$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} -j0.03 & j0.02 & j0.01 \\ j0.02 & -j0.02 & 0 \\ j0.01 & 0 & -j0.01 \end{bmatrix} \begin{bmatrix} 4200\angle 10° \\ 3800\angle 20° \\ 3600\angle 0° \end{bmatrix}$$

and we can calculate using R:

```
Y <- matrix(ncol=3, nrow=3)
Y[1,1] <- 0-3i; Y[1,2] <- 0+2i; Y[1,3] <- 0+i 0+1i
Y[2,1] <- Y[1,2]; Y[2,2] <- 0-2i; Y[2,3] <- 0+0i
Y[3,1] <- Y[1,3]; Y[3,2] <- Y[2,3]; Y[3,3] <- 0-i 0-1i
Y <- Y*0.01
Vm <- c(4.2,3.8,3.6)*1000 #kV
Va <- c(10,20,0)
V <- complex(mod=Vm, arg=Va*pi/180)
I <- Y%*%V # matrix multiplication
Im <- c(Mod(I))
Ia <- c(Arg(I)*180/pi) # convert to polar
```

We can then print the admittance, voltage, and current:

```
> print(Y);print(Vm);print(Va);print(round(Im,1)); print(round(Ia,1))
      [,1]    [,2]    [,3]
[1,] 0.00-0.03i 0+0.02i 0.03+0i 0.00+0.01i
[2,] 0.00+0.02i 0-0.02i 0.00+0i
0.00+0.01i[3,] 0.03+0.00i 0+0.00i -0.03+0i 0.00-0.01i
[1] 4200 3800 3600
[1] 10 20  0
[1] 116.5 16.1 27.2   17.2 16.1 9.1
[1] -26.9 44.7 53.7   -103.9 44.7 143.7
> >
```

17.2, 16.1, and 9.1 A

We see that the currents injected to the buses have magnitudes ~~116.5, 16.1, 27.2A~~ with angles ~~26.9°, 44.7°, 53.7°.~~

-103.9°, 44.7°, and 143.7°.

11.7 BASICS OF PER UNIT (P.U.) SYSTEM

The *per unit* (p.u.) expression relates to a ratio of a voltage, current, power, or impedance to a chosen *base* quantity. For example, if the selected base is 120 kV, then a voltage of 108 kV is expressed as $\frac{108\,\text{kV}}{120\,\text{kV}} = 0.9\,\text{p.u.}$ This concept is useful in power systems dealing with large quantities, for instance kV and kW. A base in kW or kVA is the result of base voltage in kV and base current in A.

In three-phase, the use of p.u. results in the same line and phase voltages. For example, suppose we select a base apparent total power of 30,000 kVA and a base line-to-line voltage V_l of 120 kV. Note that dividing by 3, the base apparent power per phase is 10,000 kVA. Suppose we actually have V_l = 108 kV and total S = 18,000 kVA. Then, $Vl = \frac{108}{120} = 0.9\,\text{p.u.}$ But the line-to-neutral in p.u. would be $V_{an} = \frac{108/\sqrt{3}}{120/\sqrt{3}} = 0.9\,\text{p.u.}$ because the $\sqrt{3}$ cancels. Therefore, we see that the line and phase are the same in p.u. when the system is balanced. The apparent power in p.u. is

$S = \frac{18,000}{30,000} = 0.6$ p.u. and the apparent power per phase is $S = \frac{18,000/3}{30,000/3} = 0.6$ p.u. We can see that total apparent power and apparent power per phase is the same when the system is balanced.

11.8 POWER FLOW

11.8.1 Matrix Equation

Complex *power flow* into a bus is determined by the bus voltage and injected current. This current is determined by the bus admittance $\mathbf{I} = \mathbf{YV}$. Thus, complex power is $\mathbf{S} = \mathbf{VI}^* = \mathbf{V}(\mathbf{YV})^* = \mathbf{VY}^*\mathbf{V}^*$. For each bus i the power is

$$\mathbf{S}_i = \mathbf{V}_i \sum_j \mathbf{Y}_{ji}^* \mathbf{V}_j^* \tag{11.2}$$

which will also be possible to write as

$$\mathbf{S}_i = P_i + jQ_i \tag{11.3}$$

Separating real and imaginary parts in Equation 11.2 and equating to their counterparts in Equation 11.3 constitutes the power flow equations. Note that in Equation 11.2, variables are magnitude and angle of voltage, and real and reactive power of the bus. Also, note that power in Equation 11.2 can be partitioned as $\mathbf{S}_i = \mathbf{V}_i \mathbf{Y}_{ii}^* \mathbf{V}_i^* + \mathbf{V}_i \sum_{i \neq j} \mathbf{Y}_{ij}^* \mathbf{V}_j^*$, as we did the current separating the terms related to the mutual admittances. The power flow problem consists of solving Equation 11.2 combined with Equation 11.3 subject to restrictions on the voltages, currents, and power for each bus. The solution is found numerically and is computationally challenging for networks composed of thousands of buses. In this textbook, we will only cover simple examples of a few buses.

Example 11.4

Use base 1 kV, ~~0.01 Ω,~~ 1 A, 1 ~~k~~MVA to express variables in p.u. The bus admittance from the previous example ~~on a p.u. to~~ is $\mathbf{Y} = \begin{bmatrix} -j3 & j2 & j1 \\ j2 & -j2 & 0 \\ j1 & 0 & -j1 \end{bmatrix}$ ×0.01 S and voltages $\mathbf{V} = \begin{bmatrix} 4.2\angle 10^\circ \\ 3.8\angle 20^\circ \\ 3.6\angle 0^\circ \end{bmatrix}$ on p.u. Calculate the complex power of all three buses. Answer: The matrix equation is

```
Y <- matrix(ncol=3, nrow=3)
Y[1,1] <- 0-3i; Y[1,2] <- 0+2i; Y[1,3] <- 0+1i
Y[2,1] <- Y[1,2]; Y[2,2] <- 0-2i; Y[2,3] <- 0+0i
Y[3,1] <- Y[1,3]; Y[3,2] <- Y[2,3]; Y[3,3] <- 0-1i
Y<- Y*0.01
# base 1 kV 1MW 1 MVA
Vm <- c(4.2,3.8,3.6)
Va <- c(10,20,0)
V <- complex(mod=Vm, arg=Va*pi/180)
S <- V*Conj(Y)%*%Conj(V) # matrix multiplication
```

~~round(S,1)~~
~~> round(S,1)~~
~~[,1]~~
~~[1,] -2.9+6.6i~~
~~[2,] 5.5-2.6i~~
~~[3,] -2.6-1.9i~~
~~>~~

```
> round(S,3)
        [,1]
[1,] -0.029+0.066i
[2,]  0.055-0.026i
[3,] -0.026-0.019i
>
```

$\begin{bmatrix} -0.029 \\ 0.055 \\ -0.026 \end{bmatrix}$ in p.u. or MW $\qquad \begin{bmatrix} 0.066 \\ -0.026 \\ -0.019 \end{bmatrix}$ in p.u. or MVAR

We have obtained real power $\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \cancel{\begin{bmatrix} -2.9 \\ 5.5 \\ -2.6 \end{bmatrix}}$ and reactive power $\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \cancel{\begin{bmatrix} 6.6 \\ -2.6 \\ 1.9 \end{bmatrix}}$. There is import of real power at bus 2 and export at bus 1 and 3; ~~vice versa~~, bus ~~2 exports~~ 1 imports reactive power and the other two buses ~~import~~ export it.

11.8.2 Power Flow for a Two-Bus System

Consider an extremely simple case of two buses connected by a short line with impedance $\mathbf{Z}_1 = jX$ or admittance $\mathbf{Y}_1 = -jX^{-1}$. The bus admittance matrix is $\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_{11} & \mathbf{Y}_{12} \\ \mathbf{Y}_{21} & \mathbf{Y}_{22} \end{bmatrix} = \begin{bmatrix} -jX^{-1} & jX^{-1} \\ jX^{-1} & -jX^{-1} \end{bmatrix}$ and then we write the power flow equations

$$\mathbf{S}_1 = \mathbf{V}_1\mathbf{Y}_{11}^*\mathbf{V}_1^* + \mathbf{V}_1\mathbf{Y}_{12}^*\mathbf{V}_2^*$$
$$\mathbf{S}_2 = \mathbf{V}_2\mathbf{Y}^*\mathbf{V}_1^* + \mathbf{V}_2\mathbf{Y}_{22}^*\mathbf{V}_2^*$$

Substitute admittance $\mathbf{Y}_{ii}$ according to the matrix and assume $\mathbf{V}_i = V_i \angle \theta_i$ for $i = 1, 2$

$$\mathbf{S}_1 = jX^{-1}V_1^2 - jX^{-1}V_1(\cos\theta_1 + j\sin\theta_1)V_2(\cos\theta_2 - j\sin\theta_2)$$
$$\mathbf{S}_2 = jX^{-1}V_2^2 - jX^{-1}V_1(\cos\theta_1 - j\sin\theta_1)V_2(\cos\theta_2 + j\sin\theta_2).$$

Expand

$$\mathbf{S}_1 = jX^{-1}V_1^2 - jX^{-1}V_1V_2[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + j(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)]$$
$$\mathbf{S}_2 = jX^{-1}V_2^2 - jX^{-1}V_1V_2[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + j(-\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)].$$

Collect real and imaginary terms in each equation

$$\mathbf{S}_1 = X^{-1}V_1V_2(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2) + j\left[X^{-1}V_1^2 - jX^{-1}V_1V_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)\right]$$
$$\mathbf{S}_2 = -X^{-1}V_1V_2(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2) + j\left[X^{-1}V_2^2 - jX^{-1}V_1V_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)\right]$$

For each equation, use Equation 11.3; make the real part equal to P_i and the imaginary part equal to Q_i:

$$\begin{aligned} &X^{-1}V_1V_2(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2) = P_1 \\ &X^{-1}V_1{}^2 - X^{-1}V_1V_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) = Q_1 \\ -&X^{-1}V_1V_2(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2) = P_2 \\ &X^{-1}V_2{}^2 - X^{-1}V_1V_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) = Q_2 \end{aligned} \tag{11.4}$$

Observe that $P_1 = -P_2 = X^{-1}V_1V_2(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)$ and that each equation can be partitioned into a term $X^{-1}V_i^2$ determined by the main diagonal terms and a term determined by the sum of mutual conductance terms $-X^{-1}V_1V_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$. Rewrite Equation 11.4 as

$$\begin{aligned} &X^{-1}V_1V_2(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2) = P_1 = -P_2 \\ &X^{-1}V_1[V_1 - V_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)] = Q_1 \\ &X^{-1}V_2[V_2 - V_1(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)] = Q_2 \end{aligned} \tag{11.5}$$

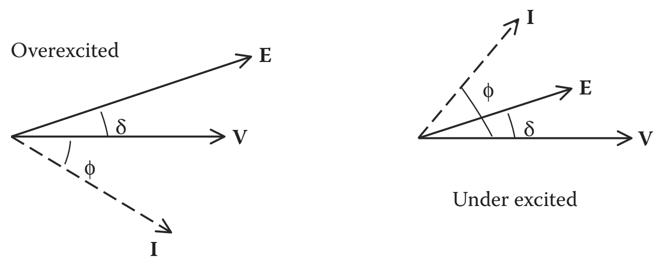

FIGURE 11.15 Phasor diagram of infinite bus conditions.

For a given E, the critical value δ_{cr} of the angle δ at which we change mode is found from $E\cos\delta_{cr} = V$ to get $\delta_{cr} = \cos^{-1}(V/E)$. For values $\delta < \delta_{cr}$, the generator is in overexcited mode, whereas for values $>$ $\delta \cancel{<} \delta_{cr}$, the generator is in underexcited mode. For a given δ, the critical value E_{cr} of E, in which there is change mode is found from $E_{cr}\cos\delta = V$ to get $E_{cr} = V/\cos\delta$. For values $E > E_{cr}$, the generator is in overexcited mode, whereas for values $E > E_{cr}$, the generator is in underexcited mode.

Example 11.7

Assume infinite bus voltage V = 13.8 kV and consider two cases: (a) We increase the field current to achieve $E = 15$ kV. What is the critical δ? (b) We increase the torque of the prime mover to achieve $\delta = 10°$. What is the critical E? Answer: (a) $\delta_{cr} = \cos^{-1}(V/E) = \cos^{-1}(13.8/15) = 23.073°$, we cannot get past this angle to stay in overexcited mode. (b) $E_{cr} = V/\cos\delta = 13.8/\cos(10°) = 14.01$ kV, we have to increase the field current to at least this value to stay in overexcited mode.

Of interest is to calculate the power flow or power at the sending (generator) and receiving (bus) ends. We will work with the conjugate of complex power because in this case it is easier to see the conjugate of voltage than the conjugate of current. In general for a voltage **V** and a current **I**, complex power is $\mathbf{S} = \mathbf{VI}^* = P + jQ$ and the conjugate of complex power is $\mathbf{S}^* = \mathbf{V}^*\mathbf{I} = P - jQ$. Note, that calculating the conjugate of complex power yields an opposite sign of reactive power.

The conjugate of complex power at the generator is $\mathbf{S}_1^* = \mathbf{E}^*\mathbf{I} = P_1 - jQ_1$.

$$\mathbf{S}_1{*} = \mathbf{E}{*}\mathbf{I} = (E(\cos\delta - j\sin\delta))\left(\frac{E(\cos\delta + \mathrm{j}\sin\delta) - V}{jX}\right) = \left(\frac{E^2}{jX}\right) - \left(\frac{E(\cos\delta - j\sin\delta)V}{jX}\right)$$

therefore,

$$\mathbf{S}_1{*} = \left(-j\frac{E^2}{X}\right) - \left(\frac{EVj\cos\delta - EV\sin\delta}{X}\right) = P_1 - \mathrm{j}Q_1$$

The real part is $P_1 = \dfrac{EV}{X}\sin\delta$ and the imaginary part is $Q_1 = \dfrac{E^2}{X} - \dfrac{EV}{X}\cos\delta$. Likewise, the conjugate of complex power at the infinite bus is $\mathbf{S}_2^* = \mathbf{V}^*\mathbf{I} = P_2 - jQ_2$

$$\mathbf{S}_2{*} = \mathbf{V}{*}\mathbf{I} = (V + j0)\left(\frac{E(\cos\delta + j\sin\delta) - V}{jX}\right) = \frac{EV}{X}(\sin\delta - j\cos\delta) + \left(jV^2\right) = P_2 - jQ_2,$$

the real part is $P_2 = \dfrac{EV}{X}\sin\delta$ and the imaginary part is $Q_2 = \dfrac{EV}{X}\cos\delta - \dfrac{V^2}{X}$.