Maximum Likelihood Estimation Examples

Bernoulli Distribution - Maximum Likelihood Estimate of *p*

$$P(X = x | x \in \{0, 1\} = p^x (1-p)^{1-x}$$

$$L(p|x_1, x_2, x_3, ..., x_n) = \left(p^{x_1}(1-p)^{1-x_1}\right)\left(p^{x_2}(1-p)^{1-x_2}\right)\left(p^{x_3}(1-p)^{1-x_3}\right)\cdots\left(p^{x_n}(1-p)^{1-x_n}\right)$$

$$= \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}$$

$$= p^{\sum_{i=1}^{n} x_i}(1-p)^{\sum_{i=1}^{n} 1-x_i}$$

$$= p^{\sum_{i=1}^{n} x_i}(1-p)^{n-\sum_{i=1}^{n} x_i}$$

$$\ell(x_1, x_2, x_3, ..., x_n) = \sum_{i=1}^{n} x_i \ln(p) - \left(n - \sum_{i=1}^{n} x_i\right)(1-p)$$

$$\frac{d\ell}{dp} = \frac{\sum_{i=1}^{n} x_i}{p} - \frac{n - \sum_{i=1}^{n} x_i}{1-p}$$

Set the derivative equal to 0.

$$\frac{\sum_{i=1}^{n} x_i}{p} - \frac{n - \sum_{i=1}^{n} x_i}{1-p} = 0$$

$$\frac{\sum_{i=1}^{n} x_i}{p} = \frac{n - \sum_{i=1}^{n} x_i}{1-p}$$

$$\sum_{i=1}^{n} x_i - p\sum_{i=1}^{n} x_i = np - p\sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n} x_i = np$$

$$p = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\hat{p}_{MLE} = \bar{x}$$

The maximum likelihood estimator of *p* is the sample mean.

Poisson Distribution - Maximum Likelihood Estimate of λ

$$P(X = x | x \in N_0 \equiv \{0, 1, 2, 3, ...\}) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$$L(\lambda | x_1, x_2, x_3, ..., x_n) = \left(\frac{\lambda^{x_1} e^{-\lambda}}{x_1!}\right)\left(\frac{\lambda^{x_2} e^{-\lambda}}{x_2!}\right)\left(\frac{\lambda^{x_3} e^{-\lambda}}{x_3!}\right)\cdots\left(\frac{\lambda^{x_n} e^{-\lambda}}{x_n!}\right)$$

$$= \frac{\lambda^{\sum_{i=1}^{n} x_i} e^{-\lambda}}{\prod_{i=1}^{n} x_i!}$$

$$\ell(\lambda | x_1, x_2, x_3, ..., x_n) = \sum_{i=1}^{n} x_i \ln(\lambda) - n\lambda - \ln\left(\prod_{i=1}^{n} x_i!\right)$$

$$= \sum_{i=1}^{n} x_i \ln(\lambda) - n\lambda - \sum_{i=1}^{n} \ln(x_i!)$$

$$\frac{d\ell}{d\lambda} = \frac{\sum_{i=1}^{n} x_i}{\lambda} - n$$

Set the derivative equal to 0.

$$\frac{\sum_{i=1}^{n} x_i}{\lambda} - n = 0$$

$$\frac{\sum_{i=1}^{n} x_i}{\lambda} = n$$

$$\lambda = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\hat{\lambda}_{MLE} = \overline{x}$$

The maximum likelihood estimator of λ is the sample mean.