SIO 209: Signal Processing for Ocean Sciences
Class 10

Florian Meyer

Scripps Institution of Oceanography
Electrical and Computer Engineering Department
University of California San Diego

SCRIPPS INSTITUTION OF OCEANOGRAPHY

UC San Diego
JACOBS SCHOOL OF ENGINEERING

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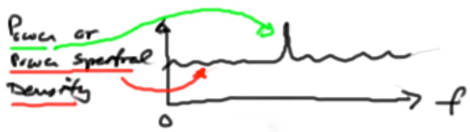
Recovering Sinusoid Power from FFT



Power of Sinusoid: $A^2/2$

$$x[n] = A\sin\omega n$$
$$= A\left\{\frac{e^{j\omega n} - e^{-j\omega n}}{2j}\right\}$$

let $\omega = \dfrac{2\pi}{M}k'$

M length of FFT

$$X(k) = \sum_{n=0}^{M-1} w[n]x[n]e^{-j\frac{2\pi}{M}nk}$$
$$= A\sum_{n=0}^{M-1} w[n]\left\{\frac{e^{j\frac{2\pi}{M}nk'} - e^{-j\frac{2\pi}{M}nk'}}{2j}\right\}e^{-j\frac{2\pi}{M}nk}$$
$$= \frac{A}{2j}\sum_{n=0}^{M-1} w[n] \quad \text{for } k = k' \text{ (sinusoid is bin centered)}$$

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Recovering Sinusoid Power from FFT

Amplitude and Power of Bin-Centered Sinusoid:

$$X(k) = \frac{A}{2j} \sum_{n=0}^{M-1} w[n] \quad k = k'$$

sinusoid amplitude $$A = \frac{2}{\sum_{n=0}^{M-1} w[n]} |X(k')|$$

True regardless of the phase of sinusoid, since we use $|X(k')|$ which is insensitive to phase

Positive and negative frequency power

sinusoid power $$\frac{A^2}{2} = \frac{\boxed{2}}{\left(\sum_{n=0}^{M-1} w[n]\right)^2} |X(k')|^2$$

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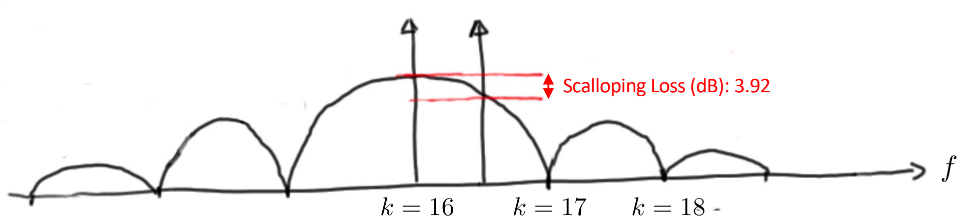
Scalloping Loss (dB)

Example: Rectangular Window



- Comparison between sinusoid at bin center (e.g., $k = 16$) vs. bin boundary (e.g., halfway between $k = 16$ and $k = 17$)
- Scalloping loss of Kaiser-Bessel window is 1.2 dB for $\alpha = 2.5$ ($\beta = \pi\alpha = 7.85$)

F. J. Harris, "On the use of windows for harmonic analysis with the discrete Fourier transform." Proc. IEEE, 1978

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Recovering Sinusoid Power from PSD Estimate

- Recall 2-sided power spectral density (power per unit frequency)

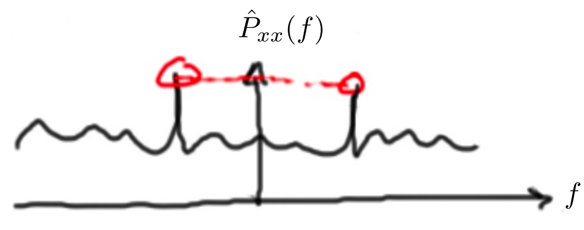
$$\hat{P}_{xx}(f_k) = \frac{1}{MUf_s}\overline{|X(k)|^2}$$

averaged over potentially overlapping and windowed records

in "Power"/Hz

$$U = \frac{1}{M}\sum_{n=0}^{M-1} w^2[n]$$

normalization constant



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Recovering Sinusoid Power from PSD Estimate

Sinusoid Power

$$\frac{A^2}{2} = \frac{2}{\left(\sum_{n=0}^{M-1} w[n]\right)^2}\overline{|X(k)|^2}$$

$$= \frac{2\, f_s M U \hat{P}_{xx}(f_k)}{\left(\sum_{n=0}^{M-1} w[n]\right)^2}$$

Positive and negative frequency power

$$= \frac{\boxed{2}\, f_s}{M}\boxed{\frac{M\sum_{n=0}^{M-1} w^2[n]}{\left(\sum_{n=0}^{M-1} w[n]\right)^2}}\boxed{\hat{P}_{xx}(f_k)}$$

Equivalent noise bandwidth (see table in *F. Harris, 1978*)

2-sided power spectral density

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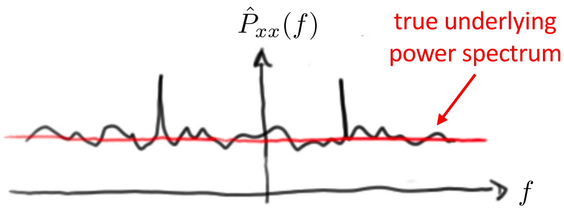
Confidence Intervals

- 90 % confidence intervals for conventional power spectral density estimation



Number of Averages K

	4	8	16	32	64
Upper Limit (dB)	+4.7	+3.0	+2.0	+1.4	+1.0
Lower Limit (dB)	-2.9	-2.2	-1.6	-1.2	-0.8

- Note that each sample of the power spectral density, follows a Chi-squared distribution with $2K$ degrees of freedom

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