SHAHID BEHESHTI UNIVERSITY

Problem Set 3

Course: *Mathematical Physics II (Winter 2022)*

Due Date: *5:00 PM - Thursday, 19th of Esfand 1400*

Problem 1. Consider the Hilbert space $\mathbb{C}^2$ and the vectors

$$|0\rangle = \begin{pmatrix} i \\ i \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Normalize these vectors and then calculate the probability $|\langle 0|1\rangle|^2$.

Problem 2. Consider the Hilbert space $\mathbb{R}^n$. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Show that

$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)$$

Note that

$$\|\mathbf{x}\|^2 := \langle \mathbf{x}|\mathbf{x}\rangle .$$

Problem 3. Consider the three vectors:

$$\begin{aligned}
|\psi_1\rangle &= \frac{1}{\sqrt{3}}|+\rangle + i\frac{\sqrt{2}}{\sqrt{3}}|-\rangle \\
|\psi_2\rangle &= \frac{1}{\sqrt{5}}|+\rangle - \frac{2}{\sqrt{5}}|-\rangle \\
|\psi_3\rangle &= \frac{1}{\sqrt{2}}|+\rangle + e^{i\pi/4}\frac{1}{\sqrt{2}}|-\rangle
\end{aligned}$$

Use bra-ket notation (not matrix notation) to solve the following problems. Note that $\langle +|+\rangle = 1$, $\langle -|-\rangle = 1$, $\langle +|-\rangle = 0$.

1. For each of the ψ_i above, find the normalized vector ϕ_i that is orthogonal to it.
2. Calculate the inner products $\langle \psi_i|\psi_j\rangle$ for i and j = 1, 2, 3.

Problem 4. Let $|0\rangle$, $|1\rangle$ be an orthonormal basis in the Hilbert space $\mathbb{C}^2$.
Let

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$$

Where $\theta, \phi \in \mathbb{R}$.

1. Find $\langle\psi|\psi\rangle$.
2. Find the probability $|\langle 0|\psi\rangle|^2$.
3. Assume that

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Find the 2×2 matrix $|\psi\rangle\langle\psi|$ and calculate the eigenvalues.

Problem 5. Consider the Hilbert space $\mathcal{H}$ of the 2×2 matrices over the complex numbers with the scalar product

$$\langle A|B\rangle := tr(AB^\dagger), \quad A, B \in \mathcal{H}.$$

Show that the rescaled Pauli matrices $\mu_j := \frac{1}{\sqrt{2}}\sigma_j$, $j = 1, 2, 3$

$$\mu_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mu_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mu_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

plus the rescaled 2×2 identity matrix

$$\mu_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

form an orthonormal basis in the Hilbert space $\mathcal{H}$.