

Algebraic and Geometric Structures in de Sitter Quantum Gravity

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1 Algebraic Structure

In the limit $G_N \rightarrow 0$, the GNS Hilbert space associated with the de Sitter vacuum state $|\Psi_{dS}\rangle$ reduces to the Fock space¹:

$$\mathcal{H}_{\Psi_{dS}}^{GNS} \cong \mathcal{H}^{\text{Fock}}. \quad (1)$$

The state $|\Psi_{dS}\rangle$ is cyclic and separating, and can be expressed as the Hartle–Hawking state:

$$|\Psi_{dS}\rangle = |HH\rangle = \int \mathcal{D}\phi \, e^{-I[\phi]}. \quad (2)$$

In quantum field theory on de Sitter spacetime, local operator algebras are **von Neumann algebras of type III**. The algebra \mathcal{A} acting on $\mathcal{H}_{\Psi_{dS}}^{GNS}$ is therefore a **type III** factor. By the Tomita–Takesaki theorem, the modular operator is defined as

$$\Delta_{\Psi_{dS}} \equiv \Delta_{\Psi} = e^{-K_{\Psi}}, \quad (3)$$

where K_{Ψ} is the **modular Hamiltonian**.

Now consider coupling the system to a one-dimensional quantum mechanical degree of freedom with Hamiltonian \hat{q} . The extended Hilbert space becomes

$$\hat{\mathcal{H}} = \mathcal{H}_{\Psi_{dS}}^{GNS} \otimes L^2(\mathbb{R}), \quad (4)$$

¹Witten said the hilbert space can be veiwed as a fock space, but there is no natural notion of creation and annihilation operators

and the full operator algebra is

$$\mathcal{M} = \mathcal{A} \otimes \mathcal{B}(L^2(\mathbb{R})). \quad (5)$$

Define the constraint operator

$$\hat{C} = K_\Psi + \hat{q}. \quad (6)$$

The commutant subalgebra $\widehat{\mathcal{M}}$ consists of all operators in \mathcal{M} that commute with \hat{C} :

$$\widehat{\mathcal{M}} = \{\hat{A} \in \mathcal{M} \mid [\hat{A}, \hat{C}] = 0\}. \quad (7)$$

Physical states are obtained by group averaging over the constraint [1, 2, 3]:

$$|\hat{\Psi}\rangle_{\text{Phys}} = \int_{-\infty}^{\infty} ds \, e^{-is\hat{C}} |\Psi\rangle \in \hat{\mathcal{H}} \equiv \mathcal{H}_{\text{Phys}}. \quad (8)$$

The inner product on physical states is

$$(\hat{\Psi}, \hat{\Phi}) = \langle \hat{\Psi} | \int_{-\infty}^{\infty} ds \, e^{-is\hat{C}} | \hat{\Phi} \rangle. \quad (9)$$

The algebra $\widehat{\mathcal{M}}$, known as the ****crossed product****, admits a trace

$$\text{tr}(\hat{A}) = \langle \hat{\Psi} | \hat{A} | \hat{\Psi} \rangle, \quad (10)$$

where the normalized physical state is

$$|\hat{\Psi}\rangle = |\Psi_{dS}\rangle \otimes e^{-q/2} |q\rangle. \quad (11)$$

The trace of the identity is

$$\text{tr}(\hat{I}) = \langle \hat{\Psi} | \hat{\Psi} \rangle = \int dq \, e^{-q} \int_{-\infty}^{\infty} ds \, e^{-is\hat{C}} \langle \Psi_{dS} | \Psi_{dS} \rangle. \quad (12)$$

Using (2), this becomes proportional to a contour integral:

$$\text{tr}(\hat{I}) \propto \int_{-i\infty}^{i\infty} \frac{d\beta}{2\pi i} \int dq \, e^{-\beta q} \int dE \, \rho(E) \, e^{-\beta E}. \quad (13)$$

Thus,

$$\boxed{\text{tr}(\hat{I}) \propto \int_{-i\infty}^{i\infty} \frac{d\beta}{2\pi i} \int dq \, e^{-\beta q} \int dE \, \rho(E) \, e^{-\beta E} \propto 1.} \quad (14)$$

2 Geometric Structure

Prior to coupling, the cyclic and separating state is given by (2). After including the one-dimensional system, it takes the form (11). As discussed in [3], this construction admits a geometric interpretation known as a **quantum wormhole**.

The physical state can be represented path-integrally as

$$|\hat{\Psi}\rangle_{\text{Phys}} = |\widetilde{HH}\rangle = \int \mathcal{D}\phi \mathcal{D}q \mathcal{D}p e^{-I[\phi] - I[p,q]}. \quad (1)$$

For each fixed q , the corresponding state is cyclic and separating within its sector. This infinite family of states emerges naturally in the limit $N \rightarrow \infty$.



Figure 1: Schematic of the quantum wormhole geometry induced by coupling to a 1D quantum system (valid in the $N \rightarrow \infty$ limit).

Under the isomorphism

$$\mathcal{H}_{\Psi_{dS}}^{GNS} \otimes L^2(\mathbb{R}) \cong \mathcal{H}_{\text{New}}^{\text{Fock}}, \quad (2)$$

the trace identity becomes

$$\boxed{\text{tr}(\hat{I}) = \langle \hat{\Psi} | \hat{\Psi} \rangle = \langle \widetilde{HH} | \widetilde{HH} \rangle \propto 1}. \quad (3)$$

From the geometric perspective [4], the partition function of the sphere in the presence of an observer is

$$\mathcal{Z} = (\text{HH}, \widetilde{\text{HH}}) \propto \int_{-i\infty}^{i\infty} d\beta \int dq e^{-\beta q} \int dE \rho(E) e^{-\beta E}. \quad (4)$$

Consequently, the **state-counting partition function** is

$$\boxed{Z_{\text{count}} = \int_{-i\infty}^{i\infty} \frac{d\beta}{2\pi i} \int dq e^{-\beta q} \int dE \rho(E) e^{-\beta E} \propto 1}, \quad (5)$$

in agreement with [5] and contrary to [6].

3 Summary of Results

1. Under the Hilbert space isomorphism,

$$\boxed{\text{tr}(\hat{I}) \propto Z_{\text{count}}}, \quad (1)$$

linking the algebraic trace (LHS) to the geometric partition function (RHS).

2. The norm of the wormhole state satisfies

$$\boxed{\langle \widetilde{HH} | \widetilde{HH} \rangle \propto Z_{\text{count}}}, \quad (2)$$

consistent with [7].

3. The full partition function of the S^D system with an observer is

$$\boxed{\mathcal{Z} \propto (\text{HH}, \widetilde{\text{HH}}) \propto -i Z_{\text{count}}}. \quad (3)$$

4 Holographic Interpretation via dS/CFT

The equivalence $\text{tr}(\hat{I}) \propto Z_{\text{count}}$ admits a natural holographic dual in the dS/CFT correspondence [8]. The algebraic trace on the observer-dependent Type II₁ algebra $\widehat{\mathcal{M}}$ computes the partition function of the dual CFT on the boundary sphere \mathcal{I}^+ :

$$\text{tr}(\hat{I}) = Z_{\text{CFT}} \propto \int_{-i\infty}^{i\infty} \frac{d\beta}{2\pi i} \int dq e^{-\beta q} \int dE \rho(E) e^{-\beta E}, \quad (1)$$

where the modular flow (from Δ_Ψ) encodes the boundary modular invariance [3, 9].

On the geometric side, Z_{count} is the bulk gravitational partition function, given by the on-shell action of the wormhole geometry dual to the CFT state-counting:

$$Z_{\text{count}} = \int \mathcal{D}\phi \mathcal{D}q \mathcal{D}p e^{-I[\phi] - I[p,q]} \propto 1. \quad (2)$$

Holography equates the two via $\log Z_{\text{CFT}} = -I_{\text{grav}}$ (up to the dS entropy constant $S_{\text{dS}} = \pi \ell^2 / G_N$), resolving finite microstate counting in the observer's static patch [5].

This identifies the left-hand side (algebraic trace) as the **boundary holographic dual** and the right-hand side (geometric count) as the **bulk gravitational partition function**.

References

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