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Contents

Problems

First Round

1. (Morteza Adl) a and b are two integers such that $a > b$. If $ab - 1$ and $a + b$ are relatively prime, and so $ab + 1$ and $a - b$; prove that $(ab + 1)^2 + (a - b)^2$ is not a perfect square.

2. (Mohsen Jamali) There are n points in the plane that no three of them are collinear. Prove that the total number of triangles whose vertices are from these n points and having area equal to 1, is not greater than $\frac{2}{3}(n^2 - n)$.

3. (Davood Vakili) Two circles W_1 and W_2 intersect at D and P. A and B are two points on W_1 and W_2 respectively such that AB is their common tangent. Suppose D is closer to AB than P. If AD intersects W_2 again in C, and M be the midpoint of BC, prove that:

$$\angle DPM = \angle BDC$$

4. (Arashk Hamidi) The coefficients of $P(x) = ax^3 + bx^2 + cx + d$ are real and

$$min\{d, b + d\} > max\{|c|, |a + c|\}$$

prove that the equation $P(x) = 0$ does not have a solution in $[-1, 1]$.

5. (Davood Vakili) In triangle ABC, $\angle A = 60°$. Let E and F be points on the extensions of AB and AC such that $BE = CF = BC$. The circumcircle of ACE intersects EF in K (different from E). prove that K lies on the bisector of $\angle BAC$.

6. (Mohammad Ali Karami) A school has n students and there are some extra classes provided for them so that each student can participate in any number of them. We know that there are at least two participants in any class. We also know that if two different classes have two common students, then the number of their participants are different. Prove that the total number of classes is not greater than $(n - 1)^2$.

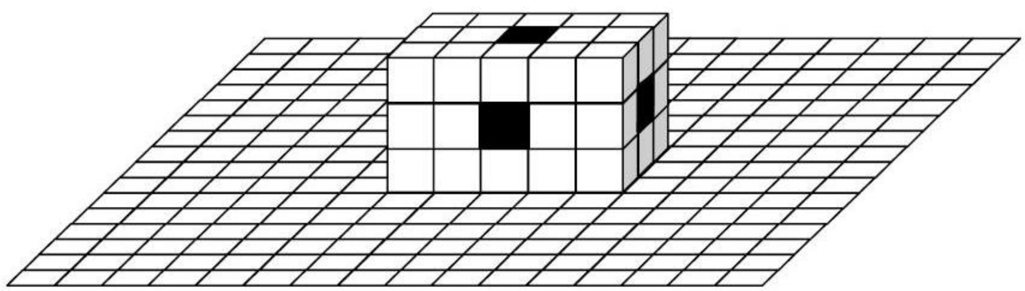
Second Round

1. (Mostafa Eynollahzadeh) Let $P(x, y)$ be a polynomial in two variables with real coefficients. The degree of a monomial is sum of the powers of x and y in it. Let $Q(x, y)$ be sum of the expressions in $P(x, y)$ with highest degree. (For example if $P(x, y) = 3x^4y - 2x^2y^3 + 5xy^2 + x - 5$ then $Q(x, y) = 3x^4y - 2x^2y^3$.) Suppose there exists real numbers x_1, x_2, y_1 and y_2 such that

$$Q(x_1, y_1) > 0 \ , \ Q(x_2, y_2) < 0$$

prove that the set $\{(x, y)|P(x, y) = 0\}$ is not bounded. (A subset S of the plane is bounded if there exists a positive number M such that the distance of the elements of S from origin is less than M.)

2. (Sina Jalali) We have put a rectangular parallelepiped with dimensions $(2a+1) \times (2b+1) \times (2c+1)$ on an infinite lattice plane with unit squares, where the vertices of its bottom face are lattice points, and a, b and c are positive integers. We can roll the parallelepiped in every four directions, and its faces are divided into unit squares while the square in the middle of each face is black colored. If this black square lies on a square of lattice, then it would become black as well.
Prove that if the numbers $2a + 1$, $2b + 1$ and $2c + 1$ are pairwise coprime, then we can make every square of the lattice black by rolling the parallelepiped.

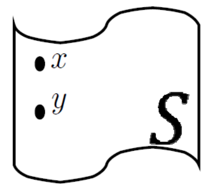


3. (Sepehr Ghazi Nezami) A is a set of n points in the plane. A version of A is a set of points that can be reached from A by translation, rotation, homothety or any combination of these transformations. We want to place n versions of A in the plane so that the intersection of every two of them be a single point, and the intersection of every three of them be empty.

   (a) Prove that if no four points of A form a parallelogram, we can do this using only translations. (Note that A does not contain a parallelogram with zero angle, or a parallelogram that its opposite vertices coincide.)

   (b) Prove that we can always do this using above transformations.

4. (Sepehr Ghazi Nezami) Suppose S be a figure in the plane that its border does not contain any lattice point and let $x, y \in S$ be two lattice points with unit distance from one another. A copy of S is a combination of rotations and reflections of S. Assume that there exists a tiling of the plane with copies of S such that x and y lie on lattice points for every copy of S in this tiling. Prove that the area of S is equal to the number of lattice points in it.



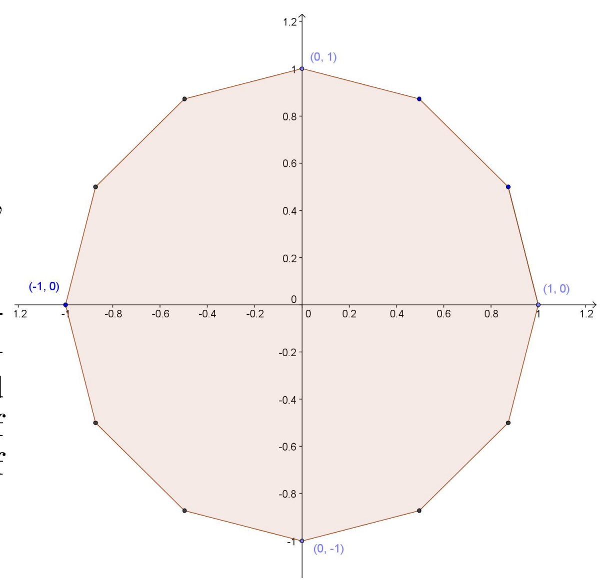
5. (M.H.Shafinia, M.Mansouri, S.Dashmiz) Let n be a positive integer and $x_1, x_2, ...$ a sequence of 1 and -1 with these properties:

   - is periodic with $2^n - 1$ as its fundamental period. (i.e. for every positive integer j we have $x_{j+2^n-1} = x_j$ and $2^n - 1$ is the minimum number with this property.)
   - there exist distinct integers $0 \leq t_1 < t_2 < ... < t_k < n$ such that for every j we have
$$x_{j+n} = x_{j+t_1} \times x_{j+t_2} \times ... \times x_{j+t_k}$$

   prove that for every positive integer $s < 2^n - 1$ we have
$$\sum_{i=1}^{2^n-1} x_i x_{i+s} = -1$$

6. (Sepehr Ghazi Nezami) We define a 12-gon in the plane to be **good** if it has these conditions:
   - be regular.
   - be filled.
   - the origin be its center.
   - its vertices contain points $(0,1)$, $(1,0)$, $(-1,0)$ and $(0,-1)$.

   Find the number of faces of the polyhedral with maximum volume that its projections on xy , yz and zx planes are good 12-gons. (It is clear that the centers of these 12-gons coincide and is the origin of the 3D-space.)



7. (Bijan Aghdasi) S is a set with n elements, and $P(S)$ is the set of all subsets of S. Let

$$f : P(S) \longrightarrow \mathbb{N}$$

be a function with these characteristics:

- For each $A \subseteq S$ we have $f(A) = f(S - A)$.
- For every two $A, B \subseteq S$ we have $max(f(A), f(B)) \geq f(A \cup B)$.

prove that the number of elements in the range of f is not greater than n.

8. (Mohsen Jamali) Show that there exist infinitely many positive integers n such that $n^2 + 1$ has no divisor of the form $k^2 + 1$ except for $n^2 + 1$.

Third Round

1. (Hesam Rajabzadeh) In non-isosceles triangle ABC, let M be the midpoint of BC, and let D and E be the foot of perpendiculars from C and B to AB and AC respectively. If L and K be the midpoints of MD and ME, and T be a point on LK such that $AT||BC$, prove that $TA = TM$.

2. (Morteza Saghafian) For what integer values of $n > 2$, there exist numbers a_1, a_2, ..., $a_n \in \mathbb{N}$ such that the following sequence is a nonconstant arithmetic progression?
$$a_1a_2, a_2a_3, ..., a_na_1$$

3. (Mohsen Jamali) There are n points on a circle ($n > 1$). We define *interval* to be an arc between any two of these n points. Let $\mathcal{F}$ be a family of these intervals such that every element of $\mathcal{F}$ is a subset of at most one other element of $\mathcal{F}$. An interval is called *maximal* if it is not a subset of any other interval. Let m denote the number of maximal elements of $\mathcal{F}$, and a be the number of its non-maximal elements, prove that:
$$m + \frac{a}{2} \leq n$$

4. (Mohammad Mansouri) We define a finite subset of $\mathbb{N}$ like A to be *good* if it has the following properties:

   - For every three distinct $a, b, c \in A$ we have $\gcd(a, b, c) = 1$.
   - For every two distinct $b, c \in A$, there exists $a \in A$ that $a \neq b, c$ and $a|bc$.

   find all *good* sets.

5. (Mohammad Jafari) Find all surjective functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that for every $x, y \in \mathbb{R}$ we have:
$$f(x + f(x) + 2f(y)) = f(2x) + f(2y)$$

6. (Sepehr Ghazi Nezami) Let OBC be an isosceles triangle ($OB = OC$). Consider circle ω with center O and radius OB. Tangents to this circle from B and C intersect at A. Consider circle ω_1 inside triangle ABC which is tangent to ω, and tangent to AC at H. Circle ω_2 is also inside triangle ABC and is tangent to ω, tangent to ω_1 at J, and tangent to AB at K. Prove that the bisector of $\angle KJH$ passes through I, the incenter of triangle OBC.

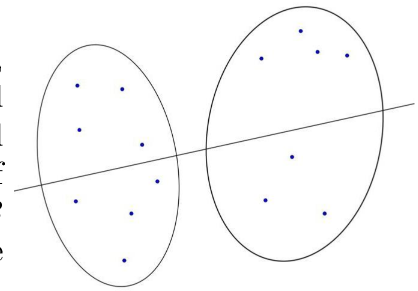
7. (Ali Khezeli) Find the locus of all points P inside an equilateral triangle having the following property
$$\sqrt{h_1} + \sqrt{h_2} = \sqrt{h_3}$$
where h_1, h_2, h_3 are the distances of P from sides of the triangle.

8. (Mahyar Sefidgaran) p is a prime number and $k \leq p$. $f(x)$ is a polynomial with integer coefficients that for every integer x, $f(x)$ is divisible by p^k. Prove that there exist polynomials $A_0(x), A_1(x), ..., A_k(x)$ with integer coefficients such that

$$f(x) = \sum_{i=0}^{k} (x^p - x)^i p^{k-i} A_i(x)$$

also prove that the statement is not correct for $k > p$.

9. (Morteza Saghafian) There are n points in the plane, not all on the same line. A line l in the plane is called *wise* if one can divide these n points into two sets A and B ($A \cap B = \emptyset$) such that sum of distances of points of A from l is equal to the sum of distances of points of B from l. Prove that there are infinite points in the plane such that $n+1$ *wise* lines pass through them.



10. (Mohammad Jafari) Find the smallest real k such that for every $a, b, c, d \in \mathbb{R}$ we have:

$$\sqrt{(a^2+1)(b^2+1)(c^2+1)} + \sqrt{(b^2+1)(c^2+1)(d^2+1)}$$
$$+\sqrt{(c^2+1)(d^2+1)(a^2+1)} + \sqrt{(d^2+1)(a^2+1)(b^2+1)}$$
$$\geq 2(ab+bc+cd+da+ac+bd) - k$$

11. (Sepehr Ghazi Nezami) Let A', B' and C' denote the midpoints of sides BC, AC and AB of triangle ABC respectively. Assume that P and P' are two variable points such that $PA = P'A'$, $PB = P'B'$ and $PC = P'C'$. Prove that the line PP' passes through a fixed point.

12. (Mohsen Jamali) Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ be a function such that for every $a, b \in \mathbb{N}$, $af(a) + bf(b) + 2ab$ is a perfect square. Prove that for every $a \in \mathbb{N}$ we must have $f(a) = a$.

Solutions

First Round

1. We know that

$$(a+b)^2+(ab-1)^2=a^2b^2+a^2+b^2+1 \Longrightarrow c^2=(a^2+1)(b^2+1)$$

now we prove that $(a^2+1, b^2+1)=1$. Suppose the contrary, hence there exists a prime number p such that:

$$p|a^2+1, p|b^2+1 \Longrightarrow p|a^2-b^2$$

$$\Longrightarrow p|a-b \quad \text{or} \quad p|a+b$$

If $p|a-b$, then $p|ab-b^2$ and since $p|b^2+1$ then $p|ab+1$ which is a contradiction as we know $(ab+1, a-b)=1$. The case $p|a+b$ similarly reaches contradiction. Now that a^2+1 and b^2+1 are relatively prime and their multiplication is a perfect square, both should be perfect squares which is impossible because a^2+1 can not be one and the proof is complete.

2. **lemma 1:** Let points A and B in the plane (with distance d), if there is a point C such that the area of triangle ABC is 1, then the distance of C from line AB is $\frac{2}{d}$. i.e. the locus of such points consist of two lines parallel to AB at a $\frac{2}{d}$ distance from it.
proof: trivial.
lemma 2: The number of triangles with vertices A and B having area 1 is at most 4.
proof: If there are more than 4 triangles with this property, then from lemma 1 and pigeonhole principle it concludes that 3 of the points must lie on one of the two lines at distance $\frac{2}{d}$ from AB which is a contradiction.
Now we count the number of triangles (T) in this way that for each pair of points there can be at most 4 triangles and we have counted each point at least 3 times, hence

$$T \leq \frac{\binom{n}{2} \times 4}{3} \Longrightarrow T \leq \frac{2}{3} \times n(n-1)$$

3. Let the extension of PD intersect AB at N. Since $\angle BDC = \angle BPC$, it is enough to prove that $\angle DPM = \angle BPC$ or equivalently $\angle DPB = \angle MPC$. N is the midpoint of AB (because $NA^2 = ND \times NP = NB^2$).
Now we have

$$\angle PBC = \angle PDC = \frac{\overset{\frown}{AP}}{2} = \angle PAB$$

$$\angle PCB = \frac{\overset{\frown}{PB}}{2} = \angle PBA$$

hence $\triangle PCB$ and $\triangle PBA$ are similar, and since PM and PN are their medians, the angles $\angle DPB = \angle NPB = \angle MPC$.

4. Suppose that α be a real root for $P(x)$ such that $|\alpha| \leq 1$, hence

$$P(\alpha) = 0 \Longrightarrow a\alpha^3 + b\alpha^2 + c\alpha + d = 0 \Longrightarrow a\alpha^3 + c\alpha = -(b\alpha^2 + d)$$

$$\Longrightarrow |\alpha||a\alpha^2 + c| = |b\alpha^2 + d| \Longrightarrow |a\alpha^2 + c| \geq |b\alpha^2 + d| \quad (*)$$

on the other hand $f(x) = |ax + c|$ takes its maximum over the interval $[0, 1]$ in one of its boundary points, i.e. $f(x) \leq max\{|c|, |a + c|\}$, so

$$|a\alpha^2 + c| \leq max\{|c|, |a + c|\}$$

it follows from the hypothesis that d and $b + d$ are positive, so $bx + d > 0$ for each $0 \leq x \leq 1$, hence similarly we can say that (considering minimum of $g(x) = |bx + d|$)

$$|b\alpha^2 + d| \geq min\{|d|, |b + d|\}$$

from these inequalities and $(*)$ we have

$$max\{|c|, |a + c|\} \geq |a\alpha^2 + c| \geq |b\alpha^2 + d| \geq min\{|d|, |b + d|\} \geq min\{d, b + d\}$$

hence

$$max\{|c|, |a + c|\} \geq min\{d, b + d\}$$

and this is a contradiction, so there is no α with such property.

5. Let T be the intersection of BF and CE. We have $\angle BCE + \angle BEC = \angle ABC$, since $BC = BE$ we have $\angle BCE = \frac{\angle ABC}{2}$. similarly $\angle CBF = \frac{\angle ACB}{2}$. Hence

$$\angle CTF = \angle BCE + \angle CBF = \frac{\angle ABC + \angle ACB}{2} = 60^\circ$$

and the quadrilateral $ABTC$ is cyclic, so $\angle EBF = \angle ACE = \angle AKE$ and then $\angle ABF = 180 - \angle EBF = 180 - \angle AKE = \angle AKF$, thus $ABKF$ is cyclic.
Having $\angle EBK = \angle CFK$ and $\angle BEK = \angle FCK$, and from $BE = FC$ we conclude that $\triangle KBE$ and $\triangle KFC$ are congruent, so $KC = KE \Longrightarrow \overset{\frown}{KC} = \overset{\frown}{KE}$, thus AK is the bisector of $\angle BAC$.

6. Suppose $A_i \quad (2 \leq i \leq n)$ be the set of all classes with i participants. We will show that $|A_i| \leq \frac{n(n-1)}{i(i-1)}$. Since every 2-element subset of students are together in at most one class of each A_i, then the number of 2-element subsets of A_i multiply the elements of A_i must be less than or equal to the number of 2-element subsets of students, hence $|A_i| \leq \frac{\binom{n}{2}}{\binom{i}{2}} = \frac{n(n-1)}{i(i-1)}$. On the other hand $m = |A_2| + ... + |A_n|$, hence we have

$$m \leq n(n-1)(\frac{1}{2(2-1)} + ... + \frac{1}{n(n-1)}) = n(n-1)(1 - \frac{1}{n}) = (n-1)^2$$

Second Round

1. First note some points: (in the proof by the ray passing through origin and (x, y), we mean the ray with origin as its start point)

   - We can suppose that $(0,0)$, (x_1, y_1) and (x_2, y_2) are not collinear, otherwise choose another point (x, y) out of this line such that $Q(x, y) \neq 0$ (if such point doesn't exist then Q must be zero on all plane except a line, hence it must be zero overall.) and with respect to the fact that the sign of $Q(x, y)$ is different from the sign of either $Q(x_1, y_1)$ or $Q(x_2, y_2)$, let it be $Q(x_1, y_1)$, now we can replace (x_2, y_2) by (x, y).
   - We know that Q is homogeneous so if we put $k = \deg(Q)$ we have $Q(tx, ty) = t^k Q(x, y)$ and we have $t^k > 0$ for $t > 0$ so $Q(tx, ty)$ and $Q(x, y)$ have the same sign for $t > 0$.
   - If two rays passing through origin doesn't form a line together, then there exists point (s_1, s_2) on the first ray, and point (t_1, t_2) on the second ray such that the distance of origin from the line passing through these points is greater than an arbitrary $M \in \mathbb{R}^+$. (this is trivial.)

   lemma. For every $(x, y) \in \mathbb{R}^2$ that $Q(x, y) \neq (0, 0)$, there exists $S \in \mathbb{R}^+$ such that if $A > S$ then $P(Ax, Ay)$ and $Q(x, y)$ are of the same sign.
   proof of lemma. Note that if $degQ = k$ then there exist homogeneous polynomials $Q_0, ..., Q_{k-2}, Q_{k-1}$ such that $degQ_i = i$ $(0 \leq i \leq k-1)$ and

   $$P = Q + Q_{k-1} + Q_{k-2} + ... + Q_0$$

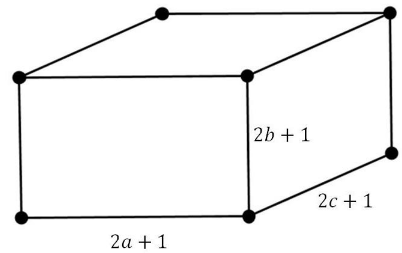
   let $L(A) = P(Ax, Ay)$, $F(A) = Q(Ax, Ay)$ and $F_i(A) = Q_i(Ax, Ay)$, now since Q and Q_i are homogeneous we have

   $$L(A) = P(Ax, Ay) = A^k Q(x, y) + A^{k-1} Q_{k-1}(x, y) + ... + Q_0(x, y)$$

   that $Q_i(x, y)$ $(1 \leq i \leq k-1)$ and $Q(x, y)$ are constant numbers, so when $A \to +\infty$ the sign of $L(A)$ is the sign of the coefficient of A^k which is $Q(x, y)$. (we assumed that $Q(x, y) \neq 0$) So the lemma is proved.

   Now with respect to above notes there is a point on the ray passing through origin and (x_1, y_1) like (l_1, l_2), and another point on the ray passing through origin and (x_2, y_2) like (m_1, m_2) that $P(l_1, l_2) > 0, P(m_1, m_2) < 0$, and the distance of origin from the line passing through these points is greater than an arbitrary $M \in \mathbb{R}^+$. Now since the restriction of P to this line is a single variable polynomial (by replacing $x = ay + b$ or $y = ax + b$) that is negative in a point and positive in another, so P has a root on this line and the problem is solved.

2. Suppose that at the beginning the cuboid lies on the plane like the figure. By rolling it in the directions down, right, up, left, up, right respectively, it is translated by the vector $(2a+1, 2a+1)$ and lies on the same face. Now with respect to the symmetry between a and c, we can translate it by the vector $(2c+1, 2c+1)$, lying on the same face as well. Now we show that this can also be achieved for b by rolling it upward, translating by the vector $(2b+1, 2b+1)$, then rolling it downward.



Now since the center of the cuboid face is of integer values, if we can color a cell of the lattice plane with (x, y) coordinates, we can color these cells as well

$$(x+2a+1, y+2a+1)$$

$$(x+2b+1, y+2b+1)$$

$$(x+2c+1, y+2c+1)$$

now we prove that we can color $(x+1, y+1)$ too. As $2a+1$ and $2c+1$ are coprime, there exists $r, s \in \mathbb{Z}$ such that $r(2a+1)+s(2c+1) = 1$, now we can write

$$(x+1, y+1) = (x + r(2a+1) + s(2c+1), y + r(2a+1) + s(2c+1))$$

so this cell can be colored. If the center of the bottom face of the cuboid lies on the $(0, 0)$ cell of the plane, we can color all (x, y) cells where $x+y$ is even. Without loss of generality assume that a and b have the same parity and $a+b$ is even, so $a+b+1$ is odd. Now by rolling the cuboid in the shown direction we can color a cell that sum of its coordinates are different from the previous cell, and with respect to what is said, we can color all of the plane cells.
Remark. The statement also holds for the hypothesis $(2a+1, 2b+1, 2c+1) = 1$.

3. (a) Let an arbitrary point of the plane be origin, and suppose that the points of the set A be $\{a_1, a_2, ..., a_n\}$. We define $S + b = \{x + b | x \in S\}$ where $S \subseteq \mathbb{R}^2$, $b \in \mathbb{R}^2$. Now we prove that the sets $A + a_1$, $A + a_2$, ... and $A + a_n$ have this property. First note that these are n translations of A. Let $B_i = A + a_i$ $(1 \leq i \leq n)$, since $a_j + a_i \in B_i$ and $a_i + a_j \in B_j$, so $x_{ij} = a_i + a_j \in B_i \cap B_j$. Now if $x \in B_i \cap B_j$ and $x \neq x_{ij}$, then there must exist $j_1 \neq j$ and $i_1 \neq i$ that $a_{i_1} + a_j = x = a_{j_1} + a_i$, hence $x - x_{ij} = a_{j_1} - a_j = a_{i_1} - a_i$, so a_i,a_{i_1},a_j and a_{j_1} form a parallelogram, a contradiction, therefore $B_i \cap B_j$ has a single element.
Clearly, with respect to previous part, the intersection of every three of them is empty, because if $x \in B_{i_1} \cap B_{i_2} \cap B_{i_3}$, then we must have $a_{i_1} + a_{i_2} = a_{i_1} + a_{i_3}$, so $a_{i_2} = a_{i_3}$ and $i_2 = i_3$ a contradiction. Therefore part (a) is complete.

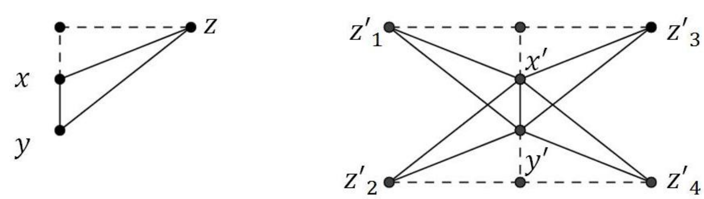
   (b) Let an arbitrary point apart from $a_1, ..., a_n$ be origin. Assume that $a_1, ..., a_n$ are complex numbers in this plane. Similar to the previous part we define

$S.b = \{xb | x \in S\}$ where $b \in \mathbb{C}$ and $S \subseteq \mathbb{C}$ is arbitrary. Let $B_i = A.a_i$, $(1 \leq i \leq n)$, similar to part (a) we have

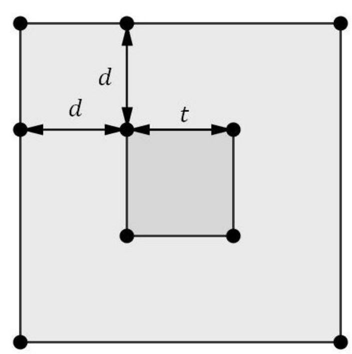
$$a_i.a_j = a_j.a_i \in B_i \cap B_j$$

now for every i_1,i_2,i_3 and i_4 that at least three of them are distinct we must have $a_{i_1}a_{i_2} \neq a_{i_3}a_{i_4}$, otherwise $\frac{a_{i_1}}{a_{i_3}} = \frac{a_{i_4}}{a_{i_2}}$ and it means that the origin is the center of a rotational homothety that brings the segment $a_{i_1}a_{i_3}$ to the segment $a_{i_2}a_{i_4}$. Clearly since the ways of choosing i_1,i_2,i_3 and i_4 are finite, we can choose O in such a way that this won't happen, and the $B_i \cap B_j$ would have one element in this case. Note that each one of the $B_1, ..., B_n$ is a version of S, since it is obtained from S by a rotational homothety about the origin. (note that it is required that no two of a_is are collinear with origin for taking it as center of this rotational homothety)

4. As said in the hypothesis, there are two points x, y with unit distance on a tile that their corresponding points on other tiles are lattice points. Let z be another lattice point in S, and let S' be a copy of S. Also let $x', y', z' \in S'$ be the corresponding points to x, y, z. There are four possible places for z' shown in below figure, and all of them are lattice points, therefore the total number of lattice points of each of the copies of S are equal, let this number be k.



Let $d \in \mathbb{N}$ be greater than the maximum distance between points of S. Consider the following squares of size $t \times t$ and $(t + 2d) \times (t + 2d)$.



Assume that u be the number of tiles that has intersection with $t \times t$ square, since every tile has at most k lattice points of that square we have $u \geq \frac{(t+1)^2}{k}$, on the other hand none of the tiles has any point outside the $(t+2d) \times (t+2d)$ square, so we have $u \leq \frac{(t+2d+1)^2}{k}$. Now let s be the area of S, since the inner square is totally covered we have $us \geq t^2$ and $us \leq (t+2d)^2$, hence

$$\left.\begin{array}{rcccl} (t+1)^2 & \leq & uk & \leq & (t+2d+1)^2 \\ t^2 & \leq & us & \leq & (t+2d)^2 \end{array}\right\} \Rightarrow \frac{(t+1)^2}{(t+2d)^2} \leq \frac{k}{s} \leq \frac{(t+2d+1)^2}{t^2}$$

$$\Rightarrow \lim_{t\to\infty} \frac{(t+1)^2}{(t+2d)^2} \leq \frac{k}{s} \leq \lim_{t\to\infty} \frac{(t+2d+1)^2}{t^2} \Rightarrow k = s$$

so the proof is complete.

5. We define $[x]_i = x_i$ where $x = (x_1, \cdots, x_n)$.
Lemma. Let n, k, $0 < i_1 < i_2 < \cdots < i_k \leq n$ be integers, then we have $\sum_{x\in X'} [x]_{i_1}[x]_{i_2}\cdots[x]_{i_k} = -1$ where $X' = X - \{\underbrace{(1,1,\cdots,1)}_{n}\}$ and X is the set of all n-tuples of $\{-1, 1\}$.
Proof of lemma. Since for every n-tuple x, there exists a n-tuple y such that $[x]_{i_1} \neq [y]_{i_1}$ and $[x]_s = [y]_s$ for all $s \neq i_1$, then we have $\sum_{x\in X}[x]_{i_1}[x]_{i_2}\cdots[x]_{i_k} = 0$.
This proves the lemma.
Because $2^n - 1$ is the minimum period of the sequence, the n-tuples $(x_i, x_{i+1}, \cdots, x_{i+n-1})$ for $1 \leq i \leq 2^n - 1$ are distinct. ($\underbrace{(1,1,\cdots,1)}_{n}$ is not among them) Note that by defining $x_i = x_{2^n-1+i}$ we can expand the sequence to negative indices. Also note that $x_i x_{i+s}$ can be written in the form of product of $x_{i-1}, \cdots, x_{i-n}$, because from the hypothesis we have $x_i = x_{i-n+t_1} \times \cdots \times x_{i-n+t_k}$, x_i can be written in the form of product of $x_{i-1}, \cdots, x_{i-n}$, now x_{i+s} can be written in the form of product of $x_{i+s-1}, \cdots, x_{i+s-n}$, x_{i+s-1} can be written in the form of product of $x_{i+s-2}, \cdots, x_{i+s-n-1}$ and so x_{i+s} can be written in the form of $x_{i-1}, \cdots, x_{i-n}$. Indeed every member of the sequence can be evaluated using n consecutive members of this sequence, now since $x_l^2 = 1$ for every $l \in \mathbb{Z}$, according to what we said, there exist $0 < u_1 < u_2 < \cdots < u_p \leq n$ such that for every $l \in \mathbb{Z}$ we have

$$x_p = x_{l-u_1} \times x_{l-u_2} \times \cdots \times x_{l-u_p}$$

hence $\sum_{i=1}^{2^n-1} x_i x_{i+2}$ is in the form of one of the summations stated in the lemma, because i can be changed from $n+1$ to $n+1+2^n-2$ and $x_i x_{i+s}$ can be evaluated by n members before x_i, and these n members form a n-tuple and we know that by changing i these sequences will contain all n-tuples of $\{-1, 1\}$ except $\underbrace{(1,1,\cdots,1)}_{n}$ so using lemma, this summation is equal to -1.

6. We show that the desired polyhedron has 36 faces. It is trivial that this polyhedron is in the intersection of three 12-gonal prisms that their projections on the axis planes are mentioned 12-gonals. Now for constructing the shape with maximum

volume we should consider the intersection of these three prisms. First consider the prism corresponding to xy plane, the resulting shape is a 12-gon, now consider the intersection of this shape with the prism corresponding to xz plane, we will show that each face of the recent prism, adds exactly one face to the shape and the intersection of this two prisms forms a polyhedron with 24 faces. For showing this it is sufficient to show that each face of the xz prism, intersects the xy prism, because the xy prism and every face are convex, the desired intersection is also convex, so it is connected and only adds one face. (because the projection of xy prism on xz plane is a finite strip, it contains the 12-gon on xz plane)
So the intersection of these two prisms is a polyhedron with 24 faces which is convex, because it is the intersection of two convex shapes. On the other hand this polyhedron contains the points $(x, y, z) = (0, 1, 1), (0, 1, -1), (0, -1, 1), (0, -1, -1)$, so it contains a square with vertices at these points, and the projection of this polyhedron on the yz plane totally contains the 12-gon corresponding to yz plane.
Similarly to the previous part, if we consider the intersection of this polyhedron and the yz prism, firstly, every face of this prism, intersects the polyhedron, secondly since the polyhedron is convex, every face intersects the polyhedron in one region and adds one face to the shape.
So for every face of the 12-gonal prism, one face adds to the polyhedron, therefore the resulting shape has $24 + 12 = 36$ faces.

7. **lemma.** If B is obtained by intersection and complement from A_1, A_2, ..., A_m, then

$$f(B) \leq max\{f(A_1), f(A_2), ..., f(A_m)\}$$

proof. It is sufficient to prove that $f(A \cap B) \leq max\{f(A), f(B)\}$,

$$f(A \cap B) = f((A \cup B)^c) = f(A^c \cap B^c)$$

$$\leq max\{f(A^c), f(B^c)\} = max\{f(A), f(B)\}$$

and the proof is complete.
First Solution. Suppose that $f(A_1) < f(A_2) < ... < f(A_m)$, using lemma, it concludes that A_m is not obtained from A_1,A_2,...,A_{m-1}. (by intersection and complement) Since A_m is obtained from single-element sets of itself, so there exists $i \in A_m$ such that $\{i\}$ is not obtained from A_1,A_2,...,A_{m-1}. By replacing some of A_js $(1 \leq j \leq m-1)$ with their complement we can assume that $i \in A_1 \cup A_2 \cup ... \cup A_{m-1}$, and since i is not obtained from A_1,A_2,...,A_{m-1}, so $\{i\} \neq A_1 \cup A_2 \cup ... \cup A_m$, therefore there exists $l \neq i$ such that $l \in A_1 \cup A_2 \cup ... \cup A_m$. Now let $a = \{i, l\}$ (indeed, construct a new element by joining i and l), if we restrict f to the subsets of S that either have both i and l or none of them, then the function $f' : P((S - \{i, l\}) \cup \{a\}) \longrightarrow \mathbb{N}$ is obtained and clearly has the hypothesis properties. Hence, because $A_1, A_2, ..., A_{m-1} \in Domain(f)$, from the hypothesis of induction we have $m - 1 \leq n - 1$ so $m \leq n$ and the proof is complete.
Second Solution. It concludes from the lemma that $f(A \Delta B) \leq max\{f(A), f(B)\}$,

now note that $P(S)$ with operation Δ forms a group ($\emptyset$ as identity and A itself as its inverse), with respect to this the sets

$$B_i = \{x|f(x) \leq i\} \ (1 \leq i \leq m)$$

are subgroups of $P(S)$ (let $\{1, 2, .., m\}$ be the range of f). Therefore by Lagrange's Theorem $|B_i|\big||P(S)|$, so $|B_i|$ is a power of 2, and since $B_1 \subsetneq B_2 \subsetneq \ldots \subsetneq B_m = P(S)$ and B_1 has at least two elements ($x \in B_1 \Rightarrow x^c \in B_1$) hence

$$|B_1| \geq 2^1, |B_2| \geq 2^2, ..., |B_m| \geq 2^m \Longrightarrow 2^n \geq 2^m \Longrightarrow m \leq n$$

therefore the proof is complete.
Note that there is a f that its range has exactly n elements, let

$$f(A) := max\{min(A), min(A^c)\}$$

and by $min\{X\}$ we mean the minimum element of X.

8. We claim that for every positive integer n, n^2+1 has a divisor of the form k^2+1 that satisfies the statement of the problem. If n^2+1 has no divisor of the form k^2+1 (except n^2+1) we are done. Otherwise suppose that $k^2+1|n^2+1$ for some positive integer k. If k^2+1 has no divisor of this form we are done. Otherwise $l^2+1|k^2+1$, by continuing this way our claim is proved.
Now it is sufficient to consider an infinite sequence a_n of positive integers of the form k^2+1 such that $(a_m, a_n) = 1$ whenever $m \neq n$. As an example consider $a_n = 2^{2^n}+1$.

Third Round

1. Let H be the orthocenter of ABC, and let ω be the circle with diameter AH. Since $TA \perp AH$, TA is tangent to ω in A. On other hand $BM = EM \Rightarrow \angle MEB = \angle MBE = 90 - \angle C = \angle HAC = \frac{\overset{\frown}{EH}}{2}$ (of ω), hence ME is tangent to ω and so is MD. Thus LK is the radical axis of point M and circle ω, and so the powers of T with respect to M and ω are equal, which means

$$TA^2 = TM^2 \Longrightarrow TA = TM$$

2. Let n be even, in this case no such numbers exist because if $n = 2k$ we have $a_1a_2 < a_2a_3 < ... < a_{2k-1}a_{2k} < a_{2k}a_1$ hence $a_1 < a_3 < ... < a_{2k-1} < a_1$ which is a contradiction.
Now let n be odd, we construct such progression with $a_1, ..., a_{2k+1} \in \mathbb{R}^+$, assume $b_1 = x$ and

$$b_1b_2 = 1, b_2b_3 = 2, ..., b_{2k+1}b_1 = 2k+1$$

$$b_1b_2 = 1 \Rightarrow b_2 = \frac{1}{x} \Rightarrow b_3 = 2x \Rightarrow b_4 = \frac{3}{2x}, \cdots$$

$$\begin{array}{rcll} b_{2i} & = & \frac{1\times3\times\cdots\times(2i-1)}{2\times4\times\cdots\times(2i-2)} \times \frac{1}{x} & 2 \le i \le k \\ b_{2i+1} & = & \frac{2\times4\times\cdots\times(2i)}{1\times3\times\cdots\times(2i-1)}x & 1 \le i \le k \end{array}$$

but $b_{2k+1}b_1 = 2k+1$ so

$$\frac{2\times4\times\cdots\times(2k)}{1\times3\times\cdots\times(2k-1)}x^2 = 2k+1 \Rightarrow x = \sqrt{\frac{1\times3\times\cdots\times(2k+1)}{2\times4\times\cdots\times(2k)}}$$

now let $a_i = b_i\sqrt{(2k+1)!}$, thus it is trivial that $a_1a_2, a_2a_3, \cdots, a_na_1$ forms an arithmetic progression too. On the other hand

$$a_{2i+1} = \frac{2\times4\times\cdots\times(2i)}{1\times3\times\cdots\times(2i-1)}\times x\times\sqrt{(2k+1)!} = \Big(2\times\cdots\times(2i)\Big)\Big((2i+1)\times\cdots\times(2k+1)\Big)$$

$$a_{2i} = \frac{1\times3\times\cdots\times(2i-1)}{2\times4\times\cdots\times(2i-2)}\times\frac{1}{x}\times\sqrt{(2k+1)!} = \Big(1\times3\times\cdots\times(2i-1)\Big)\Big((2i)\times\cdots\times(2k)\Big)$$

therefore all of a_i's are distinct natural numbers, and the statement is true for all $n = 2k+1$.

3. Let the points be $1, 2, ..., n$. We denote an interval by $[i, j]$ where $i, j \in \{1, 2, ..., n\}$ and $i < j$. Let $A = \{[x_1, y_1], [x_2, y_2], ..., [x_m, y_m]\}$ be the set of maximal intervals of $\mathcal{F}$, we claim that $\forall 1 \le i, j \le m : x_i \ne x_j$. If $x_i = x_j$ then one of their corresponding intervals must contain the other one, so the claim is proved. Similarly $\forall 1 \le i, j \le m : y_i \ne y_j$. Let X, Y denote the multi-set (the members of a multi-set need not to be distinct) of all lower and upper endpoints of all intervals of $\mathcal{F}$ respectively, and let $[x, y]$ be a non-maximal interval of $\mathcal{F}$, we show that either $x \notin X - \{x\}$

or $y \notin Y - \{y\}$. If $x \in X - \{x\}$ and $y \in Y - \{y\}$, there must exist intervals $[x, a]$ and $[b, y]$ in $\mathcal{F}$ different from $[x, y]$. Let

$$B = [x, y], C = [x, a], D = [b, y]$$

if $B \subseteq C, B \subseteq D$ then two intervals contain B, a contradiction. If $B \subseteq C, B \nsubseteq D$, then we must have $D \subseteq B$ and so $D \subseteq B, C$, hence two intervals contain D, a contradiction. Similarly $B \nsubseteq C, B \subseteq D$ is not possible. And if $B \nsubseteq D, B \nsubseteq C$ which means $D \subseteq B, C \subseteq B$, therefore C, D are not maximal and since B is not maximal either there exists $E \in A$ such that $D \subseteq B \subseteq E$, so D is subset of two intervals, a contradiction. Now let A_1 denote the set of non-maximal intervals that their upper endpoints are different from the upper endpoints of other elements of $\mathcal{F}$, and let A_2 denote the set of non-maximal intervals that their lower endpoints are different from the lower endpoints of other elements of $\mathcal{F}$, so $|A_1| + |A_2| \geq a$. This results that the upper endpoints of intervals in A and A_1 are distinct, so we have $|A| + |A_1| \leq n$, hence $m + |A_1| \leq n$. Similarly $m + |A_2| \leq n$, and since $|A_1| + |A_2| \geq a$, either $|A_1| \geq \frac{a}{2}$ or $|A_2| \geq \frac{a}{2}$, so $m + \frac{a}{2} \leq n$.

4. First note that $1 \notin A$ because if $1 \in A$ and a be the minimum element of $A - \{1\}$, then there must exists $b \in A, b \neq 1, a$ that $b|a$ and this means that $1 < b < a$ which is a contradiction.
   Now we prove that if $a, b \in A$ and $|A| \geq 4$, then $a \nmid b$. Assume that $a|b$, then for every $c \in A - \{a, b\}$ we must have

   $$(a, c) = (a, b, c) = 1$$

   now there exists $d \in A$ such that $d|ac$, we claim that $b|ac$, if $b \nmid ac$, since $d \in A - \{a, b\}$ and $(d, a) = 1$, hence $d|c$. Similarly there exists $e \in A$ that $e|ad$, if $e = b$ then

   $$b|ad|ac$$

   which is contradiction, so our claim is proved, therefore $e \in A - \{a, b\}$ so $e|d$, hence if $e = c$ then $c = d$ which can't be true, otherwise $e|d|c$ which means $(e, d, c) = e > 1$, a contradiction (note that the existence of e needs A to have at least 4 elements, for $|A| = 4$ there isn't such e and we lead to contradiction sooner) Thus $b|ac$ so if c and d be two elements of $A - \{a, b\}$ we have

   $$\left.\begin{array}{l} b|ac \\ b|ad \\ b|ab \end{array}\right\} \Rightarrow b|a(b, c, d) = a \Rightarrow b|a \Rightarrow a = b$$

   which is a contradiction. So assume that $|A| \geq 4$, in this case $a \in A$ exists such that for two pairs $\{b, c\}$ and $\{d, e\}$ we have

   $$a|bc, a|de$$

   because we can choose these pairs in $\binom{n}{2}$ ways, and if $n \geq 4$ we have $\binom{n}{2} > n$. Now if b, c, d, e be distinct, and let p be a prime factor of a, hence $p|d$ or $p|e$, suppose

$p|d$, similarly suppose that $p|c$, therefore $p|(a,c,d)$ which is a contradiction, so e.g we must have $b=d$, hence

$$\left.\begin{array}{l} a|bc \\ a|be \\ a|ba \end{array}\right\} \Rightarrow a|b(c,e,a)=b$$

which can't be true. Therefore $|A|\leq 3$, now assume that $A=\{a,b,c\}$ be good, so $(a,b,c)=1$, $a|bc$, $b|ac$ and $c|ab$, thus $a=pq$ such that $q|b$ and $p|c$. Let $b=qr$ and $c=pr'$, we have

$$c|ab \Rightarrow pr'|pq.qr \Rightarrow r'|q^2r$$

note that if r' and q have common factors, so do a,b,c unless $q=1$ and it means $a|c$, so $(a,b)=(a,b,c)=1$, now because $b|ac$ we have $b|c$ which means that a,b are two coprime factors of c, yet $c|ab$ so $c=ab$. Therefore one condition is that $c=ab$ and $(a,b=1$, if $q>1$ we must have $(r',q)=1$ so $r'|r$, similarly $r|r'$, so $r=r'$ which means that $a=pq$, $b=qr$ and $c=pr$, but in this situation we must have $(p,q,r)=1$ because if a prime number $s|(p,q,r)$ then $s|(a,b,c)$. Thus we all sets are of the form

$$A=\{a,b,ab\} \quad (a,b)=1 \qquad OR \qquad A=\{pq,qr,rp\} \quad (p,q,r)=1$$

5. Since $f(x)$ is surjective, there exists $t\in\mathbb{R}$ such that $f(t)=0$, now let

$$x=y=t \Rightarrow f(t)=2f(2t) \Rightarrow f(2t)=0 \quad (1)$$

$$\left.\begin{array}{l} x=0 \Rightarrow f(f(0)+2f(y))=f(0)+f(2y) \\ x=t \overset{(1)}{\Rightarrow} f(t+2f(y))=f(2y) \end{array}\right\} \Rightarrow \begin{array}{l} f(f(0)+2f(y)) \\ -f(t+2f(y)) \\ =f(0) \end{array}$$

but $2f(y)$ takes all real values, so for every $c\in\mathbb{R}$ we have

$$f(f(0)+c)-f(t+c)=f(0)$$

on the other hand

$$\begin{array}{rcccll} c=t & \overset{(1)}{\Rightarrow} & f(f(0)+t) & = & f(0) & (2) \\ c=0 & \Rightarrow & f(f(0)) & = & f(0) & (3) \\ c=f(0) & \overset{(2)}{\Rightarrow} & f(2f(0)) & = & 2f(0) & (4) \\ x=0,y=0 & \Rightarrow & f(3f(0)) & = & 2f(0) & (5) \end{array}$$

now by putting $x=0,y=f(0)$ in the first equation, we obtain from $(3),(4)$ that $f(3f(0))=3f(0)\overset{(5)}{\Rightarrow} f(0)=0$ now if we put $x=0$, it results that $f(2f(y))=f(2y)$, so $f(x+f(x)+2f(y))=f(2x)+f(2f(y))$. Since $2f(y)$ takes all values in $\mathbb{R}$ we can conclude that

$$f(x+f(x)+y)=f(2x)+f(y) \quad (6)$$

by putting $y=0$ we have

$$f(x+f(x)) = f(2x) \quad (7)$$

also putting $y = f(x+f(x)) = f(2x)$ in (6) results

$$f(x+f(x)+y) = f(x+f(x)+f(x+f(x))) = f(2x+2f(x))$$

$$= f(x+f(x)+(x+f(x))) = f(2x)+f(x+f(x)) = 2f(2x)$$

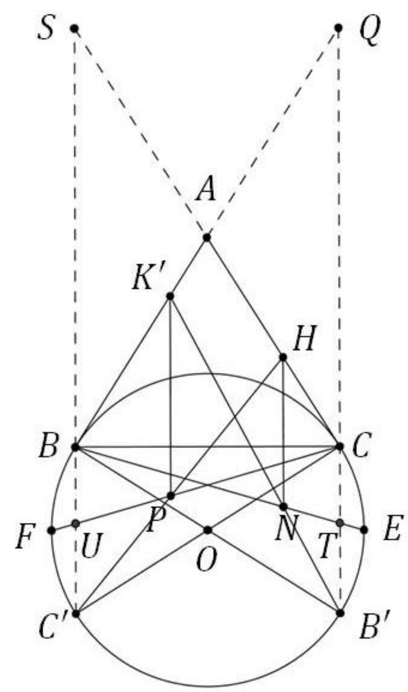
on the other hand

$$f(x+f(x)+y) = f(x+f(x)+f(2x)) = f(2x)+f(f(2x))$$

so $f(f(2x)) = f(2x)$, and since $f(2x)$ takes all real values we have

$$\forall a \in \mathbb{R} : f(a) = a$$

6. **Lemma.** In the next figure BB' and CC' are diagonals of ω and E and F are midpoints of arcs $B'C$ and BC' respectively. N is a point on BE and P is a point on CF such that $K'P$ and HN are perpendicular to BC. We prove that N lies on $K'B'$ if and only if P lies on HC'.



Proof of lemma. Using Menelaus' theorem in triangle BTQ and secant secant $K'B'$:

$$\frac{BK'}{K'Q}.\frac{QB'}{B'T}.\frac{TN}{NB} = 1$$

but $\frac{TN}{NB} = \frac{CH}{HS'}$, so

$$\frac{BK'}{K'Q}.\frac{CH}{HS} = \frac{B'T}{QB'}$$

similarly the condition that P lies on $C'H$, leads to

$$\frac{BK'}{K'Q}.\frac{CH}{HS} = \frac{C'U}{SC'}$$

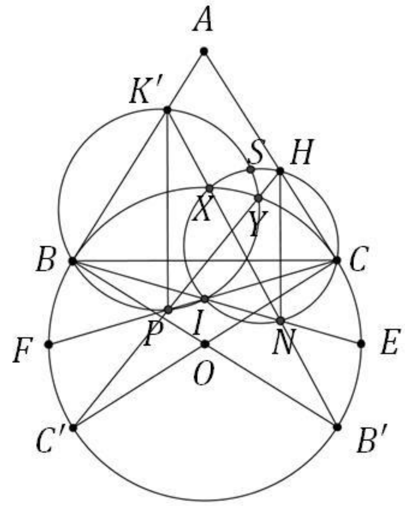
and considering the identity $\frac{B'T}{QB'} = \frac{C'U}{SC'}$ the lemma is proved.

Now, draw a perpendicular from H to BC such that intersects BE at N and $B'N$ intersects with circle at X and with AB at K'. HC' intersects line CF at P and circle at Y. According to above lemma, $K'P$ is perpendicular to BC. Also

$\angle CHN = \angle CXN = \angle CIN$, because all are equal to half of arc $B'C$. So the pentagon $XHCNI$ and similarly the pentagon $BPIYK'$ are cyclic. Suppose that J' is the second intersection point of circumcircles of these pentagons. We have

$$\angle J'HA = \angle J'IC = \angle J'YH$$

so circumcircle of $J'YH$ is tangent to AC. On the other hand we know that if a circle is tangent to AC at H and to ω at Y', then Y' is the inverse homothetic center of these two circles and since C' and H are homothetic so Y' should be on $C'H$, therefore $Y = Y'$. Hence this circle is exactly the circumcircle of $J'YH$. So circumcircle of $J'YH$ is tangent to ω. Similarly circumcircle of $K'XJ'$ is also tangent to AB and ω. We have

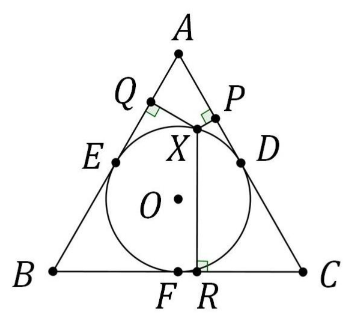


$$\angle K'J'H = \angle ACI + \angle ABI = 180° - \frac{\angle BAC}{2}$$

$$= \angle BIC = \angle AHJ' + \angle AK'J'$$

and since AH and AK' are tangent to circumcircles of $J'YH$ and $K'J'X$, it follows that circumcircles of $J'YH$ and $K'J'X$ are also tangent to each other. So circumcircle of $J'YH$ is ω_1 and circumcircle of $K'J'X$ is ω_2. Also $J' = J$ and $K' = K$ and $\angle IJH = 180° - \angle ICH = 180° - \angle IBK = \angle IJK$, so IJ is bisector.

7. We claim that the answer is the incircle. We have $\angle XDP + \angle XEQ = \frac{\overset{\frown}{DX}}{2} + \frac{\overset{\frown}{XE}}{2} = \frac{\angle DOE}{2} = 60°$ $\angle XFR - \angle XDP = \frac{\overset{\frown}{FD}+\overset{\frown}{DX}}{2} - \frac{\overset{\frown}{DX}}{2} = \frac{\overset{\frown}{FD}}{2} = \angle FOD = 60°$ so if $\angle XDP = \alpha$ we have $\angle XEQ = 60° - \alpha$ and $\angle XFR = 60° + \alpha$. According to the law of sines



$$\begin{array}{rclclcl} PX &=& XD\sin\alpha &=& (2r\sin\alpha)\sin\alpha &=& 2r\sin^2\alpha \\ RX &=& XF\sin\angle XFR &=& XF\sin(60+\alpha) &=& 2r\sin^2(60+\alpha) \\ QX &=& XE\sin\angle XEQ &=& XE\sin(60-\alpha) &=& 2r\sin^2(60-\alpha) \end{array}$$

$$\Rightarrow \sqrt{RX} = \sqrt{2r}\sin(60+\alpha) = \sqrt{2r}(\sin\alpha + \sin(60-\alpha)) = \sqrt{PX} + \sqrt{QX}$$

Now if this property holds for a point not on the incircle, without loss of generality we assume that the point is in the quadrilateral $AEOD$, let X' denote the point of intersection of AX with the incircle (closer to A), a homothety with center A and $\frac{AX}{AX'}$ as its coefficient, maps X', B, C to X, B', C' respectively, according to previous part since the incircle of $A'B'C'$ passes through X, we have $\sqrt{XR'} = \sqrt{XQ} + \sqrt{XP}$,

on the other hand we know that $\sqrt{XR} = \sqrt{XQ} + \sqrt{XP}$ which suffices that $XR' = XR$, hence X and X' coincide, which completes the proof.
Remark. For an arbitrary triangle ABC, the desired locus is an ellipse tangent to sides of the triangle.

8. We prove the first part by induction ,for $k = 1$ the problem is well-known. Now we add the constraint that $\deg(A_i) \le p-1$ for $i \le k-1$. Using division algorithm we can write

$$f(x) = (x^p - x)^k A_k(x) + g(x), \quad \deg(g) \le kp - 1$$

now the hypothesis of induction for $k-1$ on $g(x)$ results that

$$f(x) = (x^p - x)^k A_k(x) + (x^p - x)^{k-1} A_{k-1}(x) + \cdots + p^{k-1} A_0(x) \qquad \text{(I)}$$

$$\Rightarrow p|(\frac{x^p - x}{p})^{k-1} A_{k-1}(x) + (\frac{x^p - x}{p})^{k-2} A_{k-2}(x) + \cdots + A_0(x)$$

if we let $h(x) = \frac{x^p - x}{p}$, it is obvious that h has integer value but it does not have integer coefficients. We have

$$g(x + ap) \equiv \frac{(x + ap)^p - (x + ap)}{p} \equiv \frac{x^p - x - ap}{p} \equiv h(x) - a \pmod p$$

and $A_i(x + ap) \equiv A_i(x) \pmod p$. Hence $p|(h(x) - a)^{k-1} A_{k-1}(x) + \cdots$, so for a constant x, a takes all values module p, therefore

$$p|y^{k-1} A_{k-1}(x) + \cdots + A_0(x) \qquad \text{(II)}$$

now let x be fixed, so we have a polynomial in terms of y whose degree is at most $k - 1 \le p - 1$, thus its coefficients are divisible by p, hence $\forall x, i : A_i(x) \equiv 0 \pmod p$. Now since the degree of A_i is less than or equal to $p-1$, we have $A_i(x) = pB_i(x)$ (III) in which the coefficients of $B_i(x)$ are integer. By replacement of (III) in (I) the statement results.
For $k > p$ the counterexample $f(x) = (x^p - x)^{k-1} - p^{p-1}(x^p - x)^{k-1}$ works.

9. **lemma.** Suppose that we have a set of n points is the plane and G be their centroid, if a line l passes through G then sum of the signed distances of these n points from l is zero. (i.e. sum of the distances of points in each side of the line are equal)
proof of lemma. Let G be the origin of the plane and l be the x-axis, since G is the average of all of the points we have

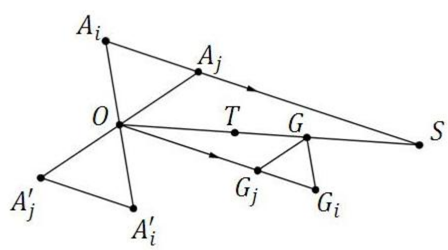
$$0 = y_G = \frac{y_{A_1} + y_{A_2} + \ldots + y_{A_n}}{n} \Rightarrow y_{A_1} + y_{A_2} + \ldots + y_{A_n} = 0$$

so the proof is complete.
Let $A = \{A_1, A_2, \ldots, A_n\}$ be the points. Now we define $n + 1$ points $G, G_1, \ldots, G_n$

in this way: let G be the centroid of these n points and assume that O be an arbitrary point in the plane, let A'_i be the reflection of A_i over O, we define G_i to be the centroid of $A_1, ..., A_{i-1}, A'_i, A_{i+1}, ..., A_n$. According to what we said, the line OG is a wise line for the point O, if we divide the points into two sets by OG. (we can assign the points on OG to each of the sets) We claim that OG_i is a wise line, if A'_i be in one side of OG_i with points $\{A_{i_1}, A_{i_2}, ..., A_{i_k}\}$, then let $X = \{A_i, A_{i_1}, A_{i_2}, ..., A_{i_k}\}$ be one of the sets and $Y = A - X$ be the other set, hence we have found $n+1$ wise lines. Now we have to prove that there exists infinite points like O that these lines are pairwise distinct for them.
Suppose that for a point O, G and G_i are collinear with O, we have

$$\left.\begin{array}{l} \vec{G} = \frac{\vec{A_1}+\vec{A_2}+...+\vec{A_n}}{n} \\ \vec{G_i} = \frac{\vec{A_1}+\vec{A_2}+...+\vec{A_n}-\vec{A_i}+\vec{A'_i}}{n} \end{array}\right\} \Rightarrow \vec{G} - \vec{G_i} = \frac{\vec{A_i} - \vec{A'_i}}{n} = \frac{2}{n}(\vec{A_i} - \vec{O})$$

this means that O, A_i and G are collinear, thus it is sufficient that O does not lie on lines $A_1G, A_2G, \cdots, A_nG$. Therefore the assumption that O,G and G_i are not collinear is correct.



Now, suppose that O,G_i and G_j are collinear, and $\overrightarrow{GG_i} = \frac{2}{n}\overrightarrow{A_iO}$. Hence the triangles GG_iG_j and OA_iA_j are similar with ratio $\frac{2}{n}$, and assume that the center of the homothety taking OA_iA_j to GG_iG_j is T. Now let the extension of A_iA_j intersects OG in S, thus $\frac{TG}{TO} = \frac{2}{n}$, $\frac{TO}{TS} = \frac{2}{n}$, therefore

$$\frac{GO}{GS} = \frac{GO/GT}{GS/GT} = \frac{n/2+1}{n^2/4-1} = \frac{1}{n/2-1} = \frac{2}{n-2}$$

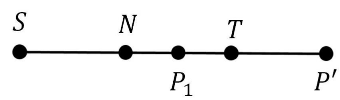
in this case a homothety with center G takes O to S and we have $\frac{OG}{GS} = \frac{2}{n-2}$. Therefore it is sufficient that O does not lie on the lines that are resulted by homothety over G with ratio $\frac{-2}{n-2}$ from the lines A_iA_j.

10. According to Cauchy-Schwartz inequality we can write

$$\begin{aligned} \sqrt{(a^2+1)(b^2+1)(c^2+1)} &= \sqrt{((a+b)^2+(ab-1)^2)(c^2+1)} \\ &\geq (a+b)c+ab-1 \\ \Rightarrow \sum_{cyc} \sqrt{(a^2+1)(b^2+1)(c^2+1)} &\geq 2\sum_{sym} ab - 4 \end{aligned}$$

and equality holds for $a = b = c = d = \sqrt{3}$, so the minimum value for k is 4.

11. **First Solution.** Let the homothety with center at G, centroid of ABC and $A'B'C'$, and ratio $\frac{-1}{2}$ maps P to P_1, then we have $\frac{P_1A'}{A'P'} = \frac{P_1B'}{B'P'} = \frac{P_1C'}{C'P'} = \frac{1}{2}$, so the circumcircle of $A'B'C'$ is the apollonius circle about P_1 and P' with ratio $\frac{1}{2}$ (i.e. the locus of the points X with the property that $\frac{XP_1}{XP'} = \frac{1}{2}$). Let N denote the center of this circle, and let $x = |P'P_1|$, we have (S and T are intersection points of the apollonius circle with PP')



$$\left.\begin{array}{l} P_1T = \frac{x}{3} \\ P_1S = x \end{array}\right\} \Rightarrow NS = \frac{P_1S + P_1T}{2} = \frac{2x}{3} \Rightarrow NP_1 = \frac{x}{3}, NP' = \frac{4x}{3}$$

note that $NT = \frac{2x}{3}$ is the radius of the apollonius circle, so x is constant, hence the locus of P_1 and P' are two circles with centers at N, on the other hand a homothety with respect to G takes P to P_1, so the locus of P is a circle centered at the circumcenter of ABC. Now since P' maps to P_1 with a homothety centered at N having ratio $\frac{1}{4}$, and P_1 maps to P with a homothety with ratio $\frac{-1}{2}$, by combining these two homotheties P' maps to P with a homothety with ratio $\frac{-1}{2}$, therefore PP' passes through the center of this homothety which is a fixed point.

Second Solution. Let G be the origin, we have (so by $\vec{K}$ we mean $\overrightarrow{GK}$)

$$\vec{A'} = \frac{-\vec{A}}{2}, \vec{B'} = \frac{-\vec{B}}{2}, \vec{C'} = \frac{-\vec{C}}{2}$$

the hypothesis results that

$$\begin{array}{lllllll} (\vec{P}-\vec{A})^2 & = & (\vec{P'}-\vec{A'})^2 & \Rightarrow & \vec{P}^2 + \vec{A}^2 - 2\vec{A}.\vec{P} & = & \vec{P'}^2 + \frac{\vec{A}^2}{4} + \vec{A}.\vec{P'} \quad (1) \\ (\vec{P}-\vec{B})^2 & = & (\vec{P'}-\vec{B'})^2 & \Rightarrow & \vec{P}^2 + \vec{B}^2 - 2\vec{B}.\vec{P} & = & \vec{P'}^2 + \frac{\vec{B}^2}{4} + \vec{B}.\vec{P'} \quad (2) \\ (\vec{P}-\vec{C})^2 & = & (\vec{P'}-\vec{C'})^2 & \Rightarrow & \vec{P}^2 + \vec{C}^2 - 2\vec{C}.\vec{P} & = & \vec{P'}^2 + \frac{\vec{C}^2}{4} + \vec{C}.\vec{P'} \quad (3) \end{array}$$

$$\begin{array}{lllll} (1),(2) & \Rightarrow & \frac{3}{4}(\vec{A}^2 - \vec{B}^2) & = & (2\vec{P} + \vec{P'}).\overrightarrow{BA} \\ (2),(3) & \Rightarrow & \frac{3}{4}(\vec{B}^2 - \vec{C}^2) & = & (2\vec{P} + \vec{P'}).\overrightarrow{CB} \end{array}$$

note that the vectors $\overrightarrow{BA}$ and $\overrightarrow{CB}$ are independent vectors and the dot product of them with $2\vec{P} + \vec{P'}$ is a fixed value, so $2\vec{P} + \vec{P'}$ is determined uniquely. Now let X be a point on PP' such that $\frac{XP'}{XP} = 2$, thus $\vec{X} = \frac{2\vec{P} + \vec{P'}}{3}$, hence X is the fixed point that PP' passes through.

12. First note that if p be an odd prime, putting $a = b = p$ results that $2p(p + f(p))$ is a perfect square, so $p | p + f(p)$ and $p | f(p)$. Now suppose that for $a \in \mathbb{N}$, $af(a)$ is not a perfect square, therefore for a prime p we have $p^{2k-1} \| af(a)$ that $k \in \mathbb{N}$. Let $af(a) = p^{2k-1}s$ where $s \in \mathbb{N}, p \nmid s$, now put $b = p^{2k}$ so $p^{2k-1}s + p^{2k}(f(p^{2k}) + 2a)$ is

a perfect square, which means $p^{2k-1}(s+p(f(p^{2k})+2a))$ is a perfect square but the term in the parenthesis does not have p factor which is a contradiction, so $af(a)$ is a perfect square. Now assume that $p > f(1)$ be an odd prime, since $p|f(p)$ so $p \leq f(p)$, hence $pf(p) + f(1) + 2p < (\sqrt{pf(p)} + 2)^2 \Leftrightarrow f(1) + 2p < 4\sqrt{pf(p)} + 4$, because $p \leq f(p)$ so $p \leq \sqrt{pf(p)}$, therefore $4\sqrt{pf(p)} \geq 4p > 2p + f(1)$. So we have

$$(\sqrt{pf(p)})^2 < pf(p) + f(1) + 2p < (\sqrt{pf(p)} + 2)^2$$

so $pf(p) + f(1) + 2p = (\sqrt{pf(p)} + 1)^2$, hence

$$f(1) + 2p = 2\sqrt{pf(p)} + 1$$

now since $p|f(p)$ and $pf(p)$ is a perfect square so $f(p) = k^2p$ that $k \in \mathbb{N}$. Now if $k \geq 2$ we have

$$3p > f(1) + 2p = 2\sqrt{pf(p)} + 1 \geq 4p + 1$$

so $f(p) = p$ for any odd prime p greater than $f(1)$, but above equation simultaneously proves that $f(1) = 1$, so for every odd prime p we have $f(p) = p$. Now suppose for $a \in \mathbb{N}$ we have $f(a) \neq a$, so ($b = p$ in which p is and odd prime) $(a+p)^2 \neq af(a) + p^2 + 2ap$, hence $[af(a) + p^2 + 2ap] - (a+p)^2 = af(a) - a^2$, but the absolute value of left side of the equation is greater than or equal to $2(a+p)-1$ because the difference of $(a+p)^2$ with any perfect square except itself is at least this value, so for any odd prime we have $|af(a) - a^2| \geq 2(a+p) - 1$ which can't be true for a constant a, so $f(a) = a$.