SIO 209: Signal Processing for Ocean Sciences
Class 3

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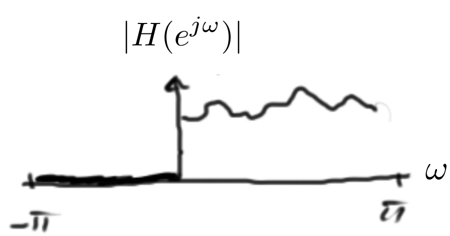
HT Relations for Complex Sequence

- We are interested in complex sequences where the real and the imaginary parts can be related through a Hilbert transform (HT)
- Here ``causality'' means that the periodic frequency domain is zero in the second half of each period defined on $0 \leq \omega < 2\pi$ (or equivalently on the first half of each period defined on $-\pi \leq \omega < \pi$)



- These HT relations are useful for the representation of real bandpass signals as complex sequences

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HT Relations for Complex Sequence

- Let us consider the complex sequence $s[n] = s_{\mathrm{R}}[n] + js_{\mathrm{I}}[n]$ with Fourier transform $S(e^{j\omega})$
- The Fourier transform of the purely real signal $s_{\mathrm{R}}[n]$ is conjugate even and can be obtained as

$$S_{\mathrm{R}}(e^{j\omega}) = \frac{1}{2}\Big[S(e^{j\omega}) + S^*(e^{-j\omega})\Big]$$

- Similarly, the Fourier transform of the purely imaginary signal $js_{\mathrm{I}}[n]$ is conjugate odd and can be calculated as

$$jS_{\mathrm{I}}(e^{j\omega}) = \frac{1}{2}\Big[S(e^{j\omega}) - S^*(e^{-j\omega})\Big]$$

- Note that $S_{\mathrm{I}}(e^{j\omega})$ is the Fourier transform of the purely real signal $s_{\mathrm{I}}[n]$

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HT Relations for Complex Sequence

- Let us assume that $S(e^{j\omega})$ is ``causal", i.e., $S(e^{j\omega}) = 0$ for $-\pi \le \omega < 0$
- We can then express $S(e^{j\omega})$ as follows

$$S(e^{j\omega}) = \begin{cases} 2S_{\mathrm{R}}(e^{j\omega}) & 0 \le \omega < \pi \\ 0 & -\pi \le \omega < 0 \end{cases} \qquad S(e^{j\omega}) = \begin{cases} 2jS_{\mathrm{I}}(e^{j\omega}) & 0 \le \omega < \pi \\ 0 & -\pi \le \omega < 0 \end{cases}$$

- Furthermore, we can write

$$S_{\mathrm{I}}(e^{j\omega}) = \begin{cases} -jS_{\mathrm{R}}(e^{j\omega}) & 0 \le \omega < \pi \\ jS_{\mathrm{R}}(e^{j\omega}) & -\pi \le \omega < 0 \end{cases}$$

- Note that here we make use of the fact that since $s_{\mathrm{R}}[n]$ and $s_{\mathrm{I}}[n]$ are real sequences, their Fourier transforms have complex conjugate symmetry

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HT Relations for Complex Sequence

- The imaginary part of the Fourier transform $S(e^{j\omega})$ can thus also be expressed as

$$S_{\mathrm{I}}(e^{j\omega}) = H(e^{j\omega})S_{\mathrm{R}}(e^{j\omega}) \qquad H(e^{j\omega}) = \begin{cases} -j & 0 \le \omega < \pi \\ j & -\pi \le \omega < 0 \end{cases}$$

- Recall that $-j = e^{-j\pi/2}$ and $j = e^{j\pi/2}$
- Thus, $H(e^{j\omega})$ is a 90° phase-shifter that is also referred to as *Hilbert Transformer*

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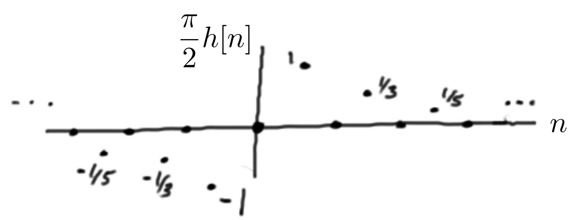
HT Relations for Complex Sequence

- The impulse response of the Hilbert Transformer $H(e^{j\omega})$ is given by

$$h[n] = \begin{cases} \dfrac{2\sin^2\left(\frac{\pi n}{2}\right)}{\pi n} & n \neq 0 \\ 0 & n = 0 \end{cases}$$



- For practical use, a finite-length Hilbert Transformer can be designed using an FIR filter design routine, e.g., "firpm"

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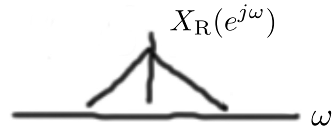
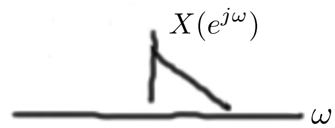
Representation of Bandpass Signals as Complex Lowpass Signals

- Let $x_{\mathrm{R}}[n]$ be a real baseband signal



- The corresponding complex lowpass signal is given by

$$\begin{aligned} x[n] &= x_{\mathrm{R}}[n] + jx_{\mathrm{I}}[n] \\ &= A[n]e^{j\phi[n]} \end{aligned}$$



where $x_{\mathrm{I}}[n]$ is the HT of $x_{\mathrm{R}}[n]$

- The signal $x[n]$ is complex with $X(e^{j\omega}) = 0$ for $-\pi \leq \omega < 0$

- Note that $x[n]$ contains the same information as $x_{\mathrm{R}}[n]$ but needs less bandwidth

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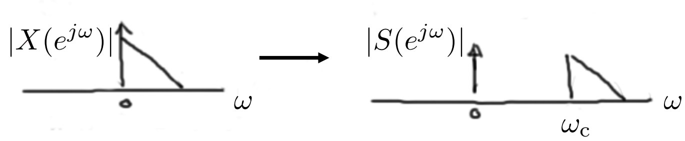
Representation of Bandpass Signals as Complex Lowpass Signals

- From the complex baseband signal $x[n]$ we can compute the complex bandpass signal as follows

$$s[n] = x[n]e^{j\omega_{\mathrm{c}}n} = A[n]e^{j(\omega_{\mathrm{c}}n + \phi[n])} = s_{\mathrm{R}}[n] + js_{\mathrm{I}}[n]$$

- Recall that multiplication by $e^{j\omega_{\mathrm{c}}n}$ in time domain results in counterclockwise rotation by ω_{c} of z-plane

- Thus, the Fourier transform of $s[n]$ is given by $S(e^{j\omega}) = X(e^{j(\omega - \omega_{\mathrm{c}})})$



- Note that $s[n]$ and $x[n]$ are complex

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Representation of Bandpass Signals as Complex Lowpass Signals

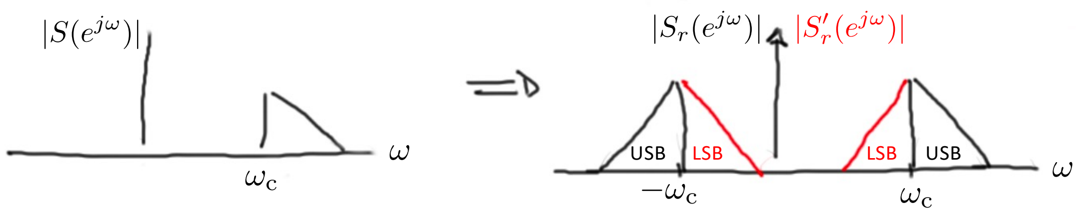
- From the complex bandpass signal $s[n]$, we can compute a real bandpass signal as

$$s_{\mathrm{R}}[n] = \mathrm{Re}\{s[n]\} = A[n]\cos(\omega_c n + \phi[n]) = x_{\mathrm{R}}[n]\cos(\omega_c n) - x_{\mathrm{I}}[n]\sin(\omega_c n)$$

- The signal $s_{\mathrm{R}}[n]$ is also called the ***upper sideband (USB)*** real bandpass signal of $x_{\mathrm{R}}[n]$

- Note that a ***lower sideband (LSB)*** real bandpass signal of $x_{\mathrm{R}}[n]$ is given by

$$s'_{\mathrm{R}}[n] = x_{\mathrm{R}}[n]\cos(\omega_c n) + x_{\mathrm{I}}[n]\sin(\omega_c n)$$



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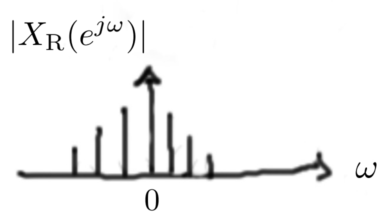
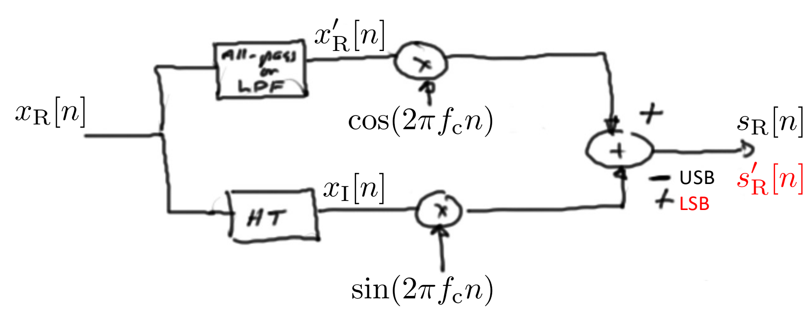
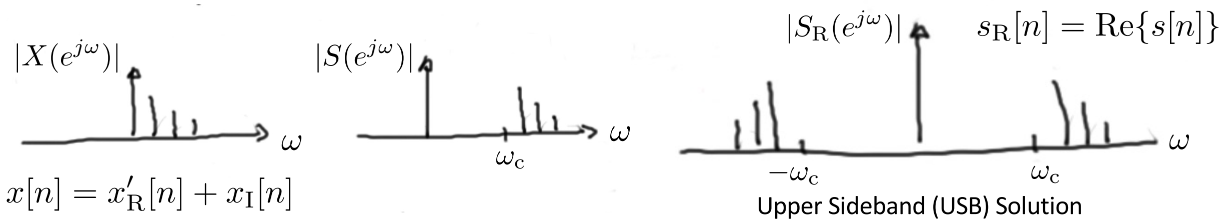
Homework 3: Single Sideband Generation







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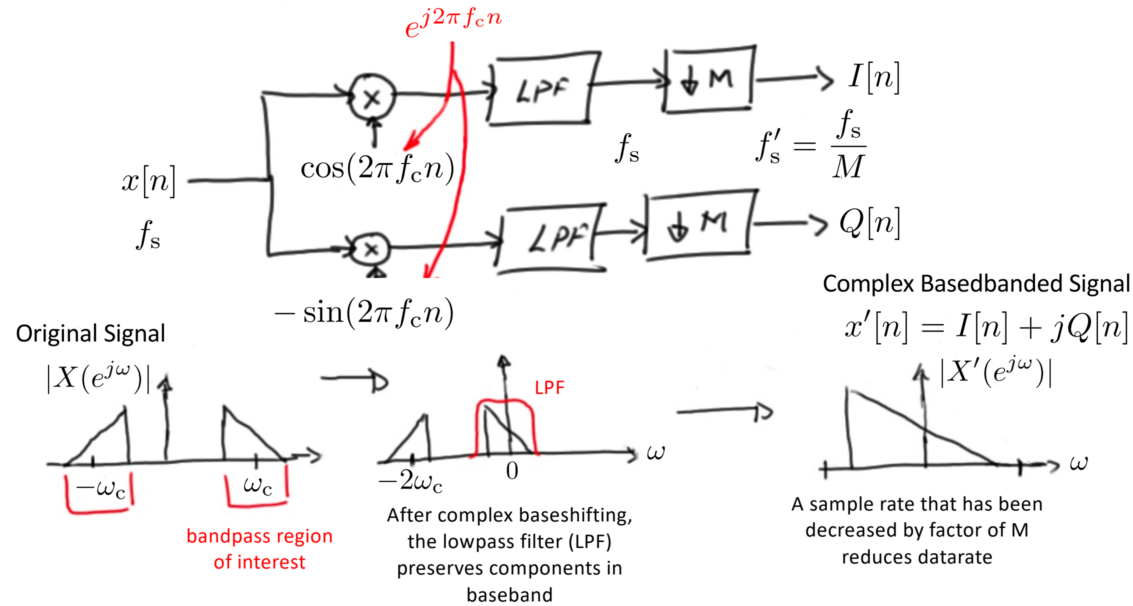
Complex Basebanding



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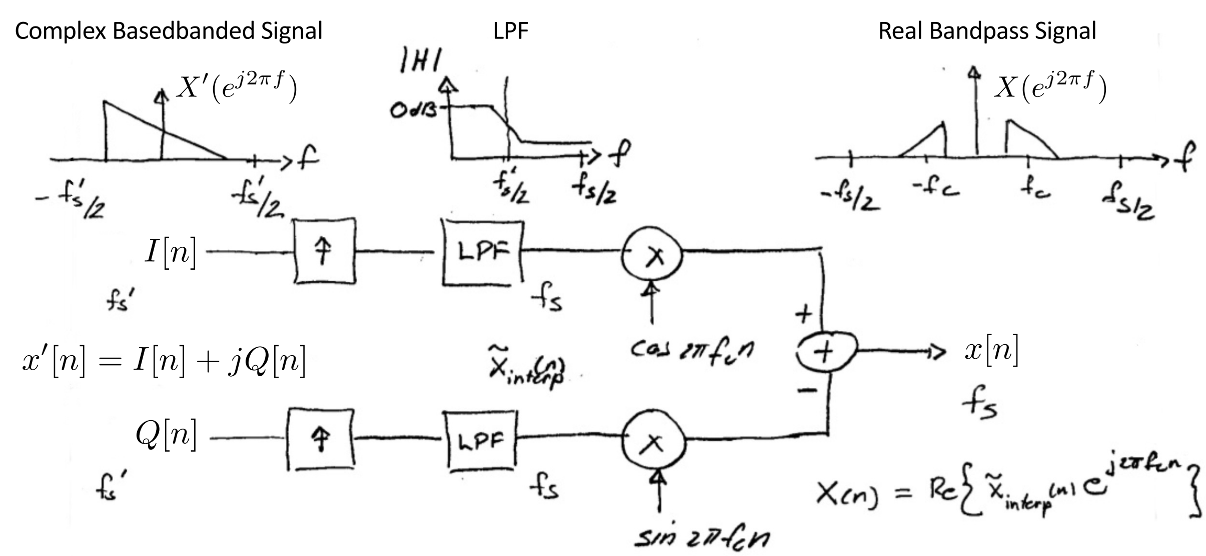
Quadrature Modulation



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