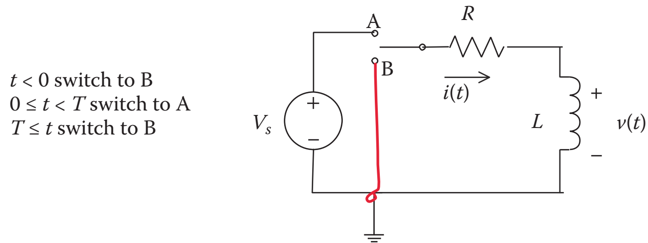

FIGURE 5.9 RL circuit charging and discharging.

Our next task is to see what happens to current $i(t)$ through an inductor as it charges. Refer to the circuit in Figure 5.9. You can see that the voltage source V_s produces a current through the resistor and inductor, and that the voltage drop across each one of these elements should add up to V_s by KVL. Denote the current by $i(t)$, and $v(t)$ the voltage across the coil. We can write $Ri(t) + v(t) = V_s$. Now use $v(t) = L\frac{di(t)}{dt}$ to obtain $L\frac{di(t)}{dt} + Ri(t) = V_s$ or equivalently $\frac{di(t)}{dt} + \frac{R}{L}i(t) = \frac{1}{L}V_s$, which is a first-order ODE. Define the quantity L/R as a *time constant* $\tau = L/R$. It has units of s when L is in H and R in Ω. Rewrite the ODE as

$$\frac{di(t)}{dt} + ai(t) = \frac{a}{R}V_s \tag{5.11}$$

where a is the inverse of the time constant $a = 1/\tau$.

We already know by analogy to the capacitor how to solve this ODE and find the unknown constants. The solution to this ODE is $i(t) = K_1 + K_2 \exp(-at)$ with unknowns K_1, K_2. Evaluate the solution at $t = 0$ to get $i(0) = K_1 + K_2$. For simplicity assume the inductor is discharged at $t = 0$, then $K_1 = -K_2$; evaluate the solution at $t = \infty$, that is, $v(\infty) = K_1$. After a long time the inductor charges to V_s/R and the voltage of the inductor becomes zero, that is, the inductor behaves as a short circuit, therefore $K_1 = V_s/R$ and consequently $K_2 = -V_s/R$. So we can write, $i(t) = \frac{V_s}{R}(1 - \exp(-at))$. Because $a = 1/\tau$, we can rewrite the solution as

$$i(t) = \frac{V_s}{R}(1 - \exp(-t/\tau)) \tag{5.12}$$

Finally, let us see what happens to current $i(t)$ as the inductor is discharging. Refer to the circuit in Figure 5.9. The inductor has been charged to V_s/R and then discharges completely to 0 A. The ODE of Equation 5.11 still holds but the right-hand side is 0. Therefore, the solution is simply $i(t) = K_2 \exp(-at)$. Evaluating at $t = 0$ we get $K_2 = V_s/R$, and therefore

$$i(t) = \frac{V_s}{R}\exp(-t/\tau) \tag{5.13}$$

Example 5.5

Consider the circuit in Figure 5.9 with $V_s = 12\ V$, $R = 20\ \cancel{k}\Omega$, $L = 1\ \cancel{m}H$. Calculate charging and discharge dynamics. Answer: The time constant is $\tau = \frac{L}{R} = \frac{1\cancel{m}H}{20\cancel{k}\Omega} = 0.05$ s. The charging dynamics is given by $\cancel{v}\,i(t) = \frac{12}{20}(1 - \exp(-t/0.05))\cancel{V}$ A. The discharge is $\cancel{v}\,i(t) = \frac{12}{20}\exp(-t/\cancel{0.2}\ 0.05)\cancel{V}$ A.

Similarly, as for the RC circuit, we can visualize these trajectories using function transient() of renpow:

```
R=20;L=1; # kohm and mH
transient(ys=12/R,tau=L/R,ylabel="iL(t) [A]",yslabel="Vs/R [V] A")
```

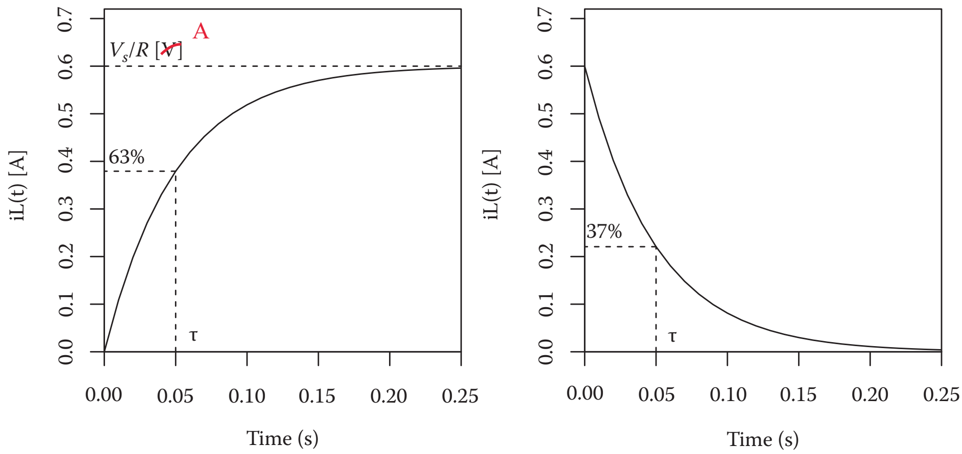

FIGURE 5.10 RL circuit transient response.

See the graphs of Figure 5.10. Note that as expected, after ~~1~~ 0.25s s (5 times the time constant), the ~~capacitor~~ coil is close to fully charged or fully discharged.

5.1.5 Combining Capacitors and Inductors

We can use Kirchhoff's current law (KCL) to calculate the equivalent capacitance of capacitors in parallel. Refer to Figure 5.11, the voltage $v(t)$ is common to both capacitors and the currents through each one, calculated from $i(t) = C\frac{dv(t)}{dt}$, add up to the total current; that is to say,

$$i(t) = i_1(t) + i_2(t) = C_1\frac{dv(t)}{dt} + C_2\frac{dv(t)}{dt} = (C_1 + C_2)\frac{dv(t)}{dt} = C\frac{dv(t)}{dt}$$

Therefore, the equivalent capacitance C is the sum of capacitances $C = C_1 + C_2$. This can easily be generalized to write the equivalent C of many, say N, capacitors as

$$C = C_1 + C_2 + \ldots + C_N \tag{5.14}$$

Now, consider the capacitors in series (Figure 5.11). We can use KVL to calculate the equivalent capacitance of capacitors in series. This time, the current $i(t)$ is common to both capacitors and the

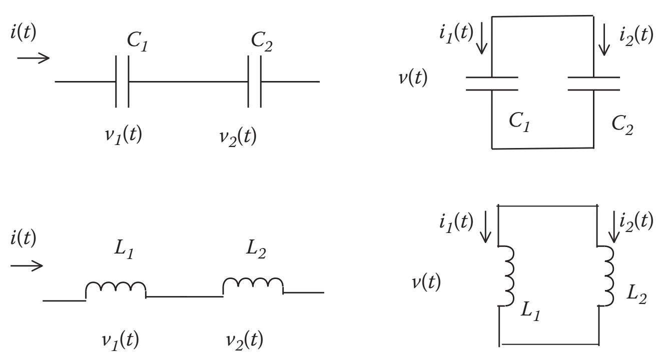

FIGURE 5.11 Capacitors and inductors in parallel and in series.

With a filter, it should be approximately 170 V. The peak-to-peak ripple is $V_{ripple} = \frac{I_{load}}{2\text{x} f_{ripple} \times C} \simeq \frac{170/(10 \times 10^3)}{60 \times 2 \times 100 \times 10^{-6}} = 1.42\text{V}$. As a percent the output is 1.42/170 = 0.83%.

5.4.2 DC-DC Converters

The simplest DC-DC converters are the *buck* or step-down type of converter shown in Figure 5.32 (left-hand side) and the *boost* or step-up type of converter (Figure 5.32, right-hand side). The transistor switch utilized in both circuits represents a device made from an *insulated gate bipolar transistor (IGBT)* and driven by a pulse waveform produced by *pulse width modulation (PWM)* (Figure 5.32, bottom).

The IGBT is a hybrid between a field effect and a bipolar transistor; the gate G is controlled by the voltage from the PWM. When the gate voltage is high enough to turn the transistor on, there is flow of large current between the collector C and the emitter E. Otherwise, the switch is off and there is negligible current from C to E. The periodic pulse waveform or signal is composed of a pulse in the on state that lasts a fraction D of the period T. For the remainder time $(1 - D)T$, the signal is in the off state. The switch receives this signal at its gate G. The switch closes when the signal is on, and it closes when the signal is off. The fraction D is called the *duty cycle* of the pulse wave.

Referring to the buck converter, when the switch is closed, the diode is an open circuit, and the input voltage drives a current through the load that obeys the differential equation $\frac{di(t)}{dt} + \frac{R}{L} i(t) = \frac{1}{L} V_{in}$ with solution $i(t) = \frac{V_s}{R}(1 - \exp(-t/\tau))$. Recall from Section 5.1.4 that $\tau = L/R$. When the switch opens, the current from the source is zero, but the current through the load will decrease according to $i(t) = \frac{V_s}{R}\exp(-(t - DT)/\tau)$. For a time constant much larger than the width DT, the current through the load will approximate a triangular wave, whereas the current from the source is triangular when the pulse is on and zero when the pulse is off.

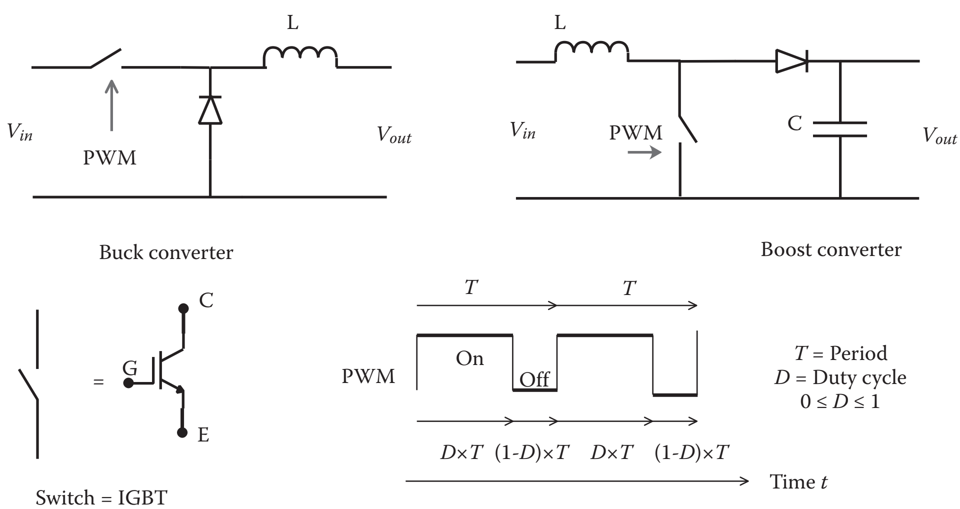

FIGURE 5.32 DC-DC converters: buck converter and boost converter. PWM signal.

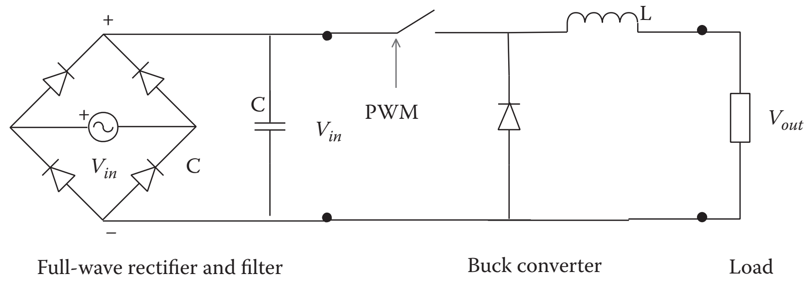

FIGURE 5.33 Switching power supply. V_{in} AC and V_{out} DC with value controlled by the duty cycle.

The average input current is $\langle I_{out}\rangle = \frac{1}{D}\langle I_{in}\rangle$ where the angular braces denote time average and therefore we must have $\langle V_{out}\rangle = DV_{in}$ in order for the output power to be approximately the same as input power (minus some losses) $V_{in}\ \langle I_{in}\rangle \simeq \langle I_{out}\rangle\langle V_{out}\rangle$. In other words, the output voltage is a fraction D of the input voltage and the output current increases by the reciprocal $\approx 1/D$.

For a boost converter, when the switch is closed the input source charges the coil. When the switch opens, the coil injects its current to the capacitor charging it. Then when the switch closes again, the capacitor holds the charge, which keeps building up, thus achieving a higher voltage at the output. In this case $\langle V_{out}\rangle = \frac{1}{1-D}V_{in}$ and the current is $\langle I_{out}\rangle = (1-D)\langle I_{in}\rangle$.

Example 5.16

What would be the duty cycle for a buck converter to step down the voltage from 15 to 10 V? What would be the duty cycle for a boost converter to step up from 12 to 24 V? Answer: For a buck converter $\langle V_{out}\rangle = DV_{in}$, thus $D = \langle V_{out}\rangle/V_{in} = 10/15 = 0.66$ or the pulse will be on 66.66% of the period. For a boost converter $\langle V_{out}\rangle = \frac{1}{1-D}V_{in}$, then $D = 1 - \frac{V_{in}}{\langle V_{out}\rangle} = 1 - \frac{12}{24} = 0.5$ or the pulse should be on 50% of the period.

5.4.3 Switching Power Supply

A full-wave rectifier and a buck converter are used together to make a power supply (Figure 5.33). For instance, we may take AC from the consumer side of the grid (170 V amplitude and 60 Hz), rectify and filter to DC 170 V with little ripple, and then convert to 24 V using the buck converter set with appropriate value of D. The switching power supply is very common nowadays in many electronic devices; for instance, it is the basic component of AC-DC adapters or of a battery charger.

EXERCISES

5.1. What is the required path length to make an air-gapped inductor with reluctance 4 MA-turns/Wb using an iron core of 2 mm diameter and relative permeability to 1000? How many turns would be required to make a 10 mH inductor with this core?

5.2. What is the capacitance required to store 10 mJ of energy when a capacitor is fully charged at 10 V?

5.3. Consider the circuit in Figure 5.7 with $V_s = 12$ V, $R = 200$ kΩ. What value of capacitance yields a discharge of $v(t) = 12\exp(-t/0.1)$V?

5.4. Consider the circuit in Figure 5.9 with $V_s = 12$ V, $R = 10\cancel{k}\Omega$. What value of inductance yields charging dynamics $\cancel{v}(t) = 1.2(1-\exp(-t/0.1))\cancel{V}$?
i

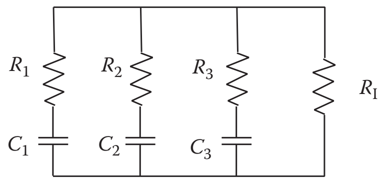

FIGURE 5.34 Supercapacitor bank.

5.5. Figure 5.34 illustrates a model for one cell of an ultracapacitor or supercapacitor. A module of the "ultracap" is made of N identical cells. For energy storage applications, it is of interest to know how quickly the supercap charges and discharges. Rl = 1 KΩ is leakage. Assume R1 = 500 μΩ, C1 = 3 kF, R2 = 1 Ω, C2 = 0.3 kF, R3 = 3 Ω, C3 = 0.6 kF. Calculate the time constant τ of each one of the three RC branches. Which one is the slow τ branch? Which one is the medium τ branch? Which one is the fast τ branch? Assuming the cell is fully charged to 12 V, what amount of energy is stored in one cell? Assuming N = 10 cells in a module, how much energy is stored in a module if all cells are fully charged? Ignore leakage resistance

5.6. Assume f = 60 Hz, write time domain of voltage wave with V_m = 170 V and consider two phase angles, 0° and 90°. Write these voltages as phasors (frequency domain). Draw phasor diagrams.

5.7. An inductor is driven by a 60 Hz AC current modeled by a sine wave $i(t) = I_m \cos(\omega t)$. Determine the voltage. What is the phase difference between voltage and current? Draw the time functions for voltage and current. Why do we say that voltage "leads" the current?

5.8. Consider a 4-pole generator. How fast (in rpm) do you have to rotate the shaft to get a 60 Hz wave? Use renpow to plot one cycle of single phase 60 Hz sine wave voltage.

5.9. Using the renpow complex number functions calculate $\mathbf{I} = \dfrac{\mathbf{I}_1}{\mathbf{I}_1 + \mathbf{I}_2}$ given ~~voltages~~ currents $\mathbf{I}_1$ = $15\angle 45°$ and $\mathbf{I}_2 = 10\angle 30°$.

5.10. Determine the phase angle between a voltage $v(t) = 170\cos(377t + 45°)$ V and a current $i(t) = 60\cos(377t + 25°)$ A. Which leads which? Write them as phasors. Draw a time-domain plot and a phasor plot.

5.11. Suppose R = 1 kΩ, C = 100 μF, and L = 1 mH. What is the impedance of each element at f = 60 Hz?

5.12. Suppose the input voltage to a full bridge rectifier is 170 V amplitude at 60 Hz and the load is 1 kΩ. How large should the filter capacitor be to have less than 1% ripple?

5.13. What would be the duty cycle for a buck converter to step down the voltage from 24 to 12 V? What would be the duty cycle for a boost converter to step up from 10 to 15 V?

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