Algebraic and Geometric Structures in de Sitter Quantum Gravity

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1 Algebraic Structure

In the limit $G_N \to 0$, the GNS Hilbert space associated with the de Sitter vacuum state $|\Psi_{dS}\rangle$ reduces to the Fock space[1]:

$$\mathcal{H}^{GNS}_{\Psi_{dS}} \cong \mathcal{H}^{\text{Fock}}. \tag{1}$$

The state $|\Psi_{dS}\rangle$ is cyclic and separating, and can be expressed as the Hartle–Hawking state:

$$|\Psi_{dS}\rangle = |HH\rangle = \int \mathcal{D}\phi \; e^{-I[\phi]}. \tag{2}$$

In quantum field theory on de Sitter spacetime, local operator algebras are **von Neumann algebras of type III**. The algebra $\mathcal{A}$ acting on $\mathcal{H}^{GNS}_{\Psi_{dS}}$ is therefore a **type III** factor. By the Tomita–Takesaki theorem, the modular operator is defined as

$$\Delta_{\Psi_{dS}} \equiv \Delta_{\Psi} = e^{-K_{\Psi}}, \tag{3}$$

where K_{Ψ} is the **modular Hamiltonian**.

Now consider coupling the system to a one-dimensional quantum mechanical degree of freedom with Hamiltonian $\hat{q}$. The extended Hilbert space becomes

$$\hat{\mathcal{H}} = \mathcal{H}^{GNS}_{\Psi_{dS}} \otimes L^2(\mathbb{R}), \tag{4}$$

[1] Witten said the hilbert space can be veiwed as a fock space, but there is no natural notion of creation and annihilation operators

and the full operator algebra is

$$\mathcal{M} = \mathcal{A} \otimes \mathcal{B}(L^2(\mathbb{R})). \tag{5}$$

Define the constraint operator

$$\hat{C} = K_\Psi + \hat{q}. \tag{6}$$

The commutant subalgebra $\widehat{\mathcal{M}}$ consists of all operators in $\mathcal{M}$ that commute with $\hat{C}$:

$$\widehat{\mathcal{M}} = \{\hat{A} \in \mathcal{M} \mid [\hat{A}, \hat{C}] = 0\}. \tag{7}$$

Physical states are obtained by group averaging over the constraint [1, 2, 3]:

$$|\hat{\Psi}\rangle_{\text{Phys}} = \int_{-\infty}^{\infty} ds\; e^{-is\hat{C}}\, |\Psi\rangle \in \hat{\mathcal{H}} \equiv \mathcal{H}_{\text{Phys}}. \tag{8}$$

The inner product on physical states is

$$(\hat{\Psi}, \hat{\Phi}) = \langle\hat{\Psi}| \int_{-\infty}^{\infty} ds\; e^{-is\hat{C}} |\hat{\Phi}\rangle\, . \tag{9}$$

The algebra $\widehat{\mathcal{M}}$, known as the **crossed product**, admits a trace

$$\text{tr}(\hat{A}) = \langle\hat{\Psi}|\hat{A}|\hat{\Psi}\rangle\, , \tag{10}$$

where the normalized physical state is

$$|\hat{\Psi}\rangle = |\Psi_{dS}\rangle \otimes e^{-q/2}\, |q\rangle\, . \tag{11}$$

The trace of the identity is

$$\text{tr}(\hat{I}) = \langle\hat{\Psi}|\hat{\Psi}\rangle = \int dq\; e^{-q} \int_{-\infty}^{\infty} ds\; e^{-is\hat{C}}\, \langle\Psi_{dS}|\Psi_{dS}\rangle\, . \tag{12}$$

Using (2), this becomes proportional to a contour integral:

$$\text{tr}(\hat{I}) \propto \int_{-i\infty}^{i\infty} \frac{d\beta}{2\pi i} \int dq\; e^{-\beta q} \int dE\; \rho(E)\; e^{-\beta E}. \tag{13}$$

Thus,

$$\boxed{\text{tr}(\hat{I}) \propto \int_{-i\infty}^{i\infty} \frac{d\beta}{2\pi i} \int dq\; e^{-\beta q} \int dE\; \rho(E)\; e^{-\beta E} \propto 1}. \tag{14}$$

2 Geometric Structure

Prior to coupling, the cyclic and separating state is given by (2). After including the one-dimensional system, it takes the form (11). As discussed in [3], this construction admits a geometric interpretation known as a **quantum wormhole**.

The physical state can be represented path-integrally as

$$|\hat{\Psi}\rangle_{\text{Phys}} = |\widetilde{HH}\rangle = \int \mathcal{D}\phi \, \mathcal{D}q \, \mathcal{D}p \, e^{-I[\phi]-I[p,q]}. \tag{1}$$

For each fixed q, the corresponding state is cyclic and separating within its sector. This infinite family of states emerges naturally in the limit $N \to \infty$.

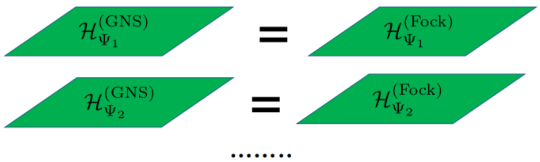

Figure 1: Schematic of the quantum wormhole geometry induced by coupling to a 1D quantum system (valid in the $N \to \infty$ limit).

Under the isomorphism

$$\mathcal{H}^{GNS}_{\Psi_{dS}} \otimes L^2(\mathbb{R}) \cong \mathcal{H}^{\text{Fock}}_{\text{New}}, \tag{2}$$

the trace identity becomes

$$\boxed{\operatorname{tr}(\hat{I}) = \langle \hat{\Psi} | \hat{\Psi} \rangle = \langle \widetilde{HH} | \widetilde{HH} \rangle \propto 1}. \tag{3}$$

From the geometric perspective [4], the partition function of the sphere in the presence of an observer is

$$\mathcal{Z} = (\text{HH}, \widetilde{\text{HH}}) \propto \int_{-i\infty}^{i\infty} d\beta \int dq \, e^{-\beta q} \int dE \, \rho(E) \, e^{-\beta E}. \tag{4}$$

Consequently, the **state-counting partition function** is

$$\boxed{Z_{\text{count}} = \int_{-i\infty}^{i\infty} \frac{d\beta}{2\pi i} \int dq \, e^{-\beta q} \int dE \, \rho(E) \, e^{-\beta E} \propto 1}, \tag{5}$$

in agreement with [5] and contrary to [6].

3 Summary of Results

1. Under the Hilbert space isomorphism,
$$\boxed{\mathrm{tr}(\hat{I}) \propto Z_{\mathrm{count}}}, \tag{1}$$
linking the algebraic trace (LHS) to the geometric partition function (RHS).

2. The norm of the wormhole state satisfies
$$\boxed{\langle \widetilde{HH} | \widetilde{HH} \rangle \propto Z_{\mathrm{count}}}, \tag{2}$$
consistent with [7].

3. The full partition function of the S^D system with an observer is
$$\boxed{\mathcal{Z} \propto (\mathrm{HH}, \widetilde{\mathrm{HH}}) \propto -iZ_{\mathrm{count}}}. \tag{3}$$

4 Holographic Interpretation via dS/CFT

The equivalence $\mathrm{tr}(\hat{I}) \propto Z_{\mathrm{count}}$ admits a natural holographic dual in the dS/CFT correspondence [8]. The algebraic trace on the observer-dependent Type II_1 algebra $\widehat{\mathcal{M}}$ computes the partition function of the dual CFT on the boundary sphere $\mathcal{I}^+$:

$$\mathrm{tr}(\hat{I}) = Z_{\mathrm{CFT}} \propto \int_{-i\infty}^{i\infty} \frac{d\beta}{2\pi i} \int dq\, e^{-\beta q} \int dE\, \rho(E)\, e^{-\beta E}, \tag{1}$$

where the modular flow (from Δ_Ψ) encodes the boundary modular invariance [3, 9].

On the geometric side, Z_{count} is the bulk gravitational partition function, given by the on-shell action of the wormhole geometry dual to the CFT state-counting:

$$Z_{\mathrm{count}} = \int \mathcal{D}\phi\, \mathcal{D}q\, \mathcal{D}p\, e^{-I[\phi]-I[p,q]} \propto 1. \tag{2}$$

Holography equates the two via $\log Z_{\mathrm{CFT}} = -I_{\mathrm{grav}}$ (up to the dS entropy constant $S_{\mathrm{dS}} = \pi\ell^2/G_N$), resolving finite microstate counting in the observer's static patch [5, 10].

This identifies the left-hand side (algebraic trace) as the **boundary holographic dual** and the right-hand side (geometric count) as the **bulk gravitational partition function**, consistent with covariant observer algebras in dS [11].

References

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