SIO 209: Signal Processing for Ocean Sciences
Class 5

Florian Meyer

Scripps Institution of Oceanography
Electrical and Computer Engineering Department
University of California San Diego

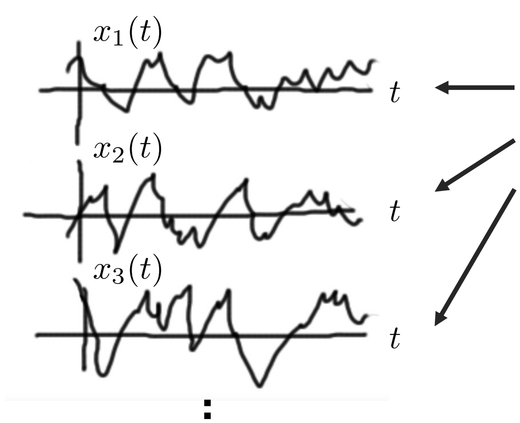
SCRIPPS INSTITUTION OF OCEANOGRAPHY

UC San Diego
JACOBS SCHOOL OF ENGINEERING

0

Discrete Random Sequences

Random Process



Sample Functions
of Random Process

A random process is an ensemble of sample functions

see also Chapter 2.1 and Appendix A in *Oppenheim & Schafer, 2009*

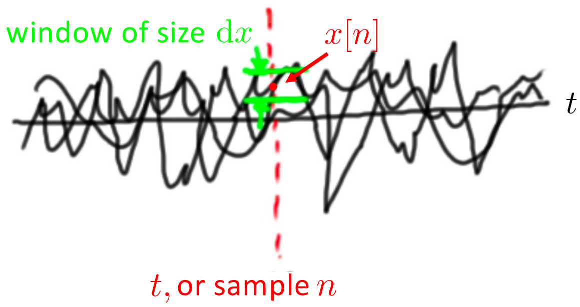
1

Discrete Random Sequences

Random Process Description

We are interested in characterizing the probability density function both of a single point in time as well as the joint characteristics at different points in time.

An individual random variable $x[n]$ is described by the probability density function $p(x[n])$



Histogram

$$\tilde{p}(x[n]) = \frac{k}{K}$$

K total number of sample functions in the ensemble

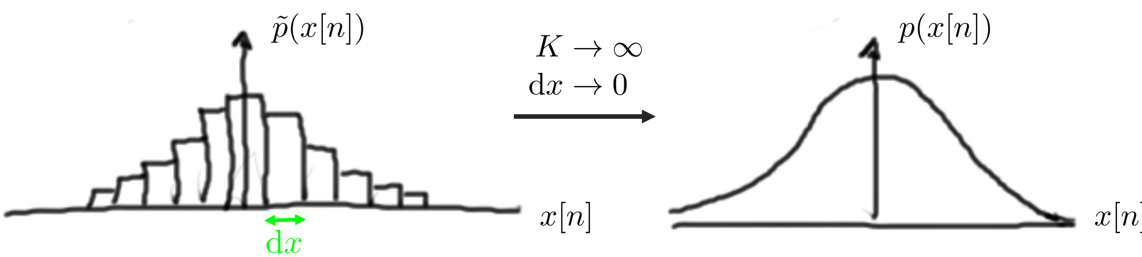
k number of sample functions that go through window at $x[n]$

2

Discrete Random Sequences

Random Process Description (cnt.)

As the size of the window becomes smaller and smaller and the number of sample functions becomes larger and larger, the histogram transitions to a continuous probability density function



Histogram

Probability Density Function

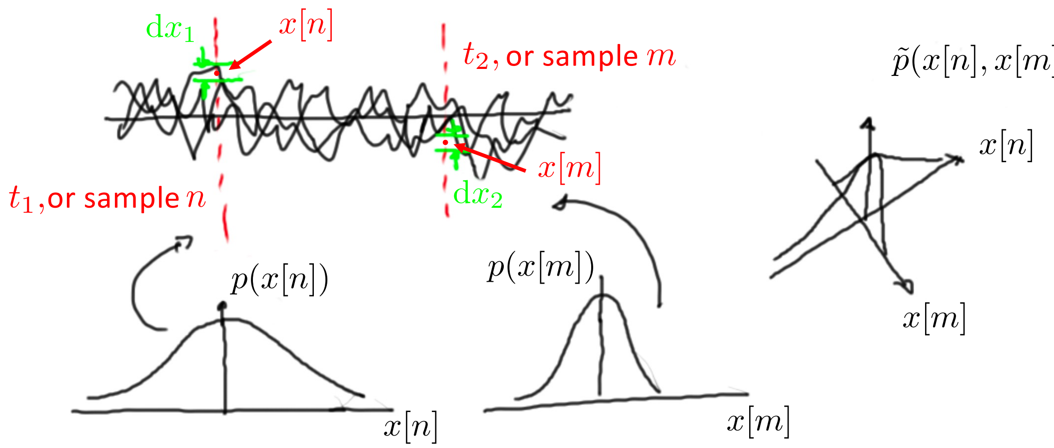
3

Discrete Random Sequences

number of sample functions that go through both windows

$$\tilde{p}(x[n], x[m]) = \frac{k_{12}}{K}$$

Random Process Description (cnt.)



Statistical Independence: $p(x[n], x[m]) = p(x[n])p(x[m])$

4

Real Scalar Random Variables

Ensembles Averages

- The average or mean of the process is defined as (for simplicity we drop time index n)

$$\mu = \mathrm{E}[x] = \int_{-\infty}^{\infty} x\, p(x)\mathrm{d}x$$

where $\mathrm{E}[\,\cdot\,]$ denotes expectation

- Similarly, we can determine $\mathrm{E}[g(x)]$ where $g(x)$ is a single-valued function of $\mathrm{d}x$

$$\mathrm{E}[g(x)] = \int_{-\infty}^{\infty} g(x)\, p(x)\mathrm{d}x$$

5

5

Real Scalar Random Variables

- Simple properties

$$E[x+y] = E[x] + E[y]$$
$$E[ax] = aE[x]$$

x and y are random
a is deterministic

- In general, the average of a product is not equal to the product of the averages
- The average of the product is equal to the product of the averages if the two random variables are uncorrelated
- A pair of random variables can have the following properties
  - orthogonal $E[xy] = 0$
  - uncorrelated $E[xy] = E[x]E[y]$
  - independent $p(x,y) = p(x)p(y)$

Note that $p(x,y) = p(x)p(y) \implies E[xy] = E[x]E[y]$

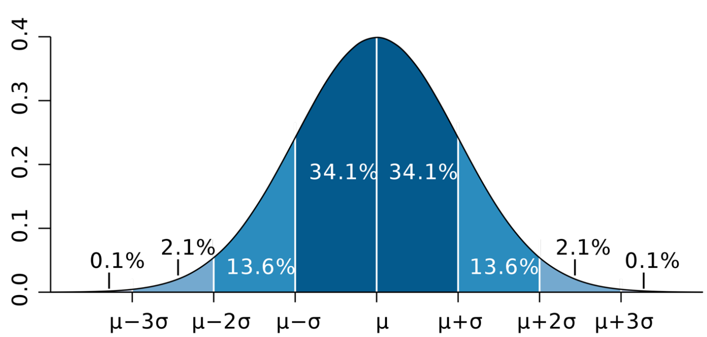
6

6

The Gaussian Distribution

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

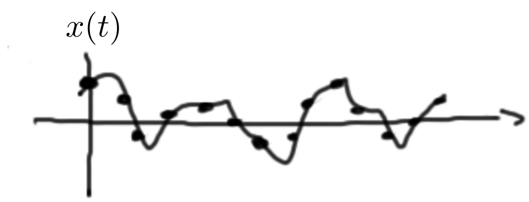
mean $\mu = E[x]$
variance $\sigma^2 = E[(x-\mu)^2]$



7

7

Real Random Vectors



$\Longrightarrow$

sampled at f_s

$$\boldsymbol{x} = \begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{pmatrix}$$

Real Random Vectors

The Multivariate Gaussian Distribution

$$p(\boldsymbol{x}) = (2\pi)^{-\frac{N}{2}} \det(\boldsymbol{\Sigma})^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\Big)$$

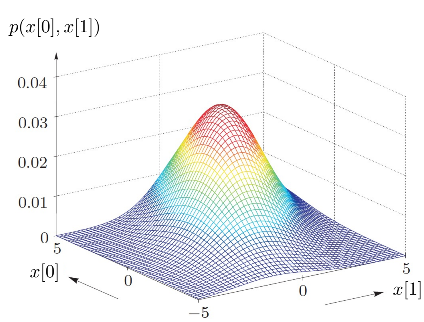
mean $\boldsymbol{\mu} = \mathrm{E}[\boldsymbol{x}]$

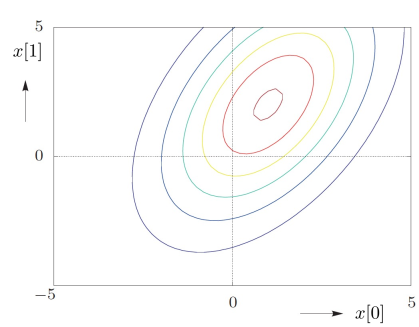
covariance matrix
$\boldsymbol{\Sigma} = \mathrm{E}[(\boldsymbol{x}-\boldsymbol{\mu})(\boldsymbol{x}-\boldsymbol{\mu})^{\mathrm{T}}]$

- Example ($N=2$):

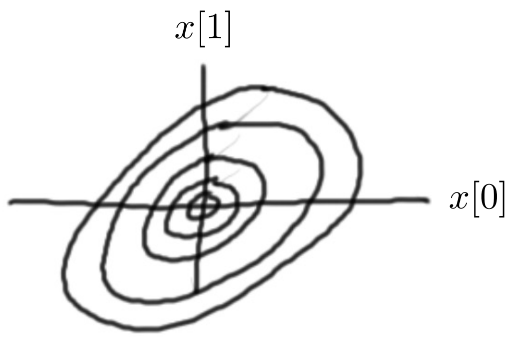
$$\boldsymbol{\mu} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix}$$

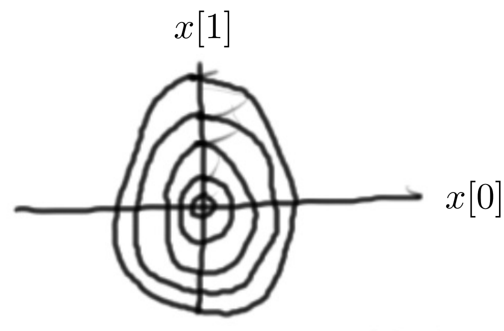




The Multivariate Gaussian Distribution



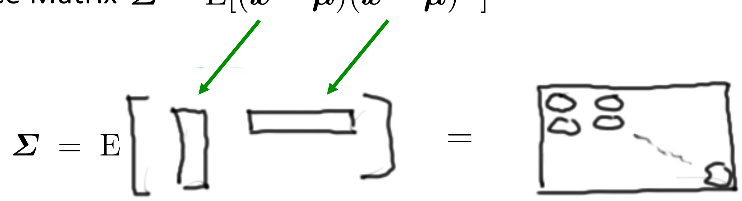
$x[0]$ and $x[1]$ are correlated



$x[0]$ and $x[1]$ are uncorrelated

The Multivariate Gaussian Distribution

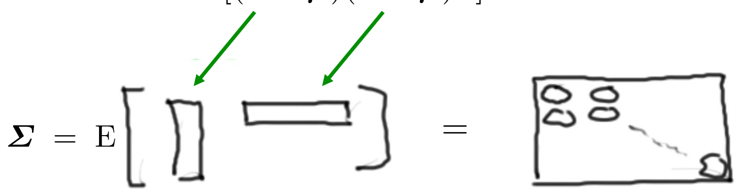
- Covariance Matrix $\boldsymbol{\Sigma} = \mathrm{E}[(\boldsymbol{x} - \boldsymbol{\mu})(\boldsymbol{x} - \boldsymbol{\mu})^{\mathrm{T}}]$



$$\boldsymbol{\Sigma} = \begin{bmatrix} \mathrm{E}\{(x[0]-\mu_0)^2\} & \mathrm{E}\{(x[0]-\mu_0)(x[1]-\mu_1)\} & \cdots & \mathrm{E}\{(x[0]-\mu_0)(x[N-1]-\mu_{N-1})\} \\ \mathrm{E}\{(x[1]-\mu_1)(x[0]-\mu_0)\} & \mathrm{E}\{(x[1]-\mu_1)^2\} & \cdots & \mathrm{E}\{(x[1]-\mu_1)(x[N-1]-\mu_{N-1})\} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{E}\{(x[N-1]-\mu_{N-1})(x[0]-\mu_0)\} & \mathrm{E}\{(x[N-1]-\mu_{N-1})(x[1]-\mu_1)\} & \cdots & \mathrm{E}\{(x[N-1]-\mu_{N-1})^2\} \end{bmatrix}$$

The Multivariate Gaussian Distribution

- Covariance Matrix $\boldsymbol{\Sigma} = \mathrm{E}[(\boldsymbol{x} - \boldsymbol{\mu})(\boldsymbol{x} - \boldsymbol{\mu})^{\mathrm{T}}]$



$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_0^2 & \mathrm{E}\{(x[0]-\mu_0)(x[1]-\mu_1)\} & \cdots & \mathrm{E}\{(x[0]-\mu_0)(x[N-1]-\mu_{N-1})\} \\ \mathrm{E}\{(x[1]-\mu_1)(x[0]-\mu_0)\} & \sigma_1^2 & \cdots & \mathrm{E}\{(x[1]-\mu_1)(x[N-1]-\mu_{N-1})\} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{E}\{(x[N-1]-\mu_{N-1})(x[0]-\mu_0)\} & \mathrm{E}\{(x[N-1]-\mu_{N-1})(x[1]-\mu_1)\} & \cdots & \sigma_{N-1}^2 \end{bmatrix}$$

12

12

The Multivariate Gaussian Distribution

- Off-diagonal terms in $\boldsymbol{\Sigma}$

$$\begin{aligned} &\mathrm{E}\{(x[n]-\mu_n)(x[m]-\mu_m)\} \\ &\quad = \mathrm{E}\{x[n]x[m] - \mu_n x[m] - \mu_m x[n] + \mu_n\mu_m)\} \\ &\quad = \mathrm{E}\{x[n]x[m]\} - \mu_n\mu_m \end{aligned}$$

- If $x[n]$ and $x[m]$ are uncorrelated, $\mathrm{E}\{x[n]x[m]\} = \mathrm{E}\{x[n]\}\mathrm{E}\{x[m]\}$ and $\mathrm{E}\{(x[n]-\mu_n)(x[m]-\mu_m)\} = 0$

- Multivariate Gaussian random vectors have the (very special) property that uncorrelatedness implies independence, i.e.,

$$\mathrm{E}\{x[n]x[m]\} = \mathrm{E}\{x[n]\}\mathrm{E}\{x[m]\} \implies p(x[n], x[m]) = p(x[n])p(x[m])$$

13

13