Optimal Linear Phase FIR Filter Design

Lecture By Prof. Meyer

ECE 161A

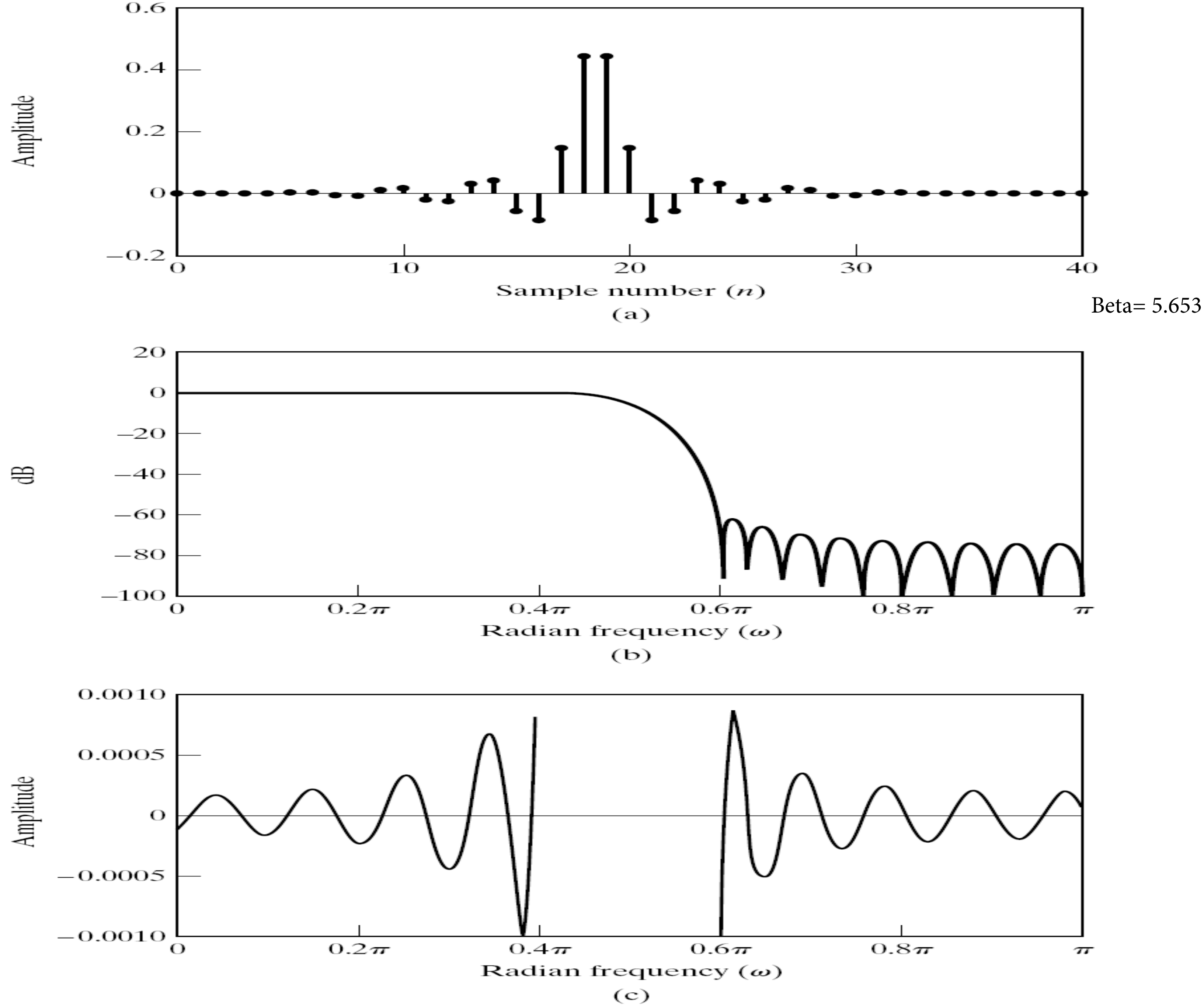


Figure 7.25 Response functions for Example 7.8. (a) Impulse response ($M = 37$). (b) Log magnitude. (c) Approximation error.

Pros and Cons of Window based Design

- Advantages
  - Easy to design
  - Can be applied to general linear system design
- Disadvantages
  - Exceeds the specs everywhere except at the edges of the passband and stopband
  - δ_1 and δ_2 cannot be independently controlled. Have to design more conservatively for the smaller of the two

Objectives of Optimal Design

- Control δ_1 and δ_2 separately
- Spread the ripples out over all of the passband and stopband

Optimal Filter Design

- Design Specifications

$$(\omega_p, \omega_s, M, \delta_1, \delta_2) \Rightarrow (\omega_p, \omega_s, M, K(=\frac{\delta_1}{\delta_2}), \delta(=\delta_2))$$

- Parks and McClellan Approach: given

(ω_p, ω_s, M, K) minimize δ

- Type I filter Design: (M even)

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\frac{\omega M}{2}}$$

$$A_e(e^{j\omega}) = \sum_{n=-L}^{L} h_e[n]e^{-j\omega n}, \quad L = \frac{M}{2}$$

$$h_e[n] = h_e[-n] \quad and \quad h[n] = h_e[n-L], \; 0 \le n \le M$$

Optimal Filter Design (Cont.)

Using the symmetry in the coefficients

$$A_e(e^{j\omega}) = h_e[0] + 2\sum_{n=1}^{L} h_e[n]\cos\omega n = \sum_{K=0}^{L} a_k(\cos\omega)^k, \ -\pi \le \omega \le \pi$$

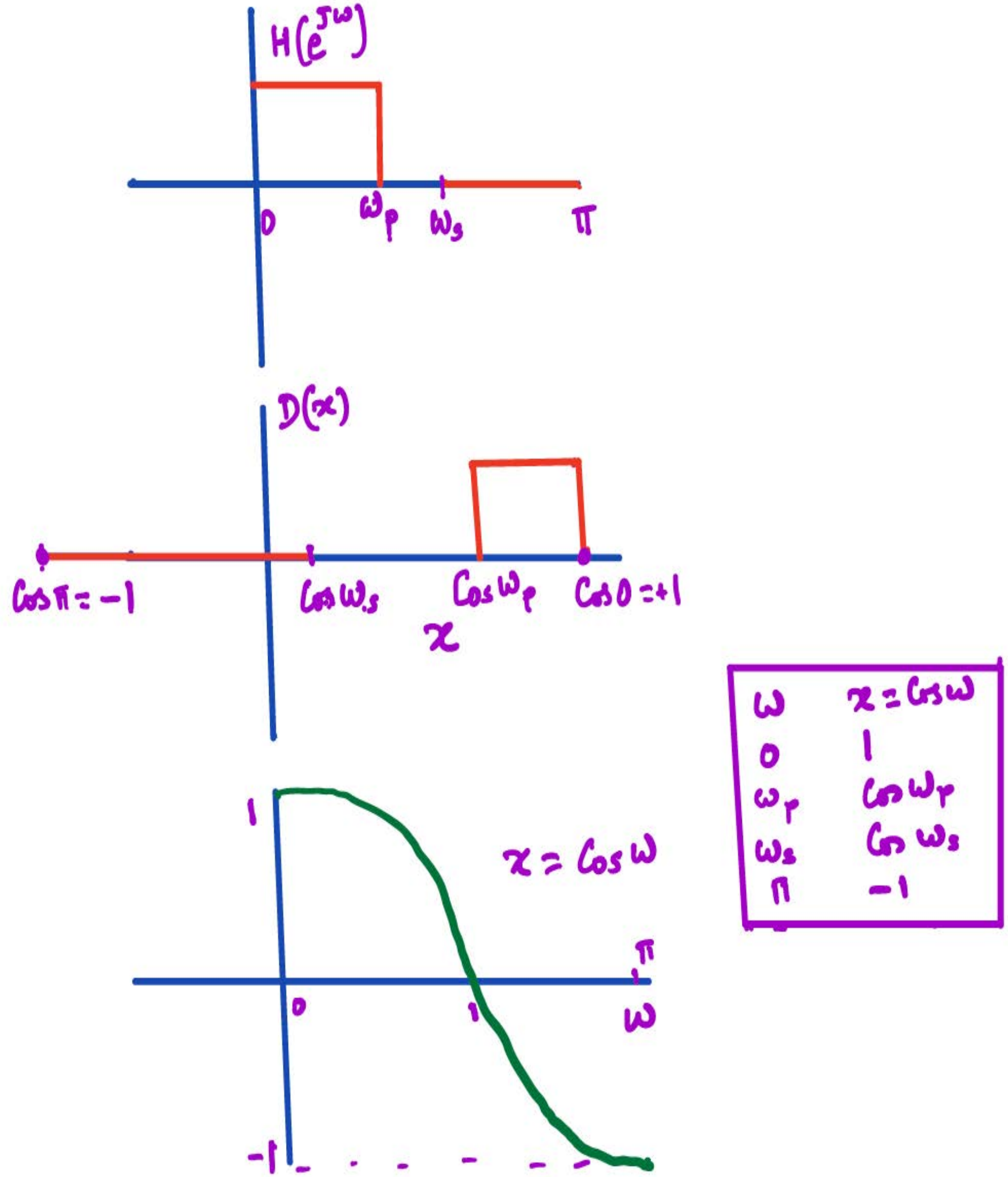
$$= \sum_{k=0}^{L} a_k x^k \Big|_{x=\cos\omega} = P(x), \ -1 \le x \le 1$$

- Can View Filter Design as a Polynomial Approximation problem

Desired Filter

$$H_d(e^{j\omega}) = \begin{cases} 1, & 0 \le \omega \le \omega_p \\ 0, & \omega_s \le \omega \le \pi \end{cases}$$

$$D(x) = \begin{cases} 1, & \cos\omega_p \le x \le 1 \\ 0, & -1 \le x \le \cos\omega_s \end{cases}$$



Problem Statement

Define the following closed sets

$$F_\omega = [0, \omega_p] \cup [\omega_s, \pi]$$

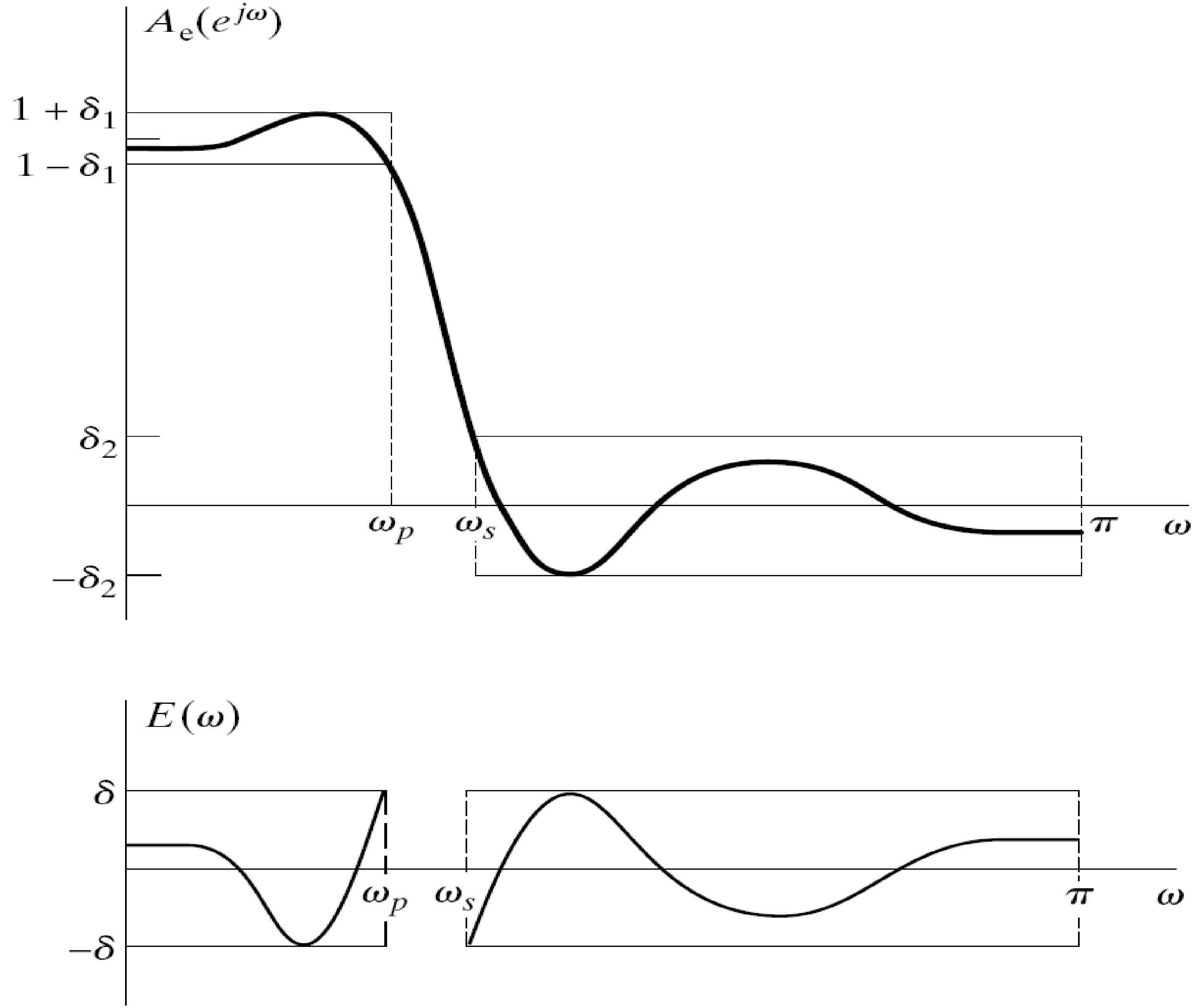
$$F_x = [-1, \cos\omega_s] \cup [\cos\omega_p, 1]$$

Error Functions

$$E(\omega) = W(\omega)[H_d(e^{j\omega}) - A_e(e^{j\omega})]$$

$$E(x) = W(x)[D(x) - P(x)]$$

$$where\ \bar{W}(\omega) = \begin{cases} \dfrac{1}{K}, & 0 \le \omega \le \omega_p \\ 1, & \omega_s \le \omega \le \pi \end{cases} \quad and\ K = \frac{\delta_1}{\delta_2}$$


$A_e(e^{j\omega})$
$1+\delta_1$
$1-\delta_1$
δ_2
$-\delta_2$
ω_p
ω_s
π
ω
$E(\omega)$
δ
$-\delta$
ω_p
ω_s
π
ω

Optimization Problem

Max Approximation Error

$$M(h_e[n]) = \max_{\omega \in F_\omega} |E(\omega)| = \max_{\omega \in F_\omega} W(\omega)\left|H_d(e^{j\omega}) - A_e(e^{j\omega})\right|$$

$$M(a_k) = \max_{x \in F_x} |E(x)| = \max_{x \in F_x} W(x)\left|D(x) - P(x)\right|$$

Mini-Max Criterion (Cost Function)

$$\min_{h_e[n]} M(h_e[n]) = \min_{h_e[n]}[\max_{\omega \in F_\omega} W(\omega)\left|H_d(e^{j\omega}) - A_e(e^{j\omega})\right|]$$

$$\min_{a_k} M(a_k) = \min_{a_k}[\max_{x \in F_x} W(x)\left|D(x) - P(x)\right|]$$

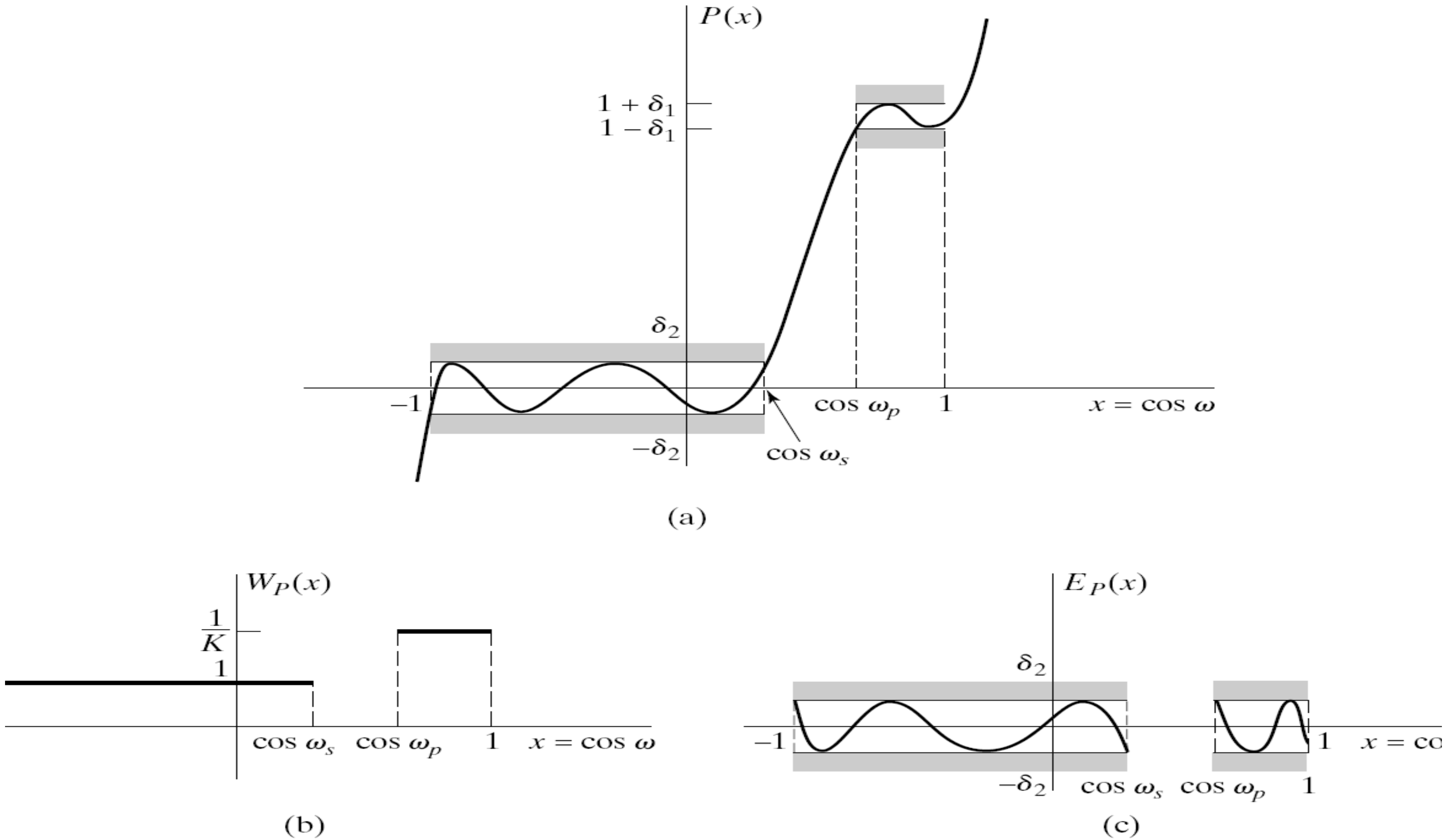


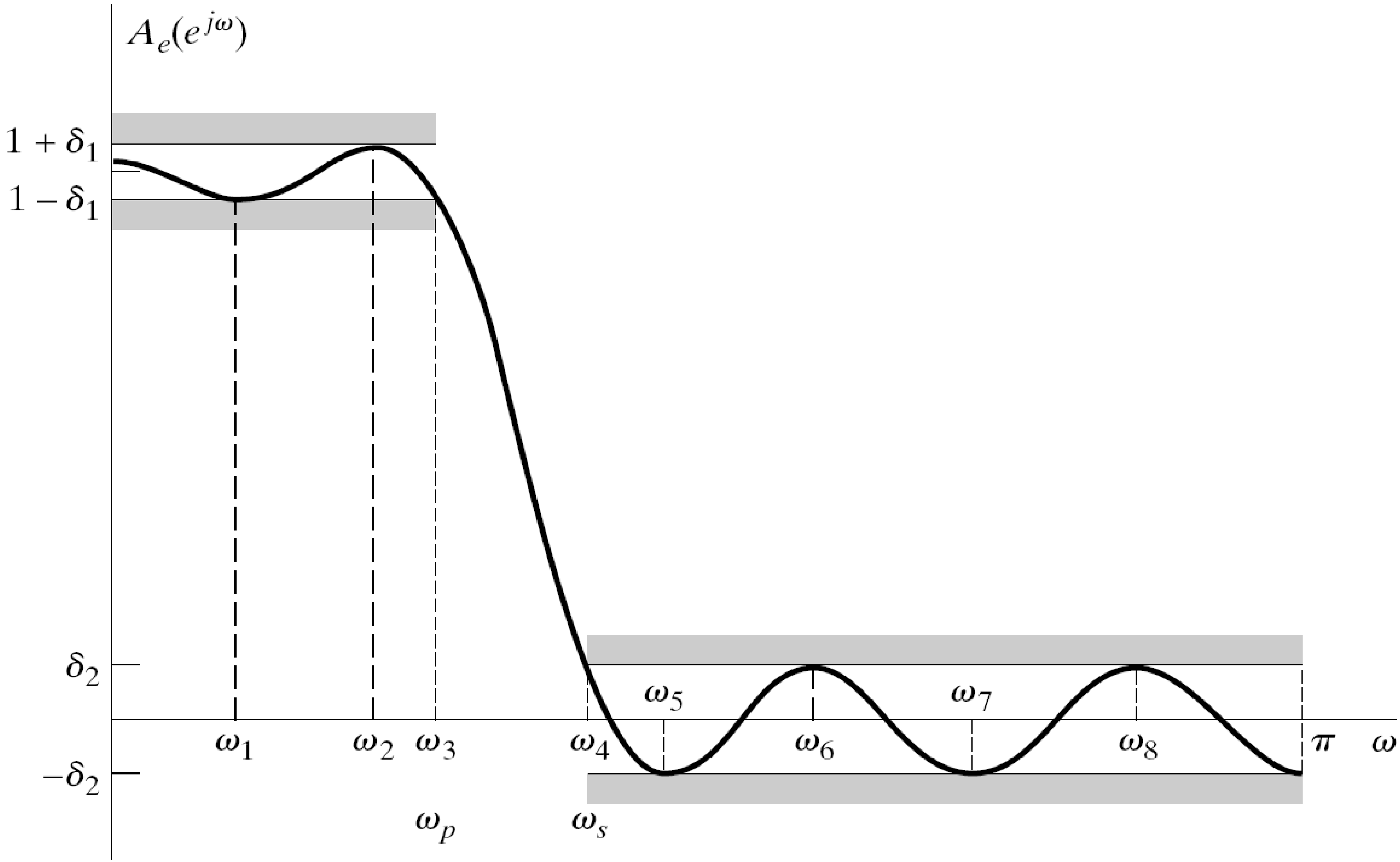
Figure 7.36 Equivalent polynomial approximation functions as a function of $x = \cos\omega$. (a) Approximating polynomial. (b) Weighting function. (c) Approximation error.

Alternation Theorem

- Alternation Theorem: Minimax Cost function has an unique minimum, and the minimum has at least (L+2) alternations.
- If $\delta = \min_{a_k} [\max_{x \in F_x} |E(x)|]$ then we can find

$$x_1 < x_2 < .. < x_{L+2} \; where \; E(x_i) = -E(x_{i+1})$$

and $\left|E(x_i)\right| = \delta$



Typical example of a lowpass filter approximation that is optimal according to the alternation theorem for L=7

Alternation Points

$$P(x) = \sum_{k=0}^{L} a_k x^k$$

$$\frac{dP(\cos\omega)}{d\omega} = \frac{\partial P(x)}{\partial x}\frac{dx}{d\omega} = \sum_{k=0}^{L} k a_k x^{k-1}(-\sin\omega)$$

$$= \sum_{k=0}^{L-1}(k+1)a_{k+1}x^k(-\sin\omega) = -P'(x)\sin\omega$$

Alternation points can be found by examining where the derivative is zero

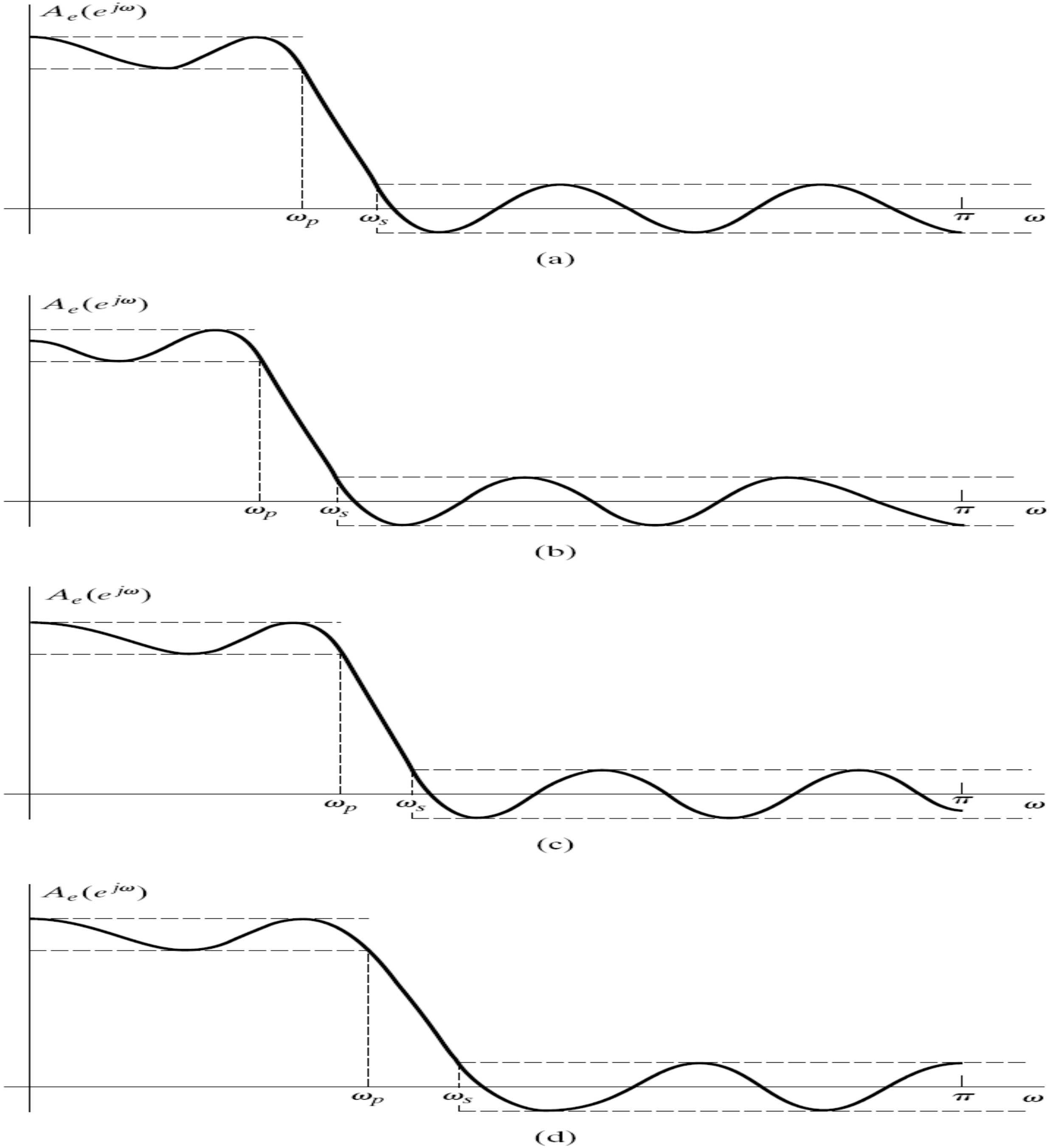
Potential Alternation Points

- P'(x) is a polynomial of degree (L-1) and so has (L-1) zeros which can correspond to the alternation points

- Error can be δ at the passband and stopband edges. So ω_p and ω_s can be alternation points

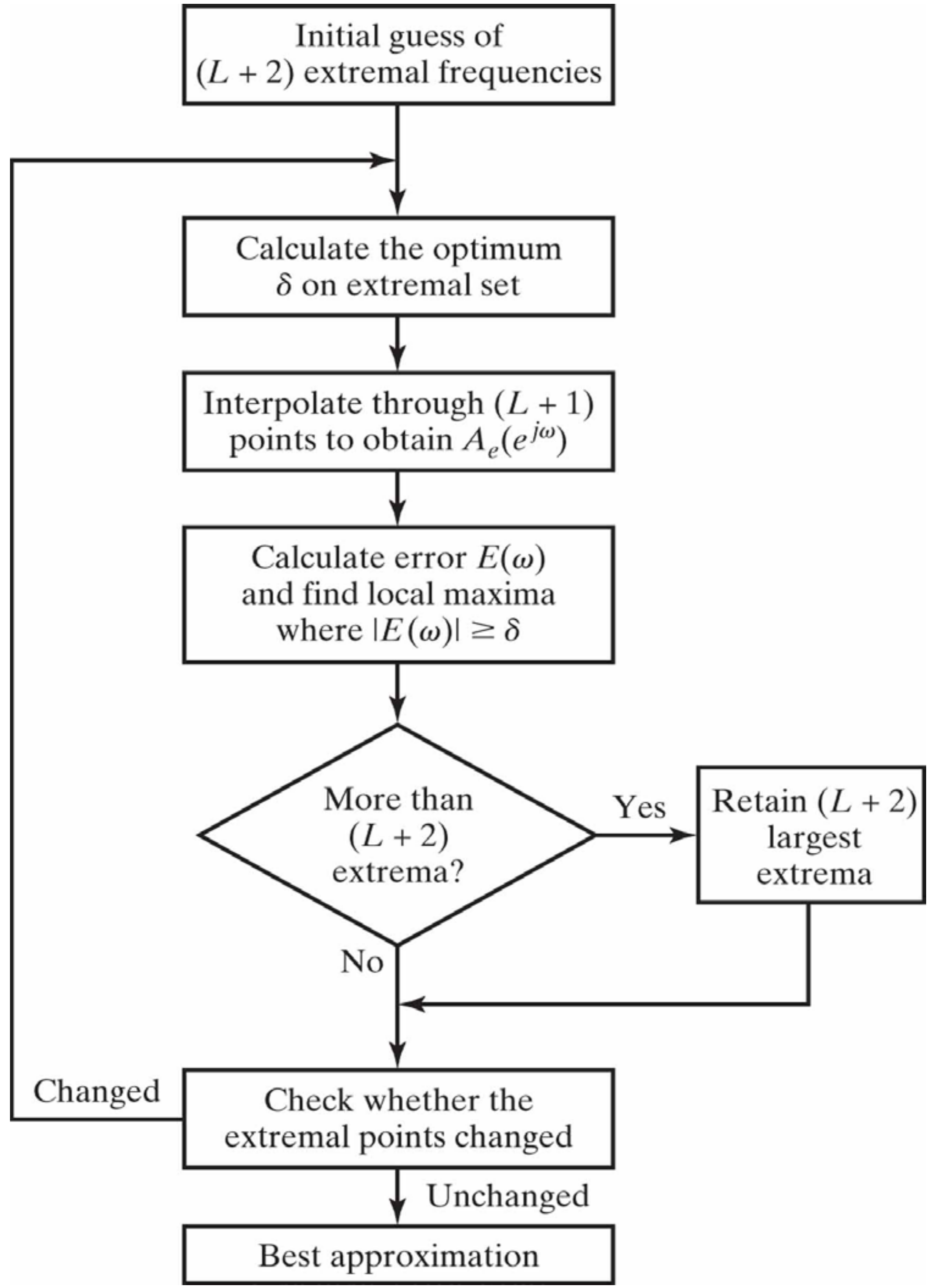
- $Sin\omega$ is zero at $\omega = 0$ and $\omega = \pi$ and so they are candidates for alternation points.

Maximum number of alternation points = (L-1) +2 +2 = L+3



Possible optimum lowpass filter approximations for L=7

Figure 7.50 Flowchart of Parks–McClellan algorithm.



$$A_e(e^{j\omega}) = P(x)\Big|_{x=\cos\omega} = \sum_{k=0}^{L} a_k x^k$$

Note: $A_e(e^{j\omega})$ and $P(x)$ are uniquely defined by their values at (L+1) points.

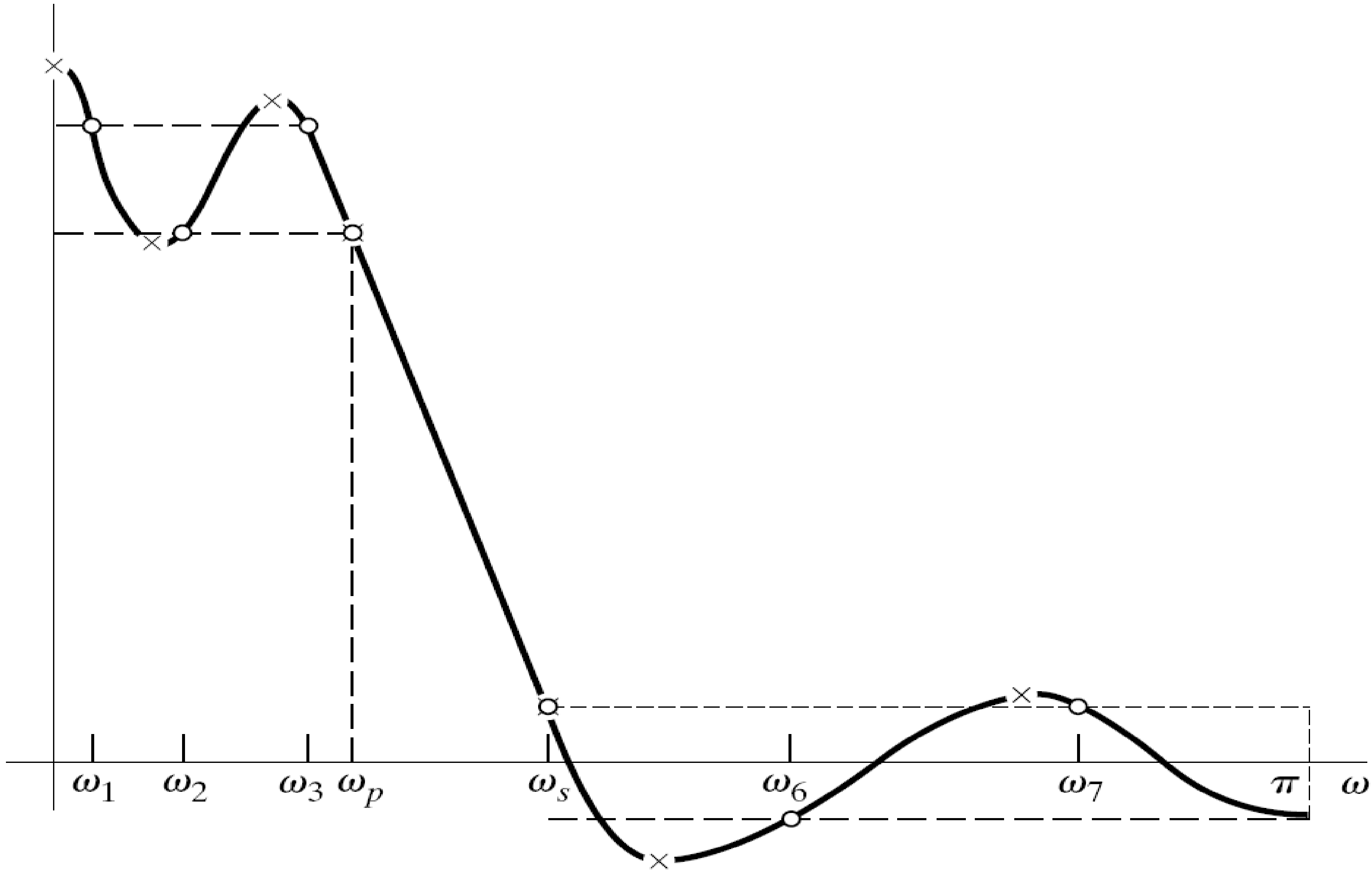


Figure 7.40 Illustration of the Parks–McClellan algorithm for equiripple approximation.

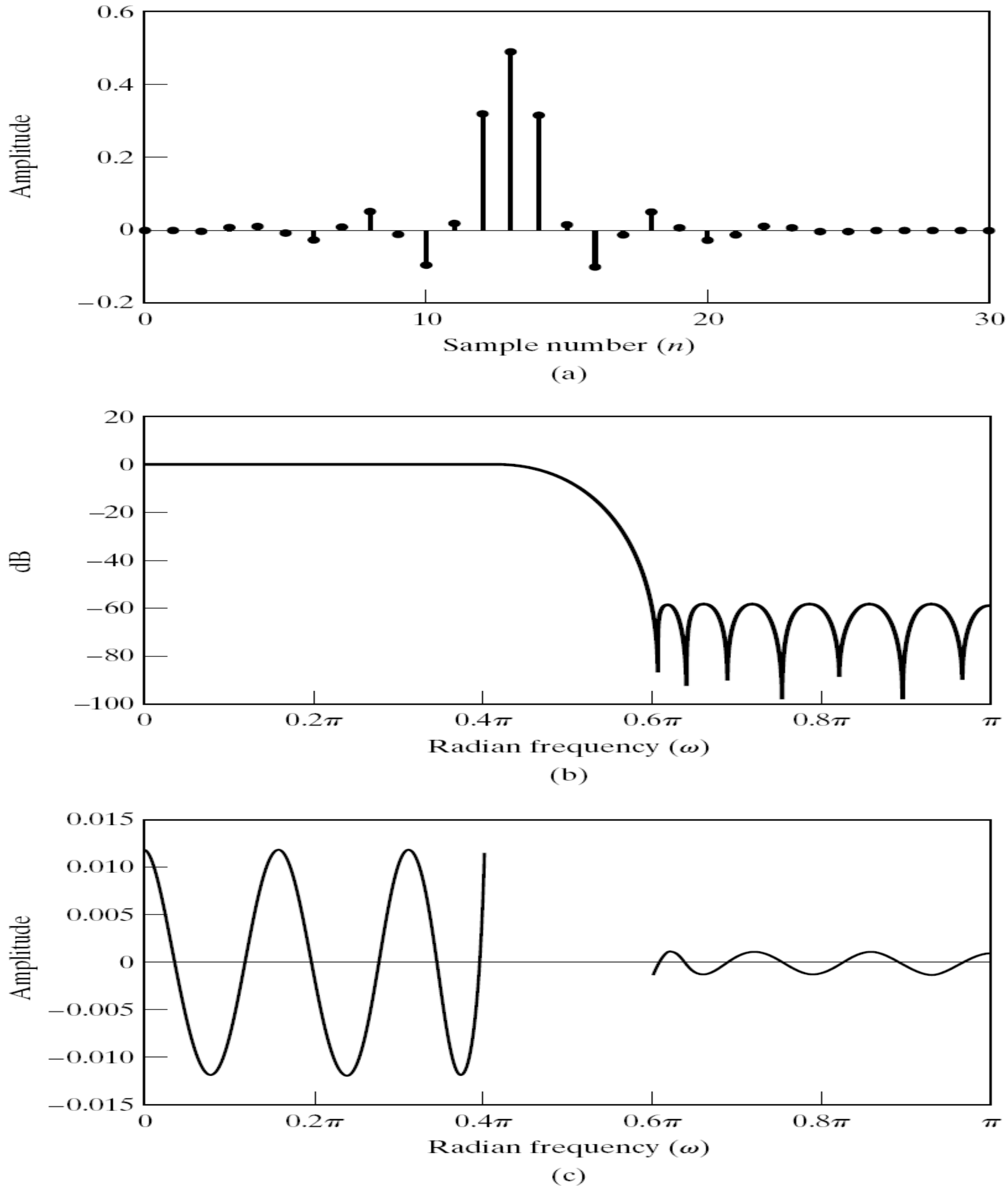
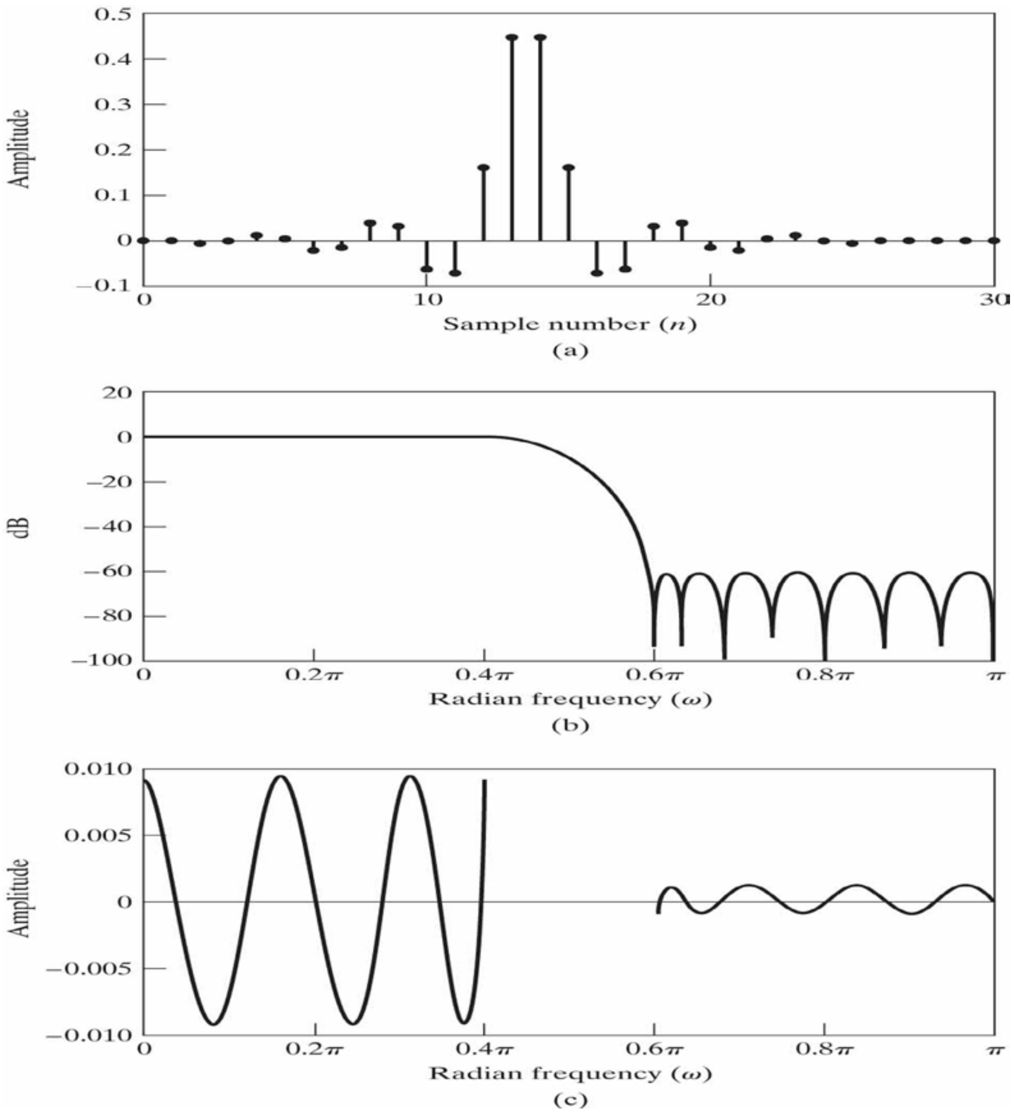


Figure 7.43 Optimum type I FIR lowpass filter for $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$, $K = 10$, and $M = 26$. (a) Impulse response. (b) Log magnitude of the frequency response. (c) Approximation error (unweighted).

Figure 7.53 Optimum type II FIR lowpass filter for $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$, $K = 10$, and $M = 27$. (a) Impulse response. (b) Log magnitude of frequency response. (c) Approximation error (unweighted).



$\omega_p = .1\pi, \omega_s = .15\pi, K = 1\ (\delta_1 = \delta_2)$