Z-Transform: Supplementary Material

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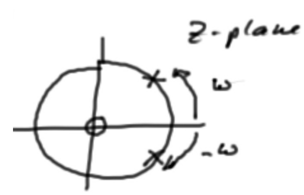
Z-Transform of Sinusoidal Sequence

- We consider the sinusoidal sequence $x[n] = (\sin \omega n) u[n]$; note that $\sin \omega n = \left(e^{j\omega n} - e^{-j\omega n}\right)/2j$

$$
\begin{aligned}
X(z) &= \sum_{n=0}^{\infty} \sin[\omega n] z^{-n} \\
&= \sum_{n=0}^{\infty} \left(\frac{e^{jwn} - e^{-jwn}}{2j} \right) z^{-n} \\
&= \frac{1}{2j} \sum_{n=0}^{\infty} \overset{a}{\boxed{e^{j\omega}}} {}^{n} z^{-n} - \frac{1}{2j} \sum_{n=0}^{\infty} \boxed{e^{-j\omega}} {}^{n} z^{-n} \\
&= \frac{1}{2j} \left(\frac{z}{z - e^{j\omega}} \right) - \frac{1}{2j} \left(\frac{z}{z - e^{-j\omega}} \right) \\
&= \frac{z \sin \omega}{(z - e^{j\omega})(z - e^{-j\omega})}
\end{aligned}
$$



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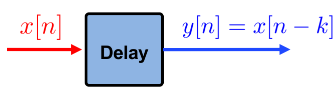
Z-Transform Pairs and Properties

- **A:** Linearity

$$x[n] = a_1 x_1[n] + a_2 x_2[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} \{a_1 x_1[n] + a_2 x_2[n]\} z^{-n}$$

- **B:** Shifting (delay/advance)



$$Y(z) = \sum_{n=-\infty}^{\infty} y[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n-k] z^{-n}$$

let $l = n - k$
$\rightarrow n = l + k$

$$= \sum_{l=-\infty}^{\infty} x[l] z^{-(l+k)} = z^{-k} \sum_{l=-\infty}^{\infty} x[l] z^{-l}$$

$$= z^{-k} X(z)$$

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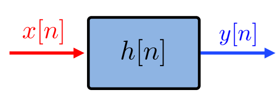
System Input/Output Description

- **C:** System Input/Output Description

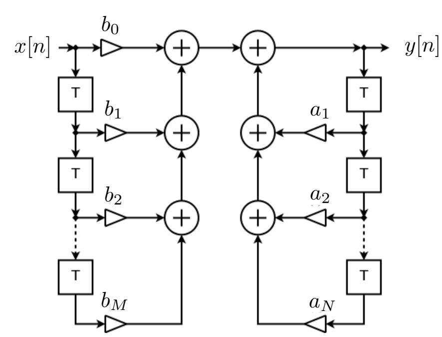


$$Y(z) = \sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{r=0}^{M} b_r z^{-r} X(z)$$

$$= H(z) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{r=0}^{M} b_r z^{-r}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

$$= \frac{b_0 z^M + b_1 z^{M-1} + \cdots + b_M}{z^N + a_1 z^{N-1} + \cdots + a_N}$$



$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{r=0}^{M} b_r x[n-r]$$

$$= \sum_{k=-\infty}^{\infty} h[k] x[n-k] \quad \text{convolution summation}$$

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Multiplication

- **D:** Multiplication

$$y[n] = x_1[n]x_2[n]$$

$$Y(z) = \sum_{n=\infty}^{\infty} x_1[n]x_2[n]z^{-n}$$

- We restrict our interest to the unit circle, i.e., $z = e^{j\omega}$ and assume a Fourier transform

$$\begin{aligned} Y(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x_1[n]x_2[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} x_1[n]\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} X_2(e^{j\omega'})e^{j\omega' n}d\omega'\right\}e^{-j\omega n} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_2(e^{j\omega'})\sum_{n=-\infty}^{\infty} x_1[n]e^{-j(\omega-\omega')n}\,\mathrm{d}\omega' \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_2(e^{j\omega'})X_1(e^{j(\omega-\omega')})\mathrm{d}\omega' \end{aligned}$$

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Multiplication

- Multiplication in time domain is convolution is frequency domain, i.e.,

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_2(e^{j\omega'})X_1(e^{j(\omega-\omega')})\mathrm{d}\omega'$$

- *__Note:__*

1. This is a periodic or circular convolution (not linear)

$$Y(e^{j\omega}) = X_1(e^{j\omega}) \oplus X_2(e^{j\omega})$$

2. Think of circular convolution as a divided cylinder with Fourier transforms pointed on them

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Multiplication

- **E.** Multiplication by a^n

$$x_1[n] = a^n x[n]$$

$$\begin{aligned} X_1(z) &= \sum_{n=-\infty}^{\infty} \big(a^n x[n]\big) z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x[n] \left(a^{-1}z\right)^{-n} \qquad \text{let } z' = a^{-1}z \\ &= \sum_{n=-\infty}^{\infty} x[n]\, z'^{-n} \\ &= X(z') \\ &= X\left(\frac{z}{a}\right) \end{aligned}$$

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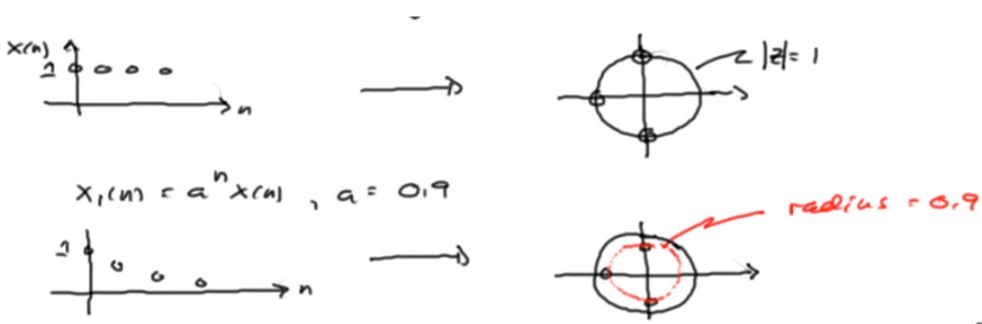
Examples

- Let a be real and positive with $a \leq 1$
- This has the effect of drawing the roots inward on radial paths



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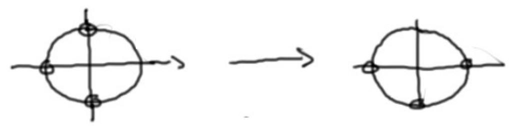
Examples

- Let a be complex and on the unit circle, i.e., $a = e^{j\omega_c}$
- This has the effect of rotating the original z-transform

$x_1[n] = a^n x[n] = e^{j\omega n} x[n]$



$a = e^{j\frac{\pi}{2}}$

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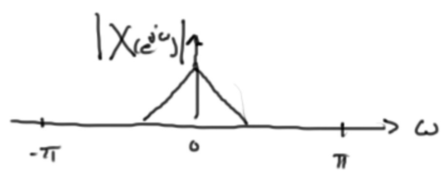
Modulation

$x[n]$ is a low pass audio process



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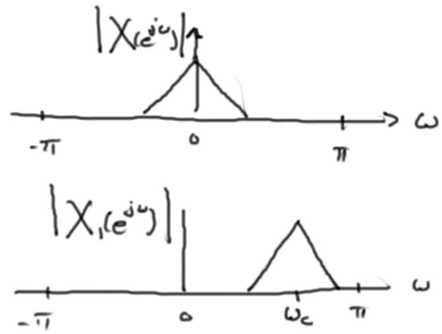
Modulation

$x[n]$ is a low pass audio process

$x_1[n] = a^n x[n]$ where $a = e^{j\omega_c}$



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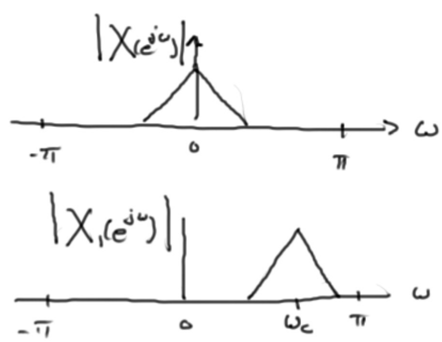
Modulation

$x[n]$ is a low pass audio process

$x_1[n] = a^n x[n]$ where $a = e^{j\omega_c}$



amplitude modulation $x_1[n] = \cos(\omega_c n)x[n]$

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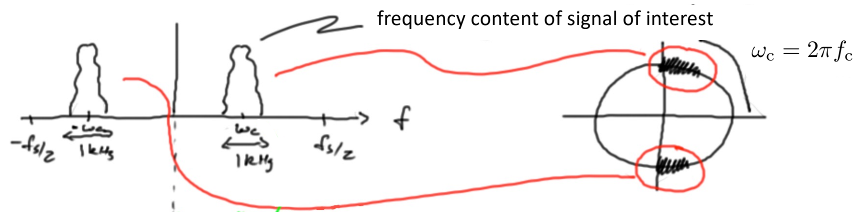
Complex Basebanding



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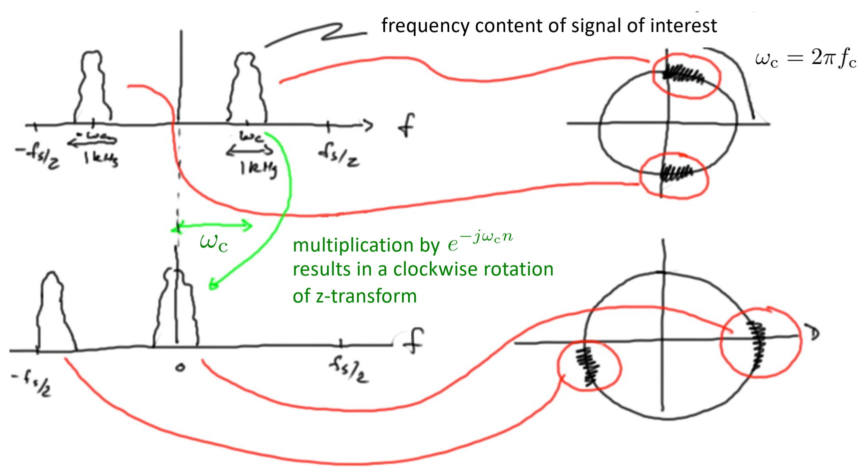
Complex Basebanding



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Z-Transform of Finite Length Sequences

$N = 4$
$a = 1$

- Assume sequence is of length N

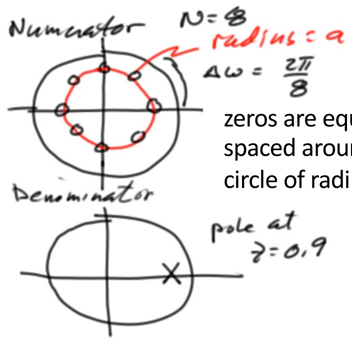
$N = 4$
$a = 0.9$

$$X(z) = \sum_{n=0}^{N-1} x[n] z^{-n} \qquad x[n] = a^n \text{ for } n = 0, \ldots, N-1$$

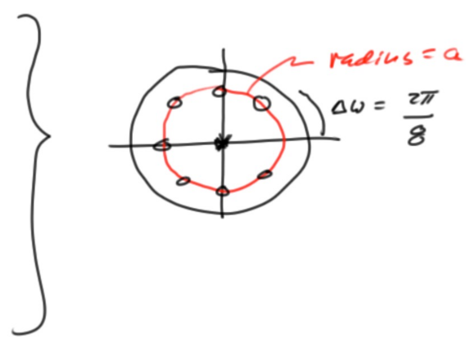
$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha} = \frac{1-(az^{-1})^N}{1-(az^{-1})} = \left(\frac{z}{z-a}\right)\left(\frac{z^N - a^N}{z^N}\right)$$

Z-Transform of Finite Length Sequences

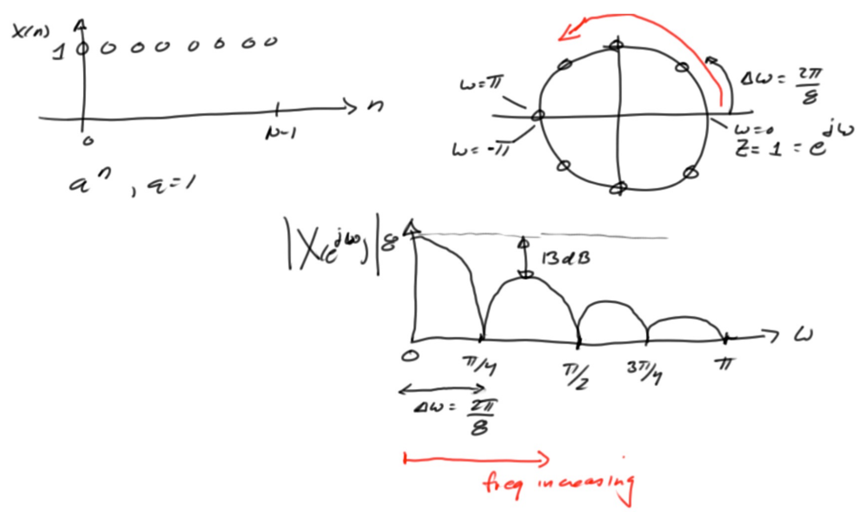
$$X(z) = \left(\frac{z}{z-a}\right)\left(\frac{z^N - a^N}{z^N}\right) \qquad x[n] = a^n \text{ for } n = 0, \ldots, N-1$$



zeros are equally spaced around circle of radius a



Z-Transform of Finite Length Sequences



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