ECE 161A: Circular Convolution and Linear Convolution

Florian Meyer
University of California, San Diego
Email: flmeyer@ucsd.edu

Circular Convolution versus Linear Convolution

$h[n]$, the filter is of duration P, i.e. $h[n]$ non-zero over interval $0 \leq n \leq P-1$.

$x[n]$, the input sequence is of duration L, i.e. $x[n]$ non-zero over interval $0 \leq n \leq L-1$. Assume $L \geq P$.

$$x[n] \overset{N}{\longleftrightarrow} X[k] \text{ and } h[n] \overset{N}{\longleftrightarrow} H[k].$$

with $N \geq \max(P, L) = L$. Then

$$y[n] = \sum_{m=0}^{N-1} x[m]h[((n-m))_N] = x[n] \circledN h[n] \overset{N}{\longleftrightarrow} Y[k] = X[k]H[k].$$

$x[n] = x_1[n] \circledN x_2[n], n = 0, 1, \ldots, N-1$ is a sequence of length N. N is a parameter of choice.

Linear Convolution: $y_{lc}[n] = x[n] * h[n]$ and $Y_{lc}(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$.

Question: What is the relationship between $y_{lc}[n]$ and $y[n]$ and when are they equal?

Relationship

$y[n] = x[n] \textcircled{N} h[n] \overset{N}{\longleftrightarrow} Y[k] = X[k]H[k]$

$y_{lc}[n] = x[n] * h[n]$ and $Y_{lc}(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$.
Now

$$\begin{aligned} Y[k] &= H[k]X[k] = H(e^{j\omega})|_{\omega=\frac{2\pi}{N}k} X(e^{j\omega})_{\omega=\frac{2\pi}{N}k}, k = 0, 1, .., N-1. \\ &= Y_{lc}(e^{j\omega})|_{\omega=\frac{2\pi}{N}k}, k = 0, 1, .., N-1 \end{aligned}$$

Main Result:

1. $y_{lc}[n]$ is of duration L_c, where $L_c = L + P - 1$. (Please prove)
2. By Frequency Sampling Theorem

$$\begin{aligned} y[n] &= \sum_{r=-\infty}^{\infty} y_{lc}[n + rN] \\ &= ... + y_{lc}[n - N] + y_{lc}[n] + y_{lc}[n + N] + ..., 0 \leq n \leq N - 1 \end{aligned}$$

Choice of N and Implications

$y[n] = \sum_{r=-\infty}^{\infty} y_{lc}[n+rN] = ...+y_{lc}[n-N]+y_{lc}[n]+y_{lc}[n+N]+..., 0 \leq n \leq N-1$

$N \geq \max(P, L) = L$. Minimum choice is L.

For what choice of N is there no aliasing. This implies $y[n] = y_{lc}[n], 0 \leq n \leq N-1$. and the whole linear convolution is obtained. Also range $0 \leq n \leq N-1$ is of interest.

Answer: $N \geq L_c = L + P - 1$, where L_c is the duration of $y_{lc}[n]$.

Proof: Follows from Frequency Sampling Theorem. Summarized below.

$y_{lc}[n]$ is non-zero between $0 \leq n \leq L_c - 1$.

$y_{lc}[n-N]$ is non-zero between $N \leq n \leq N + L_c - 1$.

$y_{lc}[n+N]$ is non-zero between $-N \leq n \leq -N + L_c - 1$.

Need $N \geq L_c$ and $-N + L_c - 1 < 0$. Satisfied by $N \geq L_c$. Since the shifted copies do not overlap, $y_{lc}[n]$ can be obtained from $y[n]$ by retaining samples $0 \leq n \leq N-1$.

Choice of $L \leq N \leq L_c - 1$ and Implications

$y[n] = \sum_{r=-\infty}^{\infty} y_{lc}[n+rN] = ...+y_{lc}[n-N]+y_{lc}[n]+y_{lc}[n+N]+..., 0 \leq n \leq N-1$

Since the samples of $y[n]$ of interest are in the range $0 \leq n \leq N-1$, we do not need to pay attention to values of n outside this range.

$y_{lc}[n]$ is non-zero between $0 \leq n \leq L_c - 1$.

$y_{lc}[n-N]$ is non-zero between $N \leq n \leq N + L_c - 1$. Starts at N and so outside the range of interest of values for n. Can Ignore.

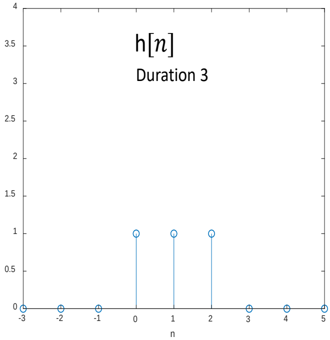
$y_{lc}[n+N]$ is non-zero between $-N \leq n \leq -N + L_c - 1$. Since $N \leq L_c - 1$, we have $-N + L_c - 1 \geq 0$. Aliasing problem

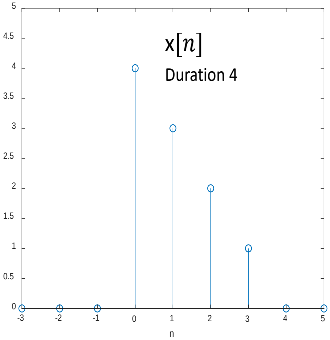
Samples $0 \leq n \leq -N + L_c - 1$ are aliased and
$y[n] = y_{lc}[n], n = -N + L_c, \ldots, N - 1$.

Conclusion: Partial success. Early samples are aliased and corrupted but later samples correspond to linear convolution.

Special Case of $N = L$: We have $-N + L_c - 1 = -L + L + P - 1 - 1 = P - 2$.
Samples $0 \leq n \leq P - 2$ are aliased and
$y[n] = y_{lc}[n], n = P - 1, \ldots, N - 1 = L - 1$.

Circular Convolution $y_{lc}[n] = \sum_{m=0}^{N-1} h[m]x[((n-m))_N]$: Example





$P = 3$ and $L = 4$. Hence $L_c = L + P - 1 = 6$.
Linear Convolution: $y_{lc}[n] = \{4, 7, 9, 6, 3, 1, 0, 0..\}$. Duration is 6 as expected.

Circular Convolution

$P = 3$ and $L = 4$. Hence $L_c = L + P - 1 = 6$.
Linear Convolution: $y_{lc}[n] = \{4, 7, 9, 6, 3, 1, 0, 0..\}$.

$N = 8$. Since $N > L_c$, no aliasing and $y[n] = h[n] \, 8 \, x[n] = y_{lc}[n]$.

Only concerned with $0 \leq n \leq 7$.
Check: $y[n] = .. + y_{lc}[n-8] + y_{lc}[n] + y_{lc}[n+8] + .. = y_{lc}[n], 0 \leq n \leq 7$.

$N = L = 4$. This implies $y[n] = h[n] \, 4 \, x[n]$. $0 \leq n \leq P - 2 = 1$ will be aliased and $y[n], P - 1 = 2 \leq n \leq N - 1 = 3$ correspond to linear convolution.

Check: Only concerned about $0 \leq n \leq N - 1 = 3$.
$y[n] = y_{lc}[n-4] + y_{lc}[n] + y_{lc}[n+4] =$
$\{\underbrace{0}_{n=0}, 0, 0, 0\} + \{\underbrace{4}_{n=0}, 7, 9, 6\} + \{\underbrace{3}_{n=0}, 1, 0, 0\} = \{7, 8, 9, 6\}$

Summary of Linear Convolution using DFT

Assumptions: $h[n]$ of duration P and $x[n]$ of duration L and $N \geq L_c = L + P - 1$

1. Compute $H[k]$ and $X[k]$ where

$$h[n] \overset{N}{\leftrightarrow} H[k] \text{ and } x[n] \overset{N}{\leftrightarrow} X[k]$$

   Often the filter is known before hand and one can compute and store $H[k]$.

2. Compute $Y[k]$, where $Y[k] = H[k]X[k], k = 0, 1, \ldots, N-1$.
3. Compute $y[n]$, the IDFT of $Y[k]$, i.e. $y[n] = \mathcal{IDFT}(Y[k])$
4. The linear convolution $y_{lc}[n] = y[n], n = 0, 1, \ldots, N-1$

If N is chosen differently, i.e. $L \leq N \leq L_c - 1$, then the correct values of $y[n]$ are extracted to get samples of the linear convolution

$$y_{lc}[n] = y[n], -N + L_c \leq n \leq N - 1.$$