Generalized Linear Phase

A filter with generalized linear phase has the following form

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega}e^{j\beta}$$

$A(e^{j\omega})$ is a real function of frequency

Theorem: A linear phase Real, Causal, Stable, Rational (RCSR) filter is necessarily FIR. The group delay of such a filter is half its order and satisfies either the symmetry or antisymmetric relation (h[n] = ± h[M-n]).

Generalized Linear Phase Filters

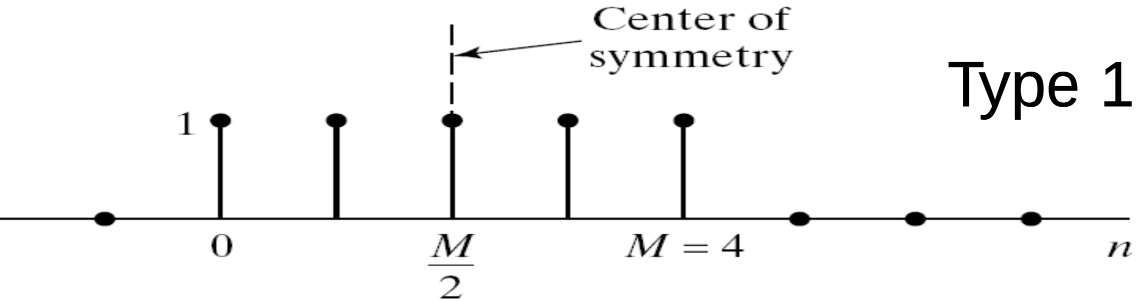
We will consider the design of Finite Impulse Response (FIR) filters with real impulse response

$$H(z) = \sum_{n=0}^{M} h[n] z^{-n}$$

Four types of linear phase FIR filters.

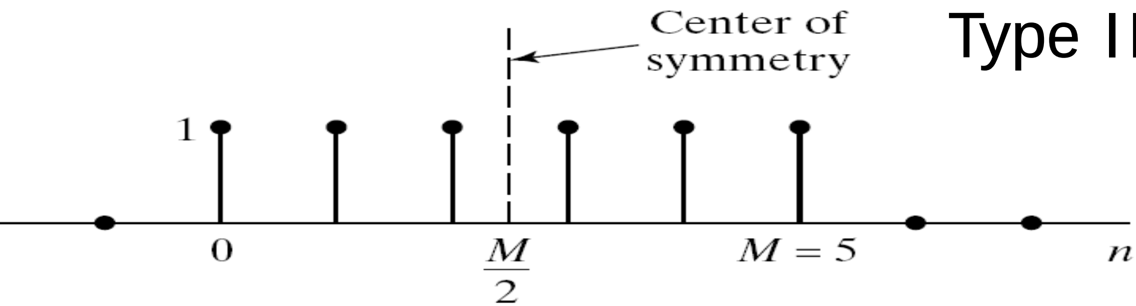
- Type I $h[n] = h[M-n]$, M even
- Type II $h[n] = h[M-n]$, M odd
- Type III $h[n] = -h[M-n]$, M even
- Type IV $h[n] = -h[M-n]$, M odd

Type 1, M even, h[n]=h[M-n]



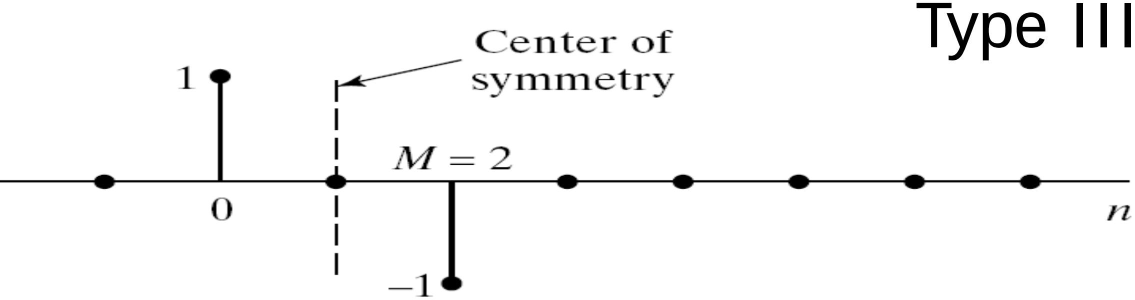
(a)

Type II, M odd, h[n]=h[M-n]



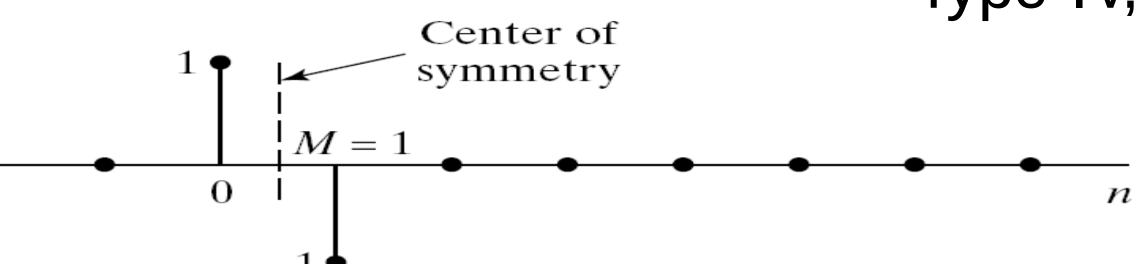
(b)

Type III, M even, h[n]=-h[M-n]



(c)

Type IV, M odd, h[n]=-h[M-n]



(d)

Zeros of Linear Phase FIR Filters

For Type I and II FIR Filters $H(z) = z^{-M} H(z^{-1})$

- Type II: M odd (Proof)

$$H(z) = \sum_{n=0}^{M} h[n] z^{-n} = \sum_{n=0}^{\frac{M-1}{2}} h[n] z^{-n} + \sum_{n=\frac{M+1}{2}}^{M} h[n] z^{-n}$$

$$= \sum_{n=0}^{\frac{M-1}{2}} h[n] z^{-n} + \sum_{n=0}^{\frac{M-1}{2}} h[M-n] z^{-(M-n)}$$

$$= \sum_{n=0}^{\frac{M-1}{2}} h[n] (z^{-n} + z^{-M+n}) = z^{-M} H(z^{-1})$$

- Type I: M even can be shown similarly

$$H(z) = \sum_{n=0}^{L} h[n](z^{-n} + z^{-(M-n)}), L = \frac{M-1}{2}$$

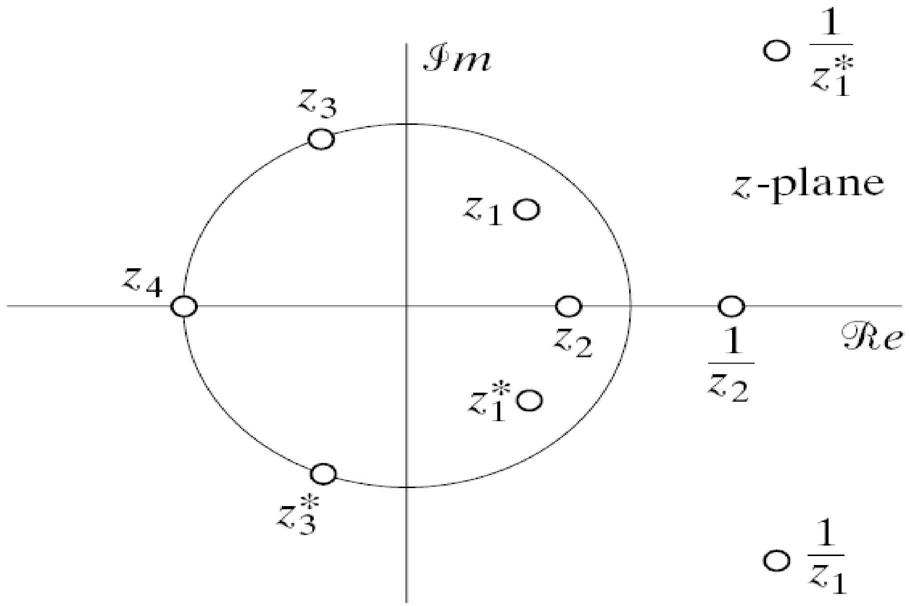
$$z^{-M}H(z^{-1}) = z^{-M}\sum_{n=0}^{L} h[n](z^{n} + z^{M-n}) = \sum_{n=0}^{L} h[n](z^{-(M-n)} + z^{-n}) = H(z)$$

Zeros of Linear Phase FIR Filters Cont'd

- For Type III and Type IV filters

$$H(z) = -z^{-M} H(z^{-1})$$

- NOTE: if z_i, is a zero of system H(z), then $1/z_i$ is a zero of H(z). True for all four types.



More on the Zeros

- Type I and Type II: Since $H(z) = z^{-M} H(z^{-1})$

$$H(-1) = (-1)^{-M} H(-1)$$

- For Type II filters, M odd and hence H(-1)=0. Therefore, for Type II systems, z=-1 (or $\omega = \pi$) is a zero of H(z).

More on the Zeros (cont.)

Type III and IV: Since $H(z) = -z^{-M} H(z^{-1})$

$$H(1) = -(1)^{-M} H(1)$$

and

$$H(-1) = -(-1)^{-M} H(-1)$$

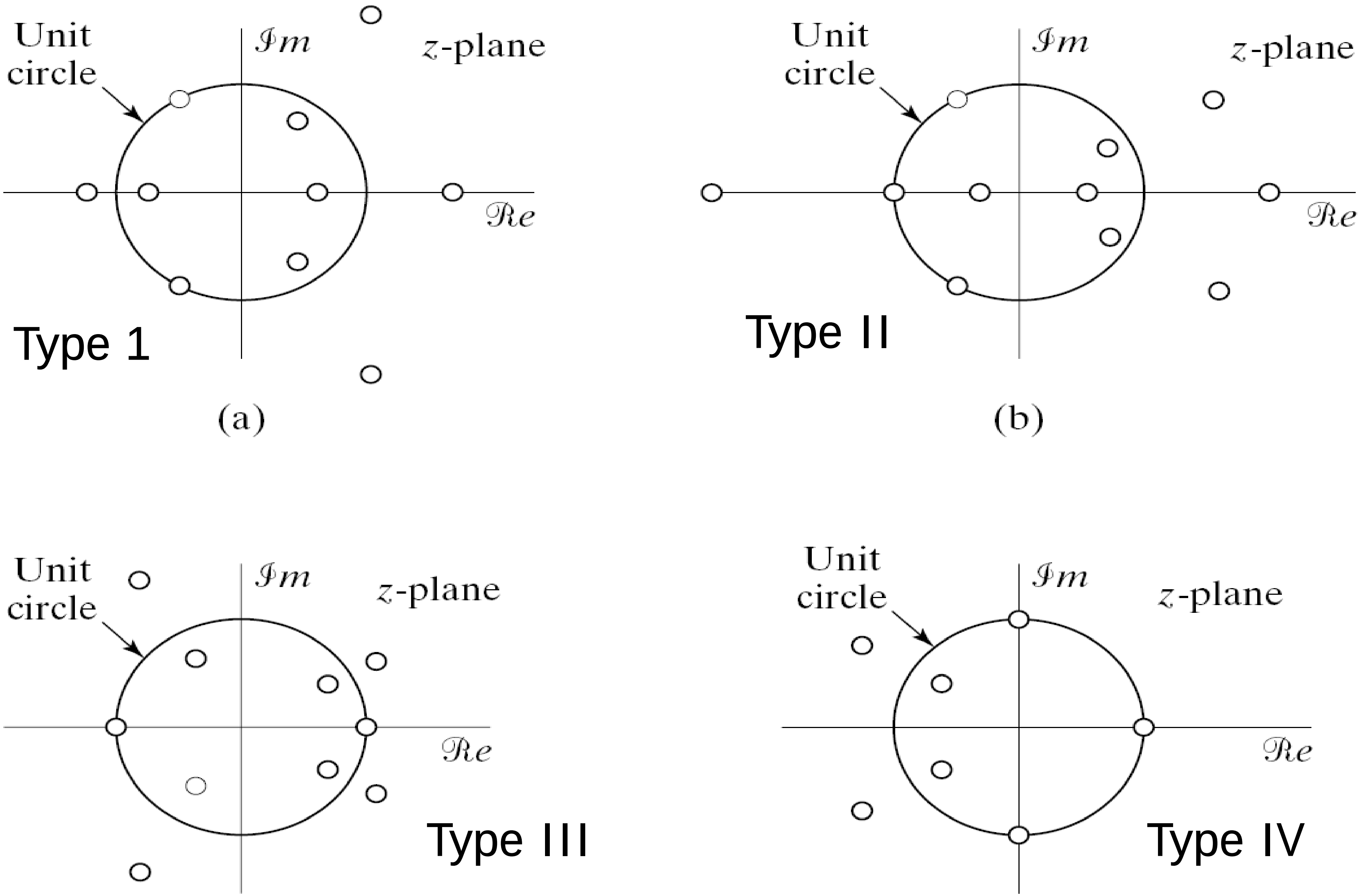
z=1 is a zero of both Type III and Type IV systems
z=-1 is a zero of Type III system

Summary on the Zeros

- For both Type III, and IV systems, H(1)=0 and hence z=1 (or $\omega = 0$) is a zero of H(z)

- For Type II and III systems
  H(-1)=0 and hence z=-1 (or $\omega = \pi$) is also a zero of the H(z)

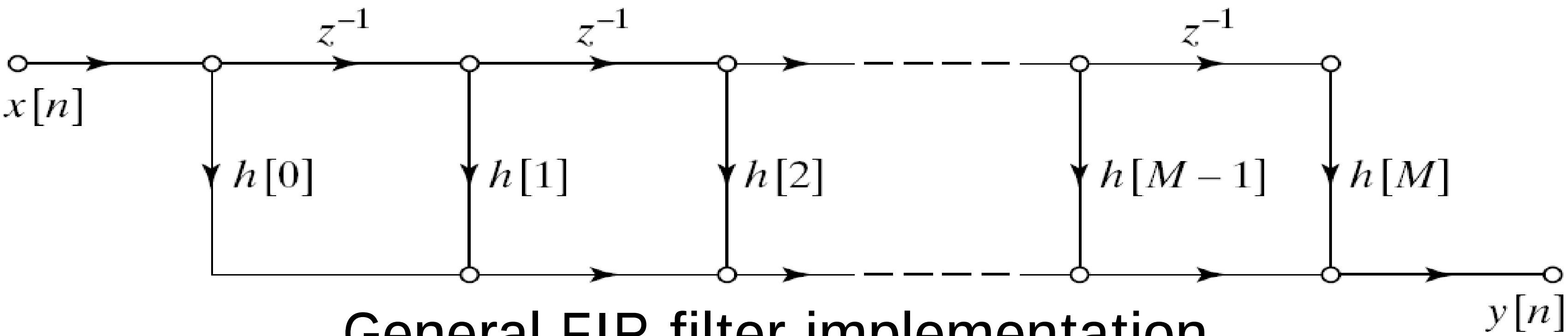
Typical Zero Plots for Linear Phase Systems



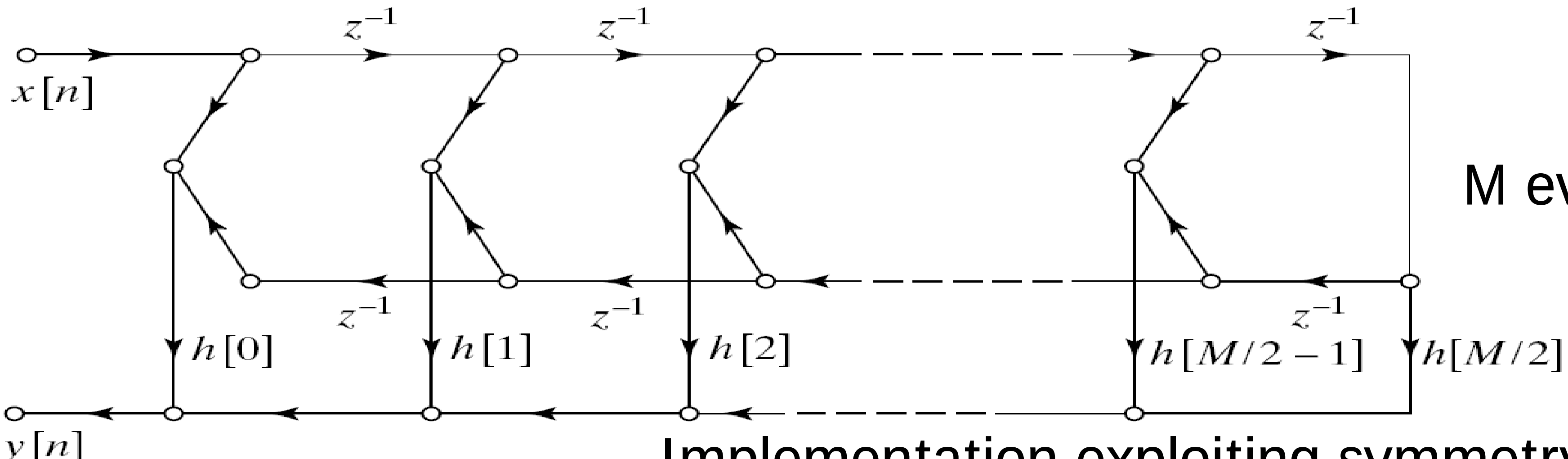
TYPE	1	2	3	4
Symmetry	$h(n) = h(M-n)$	$h(n) = h(M-n)$	$h(n) = -h(M-n)$	$h(n) = -h(M-n)$
Parity of M	M even	M odd	M even	M odd
Expression for frequency response	$e^{-j\omega M/2} H_R(\omega)$	$e^{-j\omega M/2} H_R(\omega)$	$je^{-j\omega M/2} H_R(\omega)$	$je^{-j\omega M/2} H_R(\omega)$
Amplitude response or zero-phases response	$\sum_{k=0}^{M_1} a[k]\cos(\omega k)$ $M_1 = M/2$	$\sum_{k=1}^{M_1} b[k] \cos(\omega(k-1/2))$ $M_1 = (M+1)/2$	$\sum_{k=1}^{M_1} c[k] \sin(\omega k)$ $M_1 = M/2$	$\sum_{k=1}^{M_1} d[k]\sin(\omega(k-1/2))$ $M_1 = (M+1)/2$
Special Features		Zero at $\omega = \pi$	Zero at $\omega = 0$ and π	Zero at $\omega = 0$

Filter Implementation

$$H(z)=\sum_{n=0}^{M}h[n]z^{-n}$$



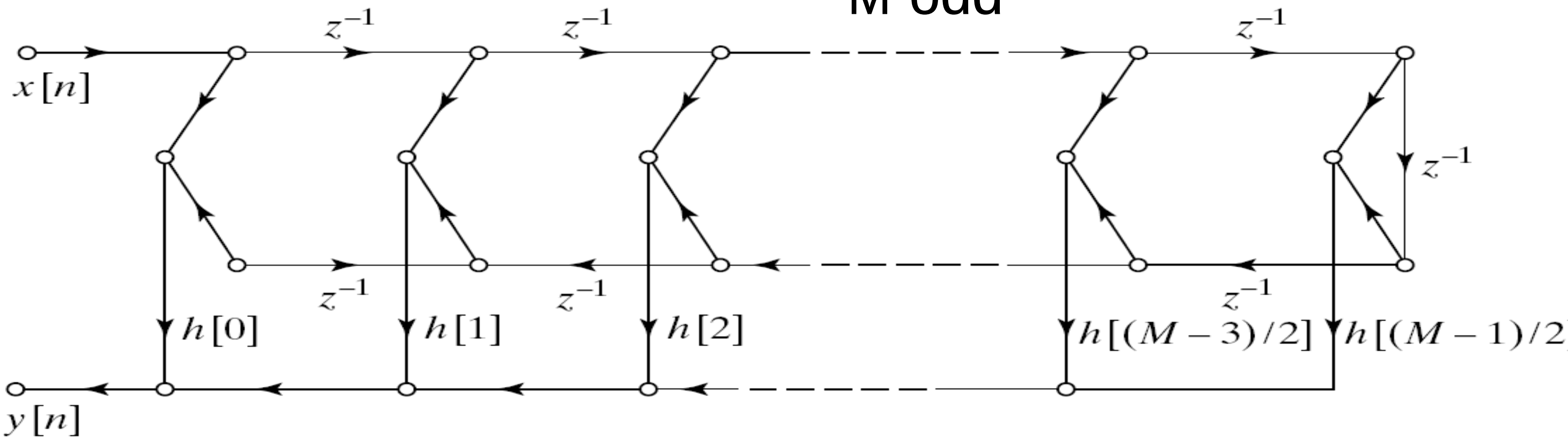
General FIR filter implementation



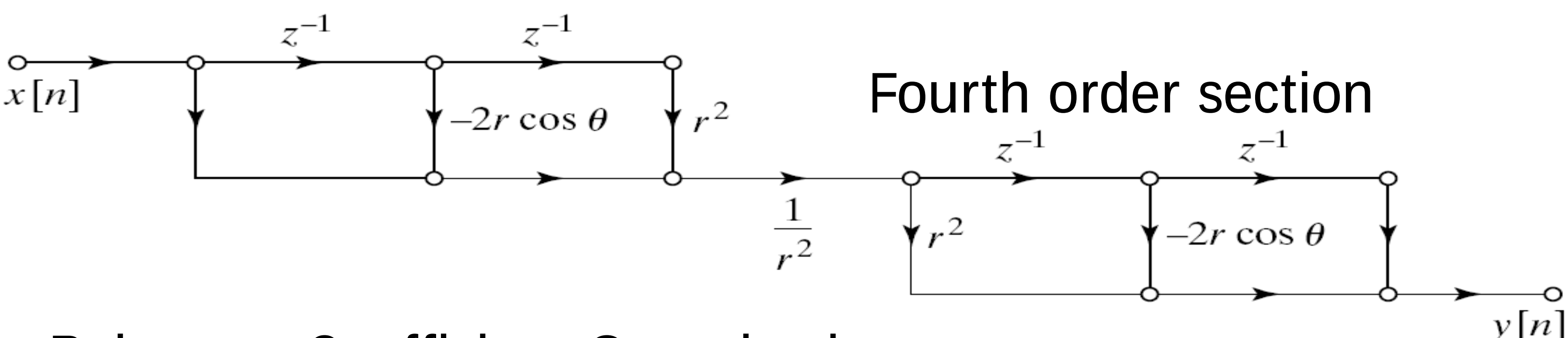
M even

Implementation exploiting symmetry

M odd



$$H(z) = (1 - 2r\cos\theta \; z^{-1} + r^2 z^{-2})\frac{1}{r^2}(r^2 - 2r\cos\theta \; z^{-1} + z^{-2})$$



Fourth order section

Robust to Coefficient Quantization