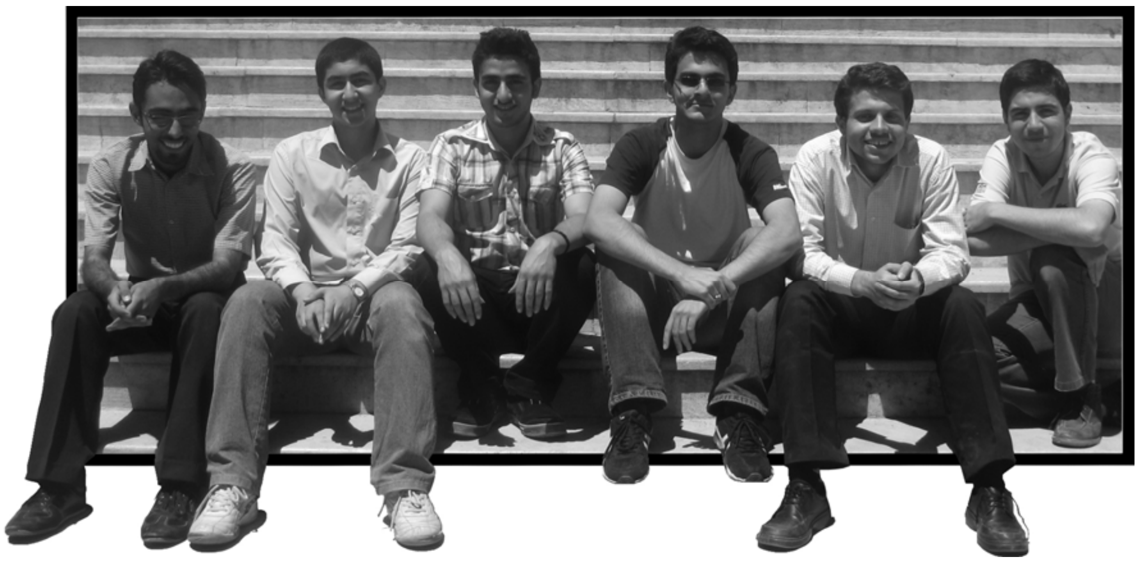

Iranian Team Members (from left to right):

- Saeed Tayyebi
- Javad Abedi
- Morteza Ashrafi-jou
- Sina Rezaei
- Hessamoddin Rajab-Zadeh
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This booklet is prepared by Meysam Aghighi and Morteza Saghafian.

27th

Iranian Mathematical Olympiad

2009-2010

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Problems

First Round

1. Let $p(x)$ be a polynomial of degree 2 such that $|p(x)| \leq 1$ holds for $x \in \{-1,0,1\}$. Show that for every $x \in [-1,1]$
$$|p(x)| \leq \frac{5}{4}$$
2. (Meysam Aghighi) We have a 50×50 garden which is divided into 1×1 pieces and in some pieces an apple, orange or peach tree is planted. We know that there is at least one apple tree adjacent to every orange tree. There are at least one apple tree and one orange tree adjacent to every peach tree and at least one apple tree, one orange tree and one peach tree adjacent to every empty piece (a piece without any tree). (We say that two pieces are adjacent if they have a common side.)
Show that the number of empty pieces is not greater than 1000.

3. (Meysam Aghighi) Let the inner bisector of the angle A of triangle ABC intersects BC and the circumcircle of ABC in D and M respectively. We draw a line through D so that it intersects the rays MB and MC (with M the startpoint) in P and Q. Show that $\widehat{PAQ} \geq \hat{A}$.
4. (Erfan Salavati) There are $n(n+2)$ soldiers standing in n equal columns next to each other such that the distance between every two of them is one step. When commander commands each soldier either doesn't move or takes a step in one of the four directions! After the command, the soldiers are standing in $n+2$ equal columns, such that the first and last rows are omitted and two columns are added in left and right. Prove that n is even.

5. (Mohsen Jamali) Natural numbers $a_1 < a_2 < \cdots < a_n$ have this property that for every distinct i and j, a_i is divisible by $a_j - a_i$. Prove that for every $i < j$,

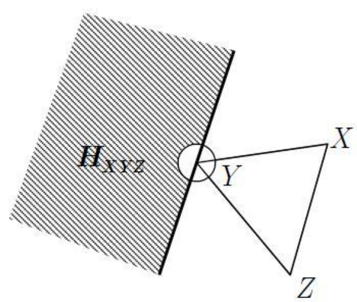
$$ia_j \leq ja_i$$

6. (Omid Naghshineh) There are 11 men sitting around a circular table with equal distances and 11 cards with numbers $1, 2, \dots, 11$ on them are dealt among them. It is possible that one has no cards and the other has more than one. In each step one can give one of his cards to his adjacent individual if the card number i has the following property: before and after this stage, the places of the cards with numbers $i-1$, i and $i+1$ are not the vertices of an acute-angled triangle. (card 0 is the same as card 11, and card 12 is the same as card 1)

   At the beginning the cards 1 to 11 are dealt among them counter-clockwise. (everyone has exactly one card) Prove that there will never be a man who has all the cards.

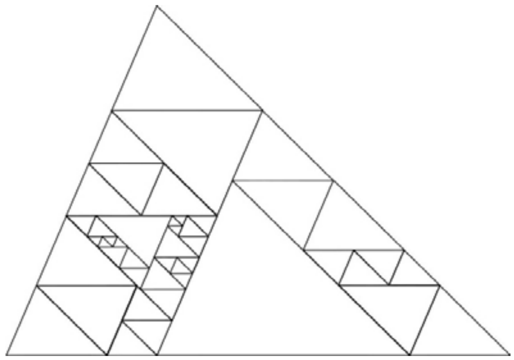
Second Round

1. (Ali Khezeli) Let $n > 2$ and $A_1, A_2, \dots, A_n$ be n points on the plane that no three of them are collinear.
   a. Let $M_1, M_2, \dots, M_n$ be n points on the segments $A_1A_2, A_2A_3, \dots, A_nA_1$ respectively. Show that if $B_1, B_2, \dots, B_n$ be n points in the triangles $M_nA_1M_1$, $M_1A_2M_2, \dots, M_{n-1}A_nM_n$ respectively, then the following holds
   $$|B_1B_2| + |B_2B_3| + \cdots + |B_nB_1| \leq |A_1A_2| + |A_2A_3| + \cdots + |A_nA_1|$$
   Where $|XY|$ means the length of the segment XY.
   b. We define H_{XYZ} the half-plane with the outer bisector of angle $\widehat{XYZ}$ as its boundary which does not contain the inner bisector of $\widehat{XYZ}$.
   Show that if $C_1, C_2, \dots, C_n$ be n points in the half-planes $H_{A_nA_1A_2}, H_{A_1A_2A_3}, \dots, H_{A_{n-1}A_nA_1}$ respectively, then,
   $$|A_1A_2| + |A_2A_3| + |A_nA_1| \leq |C_1C_2| + |C_2C_3| + \cdots + |C_nC_1|$$
   Suggestion: for part (b), first solve the problem assuming that C_i is on the outer bisector of the corresponding angle.



2. (Mohsen Jamali) We call the permutation π of $\{1,2,\dots,n\}$ *consistent* if the set $\{\pi(k) - k | k = 1,2,\dots,n\}$ has 2 members. Prove that the total number of *consistent* permutations is $\sigma(n) - \tau(n)$, where $\sigma(n)$ is the sum of the positive divisors of n and $\tau(n)$ is the number of positive divisors of n.

3. We have partitioned a triangle into similar triangles with itself, such that every side of a small triangle is parallel to one of the sides of the main triangle. In



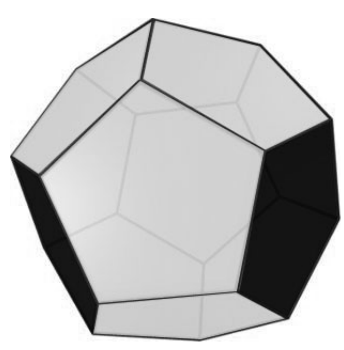
this case every small triangle is derived by a homothety (with a positive or negative ratio) from the main triangle. Prove that the sum of all of these ratios is 1.

4. (Amir Ja'fari) Do there exist two functions $f, g: \mathbb{R} \to \mathbb{R}$ such that for all $x \neq y$ the following inequality holds?
$$|f(x) - f(y)| + |g(x) - g(y)| > 1$$

5. Consider a spherical ball on a plane and a point marked on it. We want to roll the ball on a closed polygon of the plane to bring the mark to the top of the ball while the ball is on its first place. Note that the ball must not roll in its place (it means rolling without moving on the plane). Prove that this is possible.

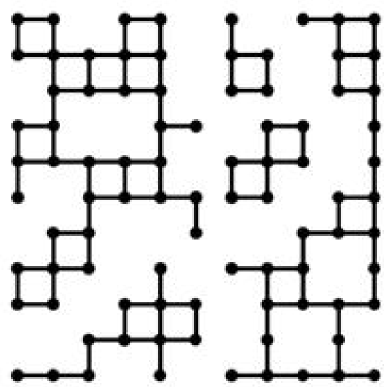
6. Let m, n be two relatively prime integers. Prove that the following equation has an infinite number of solutions:
$$x^m t^n + y^m s^n = v^m r^n$$

7. P is a circumscribed polyhedral. Its faces are colored with black and white such that there isn't any pair of black faces with an edge in common.



Prove that sum of the black areas is not greater than sum of the white areas.

8. (Omid Naghshineh) Consider that we have omitted some of the points of a 2d-lattice (same as $\mathbb{Z}^2$). Look at these points as a graph, two points (vertices) are connected if they are the same in one of their coordinates and differ 1 in another. Each connected component is called a *cluster*. Suppose that for all $n \in \mathbb{N}$, the number of omitted points inside the horizontal square with side length $2n + 1$ and the origin as its center, be less than $n/2$. Prove that the non-omitted points contain exactly one infinite *cluster*.



Third Round

1. (Naser Talebizadeh) Prove that for every natural number m, there exists a natural number N such that for every natural number b that $2 \leq b \leq 1389$, the sum of digits of N in base b, is more than m.
2. (Sepehr Ghazinezami) In triangle ABC, O is the circumcenter and H is the orthocenter. M and N are the midpoints of BH and CH respectively and BB' is a diameter of the circumcircle. If $HONM$ be an inscribed quadrilateral, prove that
$$B'N = \frac{1}{2}AC$$
3. (Morteza Saghafian) There is a $m \times n$ board divided into mn unit squares, and we have drawn one of the two diagonals of each unit square. Prove that there is a path using only these diagonals that either connects the upper side of the board to the lower side or connects the left side of it to the right side.
4. (Morteza Saghafian) We have drawn a polygon with 2n sides without picking up the pencil from the paper. If we name the sides from 1 to $2n$ with the order of drawing them, then the odd sides are all vertical and are drawn bottom-up. Prove that this polygon intersects itself.
5. (Davood Vakili) In the isosceles triangle ABC, we have $AB = AC$ and $BC > AB$. D and M are the midpoints of BC and AB respectively. X is a point that $BX \perp AC$ and $XD \parallel AB$. H is the intersection of BX and AD. If P be the intersection of DX with the circumcircle of AHX (not X), prove that the tangent line in A to the circumcircle of AMP is parallel to BC.
6. (Ali Khezeli) We call a sequence $a_0, \dots, a_{1389}$ of real numbers *concave* if for every $0 < i < 1389$ we have $a_i \geq \frac{a_{i-1}+a_{i+1}}{2}$. Find the maximum number c such that for every *concave* sequence of nonnegative numbers, we have
$$\sum_{i=0}^{1389} i a_i^2 \geq c \sum_{i=0}^{1389} a_i^2$$
7. (Mehdi E'tesami) M is an arbitrary point on the side BC of the triangle ABC. ω is a circle tangent to the segments AB and BM in T and K respectively, also is tangent to the circumcircle of AMC in P. If $TK \parallel AM$, prove that the circumcircles of APT and KPC are tangent to each other.

8. (Mir Omid Hajimirsadeghi) S and T are two trees without having a vertex of degree 2. Each edge of them has a positive number named the *length* of the edge. The *distance* between two vertices is the sum of the *length* of the edges of the path between them. We call the vertices with degree 1, *leaf*. f is an injective and surjective function from the set of *leaves* of S to the set of *leaves* of T with this property that for every two *leaves* u and v in S, the *distance* between u and v in S is equal to the *distance* between $f(u)$ and $f(v)$ in T. Prove that there exists an injective and surjective function g from the set of vertices of S to the set of vertices of T such that for every two vertices u and v in S, the *distance* between u and v in S, is equal to the *distance* between $f(u)$ and $f(v)$ in T.
9. (Mohammad Ja'fari) Find all increasing functions $f: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$, such that for every $x, y \in \mathbb{R}^+ \cup \{0\}$ we have
$$f\left(\frac{x + f(x)}{2} + y\right) = 2x - f(x) + f(f(y))$$
(note that f is not necessarily strictly increasing.)
10. (Ali Khezeli & Morteza Saghafian) Find all polynomials $P(x, y)$ with real coefficients such that for every $a, b, c \in \mathbb{R}$,
$$P(ab, c^2 + 1) + P(bc, a^2 + 1) + P(ca, b^2 + 1) = 0$$
11. (Ehsan Azarmsa) $f: \mathbb{N} \to \mathbb{N}$ is an increasing function and n is a natural number. Suppose that there exist prime numbers $p_1, p_2, \dots, p_n$ and natural numbers $s_1, s_2, \dots, s_n$ such that for every $1 \le i \le n$ the set $\{f(p_i r + s_i) | r = 1,2, \dots\}$ be an infinite arithmetic progression. Prove that there exists natural number a such that
$$f(a + 1), f(a + 2), \dots, f(a + n)$$
is an arithmetic progression.
12. (Davood Vakili) Two circles ω_1 and ω_2 intersect at P and K. XY is the common tangent of them near to P such that X is on ω_1 and Y is on ω_2. XP intersects ω_2 for the second time at C, and YP intersects ω_1 for the second time at B. A is the intersection of BX and CY. If Q be the second intersection point of circumcircle ABC and circumcircle AXY, prove that
$$\widehat{QXA} = \widehat{QKP}$$

Solutions

First Round

1. Notice that
$$p(x) = ax^2 + bx + c = \frac{x(1+x)}{2}p(+1) - \frac{x(1-x)}{2}p(-1) + (1-x^2)p(0)$$
If $0 \leq x \leq +1$, we see that
$$|ax^2 + bx + c| \leq +\frac{x(1+x)}{2} + \frac{x(1-x)}{2} + (1-x^2) = \frac{5}{4} - \left(\frac{1}{2} - x\right)^2 \leq \frac{5}{4}$$
If $-1 \leq x \leq 0$, we see that
$$|ax^2 + bx + c| \leq -\frac{x(1+x)}{2} - \frac{x(1-x)}{2} + (1-x^2) = \frac{5}{4} - \left(\frac{1}{2} + x\right)^2 \leq \frac{5}{4}$$
This proves that $|p(x)| \leq \frac{5}{4}$ for all $x \in [-1, +1]$. The conditions for equality are
   a. $p(x) = \pm(x^2 - x - 1)$ at $x = +\frac{1}{2}$, and
   b. $p(x) = \pm(x^2 + x - 1)$ at $x = -\frac{1}{2}$.
2. Let a_1, a_2, a_3, a_4 be the total number of apple, orange and peach trees, and the total number of empty pieces respectively. We now want to count the total number of adjacencies of apple trees to an orange tree, a peach tree, or an empty piece. We know that these adjacencies are at most $4a_1$, and since every orange tree, every peach tree, and every empty piece has at least one adjacent apple tree; so the total number of adjacencies are at least $a_2 + a_3 + a_4$, hence we have
$$4a_1 \geq a_2 + a_3 + a_4$$
Now we do the same for counting the total number of adjacencies of orange trees to a peach tree or an empty space. Note that these adjacencies are at most $3a_2$, because every orange tree must have an adjacent apple tree, and it can have 3 adjacent peach tree or empty piece. On the other hand since every peach tree and every empty piece has an adjacent orange tree, so the total number of adjacencies are at least $a_3 + a_4$, so we have
$$3a_2 \geq a_3 + a_4$$
Similarly, we have the following inequality
$$2a_3 \geq a_4$$

Now, noting that $a_1 + a_2 + a_3 + a_4 = 50 \times 50 = 2500$, and using the above inequalities, we have

$$2a_3 \geq a_4 \Rightarrow a_3 \geq \frac{a_4}{2}$$

$$3a_2 \geq a_3 + a_4 \geq \frac{a_4}{2} + a_4 = \frac{3}{2}a_4 \Rightarrow a_2 \geq \frac{a_4}{2}$$

$$4a_1 \geq a_2 + a_3 + a_4 \geq \frac{a_4}{2} + \frac{a_4}{2} + a_4 = 2a_4 \Rightarrow a_1 \geq \frac{a_4}{2}$$

$$2500 = a_1 + a_2 + a_3 + a_4 \geq \frac{a_4}{2} + \frac{a_4}{2} + \frac{a_4}{2} + a_4 = \frac{5}{2}a_4 \Rightarrow a_4 \leq 1000$$

So the total number of empty pieces are not greater than 1000.

3. Since AM is the bisector, we have

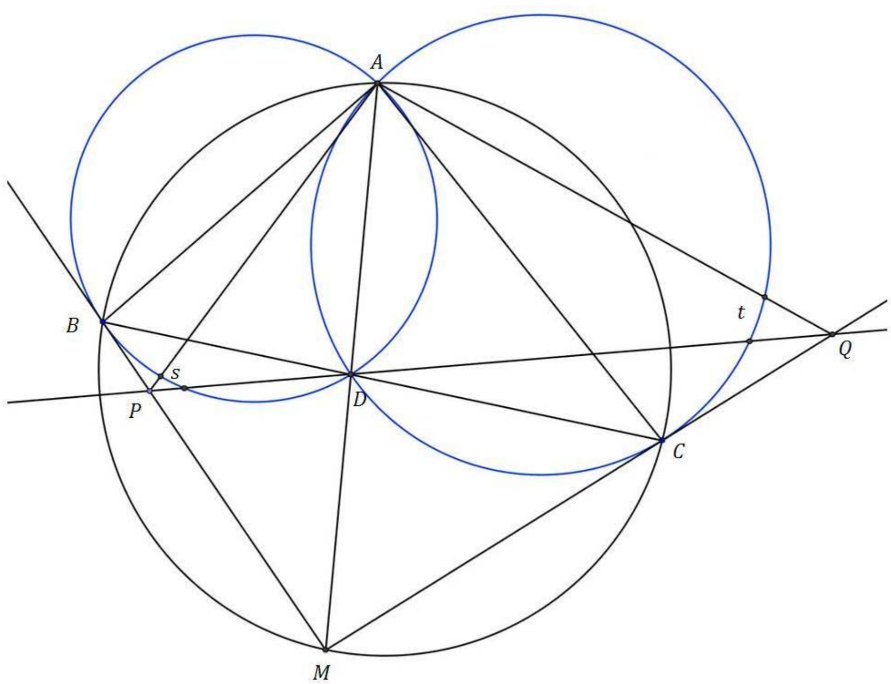
$$\angle DBM = \angle BAM = \frac{\angle A}{2}$$

So the circumcircle of ABD is tangent to BM in B. Hence if the measure of the arc AD in this circle be k, we have

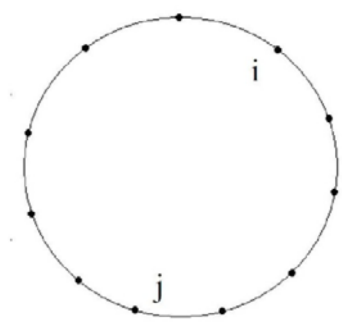
$$\angle APQ = \frac{k - s}{2} \leq \frac{k}{2} = \angle ABD = \angle B$$

Similarly we have $\angle AQP \leq \angle C$. Thus

$$\angle PAQ = 180^{\circ} - \angle APQ - \angle AQP \geq 180^{\circ} - \angle B - \angle C = \angle A$$



4. Suppose that it can be done for n. We claim that it can be done for $n-2$, and since it can't be done for 1, n must be even. Let up, down, left and right be the four directions. Note that if n columns have moved to $n+2$ columns, then the first row must've moved down, the last row must've moved up, the leftmost column must've moved left, and the rightmost column must've moved right. Now, consider the remaining soldiers, they are $n-2$ columns with n soldiers in each column, that have moved in n columns with $n-2$ soldiers in each. Thus, if it can be done for n, then n must be even.
5. For all distinct i, j that $i < j$, we have $a_j - a_i | a_j$, and
$$a_j > a_j - a_1 > a_j - a_2 > \cdots > a_j - a_i$$
On the other hand all of the above terms are divisors of a_j, so if the divisors of a_j in decreasing order be $a_j > b_1 > b_2 > \cdots > b_k$, then $a_j - a_i \leq b_i$. Since we know that $b_i \leq \frac{a_j}{i+1}$, we have
$$a_j - a_i \leq \frac{a_j}{i+1} \Rightarrow (i+1)\left(a_j - a_i\right) \leq a_j \Rightarrow i \times a_j \leq (i+1) \times a_i \leq j \times a_i$$
And the problem is solved.
6. First divide the table into 11 equal arcs. Now if the cards i, j be on the points A, B on the table, we define the distance between these two cards, the number of arcs between A, B on the table. (the smaller one) For example, the distance between i, j is 5 in the following figure:



Now, after each step we sum the distances between every two cards with consecutive numbers. Easily it can be proved that after each step this sum either remains invariant or varies by 2. Because if the place of card i be between $i-1$ and $i+1$, this sum doesn't change, otherwise changes by 2. So the parity of this sum remains invariant. At first it is odd $(= 11)$, and if they are all in one place this sum would be even, so it is impossible.

Second Round

1.

a. **Lemma.** *If P be a point in the triangle ABC, then: $PB + PC \leq AB + AC$.*
PROOF. We extend CP to meet AB in D. We have: (by triangle inequality)
$$PB + PC \leq BD + PD + PC = BD + CD \leq BD + DA + AC = AB + AC$$
Now we return to the problem, from the triangle inequality we have:

$$\begin{aligned} B_1B_2 + B_2B_3 + \cdots + B_nB_1 &\leq (B_1M_1 + B_2M_1) + (B_2M_2 + B_3M_2) + \cdots + (B_nM_n + B_1M_n) \\ &= (B_1M_1 + B_1M_n) + (B_2M_1 + B_2M_2) + \cdots + (B_nM_{n-1} + B_nM_n) \\ \text{(according to lemma)} \quad &\leq (A_1M_1 + A_1M_n) + (A_2M_1 + A_2M_2) + \cdots + (A_nM_{n-1} + A_nM_n) \\ &= A_1A_2 + A_2A_3 + \cdots + A_nA_1 \end{aligned}$$

b. Suppose that for all $1 \leq k \leq n$, we have $\overrightarrow{A_kA_{k+1}} = L_k\overrightarrow{u_k}$ in which L_k is the length of A_kA_{k+1} and $\overrightarrow{u_k}$ is the unit vector codirectional with $\overrightarrow{A_kA_{k+1}}$. In this case the vector $\overrightarrow{u_k} - \overrightarrow{u_{k-1}}$ is codirectional with the inner bisector of the angle $\angle A_{k-1}\, A_kA_{k+1}$.
Also suppose that $\overrightarrow{\omega_k} = \overrightarrow{A_kC_k}$, hence from the definition of C_k we can say that: $(\star)\ \overrightarrow{\omega_k} \cdot (\overrightarrow{u_k} - \overrightarrow{u_{k-1}}) \leq 0$, plus we have: (consider $C_0, \overrightarrow{\omega_0}, \overrightarrow{u_0}$ the same as $C_n, \overrightarrow{\omega_n}, \overrightarrow{u_n}$)
$$\overrightarrow{C_{k-1}C_k} = \overrightarrow{\omega_k} + L_k\overrightarrow{u_{k-1}} - \overrightarrow{\omega_{k-1}}$$
Note that $|\overrightarrow{u_{k-1}}| = 1$, we can say:
$$\begin{aligned} \left|\overrightarrow{C_{k-1}C_\iota}\right| \geq \overrightarrow{C_{k-1}C_k} \cdot \overrightarrow{u_{k-1}} &= (\overrightarrow{\omega_k} + L_k\overrightarrow{u_{k-1}} - \overrightarrow{\omega_{k-1}}) \cdot \overrightarrow{u_{k-1}} \\ &= \overrightarrow{\omega_k} \cdot \overrightarrow{u_{k-1}} + L_k - \overrightarrow{\omega_{k-1}} \cdot \overrightarrow{u_{k-1}} \end{aligned}$$
Hence we have:
$$\sum_{k=1}^{n} \left|\overrightarrow{C_{k-1}C_k}\right| \geq \sum_{k=1}^{n} L_k + \sum_{k=1}^{n} \overrightarrow{\omega_k} \cdot \overrightarrow{u_{k-1}} - \sum_{k=1}^{n} \overrightarrow{\omega_{k-1}} \cdot \overrightarrow{u_{k-1}}$$
From the other hand, with respect to $(\star)$, we have:
$$\begin{aligned} \sum_{k=1}^{n} \overrightarrow{\omega_k} \cdot (\overrightarrow{u_k} - \overrightarrow{u_{k-1}}) \leq 0 \Rightarrow &\sum_{k=1}^{n} \overrightarrow{\omega_k} \cdot \overrightarrow{u_{k-1}} \\ &\geq \sum_{k=1}^{n} \overrightarrow{\omega_k} \cdot \overrightarrow{u_k} = \sum_{k=1}^{n} \overrightarrow{\omega_{k-1}} \cdot \overrightarrow{u_{k-1}} \end{aligned}$$
So we deduce that:

$$\sum_{k=1}^{n} \left|\overrightarrow{C_{k-1}C_k}\right| \geq \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \left|\overrightarrow{A_k A_{k+1}}\right|$$

2. Since $\sum_{k=1}^{n}(\pi(k) - k) = 0$, so from the two members of $\{\pi(k) - k | k = 1,2, \dots, n\}$, one should be positive and the other one negative. Let a be the positive and $-b$ be the negative members. Also let $A = \{k | \pi(k) - k = a\}$ and $B = \{k | \pi(k) - k = -b\}$. First note that if $i \in A$ then $i + a + b \in A$. Because if $i \in A$ and $i + a + b \leq n$, then $\pi(i) = i + a$ and hence $\pi(i + a + b) \neq i + a$, and so $\pi(i + a + b) - (i + a + b) = a$, it means that $i + a + b \in A$. Similarly, we can say that if $j \in B$ (and $j - (a + b) \geq 1$) then $j - (a + b) \in B$.
   Also, note that for every $1 \leq i \leq b$, we have $i \in A$. So considering the above paragraph, if for $1 \leq i \leq n$, the remainder of i upon division by $a + b$, be less than or equal to b, then $i \in A$. Similarly for $n - a < i \leq n$, we have $i \in B$. Thus from every $a + b$ consecutive numbers, there are at least a of them in B and hence if the remainder of the division of j by $a + b$, be more than b, then $j \in B$. So easily we deduce that $d = a + b | n$. For each divisor d of n, there are $d - 1$ different pairs of (a, b) whose sum is d. So the total number of states is equal to:
   $$\sum_{d|n}(d - 1) = \sum_{d|n} d - \sum_{d|n} 1 = \sigma(n) - \tau(n)$$
3. Suppose that the bottom side of the main triangle is horizontal, hence sum of the ratios is sum of the ratios of the horizontal sides of the small triangles (with respect to the sign of the ratio) to the horizontal side of the main triangle, hence: (let h_δ be the length of the horizontal side of the triangle δ, and let Δ be the main triangle)
   $$\begin{aligned} &\mathit{sum\ of\ the\ ratios} \\ &= \sum_{\delta \in Small\ Triangles} \frac{h_\delta}{h_\Delta} \times sign\ of\ the\ Homothety \\ &= \frac{1}{h_\Delta} \times \sum_{\delta \in Small\ Triangles} (h_\delta \times sign\ of\ the\ Homothety) \end{aligned}$$
   But in the above summation, every horizontal segment (that contains no smaller horizontal segment) once appears with positive sign for its above

triangle and once appears with negative sign for its below triangle, except the segments of h_Δ, and these segments appear with the positive sign; so

$$\sum_{\delta \in Small\ Triangles} (h_\delta \times sign\ of\ the\ Homothety) = h_\Delta$$

and so sum of the ratios with respect to their signs is 1.

4. For all $x \neq y$ we have:

$$\big(f(x)-f(y)\big)^2 + \big(g(x)-g(y)\big)^2 \geq \frac{1}{2}(|f(x)-f(y)| + |g(x)-g(y)|)^2 > \frac{1}{2}$$

For every x, consider the point $(f(x), g(x))$ in the $\mathbb{R}^2$ plane. The above inequality shows that the distance between every two of these points is more than $\frac{1}{\sqrt{2}}$. For every point $(f(x), g(x))$, consider a circle with center at this point and a radius of $\frac{1}{2\sqrt{2}}$, so not two of these circles intersect. On the other hand the number of these points are uncountable, so we have an uncountable number of circles in the plane that no two of them intersect and this is impossible; because there is a point with rational coordinates in every one of the circles, hence we would have an uncountable number of points with rational coordinates and since we know that the total number of the points with rational coordinates is countable, this is a contradiction; so no two functions with this property exist.

5. Let r be the radius of the ball and O be its first position on the plane. First, we roll the ball (sphere) on its great circle that passes through the marked point and the top point of the ball, until the mark be on top of the ball. Let A be the current position of the ball on the plane. It is trivial that $OA < 2\pi r$. So there exists a point B on the plane so that $AB = OB = 2\pi r$. Now we roll the ball from A to B on the segment AB, and since $AB = 2\pi r$, the mark will be at the top of the ball. Then we roll the ball from B to O on BO and the mark is at the top of the ball while it is on its first position.

6. First note that if (x, t, y, s, v, r) be a solution, then

$$(x.a^{kn}, t.a^{km}, y.a^{kn}, s.a^{km}, v.a^{kn}, r.a^{km})$$

is another solution. So it is enough to find a primary solution. We know that

$$2+3=5 \Rightarrow 2^{mn+1}\times 3^{mn}\times 5^{mn} + 2^{mn}\times 3^{mn+1}\times 5^{mn}$$
$$= 2^{mn}\times 3^{mn}\times 5^{mn+1}$$

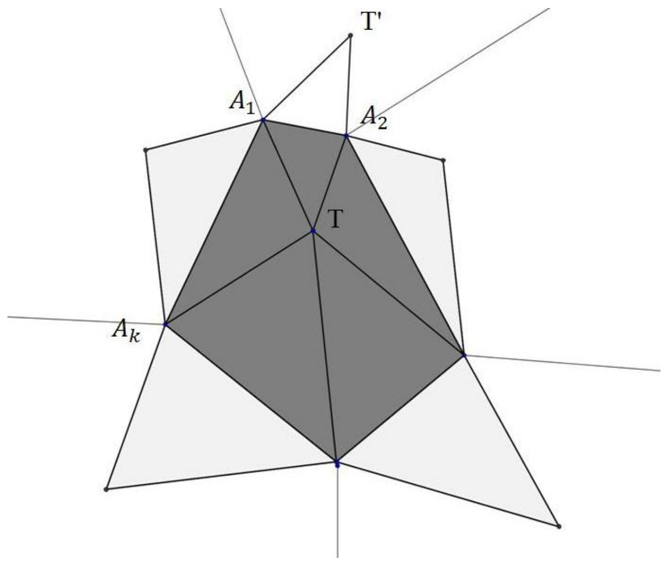
Now since $\gcd(m,n)=1$, hence there exists $1 \leq i < n$ such that $mi \equiv -1 (mod\ n)$. So $mi+1 = nd, d \in \mathbb{N}$, and hence $mn+1 = m(n-i)+mi+1 = m(n-i)+nd$. So we have

$$\left(2^{n-i}\times 3^n\times 5^n\right)^m\times(2^d)^n+\left(2^n\times 3^{n-i}\times 5^n\right)^m\times(3^d)^n$$
$$=\left(2^n\times 3^n\times 5^{n-i}\right)^m\times(5^d)^n$$

Thus, a group of infinite solutions for the mentioned equation is:

$$(x,t,y,s,v,r) = (2^{n-i}\times 3^n\times 5^n\times a^{kn}, 2^d\times a^{km}, 2^n\times 3^{n-i}\times 5^n\times a^{kn}, 3^d \times a^{km}, 2^n\times 3^n\times 5^{n-i}\times a^{kn}, 5^d\times a^{km})$$

7. For every face of the polyhedral, consider the point of tangency of the inscribed sphere with that face, and draw segments from that point to the vertices of the face. Thus, we have divided each face of the polyhedral to some triangles. Now, consider a black face $A_1A_2\ldots A_k$, and let T be the point of tangency of this face with the inscribed sphere, consider the triangle TA_1A_2, from the assumption there is a white face containing the edge A_1A_2 that the inscribed circle is tangent to it at T'. It is trivial that $A_1T=A_1T'$ and $A_2T=A_2T'$ (the length of the tangents from a point to a sphere are equal), so the triangles A_1A_2T and A_1A_2T' are congruent and have equal areas. Thus, every black triangle has a corresponding white triangle and these white triangles are distinct. So sum of the black areas is not greater than sum of the white areas.



8. We suppose that the origin is not omitted without loss of generality. Consider the cluster having the origin, if the cluster is finite, its border contains some omitted vertices that are adjacent to at least one of the vertices of the cluster. Among all of the points of the cluster having origin, consider the point having maximum horizontal distance from the origin, and then consider the point having maximum vertical distance from the origin, let k be the maximum of these two distances. So all of the points of the cluster having origin, are in the square whose center is on the origin and has side-length $2k+1$. But since, the points on the cluster's border have rounded around the origin, hence for every

$0 \leq i \leq k$ there exists a point on the border that its horizontal or vertical distance from the origin is equal to i. Thus, at least $k + 1$ points are omitted which is a contradiction. So the cluster having the origin is infinite. Now assume that there are two infinite clusters. Obviously their border contains a path of omitted points which is infinite from both directions. We extend the origin-centered square until it contains a piece of this path. From now on, by increasing n one by one, the square's side will extend by 2 and there will be 2 more vertices of the path in the square, but the value of $n/2$ will increase by $1/2$. So but increasing n, the total number of vertices of the path which is in the square are growing faster than $n/2$, and hence it cannot be always less than it. This contradiction shows that there is exactly one infinite cluster.

Third Round

1. We claim that $N = (1389!)^{m+1} - 1$ has the desired properties. Let $2 \leq b \leq 1389$, we know that there are at least $m + 1$ zeros in the rightmost of $(1389!)^{m+1}$ in base b, because $(1389!)^{m+1}$ is divisible by b^{m+1}. Hence, the $m + 1$ rightmost digits of $(1389!)^{m+1} - 1$ is $b - 1$, then sum of the digits of N in base b is at least $(m + 1)(b - 1)$ which is greater than or equal to $m + 1$.
2. Let ω be the circumcircle of the quadrilateral $HONM$. A 2 homothety at H, takes M, N to B, C respectively. So this homothety maps ω to the circumcircle of HBC. So ω is tangent to the circumcircle of HBC at H and its radius is half of the radius of the circumcircle of HBC. We know that the radius of the circumcircle of HBC is equal to the radius of the circumcircle of ABC (R). Hence ω passes through the center of the circumcircle of ABC and its radius is $\frac{R}{2}$. So ω is tangent to the circumcircle of ABC (at T). The circumcircles of ABC and HBC are symmetric across the line BC, as well as H, T (the points of tangency of these to circles with ω) and we deduce that $HT \perp BC$. On the other hand, OT is a diameter of ω $(OT = R)$, hence $HT \perp HO$ and so $HO \parallel BC$. Hence the line HO is perpendicular to $B'C$, and since $OC = OB'$, this line is the perpendicular bisector of $B'C$, so $HB' = HC$. Now let D be the midpoint of HB', then the medians $B'N, CD$ are equal in the isosceles triangle $H'BC$. Now since AH and $B'C$ are both perpendicular to BC, they are parallel. Similarly, CH and $B'A$ are both perpendicular to AB, and are parallel. So $AB'CH$ is a parallelogram and its diagonals bisect each other, from this we determine that D is the midpoint of AC and hence: $B'N = CD = \frac{AC}{2}$.
3. A connected component of diagonals is some of the diagonals that are connected to each other by a path of diagonals, and if we add another diagonal to it, it will no longer be connected. Easily, it can be seen that the diagonals that are not in this component but are on the border of this component, are connected. (until they don't cross the table sides.)

   Now, consider a path of diagonals that starts from the upper side of the table and has the maximum height among all the paths starting from the upper side. If the height of this path be m, the problem is solved, otherwise suppose that the height be less than m. Consider the connected component having this path. If this component is not connected to the right side of the table, then its rightmost border's height will be more, and this is a contradiction. Similarly if this connected component is not connected to the left side of the table, then its leftmost border's height will be more, and a contradiction. Thus, this connected component is connected to both right and left sides of the table, and hence they're connected to each other.

4. **First Solution:** If a polygon does not intersect itself, then by moving clockwise (or counterclockwise) along its sides, the inside (or outside) of the polygon must always be at our right-side. But if the said polygon doesn't intersect itself, then the right-side of its rightmost side (because it is drawn upward) is its outside, and the right-side of its leftmost side is its (because it is drawn upward) is its inside, which is a contradiction.

   Second Solution: Suppose that the leftmost side of the polygon connects the vertex A upward to the vertex B, and the rightmost side connects the vertex C upward to the vertex D. Then all of the polygon must be between the lines AB and CD. Clearly, the remaining of the polygon contains a path that connects B to C, and another one that connects D to A; but these two paths intersect.

   Third Solution: suppose that this polygon has $2n$ sides and doesn't intersect itself. In this case, total sum of the angles of this polygon is equal to $(2n-2)\times 180^{\circ} = (n-1)\times 360^{\circ}$. From the other hand each non-vertical side of this polygon is adjacent to two vertical sides that are drawn upward. So the sum of the two adjacent angles to each non-vertical side is 360°, and since we have n non-vertical sides, the total sum of all angles equals $n \times 360^{\circ}$, which is a contradiction.

5. Obviously X, P, D, M are collinear. Since $HAPX$ is an inscribed quadrilateral, then $\angle APM = \angle AHX$, and since $HCDT$ is inscribed (T is the foot of the altitude from B on AC), then $\angle AHX = \angle AHT = \angle ACD$. Hence, $\angle APM = \angle C = \angle B$. Now if we draw ray Ax parallel to BC such that $\angle CAx = \angle C$, then in the circumcircle of AMP: (let s be the measure of the arc AM)

$$\angle MAx = \angle C = \angle APM = \frac{s}{2}$$

   Hence we deduce that Ax is tangent to circumcircle of AMP.

6. First consider the concave sequence $a_i = (1389-i)d, 0 \leq i \leq 1389, d > 0$. Note that the maximum possible value of c for this sequence is $c_0 = \frac{\sum_{i=0}^{1389} i a_i^2}{\sum_{i=0}^{1389} a_i^2} = \frac{1389^2 - 5\times 1389}{4\times 1389 - 2}$.

Now we claim that for every concave sequence a_i, we have $\sum_{i=0}^{1389} i a_i^2 \geq c_0 \sum_{i=0}^{1389} a_i^2$, and since we have the above equality, c_0 must be the answer.

Consider a concave sequence $a_0, a_1, \dots, a_{1389}$, we define

$$b_i = (1389 - i)\frac{a_{\lfloor c_0 \rfloor}}{1389 - \lfloor c_0 \rfloor}$$

Indeed we are trying to make the sequence linear. Now we must prove that:

$$\sum_{i=\lceil c \rceil}^{1389} (i-c)a_i^2 \geq \sum_{i=0}^{\lfloor c \rfloor} (c-i)a_i^2$$

Considering the definition of b_i and its linearity, and the fact that $a_{\lfloor c_0 \rfloor} = b_{\lfloor c_0 \rfloor}$, $a_{1389} \geq b_{1389}$, easily it can be proved that if $i \geq \lfloor c_0 \rfloor$ then $a_i \geq b_i$, and if $i \leq \lfloor c_0 \rfloor$ then $a_i \leq b_i$. So we have:

$$\sum_{i=\lceil c_0 \rceil}^{1389} (i-c_0)a_i^2 \geq \sum_{i=\lceil c_0 \rceil}^{1389} (i-c_0)b_i^2 \geq \sum_{i=0}^{\lfloor c_0 \rfloor} (c_0-i)b_i^2 \geq \sum_{i=0}^{\lfloor c_0 \rfloor} (c_0-i)a_i^2$$

So the desired c equals to $c_0 = \frac{1389^2 - 5\times 1389}{4\times 1389 - 2}$.

7. It is obvious that the bisector of $\angle B$ is perpendicular to TK and so to the AM (by assumption). Thus $BM = BA$, and the circumcenter of AMC is on the bisector of $\angle B$ (the perpendicular bisector of AM). Since the center of ω is on the bisector of $\angle B$, and P is on the line connecting the centers of these two circles, it is on the bisector of $\angle B$, too. Hence $\angle BAP = \angle BMP = 180^\circ - \angle PMC = \angle PAC$. So P is on the bisector of $\angle A$ and hence P is the incenter of ABC. So we have:

$$\angle TPK = \angle BTK = 90 - \frac{\angle B}{2} = \frac{\angle A}{2} + \frac{\angle C}{2} = \angle TAP + \angle KCP$$

So if we draw the ray Px such that $\angle TAP = \angle TPx$ and $\angle KCP = \angle KPx$, then this ray is tangent to the circumcircles of ATP and KPC, and according to the above equation such ray exists. Thus the circles APT and KPC are tangent in P.

8. It is obvious that the total number of the leaves of S and T are equal. We solve the problem using induction. The base case for 2 leaves is trivial because there is no vertex of degree 2. Now suppose that S, T have k leaves. Let

$$S' = S - \{u\}, T' = T - \{f(u)\}$$

In which u is a leaf. It means that we omitted leaf u from S, and leaf $f(u)$ from T . Assume the induction hypothesis holds for S', T' , so there is a correspondence between their vertices, like g. Now let u' be the neighbor of u in S', and u'' be the neighbor of $f(u)$ in T'. It is enough that we prove that $u'' = g(u')$ and the length of the edge uu' is equal to the length of the edge $f(u)u''$.

First note that if v, w be two leaves in S and the path between v, u, intersects the path between v, w for the first time in vertex k, then we have: (let $l(uv)$ denote the length of the path between u and v)

$$l(uk) = \frac{l(uv) + l(uw) - l(vw)}{2}$$

And since S doesn't have a 2-degree vertex:

$$\begin{aligned} & length\ of\ edge\ uu' \\ & = \min_{v,w \in S_{leaves}} \left\{ \frac{l(uv) + l(uw) - l(vw)}{2} \right\} \\ & = \min_{f(v), f(w) \in T_{leaves}} \left\{ \frac{l\big(f(u)f(v)\big) + l\big(f(u)f(w)\big) - l\big(f(v)f(w)\big)}{2} \right\} \\ & = length\ of\ the\ edge\ f(u)u'' \end{aligned}$$

Now do the same for T. Thus, because these two values for a pair v, w and its correspondence $f(v), f(w)$, minimizes, so $u'' = f(u')$ and the length of the edge uu' is equal to the length of the edge $f(u)u''$, hence the induction and proof are complete.

9. Suppose that $f(0) = a$. Let $x = y = 0$, we have

$$f\left(\frac{a}{2}\right) = f(a) - a$$

Let $x = \frac{a}{2}, y = 0$, we have

$$f\left(\frac{f(a) - \frac{a}{2}}{2}\right) = 2a$$

Let $x = a, y = 0$, we have

$$f\left(\frac{f(a) + a}{2}\right) = 2a$$

Since f is increasing, so for every $\frac{f(a)-\frac{a}{2}}{2} \leq t \leq \frac{f(a)+a}{2}$ we have $f(t) = 2a$. Hence $f\left(\frac{f(a)+\frac{a}{2}}{2}\right) = 2a$. From the other hand by letting $x = y = \frac{a}{2}$, we have

$$f\left(\frac{f(a)+\frac{a}{2}}{2}\right) = 2a - f(a) + f(f(a) - a)$$

Hence we have

$$2a = 2a - f(a) + f(f(a) - a) \Rightarrow f(a) = f(f(a) - a)$$

Letting $x = 0, y = \frac{a}{2}$ in the main equation results in

$$f(a) = -a + f(f(a) - a) \Rightarrow -a = 0 \Rightarrow a = 0 \Rightarrow f(0) = 0$$

Now letting $x = 0$ in the main equation results in

$$f(y) = f(f(y))$$

Then let $x = y$, and we have

$$f\left(\frac{3}{2}x + \frac{1}{2}f(x)\right) = 2x - f(x) + f\big(f(x)\big) = 2x$$

So f is surjective. Hence in the equation $f(y) = f(f(y))$, $f(y)$ can be all values in domain, so f is the identity function.

10. Let $Q(x,y) = P(x, y+1)$. So we must find all polynomials $Q(x,y)$ such that for all $a, b, c \in \mathbb{R}$:

$$(\star) \qquad Q(ab, c^2) + Q(bc, a^2) + Q(ca, b^2) = 0$$

Suppose that $Q(x,y)$ be the polynomial with the least degree that satisfies the above equation, we reach the contradiction by finding a polynomial with lesser degree that satisfies the above equation, concluding that Q can only be zero. Let $a = b = c = 0$, so $Q(0,0) = 0$. Now let $a = b = 0$, it results that $Q(0, c^2) = 0$, hence we can say that for every y, we have $\underline{Q(0,y) = 0}$. Then by letting $a = 0$ we conclude that $Q(bc, 0) = 0$. Thus for all x, we have $\underline{Q(x,0) = 0}$, so we can consider $Q(x,y) = xyR(x,y)$. By applying it to the above equation, we have:

$$abc^2R(ab, c^2) + a^2bcR(bc, a^2) + ab^2cR(ca, b^2) = 0$$
$$\Rightarrow cR(ab, c^2) + aR(bc, a^2) + bR(ca, b^2) = 0$$

If we let $a = b = 0$, it concludes that $R(0, c^2) = 0$. So for all y, $R(0,y) = 0$, it means that $R(x,y) = xQ'(x,y)$. It is seen that Q' satisfies the $(\star)$, so we reached the contradiction, and hence $P \equiv 0$.

11. Suppose that $\{f(p_i r + s_i) | r \in \mathbb{N}\}$ be an arithmetic progression with common difference d_i. Since f is increasing, d_i must be positive. (even if for one i, we have $d_i = 0$, the problem is easy) so we know that $f(p_i r_1 + s_i) - f(p_i r_2 + s_i) = d_i(r_1 - r_2)$. Now by using Chinese Remainder Theorem, we conclude that exists x_{ij} such that $x_{ij} \equiv s_i (mod\ p_i)$ and $x_{ij} \equiv s_j (mod\ p_j)$. This means that x_{ij} can be displayed as both forms $p_i r + s_i$ and $p_j r' + s_j$. Now it is seen that $x_{ij} + p_i p_j$ is in both progressions, so:

$$f(x_{ij} + p_i p_j) = f(x_{ij}) + d_i p_j$$
$$f(x_{ij} + p_i p_j) = f(x_{ij}) + d_j p_i$$

So $d_i p_j = d_j p_i$. And hence $p_i | d_i$ for every i, and also $\frac{d_i}{p_i}$ is constant for every i. Name this constant value d ($d_i = d p_i$). From the other hand we can find X such that $\forall i: X \equiv s_i (mod\ p_i)$, it means X appears in all of the progressions. Also we can find Y such that $\forall i: Y \equiv -i (mod\ p_i)$, it means $\forall i: p_i | Y + i$. (X, Y are found using the Chinese Remainder Theorem) so for every i, X and $X + Y + i$ are in both $p_i r + s_i$ progressions, hence:

$$\forall i: f(X + Y + i) = f(X) + \frac{Y + i}{p_i} \times d_i = f(X) + d(Y + i)$$

And this means that if $a = X + Y$, then $f(a + 1), f(a + 2), \dots f(a + n)$ form an arithmetic progression.

12. Since Q is on both circumcircles of ABC and AXY, so there is a spiral similarity about Q carrying one of these circles to another such that it carries X to B, and Y to C. Suppose this similarity carries K to a point T. It is enough that we prove that the points P, K, T are collinear, because then, considering the similarity of QKT and QXB we have:

$$\angle QXA = 180^{\circ} - \angle QXB = 180^{\circ} - \angle QKT = \angle QKP$$

Now we prove that P, K, T are collinear:

The spiral similarity about Q, carries X, Y, K to B, C, T respectively. So the triangles XYK and BCT are similar. From the other hand: (let $m(AB)$ denote the measure of the arc AB)

$$\angle BXK = \angle BPK = \frac{m(YK)}{2} = \angle XYK$$
$$\angle XBK = \frac{m(KX)}{2} = \angle YXK$$

Thus the triangles BXK and XYK are similar, and hence BXK and BCT are similar. So $\frac{BX}{BK} = \frac{BC}{BT}$, also $\angle XBK = \angle CBT \Rightarrow \angle XBC = \angle KBT$. From which we deduce that BKT and BXC are similar. So $\angle BXC = \angle BKT$, and:

$$\Rightarrow \angle BKT + \angle BKP = \angle BXC + \angle BKP = 180^{\circ} \Rightarrow K, P, T \ are \ collinear$$