbol{x}|\boldsymbol{y})$

- Approximate MMSE estimate: $\hat{\boldsymbol{x}}^{\mathrm{MMSE}} = \sum_{j=1}^{J} \boldsymbol{x}^{(j)} w^{(j)}$

Particle Resampling

- We can consider a particle representation $\{\boldsymbol{x}^{(j)}, w^{(j)}\}_{j=1}^{J} \simeq f(\boldsymbol{x})$ as probability mass function and draw J' samples from it

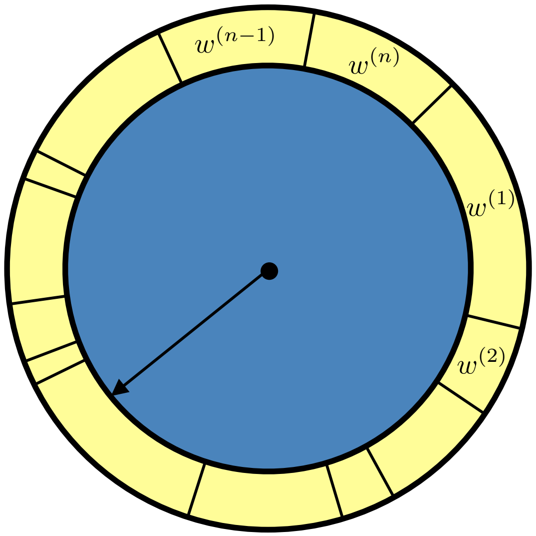
- In this way, we get a new particle representation $\{\overline{\boldsymbol{x}}^{(j)}\}_{j=1}^{J'} \simeq f(\boldsymbol{x})$

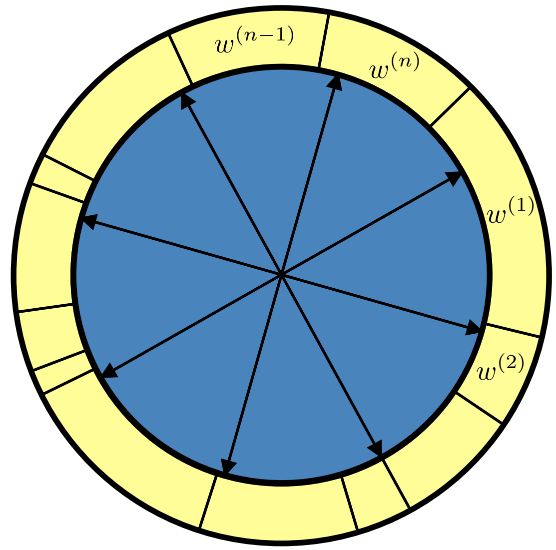
- This is motivated by the approximation ($\delta(\cdot)$ is the Dirac delta function)

$$f(\boldsymbol{x}) \approx \sum_{j=1}^{J} w^{(j)} \delta(\boldsymbol{x} - \boldsymbol{x}^{(j)})$$
$$= \tilde{f}(\boldsymbol{x})$$

$$\{\overline{\boldsymbol{x}}^{(j)}\}_{j=1}^{J'} \sim \tilde{f}(\boldsymbol{x})$$

Particle Resampling





- ``Roulette wheel''
- Binary search, complexity $\mathcal{O}(J \log J)$

- Systematic resampling
- Complexity $\mathcal{O}(J)$
- Easy to implement, low variance

Particle Marginalization

- If we have a particle representation of a PDF $f(\boldsymbol{x}) = f(x_1, x_2)$, i.e.,

$$\big\{(\boldsymbol{x}^{(j)}, w^{(j)})\big\}_{j=1}^{J} = \big\{(x_1^{(j)}, x_2^{(j)}, w^{(j)})\big\}_{j=1}^{J} \simeq f(\boldsymbol{x})$$

  we immediately have particle representations of the marginals $f(x_1)$ and $f(x_2)$, i.e.,

$$\big\{\big(x_1^{(j)}, w^{(j)}\big)\big\}_{j=1}^{J} \simeq f(x_1) \qquad \big\{\big(x_2^{(j)}, w^{(j)}\big)\big\}_{j=1}^{J} \simeq f(x_2)$$

- This is motivated by

$$\begin{aligned} f(x_1) \approx \int \tilde{f}(\boldsymbol{x})\,\mathrm{d}x_2 &= \sum_{j=1}^{J} w^{(j)} \int \delta(\boldsymbol{x} - \boldsymbol{x}^{(j)})\,\mathrm{d}x_2^{(j)} \\ &= \sum_{j=1}^{J} w^{(j)} \delta(x_1 - x_1^{(j)}) \end{aligned}$$

The Particle Filter

- Recap sequential Bayesian estimation

Prediction Step:

$$\underbrace{f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1})}_{\substack{\text{Predicted}\\ \text{posterior pdf}}} = \int \underbrace{f(\boldsymbol{x}_n|\boldsymbol{x}_{n-1})}_{\substack{\text{State-transition}\\ \text{pdf}}} \underbrace{f(\boldsymbol{x}_{n-1}|\boldsymbol{y}_{1:n-1})}_{\substack{\text{Previous}\\ \text{posterior pdf}}} \mathrm{d}\boldsymbol{x}_{n-1}$$

Update Step:

$$\underbrace{f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n})}_{\text{Posterior pdf}} \propto \underbrace{f(\boldsymbol{y}_n|\boldsymbol{x}_n)}_{\substack{\text{Likelihood}\\ \text{function}}} \underbrace{f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1})}_{\substack{\text{Predicted}\\ \text{posterior pdf}}}$$

- Approximate implementation by means of importance sampling, particle marginalization, and resampling

- Particle representation of posterior PDFs

$$\{\boldsymbol{x}_n^{(j)}\}_{j=1}^{J} \simeq f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1}) \qquad \{\overline{\boldsymbol{x}}_n^{(j)}\}_{j=1}^{J} \simeq f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n})$$

B. Ristic, S. Arulampalam, and N. Gordon, *Beyond the Kalman Filter: Particle Filters for Tracking Applications*, Artech House, 2004.

Prediction Step

- **Given:** Particles $\{\overline{\boldsymbol{x}}_{n-1}^{(j)}\}_{j=1}^{J} \simeq f(\boldsymbol{x}_{n-1}|\boldsymbol{y}_{1:n-1})$ representing the previous posterior PDF

- **Wanted:** Particles $\{\boldsymbol{x}_{n}^{(j)}\}_{j=1}^{J} \simeq f(\boldsymbol{x}_{n}|\boldsymbol{y}_{1:n-1})$ representing the predicted posterior PDF

- Calculate a particle representation of $f(\boldsymbol{x}_n, \boldsymbol{x}_{n-1}|\boldsymbol{y}_{1:n-1}) = f(\boldsymbol{x}_n|\boldsymbol{x}_{n-1})f(\boldsymbol{x}_{n-1}|\boldsymbol{y}_{1:n-1})$ by repeating J times
  - draw sample from driving noise PDF $\boldsymbol{u}_n^{(j)} \sim f(\boldsymbol{u}_n)$
  - draw sample from $\boldsymbol{x}_n^{(j)} \sim f(\boldsymbol{x}_n|\overline{\boldsymbol{x}}_{n-1}^{(j)})$ by applying state transition model $\boldsymbol{x}_n^{(j)} = g_n(\overline{\boldsymbol{x}}_{n-1}^{(j)}, \boldsymbol{u}_n^{(j)})$

- Perform particle marginalization to get $\{\boldsymbol{x}_{n}^{(j)}\}_{j=1}^{J} \simeq f(\boldsymbol{x}_{n}|\boldsymbol{y}_{1:n-1})$ from $\big\{(\boldsymbol{x}_n^{(j)}, \overline{\boldsymbol{x}}_{n-1}^{(j)})\big\}_{j=1}^{J} \simeq f(\boldsymbol{x}_n, \boldsymbol{x}_{n-1}|\boldsymbol{y}_{1:n-1})$

Update Step

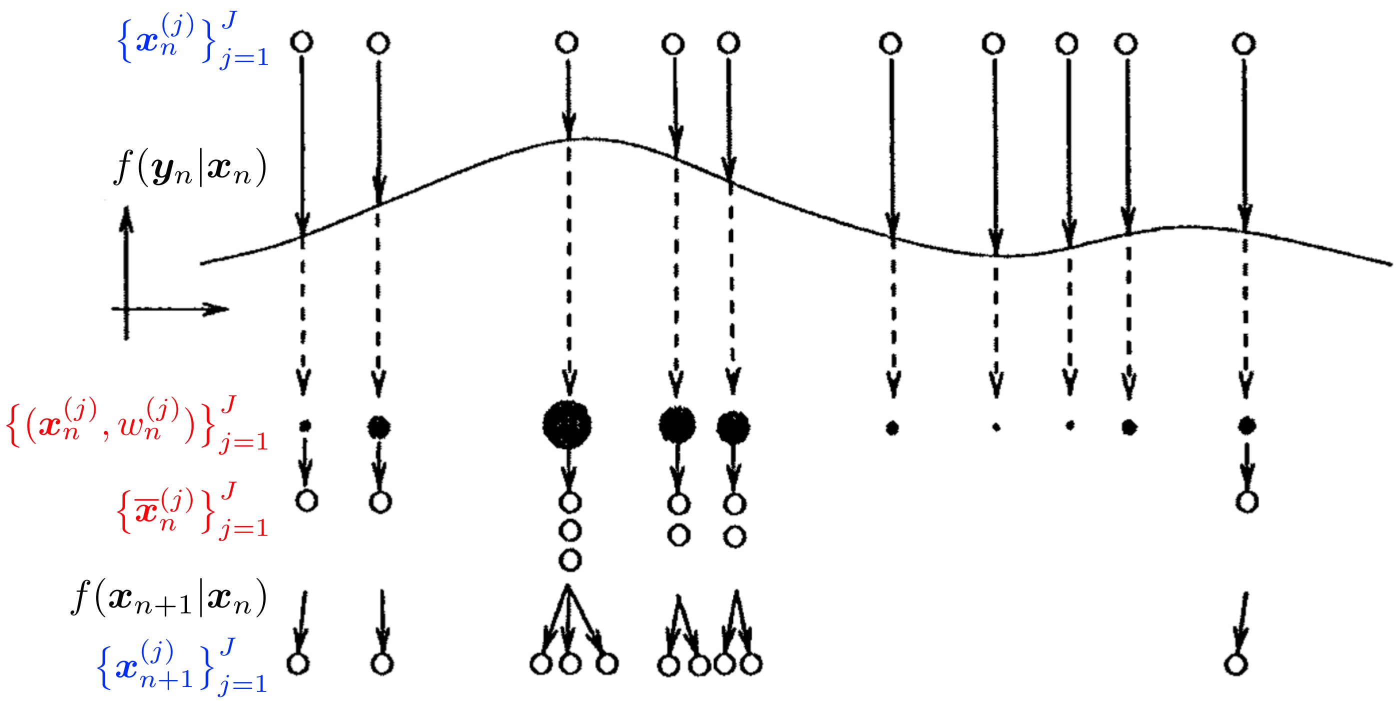
- **Given:** Particles $\{(\boldsymbol{x}_n^{(j)})\}_{j=1}^{J} \simeq f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1})$ representing the predicted posterior PDF

- **Wanted:** Particles $\{(\overline{\boldsymbol{x}}_n^{(j)})\}_{j=1}^{J} \simeq f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n})$ representing the posterior PDF

- Perform importance sampling with proposal distribution $f_{\mathrm{p}}(\boldsymbol{x}_n) = f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1})$ and target distribution $f_{\mathrm{t}}(\boldsymbol{x}_n) \propto f(\boldsymbol{y}_n|\boldsymbol{x}_n) f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1})$
  – calculate unnormalized weights $\tilde{w}_n^{(j)} = f(\boldsymbol{y}_n|\boldsymbol{x}_n^{(j)}) \propto f_{\mathrm{t}}(\boldsymbol{x}_n^{(j)})/f_{\mathrm{p}}(\boldsymbol{x}_n^{(j)})$
  – normalize weights $w_n^{(j)} = \tilde{w}_n^{(j)} / \sum_{j'=1}^{J} \tilde{w}_n^{(j')}, \quad j = 1, \ldots, J$

- Perform resampling to get $\{(\overline{\boldsymbol{x}}_n^{(j)})\}_{j=1}^{J} \simeq f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n})$ from $\{(\boldsymbol{x}_n^{(j)}, w_n^{(j)})\}_{j=1}^{J} \simeq f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n})$

Particle Filter Illustration



B. Ristic, S. Arulampalam, and N. Gordon, *Beyond the Kalman Filter: Particle Filters for Tracking Applications*, Artech House, 2004.

Particle Filter Simulation (i)

- We consider single object tracking
  - One mobile object, four static references at known locations
  - The state of the mobile object consists of location and velocity, i.e.,

$$\boldsymbol{x}_n = [x_{1,n}\ x_{2,n}\ \dot{x}_{1,n}\ \dot{x}_{2,n}]^{\mathrm{T}}$$

  - State-transition model for the mobile object:

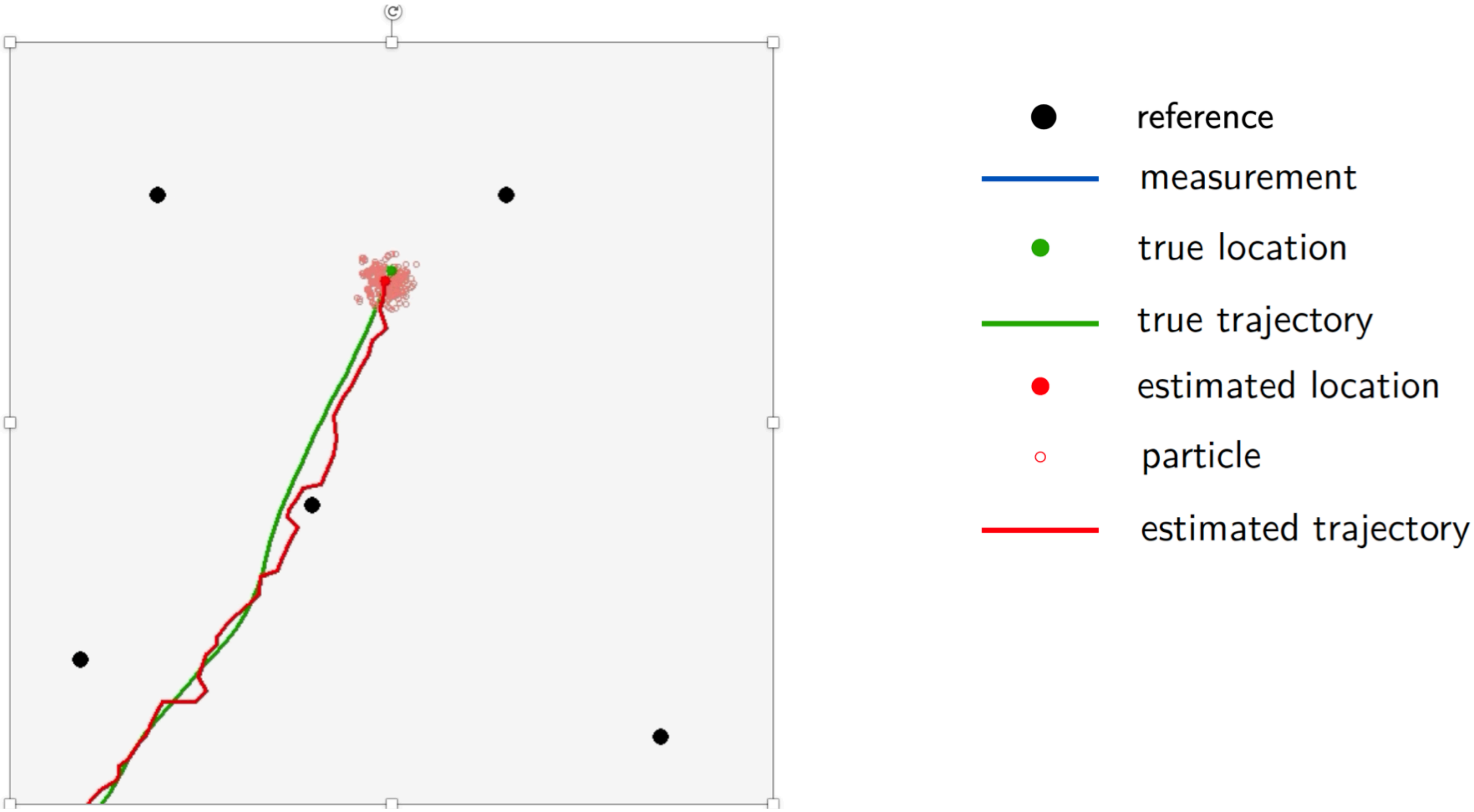
$$\boldsymbol{x}_n = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{x}_{n-1} + \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{u}_n \quad \text{with } \boldsymbol{u}_n \sim \mathcal{N}(\boldsymbol{0}, \sigma_u^2 \boldsymbol{I})$$

  - Measurement model:

$$y_{n,i} = \|[x_{1,n}\ x_{2,n}]^{\mathrm{T}} - \boldsymbol{p}_k\| + v_{n,i}, \quad i = 1, 2, 3, 4$$

with $v_i \sim \mathcal{N}(0; \sigma_v)$ and known location $\boldsymbol{p}_k$

Particle Filter Simulation (ii)



Summary

- The Unscented Kalman Filter
  - extension of the Kalman filter to moderately nonlinear systems
  - uses sigma points for linearization
  - is suitable for high-dimensional sequential Bayesian estimation problems

- The Particle Filter
  - represents posterior PDF by random particles
  - provides exact estimation results as the number of particles goes to infinity
  - is suitable for highly nonlinear sequential Bayesian estimation problem
  - is limited to low-dimensional problems (``curse of dimensionality'')