

2 The Imaginary Phase Problem in the Gravitational 3 Path Integral¹

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6 ABSTRACT: The partition function of the sphere S^D , obtained through the **Saddle Point**
7 **approximation** valid in the limit $G_N \rightarrow 0$, produces imaginary phases known as the
8 **Polchinski phase** [1]. The existence of such imaginary phases interprets of state counting
9 ill-defined.

10 To resolve this issue, by adopting the algebraic approach discussed in [2], a simple observer
11 model is introduced. Subsequently, in [3], an attempt was made to reinterpret state count-
12 ing by considering that observer model. However, that approach leads to a negative sign
13 in the counting of states, which lacks a proper physical interpretation.

14 In this report, by considering a **duality**, we attempt to propose a more physically mean-
15 ingful and consistent interpretation of state counting as discussed in [3].

¹It is written in this format only as an official example and has no further value.

16 Contents

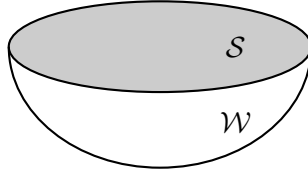
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21 1 Algebraic Structure

22 As is well known, in the limit $G_N \rightarrow 0$, we may write:

$$|\Psi_{dS}\rangle = \frac{1}{\sqrt{Z}} |\Psi_{\mathcal{W}}\rangle = \frac{1}{\sqrt{Z}} \int_{\phi_{\mathcal{W}}|_S = \phi_S} D\phi_{\mathcal{W}} e^{-I[\phi_{\mathcal{W}}]} \quad (1.1)$$

where: Thus we can assume:



23

$$\mathcal{H}_{\Psi_{dS}}^{GNS} \cong \mathcal{H}^{Fock} \quad (1.2)$$

24 where \mathcal{H}^{Fock} denotes a vacuum space, i.e., one devoid of creation and annihilation operators.

25 Since this state lies within a **Quantum Field Theory (QFT)** on a **de Sitter (dS)** background, the algebra acting on this Hilbert space, as well as on the Fock space, is a **Von Neumann Type III** algebra, denoted by \mathcal{A} .

26 From an algebraic viewpoint, according to the **Tomita–Takesaki theory**, the state $|\Psi_{dS}\rangle$ is both a **cyclic** and **separating** vector with respect to the algebra \mathcal{A} .

27 Now suppose we add a one-dimensional quantum system to the main system. Then the new Hilbert space becomes:

$$\mathcal{H}_{\Psi_{dS}}^{GNS} \rightarrow \hat{\mathcal{H}} = \mathcal{H}_{\Psi_{dS}}^{GNS} \otimes L^2(\mathcal{R}) \quad (1.3)$$

32 and the algebra acting on this extended Hilbert space is:

$$\mathcal{M} = \mathcal{A} \otimes \mathcal{B}(L^2(\mathcal{R})) \quad (1.4)$$

34 Let us consider a subalgebra of \mathcal{M} consisting of all elements commuting with the operator:

$$\hat{C} = K + \hat{q} \quad (1.5)$$

35 This subalgebra, $\widehat{\mathcal{M}}$, is defined by:

$$\widehat{\mathcal{M}} = \{\hat{A} \mid [\hat{A}, \hat{C}] = 0\} \quad (1.6)$$

36 Treating \hat{C} as a constraint, the physical states are defined as [4–6]:

$$|\hat{\Psi}\rangle_{Phys} = \int_{-\infty}^{\infty} ds e^{-isC} |\Psi\rangle \in \hat{\mathcal{H}} \equiv \mathcal{H}_{Phys} \quad (1.7)$$

37 and their inner product is:

$$(\hat{\Psi}, \hat{\Phi}) = \langle \hat{\Psi} | \int_{-\infty}^{\infty} ds e^{-isC} | \hat{\Phi} \rangle \quad (1.8)$$

38 It can be shown that the subalgebra $\widehat{\mathcal{M}}$ is a **Von Neumann Type II₁** algebra. Accord-
39 ingly, a trace map may be defined:

$$\text{tr}(\rho_{\hat{\Phi}} \hat{A}) = \langle \hat{\Phi} | \hat{A} | \hat{\Phi} \rangle \quad (1.9)$$

40 where:

$$|\hat{\Phi}\rangle \equiv |\hat{\Phi}\rangle_{Phys} = \int_{-\infty}^{\infty} ds e^{-isC} |\Psi\rangle_{dS} \otimes \int dq e^{-q/2} |q\rangle \quad (1.10)$$

41 and one can show [6] that:

$$\rho_{\hat{\Phi}} = 2\pi e^{iK\hat{p}} e^q \Delta_{\Phi\Psi} e^{-iK\hat{p}} \quad (1.11)$$

42 Since all cyclic and separating vectors with respect to \mathcal{A} in dS space coincide, we have:

$$\Delta_{\Phi\Psi} = \text{I} \quad (1.12)$$

43 Therefore:

$$\text{tr}(\rho_{\hat{\Phi}} e^{-q}) = 2\pi \text{tr}(\text{I}) = \langle \hat{\Phi} | \hat{\Phi} \rangle = (\hat{\Phi}, \int_{-\infty}^{\infty} ds e^{-isC} \hat{\Phi}) \quad (1.13)$$

44 and thus:

$$\text{tr}(\text{I}) = \frac{1}{2\pi} (\hat{\Phi}, \int_{-\infty}^{\infty} ds e^{-isC} \hat{\Phi}) \quad (1.14)$$

45 Assuming:

$$|\Psi_{dS}\rangle = \frac{1}{\sqrt{Z}} \int dE \sqrt{\rho(E)} e^{-\beta E/2} |E\rangle \quad (1.15)$$

we find:

$$\begin{aligned}\text{tr}(\mathbb{I}) &= \frac{1}{2\pi}(\hat{\Phi}, \int_{-\infty}^{\infty} ds e^{-isC} \hat{\Phi}) \\ &= \frac{1}{Z} \int_{-i\infty}^{i\infty} \frac{d\beta}{2\pi i} \int dq e^{-\beta q} \int dE \rho(E) e^{-\beta E} = 1\end{aligned}\quad (1.16)$$

Hence:

$$|\hat{\Phi}\rangle = \frac{1}{\sqrt{Z}} \int dE e^{-\beta E/2} |E\rangle \otimes |-E\rangle \quad (1.17)$$

and

$$Z = \int_{-i\infty}^{i\infty} \frac{d\beta}{2\pi i} \int dq e^{-\beta q} \int dE \rho(E) e^{-\beta E} \quad (1.18)$$

If we add a constant m to the constraint $\hat{C} = \hat{K} + \hat{q} + m\hat{I}$, then:

$$|\hat{\Phi}\rangle = \frac{1}{\sqrt{Z}} \int_0^\infty dE e^{-\beta(E-m)/2} |E+m\rangle \otimes |-E-m\rangle \quad (1.19)$$

and consequently:

$$Z = \int_{-i\infty}^{i\infty} \frac{d\beta}{2\pi i} \int dq e^{-\beta q} \int dE \rho(E) e^{-\beta E} e^{-\beta m} \quad (1.20)$$

2 Geometric Structure

As seen in 1.17, this behaves as a **ThermoField Double** (TFD) state with continuous energy levels:

$$|\hat{\Phi}\rangle = |TFD\rangle = \frac{1}{\sqrt{Z}} \sum_n e^{-\beta E_n/2} |E_n\rangle \otimes |\Theta E_n\rangle \quad (2.1)$$

Thus, adding a one-dimensional quantum system extends the Hilbert space:

$$\hat{\mathcal{H}} = \mathcal{H}_{\Psi_{dS}}^{GNS} \otimes L^2(\mathcal{R}), \quad |\hat{\Phi}\rangle = |TFD\rangle \quad (2.2)$$

and in the large N limit:

$$\hat{\mathcal{H}}_{\hat{\Phi}}^{GNS} = \mathcal{H}_{\Psi_{dS}}^{GNS} \otimes L^2(\mathcal{R}) \cong \mathcal{H}^{Fock} \otimes L^2(\mathcal{R}) = \hat{\mathcal{H}}^{Fock}, \quad (2.3)$$

$$|\hat{\Phi}\rangle = \frac{1}{\sqrt{Z}} |\tilde{\Psi}_{\mathcal{W}}\rangle \quad (2.4)$$

In this setting, an **Eternal Black Hole** exists in the bulk whose wavefunction corresponds to a generalized **Hartle–Hawking** state:

$$|\widetilde{\Psi}_{\mathcal{W}}\rangle = \int_{\phi_{\mathcal{W}}|_S = \phi_S} D\phi_{\mathcal{W}} Dq D\lambda e^{-I[\phi_{\mathcal{W}}] - I[q] - I_{LM}} \quad (2.5)$$

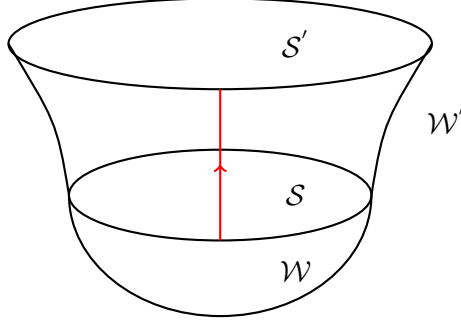
58 with Euclidean actions:

$$I[\phi_{\mathcal{W}}] = \frac{1}{16\pi G_N} \int d^d x \sqrt{g} (R - 2\Lambda), \quad (2.6)$$

$$I[q] = - \oint d\tau q = -\beta_{dS} q, \quad (2.7)$$

$$I_{LM} = \beta_{dS} \lambda (q + H) \quad (2.8)$$

Therefore:



59

$$|\tilde{\Psi}_{\mathcal{W}'}\rangle = \int Dq D\lambda e^{-\beta_{dS} q + \lambda \beta_{dS} (q+H)} |\Psi_{\mathcal{W}}\rangle \quad (2.9)$$

60 As previously discussed, by adding a one-dimensional quantum system, a bipartite structure
 61 arises consisting of two Hilbert spaces (or, in algebraic language, two modules), whose
 62 existence is determined by the geometry. In the S' construction, the manifold q is associated
 63 with one Hilbert space, while the Hamiltonian H corresponds to the other. Therefore, it is
 64 natural that, just like the states $|q\rangle$ and $|E\rangle$, these two modules are canonically conjugate
 65 with respect to energy eigenstates.

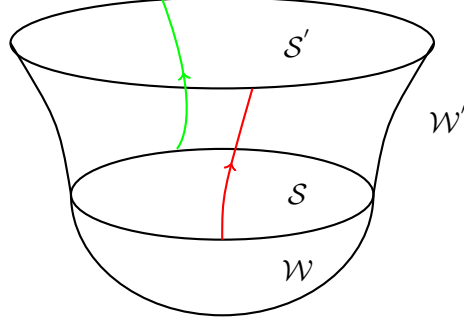
66 An important point is that the parameter q has a geometric representation: a red line
 67 indicates the flow or orientation of the module (in algebraic literature), demonstrating a
 68 preferred direction. Consequently, in the formalism, it appears as a continuous variable.

69 The existence of a state $|\hat{\Phi}\rangle$, which is both cyclic and separating, ensures that q can be
 70 interpreted consistently. Hence, it is advantageous to define all these constructions with
 71 respect to a Fock-like state $\hat{\mathcal{H}}^{Fock}$ in the extended Hilbert space. So: Now, due to the
 72 breaking of reflection symmetry, the partition function cannot be expressed simply as an
 73 inner product (or norm) of two states [7]. In other words, we have:

$$\mathcal{Z} = (\Phi, \Psi) = \langle \Phi | \hat{T} | \Psi \rangle, \quad (2.10)$$

74 where, by definition, the bilinear form is $(\ , \) : \mathcal{H}_{q_1} \times \mathcal{H}_{q_2} \rightarrow \mathbb{C}$, while the inner product,
 75 which corresponds to the norm, is defined as $\langle \ , \ \rangle : \mathcal{H}_{q_1} \times \mathcal{H}_{q_1} \rightarrow \mathbb{C}$. To compute the norm
 76 of the state $|\tilde{\Psi}_{\mathcal{W}'_q}\rangle_q$, we need to consider two states defined on the same Hilbert space. In
 77 other words, we want to evaluate:

$$Z = \langle \tilde{\Psi}_{\mathcal{W}'_q} | \tilde{\Psi}_{\mathcal{W}'_q} \rangle. \quad (2.11)$$



78 Schematically, this can be represented as:

79 Therefore:

$$\begin{aligned}
 Z &= \langle \tilde{\Psi}_{\mathcal{W}'_q} | \tilde{\Psi}_{\mathcal{W}'_q} \rangle \\
 &= \int Dq_1 D\lambda_1 e^{-\beta_{dS}(q_1+m) + \lambda \beta_{dS}(q_1+E+m)} \int Dq_2 D\lambda_2 e^{-\beta_{dS}(q_2+m) + \lambda_2 \beta_{dS}(q_2+E+m)} \langle \Psi_{\mathcal{W}} | \Psi_{\mathcal{W}} \rangle
 \end{aligned} \tag{2.12}$$

80 Given that we need to impose the constraint $q_1 - q_2 = 0$, and noting that the integral over
 81 the fields $\lambda_{1,2}$ diverges, we perform a contour rotation [1, 3]:

$$\lambda_1 \rightarrow -is_1, \quad \lambda_2 \rightarrow +is_2. \tag{2.13}$$

82 Consequently, we have:

$$\begin{aligned}
 Z &= \langle \tilde{\Psi}_{\mathcal{W}'_q} | \tilde{\Psi}_{\mathcal{W}'_q} \rangle \\
 &= \int dq_1 dq_2 e^{-\beta_{dS}(q_1+q_2+2m)} \int ds_1 ds_2 e^{-is_1(q_1+E+m)} e^{+is_2(q_2+E+m)} \langle \Psi_{\mathcal{W}} | \Psi_{\mathcal{W}} \rangle \\
 &= \int dq_1 dq_2 e^{-\beta_{dS}(q_1+q_2+2m)} \delta(q_1 + E + m) \delta(q_2 + E + m) \int dE \rho(E) e^{-\beta_{dS}E} \\
 &= \int dq_1 dq_2 dE \rho(E) e^{-\beta_{dS}(q_1+q_2+E+2m)} \delta(q_1 + E + m) \delta(q_2 + E + m) \\
 &= \int_{-\infty}^{-m} dE \rho(E) e^{\beta_{dS}(E+m)}.
 \end{aligned} \tag{2.14}$$

83 Using $\rho(E) = e^{S_{dS}} e^{\beta_{dS}E}$, we obtain:

$$\begin{aligned}
 Z &= \int_{-\infty}^{-m} dE e^{S_{dS}} e^{\beta_{dS}E} e^{\beta_{dS}(E+m)} \\
 &= e^{S_{dS}} e^{\beta_{dS}m} \int_{-\infty}^{-m} dE e^{2\beta_{dS}E} \\
 &= e^{S_{dS}} e^{\beta_{dS}m} \frac{1}{2\beta_{dS}} (e^{-2\beta_{dS}m} - 0) \\
 &= \frac{e^{S_{dS}-\beta_{dS}m}}{2\beta_{dS}} = \frac{e^{S_{dS}-\beta_{dS}m}}{4\pi}.
 \end{aligned} \tag{2.15}$$

Hence, we can write:

$$Z_{count} = \langle \tilde{\Psi}_{\mathcal{W}'_q} | \tilde{\Psi}_{\mathcal{W}'_q} \rangle. \quad (2.16)$$

This result is consistent with [8], implying that the state $|\tilde{\Psi}_{\mathcal{W}'_q}\rangle$ can be identified with the $|\widetilde{HH}\rangle$ state, i.e., the Hartle-Hawking state in the presence of an observer.

Moreover, one can establish an equivalence between the algebraic and geometric-gravitational perspectives as discussed in Eq. 2.4:

$$\hat{\mathcal{H}}_{\hat{\Phi}_q}^{GNS} \cong \hat{\mathcal{H}}_q^{Fock}, \quad |\hat{\Phi}\rangle_q = \frac{1}{\sqrt{Z}} |\tilde{\Psi}_{\mathcal{W}'_q}\rangle_q. \quad (2.17)$$

From the first part, we also observed that

$$\text{tr}(\mathbf{I}) = \langle \hat{\Phi}_q | \hat{\Phi}_q \rangle = \frac{1}{Z} \int_{-i\infty}^{i\infty} \frac{d\beta}{2\pi i} \int dq e^{-\beta q} \int dE \rho(E) e^{-\beta E} = 1, \quad (2.18)$$

and equivalently, from the geometric-gravitational structure, we have:

$$Z_{count} = \langle \tilde{\Psi}_{\mathcal{W}'_q} | \tilde{\Psi}_{\mathcal{W}'_q} \rangle. \quad (2.19)$$

Therefore, we conclude:

$$Z_{count} = \langle \tilde{\Psi}_{\mathcal{W}'_q} | \tilde{\Psi}_{\mathcal{W}'_q} \rangle = \int_{-i\infty}^{i\infty} \frac{d\beta}{2\pi i} \int dq e^{-\beta q} \int dE \rho(E) e^{-\beta E} e^{-\beta m}. \quad (2.20)$$

3 Partition Function

Now, considering that by adding a one-dimensional quantum system to the original system in the algebraic framework, one can construct a **ThermoField Double** state, and in the gravitational sector, a generalized **Hartle-Hawking** state can be defined, and furthermore, that this generalized state exhibits reflection symmetry breaking, we can compute the partition function of a sphere. As discussed in the previous section, we have:

$$Z = (\Phi, \Psi). \quad (3.1)$$

Thus, we can write:

$$Z = (\Psi_{\mathcal{W}}, \tilde{\Psi}_{\mathcal{W}'_q}) = \int Dq D\lambda e^{-\beta_{dS}(q+m) + \lambda \beta_{dS}(q+E+m)} \langle \Psi_{\mathcal{W}} | \Psi_{\mathcal{W}} \rangle. \quad (3.2)$$

Performing a contour rotation:

$$\lambda \beta_{dS} \rightarrow -is, \quad (3.3)$$

we obtain:

$$\begin{aligned}
Z &= (\Psi_{\mathcal{W}}, \tilde{\Psi}_{\mathcal{W}'_q}) \\
&= -\frac{i}{2\pi} \int_{-\infty}^{+\infty} ds \int dq \int dE \rho(E) e^{-\beta_{dS}(q+m+E)} e^{-is(q+m+E)} \\
&= -\frac{i}{2\pi} \int_{-\infty}^{+\infty} ds \int dq \int dE \rho(E) e^{-(\beta_{dS}+is)(q+m+E)} \\
&= -\int_{-i\infty+2\pi}^{+i\infty+2\pi} \frac{d\beta}{2\pi} \int dq \int dE \rho(E) e^{-\beta(q+m+E)} \\
&= -i \int_{-i\infty+2\pi}^{+i\infty+2\pi} \frac{d\beta}{2\pi i} \int dq e^{-\beta q} \int dE \rho(E) e^{-\beta E} e^{-\beta m} \\
&= -i Z_{count}.
\end{aligned} \tag{3.4}$$

Therefore, we have:

$$\boxed{Z[S^D]Z[\text{Obs}] = -i Z_{count} = -i e^{S_{dS} - \beta_{dS} m}.} \tag{3.5}$$

This result is consistent with [3], and indicates that, upon introducing an observer, the partition function of the sphere S^D , which previously carried an imaginary phase, can be interpreted as a state-counting partition function.

4 Conclusion

As is well-known, by adding a one-dimensional quantum system to the original system, one can construct a **Von Neumann Type II** algebra by considering a suitable subalgebra. In this case, a trace map tr can be defined, which in turn allows the definition of an associated entropy, as discussed in [2, 6].

Our motivation, therefore, has been to investigate whether adding a one-dimensional quantum system within the algebraic framework, which leads to a **Von Neumann Type II** algebra, has any effect in the non-algebraic or gravitational perspective.

Initially, we relied on a duality that essentially holds in the $N \rightarrow \infty$ limit, thereby establishing a correspondence between the algebraic and geometric structures. As is known, adding a one-dimensional quantum system modifies the algebra. Using the duality at our disposal, we then analyzed the structural changes in the geometry.

We found that in the algebraic framework, a **ThermoField Double** state can be constructed. Since we are in the $N \rightarrow \infty$ limit, a generalized **Hartle-Hawking** state can also be defined. The norm of this generalized state, as discussed in [8], corresponds to a state-counting partition function. This allows the proper definition of a counting partition function, thereby resolving the issue discussed in [3].

Furthermore, this result is consistent with [9], with an intriguing observation that black holes appear in both approaches. In [9], the black hole considered is a charged **Reissner-Nordstrom** solution, while in our approach, an **Eternal** black hole arises. It might be interesting to explore a possible connection between these two scenarios.

126 Finally, this approach enables a well-defined interpretation of the partition function of the
127 sphere S^D , which previously carried an imaginary phase and lacked a meaningful state-
128 counting interpretation, by including the presence of an observer.

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