ECE 286: Bayesian Machine Perception

Class 9: The Probabilistic Data Association Filter

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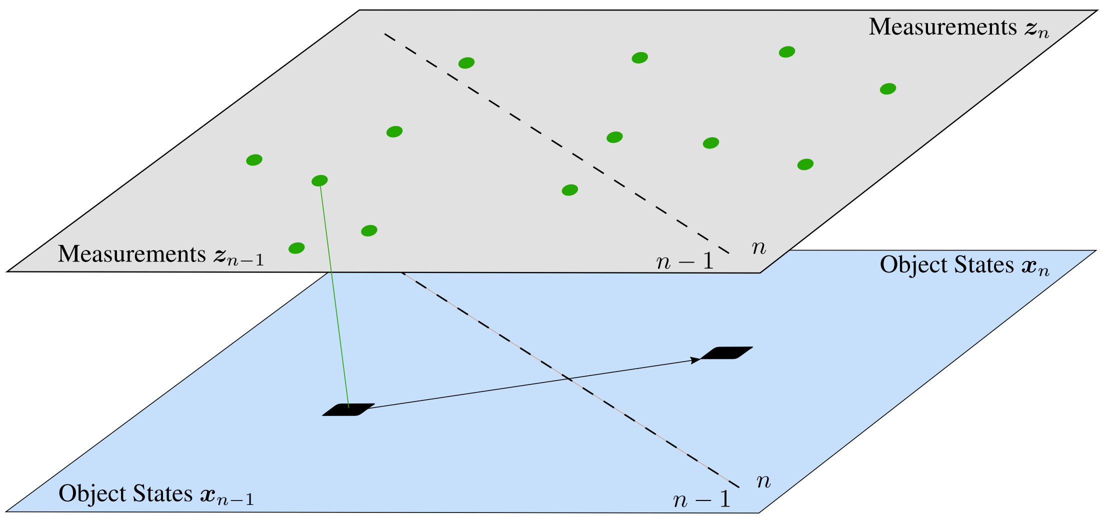
Basic Setting

- “Single object tracking in clutter” problem



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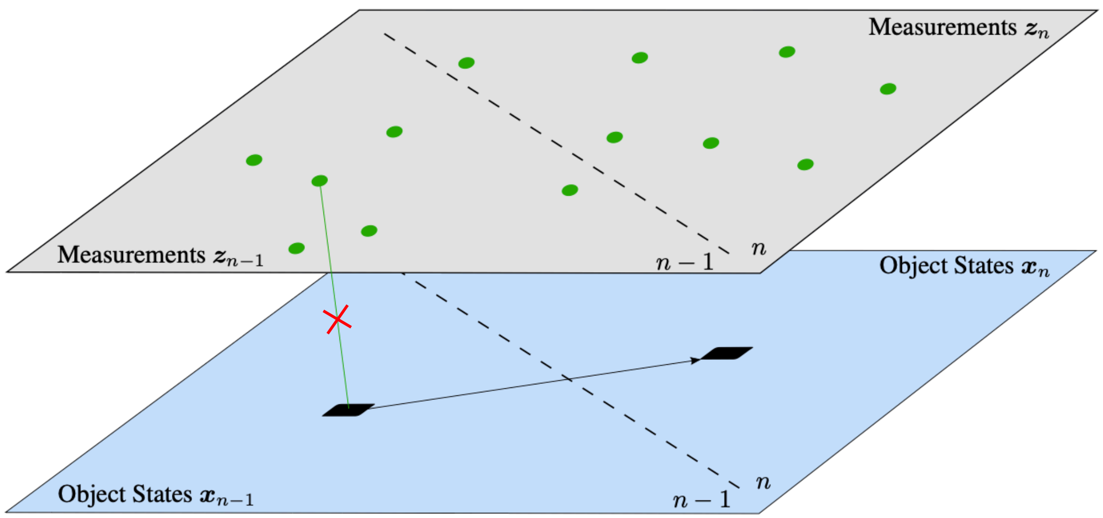
- Measurement-origin uncertainty (MOU), false clutter measurements and missed detections



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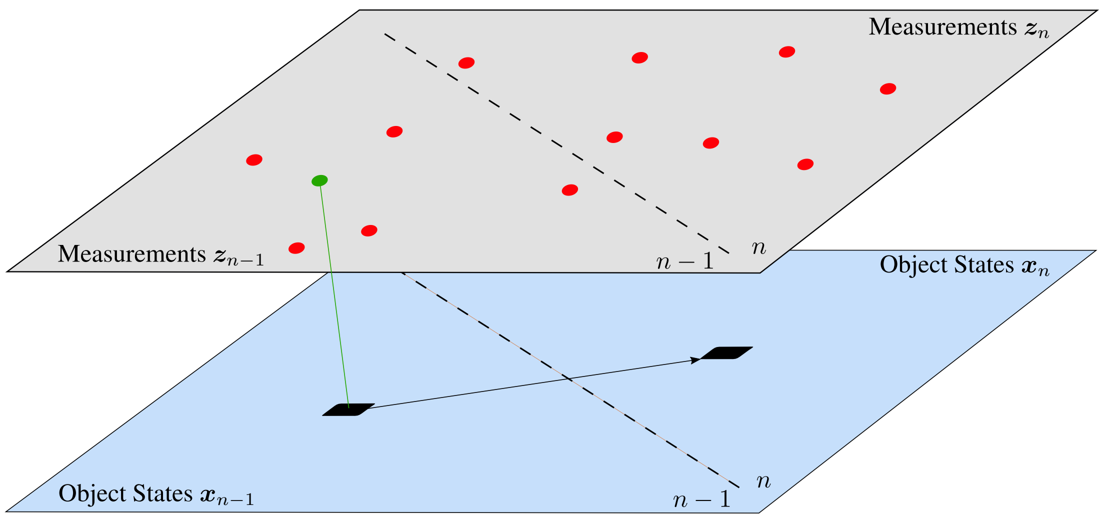

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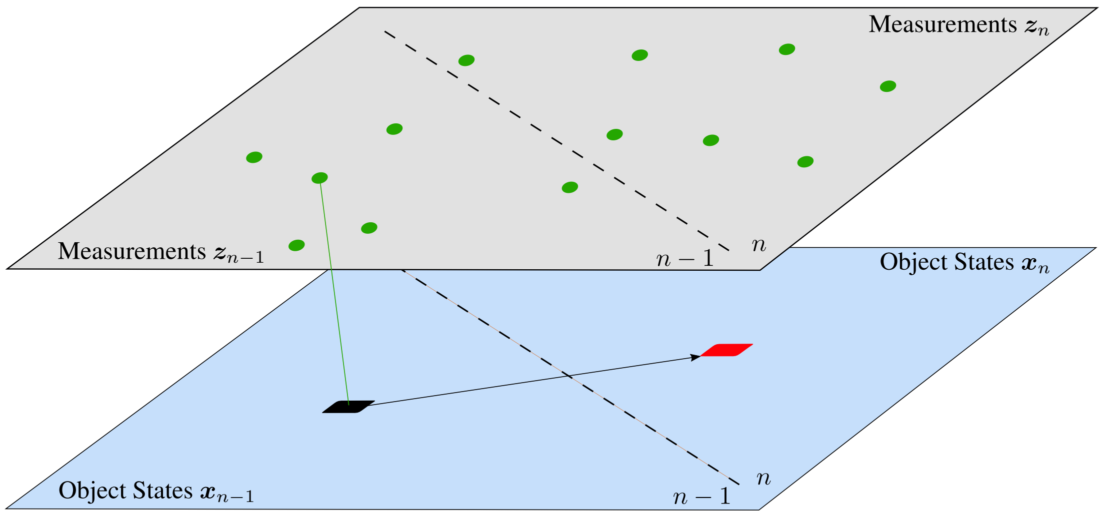
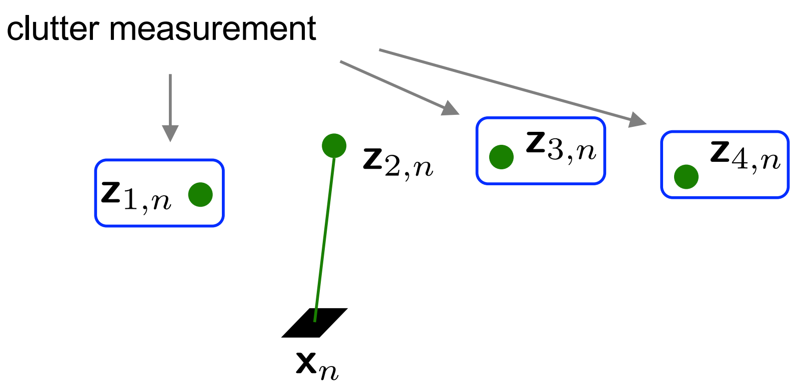


The Association Variable

- Object-oriented association variable $a_n \in \{0, 1, \ldots, \boxed{M_n}\}$ number of measurements
  - $a_n = m > 0$: at time n, the object generates the measurement with index m
  - $a_n = 0$: at time n, the object did not generate a measurement

- Example:



$M_n = 4$ $a_n = 2$

Probabilistic Data Association Filter

- **Prediction Step**

$$\underbrace{f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1})}_{\substack{\text{Predicted}\\ \text{posterior pdf}}} = \int \underbrace{f(\boldsymbol{x}_n|\boldsymbol{x}_{n-1})}_{\substack{\text{State-transition}\\ \text{pdf}}} \underbrace{f(\boldsymbol{x}_{n-1}|\boldsymbol{y}_{1:n-1})}_{\substack{\text{Previous}\\ \text{posterior pdf}}} \mathrm{d}\boldsymbol{x}_{n-1}$$

- **Updated Step**

$$\underbrace{f(\boldsymbol{x}_n|\boldsymbol{z}_{1:n})}_{\text{Posterior pdf}} \propto \underbrace{f(\boldsymbol{x}_n|\boldsymbol{z}_{1:n-1})}_{\substack{\text{Predicted}\\ \text{posterior pdf}}} \sum_{m=0}^{M_n} g_{\boldsymbol{z}_n}(\boldsymbol{x}_n, a_n = m)$$

$$= f(\boldsymbol{x}_n|\boldsymbol{z}_{1:n-1}) \left((1 - p_\mathrm{d}) + \frac{p_\mathrm{d} f(\boldsymbol{z}_{m,1}|\boldsymbol{x}_n)}{\mu_\mathrm{c} f_\mathrm{c}(\boldsymbol{z}_{m,1})} + \cdots + \frac{p_\mathrm{d} f(\boldsymbol{z}_{m,M_n}|\boldsymbol{x}_n)}{\mu_\mathrm{c} f_\mathrm{c}(\boldsymbol{z}_{m,M_n})} \right)$$

Key Parameters

- Posterior Distribution

$$f(\boldsymbol{x}_n|\boldsymbol{z}_{1:n}) \propto f(\boldsymbol{x}_n|\boldsymbol{z}_{1:n-1}) \left((1-p_\mathrm{d}) + \frac{p_\mathrm{d} f(\boldsymbol{z}_{1,n}|\boldsymbol{x}_n)}{\mu_\mathrm{c} f_\mathrm{c}(\boldsymbol{z}_{1,n})} + \cdots + \frac{p_\mathrm{d} f(\boldsymbol{z}_{M_n,n}|\boldsymbol{x}_n)}{\mu_\mathrm{c} f_\mathrm{c}(\boldsymbol{z}_{M_n,n})} \right)$$

- Probability that a measurements is generated by the object $0 < p_\mathrm{d} \leqslant 1$ (probability of detection)

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- Mean number of clutter measurements $0 < \mu_{\mathrm{c}}$

Key Parameters

- Posterior Distribution

$$f(\boldsymbol{x}_n|\boldsymbol{z}_{1:n}) \propto f(\boldsymbol{x}_n|\boldsymbol{z}_{1:n-1})\left((1-p_{\mathrm{d}}) + \frac{p_{\mathrm{d}} f(\boldsymbol{z}_{1,n}|\boldsymbol{x}_n)}{\mu_{\mathrm{c}} f_{\mathrm{c}}(\boldsymbol{z}_{1,n})} + \cdots + \frac{p_{\mathrm{d}} f(\boldsymbol{z}_{M_n,n}|\boldsymbol{x}_n)}{\mu_{\mathrm{c}} f_{\mathrm{c}}(\boldsymbol{z}_{M_n,n})}\right)$$

- Probability that a measurements is generated by the object $0 < p_{\mathrm{d}} \leqslant 1$ (probability of detection)

- Mean number of clutter measurements $0 < \mu_{\mathrm{c}}$

- Clutter pdf $0 < f_{\mathrm{c}}(\boldsymbol{z}_m)$

Linear-Gaussian State-Space Model

- Consider a sequence of states $\boldsymbol{x}_n$ and a sequence of measurements $\boldsymbol{y}_n$

State-Transition Model:

State $\boldsymbol{x}_n$ evolves according to

$$\boldsymbol{x}_n = \boldsymbol{G}_n \boldsymbol{x}_{n-1} + \underbrace{\boldsymbol{u}_n}_{\text{driving noise (white)}}$$

$\Rightarrow f(\boldsymbol{x}_n|\boldsymbol{x}_{n-1})$

with Gaussian driving noise

$$\boldsymbol{u}_n \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}_{\boldsymbol{u}_n})$$

Model for Object Generated Meas.:

Measurement $\boldsymbol{y}_n$ is generated as

$$\boldsymbol{y}_{n,m} = \boldsymbol{H}_n \boldsymbol{x}_n + \underbrace{\boldsymbol{v}_n}_{\text{measurement noise (white)}}$$

$\Rightarrow f(\boldsymbol{y}_{n,m}|\boldsymbol{x}_n)$

with Gaussian measurement noise

$$\boldsymbol{v}_n \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}_{\boldsymbol{v}_n})$$

- Prior PDF at $n = 0$, $\boldsymbol{x}_0 \sim \mathcal{N}(\boldsymbol{\mu}_{\boldsymbol{x}_0}, \boldsymbol{\Sigma}_{\boldsymbol{x}_0})$

Prob. Data Association with Linear-Gaussian Model

- Let us assume $f(\boldsymbol{x}_{n-1}|\boldsymbol{y}_{1:n-1})$ is Gaussian with mean $\boldsymbol{\mu}_{\boldsymbol{x}_{n-1}}$ and covariance $\boldsymbol{\Sigma}_{\boldsymbol{x}_{n-1}}$

- The Prediction step can be performed in closed form (as in the Kalman filter), i.e., $f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n-1})$ is Gaussian with mean $\boldsymbol{\mu}^-_{\boldsymbol{x}_n}$ and covariance $\boldsymbol{\Sigma}^-_{\boldsymbol{x}_n}$ given as

$$\boldsymbol{\mu}^-_{\boldsymbol{x}_n} = G_n \boldsymbol{\mu}_{\boldsymbol{x}_{n-1}} \qquad\qquad \boldsymbol{\Sigma}^-_{\boldsymbol{x}_n} = \boldsymbol{G}_n \boldsymbol{\Sigma}_{\boldsymbol{x}_{n-1}} \boldsymbol{G}_n^{\mathrm{T}} + \boldsymbol{\Sigma}_{\boldsymbol{u}_n}$$

- **Goal:** Closed-form solution for the update step such that $f(\boldsymbol{x}_n|\boldsymbol{y}_{1:n})$ can be represented be mean $\boldsymbol{\mu}_{\boldsymbol{x}_n}$ and covariance $\boldsymbol{\Sigma}_{\boldsymbol{x}_n}$

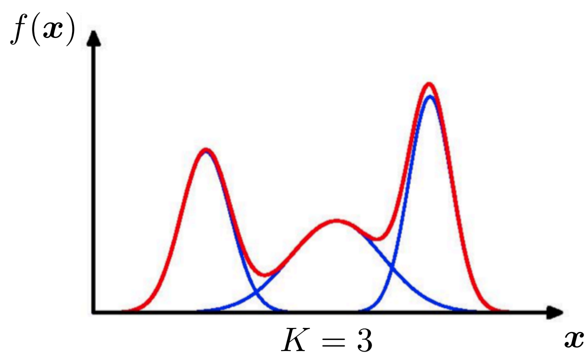
The Gaussian Mixture Distribution

- A continuous random variable $\boldsymbol{x}$ is said to have a Gaussian mixture distribution with K components and parameters w_k, $\boldsymbol{\mu}_k$, $\boldsymbol{\Sigma}_k$, $k = 1, \dots, K$ if its probability density function is given by

$$f(\boldsymbol{x}) = \sum_{k=1}^{K} w_k \, f_{\mathrm{g}}(\boldsymbol{x}; \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$$

- The $f_{\mathrm{g}}(\boldsymbol{x}; \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$ are Gaussian distributions with mean $\boldsymbol{\mu}_k$ and covariance matrix $\boldsymbol{\Sigma}_k$

- The weights $w_k > 0$ normalize to one, i.e., $\sum_{k=1}^{K} w_k = 1$



Mean and Covariance of Gaussian Mixture Distribution

- Let $f(\boldsymbol{x})$ be a Gaussian mixture distributions with parameters w_k, $\boldsymbol{\mu}_k$, $\boldsymbol{\Sigma}_k$, $k = 1, \dots, K$

- The mean of $f(\boldsymbol{x})$ is given by

$$\boldsymbol{\mu}_{\boldsymbol{x}} = w_1 \boldsymbol{\mu}_1 + w_2 \boldsymbol{\mu}_2 + \cdots + w_K \boldsymbol{\mu}_K$$

- The covariance of $f(\boldsymbol{x})$ is given by

$$\boldsymbol{\Sigma}_{\boldsymbol{x}} = \sum_{k=1}^{K} w_k \boldsymbol{\Sigma}_k + \sum_{k=1}^{K} w_k \boldsymbol{\mu}_k \boldsymbol{\mu}_k^{\mathrm{T}} - \boldsymbol{\mu}_{\boldsymbol{x}} \boldsymbol{\mu}_{\boldsymbol{x}}^{\mathrm{T}}$$

Closed-Form Update Step

- Recall posterior distribution ($\boldsymbol{z}_{1:n}$ is observed and thus fixed)

$$f(\boldsymbol{x}_n|\boldsymbol{z}_{1:n}) \propto f(\boldsymbol{x}_n|\boldsymbol{z}_{1:n-1}) \left((1-p_{\mathrm{d}}) + \frac{p_{\mathrm{d}} f(\boldsymbol{z}_{1,n}|\boldsymbol{x}_n)}{\mu_{\mathrm{c}} f_{\mathrm{c}}(\boldsymbol{z}_{1,n})} + \cdots + \frac{p_{\mathrm{d}} f(\boldsymbol{z}_{M_n,n}|\boldsymbol{x}_n)}{\mu_{\mathrm{c}} f_{\mathrm{c}}(\boldsymbol{z}_{M_n,n})} \right)$$

- **Theorem:** If the predicted posterior $f(\boldsymbol{x}_n|\boldsymbol{z}_{1:n-1})$ is Gaussian, with mean $\boldsymbol{\mu}^-_{\boldsymbol{x}_n}$ and covariance $\boldsymbol{\Sigma}^-_{\boldsymbol{x}_n}$ and the model for the object-generated measurement is linear-Gaussian, then $f(\boldsymbol{x}_n|\boldsymbol{z}_{1:n})$ is a Gaussian mixture distribution with $M_n + 1$ components and parameters

$$w_m \propto \frac{p_{\mathrm{d}} f(\boldsymbol{z}_{m,n}|\boldsymbol{z}_{1:n})}{\mu_{\mathrm{c}} f_{\mathrm{c}}(\boldsymbol{z}_{m,n})} \qquad m \in \{1, \ldots, M_n\} \qquad w_{M_n+1} \propto (1-p_{\mathrm{d}})$$

$$\boldsymbol{\mu}_m = \boldsymbol{\mu}^-_{\boldsymbol{x}_n} + \boxed{\boldsymbol{K}_n}(\boldsymbol{z}_{m,n} - \boldsymbol{H}_n \boldsymbol{\mu}^-_{\boldsymbol{x}_n}) \qquad \boldsymbol{\mu}_{M_n+1} = \boldsymbol{\mu}^-_{\boldsymbol{x}_n}$$

Kalman gain

$$\boldsymbol{\Sigma}_m = \boldsymbol{\Sigma}^-_{\boldsymbol{x}_n} - \boxed{\boldsymbol{K}_n}\boldsymbol{H}_n \boldsymbol{\Sigma}^-_{\boldsymbol{x}_n} \qquad \boldsymbol{\Sigma}_{M_n+1} = \boldsymbol{\Sigma}^-_{\boldsymbol{x}_n}$$

Closed-Form Update Step - Sketch of Proof

- Let's take a look at the single component $m \in \{1, \ldots, M_n\}$

$$\frac{p_{\mathrm{d}} f(\boldsymbol{z}_{m,n}|\boldsymbol{x}_n) f(\boldsymbol{x}_n|\boldsymbol{z}_{1:n-1})}{\mu_{\mathrm{c}} f_{\mathrm{c}}(\boldsymbol{z}_{m,n})} = \frac{p_{\mathrm{d}} f(\boldsymbol{z}_{m,n}|\boldsymbol{x}_n, \boldsymbol{z}_{1:n-1}) f(\boldsymbol{x}_n|\boldsymbol{z}_{1:n-1})}{\mu_{\mathrm{c}} f_{\mathrm{c}}(\boldsymbol{z}_{m,n})}$$

Statistical Independent Meas. & Driving Noise

$$= \frac{p_{\mathrm{d}} f(\boldsymbol{z}_{m,n}, \boldsymbol{x}_n|\boldsymbol{z}_{1:n-1})}{\mu_{\mathrm{c}} f_{\mathrm{c}}(\boldsymbol{z}_{m,n})}$$

$$= \frac{p_{\mathrm{d}} f(\boldsymbol{z}_{m,n}|\boldsymbol{z}_{1:n-1})}{\mu_{\mathrm{c}} f_{\mathrm{c}}(\boldsymbol{z}_{m,n})} f(\boldsymbol{x}_n|\boldsymbol{z}_{1:n-1}, \boldsymbol{z}_{m,n})$$

Kalman Update Step

$$= \frac{p_{\mathrm{d}} f(\boldsymbol{z}_{m,n}|\boldsymbol{z}_{1:n-1})}{\mu_{\mathrm{c}} f_{\mathrm{c}}(\boldsymbol{z}_{m,n})} f_{\mathrm{g}}(\boldsymbol{x}_n; \boldsymbol{\mu}_m, \boldsymbol{\Sigma}_m)$$

$$\boldsymbol{\mu}_m = \boldsymbol{\mu}^-_{\boldsymbol{x}_n} + \boldsymbol{K}_n(\boldsymbol{z}_{m,n} - \boldsymbol{H}_n \boldsymbol{\mu}^-_{\boldsymbol{x}_n})$$

$$\boldsymbol{\Sigma}_m = \boldsymbol{\Sigma}^-_{\boldsymbol{x}_n} - \boldsymbol{K}_n \boldsymbol{H}_n \boldsymbol{\Sigma}^-_{\boldsymbol{x}_n}$$

$$\boldsymbol{K}_n = \boldsymbol{\Sigma}^-_{\boldsymbol{x}_n} \boldsymbol{H}_n^{\mathrm{T}} (\boldsymbol{H}_n \boldsymbol{\Sigma}^-_{\boldsymbol{x}_n} \boldsymbol{H}_n^{\mathrm{T}} + \boldsymbol{\Sigma}_{\boldsymbol{v}_n})^{-1}$$

- The conditional evidence $f(z_{m,n}|z_{1:n-1})$ is given by

$$f(\boldsymbol{z}_{m,n}|\boldsymbol{z}_{1:n-1}) = f_{\mathrm{g}}(\boldsymbol{z}_{m,n}; \boldsymbol{H}_n \boldsymbol{\mu}^-_{\boldsymbol{x}}, \boldsymbol{H}_n \boldsymbol{\Sigma}^-_{\boldsymbol{x}_n} \boldsymbol{H}_n^{\mathrm{T}} + \boldsymbol{\Sigma}_{\boldsymbol{v}_n})$$

Closed-Form Update Step - Discussion

- Mean and covariances $\boldsymbol{\mu}_m, \boldsymbol{\Sigma}_m, m = 1, \dots, M_n + 1$ of the Gaussian mixture distribution are obtained by
  - performing the Kalman update step for each ``measurement component'' $m = 1, \dots, M_n$
  - keeping predicted mean and covariance for the ``missed-detection component'' $m = M_n + 1$

- Weights $\boldsymbol{\mu}_m, \boldsymbol{\Sigma}_m, m = 1, \dots, M_n + 1$ are given as

$$w_m \propto p_{\mathrm{d}} f(\boldsymbol{z}_{m,n} | \boldsymbol{z}_{1:n}) / \mu_{\mathrm{c}} f_{\mathrm{c}}(\boldsymbol{z}_{m,n}), \qquad m = 1, \dots, M_n \qquad\qquad w_{M_n+1} \propto (1 - p_{\mathrm{d}})$$

  - The probability of detection p_{d} determines the ratio between measurement component weights and missed-detection component weight
  - Large conditional evidence $f(\boldsymbol{z}_{m,n} | \boldsymbol{z}_{1:n})$ means that measurement $\boldsymbol{z}_{m,n}$ is likely to be object generated
  - Large $\mu_{\mathrm{c}} f_{\mathrm{c}}(\boldsymbol{z}_{m,n})$ means that the measurement is likely to be clutter

Example

Posterior at time $n-1$

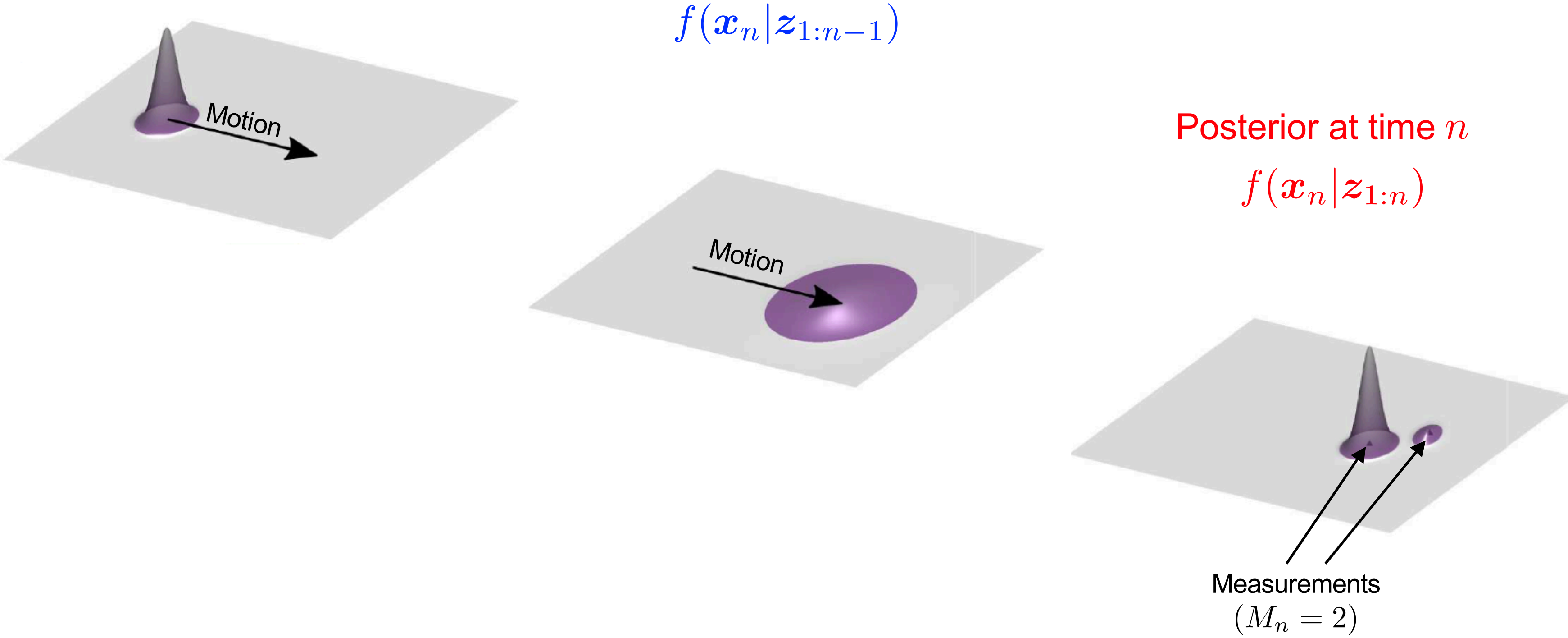
$f(\boldsymbol{x}_{n-1}|\boldsymbol{z}_{1:n-1})$

Predicted Posterior at time n

$f(\boldsymbol{x}_{n}|\boldsymbol{z}_{1:n-1})$

Posterior at time n

$f(\boldsymbol{x}_{n}|\boldsymbol{z}_{1:n})$



Closed-Form Update Step - Approximation

- Let's assume $f(\boldsymbol{x}_{n-1}|\boldsymbol{z}_{1:n-1})$ is a Gaussian mixture distribution with K components

- At time n, we could calculate a predicted posterior $f(\boldsymbol{x}_n|\boldsymbol{z}_{1:n-1})$ that has a Gaussian mixture distribution from $f(\boldsymbol{x}_{n-1}|\boldsymbol{z}_{1:n-1})$ by performing K prediction steps

- However, after the following update step, we would obtain a Gaussian mixture with $K(M_n+1)$ components $\Rightarrow$ complexity of the resulting algorithm has a computational complexity that scales exponentially with time n

- Thus, after each update step, we approximate the update posterior $f(\boldsymbol{x}_n|\boldsymbol{z}_{1:n})$ by a single Gaussian with a mean $\boldsymbol{\mu}_{\boldsymbol{x}_n}$ and covariance $\boldsymbol{\Sigma}_{\boldsymbol{x}_n}$ that are equal to the mean and covariance of its Gaussian mixture distribution (moment matching)

Closed-Form Update Step - Summary

- Step 1: Calculate means and covariances of mixture components:

$$\boldsymbol{\mu}_m = \boldsymbol{\mu}^-_{\boldsymbol{x}_n} + \boldsymbol{K}_n\big(\boldsymbol{z}_{m,n} - \boldsymbol{H}_n\boldsymbol{\mu}^-_{\boldsymbol{x}_n}\big) \qquad m = 1,\dots,M_n$$

$$\boldsymbol{\Sigma}_m = \boldsymbol{\Sigma}^-_{\boldsymbol{x}_n} - \boldsymbol{K}_n\boldsymbol{H}_n\boldsymbol{\Sigma}^-_{\boldsymbol{x}_n}$$

$$\boldsymbol{K}_n = \boldsymbol{\Sigma}^-_{\boldsymbol{x}_n}\boldsymbol{H}_n^{\mathrm{T}}\big(\boldsymbol{H}_n\boldsymbol{\Sigma}^-_{\boldsymbol{x}_n}\boldsymbol{H}_n^{\mathrm{T}} + \boldsymbol{\Sigma}_{\boldsymbol{v}_n}\big)^{-1}$$

$$\boldsymbol{\mu}_{M_n+1} = \boldsymbol{\mu}^-_{\boldsymbol{x}_n}$$

$$\boldsymbol{\Sigma}_{M_n+1} = \boldsymbol{\Sigma}^-_{\boldsymbol{x}_n}$$

- Step 2: Calculate unnormalized weights:

$$\tilde{w}_m = \frac{p_{\mathrm{d}} f_{\mathrm{g}}(\boldsymbol{z}_{m,n}; \boldsymbol{H}_n\boldsymbol{\mu}^-_{\boldsymbol{x}_n}, \boldsymbol{H}_n\boldsymbol{\Sigma}^-_{\boldsymbol{x}_n}\boldsymbol{H}_n^{\mathrm{T}} + \boldsymbol{\Sigma}_{\boldsymbol{v}_n})}{\mu_{\mathrm{c}} f_{\mathrm{c}}(\boldsymbol{z}_{m,n})} \qquad m = 1,\dots,M_n \qquad \tilde{w}_{M_n+1} = (1 - p_{\mathrm{d}})$$

- Step 3: Normalize weights: $w_m = \tilde{w}_m / \big(\sum_{m'=1}^{M_n+1} \tilde{w}_{m'}\big)$

- Step 4: Approximate Gaussian mixture by a single Gaussian with same mean and covariance (moment matching):

$$\boldsymbol{\mu}_{\boldsymbol{x}_n} = \sum_{m=1}^{M_n+1} w_m\,\boldsymbol{\mu}_m \qquad \boldsymbol{\Sigma}_{\boldsymbol{x}_n} = \sum_{m=1}^{M_n+1} w_m\,\boldsymbol{\Sigma}_m + \sum_{m=1}^{M_n+1} w_m\,\boldsymbol{\mu}_m\boldsymbol{\mu}_m^{\mathrm{T}} - \boldsymbol{\mu}_{\boldsymbol{x}_n}\boldsymbol{\mu}_{\boldsymbol{x}_n}^{\mathrm{T}}$$

- **Result:** Mean $\boldsymbol{\mu}_{\boldsymbol{x}_n}$ and covariance $\boldsymbol{\Sigma}_{\boldsymbol{x}_n}$ representing the posterior distribution $f(\boldsymbol{x}_n|\boldsymbol{z}_{1:n})$

Y. Bar-Shalom, F. Daum, and J. Huang, *The Probabilistic Data Association Filter*, IEEE Contr. Syst. Mag., 2009

Conclusion

- Single object tracking in clutter for linear-Gaussian system models
  - prediction and update steps can be performed in closed-form
  - posterior distributions are Gaussian mixture densities with a number of components that scales exponentially with time
  - to limit computational complexity, the posterior distribution is approximate by a single Gaussian after each update step