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The Imaginary Phase Problem in the Gravitational Path Integral[1]

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ABSTRACT: The partition function of the sphere S^D, obtained through the **Saddle Point approximation** valid in the limit $G_N \to 0$, produces imaginary phases known as the **Polchinski phase** [1]. The existence of such imaginary phases interprets of state counting ill-defined.
To resolve this issue, by adopting the algebraic approach discussed in [2], a simple observer model is introduced. Subsequently, in [3], an attempt was made to reinterpret state counting by considering that observer model. However, that approach leads to a negative sign in the counting of states, which lacks a proper physical interpretation.
In this report, by considering a **duality**, we attempt to propose a more physically meaningful and consistent interpretation of state counting as discussed in [3].

[1]It is written in this format only as an official example and has no further value.

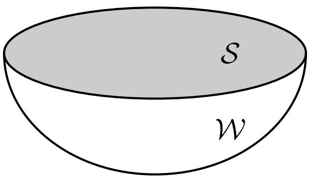
Contents

1 Algebraic Structure

As is well known, in the limit $G_N \to 0$, we may write:

$$|\Psi_{dS}\rangle = \frac{1}{\sqrt{Z}} |\Psi_{\mathcal{W}}\rangle = \frac{1}{\sqrt{Z}} \int_{\phi_{\mathcal{W}}|_{\mathcal{S}}=\phi_{\mathcal{S}}} D\phi_{\mathcal{W}}\, e^{-I[\phi_{\mathcal{W}}]} \tag{1.1}$$

where: Thus we can assume:



$$\mathcal{H}^{GNS}_{\Psi_{dS}} \cong \mathcal{H}^{Fock} \tag{1.2}$$

where $\mathcal{H}^{Fock}$ denotes a vacuum space, i.e., one devoid of creation and annihilation operators.

Since this state lies within a **Quantum Field Theory (QFT)** on a **de Sitter (dS)** background, the algebra acting on this Hilbert space, as well as on the Fock space, is a **Von Neumann Type III** algebra, denoted by $\mathcal{A}$.

From an algebraic viewpoint, according to the **Tomita–Takesaki theory**, the state $|\Psi_{dS}\rangle$ is both a **cyclic** and **separating** vector with respect to the algebra $\mathcal{A}$.

Now suppose we add a one-dimensional quantum system to the main system. Then the new Hilbert space becomes:

$$\mathcal{H}^{GNS}_{\Psi_{dS}} \to \hat{\mathcal{H}} = \mathcal{H}^{GNS}_{\Psi_{dS}} \otimes L^2(\mathcal{R}) \tag{1.3}$$

and the algebra acting on this extended Hilbert space is:

$$\mathcal{M} = \mathcal{A} \otimes \mathcal{B}(L^2(\mathcal{R})) \tag{1.4}$$

Let us consider a subalgebra of $\mathcal{M}$ consisting of all elements commuting with the operator:

$$\hat{C} = K + \hat{q} \tag{1.5}$$

This subalgebra, $\widehat{\mathcal{M}}$, is defined by:

$$\widehat{\mathcal{M}} = \{\hat{A} \,|\, [\hat{A}, \hat{C}] = 0\} \tag{1.6}$$

Treating $\hat{C}$ as a constraint, the physical states are defined as [4–6]:

$$|\hat{\Psi}\rangle_{Phys} = \int_{-\infty}^{\infty} ds\; e^{-isC}\,|\Psi\rangle \in \hat{\mathcal{H}} \equiv \mathcal{H}_{Phys} \tag{1.7}$$

and their inner product is:

$$(\hat{\Psi}, \hat{\Phi}) = \langle\hat{\Psi}| \int_{-\infty}^{\infty} ds\; e^{-isC}|\hat{\Phi}\rangle \tag{1.8}$$

It can be shown that the subalgebra $\widehat{\mathcal{M}}$ is a **Von Neumann Type II$_1$** algebra. Accordingly, a trace map may be defined:

$$\mathrm{tr}(\rho_{\hat{\Phi}}\hat{A}) = \langle\hat{\Phi}|\hat{A}|\hat{\Phi}\rangle \tag{1.9}$$

where:

$$|\hat{\Phi}\rangle \equiv |\hat{\Phi}\rangle_{Phys} = \int_{-\infty}^{\infty} ds\; e^{-isC}\,|\Psi\rangle_{dS} \otimes \int dq\; e^{-q/2}\,|q\rangle \tag{1.10}$$

and one can show [6] that:

$$\rho_{\hat{\Phi}} = 2\pi e^{iK\hat{p}} e^{q} \Delta_{\Phi\Psi} e^{-iK\hat{p}} \tag{1.11}$$

Since all cyclic and separating vectors with respect to $\mathcal{A}$ in dS space coincide, we have:

$$\Delta_{\Phi\Psi} = \mathrm{I} \tag{1.12}$$

Therefore:

$$\mathrm{tr}(\rho_{\hat{\Phi}}e^{-q}) = 2\pi\;\mathrm{tr}(\mathrm{I}) = \langle\hat{\Phi}|\hat{\Phi}\rangle = (\hat{\Phi}, \int_{-\infty}^{\infty} ds\; e^{-isC}\hat{\Phi}) \tag{1.13}$$

and thus:

$$\mathrm{tr}(\mathrm{I}) = \frac{1}{2\pi}(\hat{\Phi}, \int_{-\infty}^{\infty} ds\; e^{-isC}\hat{\Phi}) \tag{1.14}$$

Assuming:

$$|\Psi_{dS}\rangle = \frac{1}{\sqrt{Z}} \int dE\; \sqrt{\rho(E)} e^{-\beta E/2}\,|E\rangle \tag{1.15}$$

we find:

$$\mathrm{tr(I)} = \frac{1}{2\pi}(\hat{\Phi}, \int_{-\infty}^{\infty} ds\; e^{-isC}\hat{\Phi})$$
$$= \frac{1}{Z}\int_{-i\infty}^{i\infty}\frac{d\beta}{2\pi i}\int dq\; e^{-\beta q}\int dE\; \rho(E)\; e^{-\beta E} = 1 \tag{1.16}$$

Hence:

$$|\hat{\Phi}\rangle = \frac{1}{\sqrt{Z}}\int dE\; e^{-\beta E/2}\,|E\rangle \otimes |-E\rangle \tag{1.17}$$

and

$$\boxed{Z = \int_{-i\infty}^{i\infty}\frac{d\beta}{2\pi i}\int dq\; e^{-\beta q}\int dE\; \rho(E)\; e^{-\beta E}} \tag{1.18}$$

If we add a constant m to the constraint $\hat{C} = \hat{K} + \hat{q} + m\hat{I}$, then:

$$|\hat{\Phi}\rangle = \frac{1}{\sqrt{Z}}\int_{0}^{\infty} dE\; e^{-\beta(E-m)/2}\,|E+m\rangle \otimes |-E-m\rangle \tag{1.19}$$

and consequently:

$$\boxed{Z = \int_{-i\infty}^{i\infty}\frac{d\beta}{2\pi i}\int dq\; e^{-\beta q}\int dE\; \rho(E)\; e^{-\beta E}e^{-\beta m}} \tag{1.20}$$

2 Geometric Structure

As seen in 1.17, this behaves as a **ThermoField Double** (TFD) state with continuous energy levels:

$$|\hat{\Phi}\rangle = |TFD\rangle = \frac{1}{\sqrt{Z}}\sum_n e^{-\beta E_n/2}\,|E_n\rangle \otimes |\Theta E_n\rangle \tag{2.1}$$

Thus, adding a one-dimensional quantum system extends the Hilbert space:

$$\hat{\mathcal{H}} = \mathcal{H}^{GNS}_{\Psi_{dS}} \otimes L^2(\mathcal{R}), \quad |\hat{\Phi}\rangle = |TFD\rangle \tag{2.2}$$

and in the large N limit:

$$\hat{\mathcal{H}}^{GNS}_{\hat{\Phi}} = \mathcal{H}^{GNS}_{\Psi_{dS}} \otimes L^2(\mathcal{R}) \cong \mathcal{H}^{Fock} \otimes L^2(\mathcal{R}) = \hat{\mathcal{H}}^{Fock}, \tag{2.3}$$
$$|\hat{\Phi}\rangle = \frac{1}{\sqrt{Z}}\,|\widetilde{\Psi}_{\mathcal{W}}\rangle \tag{2.4}$$

In this setting, an **Eternal Black Hole** exists in the bulk whose wavefunction corresponds to a generalized **Hartle–Hawking** state:

$$|\widetilde{\Psi_{\mathcal{W}}}\rangle = \int_{\phi_{\mathcal{W}}|_{\mathcal{S}}=\phi_{\mathcal{S}}} D\phi_{\mathcal{W}} Dq D\lambda\; e^{-I[\phi_{\mathcal{W}}]-I[q]-I_{LM}} \tag{2.5}$$

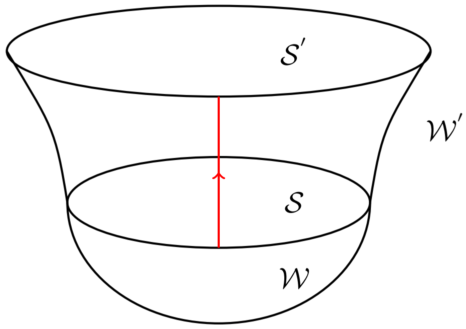
with Euclidean actions:

$$I[\phi_{\mathcal{W}}] = \frac{1}{16\pi G_N} \int d^d x \sqrt{g}(R - 2\Lambda), \tag{2.6}$$

$$I[q] = -\oint d\tau \; q = -\beta_{dS} q, \tag{2.7}$$

$$I_{LM} = \beta_{dS} \lambda (q + H) \tag{2.8}$$

Therefore:



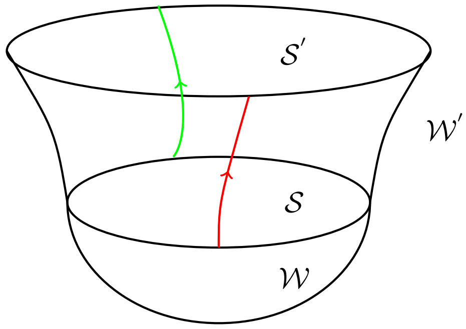
$$|\widetilde{\Psi}_{\mathcal{W}'}\rangle = \int DqD\lambda \, e^{-\beta_{dS}q + \lambda\beta_{dS}(q+H)} \, |\Psi_{\mathcal{W}}\rangle \tag{2.9}$$

As previously discussed, by adding a one-dimensional quantum system, a bipartite structure arises consisting of two Hilbert spaces (or, in algebraic language, two modules), whose existence is determined by the geometry. In the S' construction, the manifold q is associated with one Hilbert space, while the Hamiltonian H corresponds to the other. Therefore, it is natural that, just like the states $|q\rangle$ and $|E\rangle$, these two modules are canonically conjugate with respect to energy eigenstates.
An important point is that the parameter q has a geometric representation: a red line indicates the flow or orientation of the module (in algebraic literature), demonstrating a preferred direction. Consequently, in the formalism, it appears as a continuous variable.
The existence of a state $|\hat{\Phi}\rangle$, which is both cyclic and separating, ensures that q can be interpreted consistently. Hence, it is advantageous to define all these constructions with respect to a Fock-like state $\hat{\mathcal{H}}^{Fock}$ in the extended Hilbert space. So: Now, due to the breaking of reflection symmetry, the partition function cannot be expressed simply as an inner product (or norm) of two states [7]. In other words, we have:

$$\mathcal{Z} = (\Phi, \Psi) = \langle \Phi | \hat{T} | \Psi \rangle \,, \tag{2.10}$$

where, by definition, the bilinear form is $(\ ,\ ) : \mathcal{H}_{q_1} \times \mathcal{H}_{q_2} \to \mathbb{C}$, while the inner product, which corresponds to the norm, is defined as $\langle\ ,\ \rangle : \mathcal{H}_{q_1} \times \mathcal{H}_{q_1} \to \mathbb{C}$. To compute the norm of the state $|\widetilde{\Psi}_{\mathcal{W}'}\rangle_q$, we need to consider two states defined on the same Hilbert space. In other words, we want to evaluate:

$$Z = \langle \widetilde{\Psi}_{\mathcal{W}'_q} | \widetilde{\Psi}_{\mathcal{W}'_q} \rangle \,. \tag{2.11}$$



Schematically, this can be represented as:
Therefore:

$$
\begin{aligned}
Z &= \langle \widetilde{\Psi}_{\mathcal{W}'_q} | \widetilde{\Psi}_{\mathcal{W}'_q} \rangle \\
&= \int Dq_1 D\lambda_1 e^{-\beta_{dS}(q_1+m)+\lambda\beta_{dS}(q_1+E+m)} \int Dq_2 D\lambda_2 e^{-\beta_{dS}(q_2+m)+\lambda_2\beta_{dS}(q_2+E+m)} \langle \Psi_{\mathcal{W}} | \Psi_{\mathcal{W}} \rangle
\end{aligned} \tag{2.12}
$$

Given that we need to impose the constraint $q_1 - q_2 = 0$, and noting that the integral over the fields $\lambda_{1,2}$ diverges, we perform a contour rotation [1, 3]:

$$
\lambda_1 \to -is_1, \quad \lambda_2 \to +is_2. \tag{2.13}
$$

Consequently, we have:

$$
\begin{aligned}
Z &= \langle \widetilde{\Psi}_{\mathcal{W}'_q} | \widetilde{\Psi}_{\mathcal{W}'_q} \rangle \\
&= \int dq_1\, dq_2\, e^{-\beta_{dS}(q_1+q_2+2m)} \int ds_1\, ds_2\, e^{-is_1(q_1+E+m)} e^{+is_2(q_2+E+m)} \langle \Psi_{\mathcal{W}} | \Psi_{\mathcal{W}} \rangle \\
&= \int dq_1\, dq_2\, e^{-\beta_{dS}(q_1+q_2+2m)} \delta(q_1+E+m)\delta(q_2+E+m) \int dE\, \rho(E)\, e^{-\beta_{dS}E} \\
&= \int dq_1\, dq_2\, dE\, \rho(E)\, e^{-\beta_{dS}(q_1+q_2+E+2m)} \delta(q_1+E+m)\delta(q_2+E+m) \\
&= \int_{-\infty}^{-m} dE\, \rho(E)\, e^{\beta_{dS}(E+m)}.
\end{aligned} \tag{2.14}
$$

Using $\rho(E) = e^{S_{dS}} e^{\beta_{dS}E}$, we obtain:

$$
\begin{aligned}
Z &= \int_{-\infty}^{-m} dE\, e^{S_{dS}} e^{\beta_{dS}E} e^{\beta_{dS}(E+m)} \\
&= e^{S_{dS}} e^{\beta_{dS}m} \int_{-\infty}^{-m} dE\, e^{2\beta_{dS}E} \\
&= e^{S_{dS}} e^{\beta_{dS}m} \frac{1}{2\beta_{dS}} \left(e^{-2\beta_{dS}m} - 0 \right) \\
&= \frac{e^{S_{dS}-\beta_{dS}m}}{2\beta_{dS}} = \frac{e^{S_{dS}-\beta_{dS}m}}{4\pi}.
\end{aligned} \tag{2.15}
$$

Hence, we can write:

$$\boxed{Z_{count} = \langle \widetilde{\Psi}_{\mathcal{W}'_q} | \widetilde{\Psi}_{\mathcal{W}'_q} \rangle}. \tag{2.16}$$

This result is consistent with [8], implying that the state $|\widetilde{\Psi}_{\mathcal{W}'_q}\rangle$ can be identified with the $|\widetilde{HH}\rangle$ state, i.e., the Hartle-Hawking state in the presence of an observer. Moreover, one can establish an equivalence between the algebraic and geometric-gravitational perspectives as discussed in Eq. 2.4:

$$\hat{\mathcal{H}}^{GNS}_{\hat{\Phi}_q} \cong \hat{\mathcal{H}}^{Fock}_q, \quad |\hat{\Phi}\rangle_q = \frac{1}{\sqrt{Z}} |\widetilde{\Psi}_{\mathcal{W}'}\rangle_q \,. \tag{2.17}$$

From the first part, we also observed that

$$\mathrm{tr}(\mathrm{I}) = \langle \hat{\Phi}_q | \hat{\Phi}_q \rangle = \frac{1}{Z} \int_{-i\infty}^{i\infty} \frac{d\beta}{2\pi i} \int dq \, e^{-\beta q} \int dE \, \rho(E) \, e^{-\beta E} = 1, \tag{2.18}$$

and equivalently, from the geometric-gravitational structure, we have:

$$Z_{count} = \langle \widetilde{\Psi}_{\mathcal{W}'_q} | \widetilde{\Psi}_{\mathcal{W}'_q} \rangle \,. \tag{2.19}$$

Therefore, we conclude:

$$\boxed{Z_{count} = \langle \widetilde{\Psi}_{\mathcal{W}'_q} | \widetilde{\Psi}_{\mathcal{W}'_q} \rangle = \int_{-i\infty}^{i\infty} \frac{d\beta}{2\pi i} \int dq \, e^{-\beta q} \int dE \, \rho(E) \, e^{-\beta E} e^{-\beta m}}. \tag{2.20}$$

3 Partition Function

Now, considering that by adding a one-dimensional quantum system to the original system in the algebraic framework, one can construct a **ThermoField Double** state, and in the gravitational sector, a generalized **Hartle-Hawking** state can be defined, and furthermore, that this generalized state exhibits reflection symmetry breaking, we can compute the partition function of a sphere. As discussed in the previous section, we have:

$$Z = (\Phi, \Psi). \tag{3.1}$$

Thus, we can write:

$$Z = (\Psi_{\mathcal{W}}, \widetilde{\Psi}_{\mathcal{W}'_q}) = \int DqD\lambda \, e^{-\beta_{dS}(q+m)+\lambda\beta_{dS}(q+E+m)} \, \langle \Psi_{\mathcal{W}} | \Psi_{\mathcal{W}} \rangle \,. \tag{3.2}$$

Performing a contour rotation:

$$\lambda\beta_{dS} \to -is, \tag{3.3}$$

we obtain:

$$
\begin{aligned}
Z &= (\Psi_{\mathcal{W}}, \widetilde{\Psi}_{\mathcal{W}'_q}) \\
&= -\frac{i}{2\pi}\int_{-\infty}^{+\infty} ds \int dq \int dE\ \rho(E)\ e^{-\beta_{dS}(q+m+E)} e^{-is(q+m+E)} \\
&= -\frac{i}{2\pi}\int_{-\infty}^{+\infty} ds \int dq \int dE\ \rho(E)\ e^{-(\beta_{dS}+is)(q+m+E)} \\
&= -\int_{-i\infty+2\pi}^{+i\infty+2\pi} \frac{d\beta}{2\pi} \int dq \int dE\ \rho(E)\ e^{-\beta(q+m+E)} \\
&= -i\int_{-i\infty+2\pi}^{+i\infty+2\pi} \frac{d\beta}{2\pi i} \int dq\ e^{-\beta q} \int dE\ \rho(E)\ e^{-\beta E} e^{-\beta m} \\
&= -iZ_{count}.
\end{aligned} \tag{3.4}
$$

Therefore, we have:

$$
\boxed{Z[S^D]Z[\text{Obs}] = -iZ_{count} = -i\, e^{S_{dS}-\beta_{dS}m}.} \tag{3.5}
$$

This result is consistent with [3], and indicates that, upon introducing an observer, the partition function of the sphere S^D, which previously carried an imaginary phase, can be interpreted as a state-counting partition function.

4 Conclusion

As is well-known, by adding a one-dimensional quantum system to the original system, one can construct a **Von Neumann Type II** algebra by considering a suitable subalgebra. In this case, a trace map tr can be defined, which in turn allows the definition of an associated entropy, as discussed in [2, 6].

Our motivation, therefore, has been to investigate whether adding a one-dimensional quantum system within the algebraic framework, which leads to a **Von Neumann Type II** algebra, has any effect in the non-algebraic or gravitational perspective.

Initially, we relied on a duality that essentially holds in the $N \to \infty$ limit, thereby establishing a correspondence between the algebraic and geometric structures. As is known, adding a one-dimensional quantum system modifies the algebra. Using the duality at our disposal, we then analyzed the structural changes in the geometry.

We found that in the algebraic framework, a **ThermoField Double** state can be constructed. Since we are in the $N \to \infty$ limit, a generalized **Hartle-Hawking** state can also be defined. The norm of this generalized state, as discussed in [8], corresponds to a state-counting partition function. This allows the proper definition of a counting partition function, thereby resolving the issue discussed in [3].

Furthermore, this result is consistent with [9], with an intriguing observation that black holes appear in both approaches. In [9], the black hole considered is a charged **Reissner-Nordstrom** solution, while in our approach, an **Eternal** black hole arises. It might be interesting to explore a possible connection between these two scenarios.

Finally, this approach enables a well-defined interpretation of the partition function of the sphere S^D, which previously carried an imaginary phase and lacked a meaningful state-counting interpretation, by including the presence of an observer.

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