Algebraic and Geometric Structures in de Sitter Quantum Gravity

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1 Algebraic Structure

In the limit $G_N \to 0$, the GNS Hilbert space associated with the de Sitter vacuum state $|\Psi_{dS}\rangle$ reduces to the Fock space:

$$\mathcal{H}^{GNS}_{\Psi_{dS}} = \mathcal{H}^{\text{Fock}}. \tag{1}$$

The state $|\Psi_{dS}\rangle$ is cyclic and separating, and can be expressed as the Hartle–Hawking state:

$$|\Psi_{dS}\rangle = |HH\rangle = \int \mathcal{D}\phi \, e^{-I[\phi]}. \tag{2}$$

In quantum field theory on de Sitter spacetime, local operator algebras are von Neumann algebras of **type III**. The algebra $\mathcal{A}$ acting on $\mathcal{H}^{GNS}_{\Psi_{dS}}$ is therefore a **type III** factor. By the Tomita–Takesaki theorem, the modular operator is defined as

$$\Delta_{\Psi_{dS}} \equiv \Delta_\Psi = e^{-K_\Psi}, \tag{3}$$

where K_Ψ is the **modular Hamiltonian**.

Now consider coupling the system to a one-dimensional quantum mechanical degree of freedom with Hamiltonian $\hat{q}$. The extended Hilbert space becomes

$$\hat{\mathcal{H}} = \mathcal{H}^{GNS}_{\Psi_{dS}} \otimes L^2(\mathbb{R}), \tag{4}$$

and the full operator algebra is

$$\mathcal{M} = \mathcal{A} \otimes \mathcal{B}(L^2(\mathbb{R})). \tag{5}$$

Define the constraint operator

$$\hat{C} = K_\Psi + \hat{q}. \tag{6}$$

The commutant subalgebra $\widehat{\mathcal{M}}$ consists of all operators in $\mathcal{M}$ that commute with $\hat{C}$:

$$\widehat{\mathcal{M}} = \{\hat{A} \in \mathcal{M} \mid [\hat{A}, \hat{C}] = 0\}. \tag{7}$$

Physical states are obtained by group averaging over the constraint [1, 2, 3]:

$$|\hat{\Psi}\rangle_{\text{Phys}} = \int_{-\infty}^{\infty} ds\ e^{-is\hat{C}}\ |\Psi\rangle \in \hat{\mathcal{H}} \equiv \mathcal{H}_{\text{Phys}}. \tag{8}$$

The inner product on physical states is

$$(\hat{\Psi}, \hat{\Phi}) = \langle\hat{\Psi}| \int_{-\infty}^{\infty} ds\ e^{-is\hat{C}} |\hat{\Phi}\rangle\ . \tag{9}$$

The algebra $\widehat{\mathcal{M}}$, known as the **crossed product**, admits a trace

$$\text{tr}(\hat{A}) = \langle\hat{\Psi}|\hat{A}|\hat{\Psi}\rangle\ , \tag{10}$$

where the normalized physical state is

$$|\hat{\Psi}\rangle = |\Psi_{dS}\rangle \otimes e^{-q/2}\ |q\rangle\ . \tag{11}$$

The trace of the identity is

$$\text{tr}(\hat{I}) = \langle\hat{\Psi}|\hat{\Psi}\rangle = \int dq\ e^{-q} \int_{-\infty}^{\infty} ds\ e^{-is\hat{C}}\ \langle\Psi_{dS}|\Psi_{dS}\rangle\ . \tag{12}$$

Using (2), this becomes proportional to a contour integral:

$$\text{tr}(\hat{I}) \propto \int_{-i\infty}^{i\infty} \frac{d\beta}{2\pi i} \int dq\ e^{-\beta q} \int dE\ \rho(E)\ e^{-\beta E}. \tag{13}$$

Thus,

$$\boxed{\text{tr}(\hat{I}) \propto \int_{-i\infty}^{i\infty} \frac{d\beta}{2\pi i} \int dq\ e^{-\beta q} \int dE\ \rho(E)\ e^{-\beta E} \propto 1}. \tag{14}$$

2 Geometric Structure

Prior to coupling, the cyclic and separating state is given by (2). After including the one-dimensional system, it takes the form (11). As discussed in [3], this construction admits a geometric interpretation known as a **quantum wormhole**.

The physical state can be represented path-integrally as

$$|\hat{\Psi}\rangle_{\text{Phys}} = |\widetilde{HH}\rangle = \int \mathcal{D}\phi \, \mathcal{D}q \, \mathcal{D}p \, e^{-I[\phi]-I[p,q]}. \tag{1}$$

For each fixed q, the corresponding state is cyclic and separating within its sector. This infinite family of states emerges naturally in the limit $N \to \infty$.

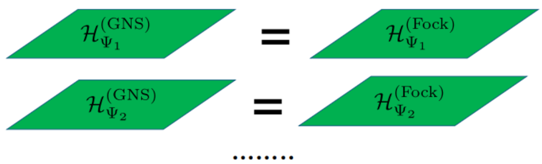

Figure 1: Schematic of the quantum wormhole geometry induced by coupling to a 1D quantum system (valid in the $N \to \infty$ limit).

Under the isomorphism

$$\mathcal{H}^{GNS}_{\Psi_{dS}} \otimes L^2(\mathbb{R}) \cong \mathcal{H}^{\text{Fock}}_{\text{New}}, \tag{2}$$

the trace identity becomes

$$\boxed{\text{tr}(\hat{I}) = \langle\hat{\Psi}|\hat{\Psi}\rangle = \langle\widetilde{HH}|\widetilde{HH}\rangle \propto 1}. \tag{3}$$

From the geometric perspective [4], the partition function of the sphere in the presence of an observer is

$$\mathcal{Z} = (\text{HH}, \widetilde{\text{HH}}) \propto \int_{-i\infty}^{i\infty} d\beta \int dq \, e^{-\beta q} \int dE \, \rho(E) \, e^{-\beta E}. \tag{4}$$

Consequently, the **state-counting partition function** is

$$\boxed{Z_{\text{count}} = \int_{-i\infty}^{i\infty} \frac{d\beta}{2\pi i} \int dq \, e^{-\beta q} \int dE \, \rho(E) \, e^{-\beta E} \propto 1}, \tag{5}$$

in agreement with [5] and contrary to [6].

2.1 Summary of Results

1. Under the Hilbert space isomorphism,

$$\boxed{\mathrm{tr}(\hat{I}) \propto Z_{\mathrm{count}}}, \tag{6}$$

linking the algebraic trace (LHS) to the geometric partition function (RHS).

2. The norm of the wormhole state satisfies

$$\boxed{\langle \widetilde{HH} | \widetilde{HH} \rangle \propto Z_{\mathrm{count}}}, \tag{7}$$

consistent with [7].

3. The full partition function of the S^D system with an observer is

$$\boxed{\mathcal{Z} \propto (\mathrm{HH}, \widetilde{\mathrm{HH}}) \propto -iZ_{\mathrm{count}}}. \tag{8}$$

References

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