# ECE 161A: The Z-Transform

Florian Meyer
University of California, San Diego
Email: flmeyer@ucsd.edu

# Z-Transform Definition

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Will use the notation $X(z) = \mathcal{Z}(x[n])$.

Definition not complete without specifying the Region Of Convergence (ROC)

$$ROC : \{z : \sum_{n=-\infty}^{\infty} |x[n]z^{-n}| = \sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty\}$$

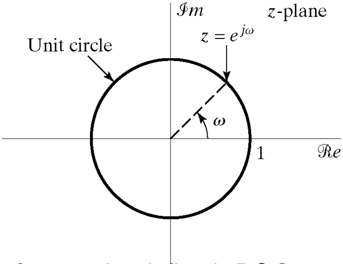
Why $Z$-transforms?

- More general: Applicable to a larger class of signals
- Easier to manipulate: Complex variable theory can be useful (Laurent series and associated results).

# Z-Transform versus DTFT

$$X(e^{j\omega}) = X(z)_{|z=e^{j\omega}}$$

Since $|e^{j\omega}| = 1$, DTFT is the z-transform evaluated on the unit circle.



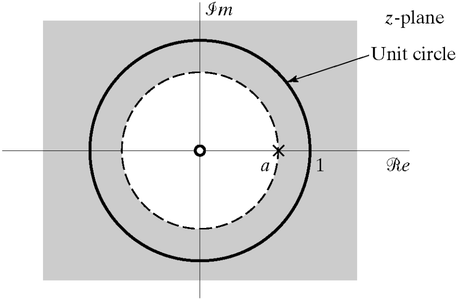

For the Fourier Transform to be defined, ROC must include the unit circle. Sequence must be absolutely summable.

# Example 1: Right Sided Sequence

Consider $x[n] = a^n u[n]$.

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} \\ &= \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}}, \quad |az^{-1}| < 1 \end{aligned}$$

$ROC : \{z : |az^{-1}| < 1\} = \{z : |z| > |a|\}$

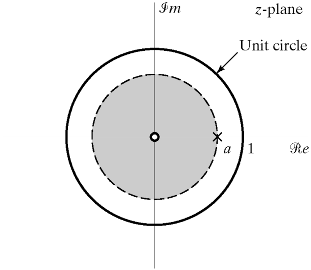


# Example 2: Left Sided Sequence

Consider $x[n] = -a^n u[-n-1]$.

$$\begin{aligned} X(z) &= -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} \\ &= -\sum_{n=1}^{\infty} a^{-n} z^n = -\sum_{n=1}^{\infty} (a^{-1}z)^n = -\frac{a^{-1}z}{1-a^{-1}z}, \quad |a^{-1}z| < 1 \\ &= -\frac{az^{-1}}{az^{-1}} \frac{a^{-1}z}{1-a^{-1}z} = -\frac{1}{az^{-1}-1} = \frac{1}{1-az^{-1}}, \quad |a^{-1}z| < 1 \end{aligned}$$

$ROC : \{z : |a^{-1}z| < 1\} = \{z : |z| < |a|\}$



# General Sequences

1. Finite Duration Sequences
2. Right Sided Sequences
3. Left Sided Sequences
4. Two Sided Sequences

# Finite Duration Sequences

1. Finite Duration Sequences: Sequences that are non-zero for $-\infty < N_1 \leq n \leq N_2 < \infty$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=N_1}^{N_2} x[n]z^{-n}$$

ROC: All $z$ except possibly $z = 0$ or $z = \infty$ because $z^{-n}$ is infinity at $z = 0$ for positive values of $n$ and similarly for $z = \infty$ for negative values of $n$.

# Right Sided Sequences

2. Right Sided Sequences: Sequences that are zero for $n < N_1 < \infty$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=N_1}^{\infty} x[n]z^{-n}$$

ROC: $\{z : |z| > r_R\}$ except possibly $z = \infty$

Reasoning: If $\sum_{n=N_1}^{\infty} |x[n]||z|^{-n}$ converges for $|z| = r_1$, it will converge for all $|z| > r_1$ since all terms with positive $n$ get smaller

$$|x[n]r_1^{-n}| = \frac{|x[n]|}{r_1^n} > \frac{|x[n]|}{r_2^n} = |x[n]r_2^{-n}|, \quad r_2 > r_1 \text{ and } n > 0$$

# Left Sided Sequences

3. Left Sided Sequences: Sequences that are zero for $n > N_2 > -\infty$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{N_2} x[n]z^{-n}$$

ROC: $\{z : |z| < r_L\}$ except possibly $z = 0$

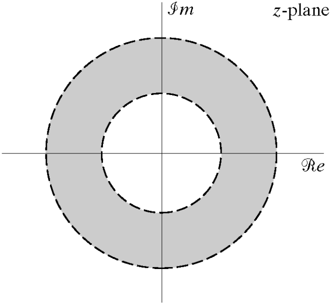
Reasoning: If $\sum_{n=-\infty}^{N_2} |x[n]||z|^{-n}$ converges for $|z| = r_1$, it will converge for all $|z| < r_1$ since all terms with negative $n$ get smaller

# Two-Sided Sequences

4. Two Sided Sequences: Sequences that are neither left-sided or right-sided, i.e. sequence defined over $-\infty < n < \infty$.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

ROC: $\{z : r_R < |z| < r_L\}$ is an annular ring
Reasoning: If $\sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{-1} x[n]z^{-n} + \sum_{n=0}^{\infty} x[n]z^{-n}$, sum of a left sided sequence and a right sided sequences. Both sequences must converge, and the ROC is the intersection.



# Summary of ROCs

1. Finite Duration Sequences. ROC: All $z$ except possibly $z = 0$ or $z = \infty$
2. Right Sided Sequences. ROC: $\{z : |z| > r_R\}$ except possibly $z = \infty$
3. Left Sided Sequences. ROC: $\{z : |z| < r_L\}$ except possibly $z = 0$
4. Two Sided Sequences. ROC: $\{z : r_R < |z| < r_L\}$ is an annular ring

**TABLE 3.1** SOME COMMON $z$-TRANSFORM PAIRS

| Sequence | Transform | ROC |
|---|---|---|
| 1. $\delta[n]$ | $1$ | All $z$ |
| 2. $u[n]$ | $\dfrac{1}{1-z^{-1}}$ | $\lvert z\rvert > 1$ |
| 3. $-u[-n-1]$ | $\dfrac{1}{1-z^{-1}}$ | $\lvert z\rvert < 1$ |
| 4. $\delta[n-m]$ | $z^{-m}$ | All $z$ except 0 (if $m>0$) or $\infty$ (if $m<0$) |
| 5. $a^n u[n]$ | $\dfrac{1}{1-az^{-1}}$ | $\lvert z\rvert > \lvert a\rvert$ |
| 6. $-a^n u[-n-1]$ | $\dfrac{1}{1-az^{-1}}$ | $\lvert z\rvert < \lvert a\rvert$ |
| 7. $na^n u[n]$ | $\dfrac{az^{-1}}{(1-az^{-1})^2}$ | $\lvert z\rvert > \lvert a\rvert$ |
| 8. $-na^n u[-n-1]$ | $\dfrac{az^{-1}}{(1-az^{-1})^2}$ | $\lvert z\rvert < \lvert a\rvert$ |
| 9. $[\cos\omega_0 n]u[n]$ | $\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$ | $\lvert z\rvert > 1$ |
| 10. $[\sin\omega_0 n]u[n]$ | $\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$ | $\lvert z\rvert > 1$ |
| 11. $[r^n\cos\omega_0 n]u[n]$ | $\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$ | $\lvert z\rvert > r$ |
| 12. $[r^n\sin\omega_0 n]u[n]$ | $\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$ | $\lvert z\rvert > r$ |
| 13. $\begin{cases} a^n, & 0\le n\le N-1,\\ 0, & \text{otherwise}\end{cases}$ | $\dfrac{1-a^Nz^{-N}}{1-az^{-1}}$ | $\lvert z\rvert > 0$ |

# Examples

Delta function:

$$x[n] = \delta[n] \leftrightarrow X(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = 1.$$

ROC: Entire z-plane

Delayed Delta function:

$$x[n] = \delta[n-n_0] \leftrightarrow X(z) = \sum_{n=-\infty}^{\infty} \delta[n-n_0] z^{-n} = z^{-n_0}.$$

ROC: Entire z-plane except $z = 0$ or $z = \infty$.

Exponential sequence: $x[n] = a^n u[n]$, has Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}}, \quad |az^{-1}| < 1$$

ROC: $\{z : |az^{-1}| < 1\} = \{z : |z| > |a|\}$

**TABLE 3.2** SOME $z$-TRANSFORM PROPERTIES

| Property Number | Section Reference | Sequence | Transform | ROC |
|---|---|---|---|---|
| | | $x[n]$ | $X(z)$ | $R_x$ |
| | | $x_1[n]$ | $X_1(z)$ | $R_{x_1}$ |
| | | $x_2[n]$ | $X_2(z)$ | $R_{x_2}$ |
| 1 | 3.4.1 | $ax_1[n] + bx_2[n]$ | $aX_1(z) + bX_2(z)$ | Contains $R_{x_1} \cap R_{x_2}$ |
| 2 | 3.4.2 | $x[n - n_0]$ | $z^{-n_0}X(z)$ | $R_x$, except for the possible addition or deletion of the origin or $\infty$ |
| 3 | 3.4.3 | $z_0^n x[n]$ | $X(z/z_0)$ | $\|z_0\|R_x$ |
| 4 | 3.4.4 | $nx[n]$ | $-z\dfrac{dX(z)}{dz}$ | $R_x$ |
| 5 | 3.4.5 | $x^*[n]$ | $X^*(z^*)$ | $R_x$ |
| 6 | | $\mathcal{R}e\{x[n]\}$ | $\frac{1}{2}[X(z) + X^*(z^*)]$ | Contains $R_x$ |
| 7 | | $\mathcal{I}m\{x[n]\}$ | $\frac{1}{2j}[X(z) - X^*(z^*)]$ | Contains $R_x$ |
| 8 | 3.4.6 | $x^*[-n]$ | $X^*(1/z^*)$ | $1/R_x$ |
| 9 | 3.4.7 | $x_1[n] * x_2[n]$ | $X_1(z)X_2(z)$ | Contains $R_{x_1} \cap R_{x_2}$ |

# Property 2: Time Shifting

$$x[n] \leftrightarrow X(z), \;\; \text{ROC: } R_x = \{z : r_R < |z| < r_L\}$$

Time Shifting: $x[n-n_0] \leftrightarrow z^{-n_0}X(z)$, ROC is $R_x$ (except the addition or deletion of 0 and $\infty$ for sequences that are not two-sided)

Proof:

$$\begin{aligned}
\mathcal{Z}(x[n-n_0]) &= \sum_{n=-\infty}^{\infty} x[n-n_0]z^{-n} \\
&= \sum_{m=-\infty}^{\infty} x[m]z^{-(m+n_0)} \;\; (\textit{Change of variables } m = n - n_d) \\
&= z^{-n_0} \sum_{m=-\infty}^{\infty} x[m]z^{-m} = z^{-n_0}X(z)
\end{aligned}$$

Due to the presence of $z^{-n_0}$, ROC is $R_x$ (except the addition or deletion of 0 and $\infty$ for sequences that are not two-sided)

# Property 3: Multiplying by an Exponential Sequence

Modulation: $z_0^n x[n] \leftrightarrow X(\frac{z}{z_0})$, with ROC $\{z : |z_0| r_R < |z| < |z_0| r_L\}$

Proof:

$$\begin{aligned} \mathcal{Z}(z_0^n x[n]) &= \sum_{n=-\infty}^{\infty} z_0^n x[n] z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x[n] (\frac{z}{z_0})^{-n} \\ &= X(\frac{z}{z_0}) \end{aligned}$$

ROC: $\{z : r_R < |\frac{z}{z_0}| < r_L\} = \{z : |z_0| r_R < |z| < |z_0| r_L\}$

# Property 9: Convolution

$x[n] * y[n] \leftrightarrow X(z)Y(z)$ and ROC contains $R_x \bigcap R_y$

Proof:

$$
\begin{aligned}
\mathcal{Z}(x[n] * y[n]) &= \mathcal{Z}(\sum_{k=-\infty}^{\infty} x[k]y[n-k]) \\
&= \sum_{k-\infty}^{\infty} x[k]\mathcal{Z}(y[n-k]) \ \ (Linearity) \\
&= \sum_{k-\infty}^{\infty} x[k]z^{-k}Y(z) \ \ (Time \ \ Shifting) \\
&= X(z)Y(z)
\end{aligned}
$$

$X(z)$ and $Y(z)$ have to be defined for the product to be defined. So ROC contains $R_x \bigcap R_y$

# Property 5

$x^*[n] \leftrightarrow X^*(z^*)$ and ROC is $R_x$

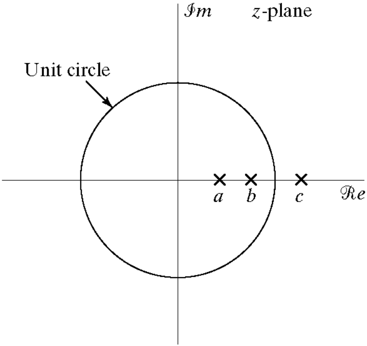
Proof:

$$\begin{aligned}
\mathcal{Z}(x^*[n]) &= \sum_{n=-\infty}^{\infty} x^*[n]z^{-n} = \sum_{n=-\infty}^{\infty} x^*[n]((z^*)^{-n})^* = \sum_{n=-\infty}^{\infty} (x[n](z^*)^{-n}))^* \\
&= \left(\sum_{n=-\infty}^{\infty} x[n](z^*)^{-n}\right)^* \\
&= (X(z^*))^* = X^*(z^*)
\end{aligned}$$

ROC: $\sum_{n=-\infty}^{\infty} |x^*[n]||z^{-n}| = \sum_{n=-\infty}^{\infty} |x[n]||z^{-n}|$, and so the ROC is $R_x$

# Example

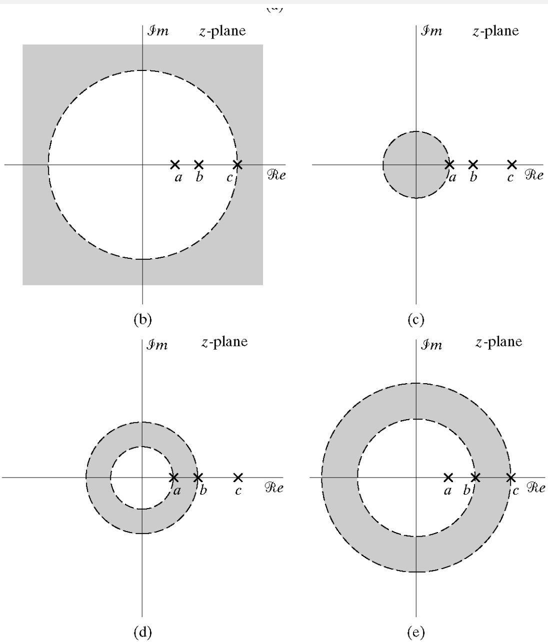
$$X(z) = \frac{1}{1 - az^{-1}} + \frac{1}{1 - bz^{-1}} + \frac{1}{1 - cz^{-1}}$$




(a)

What are the possible ROCs?

# Potential ROCs



$$X(z) = \frac{1}{1-az^{-1}} + \frac{1}{1-bz^{-1}} + \frac{1}{1-cz^{-1}}$$

Consider ROC: $\{z : a < |z| < b\}$

What is $x[n]$?

$$X(z) = \underbrace{\frac{1}{1-az^{-1}}}_{|z|>a} + \underbrace{\frac{1}{1-bz^{-1}}}_{|z|<b} + \underbrace{\frac{1}{1-cz^{-1}}}_{|z|<c}$$

$$x[n] = a^n u[n] - b^n u[-n-1] - c^n u[-n-1]$$

$$X(z) = \frac{1}{1 - az^{-1}} + \frac{1}{1 - bz^{-1}} + \frac{1}{1 - cz^{-1}}$$

Consider ROC: $\{z : |z| > c\}$

What is $x[n]$?

$$X(z) = \underbrace{\frac{1}{1 - az^{-1}}}_{|z|>a} + \underbrace{\frac{1}{1 - bz^{-1}}}_{|z|>b} + \underbrace{\frac{1}{1 - cz^{-1}}}_{|z|>c}$$

$$x[n] = a^n u[n] + b^n u[n] + c^n u[n]$$