Maximum Likelihood Estimation Examples

Bernoulli Distribution - Maximum Likelihood Estimate of p

$$egin{aligned} P(X=x|x\in\{0,1\}&=p^x(1-p)^{1-x}\ L(p|x_1,x_2,x_3,...,x_n)&=\left(p^{x_1}(1-p)^{1-x_1}
ight)\left(p^{x_2}(1-p)^{1-x_2}
ight)\left(p^{x_3}(1-p)^{1-x_3}
ight)\cdots\left(p^{x_n}(1-p)^{1-x_n}
ight)\ &=\prod_{i=1}^n p^{x_i}(1-p)^{1-x_i}\ &=p^{\sum_{i=1}^n x_i}(1-p)^{\sum_{i=1}^n 1-x_i}\ &=p^{\sum_{i=1}^n x_i}(1-p)^{n-\sum_{i=1}^n x_i}\ \ell(x_1,x_2,x_3,...,x_n)&=\sum_{i=1}^n x_i\ln(p)-\left(n-\sum_{i=1}^n x_i
ight)(1-p)\ &rac{d\ell}{dp}&=rac{\sum_{i=1}^n x_i}{p}-rac{n-\sum_{i=1}^n x_i}{1-p}\end{aligned}$$

Set the derivative equal to 0.

$$egin{aligned} rac{\sum_{i=1}^n x_i}{p} - rac{n - \sum_{i=1}^n x_i}{1 - p} &= 0 \ & rac{\sum_{i=1}^n x_i}{p} &= rac{n - \sum_{i=1}^n x_i}{1 - p} \ & \sum_{i=1}^n x_i - p \sum_{i=1}^n x_i &= np - p \sum_{i=1}^n x_i \ & \sum_{i=1}^n x_i &= np \ & p &= rac{\sum_{i=1}^n x_i}{n} \ & \hat{p}_{MLE} &= ar{x} \end{aligned}$$

The maximum likelihood estimator of p is the sample mean.

Poisson Distribution - Maximum Likelihood Estimate of λ

$$egin{aligned} P(X=x|x\in N_0\equiv\{0,1,2,3,...\}) &= rac{\lambda^x e^{-\lambda}}{x!} \ L(\lambda|x_1,x_2,x_3,...,x_n) &= \left(rac{\lambda^{x_1}e^{-\lambda}}{x_1!}
ight) \left(rac{\lambda^{x_2}e^{-\lambda}}{x_2!}
ight) \left(rac{\lambda^{x_3}e^{-\lambda}}{x_3!}
ight) \cdots \left(rac{\lambda^{x_n}e^{-\lambda}}{x_n!}
ight) \ &= rac{\lambda^{\sum_{i=1}^n x_i}e^{-\lambda}}{\prod_{i=1}^n x_i!} \ \ell(\lambda|x_1,x_2,x_3,...,x_n) &= \sum_{i=1}^n x_i \ln(\lambda) - n\lambda - \ln\left(\prod_{i=1}^n x_i!
ight) \ &= \sum_{i=1}^n x_i \ln(\lambda) - n\lambda - \sum_{i=1}^n \ln(x_i!) \ rac{d\ell}{d\lambda} &= rac{\sum_{i=1}^n x_i}{\lambda} - n \end{aligned}$$

Set the derivative equal to 0.

$$rac{\sum_{i=1}^n x_i}{\lambda} - n = 0$$
 $rac{\sum_{i=1}^n x_i}{\lambda} = n$ $\lambda = rac{\sum_{i=1}^n x_i}{n}$ $\hat{\lambda}_{MLE} = ar{x}$

The maximum likelihood estimator of λ is the sample mean.