SIO 209: Signal Processing for Ocean Sciences
Class 9

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Normalization Factors for Broadband Power Spectra

- Power Spectral Density

$$P_{xx}(\omega) = \sum_{m=-\infty}^{\infty} \phi_{xx}(m)\, e^{-j\omega m}$$

Power is obtained by integrating the power spectral density over all frequencies

$$\mu_x^2 + \sigma_x^2 = \phi_{xx}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xx}(\omega)\mathrm{d}\omega = \int_{-0.5}^{0.5} P_{xx}(f)\mathrm{d}f$$

see also Sections 10.2 and 10.5 in *Oppenheim & Schafer, 2009*

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Normalization Factors for Broadband Power Spectra

- Welch's method of windowed and overlapped FFTs

$$\hat{P}_{xx}^{L}(f_k) = \frac{1}{MUf_{\mathrm{s}}} \overline{\left|X(k)\right|^2}$$

averaged over L potentially overlapping and windowed records

in "Power"/Hz

$$U = \frac{1}{M} \sum_{n=0}^{M-1} w^2[n]$$

normalization constant

$$X^{(l)}(k) = \sum_{n=0}^{M-1} w[n]\, x^{(l)}[n]\, e^{-j\frac{2\pi}{M}nk}$$

$$l = 1, \ldots, L$$

M FFT length

$f_k = k/M$ frequency at FFT bin with index k

Normalization Factors for Broadband Power Spectra

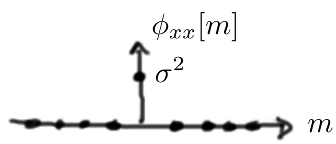
- Recall periodogram

$$I_N(\omega) = \frac{1}{N}\left|X(e^{j\omega})\right|^2 = \sum_{m=-(N-1)}^{N-1} c_{xx}[m] e^{-j\omega m}$$

$$\mathrm{E}\big[I_N(\omega)\big] = \sum_{m=-(N-1)}^{N-1} \mathrm{E}\{c_{xx}[m]\} e^{-j\omega m} = \sum_{m=-(N-1)}^{N-1} \frac{N-|m|}{N}\, \phi_{xx}[m]\, e^{-j\omega m}$$

- Consider uncorrelated (and thus IID) zero-mean Gaussian noise, i.e., $x[n] \sim \mathcal{N}(0, \sigma_x^2)$

$$\phi_{xx}[m] = \begin{cases} \sigma^2 & m = 0 \\ 0 & m \neq 0 \end{cases}$$



$$\mathrm{E}\big[I_N(\omega)\big] = \sigma_x^2$$

Normalization Factors for Broadband Power Spectra

- Consider the absolute value squared of the Fourier transform

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

$$|X(e^{j\omega})|^2 = X(e^{j\omega}) X^*(e^{j\omega}) = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x[n] x[m] e^{-j\omega n} e^{j\omega m}$$

- Assuming uncorrelated zero-mean Gaussian noise, we obtain

If window $w[n]$ had been used, this factor would be $\sum_{n=0}^{N-1} w^2[n]$

$$\mathrm{E}\{|X(e^{j\omega})|^2\} = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \mathrm{E}\{x[n]x[m]\} e^{-j\omega n} e^{j\omega m} = \sum_{n=0}^{N-1} \mathrm{E}\{x[n]^2\} = \sum_{n=0}^{N-1} \sigma^2 = \boxed{N}\sigma^2$$

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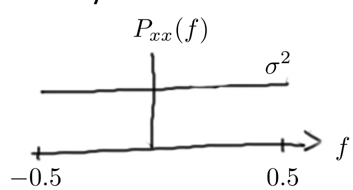
Normalization Factors for Broadband Power Spectra

- For zero-mean Gaussian noise, we have the power spectral density

$$P_{xx}(f) = \sum_{m=-\infty}^{\infty} \phi_{xx}(m)\, e^{-j2\pi f m} = \phi_{xx}(0) = \sigma_x^2$$



and the corresponding power

$$\int_{-0.5}^{0.5} P_{xx}(f)\, \mathrm{d}f = \int_{-0.5}^{0.5} \sigma_x^2\, \mathrm{d}f = \sigma_x^2$$

- Note that f is in cycles per sample or, equivalently, $f_{\mathrm{s}} = 1$

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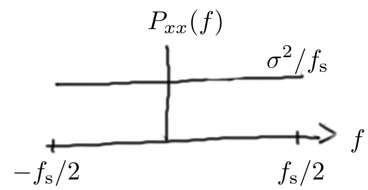
Normalization Factors for Broadband Power Spectra

- In case $f_s \neq 1$, we have

$$\frac{1}{f_s}\int_{-f_s/2}^{f_s/2} P_{xx}(f)\,\mathrm{d}f = \frac{1}{f_s}\int_{-f_s/2}^{f_s/2} \sigma_x^2\,\mathrm{d}f = \sigma_x^2$$



since total power remains unchanged

- This motivates the normalization constant used for power spectral density estimation (see second page), e.g., for $L = 1$ and $N = M$, we have

$$\hat{P}_{xx}(f_k) = \frac{1}{NUf_s}\left|X(k)\right|^2$$

U is the normalization that corrects for the distortion (tapering) of the segment time series by the window function

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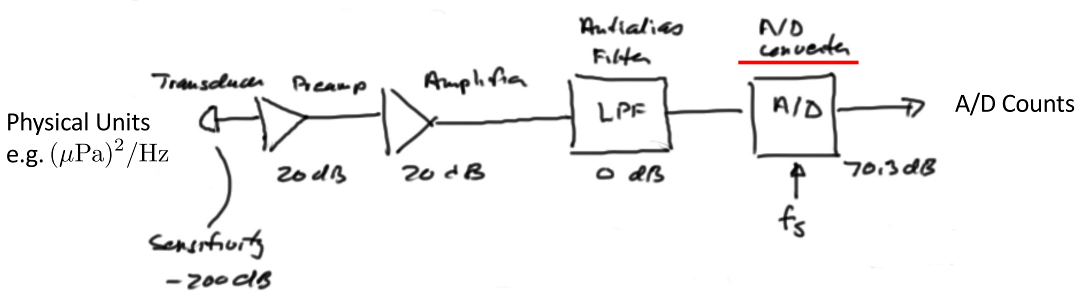
Calibration of Power Spectral Density

- **System Block Diagram**



Typical A/D Parameters:

16 bits
+/- 10 volts → 65536 counts / 20 volts →

A/D Gain:

$$20\log\frac{65536}{20} = 70.3\mathrm{dB}$$

$$\text{System Gain}: -200\mathrm{dB} + 20\mathrm{dB} + 20\mathrm{dB} + 70.3\mathrm{dB} = -89.7\mathrm{dB}$$

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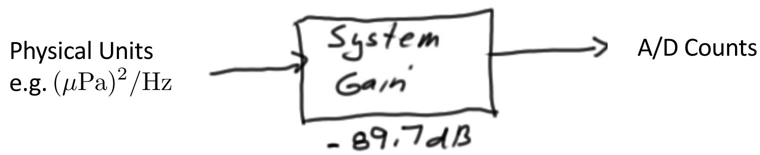
Calibration of Power Spectral Density

- For calibration purposes the system block diagram can be summarized as follows



- Thus, to go from $P_{xx}(f)$ in $(\mathrm{A/D\ counts})^2/\mathrm{Hz}$ to $P_{xx}(f)$ in $(\mu\mathrm{Pa})^2/\mathrm{Hz}$, $89.7\mathrm{dB}$ must be added