SIO 207A: Fundamentals of Digital Signal Processing
Class 15

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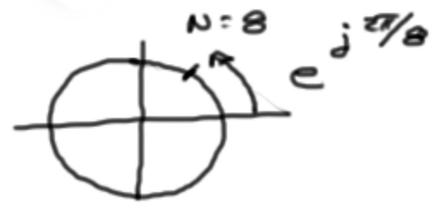
0

Discrete Fourier Transform

$$X(k) = \sum_{n=0}^{N-1} x[n] \boxed{e^{-j\frac{2\pi}{N}nk}}$$

$$= \sum_{n=0}^{N-1} x[n] w_N^{kn}$$

$$w_N = e^{-j\frac{2\pi}{N}}$$



- For every $X(k)$ there are N complex multiply-adds. Thus, for N values of $X(k)$, there are N^2 multiply-adds.

1

Decimation in Time Algorithm

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} \left[x[2n] w_N^{2nk} + x[2n+1] w_N^{(2n+1)k} \right] \qquad k = 0, 1, \ldots, N$$

even indexed points
$N/2$ points

odd indexed points
$N/2$ points

$$w_N = e^{-j2\pi/8}$$

$$w_N^2 = e^{-j(2)2\pi/8} = e^{-j2\pi/4} = w_{N/2}$$

see also Section 9.3 in
Oppenheim & Schafer, 1999

2

Decimation in Time Algorithm

$$X(k) = \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x[2n] w_{N/2}^{nk}} + w_N^k \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x[2n+1] w_{N/2}^{nk}} \qquad k = 0, 1, \ldots, N-1$$

$(N/2)^2$ multiply-adds
($N/2$ point DFT)

$(N/2)^2$ multiply-adds
($N/2$ point DFT)

3

Decimation in Time Algorithm

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x[2n]\,w_{N/2}^{nk} + w_N^k \sum_{n=0}^{\frac{N}{2}-1} x[2n+1]\,w_{N/2}^{nk} \qquad k = 0, 1, \ldots, N-1$$

$$= A(k) + w_N^k B(k)$$

$A(k)$ and $B(k)$ are N/2 point DFTs

Since DFT is periodic in k (or frequency) the value of $A(k)$ and $B(k)$ for $k < N/2$ repeat for $k \geq N/2$. Thus,

$$X(k + N/2) = A(k) + w_N^{k+N/2} B(k) \qquad k = 0, 1, \ldots, \frac{N}{2} - 1$$

and $w_N^{N/2} = e^{-j\frac{2\pi}{N}\frac{N}{2}} = -1$

4

Decimation in Time Algorithm

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x[2n]\,w_{N/2}^{nk} + w_N^k \sum_{n=0}^{\frac{N}{2}-1} x[2n+1]\,w_{N/2}^{nk} \qquad k = 0, 1, \ldots, N-1$$

$$= A(k) + w_N^k B(k)$$

$$X(k + N/2) = A(k) - w_N^k B(k) \qquad k = 0, 1, \ldots, N/2 - 1 \qquad \text{(i)}$$

$$X(k) = A(k) + w_N^k B(k) \qquad k = 0, 1, \ldots, N/2 - 1 \qquad \text{(ii)}$$

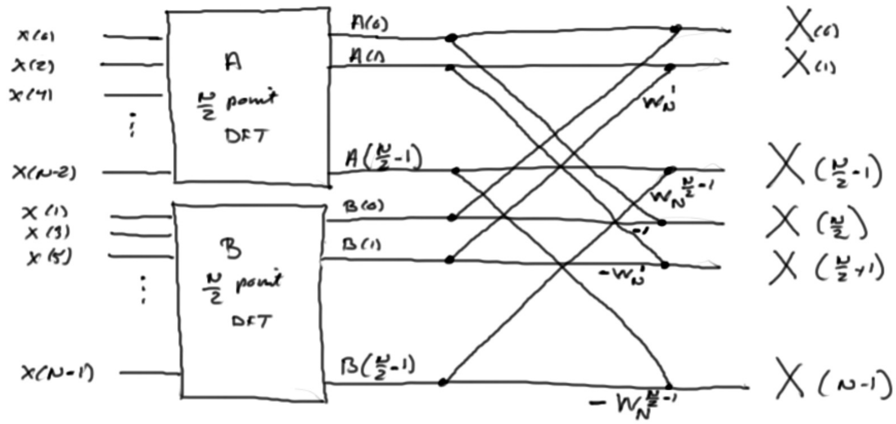
Thus, we can synthesize and N-point DFT from the N/2-point DFTs using the synthesized expressions (i) and (ii) and save half of the multiply-adds

Total Number of Operations: $(N/2)^2 + (N/2)^2 + N \approx N^2/2$ multiply adds

$\longrightarrow$ saving by a factor of two

5

Flow Graph – Decimation in Time



Basic Computational Element

Decimation in Time Algorithm

- Extend this idea to the N/2-point $A(k)$ and $B(k)$ DFTs yields pairs of N/4-points DFTs, etc.
- Fig 9.7 in *Oppenheim & Schafer, 1999* shows full algorithm diagram for $N = 8$
- Data enters full diagram in scrambled order (bit reversed order)

Index	Binary	Bit Reversed	Decimal
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	⋮
4	100	001	
5	101	101	
6	110	011	
7	111	111	

Computational Savings: Instead of N^2 we only perform $N \log_2 N$ multiply adds e.g., if

$$N = 1024 = 2^{10} \qquad N^2 = 1,048,526 \qquad N \log_2 N = 10,240$$

Flow Graph – Decimation in Time

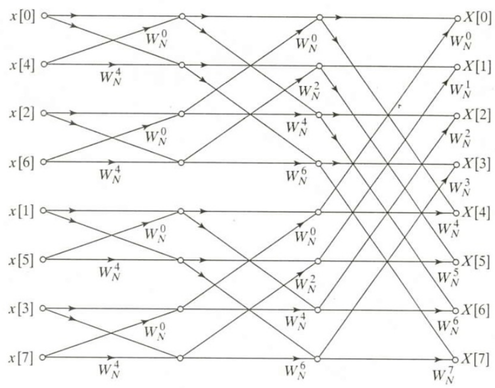


Figure 9.7 Flow graph of complete decimation-in-time decomposition of an 8-point DFT computation.

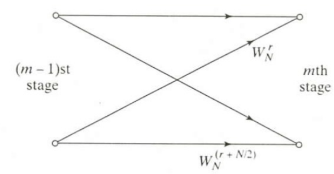


Figure 9.8 Flow graph of basic butterfly computation in Figure 9.7.

8

Decimation in Frequency – FFT Algorithm

first half of time series

second half of time series

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1}\left[\boxed{x[n]}\, w_N^{nk} + \boxed{x[n+N/2]}\, w_N^{(n+N/2)k}\right] \qquad k = 0, 1, \ldots, N-1$$

$$= \sum_{n=0}^{\frac{N}{2}-1}\left[x[n] + w_N^{(N/2)k}\, x[n+N/2]\right] w_N^{nk}$$

$$w_N^{N/2k} = 1 \qquad k = 0, 2, 4, \ldots, N-2 \qquad\qquad w_N^{nk} = w_{N/2}^{nk'} \qquad k = 2k'$$

$$w_N^{N/2k} = -1 \qquad k = 1, 3, 5, \ldots, N-1$$

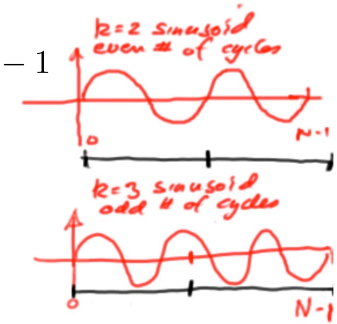
9

Decimation in Frequency – FFT Algorithm

$$X(2k') = \sum_{n=0}^{\frac{N}{2}-1} \Big[x[n] + x[n+N/2]\Big] w_{N/2}^{nk'} \qquad k' = 0, \ldots, N/2-1$$

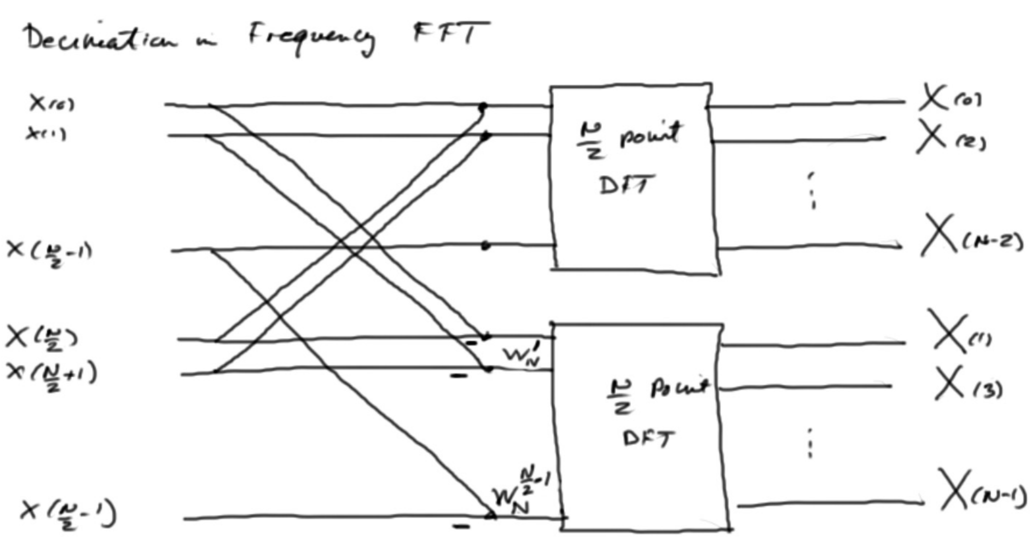
$$X(2k'+1) = \sum_{n=0}^{\frac{N}{2}-1} \Big[x[n] + w_N^{N/2(2k'+1)} x[n+N/2]\Big] w_N^{n(2k'+1)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \Big[x[n] - x[n+N/2]\Big] \underbrace{\boxed{w_N^n}}_{e^{-j\frac{2\pi}{N}n}} w_{N/2}^{nk'}$$



10

Flow Graph – Decimation in Frequency



11

Flow Graph – Decimation in Frequency

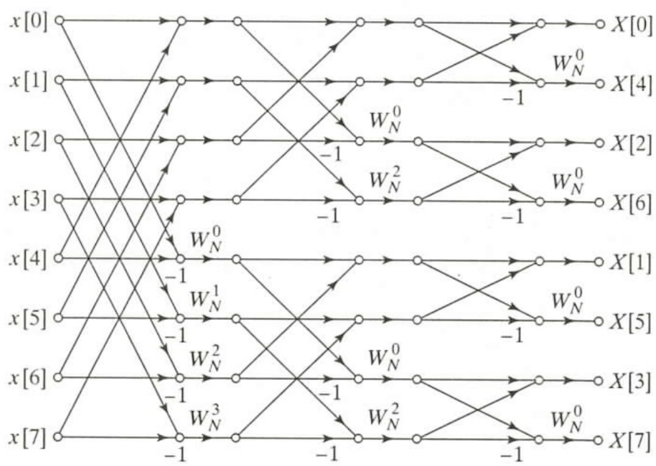


Figure 9.20 Flow graph of complete decimation-in-frequency decomposition of an 8-point DFT computation.

12