

ECE161A: Fundamentals of Digital Signal Processing

The Fast Fourier Transform (FFT)

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Discrete Fourier Transform

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} nk}$$

$$w_N = e^{-j \frac{2\pi}{N}}$$

$$= \sum_{n=0}^{N-1} x[n] w_N^{kn}$$



- For every $X(k)$ there are N complex multiply-adds. Thus, for N values of $X(k)$, there are N^2 multiply-adds.

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Decimation in Time Algorithm

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} \left[\underset{\substack{\uparrow \\ \text{even indexed points} \\ N/2 \text{ points}}}{x[2n]} w_N^{2nk} + \underset{\substack{\uparrow \\ \text{odd indexed points} \\ N/2 \text{ points}}}{x[2n+1]} w_N^{(2n+1)k} \right] \quad k = 0, 1, \dots, N$$

$$w_N = e^{-j2\pi/8}$$

$$w_N^2 = e^{-j(2)2\pi/8} = e^{-j2\pi/4} = w_{N/2}$$



see also Section 9.3 in
Oppenheim & Schaffer, 1999

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Decimation in Time Algorithm

$$X(k) = \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x[2n] w_{N/2}^{nk}}_{(N/2 \text{ point DFT})} + w_N^k \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x[2n+1] w_{N/2}^{nk}}_{(N/2 \text{ point DFT})} \quad k = 0, 1, \dots, N-1$$

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Decimation in Time Algorithm

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x[2n] w_{N/2}^{nk} + w_N^k \sum_{n=0}^{\frac{N}{2}-1} x[2n+1] w_{N/2}^{nk} \quad k = 0, 1, \dots, N-1$$

$$= A(k) + w_N^k B(k)$$

$A(k)$ and $B(k)$ are $N/2$ point DFTs

Since DFT is periodic in k (or frequency) the value of $A(k)$ and $B(k)$ for $k < N/2$ repeat for $k \geq N/2$. Thus,

$$X(k + N/2) = A(k) + w_N^{k+N/2} B(k) \quad k = 0, 1, \dots, \frac{N}{2} - 1$$

and $w_N^{N/2} = e^{-j \frac{2\pi}{N} \frac{N}{2}} = -1$

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Decimation in Time Algorithm

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x[2n] w_{N/2}^{nk} + w_N^k \sum_{n=0}^{\frac{N}{2}-1} x[2n+1] w_{N/2}^{nk} \quad k = 0, 1, \dots, N-1$$

$$= A(k) + w_N^k B(k)$$

$$X(k + N/2) = A(k) - w_N^k B(k) \quad k = 0, 1, \dots, N/2 - 1 \quad \text{(i)}$$

$$X(k) = A(k) + w_N^k B(k) \quad k = 0, 1, \dots, N/2 - 1 \quad \text{(ii)}$$

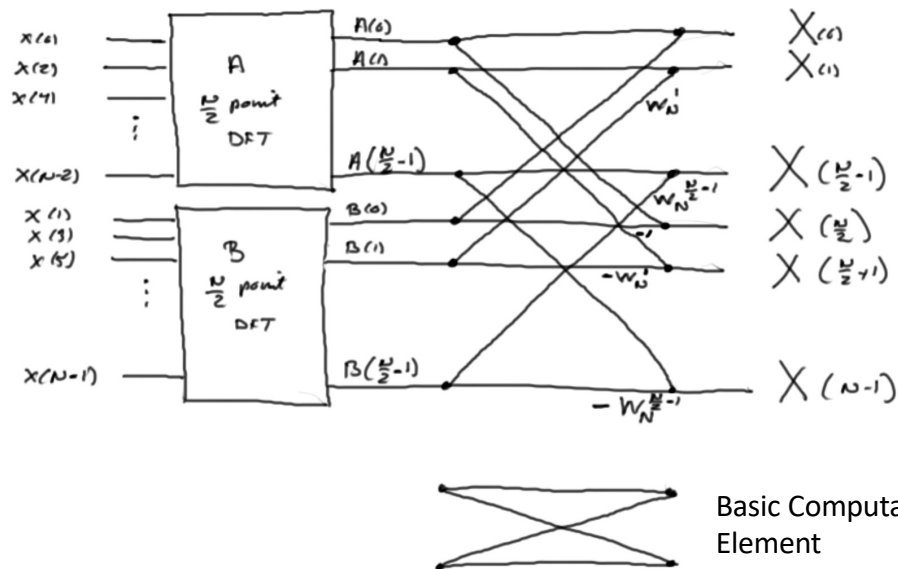
Thus, we can synthesize an N -point DFT from the $N/2$ -point DFTs using the synthesized expressions (i) and (ii) and save half of the multiply-adds

Total Number of Operations: $(N/2)^2 + (N/2)^2 + N \approx N^2/2$ multiply adds

→ saving by a factor of two

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Flow Graph – Decimation in Time



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Decimation in Time Algorithm

- Extend this idea to the $N/2$ -point $A(k)$ and $B(k)$ DFTs yields pairs of $N/4$ -points DFTs, etc.
- Data enters full diagram in scrambled order (bit reversed order)

Index	Binary	Bit Reversed	Decimal
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

Computational Savings: Instead of N^2 we only perform $N \log_2 N$ multiply adds e.g., if

$$N = 1024 = 2^{10} \quad N^2 = 1,048,526 \quad N \log_2 N = 10,240$$

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Flow Graph – Decimation in Time

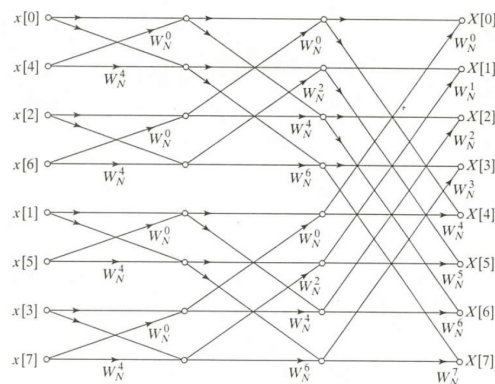


Figure 9.7 Flow graph of complete decimation-in-time decomposition of an 8-point DFT computation.

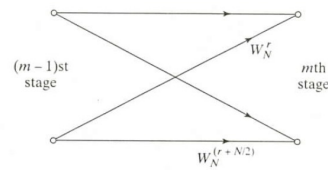


Figure 9.8 Flow graph of basic butterfly computation in Figure 9.7.

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Decimation in Frequency – FFT Algorithm

$$\begin{aligned}
 X(k) &= \sum_{n=0}^{\frac{N}{2}-1} \left[\overset{\text{first half of time series}}{\boxed{x[n]}} w_N^{nk} + \overset{\text{second half of time series}}{\boxed{x[n + N/2]}} w_N^{(n+N/2)k} \right] \quad k = 0, 1, \dots, N-1 \\
 &= \sum_{n=0}^{\frac{N}{2}-1} \left[x[n] + w_N^{(N/2)k} x[n + N/2] \right] w_N^{nk}
 \end{aligned}$$

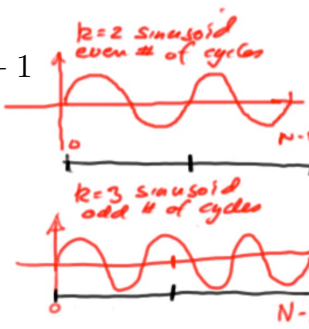
$$\begin{aligned}
 w_N^{N/2k} &= 1 & k &= 0, 2, 4, \dots, N-2 & \quad w_N^{nk} &= w_{N/2}^{nk'} & \quad k &= 2k' \\
 w_N^{N/2k} &= -1 & k &= 1, 3, 5, \dots, N-1
 \end{aligned}$$

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Decimation in Frequency – FFT Algorithm

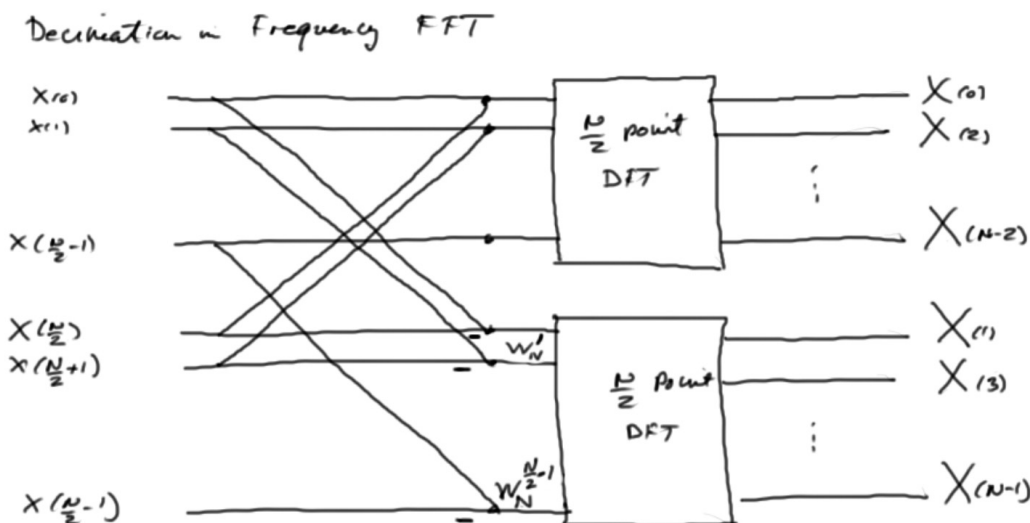
$$\begin{aligned}
 X(2k') &= \sum_{n=0}^{\frac{N}{2}-1} [x[n] + x[n + N/2]] w_{N/2}^{nk'} \quad k' = 0, \dots, N/2 - 1 \\
 X(2k' + 1) &= \sum_{n=0}^{\frac{N}{2}-1} [x[n] + w_N^{N/2(2k'+1)} x[n + N/2]] w_N^{n(2k'+1)} \\
 &= \sum_{n=0}^{\frac{N}{2}-1} [x[n] - x[n + N/2]] \boxed{w_N^n} w_{N/2}^{nk'}
 \end{aligned}$$

\uparrow
 $e^{-j\frac{2\pi}{N}n}$



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Flow Graph – Decimation in Frequency



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Flow Graph – Decimation in Frequency

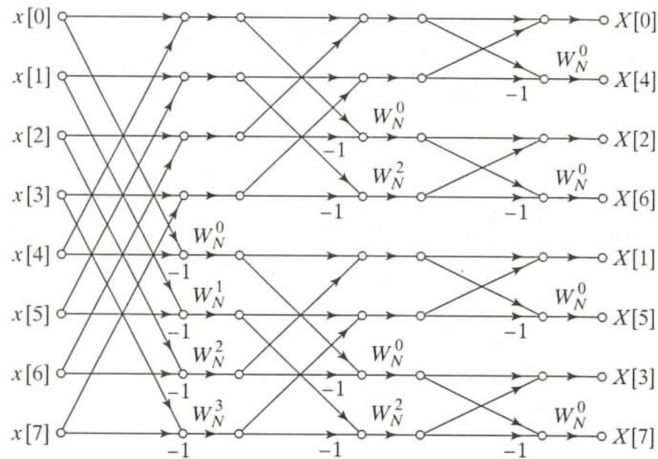


Figure 9.20 Flow graph of complete decimation-in-frequency decomposition of an 8-point DFT computation.