ECE 286 Class 2:
Sequential Bayesian Estimation

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State-Space Model

- Consider a sequence of states $\mathbf{x}_n$ and a sequence of measurements $\mathbf{y}_n$

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$$\mathbf{x}_n = \mathbf{g}_n(\mathbf{x}_{n-1}, \underbrace{\mathbf{u}_n}_{\text{Driving noise (white)}}), \quad n = 1, 2, \ldots$$

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Measurement model

Measurement $\mathbf{y}_n$ depends on state $\mathbf{x}_n$ according to

$$\mathbf{y}_n = \mathbf{h}_n(\mathbf{x}_n, \underbrace{\mathbf{v}_n}_{\text{Measurement noise (white)}}), \quad n = 1, 2, \ldots$$

This determines the likelihood function $f(\mathbf{y}_n|\mathbf{x}_n)$

Markovian Properties

- Noise sequences $\mathbf{u}_n$ and $\mathbf{v}_n$ are assumed mutually independent and independent of $\mathbf{x}_0$.
- Recall:

$$\boxed{\begin{aligned} \mathbf{x}_n &= \mathbf{g}_n(\mathbf{x}_{n-1}, \mathbf{u}_n)\,, \quad && \mathbf{u}_n \text{ is white} \\ \mathbf{y}_n &= \mathbf{h}_n(\mathbf{x}_n, \mathbf{v}_n)\,, && \mathbf{v}_n \text{ is white} \end{aligned}}$$

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- At time n, the state $\mathbf{x}_n$ summarizes all relevant information about the present and past
- Mathematically expressed by "Markovian properties":

$$\boxed{\begin{gathered} f(\mathbf{x}_n|\mathbf{x}_{n-1}, \mathbf{y}_{1:n-1}) = f(\mathbf{x}_n|\mathbf{x}_{n-1}) \\ f(\mathbf{y}_n|\mathbf{x}_n, \mathbf{y}_{1:n-1}) = f(\mathbf{y}_n|\mathbf{x}_n) \end{gathered}} \quad \text{where } \mathbf{y}_{1:n-1} \triangleq \begin{pmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_{n-1} \end{pmatrix}$$

Sequential Bayesian Estimation

- We wish to estimate the current state $\mathbf{x}_n$ from the past and current measurements $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$, i.e., from $\mathbf{y}_{1:n}$, for $n = 1, 2, \ldots$
- MMSE estimator:

$$\hat{\mathbf{x}}_n \;=\; \mathrm{E}\big\{\mathbf{x}_n | \mathbf{y}_{1:n}\big\} = \int \mathbf{x}_n \, f(\mathbf{x}_n | \mathbf{y}_{1:n}) \, \mathrm{d}\mathbf{x}_n$$

B. Ristic, S. Arulampalam, and N. Gordon, *Beyond the Kalman Filter: Particle Filters for Tracking Applications*, Artech House, 2004.

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- The posterior pdf $f(\mathbf{x}_n | \mathbf{y}_{1:n})$ can be calculated recursively/sequentially

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Sequential Bayesian Estimation

- The Markovian properties enable sequential calculation of $f(\mathbf{x}_n|\mathbf{y}_{1:n})$
- One recursion consists of two steps:

Prediction step

$$\underbrace{f(\mathbf{x}_n|\mathbf{y}_{1:n-1})}_{\text{Predicted posterior pdf}} = \int \underbrace{f(\mathbf{x}_n|\mathbf{x}_{n-1})}_{\text{State-transition pdf}} \underbrace{f(\mathbf{x}_{n-1}|\mathbf{y}_{1:n-1})}_{\text{Previous posterior pdf}} \, \mathrm{d}\mathbf{x}_{n-1}$$

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Measurement update step

$$\underbrace{f(\mathbf{x}_n|\mathbf{y}_{1:n})}_{\text{Posterior pdf}} \propto \underbrace{f(\mathbf{y}_n|\mathbf{x}_n)}_{\substack{\text{Likelihood} \\ \text{function}}} \underbrace{f(\mathbf{x}_n|\mathbf{y}_{1:n-1})}_{\substack{\text{Predicted} \\ \text{posterior pdf}}}$$

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- Unfortunately, computationally infeasible in general $\Rightarrow$ feasible approximations required
- We will discuss feasible approximations later (in a generalized setting)

Sequential Bayesian Estimation

- Consider joint posterior pdf $f(\mathbf{x}_{0:n}|\mathbf{y}_{1:n})$
- Sequential calculation of the "marginal" posterior pdf $f(\mathbf{x}_n|\mathbf{y}_{1:n})$ can be interpreted as a **factorization and marginalization** of the joint posterior pdf $f(\mathbf{x}_{0:n}|\mathbf{y}_{1:n})$

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Factorization

$$f(\mathbf{x}_{0:n}|\mathbf{y}_{1:n}) \propto f(\mathbf{x}_0) \prod_{n'=1}^{n} f(\mathbf{y}_{n'}|\mathbf{x}_{n'})\, f(\mathbf{x}_{n'}|\mathbf{x}_{n'-1})$$

Marginalization

$$f(\mathbf{x}_n|\mathbf{y}_{1:n}) \propto \int f(\mathbf{x}_0) \left(\prod_{n'=1}^{n} \left[f(\mathbf{y}_{n'}|\mathbf{x}_{n'})\, f(\mathbf{x}_{n'}|\mathbf{x}_{n'-1}) \right. \right) \mathrm{d}\mathbf{x}_{0:n-1}$$