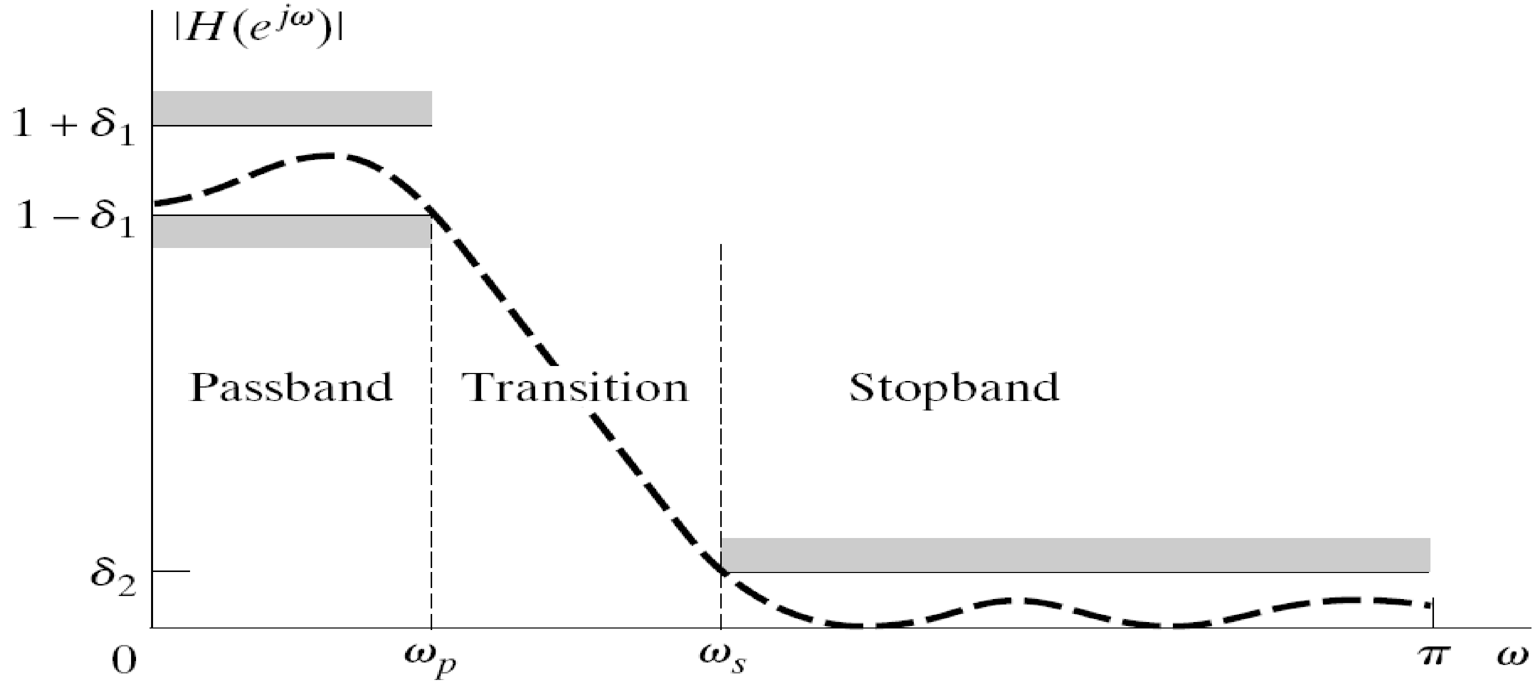
Windows and FIR Filter Design

Lecture By Prof. Meyer

ECE 161A

Filter Design: Low Pass Filter Design



Problem: Given $\delta_1, \delta_2, \omega_p$ and ω_s , find the lowest complexity filter that meets specification

Two Choices

1. IIR Filters (Infinite Impulse Response, $H(z) = \dfrac{B(z)}{A(z)}$
2. FIR Filters (Finite Impulse Response, $H(z) = B(z)$

FIR Filter Design

Approaches:

1. FIR Filters by Windowing
2. Optimal Approximation (Parks and McClellan)

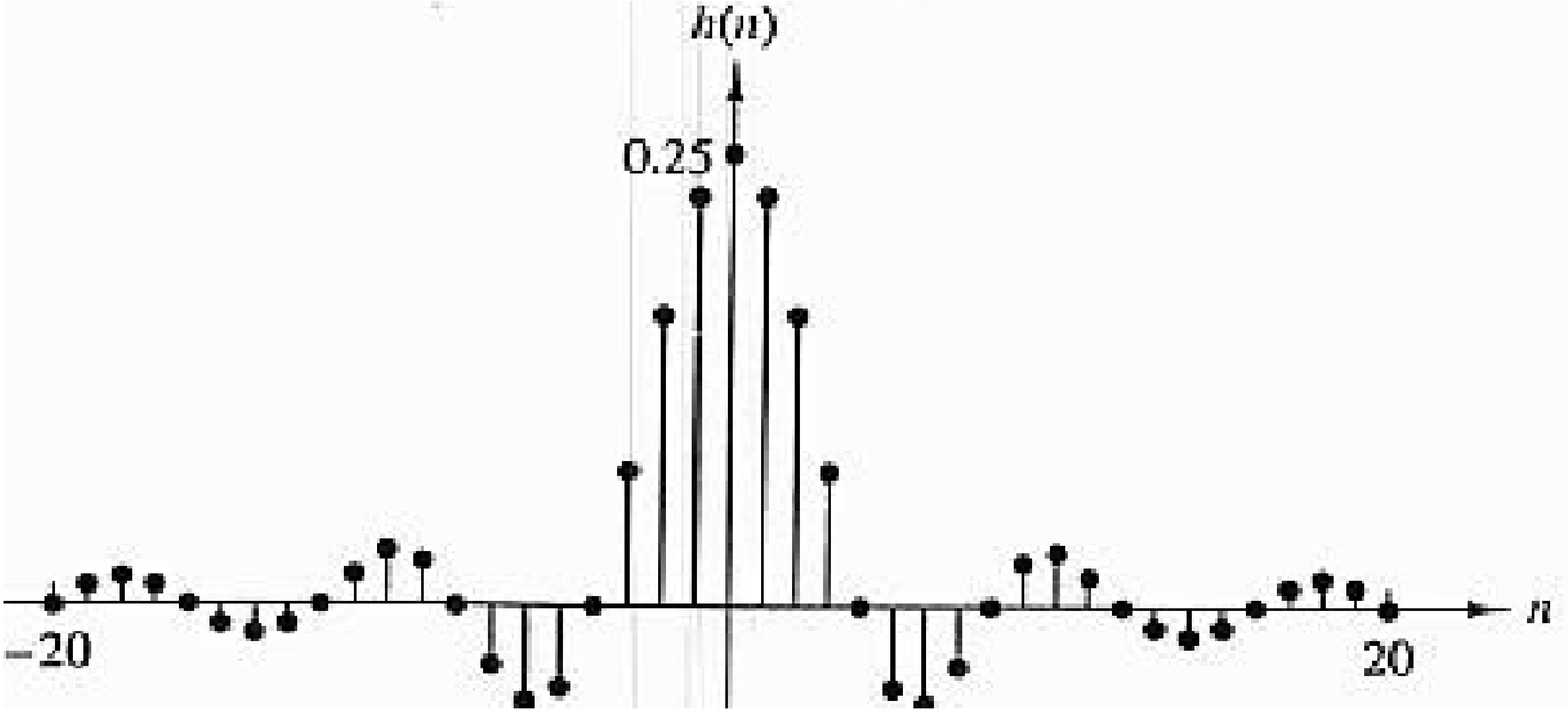
Ideal Low Pass Filter

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega}, |\omega| \le \omega_c \\ 0, \omega_c \le |\omega| \le \pi \end{cases}$$

$$h_d[n] = \frac{\sin(\omega_c(n-\alpha))}{\pi(n-\alpha)}, \alpha = \frac{M}{2}$$

FIR Filter Design (Cont.)

For Linear Phase $\alpha = \dfrac{M}{2}$



Unit sample response of an ideal lowpass filter

Rectangular Windowing

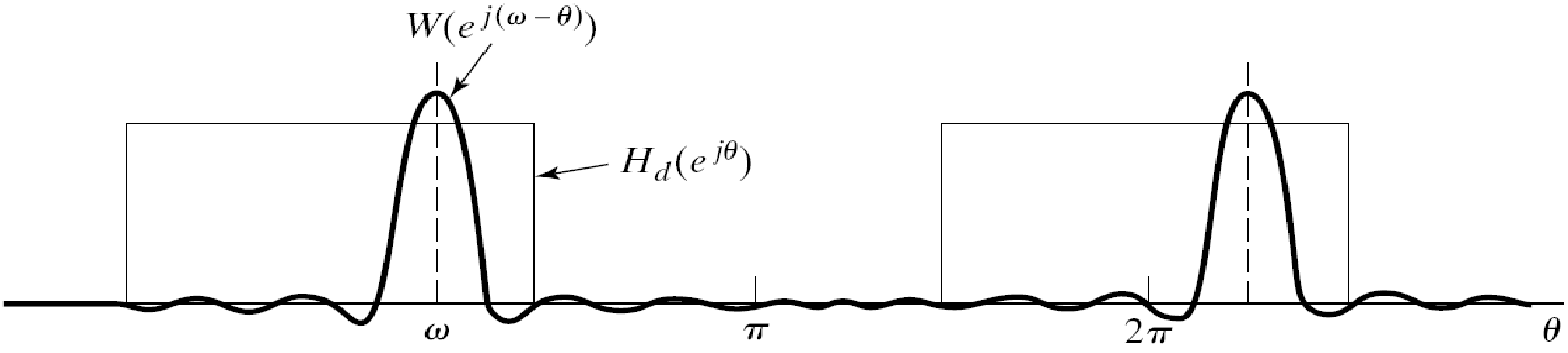
$$h[n] = h_d[n]w_r[n]$$

Where

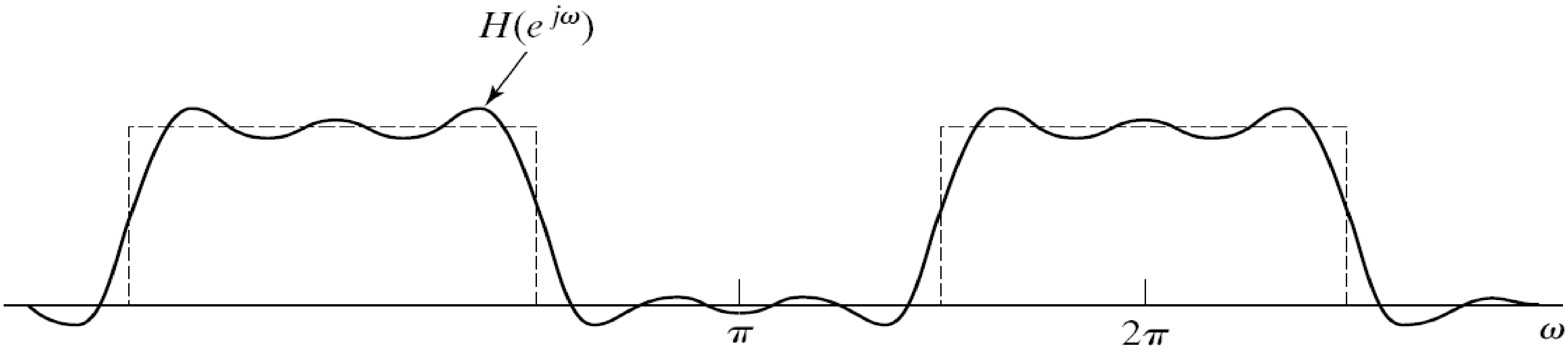
$$w_r[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & otherwise \end{cases}$$

$$H(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\theta}) W_r(e^{j(\omega-\theta)}) d\theta$$

Effects of Windowing



(a)



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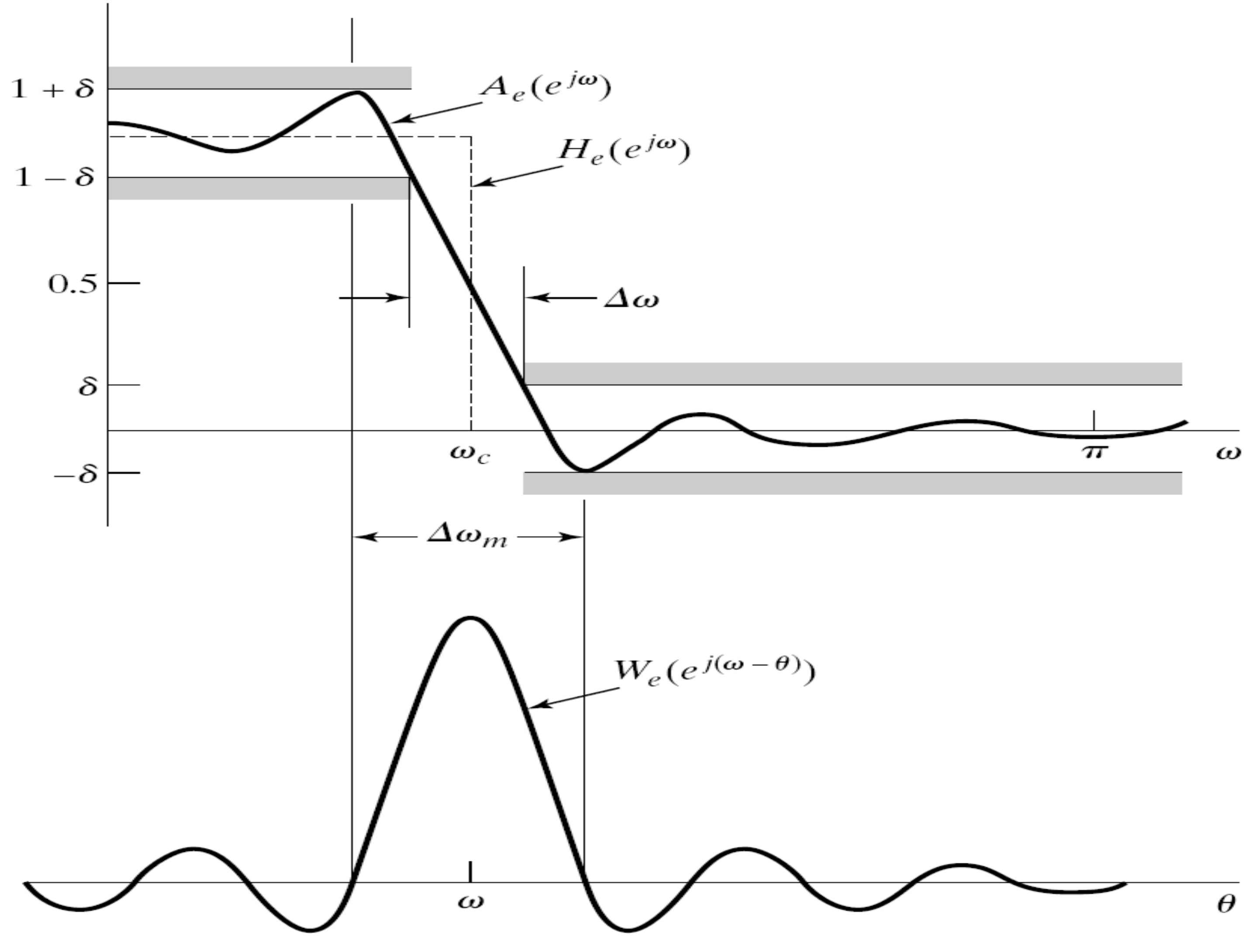
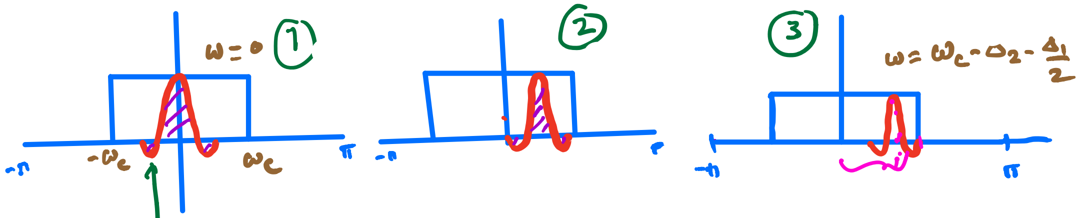
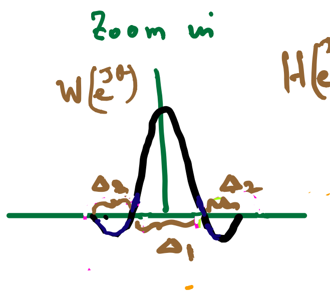


Illustration of type of approximation obtained at a discontinuity of the ideal frequency response.

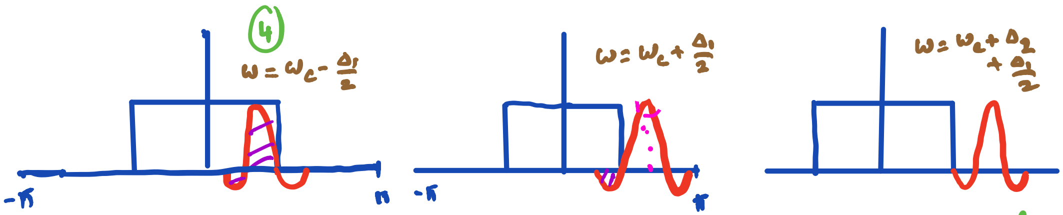




$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})\, W\left(e^{j(\omega-\theta)}\right) d\theta$$

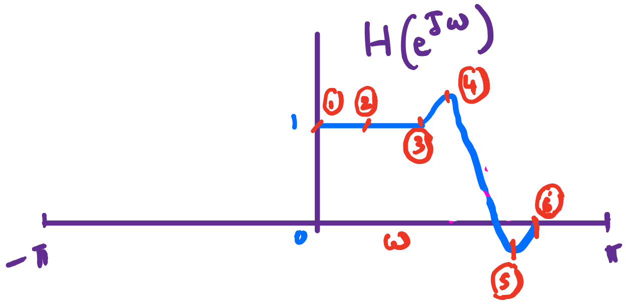
$$H(e^{j\omega}) = 1, \quad 0 \le \omega \le \omega_c - \Delta_2 - \frac{\Delta_1}{2}$$

Concentrating on $0 \le \omega \le \pi$



Reach a max $\longleftrightarrow$ Reach a min

Zero henceforth



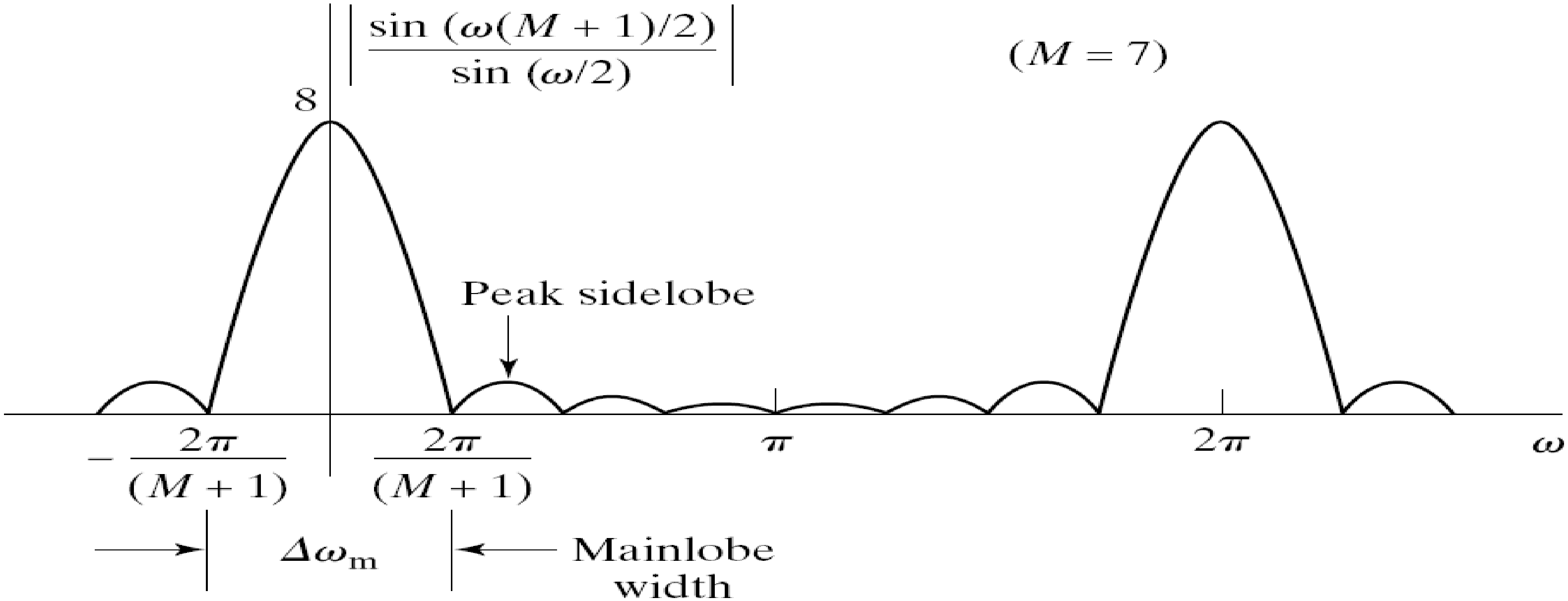
$\underbrace{\text{max to min is } \Delta_1}_{\text{mainlobe determines transition band}}$

Sidelobes determine max distortion

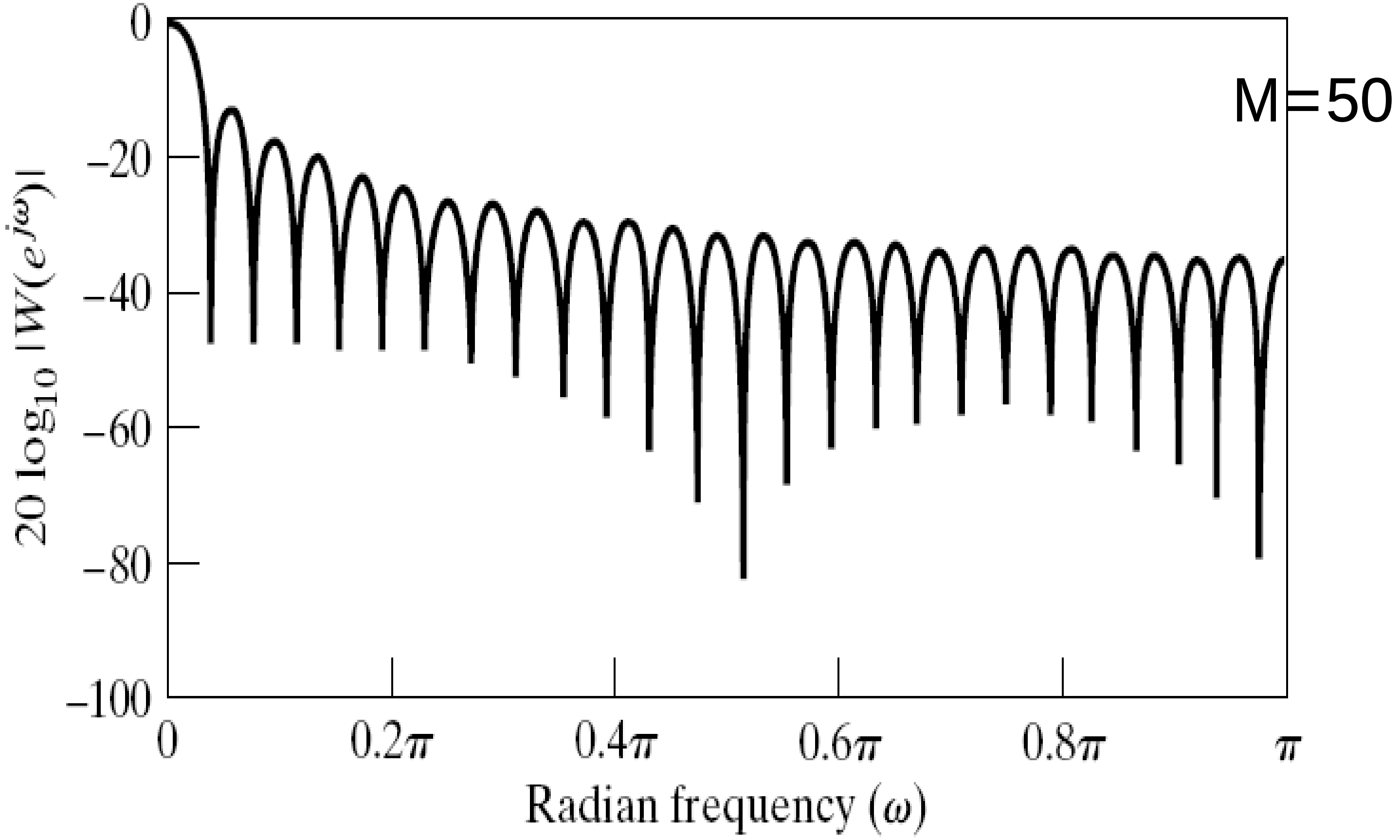
Rectangular Window

$$W_R(e^{j\omega}) = \sum_{n=0}^{M} e^{-j\omega} = e^{-\frac{j\omega M}{2}} \frac{\sin\frac{\omega(M+1)}{2}}{\sin\frac{\omega}{2}}$$

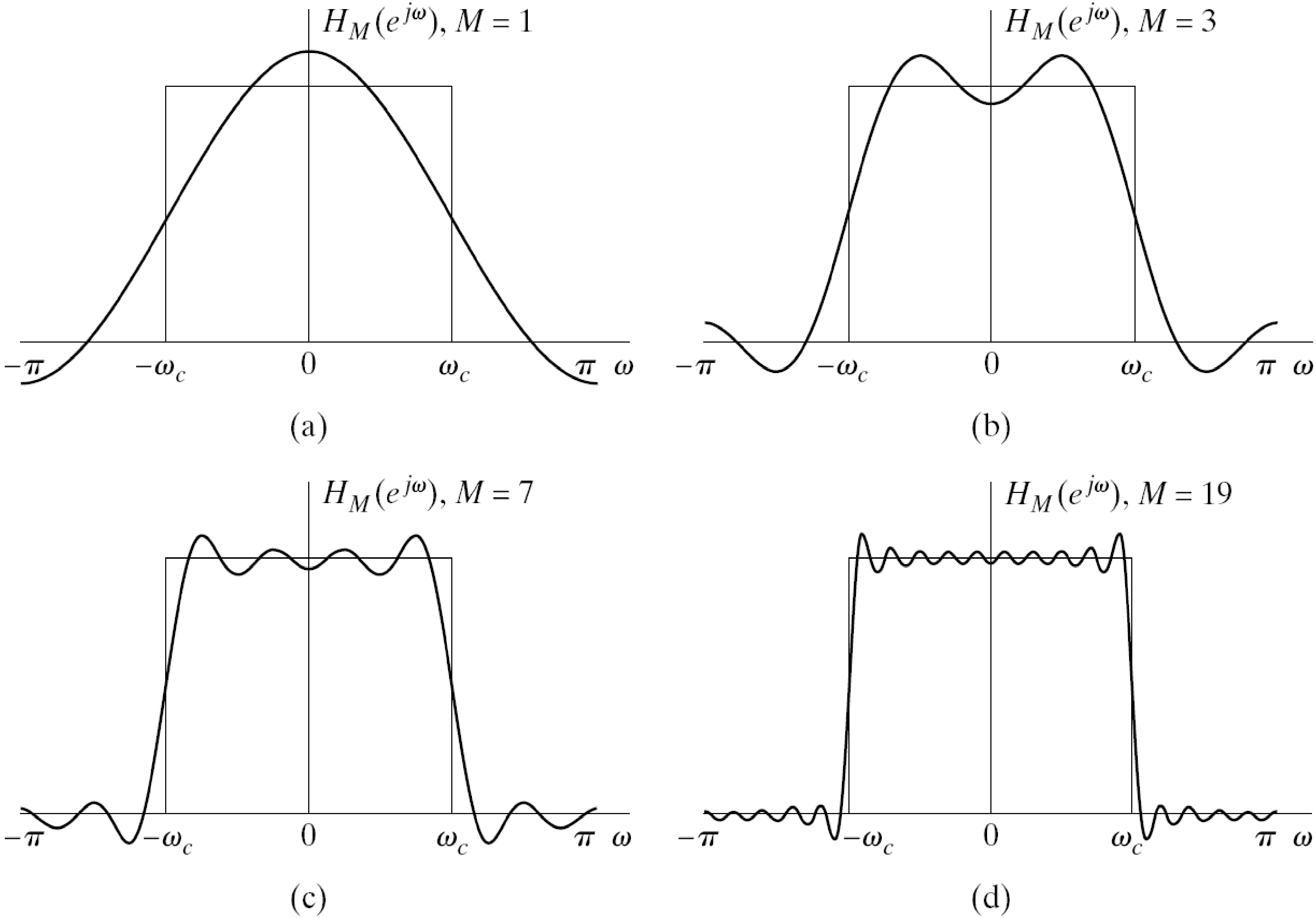
- Main Lobe Width: $\frac{4\pi}{M+1}$
- Sidelobe Magnitude= -13 db
- Stopband Attenuation=-21db



Rectangular Window

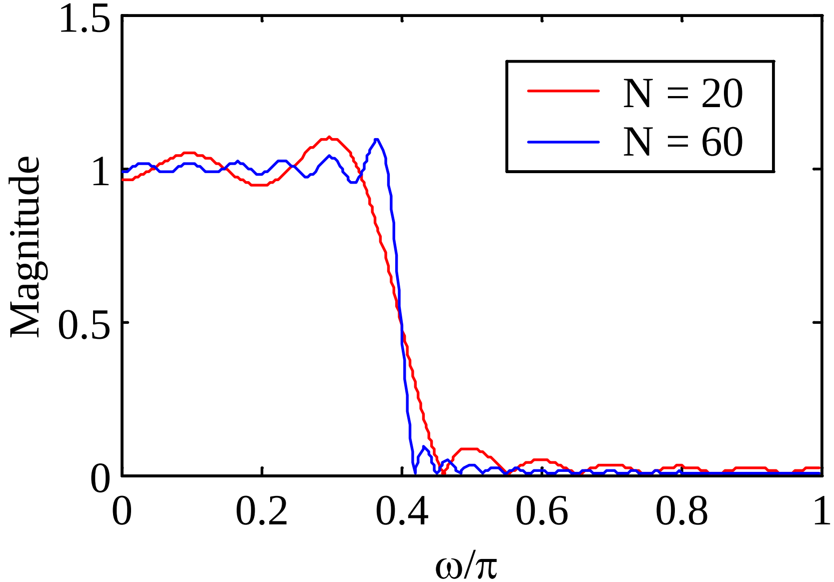


Gibbs Phenomena



Gibbs Phenomenon

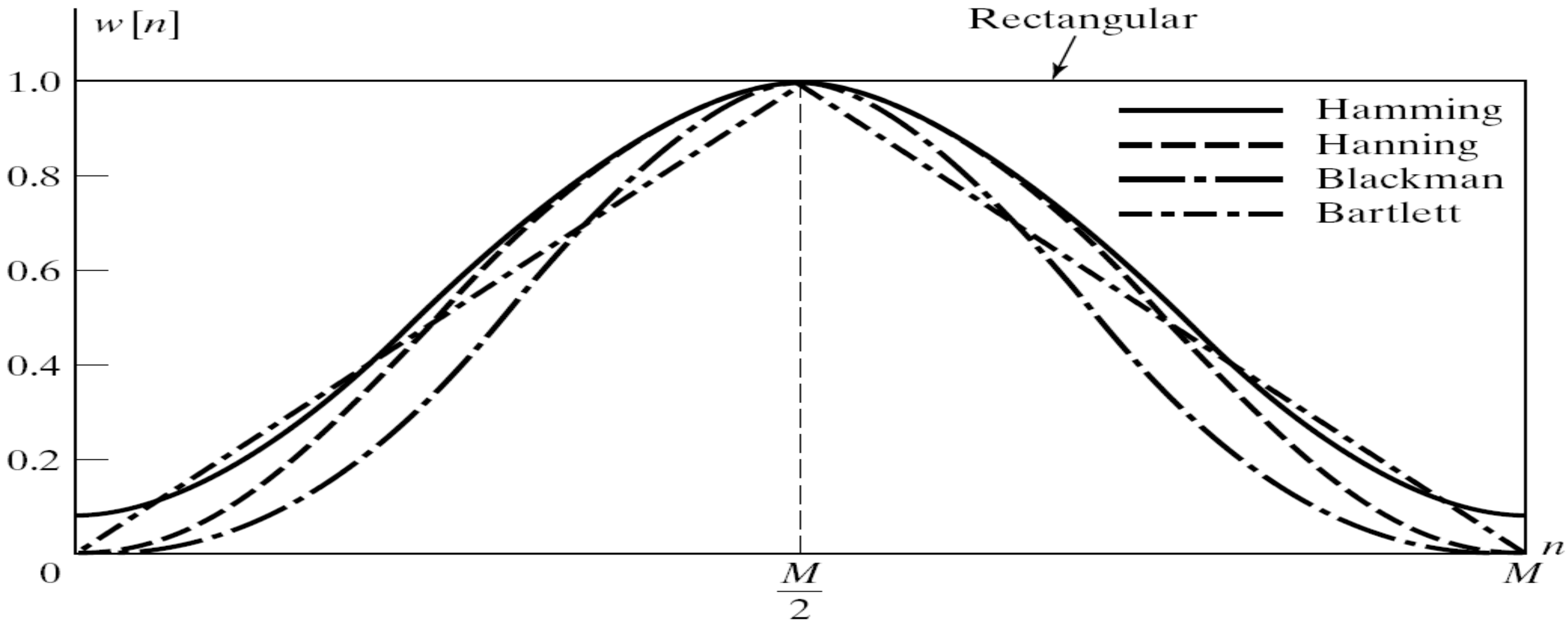
- Gibbs phenomenon - Oscillatory behavior in the magnitude responses of causal FIR filters obtained by truncating the impulse response coefficients of ideal filters



M=N

Gibbs Phenomenon

- Presence of oscillatory behavior in is basically due to:
  - 1) $h_d[n]$ is infinitely long and not absolutely summable, and hence filter is unstable $H_t(e^{j\omega})$
  - 2) Rectangular window has an abrupt transition to zero



Rectangular

$$w[n] = \begin{cases} 1, & 0 \le n \le M, \\ 0, & otherwise \end{cases}$$

Bartlett (triangular)

$$w[n] = \begin{cases} 2n/M, & 0 \le n \le M\,2 \\ 2 - 2n/M, & M/2 \le n \le M \\ 0, & otherwise \end{cases}$$

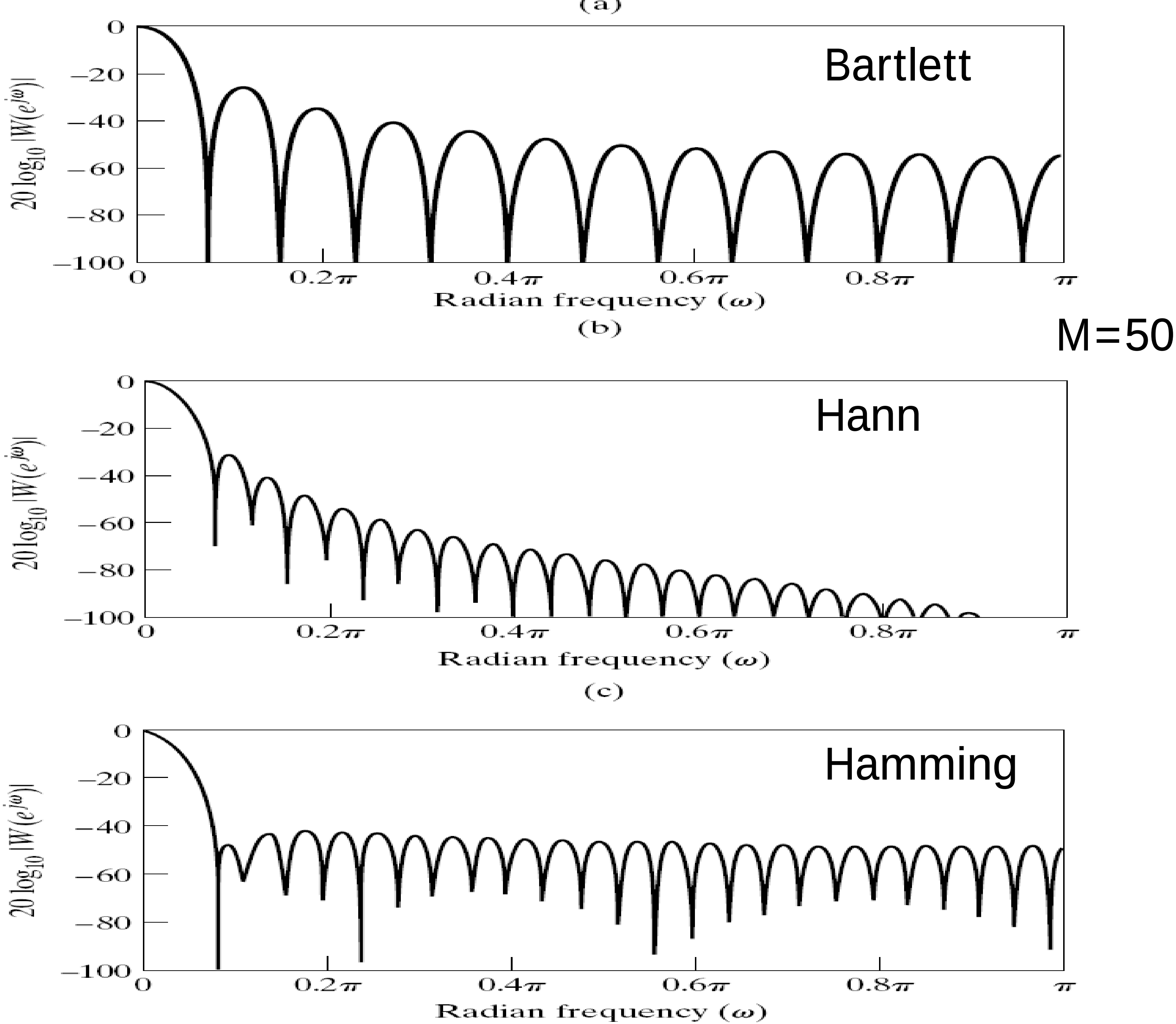
Hann

$$w[n] = \begin{cases} 0.5 - 0.5\cos(2\pi n/M), & 0 \le n \le M \\ 0, & otherwise \end{cases}$$

Hamming

$$w[n] = \begin{cases} 0.54 - 0.46\cos(2\pi n/M), & 0 \le n \le M \\ 0, & otherwise \end{cases}$$

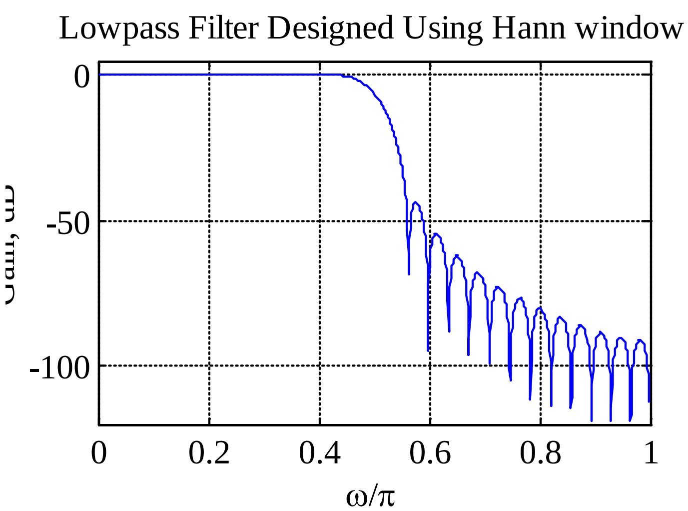
Window Type	Peak Sidelobe Amplitude (relative)	Approximate Width of Mainlobe	Peak Approximation Error 20 logδ (db)	Equivalent Kaiser Windows β	Transition Width of Equivalent Kaiser Window
Rectangular	-13	$4\pi/(M+1)$	-21	0	$1.81\pi/M$
Bartlett	-25	$8\pi/M$	-25	1.33	$2.37\pi/M$
Hann	-31	$8\pi/M$	-44	3.86	$5.01\pi/M$
Hamming	-41	$8\pi/M$	-53	4.86	$6.27\pi/M$
Blackman	-57	$12\pi/M$	-74	7.04	$9.19\pi/M$

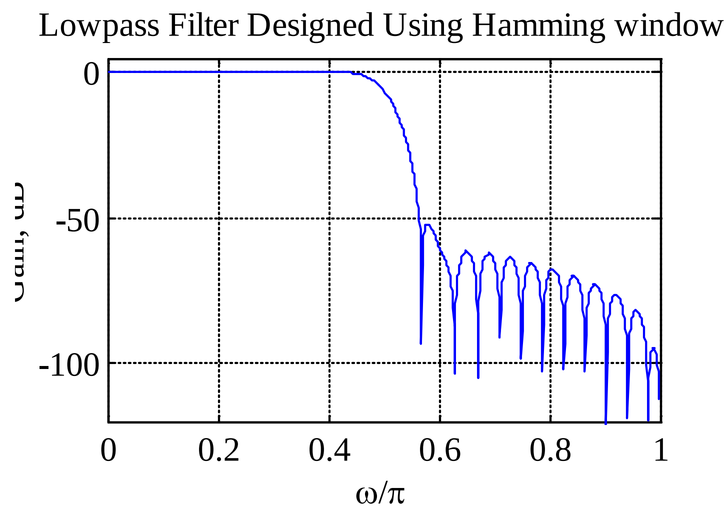


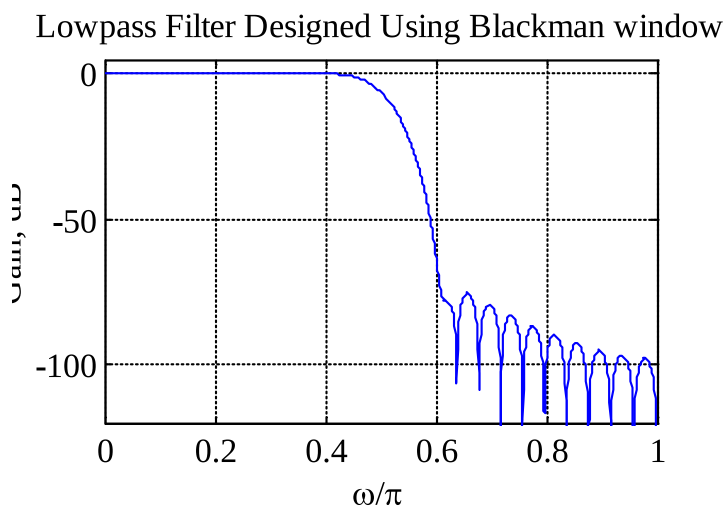
(a)
Bartlett
0
−20
−40
−60
−80
−100
$20\log_{10}|W(e^{j\omega})|$
0
0.2π
0.4π
0.6π
0.8π
π
Radian frequency (ω)
(b)
M=50
Hann
0
−20
−40
−60
−80
−100
$20\log_{10}|W(e^{j\omega})|$
0
0.2π
0.4π
0.6π
0.8π
π
Radian frequency (ω)
(c)
Hamming
0
−20
−40
−60
−80
−100
$20\log_{10}|W(e^{j\omega})|$
0
0.2π
0.4π
0.6π
0.8π
π
Radian frequency (ω)

FIR Filter Design Example

- Lowpass filter of length 51 and $\omega_c = \pi/2$







Window Based Design

Given specifications: $\delta_1, \delta_2, \omega_p$ and ω_s

We employ the following procedure

- 1. Compute $\delta = \min(\delta_1, \delta_2)$

  $\Delta\omega = \omega_p - \omega_s$

- Compute $20\log_{10}\delta$ and select window type
- Choose M, the filter order, to meet transition width
- Filter Coefficients are given by

$$h[n] = h_d[n]w[n], 0 \le n \le M$$

$$where\ \ \omega_c = (\omega_p + \omega_s)/2,$$

$$h_d[n] = \frac{\sin(\omega_c(n-\alpha))}{\pi(n-\alpha)}, \alpha = \frac{M}{2}$$

Filter Design

Filter Design

Parameters (Problem 7.15):

Filter Design

Parameters (Problem 7.15): $\delta_1 = .05$, $\delta_2 = .1$, $\omega_p = .25\pi$, and $\omega_s = .35\pi$.

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Parameters (Problem 7.15): $\delta_1 = .05$, $\delta_2 = .1$, $\omega_p = .25\pi$, and $\omega_s = .35\pi$.

$\omega_c = \frac{\omega_p + \omega_s}{2} = .30\pi$,

Filter Design

Parameters (Problem 7.15): $\delta_1 = .05$, $\delta_2 = .1$, $\omega_p = .25\pi$, and $\omega_s = .35\pi$.

$\omega_c = \frac{\omega_p + \omega_s}{2} = .30\pi$, $\delta = \min(\delta_1, \delta_2) = .05$,

Filter Design

Parameters (Problem 7.15): $\delta_1 = .05$, $\delta_2 = .1$, $\omega_p = .25\pi$, and $\omega_s = .35\pi$.

$\omega_c = \frac{\omega_p + \omega_s}{2} = .30\pi$, $\delta = \min(\delta_1, \delta_2) = .05$, and $\delta\omega = \omega_s - \omega_p = .1\pi$.

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$20 \log \delta = -26db$.

Filter Design

Parameters (Problem 7.15): $\delta_1 = .05$, $\delta_2 = .1$, $\omega_p = .25\pi$, and $\omega_s = .35\pi$.

$\omega_c = \frac{\omega_p + \omega_s}{2} = .30\pi$, $\delta = \min(\delta_1, \delta_2) = .05$, and $\delta\omega = \omega_s - \omega_p = .1\pi$.

$20 \log \delta = -26db$. Window choice $=$

Filter Design

Parameters (Problem 7.15): $\delta_1 = .05$, $\delta_2 = .1$, $\omega_p = .25\pi$, and $\omega_s = .35\pi$.

$\omega_c = \frac{\omega_p+\omega_s}{2} = .30\pi$, $\delta = \min(\delta_1, \delta_2) = .05$, and $\delta\omega = \omega_s - \omega_p = .1\pi$.

$20 \log \delta = -26db$. Window choice = Hanning window.

Filter Design

Parameters (Problem 7.15): $\delta_1 = .05$, $\delta_2 = .1$, $\omega_p = .25\pi$, and $\omega_s = .35\pi$.

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Filter Length

Filter Design

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$20 \log \delta = -26db$. Window choice = Hanning window.

Filter Length

$$\delta\omega = \frac{5.01\pi}{M} = .1\pi \quad \text{or} \quad M \approx 50.$$

Filter Design

Parameters (Problem 7.15): $\delta_1 = .05$, $\delta_2 = .1$, $\omega_p = .25\pi$, and $\omega_s = .35\pi$.

$\omega_c = \frac{\omega_p+\omega_s}{2} = .30\pi$, $\delta = \min(\delta_1, \delta_2) = .05$, and $\delta\omega = \omega_s - \omega_p = .1\pi$.

$20 \log \delta = -26db$. Window choice = Hanning window.

Filter Length

$$\delta\omega = \frac{5.01\pi}{M} = .1\pi \text{ or } M \approx 50.$$

$$h[n] = h_d[n] w_{hanning}[n], 0 \leq n \leq M = 50$$

Filter Design

Parameters (Problem 7.15): $\delta_1 = .05$, $\delta_2 = .1$, $\omega_p = .25\pi$, and $\omega_s = .35\pi$.

$\omega_c = \frac{\omega_p+\omega_s}{2} = .30\pi$, $\delta = \min(\delta_1, \delta_2) = .05$, and $\delta\omega = \omega_s - \omega_p = .1\pi$.

$20 \log \delta = -26db$. Window choice $=$ Hanning window.

Filter Length

$$\delta\omega = \frac{5.01\pi}{M} = .1\pi \quad \text{or} \quad M \approx 50.$$

$$h[n] = h_d[n] w_{hanning}[n], 0 \leq n \leq M = 50$$

where

$$h_d[n] = \frac{\sin \omega_c (n-\alpha)}{\pi(n-\alpha)}, \quad \text{where} \quad \alpha =$$

Filter Design

Parameters (Problem 7.15): $\delta_1 = .05$, $\delta_2 = .1$, $\omega_p = .25\pi$, and $\omega_s = .35\pi$.

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$20 \log \delta = -26db$. Window choice = Hanning window.

Filter Length

$$\delta\omega = \frac{5.01\pi}{M} = .1\pi \quad \text{or} \quad M \approx 50.$$

$$h[n] = h_d[n] w_{hanning}[n], 0 \leq n \leq M = 50$$

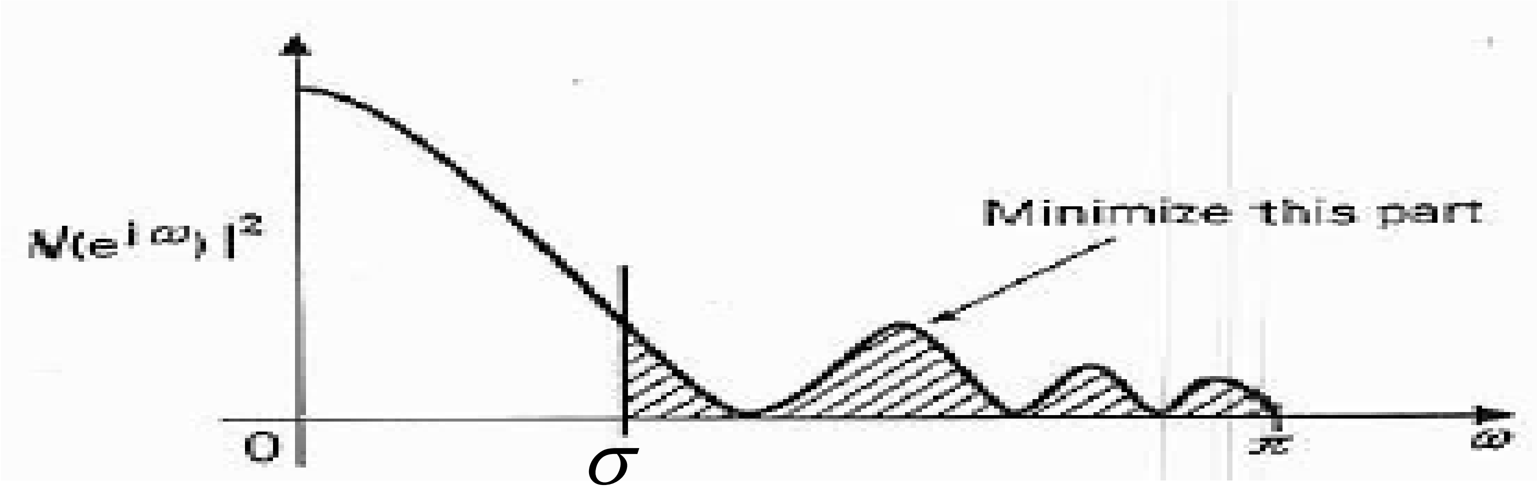
where

$$h_d[n] = \frac{\sin \omega_c(n-\alpha)}{\pi(n-\alpha)}, \quad \text{where} \quad \alpha = \frac{M}{2} = 25.$$

Optimal Window

$$W(e^{j\omega}) = \sum_{k=0}^{M} w[n] e^{-j\omega n}$$

Prolate Spheroidal Sequence $$\Phi_s = \frac{1}{\pi}\int_{\sigma}^{\pi} \left|W(e^{j\omega})\right|^2 d\omega$$



Optimization Problem: Minimize Φ_s subject to

$$\frac{1}{2\pi}\int_{0}^{\pi} \left|W(e^{j\omega})\right|^2 d\omega = 1 \quad \textbf{or} \quad \sum_{k=0}^{M} w^2[n] = 1$$

Solution to this problem is the Optimal Window.

Optimal Window

Solution: Form the matrix P=[P_{mn}] such that

$$P_{mn} = \frac{\sin(m-n)\sigma}{m-n}, \quad 0 \le m, n \le M$$

The window sequence w[n] is formed from the elements of the eigenvector corresponding to the largest eigenvalue of P. The eigenvector can be shown to be unique.

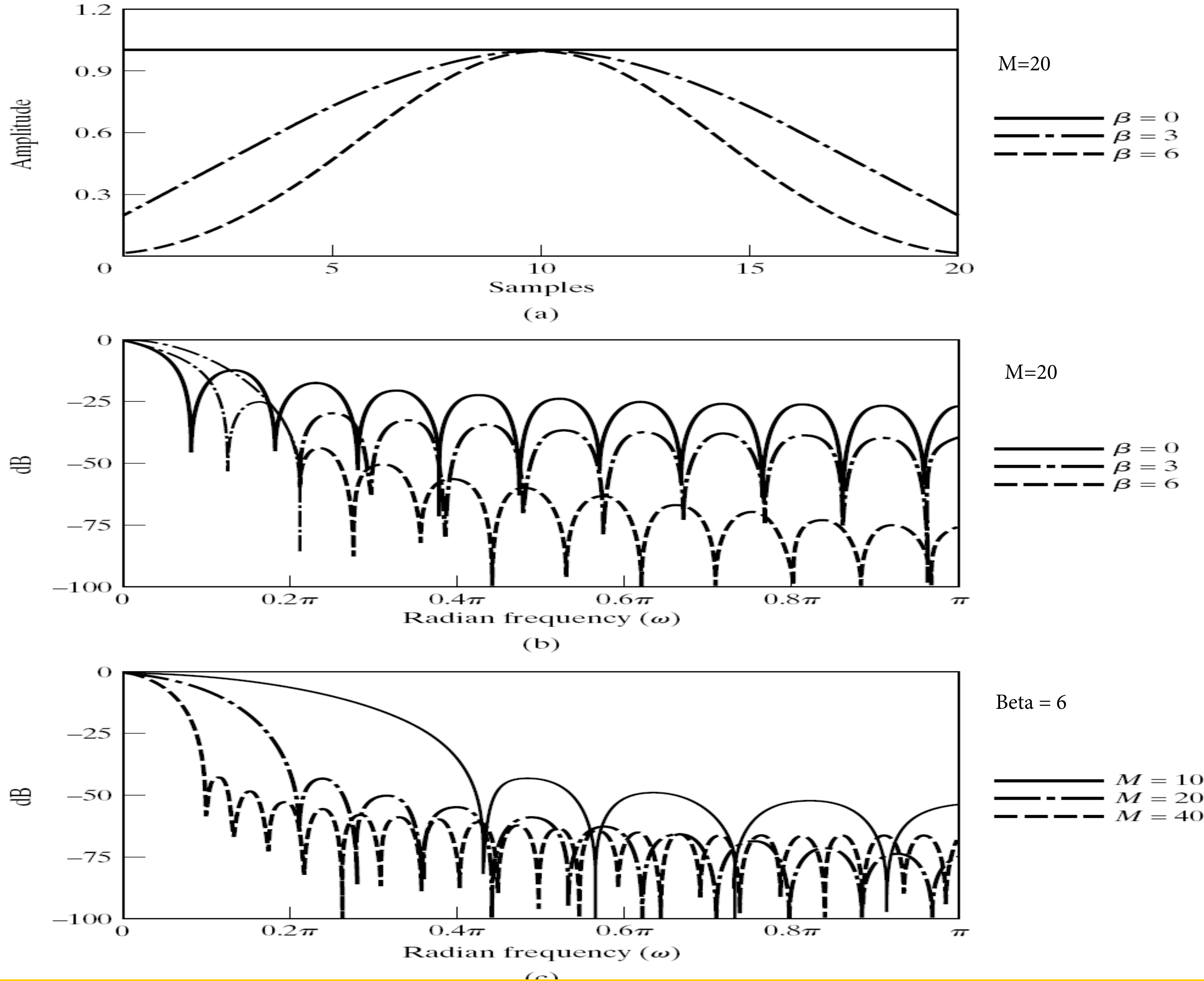
Kaiser Window

$$w[n] = \frac{I_0[\beta(1-(\frac{n-\alpha}{\alpha})^2)^{\frac{1}{2}}]}{I_0(\beta)}, \; 0 \le n \le M, \alpha = M/2$$

- $I_0(.)$ is zeroth order modified Bessel function of the First Kind

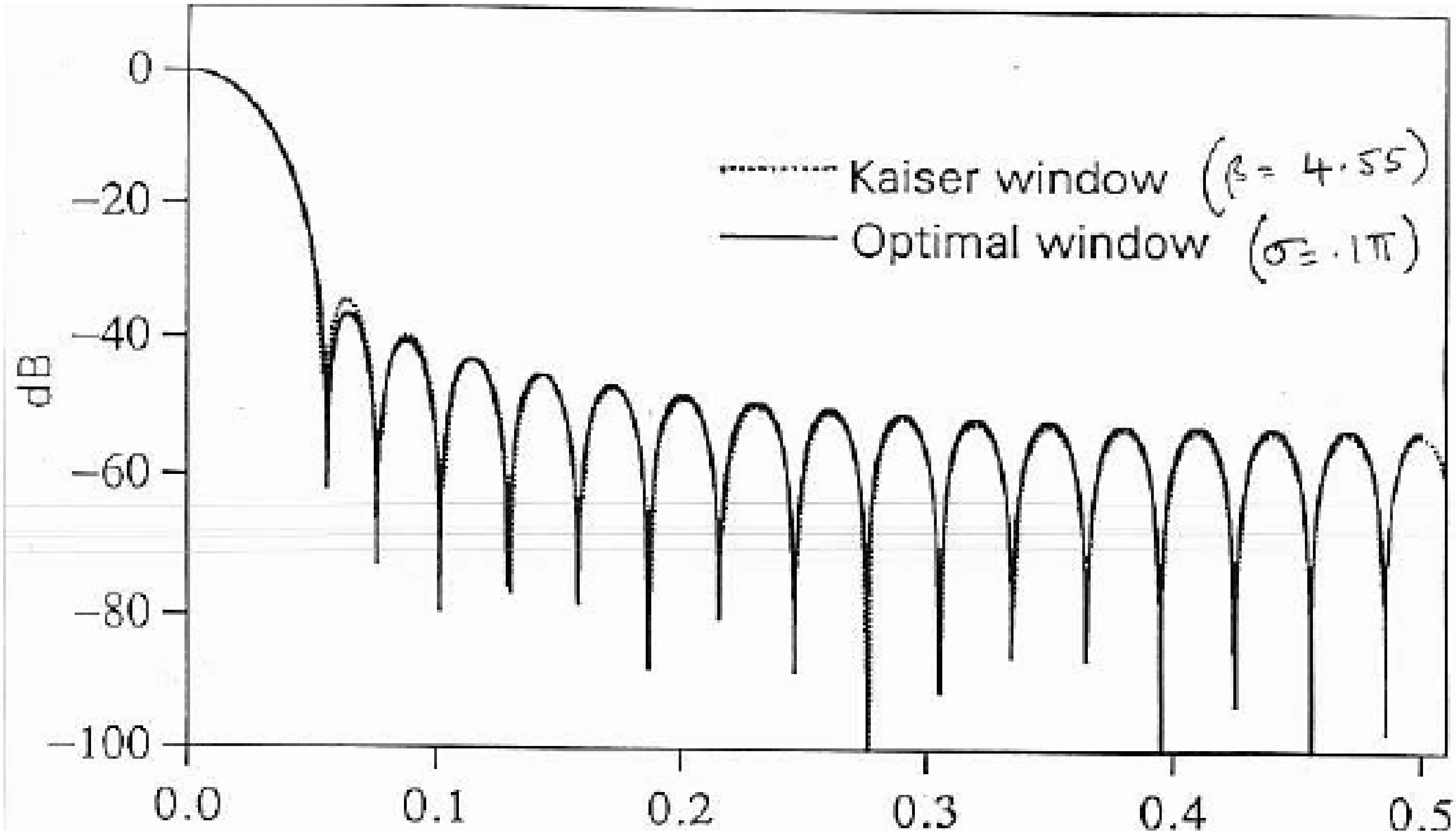
$$I_0(x) = \sum_{m=0}^{\infty}\left[\frac{(.5x)^m}{m!}\right]^2$$

- β controls sidelobe level (Stopband Attenuation)
- The filter order M controls the Mainlobe width



M=20
β = 0
β = 3
β = 6
Amplitude
1.2
0.9
0.6
0.3
0
5
10
15
20
Samples
(a)
M=20
β = 0
β = 3
β = 6
dB
0
−25
−50
−75
−100
0
0.2π
0.4π
0.6π
0.8π
π
Radian frequency (ω)
(b)
Beta = 6
M = 10
M = 20
M = 40
dB
0
−25
−50
−75
−100
0
0.2π
0.4π
0.6π
0.8π
π
Radian frequency (ω)

Normalized frequency ( )



Kaiser Window based design

- Design Method: define

$$\Delta\omega = \omega_s - \omega_p$$

$$A = -20\log_{10}\delta, \quad where\ \delta = \min(\delta_1, \delta_2)$$

- Choose β as

$$\beta = \begin{cases} .1102(A-8.7) & , A > 50 \\ .5842(A-21)^{0.4} + .07886(A-21), & 21 \le A \le 50 \\ 0, \ A < 21 \end{cases}$$

$$M = \frac{A-8}{2.285\Delta\omega}$$

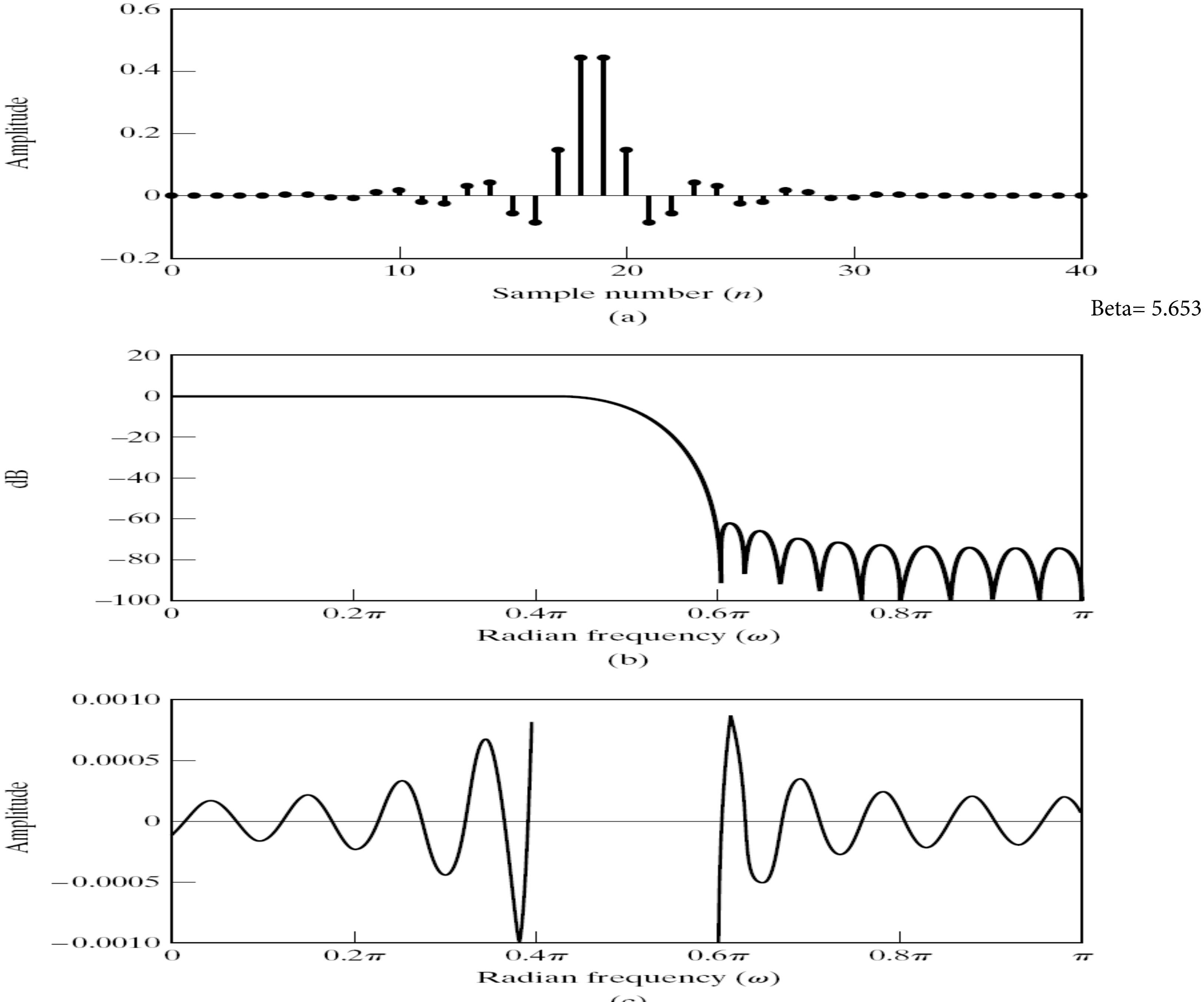


Figure 7.25 Response functions for Example 7.8. (a) Impulse response ($M = 37$). (b) Log magnitude. (c) Approximation error.

Kaiser Window based Filter Design

Kaiser Window based Filter Design

Parameters (Problem 7.15):

Kaiser Window based Filter Design

Parameters (Problem 7.15): $\delta_1 = .05$, $\delta_2 = .1$, $\omega_p = .25\pi$, and $\omega_s = .35\pi$.

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$A = -20\log\delta = 26db$.

Kaiser Window based Filter Design

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sidelobe level parameter β.

Kaiser Window based Filter Design

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$\omega_c = \frac{\omega_p + \omega_s}{2} = .30\pi$, $\delta = \min(\delta_1, \delta_2) = .05$, and $\delta\omega = \omega_s - \omega_p = .1\pi$.

$A = -20 \log \delta = 26db.$

sidelobe level parameter β.

$$\beta = .5842(A-21)^{0.4} + .07886(A-21) = .58425^{0.4} + .07886 \times 5 = 1.506$$

Kaiser Window based Filter Design

Parameters (Problem 7.15): $\delta_1 = .05$, $\delta_2 = .1$, $\omega_p = .25\pi$, and $\omega_s = .35\pi$.

$\omega_c = \frac{\omega_p + \omega_s}{2} = .30\pi$, $\delta = \min(\delta_1, \delta_2) = .05$, and $\delta\omega = \omega_s - \omega_p = .1\pi$.

$A = -20 \log \delta = 26db.$

sidelobe level parameter β.

$\beta = .5842(A-21)^{0.4} + .07886(A-21) = .58425^{0.4} + .07886 \times 5 = 1.506$

Filter Order

$$M = \frac{A-8}{2.285\delta\omega} = \frac{18}{2.285.1\pi} = \lceil 25.071 \rceil = 26$$

Kaiser Window based Filter Design

Parameters (Problem 7.15): $\delta_1 = .05$, $\delta_2 = .1$, $\omega_p = .25\pi$, and $\omega_s = .35\pi$.

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sidelobe level parameter β.

$\beta = .5842(A-21)^{0.4} + .07886(A-21) = .58425^{0.4} + .07886 \times 5 = 1.506$

Filter Order

$$M = \frac{A-8}{2.285\delta\omega} = \frac{18}{2.285.1\pi} = \lceil 25.071 \rceil = 26$$

$$h[n] = h_d[n]w_{kaiser}[n], 0 \leq n \leq M = 26$$

Kaiser Window based Filter Design

Parameters (Problem 7.15): $\delta_1 = .05$, $\delta_2 = .1$, $\omega_p = .25\pi$, and $\omega_s = .35\pi$.

$\omega_c = \frac{\omega_p + \omega_s}{2} = .30\pi$, $\delta = \min(\delta_1, \delta_2) = .05$, and $\delta\omega = \omega_s - \omega_p = .1\pi$.

$A = -20 \log \delta = 26db.$

sidelobe level parameter β.

$\beta = .5842(A - 21)^{0.4} + .07886(A - 21) = .58425^{0.4} + .07886 \times 5 = 1.506$

Filter Order

$$M = \frac{A - 8}{2.285\delta\omega} = \frac{18}{2.285.1\pi} = \lceil 25.071 \rceil = 26$$

$$h[n] = h_d[n] w_{kaiser}[n], 0 \leq n \leq M = 26$$

where

$$h_d[n] = \frac{\sin \omega_c(n - \alpha)}{\pi(n - \alpha)}, \quad \text{where} \quad \alpha =$$

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Pros and Cons of Window based Design

- Advantages
  - Easy to design
  - Can be applied to general linear system design
- Disadvantages
  - Exceeds the specs everywhere except at the edges of the passband and stopband
  - δ_1 and δ_2 cannot be independently controlled. Have to design more conservatively for the smaller of the two