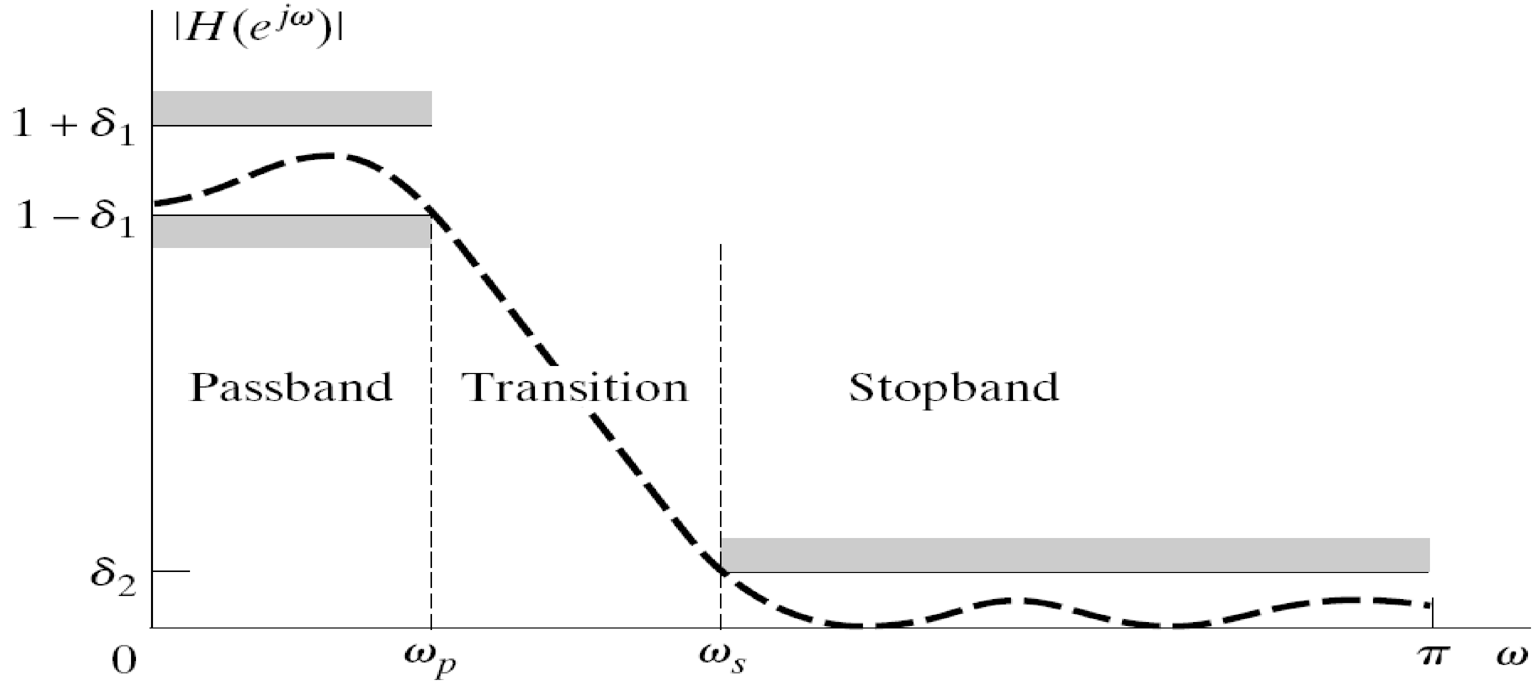
IIR Filter Design

Lecture By Prof. Meyer

ECE 161A

Filter Design: Low Pass Filter Design



Problem: Given $\delta_1, \delta_2, \omega_p$ and ω_s, find the lowest complexity filter that meets specification

Two Choices

1. IIR Filters (Infinite Impulse Response, $H(z) = \dfrac{B(z)}{A(z)}$
2. FIR Filters (Finite Impulse Response, $H(z) = B(z)$

IIR Filter Design

- Approach: Transform a continuous time filter to a discrete time filter

Rationale:

- Continuous time IIR filter design highly advanced.
- Many continuous time IIR filters have closed form design formulas
- Approximating directly in the digital domain not easy.

IIR Filter Design Methods

1. Impulse Invariance
2. Bilinear Transformation

Another useful Tool: Frequency Transformation. It allows design of other filter types (High Pass, Band Pass...) by transforming a low pass filter.

Impulse Invariance Design

Simply applies the sampling theorem and hopes that aliasing is small

$$h[n] = T_d h_c(nT_d)$$

Then

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\frac{\omega}{T_d} - j\frac{2\pi}{T_d}k\right)$$

More generally

$$H(z)\Big|_{z=e^{sT_d}} = \sum_{k=-\infty}^{\infty} H_c\left(s - j\frac{2\pi}{T_d}k\right)$$

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$$

Then

$$h_c(t) = \sum_{k=1}^{N} A_k e^{s_k t}, \quad t \geq 0$$

$$h[n] = T_d h_c(nT_d) = T_d \sum_{k=1}^{N} A_k e^{s_k T_d n}, \quad n \geq 0$$

$$H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}$$

Pole of $H_c(s)$ at s_k gets transformed to a pole of H(z) at $e^{s_k T_d}$

Stability is preserved

Problem: How to account for aliasing

Similar approach can be used for step invariance or other waveform invariance criteria

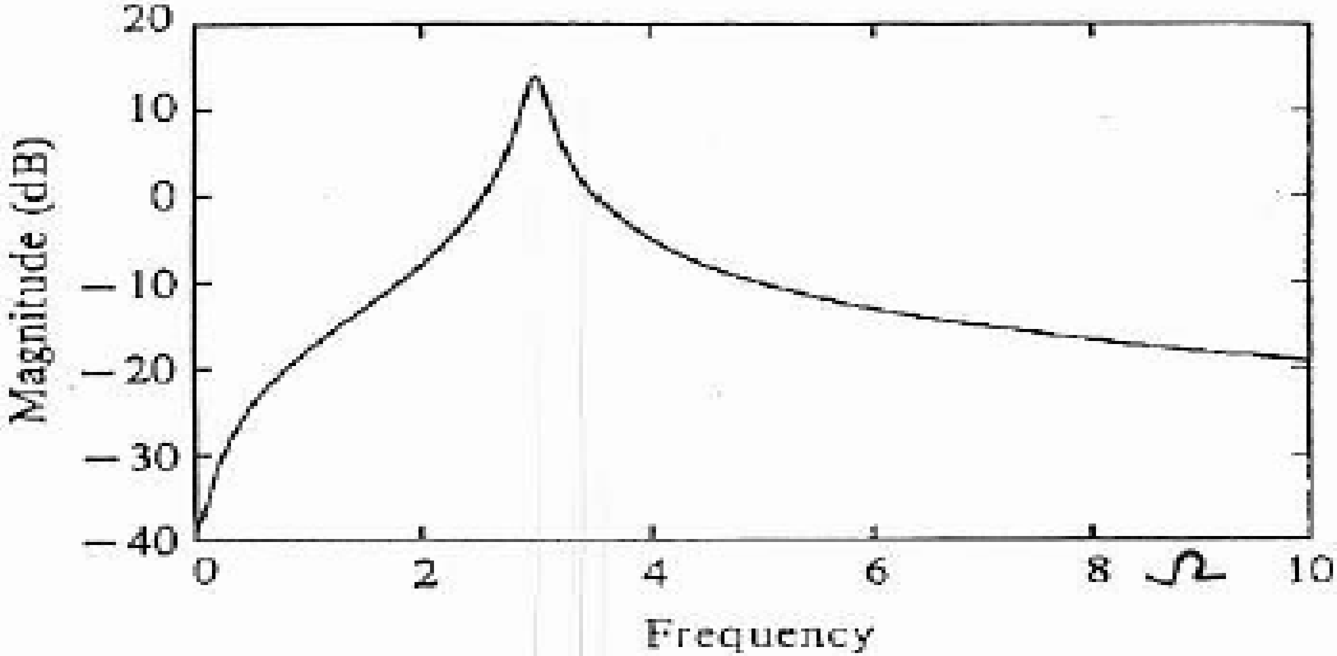
Impulse Variance Design (Cont.)

Example

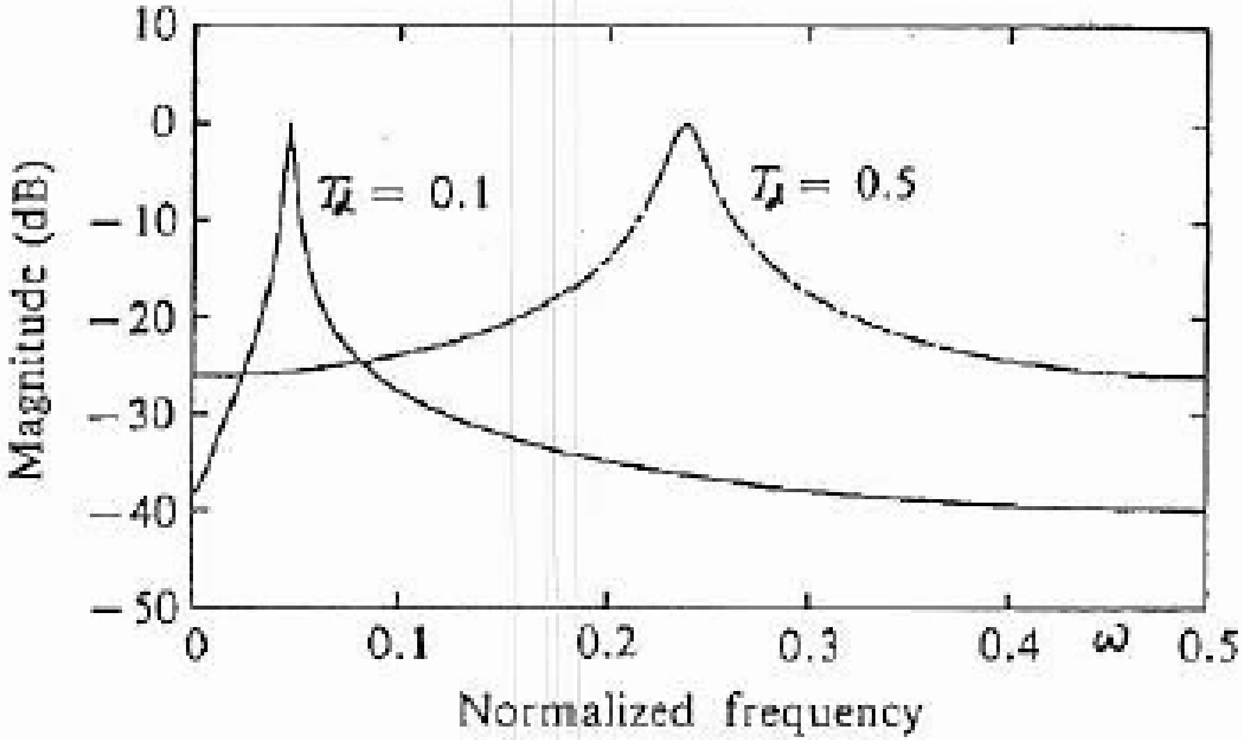
$$H_c(s) = \frac{s+.1}{(s+.1)^2+9} = \frac{.5}{s+.1-j3} + \frac{.5}{s+.1+j3}$$

$$H(z) = \frac{.5T_d}{1-e^{-.1T_d}e^{j3T_d}z^{-1}} + \frac{.5T_d}{1-e^{-.1T_d}e^{-j3T_d}z^{-1}}$$

Analog Filter



Impulse Invariance Design



Bi-Linear Transformation

$$s = \frac{2}{T_d} \frac{1 - z^{-1}}{1 + z^{-1}}$$

The digital filter is obtained by a change of variables

$$H(z) = H_c\left[\frac{2}{T_d} \frac{1 - z^{-1}}{1 + z^{-1}}\right]$$

$$z = \frac{1 + \frac{T_d}{2}s}{1 - \frac{T_d}{2}s} = \frac{1 + \frac{\sigma T_d}{2} + j\frac{\Omega T_d}{2}}{1 - \frac{\sigma T_d}{2} - j\frac{\Omega T_d}{2}} \quad or \quad |z|^2 = \frac{\left(1 + \frac{\sigma T_d}{2}\right)^2 + \frac{\Omega^2 T_d^2}{4}}{\left(1 - \frac{\sigma T_d}{2}\right)^2 + \frac{\Omega^2 T_d^2}{4}}$$

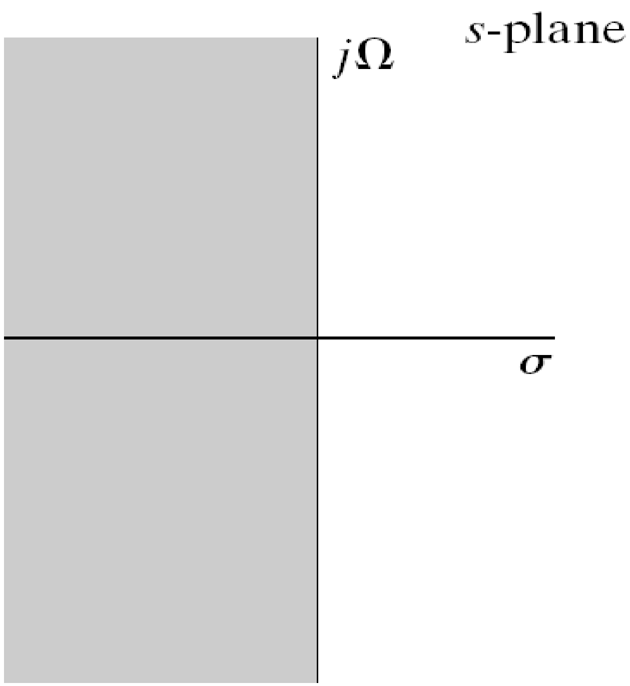
Bilinear Transformation

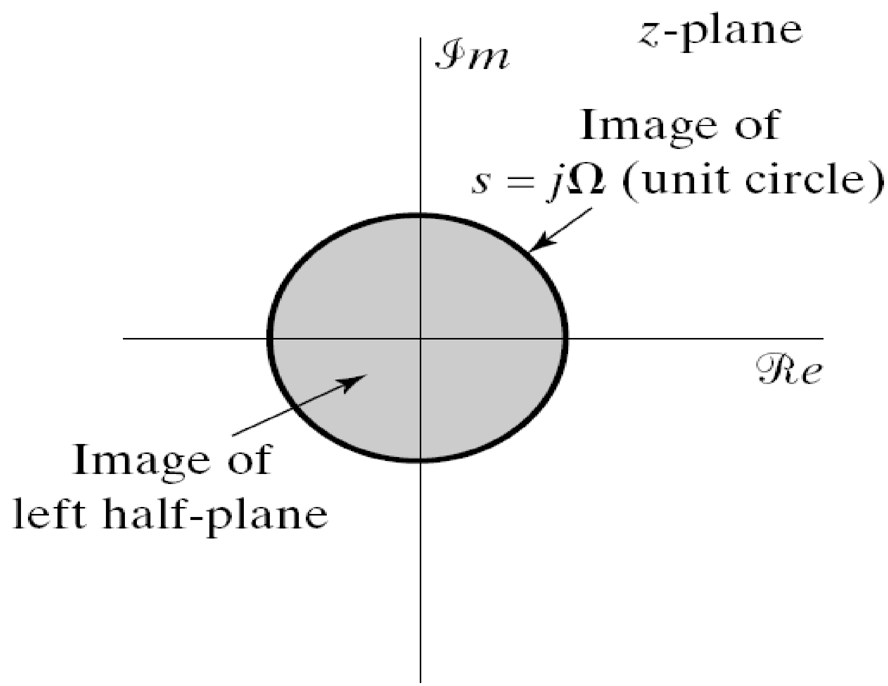
$$|z|^2 = \frac{\left(1+\frac{\sigma T_d}{2}\right)^2 + \frac{\Omega^2 T_d^2}{4}}{\left(1-\frac{\sigma T_d}{2}\right)^2 + \frac{\Omega^2 T_d^2}{4}}$$

Properties of Mapping

$\sigma < 0 \Rightarrow |z| < 1$

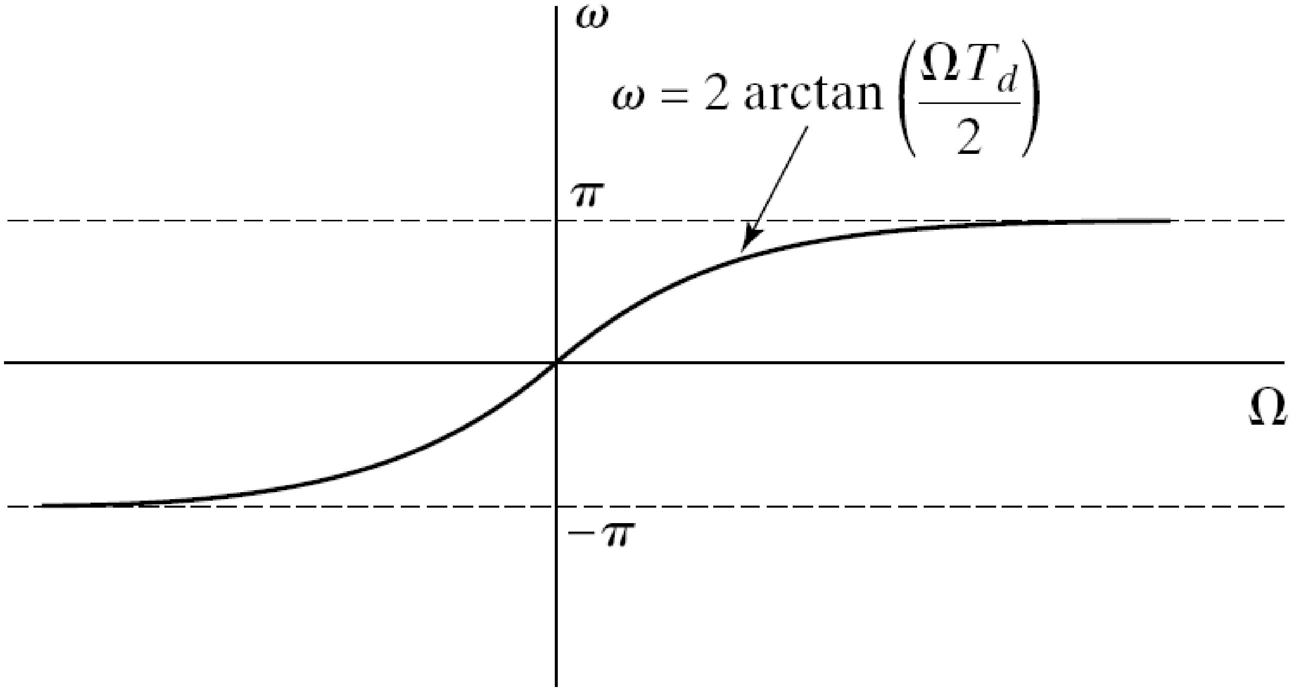
$\sigma = 0 \Rightarrow |z| = 1$ $\quad\quad$ $\sigma > 0 \Rightarrow |z| > 1$



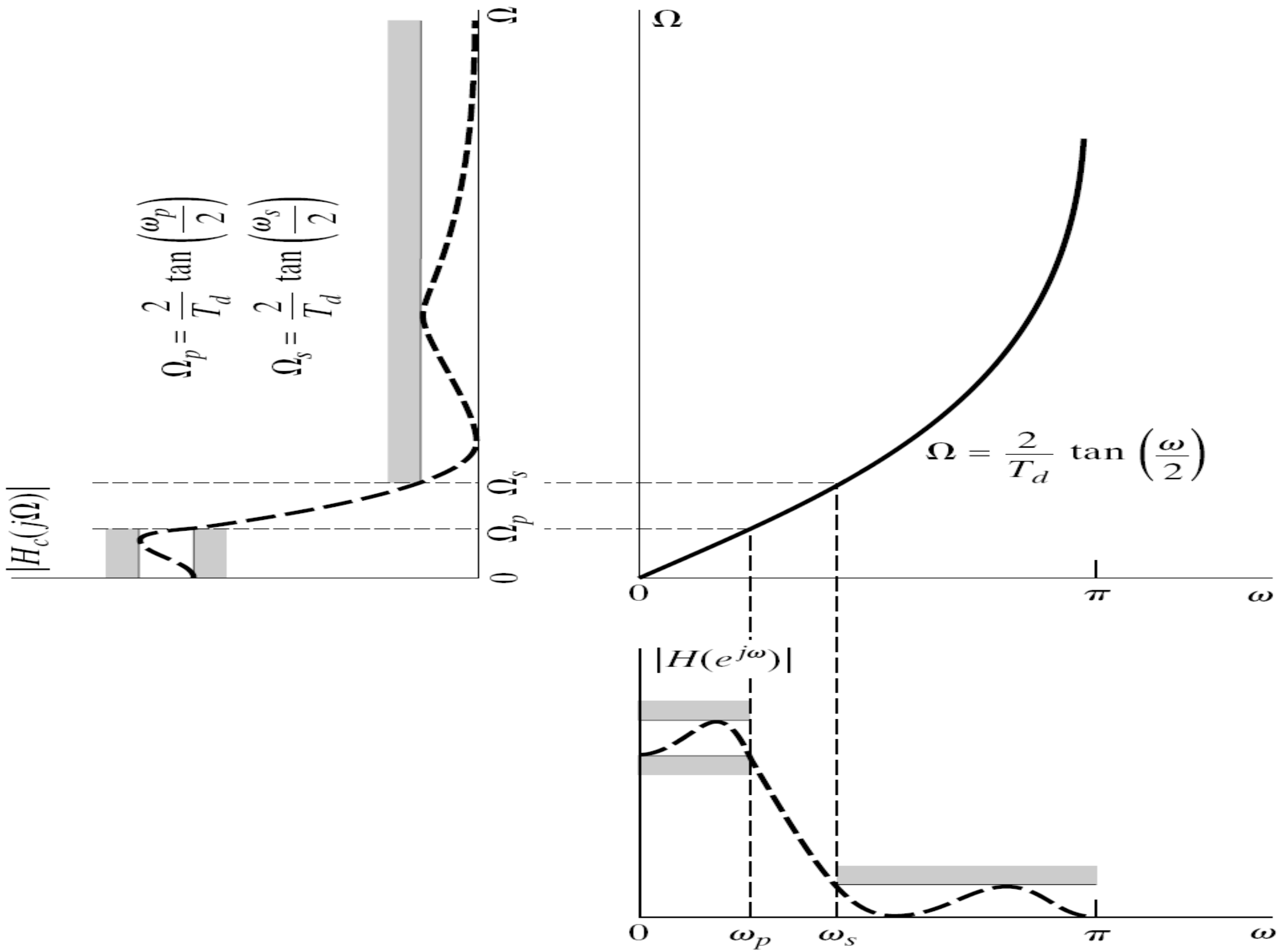


Mapping and its Role in Filter Design

$$e^{j\omega} = \frac{1 + j\dfrac{\Omega T_d}{2}}{1 - j\dfrac{\Omega T_d}{2}} \quad or \quad \Omega = \frac{2}{T_d}\tan\left(\frac{\omega}{2}\right) \quad or \quad \omega = 2\arctan\left(\frac{\Omega T_d}{2}\right)$$



Mapping between the frequency ω and Ω resulting from the Bilinear transformation



Frequency warping inherent in the bi-linear transformation of a continuous-time lowpass filter into a discrete-time lowpass filter.

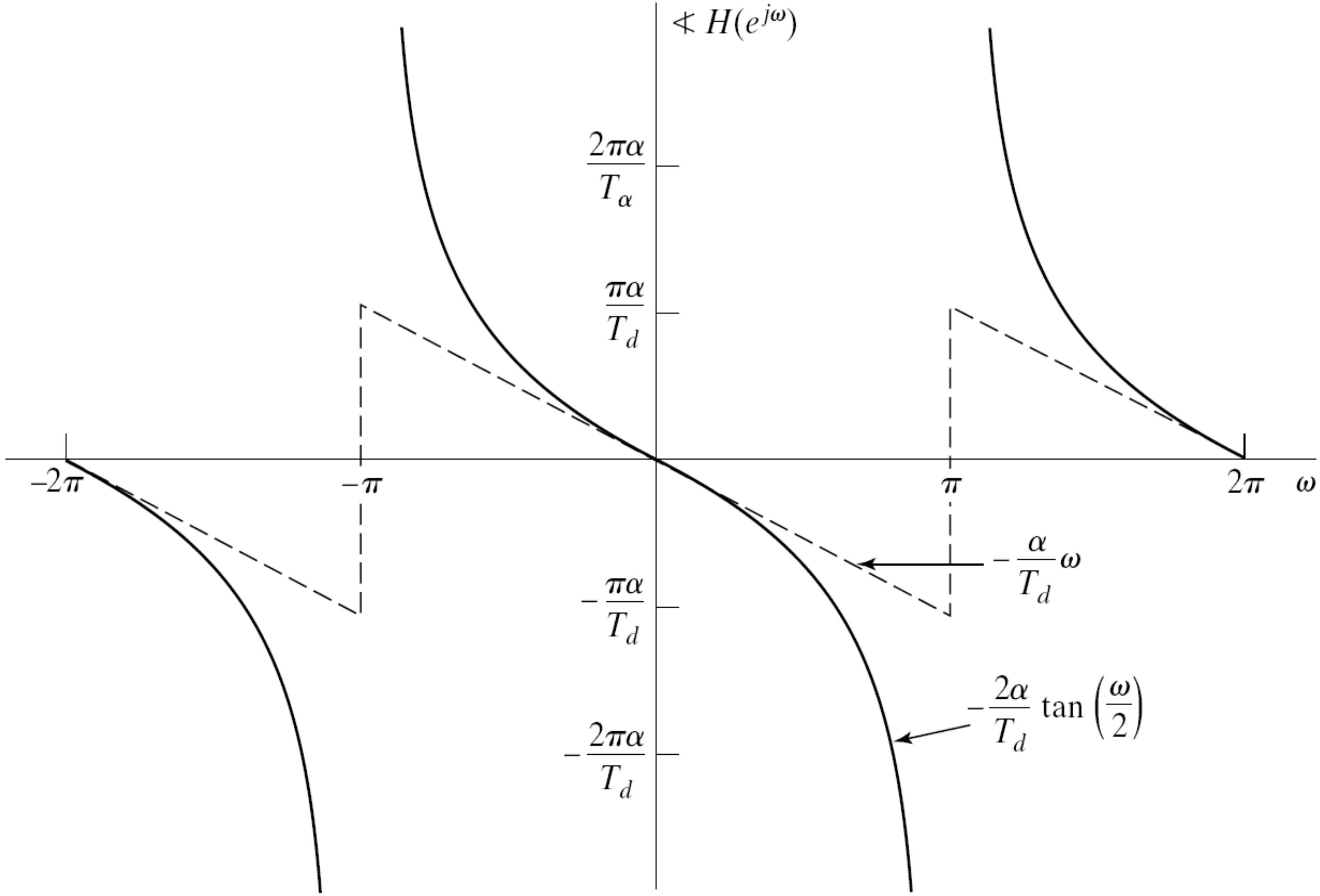


Illustration of the effect of the bi-linear transformation on a linear phase characteristic

Overall Design Steps

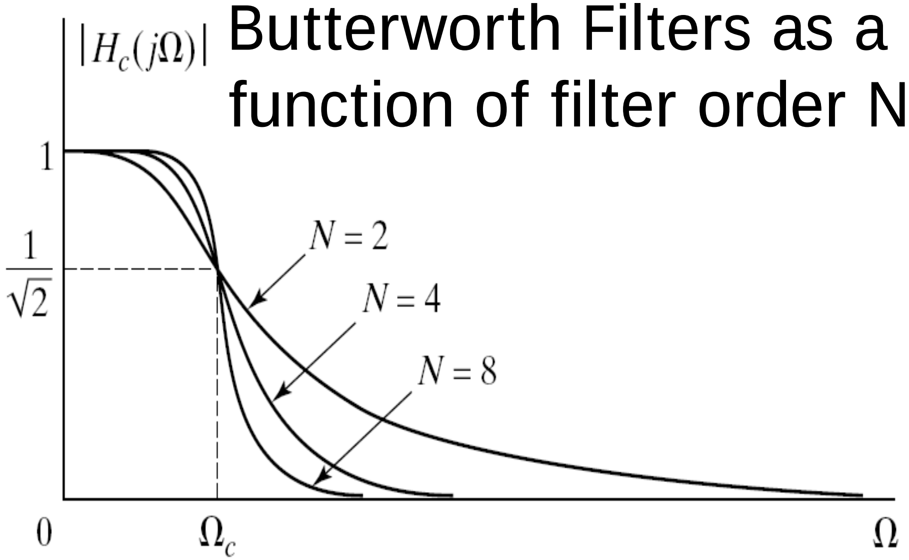
Given $\delta_1, \delta_2, \omega_{\mathrm{p}}, \omega_{\mathrm{s}}$

1. Pre-warp to get $\delta_1, \delta_2, \Omega_{\mathrm{p}}, \Omega_{\mathrm{s}}$

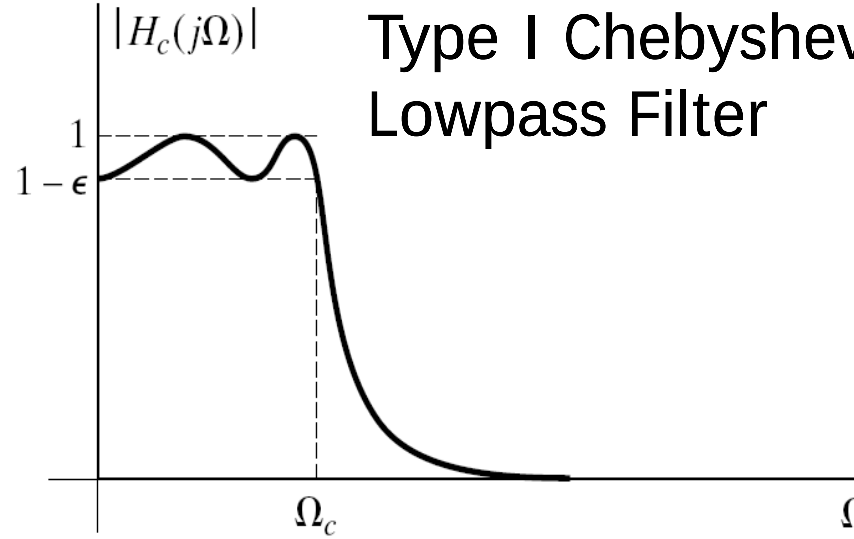
where $\Omega_i = \dfrac{2}{T_d} \tan\left(\dfrac{\omega_i}{2}\right)$

2. Using these specs design Analog Filter $H_c(j\Omega)$ *or* $H_c(s)$

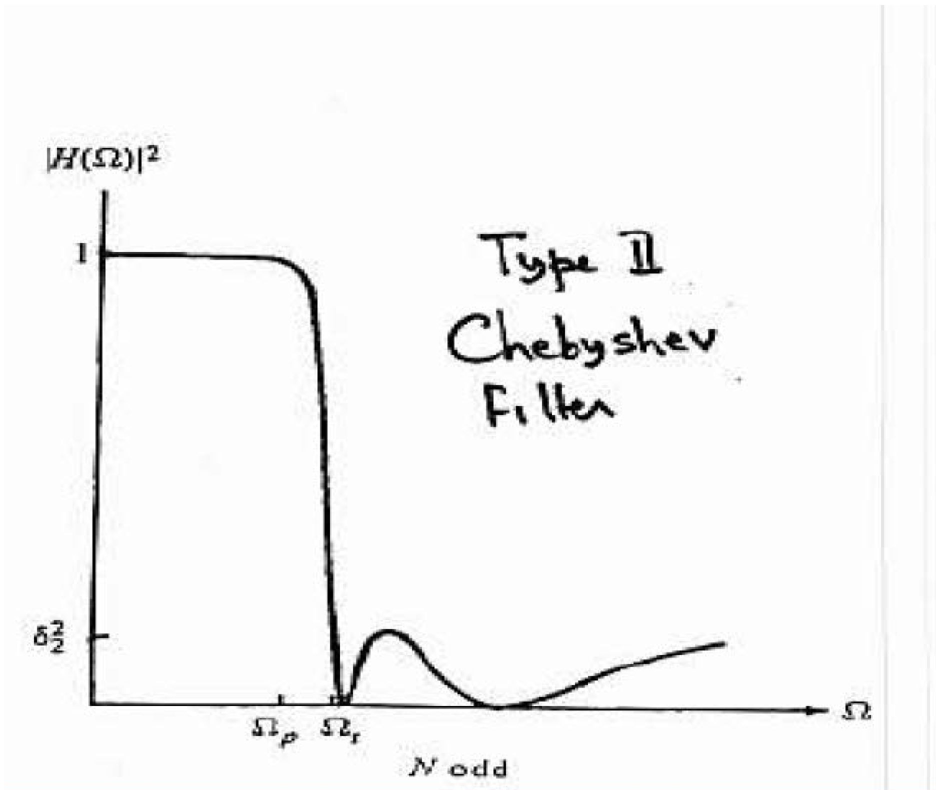
3. $H(z) = H_c\left[\dfrac{2}{T_d}\dfrac{1 - z^{-1}}{1 + z^{-1}}\right]$



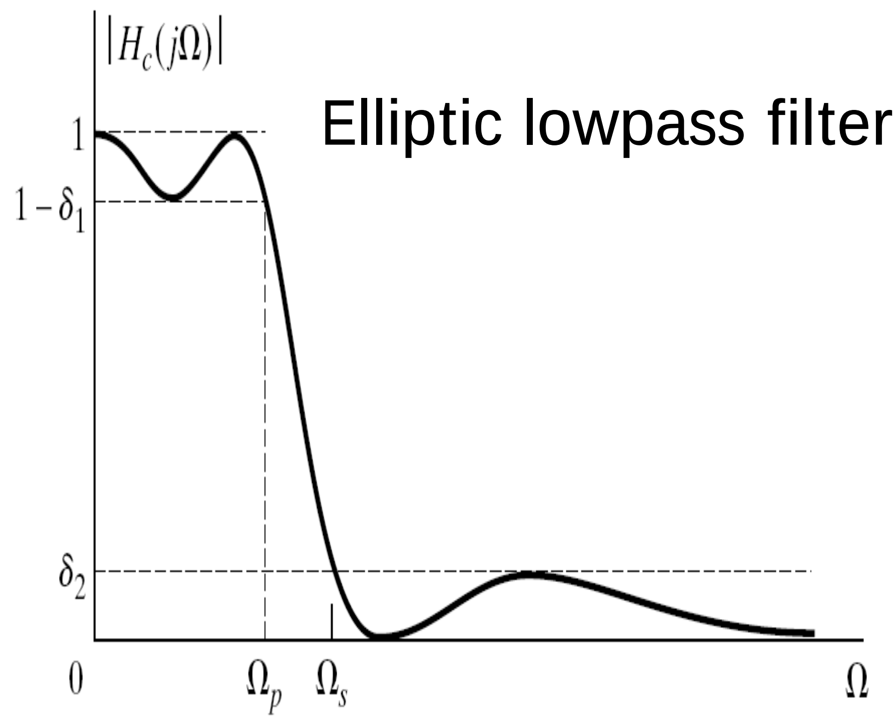
Butterworth Filters as a function of filter order N



Type I Chebyshev Lowpass Filter



Type II Chebyshev Filter



Elliptic lowpass filter