Group Delay, All-Pass Filters, and Generalized Linear Phase

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Group Delay

- In addition to magnitude, one may have some requirements of phase.
- Group delay is a useful concept for this purpose
  - $\tau(\omega) = -\frac{d}{d\omega}\arg(H(e^{j\omega}))$
  - $\tau(\omega_0)$ is a measure of the amount of delay a narrow band signal centered at frequency ω_0 will experience

Magnitude, Phase, Group-Delay of Difference Equations Based LTI systems

$$H(e^{j\omega}) = \frac{B(e^{j\omega})}{A(e^{j\omega})} = \frac{\sum_{m=0}^{M} b_m e^{-j\omega m}}{\sum_{k=0}^{N} a_k e^{-j\omega k}} = \frac{b_0}{a_0}\frac{\Pi_{m=1}^{M}(1 - z_m e^{-j\omega})}{\Pi_{k=1}^{N}(1 - p_k e^{-j\omega})}$$

Magnitude Response

$$\begin{aligned} 20\log_{10}|H(e^{j\omega})| &= 20\log_{10}|\frac{b_0}{a_0}| + \sum_{m=1}^{M} 20\log_{10}|1 - z_m e^{-j\omega}| \\ &\quad - \sum_{k=1}^{N} 20\log_{10}|1 - p_k e^{-j\omega}| \end{aligned}$$

Phase

$$\phi(\omega) = \angle H(e^{j\omega}) = \angle\frac{b_0}{a_0} + \sum_{m=1}^{M}\angle(1 - z_m e^{-j\omega}) - \sum_{k=1}^{N}\angle(1 - p_k e^{-j\omega})$$

Group-Delay

$$\tau(\omega) = -\frac{d\phi(\omega)}{d\omega} = -\sum_{m=1}^{M}\frac{d\angle(1 - z_m e^{-j\omega}}{d\omega}) + \sum_{k=1}^{N}\frac{d\angle(1 - p_k e^{-j\omega}}{d\omega})$$

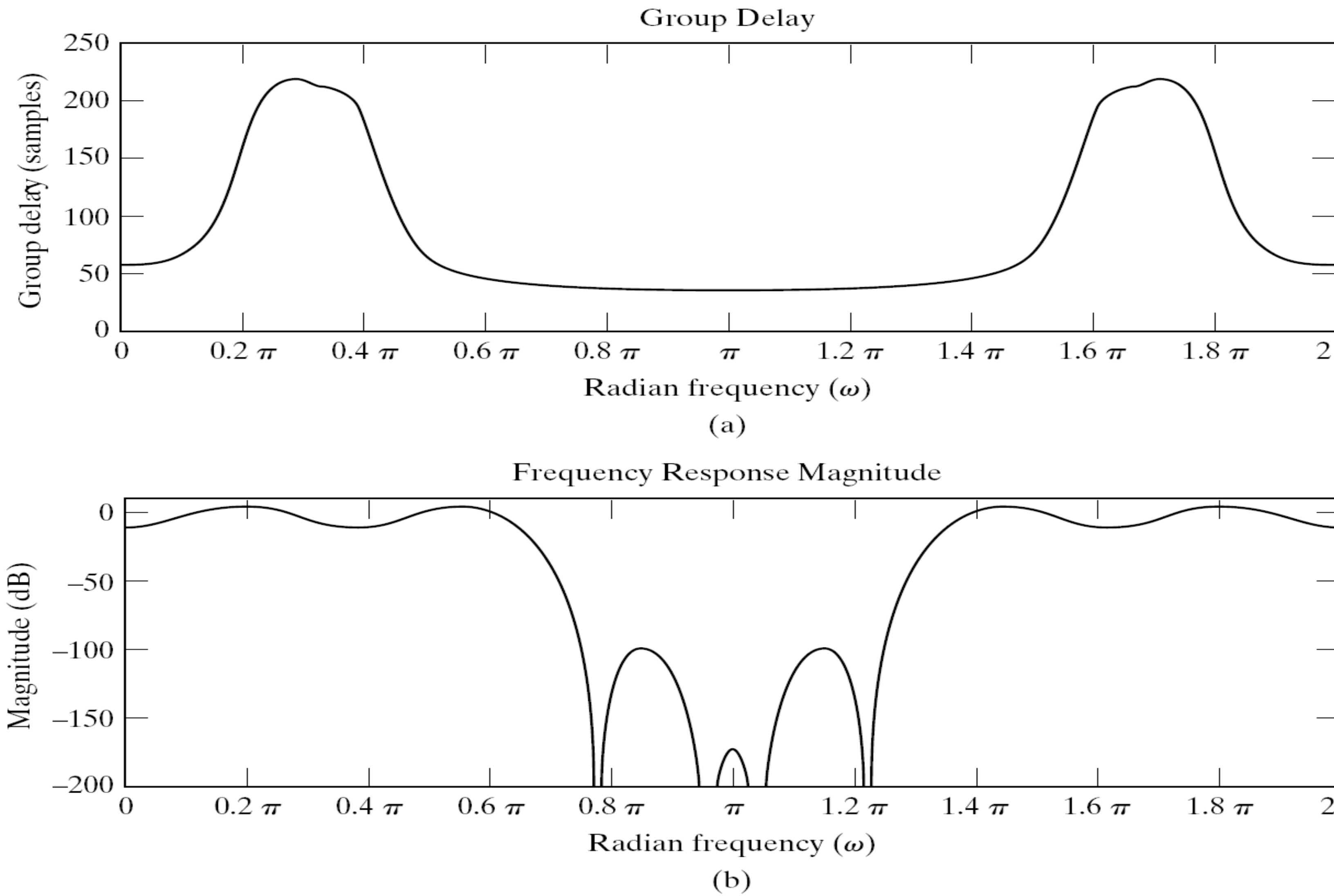


Figure 5.1 Frequency response magnitude and group delay for the filter in Example 5.1.

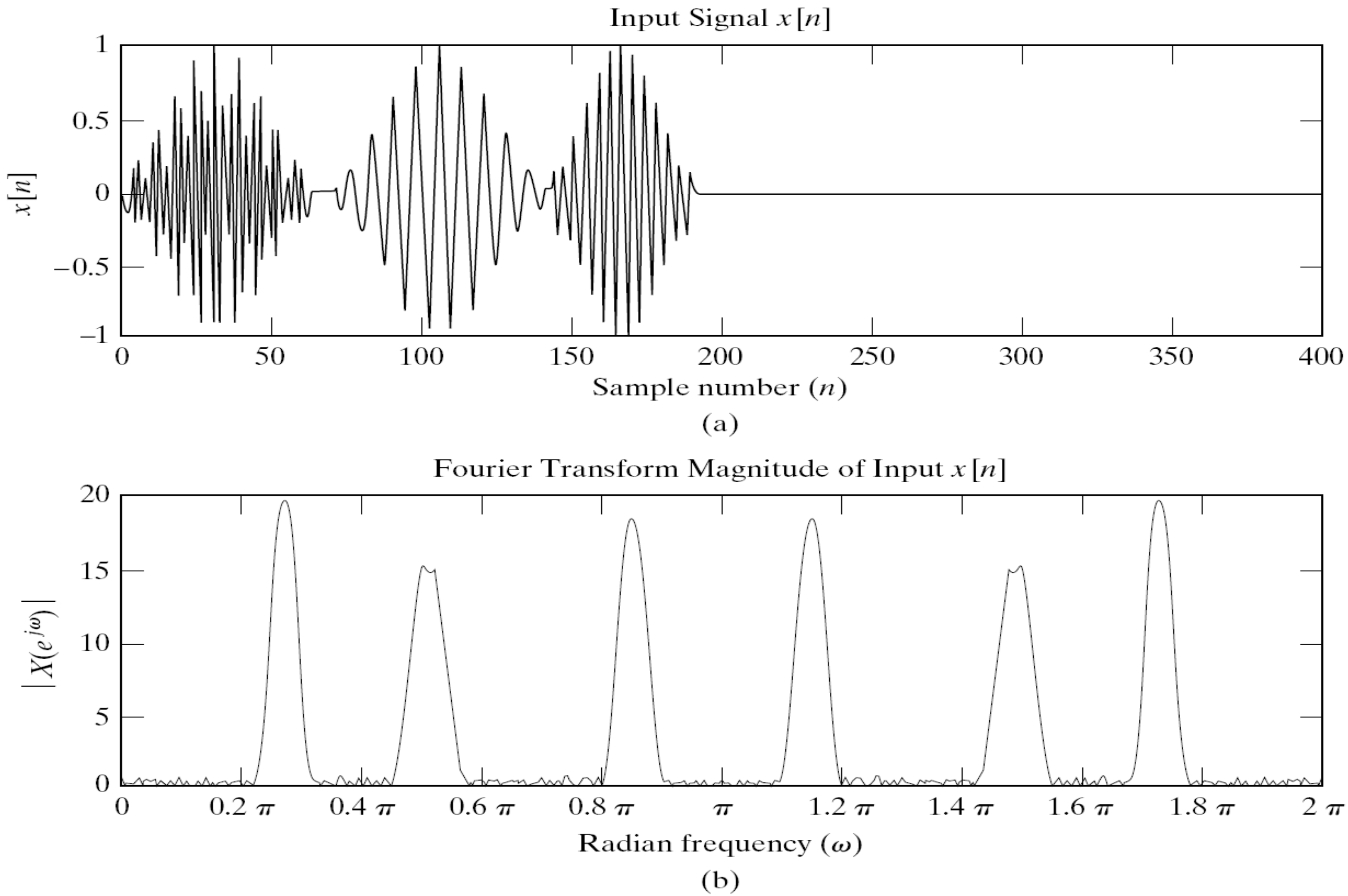


Figure 5.2 Input signal and associated Fourier transform magnitude for Example 5.1.

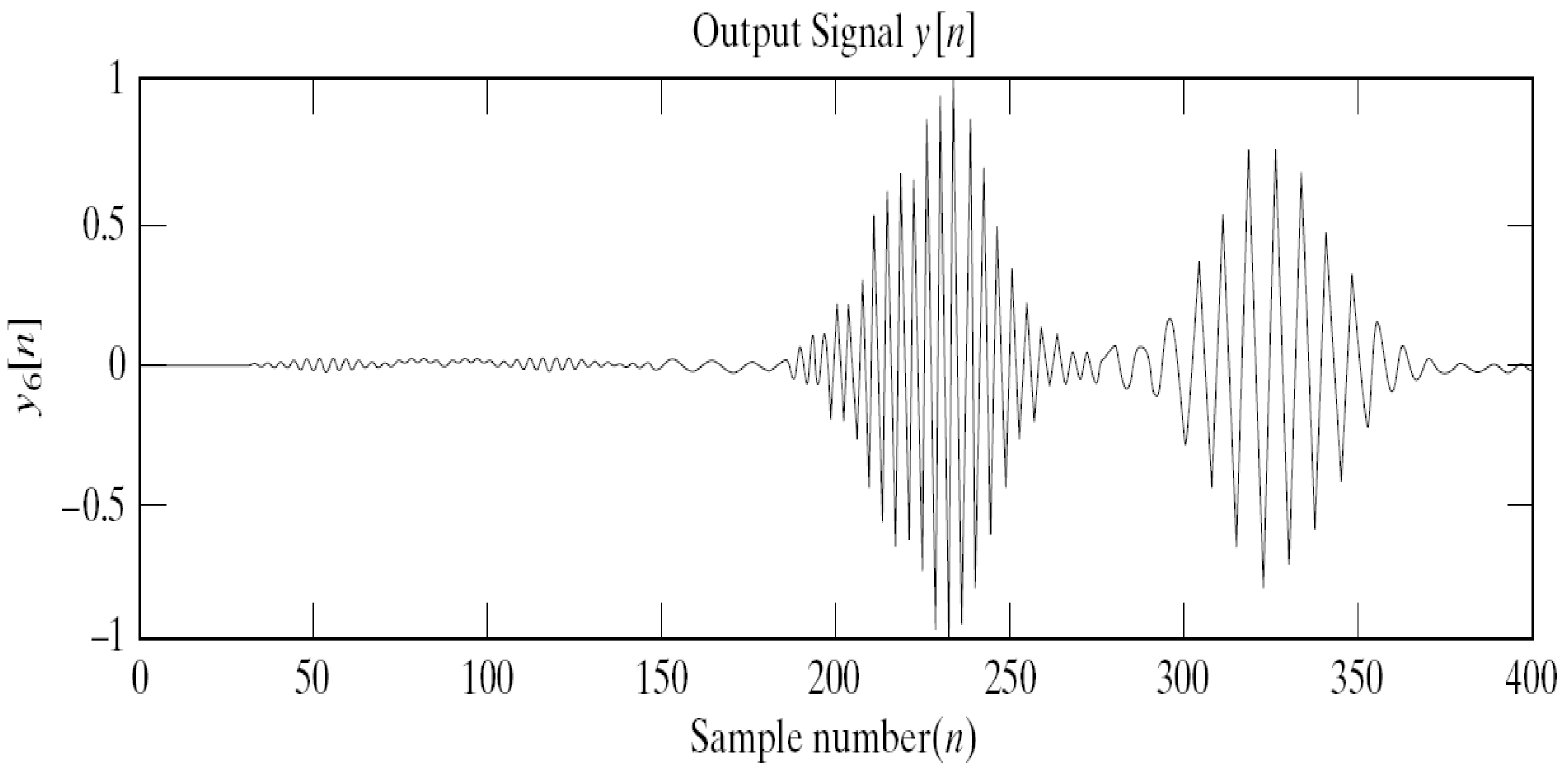


Figure 5.3 Output signal for Example 5.1.

All Pass Filters

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}, |a| < 1$$

With $a = re^{j\theta}$, can show $|H_{ap}(e^{j\omega})| = 1$, and

$$\phi(\omega) = \angle H_{ap}(e^{j\omega}) = -\omega - 2\arctan\left[\frac{r\sin(\omega - \theta)}{1 - r\cos(\omega - \theta)}\right]$$

$$\mathrm{grd}(H(e^{j\omega}) = -\frac{d\phi(\omega)}{d\omega} = \frac{1 - r^2}{1 + r^2 - 2r\cos(\omega - \theta)} = \frac{1 - r^2}{|1 - r^{j\theta}e^{-j\omega}|^2} \geq 0$$

A first order all pass filter has a pole at $a = re^{j\theta}$ and a zero at $\frac{1}{a^*} = \frac{1}{r}e^{j\theta}$.

Magnitude and Phase Plots for First Order All Pass Filter

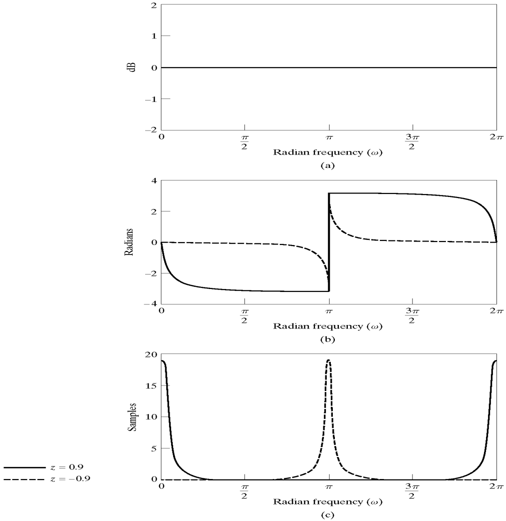
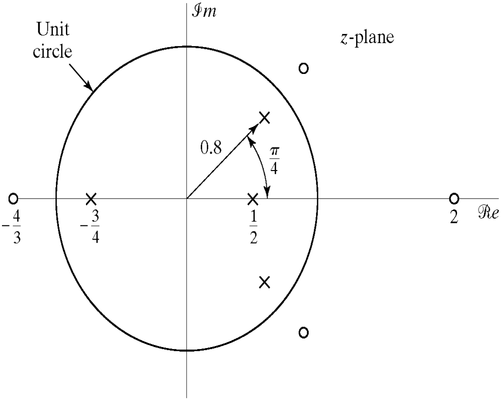


Figure 5.22 Frequency response for all-pass filters with real poles at $z = 0.9$ (solid line) and $z = -0.9$ (dashed line). (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

Poles-Zeros for a General All-Pass Filter

$$H_{ap}(z) = \Pi_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \;\; \Pi_{k=1}^{M_c} \frac{(z^{-1} - e_k)(z^{-1} - e_k^*)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}, \quad (d_k \;\; \text{real})$$



Desirable Phase

Let $x_1[n]$ be a low frequency signal (lies in pass-band) we wish to retain, and $x_2[n]$ is a high frequency signal (lies in stop-band) we wish to remove.

So the output of the filter

$$y[n] = h[n] * (x_[n] + x_2[n]) = h[n] * x_1[n] + h[n] * x_2[n] = h[n] * x_1[n]$$

Since the goal is to retain $x_1[n]$ without distortion, we want

$$y[n] = x_1[n - n_0]$$

where n_0 is the delay the signal experience. Hence $H(e^{j\omega}) = e^{-j\omega n_0}$ in the pass-band. Not concerned about phase in stop-band.

Desirable Phase: Linear Phase in the pass-band.

Generalized Linear Phase

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega}e^{j\beta},$$

where $A(e^{j\omega})$ is real, i.e. $A(e^{j\omega}) = A^*(e^{j\omega})$.

Theorem: A linear phase Real, Causal, Stable, Rational (RCSR) filter is necessarily FIR. The phase or group delay of such a filter is half its order; it satisfies either the symmetry relationship ($h[n] = h[M-n]$) or the antisymmetry relationship ($h[n] = -h[M-n]$)

Proof: 3 parts

Part 1: α is either L or $L + \frac{1}{2}$, where L is an integer.

Part 2: $\beta = 0$ or $\beta = \frac{\pi}{2}$

Part 3: $h[n]$ is FIR and $h[n] = \pm h[M-n], \;\; M = 2\alpha$.

Part 1

α is either L or $L + \frac{1}{2}$, where L is an integer.
Generalized Linear Phase

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega}e^{j\beta},$$

$$\begin{aligned}
H(e^{j(\omega+2\pi)}) &= H(e^{j\omega}) \\
\Rightarrow A(e^{j(\omega+2\pi)})e^{-j\alpha(\omega+2\pi)}e^{j\beta} &= A(e^{j\omega})e^{-j\alpha\omega}e^{j\beta} \\
\Rightarrow A(e^{j(\omega+2\pi)})e^{-j\alpha 2\pi} &= A(e^{j\omega}) \\
\Rightarrow e^{-j\alpha 2\pi} &= \frac{A(e^{j\omega})}{A(e^{j(\omega+2\pi)})}
\end{aligned}$$

Using the fact that $A(e^{j\omega})$ is real and $|A(e^{j\omega})| = |A(e^{j(\omega+2\pi)})|$, we have

$$e^{-j\alpha 2\pi} = \frac{A(e^{j\omega})}{A(e^{j(\omega+2\pi)})} = \pm 1$$

This implies that 2α is an integer, or α is either L or $L + \frac{1}{2}$, where L is an integer.

Part 2

$\beta = 0$ or $\beta = \frac{\pi}{2}$
Generalized Linear Phase

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega}e^{j\beta},$$

$$\begin{aligned}
\text{impulse response real } \Rightarrow H(e^{j\omega}) &= H^*(e^{-j\omega}) \\
\Rightarrow A(e^{j\omega})e^{-j\alpha\omega}e^{j\beta} &= (A(e^{-j\omega})e^{j\alpha\omega}e^{\dagger j\beta})^* \\
&= A(e^{-j\omega})e^{-j\alpha\omega}e^{-j\beta} \\
e^{j2\beta} &= \frac{A(e^{-j\omega})}{A(e^{j\omega})}
\end{aligned}$$

Since $A(e^{j\omega})$ is real and $|A(e^{j\omega})| = |A(e^{-j\omega})|$, we have

$$e^{j2\beta} = \frac{A(e^{-j\omega})}{A(e^{j\omega})} = \pm 1.$$

Hence $\beta = 0$ or $\beta = \pm\frac{\pi}{2}$. Consider $\beta = 0$ and $\beta = \frac{\pi}{2}$ without loss of generality.

Part 3

$h[n]$ is FIR and $h[n] = \pm h[M-n]$, $M = 2\alpha$.
Generalized Linear Phase

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega}e^{j\beta},$$

$$\begin{aligned}
A(e^{j\omega}) &= H(e^{j\omega})e^{j\alpha\omega}e^{-j\beta} \\
h[2\alpha - n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega(2\alpha-n)}d\omega = \frac{1}{2\pi}e^{j\beta}\int_{-\pi}^{\pi} A(e^{j\omega})e^{-j\alpha\omega}e^{j\omega(2\alpha-n)}d\omega \\
&= \frac{1}{2\pi}e^{j\beta}\int_{-\pi}^{\pi} A(e^{j\omega})e^{j\omega(\alpha-n)}d\omega = \left(\frac{1}{2\pi}e^{j\beta}\int_{-\pi}^{\pi} A(e^{j\omega})e^{j\omega(\alpha-n)}d\omega\right)^{*} \\
&= \frac{e^{-j\beta}}{2\pi}\int_{-\pi}^{\pi} A(e^{j\omega})e^{+j\omega(n-\alpha)}d\omega = \frac{e^{-j\beta}}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\alpha\omega}e^{-j\beta}e^{j\omega(n-\alpha)}d\omega \\
&= \frac{e^{-j2\beta}}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega = h[n]e^{-j\beta 2}
\end{aligned}$$

$$h[2\alpha - n] = h[n]e^{-2j\beta}$$

Consider $\beta = 0$, then $h[2\alpha - n] = h[n]$.

Consider $\beta = \frac{\pi}{2}$, then $h[2\alpha - n] = -h[n]$.

From causality property of $h[n]$ we have $h[2\alpha - n] = 0$ for $n > 2\alpha$. Hence

$$h[n] = \pm h[2\alpha - n] = 0, \; \forall n > 2\alpha.$$

Since $\alpha = L$ or $L + \frac{1}{2}$, where L is an integer, this implies that $h[n]$ is FIR.

Let $M = 2\alpha$, then $h[n] = \pm h[M - n]$.