# Problem Set 2

Course: *Mathematical Physics II (Winter 2022)*

Due Date: *5:00 PM - Thursday, 12th of Esfand 1400*

### Analytical Problems

**Problem 1.** Find the eigenvalues and corresponding normalized eigenvectors of the following matrix:

$$A = \begin{pmatrix} 0 & 0 & -1+i & -1-i \\ 0 & 0 & -1+i & 1+i \\ -1-i & -1-i & 0 & 0 \\ -1+i & 1-i & 0 & 0 \end{pmatrix} \tag{1}$$

**Problem 2.** Find the unitary matrices that diagonalize the following hermitian matrices:

$$A = \begin{pmatrix} 2 & -1+i \\ -1-i & -1 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 2 & 0 & i \\ 0 & -1 & -i \\ -i & i & 0 \end{pmatrix}$$

**Problem 3.** Let $|a_1\rangle \equiv \mathbf{a}_1 = (1, 1, -1)$ and $|a_2\rangle \equiv \mathbf{a}_2 = (-2, 1, -1)$

1. Construct (in the form of a matrix) the projection operators $\mathbf{P}_1$ and $\mathbf{P}_2$ that project onto the directions of $|a_1\rangle$ and $|a_2\rangle$, respectively. Verify that they are indeed projection operators
2. Construct (in the form of a matrix) the operator $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2$ and verify directly that it is a projection operator
3. Let $\mathbf{P}$ act on an arbitrary vector $(x, y, z)$. What is the dot product of the resulting vector with the vector $\mathbf{a}_1 \times \mathbf{a}_2$? Is that what you expect?

**Problem 4.** If $\sigma_x, \sigma_y$ and $\sigma_z$ are Pauli's matrices What is the result of the $e^{i\theta\sigma_x}\sigma_z e^{-i\theta\sigma_x}$

**Problem 5.** Consider matrix M and its corresponding eigenvalues $\lambda_1, \lambda_2$ and $\lambda_3$

$$M = \begin{pmatrix} cos^2\alpha & 0 & \beta cos\alpha \\ 0 & -1 & 0 \\ \beta^* cos^2\alpha & \beta^* & sin^2\alpha \end{pmatrix} \tag{2}$$

in which $\alpha$ is a real number and $\beta$ is a imaginary number. Find $\lambda_1^3 + \lambda_2^3 + \lambda_4^3$.

**Problem 6.** Show that the quadratic surface

$$5x^2 + 11y^2 + 5z^2 - 10yz + 2xz - 10xy = 4 \tag{3}$$

is an ellipsoid.

**Problem 7.** Point $\mathbf{a} = (a_1, a_2, ..., a_2) \in \mathbb{R}$ is maximum (minimum) of function

$$f(x_1, x_2, ..., x_n) \equiv f(\mathbf{r}) \tag{4}$$

if $\nabla|_{r=a} \equiv (\dfrac{\partial f}{\partial x_1}, ..., \dfrac{\partial f}{\partial x_n}) = 0$ and if for small values of $\delta = \mathbf{r} - \mathbf{a}$ differential of $f(\mathbf{r}) - f(\mathbf{a})$ is negative (positive). Show that $f(\mathbf{r})$ only has a minimum (maximum) if the n-by-n matrix $(D)_{ij} \equiv (\frac{\partial^2 f}{\partial x_i \partial x_j})_{\mathbf{a}}$ has non-negative (non-positive) eigenvalues.